Exit Dilemma*

The Role of Private Learning on Firm Survival

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Abstract

We study the exit decision of duopolists from a stochastically declining market. Over time, firms privately learn about the market conditions from observing the stochastic arrival of customers. Exit decisions are publicly observed; thus the model features both observational and private learning. A larger firm is more likely to have customers and hence has better information about the market conditions than does a smaller rival. We provide sufficient conditions for either the smaller or the larger firm to be the first to exit the market in the unique equilibrium. Because of observational learning, exiting may be a firm's dominant action since continuing operation would bring too much good news to the rival, leading it to further postpone its exit. Uniqueness then follows from iterated conditional dominance.

Keywords: Duopoly, Exit, Private Learning, War of Attrition

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1 Introduction

Divestment is an important component of a firm's strategy,[1] and contributes to shaping the evolution of the market structure of many industries, one of the central objects of study in industrial organization. Furthermore, confidential business information has been recognized to play a key role in market competition by, for example, the Federal Trade Commission.[2] We study the role of private learning in a strategic exit model, which has thus far received little attention. Specifically, our paper sets up a parsimonious model which can contribute to explaining why the type of firm exiting varies from industry to industry.

The empirical evidence on exit from a declining industry has shown that the relationship between firm size and exit patterns varies across industries and may depend on industry-specific characteristics. Some studies have found higher rates of exit for small firms (see, for example, Lieberman, 1990). Other studies have documented that in mature stages of the industry life cycle, and particularly in technically advanced industries, smaller-scale firms are not necessarily confronted with a lower likelihood of survival than their larger counterparts (see Agarwal and Audretsch, 2001).

The theoretical literature has modeled strategic exit from a declining industry using the war of attrition paradigm, predicting that a stronger firm can force a weaker firm to exit first. Starting with the seminal contributions of Ghemawat and Nalebuff (1985), Fudenberg and Tirole (1986), and Fine and Li (1989) (see also Murto, 2004), most papers have identified a firm's strength with its profit flow: a firm is stronger than its competitor if its profit flow is greater.

To the best of our knowledge, no existing model explains why in some cases, the firm that survives the industry decline has the lowest profit flow. However, when a firm privately learns about the profitability of the industry, for example, from sales data, its strategy—whether to exit—conveys information to its competitor. As a result, a firm's relative strength is determined not only by its profit flow but also by the information externalities generated by its actions.

[1] A study by Accenture (Anslinger et al., 2003) predicted that "for the next years, many companies will give far more thought to divestitures than they did in the late 1990s."

[2] See, for example, the FTC order designed to remedy the anti-competitive effects resulting from Broadcom Limited's acquisition of Brocade Communications Systems, https://www.ftc.gov/enforcement/cases-proceedings/171-0027/broadcom-limitedbrocade-communications-systems

To model dynamic selection in a declining industry, we consider an irreversible timing game. Initially, each duopolist earns positive expected profits; the industry randomly, and unbeknownst to either firm, transitions to a declining stage of its life cycle. In this declining stage, the duopoly loses viability because the expected profit of each duopolist becomes negative. Firms privately learn about the profitability of the industry by observing their customer arrivals. We focus on the case in which firms are asymmetric in that they have different customer arrival rates and hence learn at different speeds. Since customer arrivals are privately observed and exit decisions are public, the model features both private and observational learning, akin to an incomplete information war of attrition.

As a first step, we show that there always exists an equilibrium in which one of the two firms exits first with probability one. To determine which firm survives, we consider each firm's best reply to the other firm never exiting. In our model, the first exit time when playing this best reply[3] determines a firm's strength. The smaller the first exit time is, the weaker the firm. There always exists an equilibrium in which the weaker firm exits first with probability one, acting solely on the basis of its private information, and never learns anything from the stronger firm.

There is a non-monotone relationship between the speed of private learning—or, equivalently, the firm's expected profit flow, a proxy for size—and the firm's strength in the war of attrition—or, equivalently, its first exit time. The non-monotone relationship arises because of two countervailing forces. On the one hand, the higher the customer arrival rate is, the faster the firm becomes pessimistic about the market conditions if no customer arrives. On the other hand, the higher the customer arrival rate is, the higher the expected profit flow, and the stronger the incentive to remain in the market for any belief regarding the state of the industry.

Intuitively, the stronger firm has incentives to wait for the news revealed at the weaker firm's first exit time. In addition, if the weaker firm does not exit at that time, the stronger firm's incentive to remain in the market is reinforced. The weaker firm, by staying in the market, sends a signal that is against its own interest and that makes the stronger firm more optimistic about the market conditions—however, in equilibrium, it cannot avoid doing so. As a result, the weaker firm is discouraged from remaining in the market longer compared to the case in which it expects never to enjoy monopoly profits nor to benefit from observing the other firm's action.

[3]The first exit time is the earliest time when the firm exits with positive probability along the path induced by the best reply.

We then provide sufficient conditions for the equilibrium in which the weaker firm exits first to be the unique equilibrium of the game. In light of the non-monotonicity between a firm's customer arrival rate and its strength, in the unique equilibrium, either the larger or the smaller firm survives the industry decline. As both predictions have received some empirical support, our model sheds light on how industry characteristics can affect the equilibrium outcome. Roughly, if there is a high degree of uncertainty, i.e., both firms have little information about the market conditions, then the smaller firm exits first with probability one; if the larger firm has (sufficiently) precise information, it exits first with probability one.

Technically, our proof of equilibrium uniqueness relies on iterated deletion of (conditionally) dominated strategies à la Shimoji and Watson (1998). In principle, to identify dominated strategies, one needs to compute the beliefs of a firm for any given strategy of the rival. This computation proves to be difficult in our setup because higher-order beliefs play a key role, not only because a firm needs to forecast its opponent's action but also because firms' private signals are correlated. Standard techniques do not apply: since the underlying state of the world evolves over time, it is not possible to simplify the dynamic inference problem by decomposing the posterior belief into two single-dimensional statistics, i.e., a private belief and a public belief, such as in Foster and Viswanathan (1996) or Rosenberg, Solan, and Vieille (2007), as discussed in Section 4.2.

Our approach is radically different; we provide a recursive lower bound on a firm's posterior belief about the prevailing state in any equilibrium of the game. We believe that this approach can be applied to other models with private learning or private monitoring. In a first step, we compute a lower bound on the stronger firm's posterior belief conditional on the weaker firm not using a strictly dominated strategy. In a second step, we use this lower bound to identify an initial interval of time when continuing operations is a conditionally dominant strategy for the stronger firm, irrespective of its private history. In a third step, we show that exiting is initially dominant for the weaker firm whenever the time elapsed since last observing a customer is sufficiently long.

Our model relies on the canonical exponential bandit framework with inconclusive good news. In our proof of equilibrium uniqueness, we start by showing that in the special case of conclusive news, that is, the case in which no customer arrives in the declining phase of the industry, the stronger firm's inference problem exhibits a recursive structure, which allows us to generalize the dominance argument to countably many rounds of deletion, concluding that the stronger firm never exits first in equilibrium.

Inconclusive news disrupts the recursive properties of the inference problem. Our proof relies instead on a limit argument. In the limit, as the weaker firm's information becomes arbitrarily precise, along any history, the stronger firm bases its inference only on observational learning, allowing us to conclude that there exists a unique equilibrium. Moreover, we show that introducing asymmetry in other dimensions such as the discount rate, cost, or revenue does not change our main results.

1.1 Literature Review

Our work is closely related to the theoretical literature analyzing exit through the lens of the war of attrition paradigm. Ghemawat and Nalebuff (1985, 1990) study disinvestment in declining industries when demand shrinks deterministically over time. Applying a backward induction argument, Ghemawat and Nalebuff (1985) show that in the unique equilibrium the larger firm exits first because it is unable to adjust capacity and loses viability more quickly: in their model, the smaller firm is "stronger" in that when left alone, it remains profitable for longer. To say it differently, in Ghemawat and Nalebuff (1985), the smaller firm enjoys a larger profit flow. Murto (2004) shows that the main insights of Ghemawat and Nalebuff (1985) carry over to the case of stochastic market decline and a general payoff structure,[4] and there always exists an equilibrium in which the firm with the lowest expected profit flow exits first. In contrast to our model, in Murto (2004) signals about the underlying uncertainty, which is modeled as a geometric Brownian motion, are public, precluding signaling effects.

Incomplete information in a war of attrition has been studied in Fudenberg and Tirole (1986), who characterize the unique equilibrium of the exit game when firms have private information about their outside option or their cost. In contrast to Fudenberg and Tirole (1986), in our model information is interdependent; thus, higher-order beliefs are relevant not only to form beliefs about the strategy of the opponent but also to assess the prevailing state of the world. Takahashi (2015) empirically estimates the model of Fudenberg and Tirole (1986) using the US movie

[4] See also the discrete-time model of Fine and Li (1989).

theater industry to quantify the welfare loss from strategic delay. In the same spirit, our paper provides a tractable framework to estimate the welfare implications of observational learning in the presence of interdependent private information.

Within the broader literature on stopping games, our paper is closely related to Rosenberg, Solan, and Vieille (2007) and Murto and Välimäki (2011). While these papers are concerned with different questions, they feature, similar to ours, observational learning and irreversible action, but they do not incorporate payoff externalities. In contrast, Hopenhayn and Squintani (2011) and Gorno and Iachan (2020) feature independent private information and payoff externalities.

It is well known that in non-zero-sum games, receiving additional information can be to a player's advantage but also to his detriment (see, for example, Maschler et al., 2013, Ch. 5.7). Recently, Awaya and Krishna (2021) show that information can be a strategic disadvantage in an R&D race with private information. Under some parametric restrictions, in the unique equilibrium, the better-informed firm exits the race more frequently and has lower payoffs.[5] Relatedly, Moscarini and Squintani (2010) investigate the role of private information in a winner-take-all R&D race in a model that, barring the technical details of the information structure, is identical to that of Awaya and Krishna (2021). In Awaya and Krishna (2021), conditionally on the innovation being unfeasible, rivalry disappears as neither of the firms benefits from remaining in the race. In contrast, our model can account for the case in which a monopolist always enjoys positive profits. We show that even in this case, the firm with the largest profit flow and the more precise information is the first to exit in the unique equilibrium.

In our model, firms are not only asymmetric in their information, but also in their payoffs. As in canonical models of market experimentation, customers bring both information and revenues; this assumption is intended to capture a positive correlation between a firm's quality of information and its profit flow. The question is whether the strategic disadvantage generated by having more precise information can outweigh the benefit from a higher payoff and thereby overturn the prediction in the existing literature (e.g., Ghemawat and Nalebuff, 1985 and Murto, 2004), that is, that the firm with the lowest profit flow exits first. We address this question

[5] Similarly, Chen and Ishida (2021) study an asymmetric war of attrition with independent types and dynamic private learning and show that in some equilibria the less-efficient firm wins more often. See also Kim and Lee (2014), who study the effect of information acquisition in a war of attrition.

by identifying the appropriate notion of strength and showing that the relationship between strength and the speed of private learning is non-monotone, specifically, it is single-peaked.

Our model is also related to the literature on strategic experimentation with exponential bandits. As in Keller and Rady (2010), firms learn via inconclusive good news, but as in Keller and Rady (1999, 2003), the underlying state of the world changes over time.[6] Chen (2020), Rosenberg, Salomon, and Vieille (2013) and Heidhues, Rady, and Strack (2015), who analyze experimentation models with private learning, are also related to our work.

1.2 Structure of the Paper

The remainder of the paper is organized as follows. Section 2 describes the model. Section 3 analyzes two public learning benchmarks, the case of observable decline and the case of publicly observable customers, and gives a preview of the main results. Section 4 is devoted to the main results. Section 5 concludes. All proofs are provided in the Appendix and Online Appendix.

2 Model

Time is continuous and the horizon is infinite, $t \in [0, \infty)$. Two firms decide when to irreversibly exit a declining industry. Each firm's present discounted payoff from exiting the industry is normalized to 0.

The industry's profitability is determined by a state of the world ω_t that can be either good or bad, $\omega_t \in \{G, B\}$. Initially, both firms attach probability one to the industry being profitable, $\omega_0 = G$. The industry irreversibly becomes unprofitable, unbeknownst to the two firms, at some random time that is exponentially distributed with parameter $\gamma > 0$.[7]

Each active firm serves a stream of randomly arriving customers. In a duopoly, that is, as long as both firms are active, the customers of firm i arrive according

[6] See also Khromenkova (2018).

[7] Our model generalizes to the case of $\gamma = 0$ and interior prior about the state of the world ω_0 and to the case in which the transition to the bad state is not irreversible, provided that the transition rate from the bad to the good state is low enough. Moreover, the assumption that the distribution of the time when the state transitions is exponential is convenient but not essential. The results can easily be generalized, for example, to any distribution with a bounded hazard rate.

to an inhomogeneous Poisson process with intensity $\lambda^i_{\omega_t}$, where $\lambda^i_G > \lambda^i_B \geq 0$. In a monopoly, that is, after firm j exits, the customers of firm i arrive at a rate $\lambda^1_{\omega_t} + \lambda^2_{\omega_t}$. Each firm privately observes its customer arrivals, while exit decisions are public.

While active, each firm bears a flow cost c, and each customer yields a lump-sum revenue R. Firms discount the future at a common rate $r > 0$. We impose the following parametric assumptions.[8]

Assumption 1. *For* $i = 1, 2$, $\lambda^i_B R - c < 0$.

Assumption 1 states that a duopolist's flow payoff is negative whenever $\omega_t = B$: this ensures that it can be optimal to exit. Furthermore, we assume that $\lambda^1_B = \lambda^2_B = \lambda_B$, even if the assumption is not crucial for the main results, as discussed in Section 4.4.

After one of the firms has exited, the remaining firm enjoys monopoly profits until it also finds it optimal to exit, if ever. In fact, our model can accommodate both the case in which a monopolist's profit is always profitable and the case in which a monopolist's profit is negative whenever $\omega_t = B$, that is, $2\lambda_B R - c \lesseqgtr 0$.

Given a strategy profile (σ^1, σ^2), the payoff of firm i can be written as:

$$\mathbf{E}^{(\sigma^1,\sigma^2)}\left[\int_0^{\sigma^i} e^{-rt}\left(\lambda^i_{\omega_t} R + \mathbf{1}_{\{\sigma^j < t\}}\lambda^j_{\omega_t} R - c\right) \mathrm{d}t\right]. \tag{1}$$

We focus on perfect Bayesian equilibria of the stochastic timing game. However, we do not explicitly specify beliefs and behavior off the equilibrium path because they play no role in sustaining on-path behavior, as argued in the Supplementary Appendix.[9]

As in canonical bandit models (e.g., Rothschild, 1974), customers bring revenue and information. Interpreting the customer arrival rate as a proxy for the size of the firm, conceptually, our results rely on the positive correlation between the expected profit of a firm and the precision of its information. For example, one could argue that a larger firm enjoys higher profits because of higher markups and economies of

[8] We could allow the flow cost to depend on the presence of a competitor, as long as the expected profit conditional on the state is strictly higher in a monopoly than that in a duopoly. For a discussion on asymmetries, see Section 4.4.

[9] In the Supplementary Appendix of Cetemen and Margaria (2021), we show that any Nash equilibrium is outcome-equivalent to a perfect Bayesian equilibrium.

scale,[10] and that it has more precise information as a result of a more sophisticated market research department.

3 Benchmarks

To place our results into perspective, we start by discussing two benchmarks. In the first benchmark, the state transition is observable. In the second, firms do not observe the state transition, but they observe one anothers' customers.

3.1 Observable Decline

To make the problem interesting, we assume that as long as the state is good, i.e., $\omega_t = G$, it is dominant for both firms to remain in the market. That is, we assume that $\lambda_G^i > c/R$ for both firms, $i = 1, 2$. If $2\lambda_B R - c > 0$,[11] the continuation game after the state transitions, is a standard war of attrition, as in Hendricks et al. (1988), which is known to have a multiplicity of equilibria.

Specifically, it has two pure strategy asymmetric equilibria. In each of them, one of the firms exits as soon as the state transitions while the other firm never exits. There exists a mixed strategy symmetric equilibrium in which both firms exit at the constant rate

$$-r\frac{\lambda_B R - c}{2\lambda_B R - c}$$

such that the cost of waiting, $c - \lambda_B R$, equals the benefit, $\varsigma\left(2\lambda_B R - c\right)/r$, where ς is the equilibrium exit rate of each firm.

Hence, a model with observable state transition is silent about how industry characteristics affect the likelihood that the smaller or the larger firm survives the industry decline.[12] In contrast, with private learning we can give a partial answer to the question, under our parametric restrictions the equilibrium is unique, and we can identify conditions under which either the smaller or the larger firm exits first.

[10] The empirical literature finds higher markups for larger firms, e.g., De Loecker and Warzynski (2012), Edmond et al. (2018), and Boar and Midrigan (2021).

[11] Trivially, if $2\lambda_B R - c \leq 0$, in equilibrium, firms exit as soon as the state transitions.

[12] The mixed strategy equilibrium in the continuation game after the state transition is symmetric because the arrival rate of consumers of the two firms, conditional on $\omega = B$, is assumed identical that is, $\lambda_B^1 = \lambda_B^2 = \lambda_B$. If instead $\lambda_B^1 > \lambda_B^2$, firm 1 exits at a higher rate as compared to firm 2.

3.2 Public Learning

If firms observe one another's customers, the model is closely related to the model of Murto (2004). Despite the difference in the stochastic process governing the underlying state of the world, the main insights of Murto (2004) carry over to our setup, as formalized by the following proposition.

Proposition 1. *If both firms observe one another's customers and* $\lambda_G^2 > \lambda_G^1 > c/R$,[13] *there always exists an equilibrium in which firm 2 never exits first. If* λ_G^1 *is low enough,* R *is high enough (or* c *is low enough), and* $r > \lambda_G^1 + \lambda_G^2 - 2\lambda_B$*, this is the unique Nash equilibrium outcome.*

As in Murto (2004), whenever the game has a unique equilibrium, the firm with the larger profit flow forces the firm with the smaller profit flow to exit first.[14] In other words, a firm's "strength" is monotone in its profit flow: the larger λ_G^i is, the longer the firm is willing to remain in the market. When instead there exist multiple equilibria and, in particular, there exists an equilibrium in which the smaller firm never exits first, there also exists a mixed strategy equilibrium.

As we discuss in the Appendix, in the mixed strategy equilibrium, the firm with the higher customer arrival rate exits with positive probability at a certain belief; for lower beliefs, both firms exit at a positive rate. In equilibrium, the rate at which a firm exits makes the opponent indifferent between exiting and remaining in the market. Consequently, as in a nondegenerate equilibrium of a complete information war of attrition, the firm with the larger customer arrival rate has a lower probability of survival.

It has been argued that it is odd that in the mixed strategy equilibrium, the firm with the smaller customer arrival rate (i.e., the higher cost of fighting) wins more frequently. Hence, when multiple equilibria exist, the most realistic equilibrium seems to be that in which the firm with the larger customer arrival rate never exits first. (See, for example, Kornhauser et al., 1989.) In a sense, our model provides a

[13] We discuss the bound $\lambda_G^1 > c/R$ in Lemma 1 below.

[14] Two remarks in order. First, Murto (2004) demonstrates uniqueness within the class of Markov equilibria, while we provide sufficient conditions for uniqueness of a Nash equilibrium in the cases of both public learning and private learning (see Theorem 2). Second, in Murto (2004), a firm is stronger if it finds it profitable to remain in business longer in the monopoly position; consequently, a firm is stronger if it enjoys greater profits. Within the Ghemawat and Nalebuff's (1985) framework, the firm with the larger capacity is weaker because it is unable to adjust its operating cost.

rationale for the larger firm to concede more often in equilibrium: interdependent values and observational learning.

3.3 Private Learning: A Preview

Before plunging into the analysis of the main model, we provide an overview of the main result. The next section provides sufficient conditions for the game with private learning to have a unique equilibrium in which one of the firms exits first with probability one. The gist of this result can be easily understood when the arrival of a customer provides conclusive evidence that the industry is still profitable, i.e., $\lambda_B = 0$, focusing on the case of extreme asymmetry in arrival rates between the two firms. First, it is clear that regardless of the strategy used by the firm with a small customer arrival rate, the information conveyed by its behavior is of little value to the competitor with a larger arrival rate, because of the asymmetry in the precision of their signals. Second, the only reason why the larger firm may want to wait for the smaller firm to exit first is to enjoy monopoly profits. But there always exists a history along which the firm with a larger customer arrival rate has not observed any customers for some time, and hence is arbitrarily pessimistic about the state of the industry; because the rate of arrival of customers once the state transitions is zero, the value of becoming a monopolist is then nil. As a result, exiting is a dominant strategy at that history. Informally, a contagion argument across histories implies that in the unique equilibrium the firm with the large arrival rate of customers always exits first.

The crux of the paper is nailing down the main forces driving the result. As it turns out, it is the observational learning dynamics brought about by private learning that generates equilibrium uniqueness. On the one hand, we provide conditions for equilibrium uniqueness even when being a monopolist is always profitable, so that the second mover always find it profitable to remain into business. On the other hand, as a larger rate of arrival of customers not only entails more precise information but also larger revenues, it may be the smaller firm that finds it dominant to exit in the unique equilibrium of the game.

4 The Private Learning Game

This section presents our main results. We first introduce a notion of "strength" in our war of attrition game by means of analyzing a specific best-reply problem. Second, we construct two candidate equilibria. Third, we provide sufficient conditions for each of them to be the unique strategy profile that survives iterated deletion of (conditionally) dominated strategies, implying equilibrium uniqueness. Last, we discuss the main forces at play and discuss the insights that can be derived from our results.

4.1 A Best-Reply Problem

Suppose that firm j adopts the strategy of never exiting the market, and consider the best-reply problem of firm $i \neq j$. The problem of firm i can be written as a standard optimal stopping problem:

$$\sup_{\tau} \mathbf{E}\left[\int_0^{\tau} e^{-rt}\left(\lambda^i_{\omega_t} R - c\right) \mathrm{d}t\right].$$

The problem of firm i is Markov in its posterior belief about the prevailing state and the best response takes a simple form: it prescribes exiting as soon as the posterior belief falls below some cutoff $\pi^*(\lambda^i_G)$. Define $\tau^*(\lambda^i_G)$ as the earliest time firm i exits with positive probability along the path induced by the best-reply strategy, that is,[15]

$$\Pr\left[\omega_{\tau^*(\lambda^i_G)} = G \mid N^i_{\tau^*(\lambda^i_G)} = 0\right] = \pi^*(\lambda^i_G).$$

In other words, along the history with no customers, firm i exits at time $\tau^*(\lambda^i_G)$. (Recall that N^i_t denotes the inhomogeneous Poisson process of customer arrivals of firm i.)

In the special case of conclusive news, i.e., $\lambda_B = 0$, $\tau^*(\lambda^i_G)$ fully characterizes the best reply of firm i. Because the posterior belief about the prevailing state jumps to one whenever the firm observes a customer, the best reply prescribes exiting as soon as no customers have been observed for an uninterrupted amount of time of length $\tau^*(\lambda^i_G)$.

[15] Note the slight abuse of notation, as the first exit time $\tau^*(\lambda^i_G)$ does in fact depend on the other parameters of the model, but for convenience, we omit this dependence.

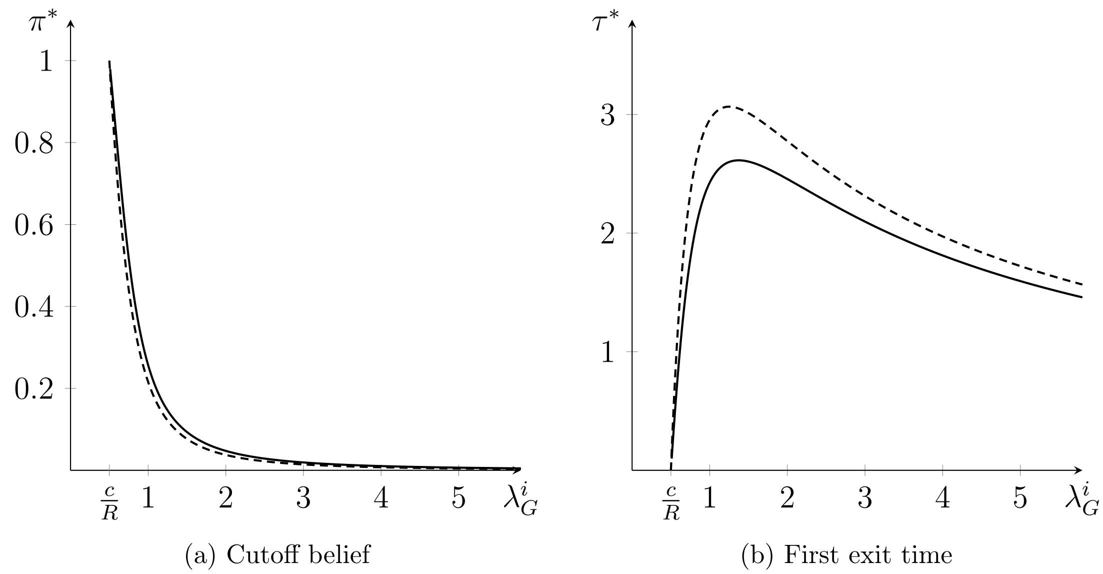


Figure 1: Best reply for $(c, R, r, \gamma) = (1/2, 1, 1/10, 1/5)$. The solid line indicates the case of conclusive news, $\lambda_B = 0$. The dashed line indicates a case of inconclusive news, $\lambda_B = 1/5$.

The following lemma characterizes how $\pi^*(\lambda_G^i)$ and $\tau^*(\lambda_G^i)$ change with λ_G^i.

Lemma 1.

(i) $\pi^*(\lambda_G^i) < 1$ *if and only if* $\lambda_G^i R > c$.

(ii) $\tau^*(\lambda_G^i)$ *is single-peaked, and* $\lim_{\lambda_G^i \to \infty} \tau^*(\lambda_G^i) = 0$.

The non-monotonicity, illustrated in the right panel of Figure 1, is due to two countervailing forces. On the one hand, the higher λ_G^i is, the higher the marginal benefit from remaining in the market at any given belief. In fact, the cutoff belief $\pi^*(\lambda_G^i)$ is decreasing in λ_G^i, as in the left panel of Figure 1. On the other hand, the higher λ_G^i is, the faster the firm becomes pessimistic about the market conditions.[16] This observation is at the core of our main results.

In contrast to the case of privately observed customers, when learning is public, the ranking of firms' first exit times coincides with the ranking of their rates of arrival of customers, that is, with their profit flow: the higher the customer arrival rate, the larger the first exit time. The single-peakedness in Lemma 1 relies on the

[16] The result is reminiscent of Halac, Kartik, and Liu (2016) and Bobtcheff and Levy (2017). Conceptually, our result generalizes theirs to a setup with inconclusive news and changing state.

fact that as λ_G^i increases, both the profit flow and the speed of learning increase.[17] With observable customers, the second force plays a limited role, in that firms learn at the same speed so changing the rate of arrival of customers of one firm is akin to a change in the discount rate, and there always exists an equilibrium in which the firm with the higher customer flow never exits first.

To simplify the exposition of our results, we say that firm i is stronger than firm $j \neq i$ if the first exit time of firm i is larger than the first exit time of firm j, that is, $\tau^*(\lambda_G^i) > \tau^*(\lambda_G^j)$. In the next section, we show that there always exists an equilibrium in which the stronger firm survives the industry decline, that is, the weaker firm exits first with probability one.

In line with standard attrition games, strength captures a firm's willingness to endure low profits: a firm is stronger if it can outlast its rival. The caveat is that, while in a war of attrition game, a stronger player also has a lower fighting cost, in our model because of the non-monotonicity stated in Lemma 1, the firm with the higher profit flow may not be the stronger firm.

4.2 A Pure Strategy Equilibrium

We now demonstrate the existence of an equilibrium in which the weaker firm exits first with probability one. The section proceeds through a sequence of observations, and the main result is formalized in Theorem 1 below.

Since $\pi^*(\lambda_G^i)$ is the optimal exit cutoff under the most pessimistic scenario in which the other firm never exits first, in equilibrium, exiting when the posterior belief is larger than $\pi^*(\lambda_G^i)$ is a dominated strategy, as formalized below.

Lemma 2. *In any equilibrium, if firm i exits with positive probability at time t along some history in which firm j is still active, then*

$$\Pr[\omega_t = G \mid (N_s^i)_{s \leq t}, \sigma^j \geq t] \leq \pi^*(\lambda_G^i).$$

In any equilibrium, a firm's belief about the underlying state of the world evolves because of private and observational learning. In light of existing results, such as those of Rosenberg et al. (2007), one may expect to be able to decompose a

[17] It can also be shown that if one normalizes the lump-sum payoff such that the expected flow is independent of the rate of arrival of news, the first exit time monotonically decreases in the learning speed. Details are available upon request.

player's posterior belief into two single-dimensional statistics, i.e., a private belief and a public belief. Unfortunately, there exist no single-dimensional statistics that, combined with the private belief, yield the posterior belief about the prevailing state of the industry.

Intuitively, because the state is not perfectly persistent, the fundamental uncertainty concerns not only whether the industry has already become unprofitable but also when that occured. In fact, conditional on the prevailing state being bad, i.e., $\omega_t = B$, players' private signals are correlated, making the standard decomposition technique inapplicable.

As a result, the relevant information for firm i cannot be summarized by its private belief about the prevailing state and the status of the other firm (active or not). The second-order belief of firm i, that is, its distribution over the private belief firm j, affects firm's i posterior about the current state of the world in a more complicated manner than in Rosenberg et al. (2007).[18]

Our first main result identifies an equilibrium in which firms' inference problems are uncomplicated. A class of strategies that generate a simple inference problem are cutoff strategies. According to a cutoff strategy, for some measurable function $p_t : [0, \infty) \to [0, 1]$, a firm exits with probability one at the first-passage time of its posterior belief under p_t. For any $p > 0$, let σ^i_p be the pure strategy according to which firm i adopts a time-independent cutoff p. Let σ^i_0 be the strategy that prescribes never exiting.

Theorem 1. *Fix firm 1's arrival rate of customers, λ^1_G. There exists a $\overline{\lambda}^2_G > \lambda^1_G$ such that for $(\lambda^1_G, \lambda^2_G)$, $\lambda^2_G > \overline{\lambda}^2_G$, there exists an equilibrium in which firm 2 (the larger firm) exits first with probability one, i.e., $(\sigma^1_0, \sigma^2_{\pi^*(\lambda^2_G)})$ is an equilibrium.*

In words, if the arrival rate of firm 2 is high enough, firm 1 is the stronger one, and survives the industry decline. The theorem provides sufficient conditions for the existence of an equilibrium in which firm 2 exits first with probability one. However, in the case of conclusive news, we can prove a stronger result: for any pair $(\lambda^1_G, \lambda^2_G)$ there always exists an equilibrium in which the weaker firm exits first with probability one. That is, either $(\sigma^1_0, \sigma^2_{\pi^*(\lambda^2_G)})$ or $(\sigma^1_{\pi^*(\lambda^1_G)}, \sigma^2_0)$ is an equilibrium,

[18] Cisternas and Kolb (2020) show that in a private monitoring setup, the second-order beliefs can be decomposed in a similar vein. However, in their setup, the first- and second-order beliefs are determined by a finite-dimensional sufficient statistic.

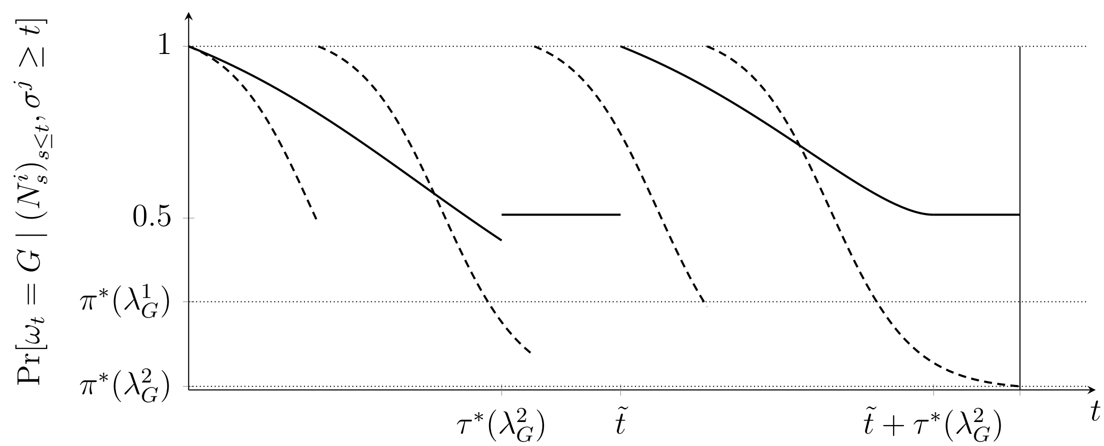


Figure 2: Example of equilibrium belief trajectories for $(c, R, r, \gamma, , \lambda^1_G, \lambda^2_G, \lambda_B) = (1/2, 1, 1/10, 1/5, 1, 4, 0)$. The solid line is the belief trajectory of firm 1. The dashed line is the belief trajectory of firm 2. The vertical line demarcates the time at which firm 2 exits.

depending on whether $\tau^*(\lambda^1_G) \gtrless \tau^*(\lambda^2_G)$. Hence, it also guarantees the existence of equilibrium in the case of conclusive news.

To gain some insight into the learning dynamics, Figure 2 illustrates a possible realization of belief paths for the equilibrium in Theorem 1 under the assumption that $\lambda^2_G > \lambda^1_G > \lambda_B = 0$ and firm 1 is the stronger firm, that is, $\tau^*(\lambda^1_G) > \tau^*(\lambda^2_G)$. First, note that in equilibrium, firm 2 never benefits from observational learning: the evolution of firm 2's belief at any point in time is uniquely driven by private learning. Second, in the interval $[0, \tau^*(\lambda^2_G))$, observational learning also plays no role for firm 1. In fact, no firm is supposed to exit, and both base their assessment of the market profitability on their private signals only.

In the equilibrium outcome illustrated in Figure 2, firm 2 does not exit at $\tau^*(\lambda^2_G)$ because it observes a customer before that time. At $\tau^*(\lambda^2_G)$, firm 1's belief jumps upward because firm 2 not exiting reveals that it has observed at least a customer in $[0, \tau^*(\lambda^2_G))$. In the example, firm 1 does not observe any customer in $[0, \tilde{t})$. As a result, at any $t \in [\tau^*(\lambda^2_G), \tilde{t})$, the belief of firm 1 about the prevailing state of the world, as well as its second-order belief, are constant.[19] (See also Figure 3.) Firm 2 not exiting at some $t \in [\tau^*(\lambda^2_G), \tilde{t})$ reveals that firm 2 has observed its last customer no earlier than $t - \tau^*(\lambda^2_G)$; otherwise, its belief would have fallen below

[19]The first- and second-order beliefs of firm 1 would not be constant in the case of inconclusive news, $\lambda_B \neq 0$.

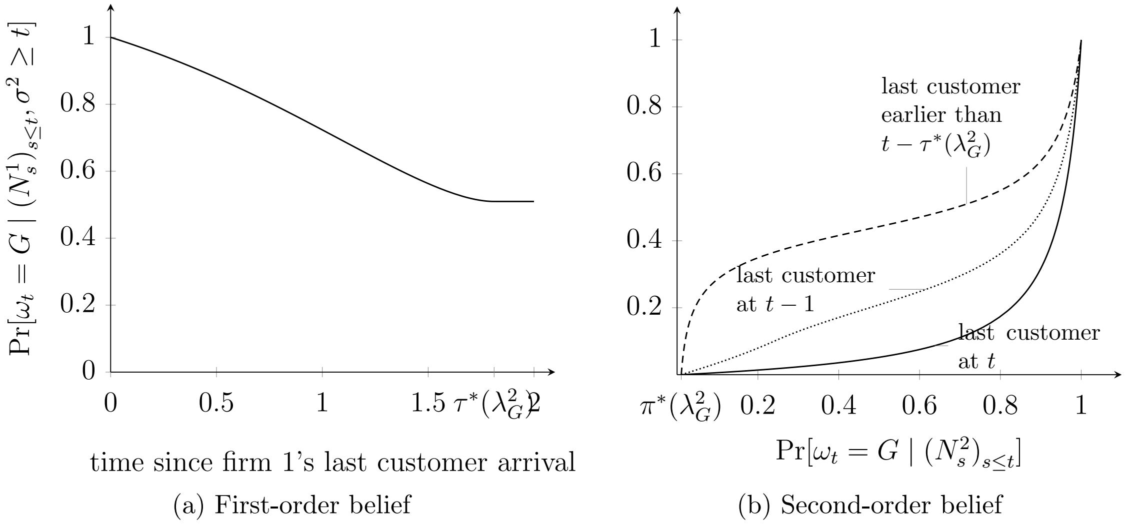


Figure 3: The first- and second- order belief of firm 1 in the equilibrium $(\sigma^1_0, \sigma^2_{\pi^*(\lambda^2_G)})$ for $(c, R, r, \gamma, \lambda^1_G, \lambda^2_G, \lambda_B) = (1/2, 1, 1/10, 1/5, 1, 4, 0)$. On the left, the posterior about the prevailing state at some time $t > \tau^*(\lambda^2_G)$ as a function of the time elapsed since last observing a customer. On the right, firm 1's distribution over firm 2's posterior belief at some $t > \tau^*(\lambda^2_G)$.

$\pi^*(\lambda^2_G)$ at some time before t. Hence, firm 2 not exiting at $t \in [\tau^*(\lambda^2_G), \tilde{t})$ also reveals that $\omega_{t-\tau^*(\lambda^2_G)} = G$. Consequently, from the perspective of view of firm 1, the lack of customers in $[0, t - \tau^*(\lambda^2_G))$ becomes irrelevant as far as its belief about the prevailing state is concerned. Intuitively, firm 1 knows that firm 2 has "fresher" news and can discard part of the information contained in its private history. This explains why the first-order belief in the left-panel in Figure 3 eventually plateaus.

Moreover, in the outcome shown in Figure 2, as soon as firm 2 exits, firm 1 follows suit. As in Moscarini and Squintani (2010), the equilibrium may display exit waves because information is revealed in a burst when a firm exits. In contrast, information is revealed gradually at any point in time after the first exit time of the weaker firm, as long as both firms remain in the market.

4.3 Equilibrium Uniqueness

In general, the equilibrium identified in Section 4.2 is not the unique equilibrium. For example, for the parameters in Figure 2, both $(\sigma^1_0, \sigma^2_{\pi^*(\lambda^2_G)})$ and $(\sigma^1_{\pi^*(\lambda^1_G)}, \sigma^2_0)$ are equilibria of the game, that is, if firms' customer arrival rates are sufficiently similar, there also exists an equilibrium in which the firm with the larger first exit time exits first with probability one. In fact, if firm 1 exits as soon as its private belief falls below the benchmark cutoff belief $\pi^*(\lambda_G^1)$, i.e., plays the strategy $\sigma^1_{\pi^*(\lambda_G^1)}$, firm 2 has incentives to remain in business at any point in time and along any history before firm 1's exit because its continuation payoff is strictly positive. In other words, firm 2, anticipating that firm 1 will eventually exit, is willing to remain in the market at beliefs below the cutoff $\pi^*(\lambda_2^G)$.[20]

When payoff externalities are absent, in light of Rosenberg et al. (2007) , it is natural to expect all the equilibria to be in cutoff strategies. The non-monotonicity of the continuation payoff (see Footnote 20) as well as the results by Murto (2004) suggest that this is not true in our setup. It is easy to construct simple mixed-strategy equilibria. For example, for some parameters, the following strategy profile is an equilibrium. Firm 2 exits with positive probability at time $\tau^*(\lambda_G^2)$ whenever $N^2_{\tau^*(\lambda_G^2)} = 0$. If it does not exit at that time, it never exits thereafter. Firm 1 adopts the strategy $\sigma^1_{\pi^*(\lambda_G^1)}$. The probability with which firm 2 exits is chosen such that firm 1's best reply to it makes firm 2 indifferent between exiting at $\tau^*(\lambda_G^2)$ and never exiting ever along the history with no customers.

Nevertheless, we are able to show that under appropriate conditions the game has a unique Nash equilibrium outcome, specifically, there exists a unique strategy profile that survives iterated deletion of conditionally dominated strategies.

Theorem 2.A. *Assume that $2\lambda_B R - c \leq 0$. In the case of both conclusive and inconclusive news, i.e., $\lambda_B \geq 0$, for any λ_G^1, there exists a $\overline{\lambda}_G^2 > \lambda_G^1$ such that for $(\lambda_G^1, \lambda_G^2)$, $\lambda_G^2 > \overline{\lambda}_G^2$, $(\sigma_0^1, \sigma^2_{\pi^*(\lambda_G^2)})$ is the unique strategy profile that survives iterated deletion of conditionally dominated strategies; hence, in the unique Nash equilibrium outcome, the larger firm exits first (with probability one).*

Theorem 2.B. *In the case of conclusive news, i.e., $\lambda_B = 0$,*[21] *there exists an open set of pairs $(\lambda_G^1, \lambda_G^2)$, $\lambda_G^2 < \lambda_G^1$, under which $(\sigma_0^1, \sigma^2_{\pi^*(\lambda_G^2)})$ is the unique strategy profile that survives iterated deletion of conditionally dominated strategies, provided that R is high enough (or c is low enough) and that r and γ are high enough; hence, in the unique Nash equilibrium outcome, the smaller firm exits first (with probability one).*

[20]Interestingly, however, on the path induced by the equilibrium $(\sigma^1_{\pi^*(\lambda_G^1)}, \sigma_0^2)$, before $\tau^*(\lambda_G^1)$, the continuation payoff of firm 2 is sometimes non-monotone in its posterior belief. In fact, as illustrated in Figure 3, the customer arrivals observed by a firm affect its second-order belief and hence the expected exit time of the rival.

[21]The last paragraph in Section 4.3.1 discusses the role of the assumption $\lambda_B = 0$, which was not needed in Theorem 2.A.

Intuitively, the lower $\tau^*(\lambda_G^2)$ is, the more influential firm 2's action, that is, the stronger the inference drawn by firm 1 by observing firm 2 not exiting the market. In other words, the lower $\tau^*(\lambda_G^2)$ is, the more firm 1 benefits from observational learning. Because observing firm 2 not exiting brings good news, this strengthens incentives for firm 1 to remain in the market. As a consequence, if this "signaling disadvantage" is large enough, in the unique equilibrium, firm 1 eventually becomes the monopolist.

In other words, private learning generates a discouragement effect. Firm 2 would be willing to remain in the market at a belief lower than the single-player cutoff only if it expected to eventually become a monopolist. However, by continuing operations, firm 2 makes the opponent more optimistic and delays its exit. As a result, anticipating a longer duopoly phase, it is discouraged from remaining in the market.

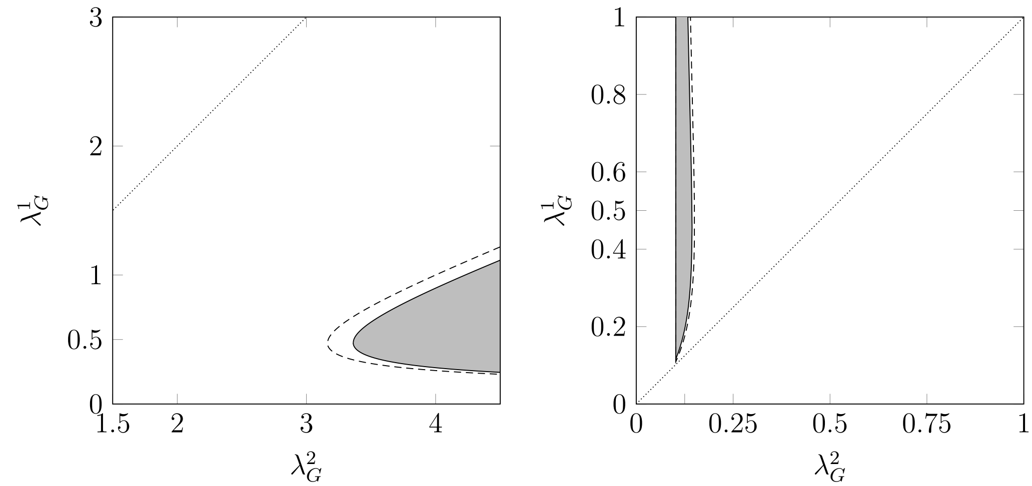
Theorem 2 provides sufficient conditions for firm 2 to be the first to exit in the unique outcome that survives iterated deletion of conditionally dominated strategies.[22] Interestingly, depending on the parameters, the larger or the smaller firm eventually becomes the monopolist.

As shown in the Appendix, our result does not rely on the assumption that a sufficiently pessimistic monopolist finds it optimal to exit, in that we can show that even when being the monopolist is always profitable, i.e., $2\lambda_B R - c > 0$, there exists a nonempty set of parameters such that the weaker firm exits first in the unique equilibrium of the game.[23] Clearly, in this case, the equilibrium never displays exit waves.

Theorem 2 can be summarized as follows: under some parametric restrictions, in the unique equilibrium, the stronger firm, as defined in Section 4.1, survives. The novel observation is the non-monotonic relationship between strength and profit flow, which may be seen as a proxy for firm size.

[22]The sufficient parametric conditions in part B are in a sense not tight. We believe that the uniqueness results extend to a larger set of parameters, but we were unable to derive tighter bounds.

[23]In fact, the statement is only slightly weakened. If $r > \gamma + \lambda_B$, we can still identify a set of pairs $\mathcal{L} \in (c/R, \infty) \times (c/R, \infty)$, $\lambda_G^2 > \lambda_G^1$, for which $(\sigma_0^1, \sigma^2_{\pi^*(\lambda_G^2)})$ is the unique equilibrium, but in contrast to Theorem 2.A, $\text{proj}_1 \mathcal{L} \neq (c/R, \infty)$.



(a) Theorem 2.A (b) Theorem 2.B

Figure 4: In gray, the sets of pairs $(\lambda_1^G, \lambda_G^2)$ for which $(\sigma_0^1, \sigma^2_{\pi^*(\lambda^1_G)})$ is the unique equilibrium for $(c, R, r, \gamma, \lambda_B) = (1/10, 1, 7, 1/5, 0)$. The dashed line identifies the sets of pairs $(\lambda_1^G, \lambda_G^2)$ for which $(\sigma_0^1, \sigma^2_{\pi^*(\lambda^1_G)})$ is the unique equilibrium when $r = 20$.

Figure 4 provides an illustration of the theorem; it identifies the set of pairs $(\lambda_G^1, \lambda_G^2)$ for which firm 2 exits first in the unique strategy profile that survives iterated deletion of dominated strategy.[24]

While delivering clear-cut predictions about the order of exit, the theorem is mute about the ranking of equilibrium payoffs. In fact, if in equilibrium, the firm with the larger arrival rate of customers exits first, it is a priori unclear which firm collects a higher ex ante expected equilibrium payoff. On the one hand, the firm with the smaller arrival rate of customers benefits from observational learning and enjoys monopoly profits; on the other hand, the firm with the larger arrival rate of customers enjoys larger revenues and can accurately time its exit to the market conditions because it has more precise information. Figure 5 illustrates how the ranking of ex ante expected equilibrium payoff depends on parameters: in the numerical example, because of monopoly profits, the equilibrium payoff of the smaller firm (solid line) is larger when the customer arrival rate are sufficiently similar. However, the ranking of payoffs is reversed when the customer arrival rate of the firm who exits first is sufficiently large.

[24] In light of the hump-shaped curve $\tau^*(\lambda_G^i)$ and Theorem 1, in the right panel of Figure 4, for any pair $(\lambda_G^1, \lambda_G^2)$ belonging to the gray area, λ_G^2 must lie to the left of $\arg\max \tau^*(\lambda_G^i)$.

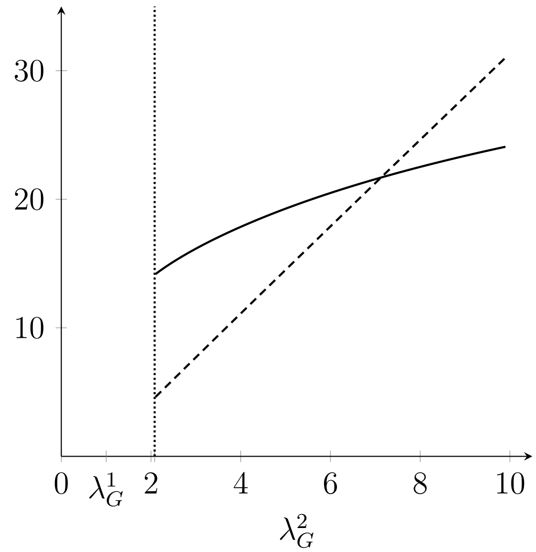


Figure 5: Ex-ante expected payoff for $(c, R, r, \gamma, \lambda_G^1) = (1/2, 1, 1/10, 1/5, 1)$. The solid and the dashed lines indicate the ex-ante expected payoff of firm 1 and firm 2, respectively, in the equilibrium $(\sigma_0^1, \sigma^2_{\pi^*(\lambda_G^2)})$. The strategy profile $(\sigma_0^1, \sigma^2_{\pi^*(\lambda_G^2)})$ is an equilibrium for any $\lambda_G^2 \geq 2.08075$, as $\tau^*(1) \simeq \tau^*(2.08075)$ (see Figure 1).

As discussed in the last paragraph of Section 4.2, depending on the parameters, the equilibrium may display exit waves. In the case of conclusive news, we simulate the probability of an exit wave using Monte Carlo methods. Figure 6 depicts the probability of an exit wave, that is, the probability that firm 1 follows suit, as a function of the customer arrival rate of firm 2; fixing the rate of arrival of firm 1, we pick the rate of arrival of firm 2 to be high enough so that, by Theorem 1, $(\sigma_0^1, \sigma^2_{\pi^*(\lambda_G^2)})$ is an equilibrium. Naturally, as the customer arrival rate of the large firm increases, more information is released at its exit and the probability of an exit wave increases.

Insights into Exit Patterns from Declining Industries

While the contribution of our paper is mainly theoretical, we believe that our framework contributes to the discussion on the relationship between firm size and the likelihood of survival. It has been argued that in markets where economies of scale play an important role and where innovative activity is dominated by larger enterprises, small firms have a lower likelihood of survival, while size does not increase the likelihood of survival in the mature stage of the industry life cycle, or in prod-

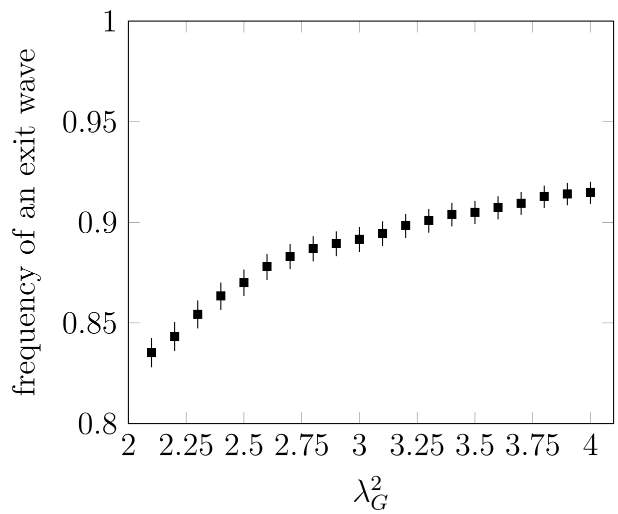


Figure 6: Frequency of exit waves for $(c, R, r, \gamma, \lambda_G^1, \lambda_B) = (1/2, 1, 1/10, 1/5, 1, 0)$ for different values of λ_2^G in the equilibrium $(\sigma_0^1, \sigma^2_{\pi^*(\lambda_G^2)})$. We simulated the probability of exit waves by running a Monte Carlo simulation with 10 000 draws. The lines indicate the 95% confidence intervals.

ucts are relatively low in technological intensity (see, for example, Audretsch, 1991, Audretsch and Mahmood, 1995, and Agarwal and Audretsch, 2001). Our results suggest a new rationale for why the technological features of an industry may affect exit patterns.

The importance of observational learning is corroborated, for example, by Goins and Gruca (2008) who document how layoff announcements affect the stock price of competitors, supporting the idea that disinvestment decisions reveal private information of the announcing firm about the future of the industry.

Roughly, our model predicts that if there is a high degree of uncertainty, that is, both firms have little information about the market conditions, the small firm exits first. (See the right panel in Figure 4.) Intuitively, if firms' private information is of little use in predicting future profits, in line with Geroski (1995), Theorem 2.B predicts that the firm with the lowest profit flow exits first in the unique equilibrium of the game. However, when the large firm has (sufficiently) precise information about the market conditions, the small firm survives the industry decline, in line with Agarwal and Audretsch (2001)'s finding that Geroski (1995)'s stylized fact does not hold in the mature phase of the industry life cycle or in high-technology industries.[25]

Merger and acquisition strategies can be a particularly important tool in mature or declining industries when it becomes desirable to quickly reduce capacity. Within our framework, the larger firm has incentives to acquire and shut down the smaller rival if allowed to do so, as was the case with GTE Sylvania and Union Electric in the 1970s (see Harrigan, 2003).

As argued by Coate and Kleit (1991), the US Federal Trade Commission's Horizontal Merger Guidelines give no special consideration to declining industry mergers, beyond arguing that the acquisition of exiting firms is unlikely to result in a reduction of consumer surplus. In contrast, European and Japanese policy makers, have implemented declining industry policies that allow firms to enter into horizontal agreements to rationalize market capacity in the face of declining demand. For example, in the late 1970s, the Japanese government organized a manufacturer association to buy and retire assets in the shipbuilding industry. In this respect, not only does our model provide further support for special treatment of declining industry mergers, but it also sheds lights on a novel inefficiency brought about by private learning.[26]

Nevertheless, in light of our extension to capacity disinvestment (see Section B.3 in the Supplementary Appendix of Cetemen and Margaria, 2021), our analysis can be viewed as suggestively complementary to Nishiwaki and Kwon (2013)'s, who analyze the efficiency tradeoffs associated with capacity-reduction behavior within multiplant firms. These authors document that in the Japanese cement industry, the less efficient firms are not more likely to reduce capacity than more efficient firms.

[25] Agarwal and Audretsch (2001) explain how the theory of strategic niches can explain this finding. We believe that our model can provide a more compelling justification for the finding.

[26] The relevance of confidential business information in merger reviews is confirmed by the recent attention that the Federal Trade Commission has devoted to implementing firewalls during the pendency of merger investigations to prevent anticompetitive information exchange between putative merger partners. (See, for example, the 2012 Statement of the Bureau of Competition of the Federal Trade Commission, *Negotiating Merger Remedies*, available at https://www.ftc.gov/system/files/attachments/negotiating-merger-remedies/merger-remediesstmt.pdf.)

4.3.1 Discussion of the Proof

The proof relies on iterated deletion of (conditionally) dominated strategies.[27] The argument is somewhat involved; here, we illustrate the main ideas. First, we derive a (uniform) lower bound on the belief of firm 1 along any path induced by a strategy profile that survives iterated deletion of conditionally strictly dominated strategies in the case of conclusive news. Next, we use a limit argument to bound the belief in the case of inconclusive news.

The argument to derive the lower bound on firm 1's belief is divided into six steps and is illustrated in Figure 7. First, recall that remaining in the market is a dominant action for firm i whenever it attaches a probability higher than the cutoff $\pi^*(\lambda_G^i)$ to the prevailing state being good. (See Lemma 2.) We show that, under the assumptions of the theorem, for any strategy of firm 2 that prescribes exiting at some belief below $\pi^*(\lambda_G^2)$, firm 2 continuing operations always brings good news.[28] As a result, the belief of firm 1 is bounded away from $\pi^*(\lambda_G^1)$ at any time before $\tau^*(\lambda_G^1)$, making exiting before $\tau^*(\lambda_G^1)$ a dominated action.

Second, we argue that if the gap between the two first exit times is large enough, specifically, if $2\tau^*(\lambda_G^2) \ll \tau^*(\lambda_G^1)$,[29] as in Figure 7, at any $t \in [0, 2\tau^*(\lambda_G^2))$, exiting is a conditionally dominant strategy for firm 2 whenever its belief falls below the cutoff $\pi^*(\lambda_G^2)$. Intuitively, firm 2 would be willing to remain in the market at a lower belief only if it expected firm 1 to exit with positive probability at some future point in time. However, by the first step, rationality implies that firm 1 does not exit until relatively late in the game, that is, until $\tau^*(\lambda_G^1)$, and hence firm 2 does not find it worthwhile to bear the expected losses to wait until then.

Third, at time $2\tau^*(\lambda_G^2)$, if firm 2 does not exit, firm 1 can infer that the belief of firm 2 never fell below the cutoff $\pi^*(\lambda_G^2)$ in $[0, 2\tau^*(\lambda_G^1))$. In the case of conclusive news, it is easy to identify the pair of firms' private histories that are consistent with them playing conditionally undominated strategies and with the public history of no exit and that, when combined, would give rise to the lowest posterior belief

[27] As explained by Shimoji and Watson (1998), the standard notion of dominance has little bite in extensive form games. In the Appendix, we explain how their definition of conditional dominance can be extended to our setup.

[28] When firm 2 uses a cutoff strategy, this is immediate. However, in contrast to Rosenberg et al. (2007), it is not necessarily true that any rationalizable strategy is a cutoff strategy (see Section 4.3). Hence, arguing that observing that the rival is still active always brings good news requires a more delicate argument.

[29] That is, if $\tau^*(\lambda_G^1) - 2\tau^*(\lambda_G^2)$ is sufficiently large.

about the state at time $2\tau^*(\lambda_G^2)$. For firm 1, the "worst" history is that without any customer. For firm 2, any "worst" history that is consistent with it playing conditionally undominated strategies and not exiting by time $2\tau^*(\lambda_G^2)$ involves observing a customer "right after" $\tau^*(\lambda_G^2)$. (See Figure 7.)

Fourth, combining the inference from these two private histories yields a lower bound on firm 1's posterior belief at time $2\tau^*(\lambda_G^2)$ along any history on the path generated by a conditionally undominated strategy. For convenience, we further approximate this bound conditioning only on $\omega_{\tau^*(\lambda_G^2)} = G$ and firm 1 not having observed any customers in $[\tau^*(\lambda_G^2), 2\tau^*(\lambda_G^2))$ to obtain the bound identified with the first red x mark in Figure 7.

Fifth, we can derive a lower bound on the belief of firm 1 at any time after $2\tau^*(\lambda_G^1)$ using the fact that the posterior belief cannot decrease faster than the private belief, that is, the belief computed while disregarding observational learning. (See Footnote 28.) When using this bound, we can show that exiting is a dominated action for firm 1 at any time before $\tau^*(\lambda_G^1) + \tau^*(\lambda_G^2)$. (See the red line in Figure 7.)

Last, at time $3\tau^*(\lambda_G^2)$ the same conditional dominance arguments apply because the problem is stationary. The stationarity hinges on the conclusive news assumptions, as if no firm exits before $3\tau^*(\lambda_G^2)$, it becomes common knowledge that $\omega_{2\tau^*(\lambda_G^2)} = G$. More precisely, for any $n = 3, 4, \ldots$, firm 2 finds it dominant to exit at any $t \in [(n-1)\,\tau^*(\lambda_G^2), n\,\tau^*(\lambda_G^2))$ as soon as its belief reaches $\pi^*(\lambda_G^2)$. Therefore, firm 1 finds it dominant not to exit at $t \in [n\,\tau^*(\lambda_G^2), n\,\tau^*(\lambda_G^2) + \tau^*(\lambda_G^1))$, irrespective of its private history.

In the case of inconclusive news, i.e., $\lambda_B > 0$, computing the lower bound for the posterior belief is more complicated since it is unclear which pair of firms' private histories would give rise to the lowest posterior belief. Nevertheless, leveraging the continuity of the posterior belief in λ_B, we show that for any $\lambda_B > 0$, as λ_G^2 goes to infinity, at any time along a history in which firm 2 does not exit, the posterior belief of firm 1 conditional on firm 2 playing undominated strategies is bounded away from the cutoff belief $\pi^*(\lambda_G^1)$.[30] Intuitively, as λ_G^2 grows large, firm 1 bases its inference mostly on observational learning, and in the limit, the fact that it learns via inconclusive bad news, instead of conclusive bad news, becomes irrelevant.

As noted, the two parts of Theorem 2 are not specular. The key step in the proof is to show that firm 2 finds it dominant to exit at $2\tau^*(\lambda_G^2)$ whenever its

[30] To be clear, the first part of Theorem 2 is valid for any $\lambda_B > 0$. We expect that the second part is valid for sufficiently low λ_B.

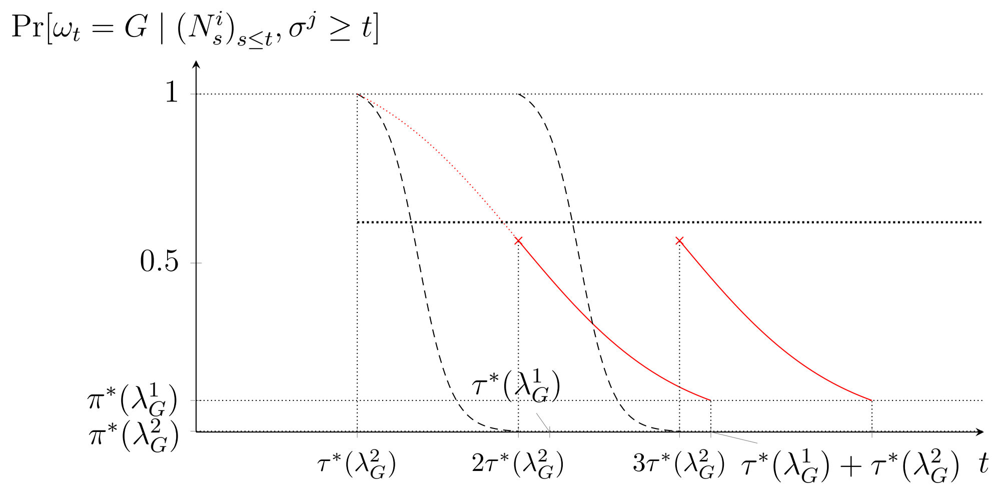


Figure 7: Illustration of the proof of Theorem 2. The black dashed line indicates the belief of firm 2 along an history with one customer at $\tau^*(\lambda^2_G)$ and one customer at $2\tau^*(\lambda^2_G)$. In red, the lower bound on firm 1's belief. In the figure, $(c, R, r, \gamma, \lambda^1_G, \lambda^2_G) = (1, 1, 1/10, 1/5, 3/2, 8)$.

belief falls short of the cutoff $\pi^*(\lambda^2_G)$. Now, when the customer arrival rate of firm 2 is sufficiently large, its cutoff belief $\pi^*(\lambda^2_G)$ is arbitrarily low. As a result, in the limit, firm 2 finds it conditionally dominant to exit for "almost any" $\tau^*(\lambda^1_G) > 2\tau^*(\lambda^2_G)$. In contrast, when firm 2 is smaller than firm 1, i.e., $\lambda^2_G < \lambda^1_G$, firm 2 finds it conditionally dominant to exit at $2\tau^*(\lambda^2_G)$ only if $\tau^*(\lambda^1_G) - 2\tau^*(\lambda^2_G)$ is sufficiently large. That is, it is not firm 2's pessimism about the state of the world that determines its incentives to exit but rather the amount of time it expects the duopoly to last. It can be shown that the maximum first exit time is increasing in the operating cost c. As a result, if firm 2 is sufficiently impatient and the operating cost is such that the first exit time of firm 1 is sufficiently high, firm 2 will find it dominant to exit at $2\tau^*(\lambda^2_G)$.

4.4 Extensions

In this section, we discuss several extensions of our stylized model and show that our results appear quite robust.

Other Asymmetries. Other types of asymmetries can easily be accommodated. Specifically, our results hold true if firms are asymmetric in their cost of operation (c), their revenues (R), their discount rate (r), or their rate of arrival of customers in the bad state (λ_B). More formally, for any set of parameters, $(c^1, c^2, R^1, R^2, r^1, r^2, \lambda_B^1, \lambda_B^2)$ and any λ_G^1, firm 2 exits first in the unique equilibrium of the game provided that λ_G^2 is large enough, even if $\lambda_B^2 < \lambda_B^1$. In this sense, the result holds true even if, as in Ghemawat and Nalebuff (1985), the larger firm bears a larger loss after the industry decline. In addition, for any set of parameters, $(c^1, c^2, R^1, R^2, r^1, r^2, \lambda_B^1, \lambda_B^2)$, there exists an open set of pairs $(\lambda_G^1, \lambda_G^2)$ such that $\lambda_G^2 < \lambda_G^1$ and firm 2 exits first in the unique equilibrium of the game provided that R^1/c^1, r^2, and γ are high enough.

In a setup with asymmetric primitive parameters, the following comparative statics result is almost immediate.

Proposition 2. *For any set of parameters* $(c^1, c^2, R^1, R^2, r^1, r^2, \lambda_B^1, \lambda_B^2)$*, the set of pairs* $(\lambda_G^1, \lambda_G^2)$ *identified in Theorem 2 is increasing (in the inclusion order) in* r^2 *and* c^2*.*

We believe also that the set of strategy profiles that survive iterated deletion of dominated strategy should be increasing in r^2 and c^2. Intuitively, for any strategy of firm 1, the expected continuation payoff of firm 2 along any history is decreasing in c^2 and r. Hence, in the iterated procedure, whenever along some history exiting is dominant for firm 2 in a game in which the operating cost of firm 2 is c^2 (the discount rate of firm 2 is r^2), exiting along that history is also dominant in a game in which the operating cost of firm 2 is $c^{2\prime} > c^2$ (the discount rate of firm 2 is $r^{2\prime} > r^2$). Deleting more strategies for firm 2 among those that prescribe remaining in the market along some history makes firm 1 more optimistic and can only increase the histories along which remaining in the market is dominant. While intuitive, this argument is difficult to formalize unless one focuses on a specific deletion procedure, as we do in Proposition 2.

Numerical simulations suggest that the comparative static with respect to the discount rate also holds in a setup with symmetric primitive parameters. (See also Figure 4.)

Public Information. Our results are to some extent robust to the introduction of background public learning. For example, in the presence of public conclusive good news, as long as the rate of arrival of public news is low enough, if the customer arrival rate of firm 2 is sufficiently high, in the unique equilibrium, firm 2 exits first with probability one. On the one hand, along any history, the additional information from the public news is of little help to firm 2 in drawing an inference about the state of the world. On the other hand, if the informativeness of the public signal is sufficiently low, the bad news from the absence of public good news cannot offset the good news from firm 2 remaining in the market. As a result, our dominance argument holds true. However, as private learning plays a key role in our proof, a sufficiently informative public signal may overturn our result by weakening the role of signaling.

Alternately, consider the case in which the state is publicly revealed at the jump times of a (state-independent) Poisson process. In this case, as soon as the public signal reveals that the market has become unprofitable, the game enters a war of attrition phase, as in Section 3.1. However, along the path with no bad conclusive news, our equilibrium analysis applies. In the same spirit, we could allow for the transition to the bad state to be publicly revealed with a deterministic delay: provided that the delay is long enough, our equilibrium analysis remains valid.[31]

Pricing. A few papers, such as Roberts (1986), have extended the Milgrom and Roberts (1982) limit-pricing model to capture post-entry predation. Conceivably, a firm may attempt to use its pricing strategy to convey some bad news about the market's profitability.[32] While we leave the analysis of dynamic private learning and signaling to future research, in this section, we discuss a simple way to introduce pricing into our model in the spirit of Diamond (1971).

Suppose that at each moment in time, each firm also sets its price. A consumer of firm i arriving at time t observes firm i's price at that time, and decides whether to purchase from firm i or to incur a cost to inspect the price posted by the other firm. (Agents are short-lived, i.e., cannot delay their purchase.) Consumers' willingness to pay is distributed according to some distribution function F, irrespective of ω_t, and that $p(1-F(p))$ is single-peaked. We now argue that regardless of whether firms observe one another's posted prices, the strategy profile in Theorem 1 is part of an equilibrium in which both firms charge the monopoly price $\arg\max_p p(1-F(p))$.

Assume first that the posted price is unobservable to the competitor. First, in line with Diamond's paradox, if customers expect both firms to charge the same price, they will not find it worthwhile to pay the search cost. Second, the weaker

[31] We thank an anonymous referee for this remark.

[32] We are not aware of pricing models with two-sided private learning about a common underlying state of the world. The paper by Sweeting et al. (2019) is an exception: they extend their finite-horizon model to allow for this possibility.

firm has no incentive to raise the price and lose a customer because this would not affect the behavior of its competitor, even if it would make it more optimistic about the market condition. The stronger firm has no incentive to lose customers to its competitor either because, if anything, this would delay the time the weaker firm exits. Last, even if firms could obverse one another's price, the equilibrium can be sustained by an appropriate choice of off-path beliefs.

5 Conclusions

We analyze a dynamic model of exit from a stochastically declining market. We investigate how private learning affects the equilibrium dynamics. We provide sufficient conditions under which the equilibrium is unique. When the equilibrium is unique, the firm that we identify to be the weaker firm exits first with probability one. We show that in our model, the strength of a firm is determined by its first exit time, a measure of its signaling disadvantage. Crucially, the first exit time is non-monotonic in firm size or, equivalently, in learning speed. As a result, our model provides a novel explanation, based on informational externalities, of the fact that in some industries, smaller firms survive the decline. Specifically, our paper offers a theory of exit that ties the industry economic primitives to exit dynamics. Furthermore, in Cetemen and Margaria (2021), we show that our model is equivalent to an investment game in which competing firms privately learn over time about the comparative profitability of an innovation and decide when to irreversibly invest in it. Therefore, our results can provide insights into the dynamics of investment in disruptive innovation à la Christensen (1997) or into the dynamics of pharmaceutical R&D (see, for example, Krieger, 2020).

We conjecture that our proof technique can be applied to other asymmetric timing games with private learning, potentially with more than two players. For example, the recursive dominance argument can be adapted to show uniqueness in some asymmetric preemption games with evolving state and private learning by identifying an appropriate bound on a player's continuation payoff if he or she does not act at the single-agent optimal cutoff.[33]

[33] Thomas (2020) proves equilibrium uniqueness in two-player preemption games with private learning (and perfectly persistent state) using a different technique.

Our results do not immediately extend to a setup in which the market profitability fluctuates over time. This generalization may have the potential to provide a novel model of shakeouts. We leave these questions for future research.

References

Agarwal, Rajshree and David B. Audretsch (2001) "Does Entry Size Matter? The Impact of the Life Cycle and Technology on Firm Survival," *The Journal of Industrial Economics*, 49, 21–43.

Anslinger, Patricia, Justin Jenk, and Ravi Chanmugan (2003) "The Art Of Strategic Divestment," *Journal of Applied Corporate Finance*, 15, 97–101.

Audretsch, David B. (1991) "New-Firm Survival and the Technological Regime," *The Review of Economics and Statistics*, 73, 441–450.

Audretsch, David B. and Talat Mahmood (1995) "New Firm Survival: New Results Using a Hazard Function," *The Review of Economics and Statistics*, 77, 97–103.

Awaya, Yu and Vijay Krishna (2021) "Startups and Upstarts," *Journal of Political Economy*, 129, 534–569.

Boar, Corina and Virgiliu Midrigan (2021) "Markups and inequality," Working Paper.

Bobtcheff, Catherine and Raphaël Levy (2017) "More Haste, Less Speed? Signaling through Investment Timing," *American Economic Journal: Microeconomics*, 9, 148–86.

Cetemen, Doruk and Chiara Margaria (2021) "Exit Dilemma: The Role of Private Learning on Firm Survival," Working Paper.

Chen, Chia-Hui and Junichiro Ishida (2021) "A War of Attrition with Experimenting Players," *The Journal of Industrial Economics*, 69, 239–269.

Chen, Yi (2020) "A revision game of experimentation on a common threshold," *Journal of Economic Theory*, 186, 104997.

Christensen, Clayton M. (1997) *The Innovator's Dilemma: When New Technologies Cause Great Firms to Fail*, Boston, MA: Harvard Business Review Press.

Cisternas, Gonzalo and Aaron Kolb (2020) "Signaling with Private Monitoring," Working Paper.

Coate, Malcolm B. and Andrew N. Kleit (1991) “Antitrust policy for declining industries,” *Journal of Institutional and Theoretical Economics (JITE) / Zeitschrift für die gesamte Staatswissenschaft*, 477–498.

Corduneanu, Constantin, Yizeng Li, and Mehran Mahdavi (2016) *Functional Differential Equations*, Hoboken, NJ: John Wiley & Sons.

De Loecker, Jan and Frederic Warzynski (2012) “Markups and Firm-Level Export Status,” *American Economic Review*, 102, 2437–71.

Décamps, Jean-Paul and Thomas Mariotti (2004) “Investment Timing and Learning Externalities,” *Journal of Economic Theory*, 118, 80–102.

Diamond, Peter A. (1971) “A Model of Price Adjustment,” *Journal of Economic Theory*, 3, 156–168.

Edmond, Chris, Virgiliu Midrigan, and Daniel Yi Xu (2018) “How costly are markups?”, Working Paper.

Fine, Charles H. and Lode Li (1989) “Equilibrium Exit in Stochastically Declining Industries,” *Games and Economic Behavior*, 1, 40–59.

Foster, F. Douglas and S. Viswanathan (1996) “Strategic Trading When Agents Forecast the Forecasts of Others,” *The Journal of Finance*, 51, 1437–1478.

Fudenberg, Drew and Jean Tirole (1986) “A Theory of Exit in Duopoly,” *Econometrica*, 54, 943–960.

Geroski, Paul A. (1995) “What do we know about entry?” *International Journal of Industrial Organization*, 13, 421–440.

Ghemawat, Pankaj and Barry Nalebuff (1985) “Exit,” *The RAND Journal of Economics*, 16, 184–194.

Ghemawat, Pankaj and Barry Nalebuff (1990) “The Devolution of Declining Industries,” *The Quarterly Journal of Economics*, 105, 167–186.

Goins, Sheila and Thomas S. Gruca (2008) “Understanding competitive and contagion effects of layoff announcements.,” *Corporate Reputation Review*, 11, 12–34.

Gorno, Leandro and Felipe S. Iachan (2020) “Competitive Real Options under Private Information,” *Journal of Economic Theory*, 185, 104945.

Halac, Marina, Navin Kartik, and Qingmin Liu (2016) “Optimal Contracts for Experimentation,” *The Review of Economic Studies*, 83, 1040–1091.

Harrigan, Kathryn R. (2003) *Declining Demand, Divestiture, and Corporate Strategy*, Washington, D.C.: Beard Books.

Heidhues, Paul, Sven Rady, and Philipp Strack (2015) “Strategic Experimentation with Private Payoffs,” *Journal of Economic Theory*, 159, 531–551.

Hendricks, Ken, Andrew Weiss, and Charles Wilson (1988) “The War of Attrition in Continuous Time with Complete Information,” *International Economic Review*, 29, 663–680.

Hopenhayn, Hugo A. and Francesco Squintani (2011) “Preemption Games with Private Information,” *The Review of Economic Studies*, 78, 667–692.

Keller, Godfrey and Sven Rady (1999) “Optimal Experimentation in a Changing Environment,” *The Review of Economic Studies*, 66, 475–507.

Keller, Godfrey and Sven Rady (2003) “Price Dispersion and Learning in a Dynamic Differentiated-Goods Duopoly,” *The RAND Journal of Economics*, 34, 138–165.

Keller, Godfrey and Sven Rady (2010) “Strategic experimentation with Poisson bandits,” *Theoretical Economics*, 5, 275–311.

Keller, Godfrey, Sven Rady, and Martin Cripps (2005) “Strategic Experimentation with Exponential Bandits,” *Econometrica*, 73, 39–68.

Khromenkova, Daria (2018) *Essays in Microeconomic Theory*, Ph.D. Dissertation, Chap. 2, University of Mannheim.

Kim, Kyungmin and Frances Zhiyun Xu Lee (2014) “Information Acquisition in a War of Attrition,” *American Economic Journal: Microeconomics*, 6, 37–78.

Kornhauser, Lewis, Ariel Rubinstein, and Charles Wilson (1989) “Reputation and Patience in the ‘War of Attrition’,” *Economica*, 56, 15–24.

Krieger, Joshua L. (2020) "Trials and Terminations: Learning from Competitors' R&D Failures," *Management Science*, forthcoming.

Lieberman, Marvin B. (1990) "Exit From Declining Industries: "Shakeout" or "Stake-out"?" *The RAND Journal of Economics*, 21, 538–554.

Maschler, Michael, Eilon Solan, and Shmuel Zamir (2013) "Game Theory," *Cambridge University Press*, 979, 4.

Milgrom, Paul and John Roberts (1982) "Limit Pricing and Entry under Incomplete Information: An Equilibrium Analysis," *Econometrica*, 50, 443–459.

Milgrom, Paul and Ilya Segal (2002) "Envelope Theorems for Arbitrary Choice Sets," *Econometrica*, 70, 583–601.

Moscarini, Giuseppe and Francesco Squintani (2010) "Competitive Experimentation with Private Information: The Survivor's Curse," *Journal of Economic Theory*, 145, 639–660.

Murto, Pauli (2004) "Exit in Duopoly Under Uncertainty," *The RAND Journal of Economics*, 35, 111–127.

Murto, Pauli and Juuso Välimäki (2011) "Learning and Information Aggregation in an Exit Game," *The Review of Economic Studies*, 78, 1426–1461.

Nishiwaki, Masato and Hyoeg Ug Kwon (2013) "Are Losers Picked? An Empirical Analysis of Capacity Divestment and Production Reallocation in the Japanese Cement Industry," *The Journal of Industrial Economics*, 61, 430–467.

Øksendal, Bernt and Agnès Sulem (2019) *Applied Stochastic Control of Jump Diffusions*, Cham: Springer, 3rd edition.

Roberts, John (1986) "A Signaling Model of Predatory Pricing," *Oxford Economic Papers*, 38, 75–93.

Rosenberg, Dinah, Antoine Salomon, and Nicolas Vieille (2013) "On Games of Strategic Experimentation," *Games and Economic Behavior*, 82, 31–51.

Rosenberg, Dinah, Eilon Solan, and Nicolas Vieille (2007) "Social Learning in One-Arm Bandit Problems," *Econometrica*, 75, 1591–1611.

Rothschild, Michael (1974) "A Two-Armed Bandit Theory of Market Pricing," *Journal of Economic Theory*, 9, 185–202.

Shimoji, Makoto and Joel Watson (1998) "Conditional Dominance, Rationalizability, and Game Forms," *Journal of Economic Theory*, 83, 161–195.

Sweeting, Andrew, James W. Roberts, and Chris Gedge (2019) "A Model of Dynamic Limit Pricing with an Application to the Airline Industry," *Journal of Political Economy*, 128, 1148–1193.

Takahashi, Yuya (2015) "Estimating a War of Attrition: The Case of the US Movie Theater Industry," *American Economic Review*, 105, 2204–41.

Thomas, Caroline (2020) "Stopping with Congestion and Private Payoffs," *Journal of Mathematical Economics*, 91, 18–4242.

A Omitted Proofs

A.1 Proofs for Section 4

Proof of Lemma 1. If $\lambda^i_G R \leq c$, even when the prevailing state is good, firm i's expected flow payoff is nonpositive. Hence, the firm finds it optimal to exit immediately, i.e., $\pi^*(\lambda^i_G) = 1$. To prove the converse, we show that if $\lambda_B = 0$, whenever $\lambda^i_G R > c$, $\pi^*(\lambda^i_G) < 1$. A fortiori $\pi^*(\lambda^i_G) < 1$ when $\lambda_B > 0$.

When $\lambda^i_G R > c$, the value function of the best-reply problem solves the following Hamilton-Jacobi-Bellman equation

$$\begin{aligned} rv(p) = \big(p\lambda^i_G + (1-p)\lambda_B\big)\left(R + v\left(j\left(p\right)\right) - v(p)\right) - c \\ - \big(p(1-p)(\lambda^i_G - \lambda_B) + p\gamma\big)\, v'(p), \end{aligned} \tag{2}$$

where

$$j(p) = \frac{p\,\lambda^i_G}{p\,\lambda^i_G + (1-p)\lambda_B},$$

denotes the belief after observing a customer.

In the case of conclusive news, i.e., $\lambda_B = 0$, by solving the Hamilton-Jacobi-Bellman equation, we find that in the continuation region,

$$v(p) = -\left(1 - \frac{\lambda^i_G}{\gamma + \lambda^i_G + r}\,p\right)\frac{c}{r} + \frac{\lambda^i_G}{\gamma + \lambda^i_G + r}\,p\left(R + v(1)\right) + p\,\Omega(p)^{1+\frac{r}{\gamma+\lambda^i_G}}\,C,$$

where C is a constant of integration. Since

$$v(1) = -\left(\frac{\gamma + r}{\gamma + \lambda^i_G + r}\right)\frac{c}{r} + \frac{\lambda^i_G}{\gamma + \lambda^i_G + r}\left(R + v(1)\right) + \gamma^{1+\frac{r}{\gamma+\lambda^i_G}}\,C,$$

we have

$$C = \gamma^{-1-\frac{r}{\gamma+\lambda^i_G}}\left(\left(\frac{\gamma + r}{\gamma + \lambda^i_G + r}\right)\left(v(1) + \frac{c}{r}\right) - \frac{\lambda^i_G}{\gamma + \lambda^i_G + r}R\right). \tag{3}$$

Using smooth-pasting condition at the cutoff $\pi^*(\lambda_G^i)$, $v'(\pi^*(\lambda_G^i) = 0$,

$$
\begin{aligned}
v(1) = &-\frac{c}{r} + \frac{\lambda_G^i}{\gamma + r} R \\
&+ \frac{\gamma + \lambda_G^i + r}{\gamma + r} \frac{\pi^*(\lambda_G^i)\lambda_G^i\gamma}{(\gamma + r)(\pi^*(\lambda_G^i)\lambda_G^i + r)\left(\frac{\Omega(\pi^*(\lambda_G^i))}{\gamma}\right)^{\frac{r}{\gamma+\lambda_G^i}} - \pi^*(\lambda_G^i)\lambda_G^i\gamma} R,
\end{aligned}
$$

where

$$
\Omega_i(p) = \frac{\gamma + (1-p)\lambda_G^i}{p}.
$$

Using this equation to replace $v(1)$ in equation (3), we obtain

$$
C = \gamma^{-\frac{r}{\gamma+\lambda_G^i}} \frac{\pi^*(\lambda_G^i)\lambda_G^i\gamma}{(\gamma + r)(\pi^*(\lambda_G^i)\lambda_G^i + r)\left(\frac{\Omega(\pi^*(\lambda_G^i))}{\gamma}\right)^{\frac{r}{\gamma+\lambda_G^i}} - \pi^*(\lambda_G^i)\lambda_G^i\gamma} R,
$$

which, replaced in the value-matching condition, $v(\pi^*(\lambda_G^i)) = 0$,

$$
\begin{aligned}
-\left(1 - \frac{\lambda_G^i}{\gamma + \lambda_G^i + r}\,\pi^*(\lambda_G^i)\right)\frac{c}{r} &+ \frac{\lambda_G^i}{\gamma + \lambda_G^i + r}\,\pi^*(\lambda_G^i)\,(R + v(1)) \\
&+ \pi^*(\lambda_G^i)\,\Omega(\pi^*(\lambda_G^i))^{1+\frac{r}{\gamma+\lambda_G^i}}\,C = 0,
\end{aligned}
$$

yields, after some manipulations,[34]

$$
\begin{aligned}
\Omega_i(\pi^*(\lambda_G^i)) = \frac{\lambda_G^i(\gamma + \lambda_G^i)(\gamma + \lambda_G^i + r)}{(\gamma + r)}\frac{R}{c} &- \frac{\lambda_G^i(\gamma + \lambda_G^i + r)}{r} \\
&+ \frac{\gamma\lambda_G^i(\gamma + \lambda_G^i)}{r(\gamma + r)\left(\Omega_i(\pi^*(\lambda_G^i))/\gamma\right)^{r/(\gamma+\lambda_G^i)}}.
\end{aligned}
\tag{4}
$$

The right-hand side is decreasing in $\pi^*(\lambda_G^i)$, and the left-hand side is increasing in $\pi^*(\lambda_G^i)$. Hence, there exists at most one root of equation (4). At $\pi^*(\lambda_G^i) = 1$, $\lambda_G^i R > c$ implies that the left-hand side if smaller than the right-hand side, while in the limit at $\pi^*(\lambda_G^i)$ goes to zero, the opposite is true. It follows that there exists

[34] It can be checked that when $\gamma = 0$, (4) reduces to the cutoff characterization of Décamps and Mariotti (2004) and Keller et al. (2005).

a unique root $\pi^*(\lambda^i_G) < 1$. Optimality follows by standard verification arguments. (See, for example Øksendal and Sulem, 2019, Theorem 3.2.).

In the general case, i.e. $\lambda_B \geq 0$ the existence and uniqueness results for functional differential equations guarantee that there exists a unique twice continuously differentiable solution to the Hamilton-Jacobi-Bellman equation given an initial guess for $v(1)$; see for example Corduneanu et al., 2016, Theorem 2.4.[35] Define the mapping $\Gamma : [0, (\lambda^i_G R - c)/r] \to \mathbf{R}$, which maps an initial guess $v(1)$ to the following function of the corresponding solution $\min_{p\in(0,1]} v(p) + |v'(p)|$. Note that $\Gamma(0) = (\lambda^i_G R - c)/r$, while $\Gamma((\lambda^i_G R - c)/r) < 0$. By Corduneanu et al., 2016, Theorem 3.6, the mapping Γ is continuous. Hence, by the intermediate value theorem, there exists a guess such that the solution to the Hamilton-Jacobi-Bellman equation satisfies $v(p) = v'(p) = 0$ for some $p \in (0, 1)$. Again, optimality and uniqueness follow by standard verification arguments.

Using the value-matching and smooth-pasting conditions, it is easily verified that the optimal cutoff $\pi^*(\lambda^i_G)$ satisfies the following equation

$$\pi^*(\lambda^i_G) = \frac{c}{(\lambda^i_G - \lambda_B)\left(R + v\left(j(\pi^*(\lambda^i_G))\right)\right)} - \frac{\lambda_B}{\lambda^i_G - \lambda_B}. \tag{5}$$

Given an increasing function v, the equation has at most one solution $\pi^*(\lambda^i_G)$. The right-hand side is decreasing in v, j, and λ^i_G. Both the value function v and the function j are increasing pointwise in λ^i_G. Hence, by the implicit function theorem,[36] we can conclude that the optimal cutoff $\pi^*(\lambda^i_G)$ is decreasing in λ^i_G.

(ii) With an abuse of notation, in the general case, i.e., $\lambda_B \geq 0$, we write

$$\Omega(p) = \frac{\gamma + (1-p)(\lambda^i_G - \lambda_B)}{p}.$$

[35] After a change of variables $q = 1 - p$, the functional differential equation (2) is a Volterra operator. After bounding the domain to $q \in [0, 1-\varepsilon)$, for arbitrarily small $\varepsilon > 0$, the assumptions of Corduneanu et al., 2016, Theorem 2.4 are satisfied.

[36] To be precise, abusing notation let $v(p, \lambda^i_G)$ and $j(p, \lambda^i_G)$ be the optimal value function and the function that describes the belief after the arrival of a customer respectively, when the rate of arrival of consumers in the good state is λ^i_G. Then, $v(p, \lambda^i_G)$ is continuously differentiable in p, and $j(p, \lambda^i_G)$ is continuously differentiable in λ^i_G. By the envelope theorem of Milgrom and Segal (2002), the value function $v(p, \lambda^i_G)$ is continuously differentiable in λ^i_G.

First, as proved above, $\pi^*(\lambda_G^i)$ is decreasing in λ_G^i. Second, note that

$$\tau^*(\lambda_G^i) = \frac{\ln\left(\Omega(\pi^*(\lambda_G^i))/\gamma\right)}{\gamma + \lambda_G^i - \lambda_B}.$$

It follows from equation (5) that

$$\pi^*(\lambda_G^i) \geq \frac{c}{(\lambda_G^i - \lambda_B)\left(R + \left(\lambda_G^i R - c\right)/r\right)} - \frac{\lambda_B}{\lambda_G^i - \lambda_B} =: \underline{\pi}(\lambda_G^i),$$

where the bound is derived by replacing $v(j(\pi^*(\lambda_G^i)))$ with $\left(\lambda_G^i R - c\right)/r$. Clearly, $v(j(\pi^*(\lambda_G^i))) < \left(\lambda_G^i R - c\right)/r$ because $\gamma > 0$ and the state market conditions eventually deteriorate.

By de l'Hôpital's rule

$$\begin{aligned}\lim_{\lambda_G^i \to \infty} \frac{\ln\left(\Omega(\underline{\pi}(\lambda_G^i))/\gamma\right)}{\gamma + \lambda_G^i - \lambda_B} &= \lim_{\lambda_G^i \to \infty} \frac{1}{\Omega(\underline{\pi}(\lambda_G^i))}\left(\frac{1 - \underline{\pi}(\lambda_G^i)}{\underline{\pi}(\lambda_G^i)} + \frac{\gamma + \lambda_G^i - \lambda_B}{\underline{\pi}(\lambda_G^i)^2}\underline{\pi}'(\lambda_G^i)\right) \\ &= \lim_{\lambda_G^i \to \infty} \frac{1 - \underline{\pi}(\lambda_G^i)}{\gamma + (1 - \underline{\pi}(\lambda_G^i))(\lambda_G^i - \lambda_B)} + \frac{\gamma + \lambda_G^i - \lambda_B}{\gamma + (1 - \underline{\pi}(\lambda_G^i))(\lambda_G^i - \lambda_B)}\frac{\underline{\pi}'(\lambda_G^i)}{\underline{\pi}(\lambda_G^i)}.\end{aligned}$$

Since $\underline{\pi}'(\lambda_G^i) = O((\lambda_G^i)^{-2})$ and $\underline{\pi}(\lambda_G^i) = O((\lambda_G^i)^{-1})$, the right-hand side converges to zero. Hence,

$$0 = \lim_{\lambda_G^i \to \infty} \frac{\ln\left(\Omega(\underline{\pi}(\lambda_G^i)/\gamma)\right)}{\gamma + \lambda_G^i - \lambda_B} \geq \lim_{\lambda_G^i \to \infty} \frac{\ln\left(\Omega(\pi^*(\lambda_G^i)/\gamma)\right)}{\gamma + \lambda_G^i - \lambda_B},$$

and $\lim_{\lambda_G^i \to \infty} \tau^*(\lambda_G^i) = 0$. We now state and prove two lemmas that are used later.

Lemma 3. *The following holds:*

$$\lim_{\lambda_G^i \to \infty} \pi^*(\lambda_G^i)\lambda_G^i \to 0.$$

Proof. First, in the case of conclusive news, i.e., $\lambda_B = 0$, the smooth-pasting condition implies

$$0 = \lambda_G^i \pi^*(\lambda_G^i)(R + v(1)) - c.$$

Moreover,

$$v(1) \geq \frac{\lambda_G^i \left(1 - e^{(\gamma+\lambda_G^i+r)\tau}\right)}{\gamma + r + \lambda_G^i e^{(\gamma+\lambda_G^i+r)\tau}} R - \frac{(\gamma+r)(\lambda_G^i+\gamma) - \lambda_G^i r e^{-(\gamma+\lambda_G^i+r)\tau} + \gamma(r+\gamma+\lambda_1)e^{-r\tau}}{r(\gamma+\lambda_G^i)\left(\gamma+r+\lambda_G^i e^{(\gamma+\lambda_G^i+r)\tau}\right)} c$$

for all $\tau \geq 0$. Hence, $v(1) \to \infty$ as $\lambda_G^i \to \infty$. As a result, $\lim_{\lambda_G^i \to \infty} \lambda_G^i \pi^*(\lambda_G^i) = 0$. Because the cutoff belief in the case of conclusive news is an upper bound on the cutoff belief in the case of inconclusive news, it follows that the result generalizes to the case of $\lambda_B > 0$. □

Lemma 4.

(i) $\pi^{*\prime}(\lambda_G^i) < 0$.

(ii) $\lim_{\lambda_G^i \to \infty} \pi^{*\prime}(\lambda_G^i) = 0$.

(iii) $\lim_{\lambda_G^i \to \infty} \pi^{*\prime\prime}(\lambda_G^i) = 0$,

Proof. (i) and (ii) As in Footnote 36, we let $v(p, \lambda_G^i)$ and $j(p, \lambda_G^i)$ be the optimal value function and the function that describes the belief after the arrival of a customer, respectively, when the rate of arrival of consumers in the good state is λ_G^i. By the implicit function theorem,

$$\pi^{*\prime}(\lambda_G^i) = -\frac{\pi^*(\lambda_G^i)}{\lambda_G^i - \lambda_B} - \frac{c}{(\lambda_G^i - \lambda_B)\left(R + v\left(j(\pi^*(\lambda_G^i), \lambda_G^i), \lambda_G^i\right)\right)^2} \left(v_p\left(j(\pi^*(\lambda_G^i), \lambda_G^i), \lambda_G^i\right) \frac{\pi^*(\lambda_G^i)(1-\pi^*(\lambda_G^i))\lambda_B}{\left(\pi^*(\lambda_G^i)\lambda_G^i + (1-\pi^*(\lambda_G^i))\lambda_B\right)^2} + v_{\lambda_G^i}\left(j(\pi^*(\lambda_G^i), \lambda_G^i), \lambda_G^i\right) \right). \tag{6}$$

A simple mimicking argument implies that $v_p(p, \lambda_G^i) > 0$ and $v_{\lambda_G^i}(p, \lambda_G^i) > 0$, that immediately implies $\pi^{*\prime}(\lambda_G^i) < 0$. For part (ii), by Lemma 3, $\pi^*(\lambda_G^i)\lambda_G^i \to 0$; it follows that $\lim_{\lambda_G^i \to \infty} \pi^{*\prime}(\lambda_G^i) = 0$. (iii) Denote the term in parentheses in equation (6) by $D_{\lambda_G^i} v\left(j(\pi^*(\lambda_G^i), \lambda_G^i)\right)$ and its derivative with respect to λ_G^i by $D^2_{\lambda_G^i} v\left(j(\pi^*(\lambda_G^i), \lambda_G^i)\right)$.

By the implicit function theorem,

$$\pi^{*\prime\prime}(\lambda_G^i) = \frac{2\pi^*(\lambda_G^i)}{(\lambda_G^i - \lambda_B)^2} + \left(\frac{2\left(\pi^*(\lambda_G^i)\lambda_G^i + (1-\pi^*(\lambda_G^i))\lambda_B\right)}{(\lambda_G^i - \lambda_B)\left(R + v\left(j(\pi^*(\lambda_G^i))\right)\right)} \right) D_{\lambda_G^i} v\left(j(\pi^*(\lambda_G^i), \lambda_G^i)\right)$$
$$+ \frac{2\left(\pi^*(\lambda_G^i)\lambda_G^i + (1-\pi^*(\lambda_G^i))\lambda_B\right)}{(\lambda_G^i - \lambda_B)\left(R + v\left(j(\pi^*(\lambda_G^i))\right)\right)^2} \left(D_{\lambda_G^i} v\left(j(\pi^*(\lambda_G^i), \lambda_G^i)\right)\right)^2$$
$$- \frac{\left(\pi^*(\lambda_G^i)\lambda_G^i + (1-\pi^*(\lambda_G^i))\lambda_B\right)}{(\lambda_G^i - \lambda_B)\left(R + v\left(j(\pi^*(\lambda_G^i))\right)\right)} D^2_{\lambda_G^i} v\left(j(\pi^*(\lambda_G^i), \lambda_G^i)\right).$$

The term on the right-hand side converges to zero, as $D^2_{\lambda_G^i} v\left(j(\pi^*(\lambda_G^i), \lambda_G^i)\right)$ is bounded. □

The proof of single-peakedness relies on the following lemma, which establishes a scale-invariance property of the optimal stopping problem.

Lemma 5. *Consider two sets of parameters* $(c, R, r, \lambda_G^i, \lambda_B, \gamma)$ *and* $(\hat{c}, R, \hat{r}, \hat{\lambda}_G^i, \hat{\lambda}_B, \hat{\gamma})$ *such that*

$$\frac{\hat{\lambda}_G^i}{\hat{\gamma}} = \frac{\lambda_G^i}{\gamma}, \qquad \frac{\hat{\lambda}_B}{\hat{\gamma}} = \frac{\lambda_B}{\gamma},$$
$$\frac{\hat{r}}{\hat{\gamma}} = \frac{r}{\gamma}, \qquad \frac{\hat{c}}{\hat{\gamma}} = \frac{c}{\gamma}.$$

The optimal value functions and the optimal cutoff beliefs in the two optimal stopping problems coincide.

Proof. First, note that the optimal value function associated with the first optimal stopping problem satisfies the Hamilton-Jacobi-Bellman equation of the second; see (2). In addition, the smooth-pasting and value-matching conditions associated with the two optimal stopping problems are identical. Since a standard verification theorem applies (see Øksendal and Sulem, 2019, Theorem 3.2), the optimal value functions and the optimal cutoff beliefs in the two optimal stopping problems coincide. □

Throughout, we fix a vector (c, R, r, λ_B). In light of Lemma 5, we consider the optimal stopping problem "scaled" by some γ, that is, we should consider how the first exit time changes with λ_G^i in the optimal stopping problem parametrized by $(\gamma c, R, \gamma r, \gamma \lambda_G^i, \gamma \lambda_B, \gamma)$, where the first element is the flow cost of remaining in business.

Define $\hat{\pi}^*(\lambda_G^i)$ to be the optimal cutoff when $\gamma = 1$, and the other parameters are $(c, R, r, \lambda_G^i, \lambda_B)$. By construction, the optimal cutoff associated with the decision problem parametrized by $(\gamma c, R, \gamma r, \gamma \lambda_G^i, \gamma \lambda_B, \gamma)$ is also $\hat{\pi}^*(\lambda_G^i)$. Define

$$\hat{\tau}^*(\Lambda_G^i; \gamma) = \frac{\ln\left(\frac{1}{\hat{\pi}^*(\Lambda_G^i)} + \frac{1-\pi^*(\Lambda_G^i)}{\pi^*(\lambda_G^i)}\left(\Lambda_G^i - \lambda_B\right)\right)}{\gamma(1 + \Lambda_G^i - \lambda_B)}$$

such that the first exit time associated with the decision problem parametrized by $(\gamma c, R, \gamma r, \gamma \lambda_G^i, \gamma \lambda_B, \gamma)$ is equal to $\hat{\tau}^*(\Lambda_G^i; \gamma)$. In other words, for any set of parameters, $\tau^*(\lambda_G^i) = \hat{\tau}^*(\lambda_G^i/\gamma; \gamma)$. Clearly, proving the single-peakedness of $\hat{\tau}^*$ is equivalent to proving the single-peakedness of τ^*.

By differentiation,

$$\begin{aligned}\hat{\tau}^{*\prime}(\Lambda_G^i; \gamma) = &\frac{1}{\gamma(1 + \Lambda_G^i - \lambda_B)}\Bigg(- \hat{\tau}^*(\Lambda_G^i; \gamma) \\ &+ \Bigg(\frac{1 - \hat{\pi}^*(\Lambda_G^i)}{1 + (1 - \hat{\pi}^*(\Lambda_G^i))(\Lambda_G^i - \lambda_B)} \\ &- \frac{1 + \Lambda_G^i - \Lambda_B}{(1 + (1 - \hat{\pi}^*(\Lambda_G^i))(\Lambda_G^i - \lambda_B))\,\hat{\pi}^*(\Lambda_G^i)}\hat{\pi}^{*\prime}(\Lambda_G^i)\Bigg)\Bigg).\end{aligned} \tag{7}$$

We now define

$$\psi(\Lambda_G^i) := \frac{1 - \hat{\pi}^*(\Lambda_G^i)}{1 + (1 - \hat{\pi}^*(\Lambda_G^i))(\Lambda_G^i - \lambda_B)} - \frac{1 + \Lambda_G^i - \Lambda_B}{(1 + (1 - \hat{\pi}^*(\Lambda_G^i))(\Lambda_G^i - \lambda_B))\,\hat{\pi}^*(\Lambda_G^i)}\hat{\pi}^{*\prime}(\Lambda_G^i).$$

Note that $\psi(\Lambda_G^i)$ does not depend on γ. The following lemma establishes two properties of the function $\psi(\Lambda_G^i)$, which we use later.

Lemma 6.

(i) $\psi'(\Lambda_G^i) < 0$ *a.e. for* $\Lambda_G^i \geq \underline{\Lambda}_G^i$, *for some* $\underline{\Lambda}_G^i \in (c/R + 1/2, \infty)$.

(ii) $\inf_{\Lambda_G^i \in (c/R+1/2, \underline{\Lambda}_G^i]} \psi(\Lambda_G^i) > 0$.

Proof. For (i), differentiating yields

$$\psi'(\Lambda_G^i) = -\left(\frac{1}{1+(1-\hat{\pi}^*(\Lambda_G^i))(\Lambda_G^i-\lambda_B)}\right)^2$$
$$\left(\left(1-\hat{\pi}^*(\Lambda_G^i)\right)^2 + 2\hat{\pi}^{*\prime}(\Lambda_G^i) + (1+\Lambda_G^i-\lambda_B)(1+(1-2\hat{\pi}^*(\Lambda_G^i))(\lambda_G^i-\lambda_B))\left(\frac{\hat{\pi}^{*\prime}(\Lambda_G^i)}{\hat{\pi}^*(\Lambda_G^i)}\right)^2\right)$$
$$-\frac{1+\Lambda_G-\lambda_B}{\hat{\pi}^*(\Lambda_G^i)\left(\gamma+(1-\hat{\pi}^*(\Lambda_G^i))(\Lambda_G^i-\lambda_B)\right)}\hat{\pi}^{*\prime\prime}(\Lambda_G^i).$$

Recall that $\hat{\pi}^{*\prime}(\Lambda_G^i) < 0$ and $\lim_{\Lambda_G^i\to\infty}\hat{\pi}^*(\Lambda_G^i) = 0$. Moreover, by Lemma 4, for sufficiently high Λ_G^i, the terms on the second and third lines are positive a.e. for $\Lambda_G^i > \underline{\Lambda}_G^i$ for some $\underline{\Lambda}_G^i < \infty$. Because $\underline{\Lambda}_G^i < \infty$, result (ii) follows.

□

To conclude, we first show that $\hat{\tau}^*(\Lambda_G^i;\gamma)$ is single-peaked for sufficiently high γ. Observe that $\hat{\tau}^*(\Lambda_G^i;\gamma)$ is (pointwise in Λ_G^i) decreasing in γ. Hence, for sufficiently high γ, for all $\Lambda_G^i \in (c/R+1/2, \underline{\Lambda}_G^i]$, $\hat{\tau}^*(\Lambda_G^i;\gamma) < \inf_{\Lambda_G^i\in(c/R+1/2,\underline{\Lambda}_G^i]}\psi(\Lambda_G^i)$. Consequently, for γ high enough, the function $\hat{\tau}^*(\Lambda_G^i;\gamma)$ is single-peaked as $\hat{\tau}^*(\Lambda_G^i;\gamma)$ crosses the function $\psi(\Lambda_G^i)$ no more than once and from below (see (7)). Therefore, $\hat{\tau}^*(\Lambda_G^i;\gamma)$ must be single-peaked for any γ, because a linear transformation of a single-peaked function is single-peaked.

□

Proof of Lemma 2. Consider firm i at time t following the private history $(N_s^i)_{s\le t}$, and assume

$$\Pr[\omega_t = G \mid (N_s^i)_{s\le t}, \sigma^j \ge t] > \pi^*(\lambda_G^i).$$

Hence, if the firm remains in the market until $t+\mathrm{d}t$, the expected payoff it collects in $[t, t+\mathrm{d}t)$ is bounded below by

$$\Bigg(-c + \left(\Pr[\omega_t = B \mid (N_s^i)_{s\le t}, \sigma^j \ge t]\lambda_B + \Pr[\omega_t = G \mid (N_s^i)_{s\le t}, \sigma^j \ge t]\lambda_G^i\right)$$
$$\cdot\left(R + v\left(\frac{\pi^*(\lambda_G^i)\lambda_G^i}{\pi^*(\lambda_G^i)\lambda_G^i + (1-\pi^*(\lambda_G^i))\lambda_B}\right)\right)\Bigg)\mathrm{d}t > 0,$$

where $v : [0,1] \to \mathbf{R}$ denotes the value function associated with the best-reply problem in Section 4.1. The bound follows from a few observations. First, the last term on the left-hand side is a lower bound on the expected continuation payoff after observing a customer: firm i can only benefit from firm j using a strategy other than never exiting. Second, in $[t, t+\mathrm{d}t)$, firm j may exit the market, but because the expected payoff in the continuation game is weakly positive, we can omit the corresponding term. Last, by the definition of $\pi^*(\lambda_G^i)$, the inequality holds for any $\Pr[\omega_t = G \mid (N_s^i)_{s \le t}, \sigma^j \ge t] > \pi^*(\lambda_G^i)$. Therefore, the result follows. □

Proof of Theorem 1. The proof is divided into two parts. We start with the case of conclusive news, $\lambda_B = 0$. We claim that if $\tau^*(\lambda_G^1) > \tau^*(\lambda_G^2)$, then $(\sigma_0^1, \sigma^2_{\pi^*(\lambda_G^2)})$ is an equilibrium. The result then follows from Lemma 1. First, by definition, $\sigma^2_{\pi^*(\lambda_G^2)}$ is a best reply to σ_0^1. Second, by the proof of Lemma 2, if for some $t \ge 0$, $\Pr[\omega_t = G \mid (N_s^1)_{s \le t}, \sigma^2 \ge t] > \pi^*(\lambda_G^1)$, it is dominant for firm 1 to remain in the market at t. Even if firm 1 did not observe any customer in $[0, \tau^*(\lambda_G^2))$, its belief $\Pr[\omega_t = G \mid (N_s^1)_{s \le t}, \sigma^2 \ge t]$ would jump upward at $\tau^*(\lambda_G^2)$ and remain at some value strictly higher than $\pi^*(\lambda_G^1)$ as long as it does not observe any arrival and firm 2 does not exit. In fact, along a history with no customers, as long as firm 2 does not exit, the belief of firm 1 is constant whenever the last customer was observed more than $\tau^*(\lambda_G^1)$ amount of time ago.[37] It follows that σ_0^1 is a best reply to $\sigma^2_{\pi^*(\lambda_G^2)}$.

We consider now the case of inconclusive news, i.e., $\lambda_B > 0$. We want to prove that for sufficiently high λ_2^G, the strategy profile $(\sigma_0^1, \sigma^2_{\pi^*(\lambda_G^2)})$ is an equilibrium of the game. First, by Lemma 1, for sufficiently high λ_2^G, $2\tau^*(\lambda_2^G) < \tau^*(\lambda_1^G)$. Now, along the path induced by this strategy profile, if firm 2 has not exited by time $t > 0$, the posterior likelihood about the prevailing state is bounded below by

$$\frac{e^{-(\gamma+\lambda_G^1)t}\Pr\left[\Pr\left[\omega_s = G \mid N_t^2\right] > \pi^*\left(\lambda_G^2\right),\ \text{for all } s \le t \mid \omega_t = G\right]}{\displaystyle\int_0^t \gamma e^{-\gamma s} e^{-\lambda_G^1 s} e^{-\lambda_B (t-s)} \left(e^{-\lambda_B (t-s)} \sum_{i=\left\lfloor \frac{t-s}{\tau^*(\lambda_G^2)} \right\rfloor}^{\infty} \frac{\left(\lambda_B^2 (t-s)\right)^i}{i!} \left(1 - e^{-\lambda_B \tau^*\left(\lambda_G^2\right)}\right)^i \right) \mathrm{d}s}$$

This is a lower bound because at any time $s \le t$, firm 2 having observed at least $\lfloor s/\tau^*(\lambda_G^2) \rfloor$ customers, with the time between two customers being no larger than $\tau^*(\lambda_G^2)$ is a necessary but not sufficient condition for it to remain in the market up to time t. Moreover in the denominator, we are writing the probability that firm 2

[37] We discuss this fact at length when explaining Figure 2.

observed sufficiently many customers once the state has transitioned to bad, but we omit the probability of observing sufficiently many customers at earlier times when the state was still good.

As shown in Lemma 7, the denominator converges to zero as $\lambda_G^2 \to \infty$, uniformly in t. It is easy to see that the numerator is always bounded away from zero. As a result, the posterior belief of firm 1 is again bounded away from the cutoff $\pi^*(\lambda_1^G)$ and by Lemma 2, σ_0^1 is a best reply to $\sigma^2_{\pi^*(\lambda_G^2)}$.

□

Lemma 7. *The following holds, uniformly in t,*

$$\lim_{\lambda_G^2 \to \infty} \int_0^t \gamma e^{(\gamma+\lambda_G^1-\lambda_B)(t-s)} \Bigg(e^{-\lambda_B(t-s)} \sum_{i=\left\lfloor \frac{t-s}{\tau^*(\lambda_G^2)} \right\rfloor}^{\infty} \frac{(\lambda_B^2(t-s))^i}{i!} \left(1 - e^{-\lambda_B \tau^*(\lambda_G^2)}\right)^i \Bigg) \mathrm{d}s = 0.$$

Proof. With a slight abuse of notation, let $\gamma(\cdot,\cdot)$ denote the lower incomplete gamma function.[38] The function above is bounded above by

$$\int_0^\infty \gamma e^{\left(\gamma+\lambda_G^1-\lambda_B\right)(t-s)} \frac{\left\lfloor \frac{t-s}{\tau^*\left(\lambda_G^2\right)} \right\rfloor}{\left\lfloor \frac{t-s}{\tau^*\left(\lambda_G^2\right)} \right\rfloor !} \exp\left(-\lambda_B(t-s)e^{-\lambda_B \tau^*(\lambda_G^2)}\right) \gamma\left(\left\lfloor \frac{t-s}{\tau^*(\lambda_G^2)} \right\rfloor, \left(1 - e^{-\lambda_B \tau^*(\lambda_G^2)}\right)(t-s)\,\lambda_B \right) \mathrm{d}s.$$

Applying a change of variable,

$$\int_0^\infty \gamma e^{\left(\gamma+\lambda_G^1-\lambda_B\right)x} \frac{\left\lfloor \frac{x}{\tau^*\left(\lambda_G^2\right)} \right\rfloor}{\left\lfloor \frac{x}{\tau^*\left(\lambda_G^2\right)} \right\rfloor !} \exp\left(-\lambda_B(x)e^{-\lambda_B \tau^*(\lambda_G^2)}\right) \gamma\left(\left\lfloor \frac{x}{\tau^*(\lambda_G^2)} \right\rfloor, \left(1 - e^{-\lambda_B \tau^*(\lambda_G^2)}\right) x\lambda_B \right) \mathrm{d}s.$$

[38] That is, $\gamma(s,x) = \int_0^x y^{s-1} e^{-y}\,\mathrm{d}y$.

Using the fact that $x > s-1$, $\gamma(s,x) < x^s e^{-x}$ and the Stirling's approximation, as for λ_G^2 high enough the first argument of the gamma function is larger than the second and $\tau^*(\lambda_G^2)$ goes to zero, the integral above is no larger than

$$\int_0^\infty \gamma e^{\left(\gamma+\lambda_G^1\right)x}\left(e\tau^*\left(\lambda_G^2\right)\right)^{x/\tau^*(\lambda_G^2)}\,\mathrm{d}x = \gamma \frac{e^{\left(1/\tau^*(\lambda_G^2)+\gamma+\lambda_G^1+\frac{\ln\tau^*(\lambda_G^2)}{\tau^*(\lambda_G^2)}\right)x}}{1/\tau^*(\lambda_G^2)+\gamma+\lambda_1^G+\frac{\ln\tau^*(\lambda_G^2)}{\tau^*(\lambda_G^2)}}\Bigg|_0^\infty$$
$$= -\frac{\gamma}{1/\tau^*(\lambda_G^2)+\gamma+\lambda_1^G+\frac{\ln\tau^*(\lambda_G^2)}{\tau^*(\lambda_G^2)}},$$

provided that λ_G^2 is enough. The term above converges to zero as $\lambda_G^2 \to \infty$, proving the result. □

The proof of Theorem 2 relies on iterated deletion of conditionally dominated strategies. The notion of conditional dominance was introduced by Shimoji and Watson (1998). Informally, according to Shimoji and Watson's (1998) definition, a strategy is conditionally dominated if one can find an information set such that the strategy is strictly dominated when one restricts attention to strategies that are consistent with reaching that information set. Iterative deletion of conditionally dominated strategies is then defined as in the case of normal form games.

Shimoji and Watson (1998) recognize that their result analysis extends to games with incomplete information but do not spell out this extension. In our model, we say that a strategy σ^i is conditionally dominated at some history if there exists another strategy $\hat{\sigma}^i$ that prescribes a different behavior at that history and potentially at some of its successors and agrees with σ^i at any other history, such that for any strategy σ^j that is consistent with that history and any system of beliefs consistent with Bayes' rule, $\hat{\sigma}^i$ yields a strictly higher expected continuation payoff than σ^i.

Proof of Theorem 2.A. We show that for sufficiently high λ_G^2, $(\sigma_0, \sigma_{\pi^*(\lambda_G^2)})$ is the unique strategy profile that survives iterated deletion of conditionally dominated strategies.

First, assume that $\lambda_B = 0$ and $2\tau^*(\lambda_G^2) < \tau^*(\lambda_G^1)$. By Lemma 1, this inequality holds for sufficiently high λ_G^2. Recall that by Lemma 2, in any equilibrium firm i continues operations as long as its belief is above $\pi^*(\lambda_G^i)$. As a result, firms' beliefs at any time before $\tau^*(\lambda_G^2)$ are uniquely determined by their private history.

We now argue that regardless of firm 1's belief about firm 2's strategy, firm 1's posterior along the history with no exit is bounded above $\pi^*(\lambda_G^1)$ at any time before $\tau^*(\lambda_G^1)$.

To this end, we start by showing that at any time $t \leq \tau^*(\lambda_G^1)$, for any strategy of firm 2 that survived our first round of deletion, the probability that firm 2's posterior belief is equal to some $p \in [0, \pi^*(\lambda_G^2))$ is higher in the bad state than in the good state, provided that λ_G^2 is taken to be arbitrary high. This implies that along the history with no exit, observational learning always brings good news, and firm 1's private belief is a lower bound on its posterior belief. The details of the argument are relegated to Section A.1.1, but here we provide some intuition.

If firm 1 expects firm 2 to play a cutoff strategy, as in Rosenberg et al. (2007) and Murto and Välimäki (2011), observational learning always brings good news, that is, firm 2 continuing operation makes firm 1 more optimistic. In fact, the distribution of firm 2's posterior belief conditional on the good state first-order stochastically dominates the distribution of firm 2's posterior belief conditional on the bad state. Once one allows for any non-cutoff strategy, this does need to be true. However, in the limit as λ_G^2 goes to infinity, two things happen. First, the distribution of firm 2's posterior beliefs conditional on either state converges to a degenerate distribution concentrated on either 0 or 1. Second, the range of beliefs for which exiting is not a dominated action shrinks, since $\pi^*(\lambda_G^2) \to 0$. As a result, even if firm 1 expects firm 2 to play a non-cutoff strategy, the probability of firm 2 exiting conditional on the prevailing state being bad is higher than the probability of firm 2 exiting conditional on the state being good. It follows that exiting before $\tau^*(\lambda_G^1)$ is a dominated action for firm 1 and that firm 2's belief at any time before $\tau^*(\lambda_G^1)$ is uniquely determined by its private history.

Consider the case in which firm 2 does not observe any customer in the interval $[0, \tau^*(\lambda_G^2))$. At time $\tau^*(\lambda_G^2)$, the expected continuation payoff of firm 2 from remaining in the market forever is bounded above by

$$\int_{\tau^*(\lambda_G^2)}^{\tau^*(\lambda_G^1)} e^{-r(t-\tau^*(\lambda_G^2))} \left(\pi^*(\lambda_G^2) e^{-\gamma(t-\tau^*(\lambda_G^2))} \lambda_G^2 R - c\right) \mathrm{d}t$$
$$+ e^{-r\left(\tau^*(\lambda_G^1)-\tau^*(\lambda_G^2)\right)} \left(\pi^*(\lambda_G^2) e^{-\gamma\left(\tau^*(\lambda_G^1)-\tau^*(\lambda_G^2)\right)} \frac{(\lambda_G^1+\lambda_G^2)R - c}{r}\right).$$

The expression above is the continuation payoff of firm 2 at time $\tau^*(\lambda_G^2)$ in the hypothetical scenario in which firm 1 exits with probability one at time $\tau^*(\lambda_G^1)$. At that time, firm 2 perfectly learns the state of the world such that, conditional on $\omega_{\tau^*(\lambda_G^1)} = B$, it exits with no delay. Clearly, this is an upper bound on the continuation payoff of firm 2 for any undominated strategy adopted by firm 1.

Observe that by the definition of $\tau^*(\lambda_G^2)$, the integrand is negative; thus the first term is negative. By Lemma 3, as $\lambda_G^2 \to \infty$, $\pi^*(\lambda_G^2)\lambda_G^2 \to 0$. Hence, for sufficiently high λ_2^G, the expected continuation payoff is negative. We can then conclude that conditional on not observing any customer in $[0, \tau^*(\lambda_G^2))$, it is dominant for firm 2 to exit at time $\tau^*(\lambda_G^2)$.

Consider now the case in which the posterior belief of firm 2 at time $2\tau^*(\lambda_G^2)$ is again $\pi^*(\lambda_G^2)$. In this case, the expected continuation payoff from remaining in the market is bounded above by

$$\int_{2\tau^*(\lambda_G^2)}^{\tau^*(\lambda_G^1)} e^{-r(t-2\tau^*(\lambda_G^2))} \left(\pi^*(\lambda_G^2)e^{-\gamma(t-2\tau^*(\lambda_G^2))}\lambda_G^2 R - c\right) \mathrm{d}t \\ + e^{-r\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)} \left(\pi^*(\lambda_G^2)e^{-\gamma\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)} \frac{(\lambda_G^1+\lambda_G^2)R - c}{r}\right). \tag{8}$$

By the same argument as above, for sufficiently high λ_G^2, the expected continuation payoff is negative and exit is dominant for firm 2. A fortiori, for any $t \in (\tau^*(\lambda_G^2), 2\tau^*(\lambda_G^2)]$, it is dominant for firm 2 to exit whenever its belief falls short of $\pi^*(\lambda_G^2)$.

In the case of conclusive news, if firm 1 does not observe an exit at $2\tau^*(\lambda_G^2)$, it is dominant for it not to exit before $\tau^*(\lambda_G^1) + \tau^*(\lambda_G^2)$. In fact, at that time, firm 1 infers that the belief of firm 2 never fell below the cutoff $\pi^*(\lambda_G^2)$ in $[0, 2\tau^*(\lambda_G^2))$. Moreover, in the worst-case scenario, firm 2 observed a customer "right after" $\tau^*(\lambda_G^2)$ (see Figure 7). Consequently, the posterior belief of firm 1 is bounded away from $\pi^*(\lambda_G^1)$ at any time before $\tau^*(\lambda_G^1)+\tau^*(\lambda_G^2)$, and by Lemma 1 remaining in the market is dominant at those times.

To show the desired result, we apply conditional dominance argument recursively. More formally, for any $n = 3, 4, \dots,$ firm 2 finds it dominant to exit at any $t \in [(n-1)\,\tau^*(\lambda_G^2), n\,\tau^*(\lambda_G^2))$ onces its belief falls short of $\pi^*(\lambda_G^2)$; in fact, for any n, the payoff from staying in the market is bounded above by

$$\int_{n\tau^*(\lambda_G^2)}^{(n-2)\tau^*(\lambda_G^2)+\tau^*(\lambda_G^1)} e^{-r(t-n\tau^*(\lambda_G^2))}\left(\pi^*(\lambda_G^2)e^{-\gamma(t-n\tau^*(\lambda_G^2))}\lambda_G^2 R - c\right)\mathrm{d}t \\ + e^{-r\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)}\left(\pi^*(\lambda_G^2)e^{-\gamma\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)}\frac{(\lambda_G^1+\lambda_G^2)R-c}{r}\right).$$

Again, for sufficiently high λ_G^2, this bound is negative and exiting is dominant for firm 2. Given this, firm 1 finds it dominant not to exit at all $t \in [(n-1)\,\tau^*(\lambda_G^2), (n-1)\,\tau^*(\lambda_G^2)+\tau^*(\lambda_G^1))$, irrespective of its private history.

For the case of inconclusive news, we can again apply the limit argument we used in the proof of Theorem 1 to show that $(\sigma_0, \sigma_{\pi^*(\lambda_G^2)})$ is the unique outcome that survives iterated deletion of conditionally dominated strategies. In this case, at time $2\tau^*(\lambda_G^2)$, the continuation payoff of firm 2 is bounded above by (omitting the dependence of π^* and τ^* on λ_B)

$$\int_{2\tau^*(\lambda_G^2)}^{\tau^*(\lambda_G^1)} e^{-r(t-2\tau^*(\lambda_G^2))} \\ \left(\left(\pi^*(\lambda_G^2)e^{-\gamma(t-2\tau^*(\lambda_G^2))}\lambda_G^2 + \left(1-\pi^*(\lambda_G^2)e^{-\gamma(t-2\tau^*(\lambda_G^2))}\right)\lambda_B^2\right)R - c\right)\mathrm{d}t \\ + e^{-r\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)}\left(\pi^*(\lambda_G^2)e^{-\gamma\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)}\frac{(\lambda_G^1+\lambda_G^2)R-c}{r}\right).$$

Recall that this is the continuation payoff at time $2\tau^*(\lambda_G^2)$ in the hypothetical scenario in which firm 1 exits with probability one at time $\tau^*(\lambda_G^1)$; at that time, firm 2 perfectly learns the state and exits with no delay if $\omega_{\tau^*(\lambda_G^1)} = B$, because by assumption, $(\lambda_B^1+\lambda_B^2)R < 0$. Again, the integrand is negative, and by Lemma 3, the second term converges to zero as $\lambda_G^2 \to \infty$. As a result, exiting is dominant for firm 2 at any time before $2\tau^*(\lambda_G^2)$ once its beliefs fall short of $\pi^*(\lambda_G^2)$. Then, by the limit argument, for sufficiently high λ_G^2, the belief of firm 1 is bounded away from $\pi^*(\lambda_G^1)$ at any time in $[\tau^*(\lambda_G^1), \tau^*(\lambda_G^1)+\tau^*(\lambda_G^2))$. Hence, it is dominant for firm 1 not to exit before $\tau^*(\lambda_G^1)+\tau^*(\lambda_G^2)$. The remainder of the proof follows from the same recursive argument as in the previous part.

□

If being a monopolist is profitable in both states, that is, $(\lambda_B^1 + \lambda_B^2)R - c > 0$, then the relevant bound becomes

$$\int_{2\tau^*(\lambda_G^2)}^{\tau^*(\lambda_G^1)} e^{-r(t-2\tau^*(\lambda_G^2))} \Big(\Big(\pi^*(\lambda_G^2) e^{-\gamma(t-2\tau^*(\lambda_G^2))} \lambda_G^2 + \Big(1 - \pi^*(\lambda_G^2) e^{-\gamma(t-2\tau^*(\lambda_G^2))} \Big) \lambda_B^2 \Big) R - c \Big) \mathrm{d}t$$
$$+ e^{-r\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)} \left(\pi^*(\lambda_G^2) e^{-\gamma\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)} \frac{(\lambda_G^1+\lambda_G^2)R-c}{r} \right)$$
$$+ e^{-r\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)} \left(\left(1 - e^{-\gamma\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)} \pi^*(\lambda_G^2)\right) \frac{2\lambda_B R - c}{r} \right).$$

Note that as $\lambda_G^2 \to \infty$, $\tau^*(\lambda_G^2) \to 0$, and $\pi^*(\lambda_G^2)\lambda_G^2 \to 0$. Hence, in the limit, as $\lambda_G^2 \to \infty$, the bound converges to

$$\frac{1 - e^{-r\tau^*(\lambda_G^1)}}{r}(\lambda_G^2 R - c) + \frac{e^{-r\tau^*(\lambda_G^1)}}{r}(2\lambda_B R - c). \tag{9}$$

From equation (5) (see also Lemma 9 below),

$$\tau^*(\lambda_G^1) \geq \frac{1}{\gamma^1 + \lambda_G^1 - \lambda_B} \ln \left(\frac{(\lambda_G^1 - \lambda_B)((\lambda_G^1 + \gamma)R - c)}{\gamma(c - \lambda_B R)} \right).$$

Replacing $\tau^*(\lambda_G^1)$ with this bound in equation (9), we obtain

$$-\frac{c - \lambda_B R}{r} + \frac{\lambda_B R}{r} \left(\frac{(\lambda_G^1 - \lambda_B)((\lambda_G^1 + \gamma)R - c)}{\gamma(c - \lambda_B R)} \right)^{-\frac{r}{\gamma + \lambda_G^1 - \lambda_B}}. \tag{10}$$

We now claim that there exists a set of λ_G^1 such that this bound is negative. Assume that $r > \gamma + \lambda_B$. Then, if $\lambda_G^1 = 2\lambda_B$, equation (10) can be shown to be strictly negative. By continuity, we can then conclude that there exists a set of pairs $\mathcal{L} \in (c/R, \infty) \times (c/R, \infty)$, $\lambda_G^2 > \lambda_G^1$, for which $(\sigma_0^1, \sigma_{\pi^*(\lambda_G^2)}^2)$ is the unique equilibrium.

me

A.1.1 Complements to the Proof of Theorem 2.A

Let σ_2 be a strategy for firm 2 that prescribes remaining in the market when the its posterior belief is strictly higher than $\pi^*(\lambda_G^2)$. We prove that for any such a strategy, for λ_G^2 large enough, observing firm 2 exiting always brings bad news to a firm 1 that has not observed any customer.

First, fix any time $t > \tau^*(\lambda_G^2)$, and assume that the strategy σ_2 prescribes exiting at t if and only if $\Pr[\omega_t = G \mid (N_s^2)_{s \le t}] = p$, where $p \in (0, \pi^*(\lambda_2^G)]$.

Lemma 8. *For any λ_G^1, there exists a $\overline{\lambda}_G^2$ such that for any $\lambda_G^2 > \overline{\lambda}_G^2$, any $p \in (0, \pi^*(\lambda_G^2)]$, and any $t \in (\tau^*(\lambda_G^2), \tau^*(\lambda_G^1)]$,*

$$\frac{\Pr\left[\omega_t = G \mid N_t^1 = 0, \Pr[\omega_t = G \mid (N_s^2)_{s \le t}] = p\right]}{\Pr\left[\omega_t = B \mid N_t^1 = 0, \Pr[\omega_t = G \mid (N_s^2)_{s \le t}] = p\right]} < \frac{\Pr\left[\omega_t = G \mid N_t^1 = 0\right]}{\Pr\left[\omega_t = B \mid N_t^1 = 0\right]}.$$

Proof. For any λ_G^2, the likelihood ratio on the left-hand side can be written as

$$\frac{\Pr\left[N_t^1 = 0, \Pr[\omega_t = G \mid (N_s^2)_{s \le t}] = p \mid \omega_t = G\right]}{\Pr\left[N_t^1 = 0, \Pr[\omega_t = G \mid (N_s^2)_{s \le t}] = p \mid \omega_t = B\right]} \frac{e^{-\gamma t}}{(1 - e^{-\gamma t})}.$$

The numerator is equal to

$$\begin{aligned} e^{-\gamma t} \Pr\left[N_t^1 = 0 \mid \omega_t = G\right] \Pr\left[\Pr[\omega_t = G \mid (N_s^2)_{s \le t}] = p \mid \omega_t = G\right] \\ = e^{-\gamma t} e^{-(\gamma + \lambda_G^1)t} \frac{p}{1-p} \Pr\left[\Pr[\omega_t = G \mid (N_s^2)_{s \le t}] = p \mid \omega_t = B\right]. \end{aligned}$$

Combining these observations, the posterior likelihood ratio is bounded above by

$$\frac{e^{-(\gamma + \lambda_G^1)t} \frac{p}{1-p} \Pr\left[\Pr[\omega_t = G \mid (N_s^2)_{s \le t}] = p \mid \omega_t = B\right]}{\Pr\left[\Pr[\omega_t = G \mid (N_s^2)_{s \le t}] = p, N_t^1 = 0 \mid \omega_t = B\right] (1 - e^{-\gamma t})}$$

which is lower than the left-hand side in the statement, that is, lower than

$$\frac{e^{-(\gamma + \lambda_G^1)t}}{\frac{\gamma}{\gamma + \lambda_G^1 - \lambda_B^1}\left(e^{-\lambda_B t} - e^{-(\gamma + \lambda_G^1)t}\right)}$$

provided that λ_G^2 is high enough, as $p \le \pi^*(\lambda_G^2)$. $\square$

Lemma 8 readily generalizes to the case when the strategy σ^2 prescribes exiting at t if and only if $\Pr[\omega_t = G \mid (N_s^2)_{s \le t}] \in E \subseteq (0, \pi^*(\lambda_2^G)]$, and to the case of mixed strategies.

These observations imply that, regardless of firm 1's belief about firm 2's strategy, firm 1's posterior along the history with no exit is bounded above $\pi^*(\lambda_G^1)$ at any time before $\tau^*(\lambda_G^1)$. Hence, along the history with no exit, remaining in the market at any time $t \leq \tau^*(\lambda_G^1)$ is a dominant action for firm 1.

As explained in the proof of Theorem 2.A, the game exhibits a recursive structure. As a result, once we prove that at any $t \leq 2\tau^*(\lambda_G^2)$, exiting is dominant for firm 2 whenever its belief is below $\pi^*(\lambda_G^2)$, Lemma 8 implies that along any history with no exit, the belief of firm 1 is bounded away from $\pi^*(\lambda_G^2)$ at any time $t \in [\tau^*(\lambda_G^2), \tau^*(\lambda_G^2) + \tau^*(\lambda_G^1))$. Therefore, remaining on the market is a dominant action. The same argument applies at each of the countably many rounds of deletion.

Proof of Theorem 2.B. First, for any c and R, we can choose λ_G^2 arbitrarily close to c/R such that $2\tau^*(\lambda_G^1) < \tau^*(\lambda_G^2)$.

Second, we show that for firm 1, exiting before $\tau^*(\lambda_1^G)$ is a dominated action. As in Theorem 2.A, we argue that firm 1's posterior along the history with no exit is bounded above $\pi^*(\lambda_G^1)$ at any time before $\tau^*(\lambda_G^1)$. The formal proof is in Section A.1.2. Here we provide an informal argument.

We show that in the limit as $R/c \to \infty$ and $\gamma \to \infty$, for λ_G^2 appropriately chosen, firm 2 not exiting always brings good news to firm 1, regardless of which strategy firm 1 expects firm 2 to play, among the strategies surviving our first round of deletion. Intuitively, the inference that firm 1 draws from observing the action of firm 2 always concerns the state at some point in time in the past, not about the prevailing state. Hence, as $\gamma \to \infty$, this inference plays a limited role in determining firm 1's posterior belief, which can be shown to be bounded away from $\pi^*(\lambda_G^1)$ at any time before the first exit time $\tau^*(\lambda_G^1)$.

Third, proceeding as in the proof of Theorem 2.A, we show that (8) is negative provided that R/c and r sufficiently high. That is,

$$\int_{2\tau^*(\lambda_G^2)}^{\tau^*(\lambda_G^1)} e^{-r(t-2\tau^*(\lambda_G^2))} \left(\pi^*(\lambda_G^2) e^{-\gamma(t-2\tau^*(\lambda_G^2))} \lambda_G^2 R - c\right) \mathrm{d}t \\ + e^{-r\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)} \left(\pi^*(\lambda_G^2) e^{-\gamma\left(\tau^*(\lambda_G^1)-2\tau^*(\lambda_G^2)\right)} \frac{(\lambda_G^1+\lambda_G^2)R-c}{r}\right) < 0.$$

Again, by the definition of $\pi^*(\lambda_G^2)$, the first term is negative. By Lemma 9, for sufficiently high R/c and $r > \lambda_G^1$, the second term converges to zero whenever λ_G^2 is chosen to be arbitrarily close to c/R. Crucially, the first integral in the equation above remains bounded away from zero as we take this limit because $\tau^*(\lambda_G^1)$

is increasing in R/c. Following the same steps as before, we can then prove that $(\sigma_0^1, \sigma^2_{\pi^*(\lambda_G^2)})$ is the unique strategy profile that survives iterated deletion of dominated strategies. □

Lemma 9. *If* $r > \lambda_G^1$,

$$e^{-(r+\gamma)\tau^*(\lambda_G^1)}\frac{\lambda_G^1 R}{r} \to 0,$$

as $R \to \infty$ *or* $c \to 0$.

Proof. We will derive a lower bound for $\tau^*(\lambda_G^1)$ by identifying an upper bound for $\pi^*(\lambda_G^1)$. From (5), replacing $v(j(\pi^*(\lambda_G^i)))$ with 0, we obtain

$$\tau^*(\lambda_G^1) \geq \frac{1}{\gamma + \lambda_G^1} \ln\left(-\frac{\lambda_G^1}{r} + \frac{\gamma + \lambda_G^1}{\gamma} \cdot \frac{\lambda_G^1 R}{c}\right).$$

Hence,

$$e^{-(r+\gamma)\tau^*(\lambda_G^1)}\frac{\lambda_G^1 R}{r} \leq \left(-\frac{\lambda_G^1}{r} + \frac{\gamma + \lambda_G^1}{\gamma} \cdot \frac{\lambda_G^1 R}{c}\right)^{-\frac{\gamma+r}{\gamma+\lambda_G^1}} \frac{\lambda_G^1 R}{r}.$$

If $r > \lambda_G^1$, the right-hand side converges to 0 as $R \to \infty$ or $c \to 0$. In fact, the necessary condition for the right-hand side to converge to 0 is that as $R \to \infty$, R/c converges to infinity in the order $O(R)$. □

A.1.2 Complements to the Proof of Theorem 2.B

Abusing notation, let $\tau^*(\lambda_G^i, \gamma)$ and $\pi^*(\lambda_G^i, \gamma)$ be the first exit time and the cutoff belief as a function of the customer arrival rate and the rate of transition to the bad state, γ. Notice that $\tau^*(\lambda_G^i, \gamma)$ is monotonically decreasing in γ.

Lemma 10. *For* γ *and* R *high enough, there exists an open set of pairs* $(\lambda_G^1, \lambda_G^2) \in (c/R, \infty) \times (c/R, \infty)$ *such that* $\lambda_G^2 < \lambda_G^1$, $\tau^*(\lambda_G^2, \gamma) < \tau^*(\lambda_G^1, \gamma)$, *and for any* $p \in (0, \pi^*(\lambda_G^2, \gamma)]$ *and any* $t \in (\tau^*(\lambda_G^2, \gamma), \tau^*(\lambda_G^1, \gamma)]$,

$$\frac{\Pr\left[\omega_t = G \mid N_t^1 = 0, \Pr[\omega_t = G \mid (N_s^2)_{s\leq t}] = p\right]}{\Pr\left[\omega_t = B \mid N_t^1 = 0, \Pr[\omega_t = G \mid (N_s^2)_{s\leq t}] = p\right]} < \frac{\Pr\left[\omega_t = G \mid N_t^1 = 0\right]}{\Pr\left[\omega_t = B \mid N_t^1 = 0\right]}.$$

Proof. Proceeding as in the proof of Lemma 8, the left-hand side is bounded above by the following

$$\frac{e^{-(\lambda_G^1+\lambda_G^2+\gamma)\tau^*(\lambda_G^2,\gamma)}}{\frac{\gamma}{\lambda_G^1+\lambda_G^2+\gamma}\left(1-e^{-(\lambda_G^1+\lambda_G^2+\gamma)\tau^*(\lambda_G^2,\gamma)}\right)}, \tag{11}$$

while the right-hand side is

$$\frac{e^{-(\lambda_G^1+\gamma)t}}{\frac{\gamma}{\lambda_G^1+\gamma}\left(1-e^{-(\lambda_G^1+\gamma)t}\right)}. \tag{12}$$

First, as argued in the proof of Theorem 2.B, λ_G^2 can be chosen to be arbitrarily close to c/R so that (11) is arbitrarily close to zero. Second, because the first exit time is strictly decreasing in γ and strictly increasing in R, for any γ, R can be chosen so that $\gamma \max_{\lambda_G^i} \tau^*(\lambda_G^i, \gamma) > k$, where $k > 0$ is an arbitrary constant. As a result, for any γ and R high enough, choosing λ_G^1 arbitrarily close to $\arg\max_{\lambda_G^i} \tau^*(\lambda_G^i, \gamma)$, (12) is bounded away from zero.

□

Proof of Proposition 2. We show that if (8) is negative for some c^2, then it is also negative for $c^{2\prime} > c^2$. Abusing notation, let $\tau^*(\lambda_G^2, c^2)$ and $\pi^*(\lambda_G^2, c^2)$ be the first exit time and the cutoff belief as a function also of the operating cost c^2. If

$$\int_{2\tau^*(\lambda_G^2,c^2)}^{\tau^*(\lambda_G^1,c^1)} e^{-r(t-2\tau^*(\lambda_G^2,c^2))}\left(\pi^*(\lambda_G^2,c^2)e^{-\gamma(t-2\tau^*(\lambda_G^2,c^2))}\lambda_G^2 R - c^2\right)\mathrm{d}t$$
$$+e^{-r\left(\tau^*(\lambda_G^1,c^1)-2\tau^*(\lambda_G^2,c^2)\right)}\left(\pi^*(\lambda_G^2,c^2)e^{-\gamma\left(\tau^*(\lambda_G^1,c^1)-2\tau^*(\lambda_G^2,c^2)\right)}\frac{(\lambda_G^1+\lambda_G^2)R-c^2}{r}\right)<0,$$

then exiting at $2\tau^*(\lambda_G^2, c^2)$ following a history with no customer is dominant for firm 2 also when its cost is $c^{2\prime} > c^2$. Hence, if firm 1 does not observe an exit at $2\tau^*(\lambda_G^2, c^2)$, it is dominant for it not to exit before $\tau^*(\lambda_G^1, c^2) + \tau^*(\lambda_G^2, c^1)$, regardless of the operating cost of firm 2. Applying the logic recursively, we can delete all strategies of firm 1 but σ_0^1. Then, any strategy of firm 2 other than $\sigma^2_{\pi^*(\lambda_G^2,c^{2\prime})}$ can be deleted. It follows that the only strategy profile that survives iterated deletion of dominated strategies is $(\sigma_0^1, \sigma^2_{\pi^*(\lambda_G^2,c^{2\prime})})$. For the case of inconclusive news, the result follows from combining the argument above and the limit in the proof of Theorem 2.A.

Last, the proof of the comparative statics result with respect to r^2 follows from a similar argument. □

A.2 Proofs for Section 3.2

With some abuse of notation, we denote by σ_p^j the pure strategy that prescribes exiting once the public belief about the prevailing state falls below a cutoff $p > 0$. Proceeding as in the Section 4.1, we start by analyzing firm i's best-reply problem to σ_0^j. In the continuation region, the value function of firm i when best replying to σ_0^j satisfies the following Hamilton-Jacobi-Bellman equation

$$rv^i(p) = p\left(\lambda_G^i + \lambda_G^j\right)\left(v^i\left(j\left(p\right)\right) - v^i(p)\right) - c + p\lambda_G^i R \\ - \left(p(1-p)(\lambda_G^i + \lambda_G^j) + p\gamma\right) v^{i\prime}(p).$$

As a result, whenever $\lambda_G^i > c/R$, the optimal cutoff of firm i satisfies the following equation

$$\mathring{\pi}^*(\lambda_G^i, \lambda_G^j) = \frac{c}{\left(\lambda_G^i + \lambda_G^j\right) v^i\left(j(\mathring{\pi}^*(\lambda_G^i, \lambda_G^j))\right) + \lambda_G^i R} \tag{13}$$

Note that in contrast to the case of unobservable customers, the optimal cutoff of firm i depends on both λ_G^i and λ_G^j. In fact, using the implicit function theorem, it is readily verified that $\mathring{\pi}^*(\lambda_G^i, \lambda_G^j)$ is decreasing both in λ_G^i and in λ_G^j, and $\lim_{\lambda_G^i \to \infty} \mathring{\pi}^*(\lambda_G^i, \lambda_G^j) = \lim_{\lambda_G^j \to \infty} \mathring{\pi}^*(\lambda_G^i, \lambda_G^j) = 0$.

It can be shown that, again, $\mathring{\pi}^*(\lambda_G^i, \lambda_G^j) < 1$ if and only if $\lambda_G^i R > c$.

Claim 1. *If $\lambda_G^2 > \lambda_G^1$, $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2) > \mathring{\pi}^*(\lambda_G^2, \lambda_G^1)$.*

Proof. The value functions of the two firms are ranked pointwise, while the function j is identical for both of them. It follows that $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2) > \mathring{\pi}^*(\lambda_G^2, \lambda_G^1)$. □

Proof of Proposition 1. By the same argument as in Lemma 2, in any equilibrium, it is dominant for firm i to exit whenever the belief is strictly above $\mathring{\pi}^*(\lambda_G^i, \lambda_G^j)$. If $\lambda_G^2 > \lambda_G^1$, then $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2) > \mathring{\pi}^*(\lambda_G^2, \lambda_G^1)$, and by the same argument as in Theorem 1 the strategy profile $(\sigma^1_{\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)}, \sigma_0^2)$ is an equilibrium.

To show that it is the unique equilibrium provided that R/c is high enough and λ_G^1 is low enough, we show that when the belief is equal to $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)$, firm 1 finds it dominant to exit. Note that the continuation payoff of firm 1 at a belief of $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)$ is bounded above by

$$\begin{aligned}
&\int_0^{\frac{\ln\left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1,\lambda_G^2))}\right)}{\gamma+\lambda_G^1+\lambda_G^2-2\lambda_B}} e^{-rt}\Big(\big(\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)e^{-\gamma t}\lambda_1^G + \big(1-\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)e^{-\gamma t}\big)\,\lambda^B\big)\,R \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - c\Big)\,\mathrm{d}t \\
&+ e^{-r\frac{\ln\left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1,\lambda_G^2))}\right)}{\gamma+\lambda_G^1+\lambda_G^2-2\lambda_B}}\left(\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)e^{-\gamma\frac{\ln\left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1,\lambda_G^2))}\right)}{\gamma+\lambda_G^1+\lambda_G^2-2\lambda_B}}\frac{(\lambda_G^1+\lambda_G^2)R-c}{r}\right. \\
&\qquad\qquad\left. + \left(1-\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)e^{-\gamma\frac{\ln\left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1,\lambda_G^2))}\right)}{\gamma+\lambda_G^1+\lambda_G^2-2\lambda_B}}\right)\frac{\max\{2\lambda_B R-c, 0\}}{r}\right)
\end{aligned} \tag{14}$$

where

$$\mathring{\Omega}(p) = \frac{\gamma + (1-p)(\lambda_G^1+\lambda_G^2-2\lambda_B)}{p}.$$

If $r > \lambda_G^1 + \lambda_G^2 - 2\lambda_B$,

$$\begin{aligned}
e^{-(r+\gamma)\frac{\ln\left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1,\lambda_G^2))}\right)}{\gamma+\lambda_G^1+\lambda_G^2-2\lambda_B}}\frac{(\lambda_G^1+\lambda_G^2)R-c}{r} &= \left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2, \lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1, \lambda_G^2))}\right)^{\frac{r+\gamma}{\gamma+\lambda_G^1+\lambda_G^2-2\lambda_B}}\frac{(\lambda_G^1+\lambda_G^2)R-c}{r} \\
&< \left(\frac{\mathring{\Omega}(c/(\lambda_G^2 R))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1, \lambda_G^2))}\right)^{\frac{r+\gamma}{\gamma+\lambda_G^1+\lambda_G^2-2\lambda_B}}\frac{(\lambda_G^1+\lambda_G^2)R-c}{r}.
\end{aligned}$$

Because

$$\begin{aligned}
&\lim_{R\to\infty}\left(\mathring{\Omega}(c/(\lambda_G^2 R))\right)^{\frac{r+\gamma}{\gamma+\lambda_G^1+\lambda_G^2-2\lambda_B}}\frac{(\lambda_G^1+\lambda_G^2)R-c}{r} \\
&\qquad\qquad = \lim_{c\to 0}\left(\mathring{\Omega}(c/(\lambda_G^2 R))\right)^{\frac{r+\gamma}{\gamma+\lambda_G^1+\lambda_G^2-2\lambda_B}}\frac{(\lambda_G^1+\lambda_G^2)R-c}{r} = 0,
\end{aligned}$$

and since $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)$ can be taken to be arbitrarily close to 1 provided that λ_G^1 is sufficiently small, the second term in equation (14) converges to zero. As a result, exiting at the cutoff belief $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)$ is dominant for firm 1, and $(\sigma^1_{\mathring{\pi}^*(\lambda_G^1,\lambda_G^2)}, \sigma_0^2)$ is the unique equilibrium.

□

It is easy to construct parametric examples for which $\lambda_G^2 > \lambda_G^1$, and $(\sigma_0^1, \sigma^2_{\mathring{\pi}^*(\lambda_G^2,\lambda_G^1)})$ is not an equilibrium because at the belief $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)$, firm one prefers to exit rather than waiting to benefit from monopoly profits. The following lemma describes the construction of the mixed strategy equilibrium we discussed in Section 3.2.

Lemma 11. *Suppose that $\lambda_G^2 > \lambda_G^1$ and $\lambda_B = 0$. If both $(\sigma_0^1, \sigma^2_{\mathring{\pi}^*(\lambda_G^2,\lambda_G^1)})$ and $(\sigma^1_{\mathring{\pi}^*(\lambda_G^1,\lambda_G^2)}, \sigma_0^2)$ are equilibria, there exists a mixed strategy equilibrium such that*

(i) *firm 2 exits with probability $q > 0$ when the posterior belief is equal to $\mathring{\pi}^*(\lambda_G^2, \lambda_G^1)$;*

(ii) *both firms exit at a positive rate when the belief belongs to $(\pi^\ddagger(\lambda_G^2+\lambda_G^1), \mathring{\pi}^*(\lambda_G^2, \lambda_G^1))$, where $\pi^\ddagger(\lambda_G^2 + \lambda_G^1) > 0$ is the belief at which a monopolist optimally exits.*

Proof. Recall that $v^i : [0, 1] \to \mathbf{R}$ denotes the payoff associated with firm i's best reply to σ_0^j. Let $W : [0, 1] \to \mathbf{R}$ denote the payoff associated with the monopolist's problem.

The equilibrium we construct yields ex-ante expected payoffs equal to $(v^1(1), v^2(1))$. The probability q is chosen such that at the belief $\mathring{\pi}^*(\lambda_G^1, \lambda_G^2)$, firm 1 is indifferent between exiting and waiting to exit as soon as the belief falls short of $\mathring{\pi}^*(\lambda_G^2, \lambda_G^1)$, provided that firm 2 does not exit then. That is, q satisfies

$$\begin{aligned}\int_0^{\frac{\ln\left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1,\lambda_G^2))}\right)}{\gamma+\lambda_G^1+\lambda_G^2}} & \mathring{\pi}^*(\lambda_G^1, \lambda_G^2) e^{-(\gamma+\lambda_G^1+\lambda_G^2)t}(\lambda_G^1+\lambda_G^2)\Bigg(\frac{\lambda_1^G}{\lambda_1^G+\lambda_2^G}e^{-rt}R + e^{-rt}v^1(1) \\ & - \frac{1-e^{-rt}}{r}c\Bigg)\,\mathrm{d}t \\ & + \left(1 - \frac{\lambda_G^1+\lambda_G^2}{\lambda_G^1+\lambda_G^2+\gamma}\mathring{\pi}^*(\lambda_G^1,\lambda_G^2)\left(1 - e^{-\left(\lambda_G^1+\lambda_G^2+\gamma\right)\frac{\ln\left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1,\lambda_G^2))}\right)}{\gamma+\lambda_G^1+\lambda_G^2}}\right)\right) \\ & \left(-\frac{c}{r}\left(1 - e^{-r\frac{\ln\left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1,\lambda_G^2))}\right)}{\gamma+\lambda_G^1+\lambda_G^2}}\right) + e^{-r\frac{\ln\left(\frac{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^2,\lambda_G^1))}{\mathring{\Omega}(\mathring{\pi}^*(\lambda_G^1,\lambda_G^2))}\right)}{\gamma+\lambda_G^1+\lambda_G^2}} qW\left(\mathring{\pi}^*(\lambda_G^2,\lambda_G^1)\right)\right) = 0.\end{aligned}$$

Clearly, exiting at some belief in $(\mathring{\pi}^*(\lambda_G^2, \lambda_G^1), \mathring{\pi}^*(\lambda_G^1, \lambda_G^2))$ is suboptimal. At any belief $p \in (\pi^\ddagger(\lambda_G^2 + \lambda_G^1), \mathring{\pi}^*(\lambda_G^2, \lambda_G^1))$, firm i exits at a rate

$$-\frac{p(\lambda_1^G+\lambda_2^G)\left(\frac{\lambda_G^j}{\lambda_1^G+\lambda_2^G}R + v^j(1)\right) - c}{W(p)}.$$

It can be verified that this rate is positive and bounded for any $p \in (\pi^{\ddagger}(\lambda_G^2 + \lambda_G^1), \mathring{\pi}^*(\lambda_G^2, \lambda_G^1))$. Consequently, along a path with no costumers, there is a strictly positive probability that both firms remain in the market once the belief reaches any given $p \in (\pi^{\ddagger}(\lambda_G^2 + \lambda_G^1), \mathring{\pi}^*(\lambda_G^2, \lambda_G^1))$. However, because the rate diverges to infinity at $\pi^{\ddagger}(\lambda_G^2 + \lambda_G^1)$, as the denominator converges to zero, no firm will remain in the market at a belief lower than the monopoly cutoff. The rate is chosen to guarantee that each firm is indifferent between exiting and remaining in the market and exiting in the next instant if no customer arrives. $\square$