Decentralized Trading with Private Information*

Michael Golosov
Princeton University

Guido Lorenzoni
Northwestern University

Aleh Tsyvinski
Yale University

February 2014

Abstract

The paper studies how asset prices are determined in a decentralized market with asymmetric information about asset values. We consider an economy in which a large number of agents trade two assets in bilateral meetings. A fraction of the agents has private information about the asset values. We show that, over time, uninformed agents can elicit information from their trading partners by making small offers. This form of experimentation allows the uninformed agents to acquire information as long as there are potential gains from trade in the economy. As a consequence, the economy converges to a Pareto efficient allocation.

*We are grateful to the editor and three anonymous referees for helpful suggestions. We also thank Daron Acemoglu, Fernando Alvarez, Manuel Amador, Abhijit Banerjee, Gadi Barlevy, V.V. Chari, Darrell Duffie, Georgy Egorov, John Geanakoplos, Veronica Guerrieri, Patrick Kehoe, Lasse Pedersen, Dimitri Vayanos, Pierre Olivier Weill, Asher Wolinsky, Pierre Yared and seminar audiences at various universities and conferences for useful comments. We are grateful to our discussants Markus Brunnermeier and Nicolae Garleanu for useful comments and suggestions.

1 Introduction

This paper studies trading and information diffusion in a decentralized market with private information. We consider an economy with two key frictions: trading takes place through bilateral meetings and some agents have private information on the value of the assets traded. In financial markets, a large number of transactions take place not in centralized exchanges, but in decentralized, over-the-counter markets. Duffie, Garleanu, and Pedersen (2005) started a literature that uses random matching and bilateral trading to model over-the-counter markets.[1] In this paper, we study a model with random matching and bilateral trading in which some market participants have private information on the value of the assets traded and analyze how information gradually spreads through the economy. In particular, we ask whether all relevant information is revealed over time and whether the allocation converges to a Pareto efficient allocation.

Our environment is as follows. Agents start with different endowments of two risky assets, match randomly, and trade in bilateral meetings. In each bilateral meeting, one of the agents makes a take-it-or-leave-it offer to the other, who can accept or reject. Therefore, apart from the presence of private information, we have a trading game in the tradition of Gale (1986, 1987).[2] Agents are risk averse so there is the potential for mutually beneficial trades of the two assets. However, before trading begins, a fraction of agents—the informed agents—receive some information about the value of the assets. Namely, they observe a binary signal that determines which one of the two assets is more valuable. The game ends at a random time, at which point the asset payoffs are revealed, and the agents consume. Uninformed agents form beliefs about the value of the two assets based on their individual trading history, which is the only information they receive during trading.

Our objective is to characterize the efficiency properties of the allocation and the value of information in the long run. Our main result is that in the long run the equilibrium converges to an ex post Pareto efficient allocation and the value of information goes to zero. Our argument is as follows. First, we focus on the informed agents and prove that their marginal rates of substitution converge. The intuition for this result is similar to the proof of Pareto efficiency in decentralized environments with common information: if two informed agents have different marginal rates of substitution, they can always find a trade that improves the utility of both. We then show that the marginal rates of substitution of uninformed agents also converge. Our argument is based on finding strategies that allow the uninformed agents to learn the signal received by the informed agents at an arbitrarily small cost. The existence of such strategies implies that two cases are possible: either uninformed agents eventually learn the signal, or

[1] See, among others, Duffie, Garleanu and Pedersen (2007), Lagos (2007), Lagos and Rocheteau (2007), Lagos, Rocheteau, and Weill (2007), Vayanos (1998), Vayanos and Weill (2007), and Weill (2007).

[2] See Gale (2000) for a general treatment of matching and bargaining games with symmetric information and Lauerman (2010) for a recent characterization.

the benefit of learning the signal goes to zero. Both cases imply that the marginal rates of substitution of all agents are equalized. We can then show that equilibrium allocations converge to ex post Pareto efficient allocations in the long run.

Our work is related to Wolinsky's (1990) seminal article on information revelation in pairwise matching environments. Wolinsky (1990) considers a game with decentralized, bilateral trading in which agents have the option to trade an indivisible good of uncertain quality, at given prices. In his game, a fraction of traders exits in each period and is replaced by new traders. He shows that steady state equilibria are possible in which some trades that would be Pareto improving under symmetric information do not take place. That is, he obtains an inefficiency result. Blouin and Serrano (2001) show that this inefficiency result survives in a version of Wolinsky's model with a fixed population of traders, which is thus closer to our environment. The crucial differences between our setup and the models in Wolinsky (1990) and Blouin and Serrano (2001) are that in our setting the good is perfectly divisible and that agents can choose at what price to trade. These assumptions lead to different implications in terms of efficiency, leading to equilibria in which information is fully revealed and allocations are ex post efficient in the long run. Intuitively, divisibility allows uninformed agents to strategically experiment by making small, potentially unprofitable trades to learn valuable information. In this strand of literature, an early paper that explores the potential for uninformed agents to learn through trading is Green (1992). Green's (1992) objective is to find sufficient conditions on equilibrium strategies and on the span of traded assets that ensure that uninformed agents can perfectly elicit the information of their trading partners in equilibrium. Although our goal here is different—to prove long run efficiency—we share his interest in characterizing the learning strategies of uninformed agents.

In the literature on asset pricing in decentralized markets a few recent papers deal with information transmission through bilateral meetings: Duffie and Manso (2007), Duffie, Giroux, and Manso (2009), Duffie, Malamud, and Manso (2009a, 2009b). These papers characterize in closed form the dynamics of beliefs in models in which agents perfectly share the information of the agents they meet. The main difference with our work is that these papers make assumptions which ensure that information is perfectly transmitted in each bilateral meeting. The setup in our paper instead is such that agents may prefer a trading behavior which is not perfectly revealing. For example, an informed agents trying to sell the less valuable asset may decide to mimic an uninformed agent with a large endowment of the same asset to trade at more favorable price. Therefore, in our environment the speed at which information is transmitted in bilateral meetings is endogenous and the ability of uninformed traders to elicit this information is at the center of our analysis. Finally, a recent related paper is Ostrovsky (2009), who studies the incentives of large, strategic traders in dynamic, centralized markets, and shows that information gets aggregated in equilibrium.

The paper is structured as follows. Section 2 describes the environment. Section 3 contains our main result on long-run efficiency. Section 4 concludes. The Appendix contains details of the proofs omitted in the paper.

2 Setup and trading game

In this section, we introduce the model and define an equilibrium.

2.1 Setup

There are two states of the world $S \in \{S_1, S_2\}$ and two assets $j \in \{1, 2\}$. Asset j is an Arrow security that pays one unit of consumption in state S_j. There is a continuum of agents with von Neumann-Morgenstern expected utility $E[u(c)]$, where E is the expectation operator. At date 0, each agent is randomly assigned a type i, which determines his initial portfolio of the two assets, denoted by the vector $x_{i,0} \equiv (x_{i,0}^1, x_{i,0}^2)$. There is a finite set of types N and each type $i \in N$ is assigned to a fraction ν_i of agents. The aggregate endowment of each asset is equal in the two states and normalized to 1:

$$\sum_{i \in N} \nu_i x_{i,0}^j = 1 \text{ for } j = 1, 2. \tag{1}$$

We make the following assumptions on preferences and endowments. The first assumption is symmetry of the endowments.

Assumption 1. **(Symmetry)** *For each type $i \in N$ there exists a type $j \in N$ of equal mass $\nu_j = \nu_i$, holding symmetric endowments $x_{j,0} = (x_{i,0}^2, x_{i,0}^1)$.*

The role of this assumption is discussed in detail in Section 4. The second assumption imposes usual properties on the utility function, as well as boundedness from above and a condition ruling out zero consumption in either state.

Assumption 2. *The utility function $u(\cdot)$ is increasing, strictly concave, twice continuously differentiable on R_{++}^2, bounded above, and satisfies $\lim_{c \to 0} u(c) = -\infty$.*

Finally, we assume that the initial endowments are interior.

Assumption 3. *The initial endowment $x_{i,0}$ is in the interior of R_+^2 for all types $i \in N$.*

At date 0, nature draws a binary signal s that takes the values s_1 and s_2 with equal probabilities. The posterior probability of S_1 conditional on s is denoted by $\phi(s)$. We assume that signal s_1 is favorable to state S_1 and that the signals are symmetric: $\phi(s_1) > 1/2$ and $\phi(s_2) = 1 - \phi(s_1)$. After s is realized, a fraction α of agents of each type privately observes the realization of s. The agents who observe s are called *informed agents*, those who do not observe it are called *uninformed agents*.

2.2 Trading

After the realization of the signal s, but before the state S is revealed, all agents engage in a trading game set in discrete time. At the beginning of each period $t \geq 1$, the game continues with probability $\gamma \in (0,1)$ and ends with probability $1-\gamma$. If the game ends, the state S is publicly revealed and the agents consume the asset payoffs.[3] If the game continues, all agents are randomly matched in pairs and a round of trading takes place. One of the two agents is selected as the *proposer* with probability 1/2. The proposer makes a take-it-or-leave-it offer $z = \left(z^1, z^2\right) \in R^2$ to the other agent, the *responder.* That is, the proposer offers to exchange z^1 of asset 1 for $-z^2$ of asset 2. The responder can accept or reject the offer. If an agent with portfolio x offers z to an agent with portfolio $\tilde{x}$ and the offer is accepted, their end-of-period portfolios are, respectively, $x - z$ and $\tilde{x} + z$. We assume that the proposer can only make feasible offers, $x - z \geq 0$, and the responder can only accept an offer if $\tilde{x} + z \geq 0$.[4]

An agent does not observe the portfolio of his opponent or whether his opponent is informed or not. Moreover, an agent only observes the trades he is involved in but not those of other agents. Therefore, both trading and information revelation take place through decentralized, bilateral meetings.

2.3 Equilibrium definition

We now define an equilibrium. First, let us introduce some notation for individual histories. At date 0, each agent is assigned the type i that determines his initial portfolio, with probability α he is informed and observes the signal s, with probability $1-\alpha$ remains uninformed. The initial history $h_0 \in N \times \{U, I_1, I_2\}$ captures this initial condition (U stands for uninformed, I_j stands for informed with signal s_j). In each period $t \geq 1$, the event $h_t = (\iota_t, z_t, r_t)$ includes the indicator variable ι_t, equal to 1 if the agent is selected as the proposer, the offer made by the proposer $z_t \in R^2$, and the indicator variable r_t, equal to 1 if the offer is accepted. The sequence $h^t = \{h_0, h_1, ..., h_t\}$ denotes the history of play up to period t for an individual agent. H^t denotes the space of all possible histories of length t and H^∞ denotes the space of all infinite histories. Letting $\Omega = \{s_1, s_2\} \times H^\infty$, a point in Ω describes an infinite history of play for an individual agent, if the game continues forever. We use (s, h^t) to denote the subset of Ω given by all the $\omega = (s, h^\infty)$ such that the first t elements of h^∞ are equal to h^t.

We can now describe strategies. If the agent is selected as the proposer at time t, his actions are given by the map:

$$\sigma_t^p : H^{t-1} \rightarrow \mathcal{P},$$

where $\mathcal{P}$ denotes the space of probability distributions over R^2 with finite support. That is, we

[3] Allowing for further rounds of trading after the revelation of S would not change our results, given that at that point only one asset has positive value and no trade will occur.

[4] The proposer only observes if the offer is accepted or rejected. In particular, if an offer is rejected the proposer does not know whether it was infeasible for the responder or the responder just chose to reject.

allow for mixed strategies and let the proposer choose the probability distribution $\sigma_t^p\left(\cdot|h^{t-1}\right)$ from which he draws the offer z.[5] If the agent is selected as the responder, his behavior is described by:

$$\sigma_t^r : H^{t-1} \times R^2 \to [0,1],$$

which denotes the probability that the agent accepts the offer $z \in R^2$ for each history h^{t-1}. A strategy is fully described by the sequence $\sigma = \{\sigma_t^p, \sigma_t^r\}_{t=1}^{\infty}$.

We focus on symmetric equilibria where all agents play the same strategy σ. We say that the probability measure P on Ω is *consistent with* σ if for each individual agent $(1-\gamma)\,\gamma^{t-1}P(s,h^{t-1})$ is the equilibrium probability that the signal s is selected, the game ends at t, and the agent's history is h^{t-1}. The sequence $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 ... \subset \mathcal{F}$ denotes the filtration generated by the information sets of the agent at the beginning of each period t. The measure P also characterizes the cross sectional distribution of individual histories in equilibrium: at the beginning of time t, the mass of agents with history h^{t-1} is equal to $P(h^{t-1}|s)$.

The probability measure P is constructed recursively in the following way. In period 0, the probability of (s,h^0) is determined by the exogenous assignment of initial portfolios and information. In any period $t \geq 1$, given that agents are randomly matched, the probability of receiving offer $z \in R^2$ for the responding agent is

$$\psi_t\left(z|s\right) = \int_{h^{t-1}} \sigma_t^p\left(z|h^{t-1}\right) dP\left(h^{t-1}|s\right),$$

and the probability that offer $z \in R^2$ is accepted for the proposing agent is

$$\chi_t\left(z|s\right) = \int_{h^{t-1}} \sigma_t^r\left(h^{t-1},z\right) dP\left(h^{t-1}|s\right).$$

Given $P(s,h^{t-1})$, $\psi_t\left(.|s\right)$ and $\chi_t\left(.|s\right)$, we can then construct $P\left(s,h^t\right)$ as follows. For an agent with history h^{t-1}, the probability of reaching history $h^t = \left(h^{t-1},(0,z,1)\right)$ at $t+1$ is

$$P\left(s,h^t\right) = \frac{1}{2}\sigma^r\left(h^{t-1},z\right)\psi_t\left(z|s\right)P\left(s,h^{t-1}\right),$$

since the probability of being selected as the responder is $1/2$, the probability of receiving offer z is $\psi_t\left(z|s\right)$, and the probability of accepting it is $\sigma^r\left(h^{t-1},z\right)$. In a similar way, we have

$$\begin{aligned}
P\left(s,h^t\right) &= \frac{1}{2}\left(1-\sigma^r\left(h^{t-1},z\right)\right)\psi_t\left(z|s\right)P\left(s,h^{t-1}\right) \text{ if } h_t = (0,z,0), \\
P\left(s,h^t\right) &= \frac{1}{2}\chi_t\left(z|s\right)\sigma^p\left(z|h^{t-1}\right)P\left(s,h^{t-1}\right) \text{ if } h_t = (1,z,1), \\
P\left(s,h^t\right) &= \frac{1}{2}\left(1-\chi_t\left(z|s\right)\right)\sigma^p\left(z|h^{t-1}\right)P\left(s,h^{t-1}\right) \text{ if } h_t = (1,z,0).
\end{aligned}$$

[5] We restrict agents to mix over a finite set of offers to simplify the measure-theoretic apparatus.

To assess whether σ is individually optimal, agents have to form expectations about their opponents' behavior. Beliefs are described by two functions:

$$\begin{aligned}\delta_t &: H^{t-1} \to [0,1]\,, \\ \delta_t^r &: H^{t-1} \times R^2 \to [0,1]\,,\end{aligned}$$

which represent, respectively, the probability assigned to signal s_1 after history h^{t-1}, at the beginning of the period, and the probability assigned to signal s_1 after history h^{t-1}, if the agent is the responder and receives offer z. The agent's beliefs are denoted compactly by $\delta = \{\delta_t, \delta_t^r\}_{t=1}^{\infty}$. At each history h^{t-1}, an agent expects that in each period $\tau \geq t$, he will face an opponent with history $\tilde{h}^{\tau-1}$ randomly drawn from the probability distribution $P(\tilde{h}^{\tau-1}|s)$, conditional on s, and he expects his opponent to play the strategy σ. This completely describes the agent's expectations about the current and future behavior of other players. For example, the probability distribution of offers expected at time $\tau \geq t$ by an agent at h^{t-1} is equal to

$$\psi_\tau\left(z|s_1\right)\delta_t\left(h^{t-1}\right) + \psi_\tau\left(z|s_2\right)\left(1 - \delta_t\left(h^{t-1}\right)\right).$$

The beliefs δ_t are required to be consistent with Bayesian updating on the equilibrium path. This implies that

$$\delta_t\left(h^{t-1}\right) = \frac{P(s_1, h^{t-1})}{\sum_s P(s, h^{t-1})},$$

for all histories h^{t-1} such that $\sum_s P(s, h^{t-1}) > 0$. The same requirement is imposed on the beliefs δ_t^r, which implies that

$$\delta_t^r\left(h^{t-1}, z\right) = \frac{\psi_t\left(z|s_1\right)P(s_1, h^{t-1})}{\sum_s \psi_t\left(z|s\right)P(s, h^{t-1})}$$

for all histories h^{t-1} and offers z such that $\sum_s \psi_t\left(z|s\right)P(s, h^{t-1}) > 0$.

This representation of the agents' beliefs embeds an important assumption: an agent who observes his opponent play an off-the-equilibrium-path action can change his beliefs about s, but maintains that the behavior of all other agents, conditional on s, is unchanged. That is, he believes that all other agents will continue to play σ in the future. This is a reasonable restriction on off-the-equilibrium-path beliefs in a game with atomistic agents and allows us to focus on the agent's beliefs about s, given that s is a sufficient statistic for the future behavior of the agent's opponents.

Moreover, the beliefs of informed agents are required to always assign probability 1 to the signal observed at date 0:

$$\begin{aligned}\delta_t\left(h^{t-1}\right) &= \delta_t^r\left(h^{t-1},z\right)=1 \text{ if } h_0=(i,I_1),\\ \delta_t\left(h^{t-1}\right) &= \delta_t^r\left(h^{t-1},z\right)=0 \text{ if } h_0=(i,I_2).\end{aligned}$$

That is, informed agents do not change their beliefs on signal s, even after observing off-the-equilibrium-path behavior from their opponents. This fact plays a useful role in the analysis, since it allows us to characterize the behavior of informed agents after any possible offer.

We are now ready to define an equilibrium.

Definition 1 *A perfect Bayesian equilibrium is given by a strategy σ, beliefs δ, and a probability space $(\Omega, \mathcal{F}, P)$, such that:*
(i) the strategy σ is individually optimal at each history h^{t-1} given the beliefs δ and given that agents expect that at each round $\tau \geq t$ they will face an opponent with history $\tilde{h}^{\tau-1}$ randomly drawn from $P(\tilde{h}^{\tau-1}|s)$ who plays σ;
(ii) the beliefs δ are consistent with Bayes' rule whenever possible;
(iii) the probability measure P is consistent with σ.

Notice that the cross sectional behavior of the economy in equilibrium is purely determined by the signal s. In other words, s is the only relevant aggregate state variable for our trading game, and, for this reason, we call it interchangeably *signal s* or *state s*.

To establish our results, we restrict attention to equilibria that satisfy two properties, which we call *symmetry across states* and *uniform market clearing*. Let us first state these two properties and then discuss their role in the analysis.

Symmetry across states means that strategies and beliefs are the same if we switch the labels of assets 1 and 2 and those of signals 1 and 2. Formally, define $\tilde{h}^t$ as the *complement* of history h^t if the following are true: (i) if $\left(x^1,x^2\right)$ is the initial endowment in h_0, then $\left(x^2,x^1\right)$ is the initial endowment in $\tilde{h}_0$; (ii) if the agent is informed and observes s_j in h_0, he is informed and observes s_{-j} in $\tilde{h}_0$; (iii) if offer $z=\left(z^1,z^2\right)$ is made/received in h_t, offer $z=\left(z^2,z^1\right)$ is made/received in $\tilde{h}_t$; (iv) responses are the same in $\tilde{h}_t$ and h_t. We can then define symmetry across states.

Definition 2 *An equilibrium satisfies symmetry across states if the strategy and beliefs σ and δ satisfy the following: (a) $\sigma^p_{t+1}(\left(z^1,z^2\right)|h^t) = \sigma^p_{t+1}(\left(z^2,z^1\right)|\tilde{h}^t)$ and $\sigma^r_{t+1}(h^t,\left(z^1,z^2\right)) = \sigma^r_{t+1}(\tilde{h}^t,\left(z^2,z^1\right))$; (b) $\delta\left(h^t\right) = 1-\delta(\tilde{h}^t)$ and $\delta^r\left(h^t,\left(z^1,z^2\right)\right) = 1-\delta^r(\tilde{h}^t,\left(z^2,z^1\right))$ for all h^t and $\left(z^1,z^2\right)$, where $\tilde{h}^t$ is the complement of h^t.*

This restriction is more stringent than the standard symmetry requirement that all agents follow the same strategy, which we also assume. Symmetry across states helps in two steps of our analysis: in the proof of Lemma 7, which is needed to prove Proposition 2, and in the proof of Theorem 1. We discuss its role in detail when we present these results.

Uniform market clearing requires that market clearing approximately holds for agents with asset holdings in an interval $[0, M]$, for M large enough.

Definition 3 *A symmetric equilibrium satisfies uniform market clearing if for all $\varepsilon > 0$ there is an M such that*

$$\int_{x_t^j(\omega) \leq M} x_t^j(\omega)\, dP(\omega|s) \geq 1 - \varepsilon,$$

for all t and for all j.

For a given t, this property is just an implication of market clearing and of the dominated convergence theorem. The additional restriction comes from imposing that the property holds uniformly over t. Notice that all equilibria in which the portfolios x_t converge almost surely satisfy uniform market clearing.[6]

Unfortunately, we do not have a general existence proof of equilibria that satisfy symmetry across states and uniform market clearing. In the Online Appendix we present two examples for which we can show, by construction, the existence of equilibria with these properties. In a companion paper, we take a computational approach and compute equilibria with these properties for a larger set of cases.

3 Long-run efficiency

In this section, we characterize the equilibrium in the long run—i.e., along the path where the game does not end. Our main result is that the equilibrium allocation converges to an ex post Pareto efficient allocation. By ex post Pareto efficient we mean Pareto efficient after s is publicly revealed but before S is revealed.[7]

After finitely many rounds of trading the allocation will not be, in general, Pareto efficient, due to the matching friction. For example, with positive probability an agent could meet only agents with his same endowment and would not be able to trade. However, if the agents keep playing the game, they will eventually meet other agents with whom profitable trades are possible. Absent informational frictions, with a long enough horizon, all potential gains from

[6]Use Theorem 16.14 in Billingsley (1995). This assumption would not be required in a trading game with a large but *finite* number of agents, as in that case there would be a natural upper bound on the assets holdings of each agent, given by the aggregate endowment. However, to extend the model to a finite number of agents is not trivial since a law-of-large-numbers argument cannot be invoked, so the aggregate state of the game is not just s. Moreover, to derive limit theorems in trading games with large but finite number of agents one usually needs to impose further restrictions on strategies, as has been shown in symmetric information environments (Rubinstein and Wolinsky, 1990, and Gale, 2000, Chapter 3).

[7]This is the standard notion of ex post efficiency as in Holmstrom and Myerson (1983). After S is revealed all allocations are trivially efficient as only one asset has positive value.

trade are eventually realized and the allocation converges to efficiency. Different versions of this result under symmetric information are discussed in Gale (2000).

With asymmetric information, it is harder to show that all profitable trades will be exhausted. Now, when two agents meet, there may be a Pareto improving trade between them conditional on s, but since s is not commonly observed the agents may not be able to credibly signal to each other the presence of this trade. For example, suppose the state is s_1 and an informed agent with a relatively large amount of asset 1 meets an informed agent with a relatively small amount of it. The informed agent would like to trade asset 1 for asset 2 at a price that is mutually beneficial conditional on s_1. But the uninformed responder may reject the offer because he is afraid that the proposer has observed s_2 and is trying to sell asset 1 because it is less valuable. Can this prevent the economy from achieving efficiency in the long run? Our main result shows that the answer is no.

3.1 Preliminary considerations

We first define and characterize the stochastic process for an agent's expected utility in equilibrium. We use the martingale convergence theorem to show that expected utility converges in the long run, conditionally on the game not ending.

Take the probability space $(\Omega, \mathcal{F}, P)$ and let $x_t(\omega)$ and $\delta_t(\omega)$ denote the portfolio and belief of the agent at the beginning of period t, at ω. Since an agent's current portfolio and belief are, by construction, in his information set at time t, $x_t(\omega)$ and $\delta_t(\omega)$ are $\mathcal{F}_t$-measurable stochastic processes on $(\Omega, \mathcal{F}, P)$. If the game ends, an agent with the portfolio-belief pair (x, δ) receives the expected payoff

$$U(x, \delta) \equiv \pi(\delta)u(x^1) + (1 - \pi(\delta))u(x^2),$$

where $\pi(\delta)$ is the probability that an agent with belief δ assigns to state S_1,

$$\pi(\delta) \equiv \delta\phi(s_1) + (1 - \delta)\phi(s_2).$$

Given the processes x_t and δ_t, we obtain the stochastic process $u_t \equiv U(x_t, \delta_t)$, which gives the equilibrium expected utility of an agent if the trading game ends in t. Finally, we obtain the stochastic process

$$v_t \equiv (1 - \gamma)E\left[\sum_{s=t}^{\infty} \gamma^{s-t}u_s \mid \mathcal{F}_t\right],$$

which gives the expected lifetime utility at the beginning of period t. This process satisfies the recursion

$$v_t = (1 - \gamma)u_t + \gamma E\left[v_{t+1} \mid \mathcal{F}_t\right]. \tag{2}$$

Notice that, since v_t is constructed using the equilibrium stochastic processes for x_t and δ_t, it represents the expected utility from following the equilibrium strategy, which is, by definition, individually optimal. Using this fact, the next lemma establishes that v_t is a bounded martingale and converges in the long run.

Lemma 1 *There exists a random variable $v^\infty(\omega)$ such that*

$$\lim_{t\to\infty} v_t(\omega) = v^\infty(\omega) \ \ a.s.$$

Proof. An agent always has the option to keep his time t portfolio x_t and wait for the end of the game, rejecting all offers and offering zero trades in all $t' \geq t$. His expected lifetime utility under this strategy is equal to u_t. Therefore, optimality implies $u_t \leq E\left[v_{t+1} \mid \mathcal{F}_t\right]$, which, combined with equation (2), gives $v_t \leq E\left[v_{t+1} \mid \mathcal{F}_t\right]$. This shows that v_t is a submartingale. It is bounded above because the utility function $u(\cdot)$ is bounded above. Therefore, it converges by the martingale convergence theorem. ■

It is useful to introduce an additional stochastic process, $\hat{v}_t$, which will be used as a reference point to study the behavior of agents who make and receive off-the-equilibrium-path offers. Let $\hat{v}_t$ be the expected lifetime utility of an agent who adopts the following strategy: (i) if selected as the proposer at time t, follow the equilibrium strategy σ; (ii) if selected as the responder, reject all offers at time t and follow an optimal continuation strategy from $t+1$ onwards. The expected utility $\hat{v}_t$ is computed at time t immediately after the agent is selected as the proposer or the responder, i.e., it is measurable with respect to $\left(h^{t-1}, \iota_t\right)$, and, by definition, satisfies $\hat{v}_t \leq E\left[v_{t+1} \mid h^{t-1}, \iota_t\right]$.

Notice that u_t is the expected utility from holding the portfolio x_t until the end of the game. The following lemma shows that, in the long run, an agent is almost as well off keeping his time t portfolio as he is under the strategy leading to $\hat{v}_t$.

Lemma 2 *Both u_t and $\hat{v}_t$ converge almost surely to v^∞:*

$$\lim_{t\to\infty} u_t(\omega) = \lim_{t\to\infty} \hat{v}_t(\omega) = v^\infty(\omega) \ \ a.s.$$

Proof. As argued in Lemma 1, v_t is a bounded submartingale and converges almost surely to v^∞. Let $y_t \equiv E[v_{t+1} \mid \mathcal{F}_t]$. Since a bounded martingale is uniformly integrable (see Williams, 1991), we get $y_t - v_t \to 0$ almost surely. Rewrite equation (2) as

$$(1-\gamma)\, u_t = \gamma \left(v_t - E\left[v_{t+1} \mid \mathcal{F}_t\right]\right) + (1-\gamma)\, v_t.$$

This gives

$$u_t - v_t = \frac{\gamma}{1-\gamma}\left(v_t - E\left[v_{t+1} \mid \mathcal{F}_t\right]\right) = \frac{\gamma}{1-\gamma}\left(v_t - y_t\right),$$

which implies $u_t - v_t \to 0$ almost surely. The latter implies $u_t \to v^\infty$ almost surely. Letting $\hat{y}_t \equiv E\left[v_{t+1} \mid h^{t-1}, \iota_t\right]$, notice that $\hat{y}_t \to v^\infty$ almost surely. Since $u_t \leq \hat{v}_t \leq \hat{y}_t$, it follows that $\hat{v}_t \to v^\infty$ almost surely. ■

3.2 Informed agents

We first focus on informed agents and show that their marginal rates of substitution converge in probability. In particular, we show that, conditional on s, the marginal rates of substitution of all informed agents converge in probability to the same sequence, which we denote $\kappa_t(s)$. In the following, we refer to $\kappa_t(s)$ as the long-run marginal rate of substitution of the informed agents.

Since this result is about informed agents, the argument is similar to the one used in decentralized markets with full information. If two informed agents have different marginal rates of substitution, they can find a trade that improves the utility of both. As their utilities converge to their long-run levels, all the potential gains from bilateral trade must be exhausted, so their marginal rates of substitution must converge.

Proposition 1 ***(Convergence of MRS for informed agents)*** *There exist two sequences $\kappa_t(s_1)$ and $\kappa_t(s_2)$ such that, conditional on each s, the marginal rates of substitution of informed agents converge in probability to $\kappa_t(s)$:*

$$\lim_{t\to\infty} P\left(\left|\frac{\phi(s)u'(x_t^1)}{(1-\phi(s))\,u'(x_t^2)} - \kappa_t(s)\right| > \varepsilon \mid \delta_t = \delta^I(s), s\right) = 0 \text{ for all } \varepsilon > 0. \tag{3}$$

Proof. We provide a sketch of the proof here and leave the details to the Appendix. Without loss of generality, suppose (3) is violated for $s = s_1$. Then, it is always possible to find a period T, arbitrarily large, in which there are two groups, of positive mass, of informed agents with marginal rates of substitution sufficiently different from each other. In particular, we can find a κ^* such that a positive mass of informed agents have marginal rates of substitution below κ^*:

$$\frac{\phi(s_1)\,u'(x_T^1)}{(1-\phi(s_1))u'(x_T^2)} < \kappa^*,$$

and a positive mass of informed agents have marginal rates of substitution above $\kappa^* + \varepsilon$:

$$\frac{\phi(s_1)\,u'(x_T^1)}{(1-\phi(s_1))u'(x_T^2)} > \kappa^* + \varepsilon,$$

for some positive ε. An informed agent in the first group can then offer to sell a small quantity ζ^* of asset 1 at the price $p^* = \kappa^* + \varepsilon/2$, that is, he can offer the trade $z^* = (\zeta^*, -p^*\zeta^*)$. Suppose this offer is accepted and the proposer stops trading afterwards. Then his utility can be approximated as follows:

$$\begin{aligned}
&\phi(s_1)\,u(x_T^1 - \zeta^*) + (1-\phi(s_1))u(x_T^2 + p^*\zeta^*) \\
&\quad\approx u_T + \left[-\phi(s_1)\,u'(x_T^1) + (1-\phi(s_1))p^*u'(x_T^2)\right]\zeta^* \\
&\quad\approx \hat{v}_T + (1-\phi(s_1))u'(x_T^2)\zeta^*\varepsilon/2,
\end{aligned}$$

where we use a Taylor expansion to approximate the utility gain and we use Lemma 2 to show that the continuation utility $\hat{v}_T$ can be approximated by the current utility u_T. By choosing T sufficiently large and the size of the trade ζ^* sufficiently small we can make the approximation errors in the above equation small enough, so that when this trade is accepted it strictly improves the utility of the proposer. All the informed responders with marginal rate of substitution above $\kappa^* + \varepsilon$ are also better off, by a similar argument. Therefore, they will all accept the offer. Since there is a positive mass of them, the strategy described gives strictly higher utility than the equilibrium strategy to the proposer, and we have a contradiction. ∎

3.3 Uninformed agents

We now turn to the characterization of equilibria for uninformed agents. The main difficulty here is that uninformed agents may change their beliefs upon observing their opponent's behavior. Thus an agent who would be willing to accept a given trade ex ante—before updating his beliefs—might reject it ex post. Moreover, updated beliefs are not determined by Bayes' rule after off-the-equilibrium-path offers, and our objective is to develop a general argument, independent of how off-the-equilibrium-path beliefs are specified. For these reasons, we need a strategy of proof different from the one used for informed agents.

Our argument is based on finding strategies that allow the uninformed agents to learn the signal s at an arbitrarily small cost. This is done in Lemma 3 below. The existence of such strategies implies that either uninformed agents eventually learn the signal or the benefit of learning goes to zero.

To build our argument, we first show that in equilibrium the marginal rates of substitution of all agents cannot converge to the same value *independently of the state* s. Since individual marginal rates of substitution determine the prices at which agents are willing to trade, this rules out equilibria in which agents, in the long run, are all willing to trade at the same price, independent of s. The fact that agents are willing to trade at different prices in the two states s_1 and s_2 will be key in constructing the experimentation strategies below. This fact will allow us to construct small trades that are accepted with different probability in the two states. By offering such trades an uninformed agent will be able to extract information on s and thus acquire the information obtained by the informed agents at date 0.

Remember that $\kappa_t(s)$ denotes the long-run marginal rates of substitution of informed agents in state s. The next proposition shows that in the long run two cases are possible: either the two values $\kappa_t(s_1)$ and $\kappa_t(s_2)$ are sufficiently far from each other, or, in each state s, there is a sufficient mass of agents with marginal rates of substitution far enough from $\kappa_t(s)$. That is, either the informed agents' marginal rates of substitution converge to different values or there are enough uninformed agents with marginal rates of substitution different from that of the informed.

Proposition 2 *Consider an equilibrium that satisfies symmetry across states and uniform market clearing. There exists a period T and a scalar $\bar{\varepsilon} > 0$ such that in all periods $t \geq T$ one of the following holds: (i) the long-run marginal rates of substitution of the informed agents are sufficiently different in the two states:*

$$|\kappa_t(s_1) - \kappa_t(s_2)| \geq \bar{\varepsilon},$$

or (ii) sufficiently many agents have a marginal rate of substitution different from $\kappa_t(s)$:

$$P\left(\left|\frac{\pi(\delta_t) u'(x_t^1)}{(1-\pi(\delta_t)) u'(x_t^2)} - \kappa_t(s)\right| \geq 2\bar{\varepsilon} \mid s\right) \geq \bar{\varepsilon}$$

for $s \in \{s_1, s_2\}$.

The proof of this proposition is in the Appendix. The argument is as follows. If both (i) and (ii) are violated, we can always find a period t in which all the agents' marginal rates of substitution are concentrated around some value κ which is independent of the state s. The point of the proof is to show that this leads to a violation of market clearing. We first show that the distribution of beliefs of the uninformed is always biased in the direction of the true signal. That is, in s_1 there are more uninformed agents with $\delta_t \geq 1/2$ than uninformed agents with $\delta_t < 1/2$. Using symmetry across states, we can then show that in s_1 market clearing holds if we sum the asset holdings of the informed agents and of the subset of uninformed agents with beliefs $\delta_t \geq 1/2$. Next, we show that if the marginal rates of substitution of all agents are the same and independent of s, this implies that in state s_1 all agents with $\delta_t \geq 1/2$ would hold weakly more of asset 1 than of asset 2, and a positive mass of informed agents with $\delta_t > 1/2$ would hold strictly more of asset 1. Since the endowments of the two assets are the same, this leads to a contradiction.

3.3.1 Experimentation

We now show how uninformed agents can experiment and acquire information on the state s by making small offers. In the proof of Proposition 3, we will construct a sequence of offers with the following property: given any $\varepsilon > 0$, if the uninformed agent makes the offers $\{\hat{z}_{t+j}\}_{j=0}^{J-1}$ at times $t, t+1, ...t+J-1$, and receives the "right" string of responses (e.g., $\{\hat{r}_{t+j}\}_{j=0}^{J-1} = \{0, 1, 1, 0, ..., 1\}$) then the probability he assigns to s_1 at time $t+J$ will be larger than $1-\varepsilon$. That is, this sequence of offers allows the experimenter to acquire arbitrarily precise information on state s_1 (a similar construction can be done for s_2). Here we make the crucial step in the construction of this sequence of offers. Namely, we find a single offer z such that if the right response is received, the proposer's belief increases by a sufficient amount.

Consider an uninformed agent who assigns probability $\delta \in (0, 1)$ to signal s_1 at the beginning of period t and makes offer z. Recall that the probability of acceptance of z, conditional on s, is $\chi_t(z|s)$. Bayes' rule implies that if the offer is accepted the agent's updated belief δ' satisfies

$$\frac{\delta'}{1-\delta'} = \frac{\delta}{1-\delta}\frac{\chi_t(z|s_1)}{\chi_t(z|s_2)},$$

while if the offer is rejected his updated belief satisfies

$$\frac{\delta'}{1-\delta'} = \frac{\delta}{1-\delta}\frac{1-\chi_t(z|s_1)}{1-\chi_t(z|s_2)}.$$

If $\chi_t(z|s_1) > \chi_t(z|s_2)$ the acceptance of offer z provides a signal in favor of s_1, if $\chi_t(z|s_1) < \chi_t(z|s_2)$ the rejection of offer z provides a signal in favor of s_1. Our objective is to find a constant $\rho > 1$, such that we can always find an offer z such that either

$$\frac{\chi_t(z|s_1)}{\chi_t(z|s_2)} > \rho$$

or

$$\frac{1-\chi_t(z|s_1)}{1-\chi_t(z|s_2)} > \rho.$$

In this way, if the agent makes offer z and receives the right response (a "yes" in the first case, a "no" in the second) his beliefs satisfy

$$\frac{\delta'}{1-\delta'} > \rho\frac{\delta}{1-\delta}.$$

Since $\rho > 1$, this ensures that we can choose a long enough sequence of offers such that, if the right responses are received, the agent's belief will converge to 1.

The following lemma shows how to construct the offer z.

Lemma 3 *Consider an equilibrium that satisfies symmetry across states and uniform market clearing. There are two scalars $\beta > 0$ and $\rho > 1$ with the following property: for all $\theta > 0$ there is a time T such that for all $t \geq T$ there exist a trade z with $\|z\| < \theta$ that satisfies either*

$$\chi_t(z|s_1) > \beta, \qquad \chi_t(z|s_1) > \rho\chi_t(z|s_2), \tag{4}$$

or

$$1-\chi_t(z|s_1) > \beta, \qquad 1-\chi_t(z|s_1) > \rho(1-\chi_t(z|s_2)). \tag{5}$$

Proof. We provide a sketch of the argument here and leave the details to the Online Appendix. We distinguish two cases. By Proposition 2 one of the following must be true in any period t following some period T: (i) either the long-run marginal rates of substitutions of informed agents $\kappa_t(s_1)$ and $\kappa_t(s_2)$ are sufficiently different from each other or (ii) there is a sufficiently large mass of agents with marginal rates of substitution sufficiently different from $\kappa_t(s)$. The proof proceeds differently in the two cases.

Case 1. Suppose that there is a large enough difference between $\kappa_t(s_1)$ and $\kappa_t(s_2)$. Assume without loss of generality that $\kappa_t(s_1) > \kappa_t(s_2)$. Suppose the uninformed agent offers to sell a small quantity ζ of asset 1 at the price $p = (\kappa_t(s_1) + \kappa_t(s_2))/2$, which lies between the two marginal rates of substitutions $\kappa_t(s_1)$ and $\kappa_t(s_2)$. That is, he offers the trade $z = (\zeta, -p\zeta)$. We now make two observations on offer z:

Observation 1. In state s_1, there is a positive mass of informed agents willing to accept offer z, provided ζ is small enough and t is sufficiently large. Combining Lemma 2 and Proposition 1, we can show that in state s_1, for t large enough, there is a positive mass of informed agents with marginal rates of substitution sufficiently close to $\kappa_t(s_1)$, who are close enough to their long-run utility. These agents are better off accepting z, as they are buying asset 1 at a price smaller than their marginal valuation.

Observation 2. Conditional on signal s_2, the offer z cannot be accepted by any agent, informed or uninformed, except possibly by a vanishing mass of agents. Suppose, to the contrary, that a positive fraction of agents accepted z in state s_2. By an argument symmetric to the one above, informed agents in state s_2 are strictly better off *making* the offer z, if this offer is accepted with positive probability, given that they would be selling asset 1 at a price higher than their marginal valuation (which converges to $\kappa_t(s_2)$ by Proposition 1). But then an optimal deviation on their part is to make such an offer and strictly increase their expected utility above its equilibrium level, leading to a contradiction.

The first observation can be used to show that the probability of acceptance $\chi_t(z|s_1)$ can be bounded from below by a positive number. The second observation can be used to show that the probability of acceptance $\chi_t(z|s_2)$ can be bounded from above by an arbitrarily small number. These two facts imply that we can make $\chi_t(z|s_1) > \beta$ for some $\beta > 0$ and $\chi_t(z|s_1)/\chi_t(z|s_2) > \rho$ for some $\rho > 1$. So in this case we can always find a trade such that (4) is satisfied, i.e., such that the acceptance of z is good news for s_1. However, when we turn to the next case this will not always be true, and we will need to allow for the alternative condition (5), i.e., rejection of z is good news for s_1.

Case 2. Consider now the second case where the long-run marginal rates of substitution of the informed agents $\kappa_t(s_1)$ and $\kappa_t(s_2)$ are close enough but there is a large enough mass of uninformed agents whose marginal rates of substitution is far from $\kappa_t(s_1)$, conditional on s_1.

This means that we can find a price p such that the marginal rates of substitution of a group of uninformed agents are on one side of p and the long-run marginal rates of substitution of informed agents $\kappa_t(s_1)$ and $\kappa_t(s_2)$ are on the other side. Consider the case where the MRS of a group of uninformed agents is greater than p, and $\kappa_t(s_1)$ and $\kappa_t(s_2)$ are smaller than p (the other case is symmetric). Then the uninformed agents in this group can make a small offer to buy asset 1 at a price p and the informed will accept this offer conditional on both signals s_1 and s_2. If the probabilities of acceptance conditional on s_1 and s_2 were sufficiently close to each other, this would be a profitable deviation for the uninformed, since then their ex post beliefs would be close to their ex ante beliefs. In other words, the uninformed would not learn from the trade but making the offer would increase their expected utility relative to their equilibrium strategy, leading to a contradiction. It follows that the probabilities of acceptance of this trade must be sufficiently different in the two states s_1 and s_2. This leads to either (4) or (5), completing the proof. ■

3.3.2 Convergence of marginal rates of substitution

We now characterize the properties of the long-run marginal rates of substitution of uninformed agents. The next proposition shows that the convergence result established for informed agents (Proposition 1) extends to uninformed agents.

In what follows, instead of looking at the ex ante marginal rate of substitution, given by $\pi(\delta_t)u'(x_t^1)/\left(1-\pi(\delta_t)\right)u'(x_t^2)$, we establish convergence for the ex post marginal rate of substitution $\phi(s)u'(x_t^1)/\left(1-\phi(s)\right)u'(x_t^2)$. This is the marginal rate of substitution at which an agent would be willing to trade asset 2 for asset 1 *if he could observe the signal s*. As we will see, this is the appropriate convergence result given our objective, which is to establish the ex post efficiency of the equilibrium allocation.

Proposition 3 ***(Convergence of MRS for uninformed agents)*** *Consider an equilibrium that satisfies symmetry across states and uniform market clearing. Conditional on each s, the marginal rate of substitution of any agent, evaluated at the full information probabilities $\phi(s)$ and $1-\phi(s)$, converges in probability to $\kappa_t(s)$:*

$$\lim_{t\to\infty} P\left(\left|\frac{\phi(s)u'(x_t^1)}{\left(1-\phi(s)\right)u'(x_t^2)}-\kappa_t(s)\right|>\varepsilon \mid s\right)=0 \text{ for all } \varepsilon>0. \tag{6}$$

Proof. We provide a sketch of the proof, leaving the details to the Appendix. Suppose condition (6) fails to hold. Without loss of generality, we focus on the case where (6) fails for $s=s_1$. This means that there is a period T in which with a positive probability an uninformed agent has ex post marginal rate of substitution sufficiently far from $\kappa_T(s_1)$ and is sufficiently close to his long-run utility. Without loss of generality, suppose his marginal rate of substitution is larger than $\kappa_T(s_1)$. To reach a contradiction, we construct a profitable deviation for this agent.

Before discussing the deviation, it is useful to clarify that, at time T, the uninformed agent has all the necessary information to check whether he should deviate or not. He can observe his own allocation x_T, compute $\phi(s_1)u'(x_T^1)/\left(1-\phi(s_1)\right)u'(x_T^2)$, and verify whether this quantity is sufficiently larger than $\kappa_T(s_1)$ (which is known, since it is an equilibrium object).

The deviation then consists of two stages:

Stage 1. This is the experimentation stage, which lasts from period T to period $T+J-1$. As stated in Lemma 3, the agent can construct a sequence of small offers $\{\hat{z}_j\}_{j=0}^{J-1}$ such that, if these offers are followed by the appropriate responses, the agent's ex post belief on signal s_1 will converge to 1. To be precise, for this to be true it must be the case that the agent does not start his deviation with a belief δ_T too close to 0. Otherwise, a sequence of J signals favorable to s_1 is not enough to bring δ_{T+J} sufficiently close to 1. Therefore, when an agent starts deviating we also require δ_T to be larger than some positive lower bound $\underline{\delta}$, appropriately defined.

Stage 2. At date $T+J$, if the agent has been able to make the whole sequence of offers $\{\hat{z}_j\}_{j=0}^{J-1}$ and has received the appropriate responses (that is, the responses which bring the probability of s_1 close to 1), he then makes one final offer z^*. This is an offer to buy a small quantity ζ^* of asset 1 at a price p^*, which is in between the agent's own marginal rate of substitution and $\kappa_T(s_1)$. By choosing T large enough, we can ensure that there is a positive mass of informed agents close enough to their long-run marginal rate of substitution, who are willing to sell asset 1 at that price.[8] Therefore, the offer is accepted with a positive probability. The utility gain for the uninformed agent, conditional on reaching Stage 2 and conditional on z^* being accepted, can be approximated by

$$U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 1) - U(x_T, 1),$$

given that, after the experimentation stage the agent's ex post belief approaches 1. Moreover, by making the final offer z^* and the experimenting offers $\hat{z}_j$ sufficiently small, this utility gain can be approximated by

$$U(x_T + z^*, 1) - U(x_T, 1) \approx \phi(s) u'(x_T^1)\zeta^* - (1 - \phi(s))\, u'(x_T^2) p^* \zeta^* > 0.$$

The last expression is positive because p^* was chosen smaller than the marginal rate of substitution $\phi(s)u'(x_T^1)/\left(1-\phi(s)\right)u'(x_T^2)$. In the Appendix we show that this utility gain is large enough that the deviation described is ex ante profitable, i.e., it is profitable from the point of view of period T. To do so, we must ensure that the utility losses that may happen along the deviating path (e.g., when some of the experimenting offers do not generate a response favorable to s_1 or when the agent is not selected as the proposer) are small enough. To establish this, we use again the fact that the experimenting offers are small. The argument in the Appendix makes use of the convergence of utility levels in Lemma 2, to show that a utility gain relative to the current utility u_t, leads to a profitable deviation relative to the expected utility $\hat{v}_t$. Since we found a profitable deviation for the uninformed agents, a contradiction is obtained which completes the argument. ■

[8] Notice that the uninformed agent is using $\kappa_T(s_1)$ as a reference point for the informed agents' marginal rate of substitution, while offer z^* is made in period $T+J$. Lemma 9 in the Appendix ensures that $\kappa_T(s_1)$ and $\kappa_{T+J}(s_1)$ are sufficiently close, so that at $T+J$ enough informed agents have marginal rate of substitution near $\kappa_T(s_1)$.

3.4 Main result

Having characterized the portfolios of informed and uninformed agents in the long run, we can finally derive our efficiency result.

Theorem 1 *All symmetric equilibrium allocations which satisfy symmetry across states and uniform market clearing converge to ex post efficient allocations in the long run, i.e.,*

$$\lim_{t\to\infty} P\left(\left|x_t^1 - x_t^2\right| > \varepsilon\right) = 0 \text{ for all } \varepsilon > 0. \tag{7}$$

The long-run marginal rates of substitution $\kappa_t(s)$ converge to the ratios of the conditional probabilities of states S_1 and S_2:

$$\lim_{t\to\infty} \kappa_t(s) = \phi(s)/(1-\phi(s)) \text{ for all } s \in \{s_1, s_2\}. \tag{8}$$

Proof. We provide a sketch of the proof and leave the formal details to the Appendix. First, suppose that $\kappa_t(s) > (1+\varepsilon)\,\phi(s)/(1-\phi(s))$ for some $\varepsilon > 0$, for infinitely many periods. Then Proposition 3 can be used to show that the agents' holdings of asset 1 will be larger than their holdings of asset 2. This, however, violates market clearing. In a similar way, we rule out the case in which $\kappa_t(s) < (1-\varepsilon)\,\phi(s)/(1-\phi(s))$ for some $\varepsilon > 0$, for infinitely many periods. This proves (8). Then, using this result and Proposition 3, we can show that $u'(x_t^1)/u'(x_t^2)$ converges in probability to 1, which implies (7). ■

This theorem establishes that equilibrium allocations converge to ex post Pareto efficient allocations. It also shows, in (8), that the prices supporting the long-run allocation are the same as the rational expectations competitive equilibrium prices of the same economy, which are proportional to the probabilities of the two states.

It is useful to clarify that our results do not necessarily imply that uninformed agents will learn the value of the signal s in the long run. The proof of Proposition 3 shows that in the long run uninformed agents have the possibility to learn the value of s with arbitrary precision at an arbitrarily low cost. However, as all agents' asset holdings converge to a perfectly diversified portfolio, the incentive to acquire this information also goes to zero. The reason is that uninformed agents know that when all asset holdings are converging to a perfectly diversified portfolio, further profitable trades are no longer possible. Therefore, our results imply that one of two possible outcomes are possible: either uninformed agents perfectly learn the signal or the value of learning the signal goes to zero. In the Online Appendix, we present two examples showing that both these outcomes are indeed possible in equilibrium depending on the model parameters.

To conclude this section, it is useful to compare our result to Wolinsky (1990). In our model, in the long run all agents are only willing to trade at a single price, which corresponds to the rational expectations competitive equilibrium price $\phi\,(s)\,/\,(1-\phi\,(s))$. Wolinsky (1990) also analyzes a dynamic trading game with asymmetric information and shows that in steady state different trades can occur at different prices, so a fraction of trades can occur at a price different from the rational expectations competitive price. That paper studies a game where a fraction of traders enter and exit at each point of time, focuses on steady-state equilibria, and takes limits as discounting goes to zero. We consider a game with a fixed set of participants and a fixed probability of ending the game γ and study long-run outcomes. The key difference is that the model in Wolinsky (1990) features an indivisible good which can only be traded once. Our environment features perfectly divisible goods (assets) which are traded repeatedly. This makes the process of experimentation by market participants very different in the two environments. In Wolinsky (1990) agents only learn if their offers are rejected. Once the offer is accepted they trade and exit the market. In our environment, agents keep learning and trading along the equilibrium play. In particular, they can learn by making small trades (as shown in Lemma 3) and then use the information acquired to make Pareto improving trades with informed agents (as shown in Proposition 3).

4 Concluding remarks

This paper analyzes long-run efficiency and the value of information in a dynamic trading game with private information. The main difficulty with our environment is that, due to private information, agents hold diverse beliefs about asset values in equilibrium and need to update these beliefs both on and off the equilibrium path. This means that standard arguments used in decentralized bargaining environments with full information cannot be applied. Nonetheless, proceeding by contradiction, we built arguments on learning and experimentation that are sufficiently powerful to characterize the long-run properties of the equilibrium without imposing additional restrictions on belief updating. To achieve this goal we had to rely on some simplifying assumptions. We conclude with some remarks on the role of these assumptions.

We derived Theorem 1 in an environment with two states and two signals, so it is useful to discuss how the logic of our argument could be extended to more states and signals. The experimentation and deviation arguments in Proposition 3 can be easily extended to the case of finitely many states and signals, as long as markets are complete and there is an Arrow security for each state S. Take a signal s that carries information about two Arrow securities which pay in states S and S'. Partition the signal space in two subsets: a singleton that includes only s and the subset of all the other signals. Then the arguments in our binary environment can be adapted to prove that the marginal rates of substitution between assets S and S' must converge to the same value for all agents, informed and uninformed, conditional on s. However, other steps used to arrive to Theorem 1 are harder to generalize. In particular, Proposition 2 shows that the agents' marginal rates of substitution cannot converge to the same value in states s_1 and s_2. The argument is by contradiction and shows that otherwise market clearing would be violated. For that argument, we use our two-signal environment and our assumption of symmetry across states to deal with the fact that along the equilibrium path uninformed agents hold, in general, a range of beliefs about the signal.[9] How to extend that argument to more general environments is an interesting open issue.

Another important simplifying assumption is that the informed agents have nothing to learn from trading, as they all obtain the only relevant piece of information at the beginning of the game. The benefit of this assumption is to have *some* agents in the economy holding fixed beliefs. Our argument is then built starting from the convergence of the marginal rates of substitution of these agents and then using these marginal rates of substitution as reference points for our experimentation steps. A challenging open question is what would happen in an environment in which different informed agents receive different pieces of information.

Finally, notice that throughout the paper we kept a fixed level of frictions in trading, by choosing a fixed value of γ. This parameter determines the random number of trading rounds before the game ends. All our long-run results implicitly depend on γ. That is, for given γ there is a large enough $T(\gamma)$ such that for all periods $t \geq T(\gamma)$ efficiency holds with probability near 1. An important open question is what happens in our model as γ goes to 1 and the economy approaches frictionless trading. The Online Appendix analyzes examples for which we can fully characterize the equilibrium for γ going to 1.

[9] This is discussed in the paragraph following Proposition 2.

References

[1] Amador, Manuel and Pierre Olivier Weill (2009). "Learning from Private and Public Observations of Other's Actions." Mimeo, UCLA.

[2] Billingsley, Patrick (1995). *Probability and Measure.* Wiley Series in Probability and Mathematical Statistics.

[3] Blouin, Max R., and Roberto Serrano (2001). "A Decentralized Market with Common Values Uncertainty: Non-Steady States," *Review of Economic Studies* 68: 323–346.

[4] Brunnermeier, Markus (2001). *Asset Pricing under Asymmetric Information.* Oxford University Press.

[5] Cho, In-Koo and David M. Kreps (1987). "Signalling Games and Stable Equilibria." *Quarterly Journal of Economics*, 102(2): 179-221.

[6] Duffie, Darrell, Nicolae Garleanu and Lasse Heje Pedersen (2005). "Over-the-Counter Markets." *Econometrica* 73: 1815–1847.

[7] Duffie, Darrell, Nicolae Garleanu and Lasse Heje Pedersen (2007). "Valuation in Over-the-Counter Markets." *Review of Financial Studies*, forthcoming.

[8] Duffie, Darrell, Gaston Giroux, and Gustavo Manso (2009). "Information Percolation," *American Economic Journal: Microeconomics*, forthcoming.

[9] Duffie, Darrell, Semyon Malamud and Gustavo Manso (2009a). "Information Percolation with Equilibrium Search Dynamics." *Econometrica*, forthcoming.

[10] Duffie, Darrell, Semyon Malamud and Gustavo Manso (2009b). "The Relative Contributions of Private Information Sharing and Public Information Releases to Information Aggregation." J*ournal of Economic Theory*, forthcoming.

[11] Duffie, Darrell, and Gustavo Manso, (2007). "Information Percolation in Large Markets," *American Economic Review, Papers and Proceedings*, 97: 203-209.

[12] Fudenberg, Drew, and David Levine (1995). "Consistency and Cautious Fictitious Play." *Journal of Economic Dynamics and Control* 19: 1065-1089.

[13] Gale, Douglas (1986, a). "Bargaining and Competition Part I: Characterization." *Econometrica*, 54 (4), pp. 785-806.

[14] Gale, Douglas (1986, b). "Bargaining and Competition Part II: Existence." *Econometrica*, 54 (4), pp. 807-818.

[15] Gale, Douglas (1987). "Limit Theorems for Markets with Sequential Bargaining." *Journal of Economic Theory* 43: 20–54.

[16] Gale, Douglas (2000). *Strategic Foundations of General Equilibrium: Dynamic Matching and Bargaining Games*. Cambridge University Press.

[17] Golosov, Micheal, Guido Lorenzoni, and Aleh Tsyvinski (2011). "Trading Dynamics and Information Diffusion in a Market with Private Information." Working paper.

[18] Green, Edward (1992): "Eliciting Traders' Knowledge in a Frictionless Asset Market," in B. Dutta et al., eds., *Game Theory and Economic Applications: Proceedings, New Delhi, 1990*, Springer, Berlin.

[19] Grossman, Sanford (1989). *The Informational Role of Prices*. MIT Press.

[20] Lagos, Ricardo (2007), "Asset Prices and Liquidity in an Exchange Economy," working paper, NYU.

[21] Lagos, Ricardo and Guillaume Rocheteau (2009). "Liquidity in Asset Markets with Search Frictions," *Econometrica*, 77(2) (March 2009): 403-426.

[22] Lagos, Ricardo, and Guillaume Rocheteau and Pierre Olivier Weill, (2007). "Crashes and Recoveries in Illiquid Markets." UCLA working paper.

[23] Lauermann, Stephan (2010). "Dynamic Matching and Bargaining Games: A General Approach." University of Michigan working paper.

[24] Milgrom, Paul R., and Nancy Stokey (1982). "Information, Trade and Common Knowledge." *Journal of Economic Theory* 26: 17–27.

[25] Ostrovsky, Michael (2009). "Information Aggregation in Dynamic Markets with Strategic Traders," Working paper.

[26] Rocheteau, Guillaume (2008). "A Monetary Approach to Asset Liquidity," Working paper.

[27] Rubinstein, Ariel and Asher Wolinsky (1990). "Decentralized Trading, Strategic Behaviour and the Walrasian Outcome." *Review of Economic Studies* 57 (1), 63-78.

[28] Wolinsky, Asher (1990), "Information Revelation in a Market with Pairwise Meetings," *Econometrica* 58: 1–23.

[29] Vayanos, Dimitri (1998). "Transaction Costs and Asset Prices: A Dynamic Equilibrium Model." *Review of Financial Studies* 11, 1–58.

[30] Vayanos, Dimitri and Pierre Olivier Weill (2007). "A Search-Based Theory of the On-the-Run Phenomenon." *Journal of Finance*, forthcoming.

[31] Weill, Pierre Olivier (2007). "Liquidity Premia in Dynamic Bargaining Markets." *Journal of Economic Theory*, forthcoming.

[32] Williams, David (1991). *Probability with Martingales.* Cambridge University Press

[33] Wolinsky, Asher (1990), "Information Revelation in a Market with Pairwise Meetings." *Econometrica*, 58, No. 1 (Jan., 1990), pp. 1-23

5 Appendix

5.1 Preliminary results

The following three lemmas are used throughout the appendix. The first is an elementary probability result. The second shows that in the long run, portfolios are in a compact set X with probability arbitrarily close to one. This set will be used to ensure that some optimization problems used in the proofs are well defined. The third lemma shows that given any two agents with portfolios in a compact set X and marginal rates of substitution that differ by at least ε, there is a trade that achieves a gain in current utility of at least Δ, for some positive Δ. The function $\mathcal{M}$ is defined as the *ex ante* marginal rate of substitution between the two assets:

$$\mathcal{M}(x,\delta) \equiv \frac{\pi(\delta)\, u'(x^1)}{(1-\pi(\delta))u'(x^2)}.$$

Lemma 4 *Take two sets $A, B \subset \Omega$ such that $P(A|s) \geq 1-\varepsilon$ and $P(B|s) > 1-\eta$ for some positive scalars ε and η. Then, $P(A \cap B|s) > 1-\varepsilon-\eta$.*

Lemma 5 *For any $\varepsilon > 0$ and any state s, there are a compact set $X \subset R^2_{++}$ and a time T such that $P(x_t \in X \mid s) \geq 1-\varepsilon$ for all $t \geq T$.*

Lemma 6 *Take a compact set $X \in R^2_{++}$. For any $\varepsilon > 0$ and $\theta > 0$ there are $\Delta > 0$ and $\zeta > 0$ with the following property. Take any two agents with portfolios $x_A, x_B \in X$ and beliefs $\delta_A, \delta_B \in [0,1]$ with marginal rates of substitution that differ by more than ε, $\mathcal{M}(x_B,\delta_B) - \mathcal{M}(x_A,\delta_A) > \varepsilon$. Choose any price sufficiently close to the middle of the interval between the two marginal rates of substitution:*

$$p \in [\mathcal{M}(x_A,\delta_A) + \varepsilon/2, \mathcal{M}(x_B,\delta_B) - \varepsilon/2].$$

Then the trade $z = (\zeta, -p\zeta)$ satisfies $\|z\| < \theta$, and the utility gains of the two agents satisfy

$$U(x_A - z,\delta_A) - U(x_A,\delta_A) \geq \Delta, \tag{9}$$

$$U(x_B + z,\delta_B) - U(x_B,\delta_B) \geq \Delta. \tag{10}$$

Moreover, there is a constant $\lambda > 0$, which depends on the set X and on the difference between the marginal rates of substitution ε, but not on the size of the trade θ, such that the potential loss in current utility associated with the trade z is bounded below by $-\lambda\Delta$ for all beliefs δ:

$$U(x_A - z,\delta) - U(x_A,\delta) \geq -\lambda\Delta \text{ for all } \delta \in [0,1]. \tag{11}$$

5.2 Proof of Proposition 1

Proof of Proposition 1. Proceeding by contradiction suppose (3) does not hold. Without loss of generality, let us focus on state s_1. If (3) is violated in s_1 then there exist an $\varepsilon > 0$ and an $\eta \in (0,1)$ such that the following holds for infinitely many periods t:

$$P\left(\left|\mathcal{M}\left(x_t, \delta_t\right) - \kappa\right| > \varepsilon, \delta_t = 1 \mid s_1\right) > \eta P\left(\delta_t = 1 \mid s_1\right) \text{ for all } \kappa. \tag{12}$$

We want to show that (12) implies a profitable deviation for informed agents. The informed agent starts deviating at some date T to be defined if three conditions are satisfied: (a) his marginal rate of substitution is below some level κ^* to be defined: $\mathcal{M}\left(x_T, \delta_T\right) < \kappa^*$; (b) his utility is close enough to its long-run level: $u_T \geq \hat{v}_T - \alpha\eta\Delta/4$, for some $\Delta > 0$ to be defined; (c) his portfolio x_T is in some compact set X to be defined. When (a)-(c) hold the agent makes an offer z^* to be defined, which is accepted with probability $\chi_T\left(z^*|s_1\right) \geq \alpha\eta/4$ and gives him a utility gain of at least Δ. The expected payoff of this strategy at time T is then

$$u_T + \chi_T\left(z^*|s_1\right)\left(U(x_T - z^*, \delta_T) - u_T\right) > u_T + \alpha\eta\Delta/4 \geq \hat{v}_T.$$

Since $\hat{v}_T$ is, by definition, the expected payoff of a proposer who follows an optimal strategy, this leads to a contradiction.

To complete the proof we need to define the scalars κ^* and Δ, the set X, the deviating period T and the offer z^* and check that they satisfy the desired properties. Applying Lemma 5, choose a compact set X such that for some T' we have $P\left(x_t \in X \mid s_1\right) \geq 1 - \alpha\eta/4$ for all $t \geq T'$. Applying Lemma 6, choose $\Delta > 0$ to be the minimal gain from trade for two agents with marginal rates of substitution that differ by at least ε with portfolios in X. Applying Lemmas 2 and 4, choose a $T'' \geq T'$ such that

$$P\left(u_t \geq \hat{v}_t - \alpha\eta\Delta/4, x_t \in X \mid s_1\right) > 1 - \alpha\eta/2 \text{ for all } t \geq T''.$$

Using (12) and the fact that there is at least α informed agents, choose $T \geq T''$ such that:

$$P\left(\left|\mathcal{M}\left(x_T, \delta_T\right) - \kappa\right| > \varepsilon, \delta_T = 1 \mid s_1\right) > \eta P\left(\delta_t = 1 \mid s_1\right) \geq \alpha\eta \text{ for all } \kappa.$$

Using Lemma 4, it follows that

$$\begin{aligned} &P\left(\left|\mathcal{M}\left(x_T, \delta_T\right) - \kappa\right| \leq \varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) < \\ &\qquad < P\left(u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) - \alpha\eta/2 \text{ for all } \kappa, \end{aligned} \tag{13}$$

Define

$$\kappa^* = \sup\left\{\kappa : P\left(\mathcal{M}\left(x_T,\delta_T\right) > \kappa + (3/2)\,\varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) \geq \alpha\eta/4\right\}.$$

This definition implies that there are less than $\alpha\eta/4$ informed agents with marginal rate of substitution above $\kappa^* + 2\varepsilon$ who satisfy (b)-(c),

$$P\left(\mathcal{M}\left(x_T,\delta_T\right) > \kappa^* + 2\varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) < \alpha\eta/4, \tag{14}$$

given that $\kappa^* + \varepsilon/2 > \kappa^*$. Consider the following chain of equalities and inequalities:

$$\begin{aligned}
&P\left(\mathcal{M}\left(x_T,\delta_T\right) \geq \kappa^*, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) = \\
&P\left(\kappa^* \leq \mathcal{M}\left(x_T,\delta_T\right) \leq \kappa^* + 2\varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) \\
&P\left(\mathcal{M}\left(x_T,\delta_T\right) > \kappa^* + 2\varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) < \\
&\qquad < P\left(u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) - \alpha\eta/4,
\end{aligned}$$

where the equalities are immediate and the inequality follows from (13) (with $\kappa = \kappa^* + \varepsilon$) and (14). This implies

$$P\left(\mathcal{M}\left(x_T,\delta_T\right) < \kappa^*, u_T \geq \hat{v}_T - \alpha\eta\Delta/4, x_T \in X, \delta_T = 1 \mid s_1\right) > 0, \tag{15}$$

which shows that conditions (a)-(c) are met with positive probability.

To choose the deviating offer z^*, notice that, by the definition of Δ, there exists an offer $z^* = (\zeta^*, -p^*\zeta^*)$, with price $p^* = \kappa^* + \varepsilon/2$, such that

$$U(x - z^*,\delta) \;\geq\; U(x,\delta) + \Delta \text{ if } \mathcal{M}\left(x,\delta\right) < \kappa^* \text{ and } x \in X, \tag{16}$$

$$U(x + z^*,\delta) \;\geq\; U(x,\delta) + \Delta \text{ if } \mathcal{M}\left(x,\delta\right) > \kappa^* + \varepsilon \text{ and } x \in X. \tag{17}$$

Condition (16) shows that an informed proposer who satisfies (a)-(c) gains at least Δ if offer z^* is accepted.

Finally, the definition of κ^* implies that there must be at least $\alpha\eta/4$ agents with marginal rate of substitution above $\kappa^* + \varepsilon$,

$$P\left(\mathcal{M}\left(x_T,\delta_T\right) > \kappa^* + \varepsilon, u_T \geq \hat{v}_T - \alpha\eta\Delta/2, x_T \in X, \delta_T = 1 \mid s_1\right) \geq \alpha\eta/4, \tag{18}$$

given that $\kappa^* - \varepsilon/2 < \kappa^*$. Recall that $\hat{v}_t$ represents, by definition, the maximal expected utility the responder can get from rejecting all offers and behaving optimally in the future. A responder who receives z^* has the option to accept it and stop trading from then on, which yields expected utility $U(x_T + z^*, \delta_T)$. For all informed agents who satisfy $\mathcal{M}\left(x_T,\delta_T\right) \geq \kappa^* + \varepsilon$,

$u_T \geq \hat{v}_T - \alpha\eta\Delta/4$ and $x_T \in X$, we have the chain of inequalities

$$U(x_T + z^*, \delta_T) \geq u_T + \Delta > u_T + \alpha\eta\Delta/4 \geq \hat{v}_T,$$

where the first inequality follows from (17). This shows that rejecting z^* at time T is a strictly dominated strategy for these informed agents. Since there are at least $\alpha\eta/4$ of them, by (18), the probability that z^* is accepted must then satisfy $\chi_T(z|s_1) \geq \alpha\eta/4$. ■

5.3 Proof of Proposition 2

The proof makes use of the following lemma, which is proved in the Online Appendix.

Lemma 7 *Given the equilibrium measure P, for all $\varepsilon > 0$, there are a scalar M and a sequence of measures G_t on the space of portfolios and beliefs $R^2_+ \times [0,1]$ that satisfy the following properties: (i) the measure is zero for all beliefs smaller than or equal to $1/2$:*

$$G_t(x, \delta) = 0 \text{ if } \delta \leq 1/2;$$

(ii) G_t corresponds to the distribution generated by the measure P conditional on s_1 for informed agents:

$$G_t(x, 1) = P(\omega : x_t(\omega) = x, \delta_t(\omega) = 1 \mid s_1) \text{ for all } x \text{ and } t;$$

(iii) the average holdings of asset 1 exceed the average holdings of asset 2, truncated at any $m \geq M$, by less than ε:

$$\int_{x^2 \leq m} \left(x^1 - x^2\right) dG_t(x, \delta) \leq \varepsilon \text{ for all } m \geq M \text{ and all } t. \tag{19}$$

To prove Proposition 2, we proceed by contradiction and suppose that for all $\varepsilon > 0$ there are infinitely many periods t in which

$$|\kappa_t(s_1) - \kappa_t(s_2)| < \varepsilon \quad \text{and} \quad P(|\mathcal{M}(x_t, \delta_t) - \kappa_t(s)| < 2\varepsilon \mid s) > 1 - \varepsilon \quad \text{in some } s. \tag{20}$$

By symmetry, the long-run marginal rates of substitutions of informed agents in states s_1 and s_2 are one the inverse of the other $\kappa_t(s_1) = 1/\kappa_t(s_2)$. Some algebra shows that $|\kappa_t(s_1) - \kappa_t(s_2)| < \varepsilon$ implies $|\kappa_t(s_1) - 1| < \varepsilon$. Moreover, by the triangle inequality, $|\mathcal{M}(x_t, \delta_t) - \kappa_t(s_1)| < 2\varepsilon$ and $|\kappa_t(s_1) - 1| < \varepsilon$ imply $|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon$. Therefore, (20) implies

$$P(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon \mid s) > 1 - \varepsilon. \tag{21}$$

Without loss of generality, we focus on the case in which this condition holds for infinitely many periods in state s_1.

We want to show that, for some appropriately chosen positive scalars m and ζ, the following inequality holds for some t^*

$$\int_{x^2 \leq m} \left(x^1 - x^2\right) dG_{t^*} > \zeta, \tag{22}$$

and then showing that this contradicts (19). The argument proceeds in two steps.

Step 1. Since there is at least a mass α of informed agents, using Lemmas 4 and 5, we can find a compact set $X \subset R^2_{++}$ and a time T such that for all $\varepsilon > 0$ we have

$$P\left(\left|\mathcal{M}\left(x_t, \delta_t\right) - 1\right| < 3\varepsilon, \delta_t = 1, x_t \in X \mid s_1\right) > (5/6)\,\alpha - \varepsilon \tag{23}$$

for all periods $t \geq T$ in which $P\left(\left|\mathcal{M}\left(x_t, \delta_t\right) - 1\right| < 3\varepsilon \mid s_1\right) > 1 - \varepsilon$. Consider the minimization problem:

$$d_I\left(\varepsilon\right) = \min_{x \in X} \left(x^1 - x^2\right) \qquad \text{s.t.} \qquad \left|\mathcal{M}\left(x, 1\right) - 1\right| \leq 3\varepsilon.$$

Notice that $d_I\left(\varepsilon\right)$ is continuous, from the theorem of the maximum. Consider this problem at $\varepsilon = 0$. Let us prove that $d_I\left(0\right) > 0$. If $x^1 \leq x^2$, then $u'\left(x^1\right) \geq u'\left(x^2\right)$ and, therefore, the marginal rate of substitution

$$\mathcal{M}\left(x, 1\right) = \frac{\pi(1) u'(x^1)}{\left(1 - \pi(1)\right) u'(x^2)} \geq \frac{\pi(1)}{1 - \pi(1)} > 1.$$

Therefore, all x that satisfy $\left|\mathcal{M}\left(x, 1\right) - 1\right| \leq 0$ must also satisfy $x^1 > x^2$. In other words, given that informed agents have a signal favorable to state 1, if their marginal rate of substitution is exactly 1 they must hold strictly more of asset 1. We can now define the constant ζ (to be used in expression (22)), as

$$\zeta = \frac{\alpha}{6} d_I\left(0\right).$$

Next, we define the quantity m. Applying uniform market clearing and Lemma 7, we can find an $m \geq d_I\left(0\right)$ such that the following inequalities hold for all t:

$$\int_{x_t^2 > m} x_t^2 dP(\omega | s_1) \leq \zeta \tag{24}$$

and

$$\int_{x^2 \leq m} \left(x^1 - x^2\right) dG_t \leq \zeta. \tag{25}$$

From (24), we have

$$mP\left(x_t^2\left(\omega\right) > m\right) \leq \int_{x_t^2(\omega) > m} x_t^2\left(\omega\right) dP(\omega | s_1) \leq \zeta \text{ for all } t,$$

which, given the definition of ζ and the fact that $m \geq d_I(0)$, implies

$$P\left(x_t^2(\omega) > m\right) \leq \frac{\alpha}{6}\frac{d_I(0)}{m} \leq \frac{\alpha}{6} \text{ for all } t.$$

We then obtain the following chain of equalities and inequalities,

$$\begin{aligned}
& P\left(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t \in X \mid s_1\right) = \\
& P\left(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t^2 \leq m, x_t \in X \mid s_1\right) \\
& \quad + P\left(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t^2 > m, x_t \in X \mid s_1\right) \\
\leq \quad & P\left(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t^2 \leq m, x_t \in X \mid s_1\right) + \alpha/6,
\end{aligned}$$

and combine it with (23) to conclude that

$$P\left(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon, \delta_t = 1, x_t^2 \leq m, x_t \in X \mid s_1\right) > (2/3)\,\alpha - \varepsilon \tag{26}$$

for all $t \geq T$ in which $P\left(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon \mid s_1\right) > 1 - \varepsilon$.

Step 2. Consider the problem

$$d_U(\varepsilon) = \min_{\substack{x^2 \leq m \\ \delta \geq 1/2}} \left(x^1 - x^2\right) \quad \text{s.t. } |\mathcal{M}(x, \delta) - 1| \leq 3\varepsilon.$$

The theorem of the maximum implies that $d_U(\varepsilon)$ is continuous. Moreover, $d_U(\varepsilon)$ is negative for all $\varepsilon > 0$ and $d_U(0) = 0$. Recall from Step 1 that $d_I(\varepsilon)$ is continuous and $d_I(0) > 0$. It is then possible to find a positive ε^*, smaller than both $\alpha/6$ and ζ/m, such that

$$\frac{\alpha}{2} d_I(\varepsilon^*) + d_U(\varepsilon^*) > \frac{\alpha}{3} d_I(0) = 2\zeta, \tag{27}$$

(the second equality comes from the definition of ζ).

Since, by construction $\varepsilon^* < \alpha/6$, it follows from (26) that the mass of informed agents with marginal rates of substitution near 1 (within $3\varepsilon^*$) and a portfolio that satisfies $x_t^2 \leq m$ and $x_t \in X$ is sufficiently high:

$$P\left(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon^*, \delta_t = 1, x_t^2 \leq m, x_t \in X \mid s_1\right) > \alpha/2 \tag{28}$$

for all $t \geq T$ in which $P\left(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon^* \mid s_1\right) > 1 - \varepsilon^*$.

Moreover, by Lemma 4, in all periods $t \geq T$ in which almost all agents have marginal rate of substitution close to 1, i.e., $P\left(|\mathcal{M}(x_t, \delta_t) - 1| < 3\varepsilon^* \mid s_1\right) > 1 - \varepsilon^*$, almost all agents with beliefs higher than 1/2 and portfolios satisfying $x_t^2 \leq m$ also have a marginal rate of substitution close to 1:

$$P\left(|\mathcal{M}(x_t,\delta_t)-1|<3\varepsilon^*,\delta_t>1/2,x_t^2\le m\mid s_1\right)>P\left(\delta_t>1/2,x_t^2\le m\mid s_1\right)-\varepsilon^*. \tag{29}$$

By hypothesis, i.e., by (21), we can choose a $t^*\ge T$ such that

$$P\left(|\mathcal{M}(x_{t^*},\delta_{t^*})-1|<3\varepsilon^*\mid s_1\right)>1-\varepsilon^*$$

so that both (28) and (29) are satisfied.

Define the following three groups of agents

$$\begin{aligned}
A_1 &= \left\{(x,\delta):|\mathcal{M}(x,\delta)-1|<3\varepsilon^*,\delta=1,x^2\le m,x\in X\right\},\\
A_2 &= \left\{(x,\delta)\notin A_1:|\mathcal{M}(x,\delta)-1|<3\varepsilon^*,\delta>1/2,x^2\le m\right\},\\
A_3 &= \left\{(x,\delta)\notin A_1\cup A_2:\delta>1/2,x^2\le m\right\}.
\end{aligned}$$

Step 3. Now we split the integral (22) in three parts, corresponding to the three sets A_j defined above and determine a lower bound for each of them. First, we have

$$\int_{A_1}\left(x^1-x^2\right)dG_{t^*}=\int_{(x_{t^*},\delta_{t^*})\in A_1}\left(x^1_{t^*}(\omega)-x^2_{t^*}(\omega)\right)dP(\omega|s_1)\ge\frac{\alpha}{2}d_I(\varepsilon^*), \tag{30}$$

where the equality follows from property (ii) of the distribution G_t (in Lemma 7) and the inequality follows from the definition of $d_I(\varepsilon^*)$ and condition (28). The definition of $d_U(\varepsilon^*)$ implies that

$$\int_{A_2}\left(x^1-x^2\right)dG_{t^*}\ge d_U(\varepsilon^*)\,P(A_2)\ge d_U(\varepsilon^*), \tag{31}$$

since $d_U(\varepsilon^*)<0$ and $P(A_2)\le 1$. Finally, the definition of the measure G_t and condition (29) imply that

$$G_{t^*}(A_3)\le P\left((x_{t^*},\delta_{t^*})\in A_3|s_1\right)\le P\left(\delta_{t^*}>1/2,x^2_{t^*}\le m|s_1\right)-P\left((x_{t^*},\delta_{t^*})\in A_1\cup A_2|s_1\right)\le\varepsilon^*<\zeta/m,$$

where the last inequality follows from the definition of ε^*. We then have the following lower bound

$$\int_{A_3}\left(x^1-x^2\right)dG_{t^*}\ge -mG_{t^*}(A_3)\ge-\zeta. \tag{32}$$

We can now combine (30), (31) and (32) and use inequality (27) to obtain a lower bound for the whole integral (22):

$$\int_{x^2\le m}\left(x^1-x^2\right)dG_{t^*}\ge\frac{\alpha}{2}d_I(\varepsilon^*)+d_U(\varepsilon^*)-\zeta>\zeta.$$

Comparing this inequality and (25) leads to the desired contradiction.

5.4 Proof of Proposition 3

The following two lemmas are used in the proof. They are proved in the Online Appendix.

Lemma 8 *For all $\varepsilon > 0$ the probability that the belief δ_t is above the threshold $\varepsilon / (1+\varepsilon)$ conditional on signal s_1 is bounded below for all t:*

$$P\left(\delta_t \geq \varepsilon / (1+\varepsilon) \mid s_1\right) > 1 - \varepsilon.$$

Lemma 9 *For any integer J, the sequence $\kappa_t(s_1)$ satisfies the property:*

$$\lim_{t \to \infty} \left|\kappa_{t+J}(s_1) - \kappa_t(s_1)\right| = 0.$$

For all $\varepsilon > 0$ and all integers J it is possible to find a T such that

$$P\left(\left|\mathcal{M}\left(x_{t+J}, \delta_{t+J}\right) - \kappa_t(s_1)\right| < \varepsilon, \delta_{t+J} = 1 \mid s\right) > \alpha - \varepsilon \textit{ for all } t \geq T.$$

To prove Proposition 3 suppose, by contradiction, that there exist an $\varepsilon > 0$ such that for some state s the following condition holds for infinitely many t:

$$P\left(\left|\mathcal{M}\left(x_t, \delta^I(s)\right) - \kappa_t(s)\right| > \varepsilon \mid s\right) > \varepsilon,$$

where $\mathcal{M}\left(x_t, \delta^I(s)\right)$ is the marginal rate of substitution of an agent (informed or uninformed) evaluated at the belief of the informed agents $\delta^I(s)$. Without loss of generality, let us focus on state s_1 and suppose

$$P\left(\mathcal{M}\left(x_t, 1\right) - \kappa_t(s_1) > \varepsilon \mid s_1\right) > \varepsilon \tag{33}$$

for infinitely many t. The other case is treated in a symmetric way.

We want to show that if (33) holds, we can construct a profitable deviation in which:

(i) The player follows the equilibrium strategy σ up to some period T.

(ii) At T, if his portfolio satisfies $\mathcal{M}(x_T, 1) > \kappa_t(s) + \varepsilon$ and his beliefs δ_T is above some positive lower bound $\underline{\delta}$ and some other technical conditions are satisfied, he moves to the experimentation stage (iii), otherwise, he keeps playing σ.

(iii) The experimentation stage lasts between T and $T + J - 1$ for some J. An agent makes a sequence of offers $\{\hat{z}_j\}_{j=0}^{J-1}$ as long as he is selected as the proposer. The "favorable" responses to the offers $\{\hat{z}_j\}_{j=0}^{J-1}$ are given by the binary sequence $\{\hat{r}_j\}_{j=0}^{J-1}$. If at any point during the experimentation stage the agent is not selected as the proposer or fails to receive response $\hat{r}_j$ after offer $\hat{z}_j$, he stops trading. Otherwise, he goes to (iv).

(iv) At time $T+J$, after making all the offers $\{\hat{z}_j\}_{j=0}^{J-1}$ and receiving responses equal to $\{\hat{r}_j\}_{j=0}^{J-1}$, if the player is selected as the proposer he makes an offer z^* and stops trading at $T+J+1$. Otherwise, he stops trading right away.

The expected payoff of this strategy, from the point of view of a deviating agent at time T, is

$$
\begin{aligned}
w \;=\; & u_T - \hat{L} + \delta_T \gamma^J 2^{-J-1} \xi_1 \chi_{T+J}\left(z^*|s_1\right) \left[U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 1) - U(x_T, 1) \right] + \\
& + \left(1-\delta_T\right) \gamma^J 2^{-J-1} \xi_2 \chi_{T+J}\left(z^*|s_2\right) \left[U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 0) - U(x_T, 0) \right], \qquad (34)
\end{aligned}
$$

where the term $\hat{L}$ captures the expected utility losses if the player makes some or all of the offers in $\{\hat{z}_j\}_{j=0}^{J-1}$ but not the last offer z^* and the following two terms capture the expected utility gains in states s_1 and s_2, if all the deviating offers, including z^*, are accepted. The factors ξ_1 and ξ_2 denote the probabilities in states s_1 and s_2, that player receives the sequence of responses $\{\hat{r}_j\}_{j=0}^{J-1}$. Notice that γ^J is the probability that the game does not end between periods T and $T+J$ and 2^{-J-1} is the probability of being selected as the proposer in all these periods.

In order to show that the strategy above is a profitable deviation, we need to show that the utility gain in the first square brackets is large enough, by choosing z^* to be a profitable trade with informed agents in s_1, and that the remaining terms are sufficiently small. In the rest of the proof, we choose the time T, the lower bound $\underline{\delta}$, and the offers $\{\hat{z}_j\}_{j=0}^{J-1}$ and z^* to achieve this goal.

Step 1. (Bounds on gains and losses for the final trade) Following steps similar to the ones in the proof of Proposition 1, we can use Lemmas 5 and 8 to find a compact set $X \subset R^2_{++}$ and a period T' such that

$$P\left(\delta_t \geq \underline{\delta}, x_t \in X \mid s_1\right) > 1 - \varepsilon/2 \qquad (35)$$

for all $t \geq T'$, where $\underline{\delta} = (\varepsilon/2)/(1+\varepsilon/2) > 0$. Pick a scalar $\theta^* > 0$ such that $x + z > 0$ when $x \in X$ and $\|z\| < \theta^*$. Using Lemma 6, we can then find a $\Delta^* > 0$ which is a lower bound for the gains from trade between two agents with marginal rates of substitution differing by at least $\varepsilon/2$ and portfolios in X, making trades of norm smaller than θ^*. This will be used as a lower bound for the gains from trading in state s_1. Define an upper bound for the potential losses of an uninformed agent who makes a trade of norm smaller than or equal to θ^* in the other state, s_2:

$$L^* \equiv - \min_{x \in X, \|z\| \leq \theta^*} \left\{ U\left(x+z, 0\right) - U\left(x, 0\right) \right\}.$$

Next, choose J to be an integer large enough that

$$\underline{\delta}\,(\alpha/2)\,\Delta^* - (1-\underline{\delta})\,\rho^{-J}L^* > 0,$$

where ρ is the scalar defined in Lemma 3. This choice of J ensures that the experimentation phase is long enough that, when offering the last trade, the agent assigns sufficiently high probability to state s_1, so that the potential gain Δ^* dominates the potential loss L^*.

Step 2. *(Bound on losses from experimentation)* To simplify notation, let

$$\tilde{\Delta} = \gamma^J 2^{-J-1}\beta^J\left(\underline{\delta}\,(\alpha/2)\,\Delta^* - (1-\underline{\delta})\,\rho^{-J}L^*\right),$$

where β is the positive scalar defined in Lemma 3. Choose a scalar $\hat{\theta} > 0$ such that for all $x \in X$, all $\|z_1\| < J\hat{\theta}$, all $\|z_2\| \leq \theta^*$, and any $\delta \in [0,1]$ the following inequality holds

$$|U(x+z_1+z_2,\delta) - U(x+z_2,\delta)| < \tilde{\Delta}/3. \tag{36}$$

Next, applying Lemma 3 we can find a time $T'' \geq T'$ such that in all $t \geq T''$ there is a trade of norm smaller than $\hat{\theta}$ that satisfies either (4) or (5). Before using this property to define the offers $\{\hat{z}_j\}_{j=0}^{J-1}$, we need to define the time period T where the deviation occurs. To do so, using our starting hypothesis (33), condition (35), and applying Lemma 2, we can find a $T''' \geq T''$ such that for infinitely many periods $t \geq T'''$ there is a positive mass of uninformed agents who have: marginal rate of substitution sufficiently above $\kappa_t(s_1)$, utility near its long-run level, beliefs sufficiently favorable to s_1, and portfolio in X; that is,

$$P\left(\mathcal{M}(x_t,1) - \kappa_t(s_1) > \varepsilon, \delta_t \geq \underline{\delta}, u_t > \hat{v}_t - \tilde{\Delta}/3, x_t \in X \mid s_1\right) > 0. \tag{37}$$

Finally, applying Lemma 9, we pick a $T \geq T'''$ so that (37) holds at $t = T$ *and*, at time $T+J$, there is a sufficiently large mass of informed agents who have: marginal rate of substitution sufficiently near $\kappa_T(s_1)$, utility near its long-run level, and portfolio in X; that is,

$$P\left(|\mathcal{M}(x_{T+J},1) - \kappa_T(s_1)| < \varepsilon/2, \delta_{T+J} = 1, u_{T+J} > \hat{v}_{T+J} - \Delta/2, x_{T+J} \in X \mid s_1\right) > \alpha/2. \tag{38}$$

Having defined T, we can apply Lemma 3 to find the desired sequence of trades $\{\hat{z}_j\}_{j=0}^{J-1}$ of norm smaller than $\hat{\theta}$, that satisfy either (4) or (5). For each trade $\hat{z}_j$, if (4) holds we set $\hat{r}_j = 1$ (accept). In this way the probability of observing $\hat{r}_j$ is $\chi_{T+j}(\hat{z}_j|s_1) > \beta$ in state s_1 and $\chi_{T+j}(\hat{z}_j|s_2) < \rho^{-1}\chi_{T+j}(\hat{z}_j|s_1)$ in state s_2. Otherwise, if (5) holds, we set $\hat{r}_j = 0$ and obtain analogous inequalities. This implies that the factors ξ_1 and ξ_2 in (34) satisfy

$$\xi_1 > \beta^J \text{ and } \xi_2 < \xi_1\rho^{-J}. \tag{39}$$

Step 3. (Define z^* and check profitable deviation) We can now define the final trade z^* to be a trade of norm smaller than θ^*, such that

$$\begin{aligned} U(x - z^*, 1) &> U(x, 1) + \Delta^* \text{ if } \mathcal{M}(x, 1) > \kappa_T(s_1) + \varepsilon \text{ and } x \in X, \\ U(x + z^*, 1) &> U(x, 1) + \Delta^* \text{ if } \mathcal{M}(x, 1) < \kappa_T(s_1) + \varepsilon/2 \text{ and } x \in X, \end{aligned}$$

which is possible given the definition of Δ^*. Finally, we check that we have constructed a profitable deviation. Let uninformed agents start deviating whenever the following conditions are satisfies at date T:

$$\mathcal{M}(x_T, 1) > \kappa_T(s_1) + \varepsilon, \delta_T \geq \underline{\delta}, u_T > \hat{v}_T - \tilde{\Delta}/3, x_T \in X.$$

Equation (37) shows that this happens with positive probability. Let us evaluate the deviating strategy payoff (34), beginning with the last two terms. The triangle inequality implies $\left\| \sum_{j=0}^{J-1} \hat{z}_j \right\| < J\hat{\theta}$. Then the definition of z^* and (36) imply that the gain from trade of the uninformed agent, conditional on s_1, is bounded below:

$$\begin{aligned} U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 1) - U(x_T, 1) &\geq U(x_T + z^*, 1) - U(x_T, 1) - \left| U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 1) - U(x_T + z^*, 1) \right| \\ &> \Delta^* - \tilde{\Delta}/3. \end{aligned}$$

The definition of L^* implies that the gain conditional on s_2 is also bounded:

$$U(x_T + \sum_{j=0}^{J-1} \hat{z}_j + z^*, 0) - U(x_T, 0) > -L^* - \tilde{\Delta}/3.$$

Moreover, condition (38) shows that the probability that informed agents accept z^* at $T + J$ satisfies $\chi_{T+J}(z^*|s_1) > \alpha/2$. These results, together with the inequalities (39) and the fact that $\chi_{T+J}(z^*|s_2) \leq 1$, imply that the last two terms in (34) are bounded below by

$$\gamma^J 2^{-J-1} \beta^J \left[\underline{\delta}(\alpha/2) \left(\Delta^* - \tilde{\Delta}/3 \right) - (1 - \underline{\delta}) \rho^{-J} \left(L^* + \tilde{\Delta}/3 \right) \right],$$

which, by the definition of $\tilde{\Delta}$, is greater than $(2/3)\tilde{\Delta}$. Finally, all the expected losses in $\hat{L}$ in (34) are bounded above by $\tilde{\Delta}/3$, thanks to (36). Therefore, $w > u_T + \tilde{\Delta}/3$. Since $u_T > \hat{v}_T - \tilde{\Delta}/3$, we conclude that $w > \hat{v}_T$ and we have found a profitable deviation.

5.5 Proof of Theorem 1

We begin from the second part of the theorem, proving (8), which characterizes the limit behavior of $\kappa_t(s)$.

Without loss of generality, let $s = s_1$. Suppose first that for infinitely many periods the long-run marginal rate of substitution $\kappa_t(s_1)$ is larger than the ratio of the probabilities $\phi(s_1)/(1-\phi(s_1))$ by a factor larger than $1+\varepsilon$:

$$\kappa_t(s_1) > (1+\varepsilon)\,\phi(s_1)/(1-\phi(s_1)) \text{ for some } \varepsilon > 0.$$

Proposition 3 then implies that for all $\eta > 0$ and T there is a t such that almost all agents have portfolios that satisfy $u'(x_t^1)/u'(x_t^2) \geq 1+\varepsilon/2$:

$$P\left(u'(x_t^1)/u'(x_t^2) \geq 1+\varepsilon/2 \mid s_1\right) > 1-\eta. \tag{40}$$

We want to show that this property violates uniform market clearing, since it implies that almost all agents hold more of asset 2 than of asset 1.

Uniform market clearing implies that for any $\zeta > 0$ we can find an M such that

$$\int_{x_t^1(\omega)\leq m} x_t^1(\omega)\, dP(\omega|s_1) \geq 1-\zeta \text{ for all } m \geq M \text{ and all } t. \tag{41}$$

Moreover, since $\int x_t^2(\omega)\, dP(\omega|s_1) = 1$, this implies that

$$\int_{x_t^1(\omega)\leq m} \left(x_t^2(\omega) - x^1(\omega)\right) dP(\omega|s_1) \leq \zeta \text{ for all } m \geq M \text{ and all } t. \tag{42}$$

The idea of the proof is to reach a contradiction by splitting the integral on the left-hand side of (42) in three pieces: a group of agents with a strictly positive difference $x_t^2 - x_t^1$, a group of agents with a non-negative difference $x_t^2 - x_t^1$, and a small residual group. The argument here follows a similar logic as the proof of Proposition 2.

Using Lemma 5, find a compact set X and a period T such that for all $t \geq T$ at least half of the agents have portfolios in X:

$$P(x_t \in X \mid s_1) \geq 1/2 \text{ for all } t \geq T. \tag{43}$$

Let us then find a lower bound for the difference between the holdings of asset 1 and 2 for agents with portfolios in X that satisfy $u'(x_t^1)/u'(x_t^2) \geq 1+\varepsilon/2$. We do so by solving the problem

$$d = \min_{x\in X}\left(x^2 - x^1\right) \quad \text{s.t. } u'(x^1)/u'(x^2) \geq 1+\varepsilon/2,$$

which gives a $d > 0$.

Let us pick $\zeta = d/5$ and find an M such that (41) and (42) hold. Condition (42) (with $\zeta = d/5$) is the market clearing condition that we will contradict below. Condition (41) is also useful, because it gives us a lower bound for $P\left(x_t^1 \le m\right)$:

$$P\left(x_t^1 \le m\right) \ge 1 - \zeta/m \text{ for all } m \ge M \text{ and all } t, \tag{44}$$

which follows from the chain of inequalities

$$mP\left(x_t^1 > m\right) \le \int_{x_t^1(\omega)>m} x_t^1(\omega)\, dP(\omega|s_1) \le \zeta.$$

Using our hypothesis (40) we know that for any $\eta > 0$ we can find a period $t \ge T$ in which more than $1-\eta$ agents satisfy $u'(x_t^1)/u'(x_t^2) \ge 1+\varepsilon/2$. Combining this with (43) and (44) (applying Lemma 4), we can always find a $t \ge T$ in which almost all agents satisfy $u'(x_t^1)/u'(x_t^2) \ge 1+\varepsilon/2$ and $x_t \le m$:

$$P\left(u'(x_t^1)/u'(x_t^2) \ge 1+\varepsilon/2, x_t \le m \mid s_1\right) > 1-\eta-\zeta/m, \tag{45}$$

and almost half of them satisfy $u'(x_t^1)/u'(x_t^2) \ge 1+\varepsilon/2$ and $x_t \le m$, and have portfolios in X:

$$P\left(u'(x_t^1)/u'(x_t^2) \ge 1+\varepsilon/2, x_t \le m, x_t \in X \mid s_1\right) > 1/2-\eta-\zeta/m. \tag{46}$$

Define the three disjoint sets

$$\begin{aligned}
A_1 &= \left\{\omega : u'(x_t^1)/u'(x_t^2) \ge 1+\varepsilon/2, x_t \in X, x_t^1 \le m\right\}, \\
A_2 &= \left\{\omega : u'(x_t^1)/u'(x_t^2) \ge 1+\varepsilon/2, x_t^1 \le m\right\}/A_1, \\
A_3 &= \left\{\omega : u'(x_t^1)/u'(x_t^2) < 1+\varepsilon/2, x_t^1 \le m\right\},
\end{aligned}$$

which satisfy $A_1 \cup A_2 \cup A_3 = \left\{\omega : x_t^1 \le m\right\}$. We can then bound from below the following three integrals:

$$\begin{aligned}
\int_{A_1} \left(x_t^2 - x_t^1\right) dP(\omega|s_1) &\ge d \cdot (1/2-\eta-\zeta/m), \\
\int_{A_2} \left(x_t^2 - x_t^1\right) dP(\omega|s_1) &\ge 0, \\
\int_{A_3} \left(x_t^2 - x_t^1\right) dP(\omega|s_1) &\ge -m \cdot (\eta+\zeta/m).
\end{aligned}$$

The first inequality follows from the definitions of d and A_1 and the fact that $P(A_1|s_1) > 1/2-\eta-\zeta/m$ from (46). The second follows from the definition of A_2 and the fact that $u'(x_t^1)/u'(x_t^2) > 1$ implies $x_t^2 > x_t^1$. The third follows from the definition of A_3 (which implies $x_t^2 - x_t^1 \ge -m$) and the fact that $P(A_3|s_1) < \eta+\zeta/m$ from (45). Summing term by term, we then obtain

$$\int_{x_t^1(\omega)\leq m} \left(x_t^2 - x_t^1\right) dP\left(\omega|s_1\right) \geq d \cdot (1/2 - \eta - \zeta/m) - m \cdot (\eta + \zeta/m).$$

Since we can choose an m arbitrarily large and an η arbitrarily close to 0 (in that order), we can make this expression as close as we want to $d/2 - \zeta$ which is strictly greater than ζ, given that $\zeta = d/5 < d/4$. This contradicts the market clearing condition (42).

In a similar way we can rule out the case in which $\kappa_t\left(s_1\right) < (1-\varepsilon)\,\phi(s_1)/(1-\phi(s_1))$ for infinitely many periods. This completes the argument for $\lim_{t\to\infty} \kappa_t\left(s_1\right) = \phi(s_1)/\left(1-\phi(s_1)\right)$. An analogous argument can be applied to s_2.

To complete the proof, we need to prove the long-run efficiency of equilibrium portfolios, i.e., property (7). Proposition 3 and $\lim \kappa_t\left(s\right) = \phi(s)/\left(1-\phi(s)\right)$, imply, by the properties of convergence in probability, that

$$\lim_{t\to\infty} P\left(\left|u'\left(x_t^1\right)/u'\left(x_t^2\right) - 1\right| > \varepsilon\right) = 0. \tag{47}$$

We want to show that negating (7) leads to a contradiction of (47).

Suppose that for some $\varepsilon > 0$ we have $P\left(\left|x_t^1 - x_t^2\right| > \varepsilon\right) > \varepsilon$ for infinitely many periods. Then, as usual, we can use Lemmas 4 and 5 to find a compact set X such that the following condition holds for infinitely many periods:

$$P\left(\left|x_t^1 - x_t^2\right| > \varepsilon, x_t \in X\right) > \varepsilon/2.$$

But then the continuity of $u'\left(\cdot\right)$ implies that there is a $\delta > 0$ such that

$$\left|u'\left(x^1\right)/u'\left(x^2\right) - 1\right| > \delta \implies \left|x^1 - x^2\right| > \varepsilon \text{ for all } x \in X$$

which implies

$$P\left(\left|u'\left(x_t^1\right)/u'\left(x_t^2\right) - 1\right| > \delta, x_t \in X\right) \geq P\left(\left|x_t^1 - x_t^2\right| > \varepsilon, x_t \in X\right) > \varepsilon/2.$$

Given that

$$P\left(\left|u'\left(x_t^1\right)/u'\left(x_t^2\right) - 1\right| > \delta\right) \geq P\left(\left|u'\left(x_t^1\right)/u'\left(x_t^2\right) - 1\right| > \delta, x_t \in X\right)$$

we conclude that there are $\varepsilon, \delta > 0$ such that

$$P\left(\left|u'\left(x_t^1\right)/u'\left(x_t^2\right) - 1\right| > \delta\right) > \varepsilon/2,$$

contradicting (47) and completing the proof.

Online Appendix for "Decentralized Trading with Asymmetric Information"

1 Proof of Lemma 5

To prove the lemma, we will find two scalars $\overline{x}$ and $\underline{u}$ such that the set

$$X = \left\{ x : x \in (0, \overline{x}]^2, U(x, \delta) \geq \underline{u} \text{ for some } \delta \in [0,1] \right\}$$

satisfies the desired properties. The proof combines two ideas: use market clearing to put an upper bound on the holdings of the two assets, that is, to show that with probability close to 1 agents have portfolios in $(0, \overline{x}]^2$; use optimality to bound their holdings away from zero, by imposing the inequality $U(x, \delta) \geq \underline{u}$.

First, let us prove that X is a compact subset of R^2_{++}. The following two equalities follow from the fact that $U(x, \delta)$ is continuous, non-decreasing in δ if $x^1 \geq x^2$, and non-increasing if $x^1 \leq x^2$:

$$\begin{aligned} \left\{ x : U(x, \delta) \geq \underline{u} \text{ for some } \delta \in [0,1], x^1 \geq x^2 \right\} &= \left\{ x : U(x, 1) \geq \underline{u}, x^1 \geq x^2 \right\}, \\ \left\{ x : U(x, \delta) \geq \underline{u} \text{ for some } \delta \in [0,1], x^1 \leq x^2 \right\} &= \left\{ x : U(x, 0) \geq \underline{u}, x^1 \leq x^2 \right\}. \end{aligned}$$

The sets on the right-hand sides of these equalities are closed sets. Then X can be written as the union of two closed sets, intersected with a bounded set:

$$X = \left(\left\{ x : U(x, 1) \geq \underline{u}, x^1 \geq x^2 \right\} \cup \left\{ x : U(x, 0) \geq \underline{u}, x^1 \leq x^2 \right\} \right) \cap (0, \overline{x}]^2,$$

and thus is compact. Notice that $x \notin X$ if $x^j = 0$ for some j because of Assumption 2 and $\underline{u} > -\infty$. Therefore, X is a compact subset of R^2_{++}.

Next, let us define $\overline{x}$ and $\underline{u}$ and the time period T. Given any $\varepsilon > 0$, set $\overline{x} = 4/\varepsilon$. Goods market clearing implies that

$$P(x_t^j > \overline{x} | s) \leq \varepsilon/4 \text{ for all } t, \text{ for } j = 1, 2. \tag{48}$$

To prove this, notice that

$$1 = \int x_t^j(\omega)\, dP(\omega|s) \geq \int_{x_t^j(\omega) > 4/\varepsilon} x_t^j(\omega)\, dP(\omega|s) \geq (4/\varepsilon)\, P(x_t^j > 4/\varepsilon | s),$$

which gives the desired inequality. Let $\overline{u}$ be an upper bound for the agents' utility function

$u(\cdot)$ (from Assumption 2). Choose a scalar $u < \overline{u}$ such that

$$\frac{\overline{u} - U(x_0, \delta_0)}{\overline{u} - u} \leq \frac{\varepsilon}{8},$$

for all initial endowments x_0 and initial beliefs δ_0. Such a u exists because $U(x_0, \delta_0) > -\infty$, as initial endowments are strictly positive by Assumption 3, and there is a finite number of types. Then notice that $U(x_0, \delta_0) \leq E\left[v_t | h^0\right]$ for all initial histories h^0, because an agent always has the option to refuse any trade. Moreover

$$E\left[v_t \mid h^0\right] \leq P\left(v_t < u \mid h^0\right) u + P\left(v_t \geq u \mid h^0\right) \overline{u}.$$

Combining these inequalities and rearranging gives

$$P\left(v_t < u \mid h^0\right) \leq \frac{\overline{u} - U(x_0, \delta_0)}{\overline{u} - u} \leq \frac{\varepsilon}{8}.$$

Taking unconditional expectations shows that $P(v_t < u) \leq \varepsilon/8$. Since $P(s) = 1/2$ it follows that

$$P(v_t < u \mid s) \leq \varepsilon/4 \text{ for all } t, \text{ for all } s. \tag{49}$$

Choose T so that

$$P(|u_t - v_t| > u/2 \mid s) \leq \varepsilon/4 \text{ for all } t \geq T. \tag{50}$$

This can be done by Lemma 2, given that almost sure convergence implies convergence in probability. We can then set $\underline{u} = u/2$.

Finally, we check that $P(x_t \in X \mid s) \geq 1 - \varepsilon$ for all $t \geq T$, using the following chain of inequalities:

$$\begin{aligned}
&P(x_t \in X \mid s) \geq P\left(x_t \in (0, \overline{x}]^2, U(x_t, \delta_t) \geq \underline{u} \mid s\right) \geq \\
&P\left(x_t \in (0, \overline{x}]^2, v_t \geq u, |u_t - v_t| \leq u/2 \mid s\right) \geq \\
&1 - \sum_j P\left(x_t^j > \overline{x} \mid s\right) - P(v_t < u \mid s) - P(|u_t - v_t| > u/2 \mid s) \geq 1 - \varepsilon.
\end{aligned}$$

The first inequality follows because $U(x_t(\omega), \delta_t(\omega)) \geq \underline{u}$ implies $U(x_t(\omega), \delta) \geq \underline{u}$ for some $\delta \in [0, 1]$. The second follows because $v_t(\omega) \geq u$ and $|u_t(\omega) - v_t(\omega)| \leq u/2$ imply $u_t(\omega) = U(x_t(\omega), \delta_t(\omega)) \geq u/2 = \underline{u}$. The third follows from repeatedly applying Lemma 4. The fourth combines (48), (49), and (50).

2 Proof of Lemma 6

The idea of the proof is as follows. We construct a Taylor expansion to compute the utility gains for any trade. Then we define the traded amount ζ and the utility gain Δ satisfying (9) and (10).

Choose any two portfolios $x_A, x_B \in X$ and any two beliefs $\delta_A, \delta_B \in [0,1]$ such that $\mathcal{M}(x_B, \delta_B) - \mathcal{M}(x_A, \delta_A) > \varepsilon$. Pick a price p sufficiently close to the middle of the interval between the marginal rates of substitution:

$$p \in [\mathcal{M}(x_A, \delta_A) + \varepsilon/2, \mathcal{M}(x_B, \delta_B) - \varepsilon/2].$$

This price is chosen so that both agents will make positive gains. Consider agent A and a traded amount $\tilde{\zeta} \leq \bar{\zeta}$ (for some $\bar{\zeta}$ which we will properly choose below). The current utility gain associated with the trade $\tilde{z} = (\tilde{\zeta}, -p\tilde{\zeta})$ can be written as a Taylor expansion:

$$\begin{aligned} & U(x_A - \tilde{z}, \delta_A) - U(x_A, \delta_A) \\ = \; & -\pi(\delta_A)u'(x_A^1)\tilde{\zeta} + (1-\pi(\delta_A))u'(x_A^2)p\tilde{\zeta} + \frac{1}{2}\left(\pi(\delta_A)u''(y^1) + (1-\pi(\delta_A))u''(y^2)p^2\right)\tilde{\zeta}^2 \\ \geq \; & (1-\pi(\delta_A))u'(x_A^2)\,(\varepsilon/2)\,\tilde{\zeta} + \frac{1}{2}\left[\pi(\delta_A)u''(y^1) + (1-\pi(\delta_A))u''(y^2)p^2\right]\tilde{\zeta}^2, \end{aligned} \tag{51}$$

for some $(y^1, y^2) \in \left[x_A^1, x_A^1 - \bar{\zeta}\right] \times [x_A^2 + p\bar{\zeta}, x_A^2]$. The inequality above follows because $p \geq \mathcal{M}(x_A, \delta_A) + \varepsilon/2$. An analogous expansion can be done for agent B.

Now we want to bound the last line in (51). To do so we first define the minimal and the maximal prices for agents with any belief in $[0,1]$ and any portfolio in X:

$$\begin{aligned} \underline{p} &= \min_{x \in X, \delta \in [0,1]} \left\{\mathcal{M}(x, \delta) + \varepsilon/2\right\}, \\ \overline{p} &= \max_{x \in X, \delta \in [0,1]} \left\{\mathcal{M}(x, \delta) - \varepsilon/2\right\}. \end{aligned}$$

These prices are well-defined as X is a compact subset of R_{++}^2 and $u(.)$ has continuous first derivative on R_{++}^2. Then, choose $\bar{\zeta} > 0$ such that for all $\tilde{\zeta} \leq \bar{\zeta}$ and all $p \in [\underline{p}, \overline{p}]$, the trade $\tilde{z} = (\tilde{\zeta}, -p\tilde{\zeta})$ satisfies $\|\tilde{z}\| < \theta$ and $x + \tilde{z}$ and $x - \tilde{z}$ are in R_+^2 for all $x \in X$. This means that the trade is small enough. Next, we separately bound from below the two terms in the last line of the Taylor expansion (51). Let

$$\begin{aligned} D_A' &= \min_{x \in X, \delta \in [0,1]} (1-\pi(\delta))u'(x^2)\varepsilon/2, \\ D_A'' &= \min_{\substack{x \in X, \delta \in [0,1], \tilde{p} \in [\underline{p}, \overline{p}], \\ y \in \left[x^1, x^1 + \bar{\zeta}\right] \times [x^2 - \tilde{p}\bar{\zeta}, x^2]}} \frac{1}{2}\left[\pi(\delta)u''(y^1) + (1-\pi(\delta))u''(y^2)\tilde{p}^2\right]. \end{aligned}$$

Note that D'_A is positive, D''_A is negative but $D''_A\tilde{\zeta}^2$ is of second order. Then, there exist some $\zeta_A \in \left(0, \bar{\zeta}\right)$ such that, for all $\tilde{\zeta} \leq \zeta_A$,

$$D'_A\tilde{\zeta} + D''_A\tilde{\zeta}^2 > 0$$

and, by construction,

$$U(x_A - \tilde{z}, \delta_A) - U(x_A, \delta_A) \geq D'_A\tilde{\zeta} + D''_A\tilde{\zeta}^2.$$

Analogously, we can find D'_B, D''_B, and ζ_B such that for all $\tilde{\zeta} \leq \zeta_B$ the utility gain for agent B is bounded from below:

$$U(x_B + \tilde{z}, \delta_B) - U(x_A, \delta_B) \geq D'_B\tilde{\zeta} + D''_B\tilde{\zeta}^2 > 0.$$

We are finally ready to define ζ and Δ. Let $\zeta = \min\{\zeta_A, \zeta_B\}$ and

$$\Delta = \min\left\{D'_A\zeta + D''_A\zeta^2, D'_B\zeta + D''_B\zeta^2\right\}.$$

By construction Δ and ζ satisfy the inequalities (9) and (10).

To prove the last part of the lemma, let

$$\lambda = \frac{1}{2}\frac{\pi(1)\min_{x\in X}\left\{u'(x^1)\right\}}{\min\left\{D'_A, D'_B\right\}},$$

which, as stated in the lemma, only depends on X and ε. Using a second-order expansion similar to the one above, the utility gain associated to $z = (\zeta, -p\zeta)$ for an agent with portfolio x_A and any belief $\delta \in [0,1]$, can be bounded below:

$$U(x_A - \tilde{z}, \delta) - U(x_A, \delta) \geq -\pi(1)\min_{x\in X}\left\{u'(x^1)\right\}\zeta + D''_A\zeta^2.$$

Therefore, to ensure that (11) is satisfied, we need to slightly modify the construction above, by choosing ζ so that the following holds

$$\frac{-\pi(1)\min_{x\in X}\left\{u'(x^1)\right\}\zeta + D''_A\zeta^2}{\Delta} > \lambda.$$

The definitions of Δ and λ and a continuity argument show that this inequality holds for some positive $\zeta \leq \min\{\zeta_A, \zeta_B\}$, completing the proof.

3 Proof of Lemma 7

For all $x \in R^2$ and all $\delta \in [0,1]$ define the measure G_t as follows

$$G_t(x,\delta) \equiv \begin{cases} P(\omega : x_t(\omega) = x, \delta_t(\omega) = \delta \mid s_1) - P(\omega : x_t(\omega) = x, \delta_t(\omega) = \delta \mid s_2) & \text{if } \delta > 1/2 \\ 0 & \text{if } \delta \leq 1/2 \end{cases}.$$

We first prove that G_t is a well defined measure and next we prove properties (i)-(iii).

Since P generates a discrete distribution over x and δ for each t, to prove that G_t is a well defined measure we only need to check that

$$P(x_t = x, \delta_t = \delta \mid s_2) \leq P(x_t = x, \delta_t = \delta \mid s_1)$$

so that G_t is non-negative. Take any $\delta > 1/2$. Bayesian rationality implies that a consumer who knows his belief is δ must assign probability δ to s_1:

$$\delta = P(s_1 \mid x_t = x, \delta_t = \delta).$$

Moreover, Bayes' rule implies that

$$\frac{P(s_2 \mid x_t = x, \delta_t = \delta)}{P(s_1 \mid x_t = x, \delta_t = \delta)} = \frac{P(x_t = x, \delta_t = \delta \mid s_2)P(s_2)}{P(x_t = x, \delta_t = \delta \mid s_1)P(s_1)}.$$

Rearranging and using $P(s_1) = P(s_2)$ and $\delta > 1/2$, yields

$$\frac{P(x_t = x, \delta_t = \delta \mid s_2)}{P(x_t = x, \delta_t = \delta \mid s_1)} = \frac{1-\delta}{\delta} < 1,$$

which gives the desired inequality.

Property (i) is immediately satisfied by construction. Property (ii) follows because $P(x_t = x, \delta_t = 1 \mid s_2) = 0$ for all x, given that $\delta_t = 1$ requires that we are at a history which arises with zero probability conditional on s_2. The proof of property (iii) is longer and involves the manipulation of market clearing relations and the use of our symmetry assumption. Using the assumption of uniform market clearing, find an M such that

$$\int_{x_t^2(\omega) \leq m} x_t^2(\omega)\, dP(\omega|s_1) \geq 1 - \varepsilon \text{ for all } m \geq M. \tag{52}$$

Notice that

$$\int_{x_t^2(\omega) \leq m} x_t^1(\omega)\, dP(\omega|s_1) \leq \int x_t^1(\omega)\, dP(\omega|s_1) = 1.$$

Which combined with (52) implies that

$$\int_{x_t^2(\omega)\leq m} \left(x_t^1(\omega) - x_t^2(\omega)\right) dP(\omega|s_1) \leq \varepsilon \text{ for all } m \geq M.$$

Decomposing the integral on the left-hand side gives

$$\int_{\substack{x_t^1>x_t^2 \\ x_t\in[0,m]^2}} \left(x_t^1 - x_t^2\right) dP(\omega|s_1) + \int_{\substack{x_t^2=x_t^1 \\ x_t\in[0,m]^2}} \left(x_t^1 - x_t^2\right) dP(\omega|s_1) + \int_{\substack{x_t^1<x_t^2 \\ x_t\in[0,m]^2}} \left(x_t^1 - x_t^2\right) dP(\omega|s_1)$$
$$+ \int_{\substack{x_t^1>m \\ x_t^2\leq m}} \left(x_t^1 - x_t^2\right) dP(\omega|s_1) \leq \varepsilon. \tag{53}$$

Let us first focus on the first three terms on the left-hand side of this expression. The second term is zero. Using symmetry to replace the third term, the sum of the first three terms can then be rewritten as

$$\int_{\substack{x_t^1>x_t^2 \\ x_t\in[0,m]^2}} \left(x_t^1 - x_t^2\right) dP(\omega|s_1) + \int_{\substack{x_t^1>x_t^2 \\ x_t\in[0,m]^2}} \left(x_t^2 - x_t^1\right) dP(\omega|s_2). \tag{54}$$

These two integrals are equal to the sums of a finite number of non-zero terms, one for each value of x and δ with positive mass. Summing the corresponding terms in each integral, we have three cases: (a) terms with $\delta_t = \delta > 1/2$ and $P(x_t = x, \delta_t = \delta|s_1) > P(x_t = x, \delta_t = \delta|s_2)$ (by Bayes' rule), which can be written as

$$\begin{aligned} \left(x^1 - x^2\right) P(x_t &= x, \delta_t = \delta|s_1) - \left(x^1 - x^2\right) P(x_t = x, \delta_t = \delta|s_2) \\ &= \left(x^1 - x^2\right) G_t(x); \end{aligned}$$

(b) terms with $\delta_t = \delta = 1/2$ and $P(x_t = x, \delta_t = \delta|s_1) = P(x_t = x, \delta_t = \delta|s_2)$ (by Bayes' rule), which are equal to zero,

$$\left(x^1 - x^2\right) P(x_t = x, \delta_t = \delta|s_1) - \left(x^1 - x^2\right) P(x_t = x, \delta_t = \delta|s_2) = 0;$$

(c) terms with $\delta_t = \delta < 1/2$ and $P(x_t = x, \delta_t = \delta|s_1) = P(x_t = x, \delta_t = \delta|s_2)$ (once more, by Bayes' rule), which can be rewritten as follows, exploiting symmetry,

$$\begin{aligned} &\left(x^1 - x^2\right) P\left(x_t = \left(x^1, x^2\right), \delta_t = \delta|s_1\right) - \left(x^1 - x^2\right) P\left(x_t = \left(x^1, x^2\right), \delta_t = \delta|s_2\right) \\ &= \left(x^1 - x^2\right) \left[P\left(x_t = \left(x^2, x^1\right), \delta_t = 1 - \delta|s_2\right) - P\left(x_t = \left(x^2, x^1\right), \delta_t = 1 - \delta|s_1\right)\right] \\ &= \left(x^2 - x^1\right) G_t\left(\left(x^2, x^1\right), 1 - \delta\right). \end{aligned}$$

Combining all these terms, the integral (54) is equal to

$$\int_{\substack{x^1>x^2,\delta>1/2\\x\in[0,m]^2}} \left(x^1-x^2\right) dG_t\left(x,\delta\right) + \int_{\substack{x^1>x^2,\delta<1/2\\x\in[0,m]^2}} \left(x^2-x^1\right) dG_t\left(\left(x^2,x^1\right),1-\delta\right)$$

$$= \int_{\substack{x^1>x^2,\delta>1/2\\x\in[0,m]^2}} \left(x^1-x^2\right) dG_t\left(x,\delta\right) + \int_{\substack{x^2>x^1,\delta>1/2\\x\in[0,m]^2}} \left(x^1-x^2\right) dG_t\left(x,\delta\right) =$$

$$= \int_{x\in[0,m]^2} \left(x^1-x^2\right) dG_t\left(x,\delta\right),$$

where the first equality follows from a change of variables and the second from the fact that G_t is zero for all $\delta \leq 1/2$. We can now go back to the integral on the right-hand side of (53), and notice that the integrand $\left(x_t^1 - x_t^2\right)$ in the fourth term is positive, so replacing the measure P with the measure G_t, which is smaller or equal than P, reduces the value of that term. Therefore the inequality (53) in terms of the measure P, leads to the following inequality in terms of the measure G_t

$$\int_{x^2\leq m} \left(x^1-x^2\right) dG_t \leq \varepsilon,$$

completing the proof of property (iii).

4 Proof of Lemma 3

We start with the usual convergence properties. Since the marginal rates of substitution of informed agents converge, by Proposition 1, and there is at least a mass α of informed agents, using Lemmas 4 and 5 we can find a compact set $X \subset R^2_{++}$ and a time T' such that there is a sufficiently large mass of informed agents with marginal rates of substitution sufficiently close to $\kappa_t\left(s\right)$ (within $\bar{\varepsilon}/2$) and portfolios in X:

$$P\left(\left|\mathcal{M}\left(x_t,\delta_t\right)-\kappa_t\left(s\right)\right| < \bar{\varepsilon}/2, \delta_t = \delta^I\left(s\right), x_t \in X \mid s\right) > \left(3/4\right)\alpha$$

for all $t \geq T'$ and for all s, where $\bar{\varepsilon}$ is defined as in Proposition 2.

Now we provide an important concept. We want to focus on the utility gains that can be achieved by small trades (of norm less than θ), by agents with marginal rates of substitution sufficiently different from each other (by at least $\bar{\varepsilon}/2$). Formally, we proceed as follows. Take any $\theta > 0$. Using Lemma 6, we can then find a lower bound for the utility gain $\Delta > 0$ from trade between two agents with marginal rates of substitution differing by at least $\bar{\varepsilon}/2$ and with portfolios in X, making trades of norm less than θ. It is important to notice that this is the gain achieved if the agents trade but do not change their beliefs. Therefore, it is also important to bound from below the gains that can be achieved by such trades if beliefs are updated in the most pessimistic way. This bound is also given by Lemma 6, which ensures that $-\lambda\Delta$ is a lower bound for the gains of the agent offering z at any possible ex post belief (where λ is a positive scalar independent of θ).

Next we want to restrict attention to agents who are close to their long-run expected utility. Per period utility u_t converges to the long-run value $\hat{v}_t$, by Lemma 2. We can then apply Lemma 4 and find a time period $T \geq T'$ such that, for all $t \geq T$ and for all s:

$$P\left(u_t \geq \hat{v}_t - \alpha\Delta/4, x_t \in X \mid s\right) > 1 - \bar{\varepsilon}/2, \tag{55}$$

and

$$P\left(\left|\mathcal{M}\left(x_t, \delta_t\right) - \kappa_t\left(s\right)\right| < \bar{\varepsilon}/4, u_t \geq \hat{v}_t - \alpha\Delta/8, \delta_t = \delta^I\left(s\right), x_t \in X \mid s\right) > \alpha/2. \tag{56}$$

Equation (55) states that there are enough agents, both informed and uninformed, close to their long-run utility. Equation (55) states that there are enough informed agents close to both their long-run utility and to their long-run marginal rates of substitution.

We are now done with the preliminary steps ensuring proper convergence and can proceed to the body of the argument.

Choose any $t \geq T$. By Proposition 2, two cases are possible: (i) either the informed agents' long-run marginal rates of substitution are far enough from each other, $|\kappa_t(s_1) - \kappa_t(s_2)| \geq \bar{\varepsilon}$; or (ii) they are close to each other, $|\kappa_t(s_1) - \kappa_t(s_2)| < \bar{\varepsilon}$, but there is a large enough mass of uninformed agents with marginal rate of substitution far from that of the informed agents, $P\left(\left|\mathcal{M}\left(x_t, \delta_t\right) - \kappa_t(s)\right| \geq 2\bar{\varepsilon} \mid s\right) \geq \bar{\varepsilon}$ for all s.

In the next two steps, we construct the desired trade z for each of these two cases, and then complete the argument in step 3.

Step 1. Consider the first case, in which $|\kappa_t(s_1) - \kappa_t(s_2)| \geq \bar{\varepsilon}$. In this case, an uninformed agent can exploit the difference between the informed agents' marginal rates of substitution in states s_1 and s_2, making an offer at an intermediate price. This offer will be accepted with higher probability in the state in which the informed agents' marginal rate of substitution is higher. In particular, suppose

$$\kappa_t\left(s_2\right) + \bar{\varepsilon} \leq \kappa_t\left(s_1\right)$$

(the opposite case is treated symmetrically). Lemma 6 and the definition of the utility gain Δ imply that there is a trade $z = (\zeta, -p\zeta)$, with price $p = (\kappa_t\left(s_1\right) + \kappa_t\left(s_2\right))/2$ and size $\|z\| < \theta$, that satisfies the following inequalities:

$$U(x_t + z, \delta_t) \quad \geq \quad u_t + \Delta \text{ if } \mathcal{M}\left(x_t, \delta_t\right) > \kappa_t\left(s_1\right) - \bar{\varepsilon}/4 \text{ and } x_t \in X, \tag{57}$$

$$U(x_t - z, \delta_t) \quad \geq \quad u_t + \Delta \text{ if } \mathcal{M}\left(x_t, \delta_t\right) < \kappa_t\left(s_2\right) + \bar{\varepsilon}/4 \text{ and } x_t \in X. \tag{58}$$

Equation (57) states that all (informed and uninformed) agents with marginal rate of substitution above $(\kappa_t\left(s_1\right) - \bar{\varepsilon}/4)$ will receive a utility gain Δ from the trade z, in terms of current utility. Equation (58) states that all (informed and uninformed) agents with marginal rate of substitution below $(\kappa_t(s_1) + \bar{\varepsilon}/4)$ will receive a utility gain Δ from the trade $-z$, in terms of current utility.

Combining conditions (56) and (57) shows that in state s_1 there is at least $\alpha/2$ informed agents with after-trade utility above the long-run utility, $U(x_t + z, \delta_t) > \hat{v}_t$. Since all these agents would accept the trade z, this implies that the probability of acceptance of the trade is $\chi_t(z|s_1) > \alpha/2$.

Next, we want to show that the trade z is accepted with sufficiently low probability conditional on s_2. In particular, we want to show that $\chi_t(z|s_2) < \alpha/4$. The key step here is to make sure that the trade is rejected not only by informed but also by uninformed agents. The argument is that if this trade were to bo accepted by uninformed agents, then informed agents should be offering z and gaining in utility. Formally, proceeding by contradiction, suppose that the probability of z being accepted in state s_2 is large: $\chi_t(z|s_2) \geq \alpha/4$. Condition (56) implies that there is a positive mass of informed agents with $\mathcal{M}(x_t, \delta_t) < \kappa_t(s_2) + \bar{\varepsilon}/2$, $x_t \in X$, and close enough to the long-run utility $u_t \geq \hat{v}_t - \alpha\Delta/8$. By(58), these agents would be strictly better off making the offer z and consuming $x_t - z$ if the offer is accepted and consuming x_t if it is rejected, since

$$(1 - \chi_t(z_t|s_2))\, U(x_t, \delta_t) + \chi_t(z_t|s_2)\, U(x_t - z, \delta_t) > u_t + \alpha\Delta/4 > \hat{v}_t.$$

Since this strategy dominates the equilibrium payoff, this is a contradiction, proving that $\chi_t(z|s_2) < \alpha/4$.

Step 2. Consider the second case, in which the long-run marginal rates of substitution of the informed agents are close to each other and there is a large enough mass of uninformed agents with marginal rate of substitution far from that of the informed agents.

The argument is as follows: with positive probability we can reach a point where it is possible to separate the marginal rates of substitution of a group of uninformed agents from the marginal rates of substitution of a group of informed agents. This means that the uninformed agents in the first group can make an offer z to the informed agents in the second group and they will accept the offer in *both* states s_1 and s_2. If the probabilities of acceptance $\chi_t(z|s_1)$ and $\chi_t(z|s_2)$ are sufficiently close to each other, this would be a profitable deviation for the uninformed, since their ex post beliefs after the offer is accepted would be close to their ex ante beliefs. In other words, in contrast to the previous case they would gain utility but not learn from the trade. It follows that the probabilities $\chi_t(z|s_1)$ and $\chi_t(z|s_2)$ must be sufficiently different in the two states, which leads to either (4) or to (5).

To formalize this argument, consider the expected utility of an uninformed agent with portfolio x_t and belief δ_t, who offers a trade z and stops trading from then on:

$$u_t + \delta_t \chi_t(z|s_1)\,(U(x_t - z, 1) - U(x_t, 1)) + (1 - \delta_t)\, \chi_t(z|s_2)\,(U(x_t - z, 0) - U(x_t, 0)),$$

where u_t is the expected utility if the offer is rejected and the following two terms are the expected gains if the offer is accepted, respectively, in states s_1 and s_2. This expected utility can be rewritten as

$$u_t + \chi_t (z|s_1) (U (x_t - z, \delta_t) - U (x_t, \delta_t)) + (1 - \delta_t) (\chi_t (z|s_2) - \chi_t (z|s_1)) (U (x_t - z, 0) - U (x_t, 0)) , \tag{59}$$

using the fact that $U (x_t, \delta_t) = \delta_t U (x_t, 1) + (1 - \delta_t) U (x_t, 0)$ (by the definition of U). To interpret (59) notice that, if the probability of acceptance was independent of the signal, $\chi_t (z|s_1) = \chi_t (z|s_2)$, then the expected gain from making offer z would be equal to the second term: $\chi_t (z|s_1) (U (x_t - z, \delta_t) - U (x_t, \delta_t))$. The third term takes into account that the probability of acceptance may be different in two states, i.e., $\chi_t (z|s_2) - \chi_t (z|s_1)$ may be different from zero. An alternative way of rearranging the same expression yields:

$$u_t + \chi_t (z|s_2) (U (x_t - z, \delta_t) - U (x_t, \delta_t)) + (1 - \delta_t) (\chi_t (z|s_1) - \chi_t (z|s_2)) (U (x_t - z, 1) - U (x, 1)) . \tag{60}$$

In the rest of the argument, we will use both (59) and (60).

Suppose that there exists a trade z and a period t which satisfy the following properties: (a) the probability that z is accepted in state 1 is large enough,

$$\chi_t (z|s_1) > \alpha/4,$$

and (b) there is a positive mass of uninformed agents with portfolios and beliefs that satisfy

$$u_t \geq \hat{v}_t - (\alpha/4) \Delta, \tag{61}$$

$$U (x_t - z, \delta_t) - U (x_t, \delta_t) \geq \Delta, \tag{62}$$

$$U (x_t - z, \delta) - U (x_t, \delta) \geq -\lambda\Delta \text{ for all } \delta \in [0, 1] , \tag{63}$$

for some $\Delta > 0$ and $\lambda > 0$. In words, the uninformed agents are sufficiently close to their long-run utility, their gains from trade at *fixed* beliefs have a positive lower bound Δ, and their gains from trade at *arbitrary* beliefs have a lower bound $-\lambda\Delta$.

Now we distinguish two cases. Suppose first that $\chi_t (z|s_2) \geq \chi_t (z|s_1)$. Then, for the uninformed agents who satisfy (61)-(63) the expected utility (59) is greater or equal than

$$\hat{v}_t - (\alpha/4) \Delta + \chi_t (z|s_1) \Delta - (\chi_t (z|s_2) - \chi_t (z|s_1)) \lambda\Delta.$$

From individual optimality, this expression cannot be larger than $\hat{v}_t$, since $\hat{v}_t$ is the maximum expected utility for a proposer in period t. We then obtain the following restriction on the acceptance probabilities $\chi_t(z|s_1)$ and $\chi_t(z|s_2)$:

$$\chi_t(z|s_1)(1+\lambda)\Delta \le \alpha\Delta/4 + \chi_t(z|s_2)\lambda\Delta.$$

Since $\chi_t(z|s_1) > \alpha/2$ and $\chi_t(z|s_1) \ge \chi_t(z|s_2)$ it follows that $\alpha/4 < (1/2)\chi_t(z|s_2)$ and we obtain

$$\chi_t(z|s_1)(1+\lambda) \le \chi_t(z|s_2)(1/2+\lambda),$$

which is equivalent to

$$\chi_t(z|s_1) \ge \frac{1+\lambda}{1/2+\lambda}\chi_t(z|s_2). \tag{64}$$

This shows that the probability of acceptance in state s_1 is larger than the probability of acceptance in state s_2 by a factor $(1+\lambda)/(1/2+\lambda)$ greater than 1.

Consider next the case $\chi_t(z|s_2) < \chi_t(z|s_1)$. Then, for the uninformed agents who satisfy (61)-(63) the expected utility (60) is greater or equal than

$$\hat{v}_t - \alpha\Delta/4 + \chi_t(z|s_2)\Delta - (\chi_t(z|s_1) - \chi_t(z|s_2))\lambda\Delta.$$

An argument similar to the one above shows that optimality requires

$$\chi_t(z|s_2) \ge \frac{1+\lambda}{1/2+\lambda}\chi_t(z|s_1).$$

Some algebra shows that this inequality and $\chi_t(z|s_1) > \alpha/2$ imply

$$1-\chi_t(z|s_1) > 1-\frac{\alpha}{2}\frac{1/2+\lambda}{1+\lambda}, \tag{65}$$

$$1-\chi_t(z|s_1) > \frac{(1-\alpha/2)(1/2+\lambda)}{(1-\alpha/2)(1/2+\lambda)-\alpha/4}(1-\chi_t(z|s_2)), \tag{66}$$

giving us a positive lower bound for the probability of rejection $1-\chi_t(z|s_1)$ and showing that $1-\chi_t(z|s_1)$ exceeds $1-\chi_t(z|s_2)$ by a factor greater than 1.

To complete this step, we show that there exists a trade z and a period t which satisfy properties (a) and (b).

Notice that $P(|\mathcal{M}(x_t,\delta_t) - \kappa_t(s_1)| \ge 2\bar{\varepsilon} \mid s_1) \ge \bar{\varepsilon}$ requires that either $P(\mathcal{M}(x_t,\delta_t) \le \kappa_t(s_1) - 2\bar{\varepsilon} \mid s_1) \ge \bar{\varepsilon}/2$ holds or $P(\mathcal{M}(x_t,\delta_t) \ge \kappa_t(s_1) + 2\bar{\varepsilon} \mid s_1) \ge \bar{\varepsilon}/2$. We concentrate on the first case, as the second is treated symmetrically. Set the trading price at $p = \min\{\kappa(s_1), \kappa(s_2)\} - \bar{\varepsilon}/2$. Lemma 6 implies that there are positive scalars Δ and λ and a trade $z = (\zeta, -p\zeta)$ with $\|z\| < \theta$ that satisfies the following inequalities:

$$U(x_t - z, \delta_t) \ge u_t + \Delta,\ U(x_t - z, \delta) \ge u_t - \lambda\Delta \text{ for all } \delta \in [0,1], \quad \text{if } \mathcal{M}(x_t,\delta_t) < p - \bar{\varepsilon}/4 \text{ and } x_t \in X, \tag{67}$$

and

$$U(x_t + z, \delta_t) \geq u_t + \Delta \text{ if } \mathcal{M}(x_t, \delta_t) > p + \bar{\varepsilon}/4 \text{ and } x_t \in X. \tag{68}$$

Since $|\mathcal{M}(x_t, \delta_t) - \kappa_t(s_1)| < \bar{\varepsilon}/4$ implies $\mathcal{M}(x_t, \delta_t) > \kappa_t(s_1) - \bar{\varepsilon}/4$ and $\kappa_t(s_1) - \bar{\varepsilon}/4$ is larger than $p + \bar{\varepsilon}/4$ by construction, conditions (56) and (68) guarantee that there is a positive mass of informed agents who accept z, ensuring that $\chi_t(z|s_1) > \alpha/2$, showing that z satisfies property (a).

Next, we want to prove that there is a positive mass of uninformed agents who gain from making offer z. To do so, notice that $|\kappa_t(s_1) - \kappa_t(s_2)| < \bar{\varepsilon}$ implies

$$p - \bar{\varepsilon}/4 = \min\{\kappa_t(s_1), \kappa_t(s_2)\} - (3/4)\bar{\varepsilon} \geq \kappa_t(s_1) - (7/4)\bar{\varepsilon} > \kappa_t(s_1) - 2\bar{\varepsilon},$$

which implies

$$P(\mathcal{M}(x_t, \delta_t) < p - \bar{\varepsilon}/4 \mid s_1) \geq P(\mathcal{M}(x_t, \delta_t) \leq \kappa_t(s_1) - 2\bar{\varepsilon} \mid s_1) \geq \bar{\varepsilon}/2.$$

This, using Lemma 4 and condition (55), implies

$$P(\mathcal{M}(x_t, \delta_t) < p - \bar{\varepsilon}/4, u_t \geq \hat{v}_t - \alpha\Delta/4, x_t \in X \mid s_1) > 0,$$

which, combined with (67), shows that the trade z satisfies property (b).

Step 3. Here we put together the bounds established above and define the scalars β and ρ in the lemma's statement. Consider the case treated in Step 1. In this case, we can find a trade z such that the probability of acceptance conditional on each signal satisfies: $\chi_t(z|s_1) > \alpha/2$ and $\chi_t(z|s_2) < \alpha/4$. Therefore, in this case condition (4) is true as long as β and ρ satisfy

$$\beta \leq \alpha/2 \text{ and } \rho \leq 2.$$

Consider the case treated in Step 2. In this case, we can find a trade z such that either $\chi_t(z|s_1) > \alpha/2$ and (64) hold or (65) and (66) hold. This implies that either condition (4) or condition (5) hold, as long as β and ρ satisfy

$$\beta \leq 1 - \frac{\alpha}{2}\frac{1/2 + \lambda}{1 + \lambda}, \rho \leq \frac{1 + \lambda}{1/2 + \lambda}, \rho \leq \frac{(1 - \alpha/2)(1/2 + \lambda)}{(1 - \alpha/2)(1/2 + \lambda) - \alpha/4}.$$

Setting

$$\begin{aligned}
\beta &= \min\left\{\alpha/2, 1 - \frac{\alpha}{2}\frac{1/2 + \lambda}{1 + \lambda}\right\} > 0, \\
\rho &= \min\left\{2, \frac{1 + \lambda}{1/2 + \lambda}, \frac{(1 - \alpha/2)(1/2 + \lambda)}{(1 - \alpha/2)(1/2 + \lambda) - \alpha/4}\right\} > 1,
\end{aligned}$$

ensures that all the conditions above are satisfied, completing the proof.

5 Proof of Lemma 8

Since $\delta_t(\omega)$ are equilibrium beliefs, Bayesian rationality requires $P(s_1 \mid \delta_t < \varepsilon/(1+\varepsilon)) < \varepsilon/(1+\varepsilon)$ for all $\varepsilon > 0$. The latter condition implies $P(s_2 \mid \delta_t < \varepsilon/(1+\varepsilon)) > 1 - \varepsilon/(1+\varepsilon)$ and thus

$$\frac{P(s_1 \mid \delta_t < \varepsilon/(1+\varepsilon))}{P(s_2 \mid \delta_t < \varepsilon/(1+\varepsilon))} < \varepsilon,$$

for all $\varepsilon > 0$. Bayes' rule implies that

$$\frac{P(s_1 \mid \delta_t < \varepsilon/(1+\varepsilon))}{P(s_2 \mid \delta_t < \varepsilon/(1+\varepsilon))} = \frac{P(\delta_t < \varepsilon/(1+\varepsilon) \mid s_1) P(s_1)}{P(\delta_t < \varepsilon/(1+\varepsilon) \mid s_2) P(s_2)}.$$

Combining the last two equations and using $P(s_1) = P(s_2) = 1/2$ yields

$$P(\delta_t < \varepsilon/(1+\varepsilon) \mid s_1) < \varepsilon P(\delta_t < \varepsilon/(1+\varepsilon) \mid s_2) \leq \varepsilon,$$

which gives the desired inequality.

6 Proof of Lemma 9

Let us begin from the first part of the lemma. Suppose, by contradiction, that

$$|\kappa_{t+J}(s_1) - \kappa_t(s_1)| > \varepsilon$$

for some $\varepsilon > 0$ for infinitely many periods. Then, at some date t, an informed agent with marginal rate of substitution close to $\kappa_t(s)$ can find a profitable deviation by holding on to his portfolio x_t for J periods and then trade with other informed agents at $t+J$. Let us formalize this argument. Suppose, without loss of generality, that

$$\kappa_{t+J}(s_1) > \kappa_t(s_1) + \varepsilon$$

for infinitely many periods (the other case is treated in a symmetric way). Next, using our usual steps and Proposition 1, it is possible to find a compact set X, a time T, and a utility gain $\Delta > 0$ such that the following two properties are satisfied: (i) in all periods $t \geq T$ there is at least a measure $\alpha/2$ of informed agents with marginal rate of substitution sufficiently close to $\kappa_t(s)$, utility close to its long-run level, and portfolio x_t in X, that is,

$$P\left(|\mathcal{M}(x_t, \delta_t) - \kappa_t(s)| < \varepsilon/3, u_t \geq \hat{v}_t - \gamma^J \alpha\Delta/2, x_t \in X, \delta_t = 1 \mid s\right) > \alpha/2, \tag{69}$$

and (ii) in all periods $t \geq T$ in which $\kappa_{t+J}(s) > \kappa_t(s) + \varepsilon$ there is a trade z such that

$$U(x - z, 1) > U(x, 1) + \Delta \text{ if } \mathcal{M}(x, 1) < \kappa_t(s) + \varepsilon/3 \text{ and } x \in X, \tag{70}$$

and

$$U(x + z, 1) > U(x, 1) + \Delta \text{ if } \mathcal{M}(x, 1) > \kappa_{t+J}(s) - \varepsilon/3 \text{ and } x \in X. \tag{71}$$

Pick a time $t \geq T$ in which $\kappa_{t+J}(s) > \kappa_t(s) + \varepsilon$ and consider the following deviation. Whenever an informed agent reaches time t and his portfolio x_t satisfies $\mathcal{M}(x_t, 1) < \kappa_t(s) + \varepsilon/3$ and $x_t \in X$, he stops trading for J periods and then makes an offer z that satisfies (70) and (71). If the offer is rejected he stops trading from then on. The probability that this offer is accepted at time $t + J$ must satisfy $\chi_{t+J}(z|s_1) > \alpha/2$, because of conditions (69) and (71). Therefore, the expected utility from this strategy, from the point of view of time t is

$$u_t + \gamma^J \chi_{t+J}(z|s_1)(U(x_t - z, 1) - u_t) > u_t + \gamma^J \alpha\Delta/2 \geq \hat{v}_t,$$

so this strategy is a profitable deviation and we have a contradiction.

The second part of the lemma follows from the first part, using Proposition 1 and the triangle inequality.

7 Examples

In this section, we present two examples in which the equilibrium can be analyzed analytically. The first objective of these examples is to show existence of equilibrium with symmetry across states and uniform market clearing in some special cases. The second objective is to study how information acquisition takes place in equilibrium. The third objective is to study how the equilibrium is affected by changing the parameter γ, which controls the probability that the game ends. In particular, we are interested in what happens when γ goes to 1.

Higher values of γ correspond to economies in which agents have the chance to do more rounds of trading before the game ends. We can interpret the limit $\gamma \to 1$ as a frictionless trading limit, in which agents have the chance to make infinite rounds of trading before the game ends. In a full information economy in which agents can trade forever (i.e., with no discounting), Gale (1986a) shows that bilateral bargaining yields a Walrasian outcome, in which agents making the first offer do not have any monopoly power, due to the fact that their partners have unlimited chances to make further trades in the future. We can then ask whether a similar result applies in our model with asymmetric information when γ goes to 1. Our first example shows that, in general, the result does not extend. In particular, in that example, agents remain uninformed even as γ goes to 1. This bounds the gains from trade that can be reaped by responders who refuse to trade in the first round. Therefore agents who are selected as proposers in the first period keep some monopoly power. In our second example, on the other hand, uninformed agents have the opportunity to acquire perfect information in equilibrium and payoffs converge to those of perfect competition.

In all the examples, there are two types, with initial portfolios $x_{1,0} = (\omega, 1-\omega)$ and $x_{2,0} = (1-\omega, \omega)$, for some $\omega \in (1/2, 1)$. Let $\phi(s_1) = \varphi > 1/2$ and recall that, by symmetry, $\phi(s_2) = 1-\varphi$. Informed agents with endowment $x_{1,0}$ are called "rich informed agents" in state s_1 and "poor informed agents" in state s_2, as their endowment's present value is greater in the first case. The opposite labels apply to informed agents with $x_{2,0}$.

For analytical tractability, we modify the setup of our model in the first round of trading and make the following assumption. In period $t = 1$, the matching process is such that agents meet other agents with complementary endowments with probability 1, that is, type 1 agents only get matched with type 2 agents. In all following periods, agents meet randomly as in the setup of Section 1. All our results from previous sections hold in this modified environment, as they only rely on the long-run properties of the game. The purpose of this assumption is to construct equilibria in which almost all trades take place in the first round.

We consider two examples. In the first example, uninformed agents do not learn anything about the state s and keep their initial beliefs at $\delta = 1/2$. In the second example, all uninformed agents learn the state s exactly in the first round of trading. For ease of exposition, we present the main results for the two examples in Sections 7.1 and 7.2 and present some more technical derivations behind the examples in Sections 7.3 and 7.4.

7.1 Example 1: an equilibrium with no learning

For this example, we introduce an additional modification to our baseline model, assuming that in period $t = 1$ all informed agents get to be proposers with probability 1. In particular, in $t = 1$ informed agents are only matched with uninformed agents and the informed agent is always selected as the proposer.[10] If two uninformed agents meet at $t = 1$, each is selected as the proposer with probability 1/2. From period $t = 2$ onwards, the matching and the selection of the proposer are as in the baseline model. That is, each agent has the same probability of meeting an informed or uninformed partner and each agent has probability 1/2 of being selected as the proposer. As pointed out above, the changes made in period $t = 1$ do not affect the long-run properties of the game and the general results of the previous sections still hold.

For a given scalar $\eta \in (0, 1)$, to be defined below, our aim is to construct and equilibrium in which strategies and beliefs satisfy:

S1. In $t = 1$, all proposers of type i offer $z_{i,E} = (\eta, \eta) - x_{-i,0}$. All responders accept.

S2. In $t = 2, 3...$, all proposers offer zero trade.

[10] This requires assuming $\alpha < 1/2$.

S3. In $t = 2, 3...$, all uninformed responders reject any offer z that satisfies

$$\min\left\{\varphi z^1 + (1-\varphi) z^2, (1-\varphi) z^1 + \varphi z^2\right\} < 0,$$

all informed responders reject any offer z that satisfies

$$\begin{aligned} \varphi z^1 + (1-\varphi) z^2 < 0 \qquad & \text{if } s = s_1, \\ \text{or } (1-\varphi) z^1 + \varphi z^2 < 0 \qquad & \text{if } s = s_2. \end{aligned}$$

B1. Uninformed responders keep their beliefs unchanged after offer $z_{i,E}$ in period $t = 1$ and after offer 0 in period $t \geq 2$.

B2. In $t = 1$, after an offer $z \neq z_{i,E}$, uninformed responders adjust their belief to $\delta = 1$ if they are of type 1 and to $\delta = 0$ if they are of type 2.

B3. In $t = 2, 3, ...$, after an offer $z \neq 0$, uninformed responders adjust their belief to $\delta = 1$ if $\varphi z^1 + (1-\varphi) z^2 < (1-\varphi) z^1 + \varphi z^2$ and to $\delta = 0$ if $\varphi z^1 + (1-\varphi) z^2 > (1-\varphi) z^1 + \varphi z^2$.

Notice that in equilibrium, all agents reach endowments on the 45 degree line after one round of trading and remain there from then on. S1-S3 and B1-B3 describe strategies and beliefs in equilibrium and along a subset of off-the-equilibrium-path histories. This is sufficient to show that we have an equilibrium, since we can prove that if other agents' strategies satisfy S1-S3 the payoff from any deviating strategy is bounded above by the equilibrium payoff. An important element of our construction is that following off-the-equilibrium-path offers at $t \geq 2$, uninformed agents hold "pessimistic" beliefs, meaning that they expect the state s to be the one for which the present value of the offer received is smaller. This, together with the fact that all agents are on the 45 degree line starting at date 2, implies that deviating agents have limited opportunities to trade after period 1.

The argument to prove that an equilibrium with these properties exists is in two steps. First, we show that zero trade from period $t = 2$ on is a continuation equilibrium. Second, we go back to period $t = 1$ and show that making and accepting the offers $z_{i,E}$ is optimal at $t = 1$.

To show that no trade is an equilibrium after $t = 2$, we use property S3. Let $V(x, \delta)$ denote the continuation utility of an agent with endowment x and belief $\delta \in [0, 1]$ at any time $t \geq 2$.[11] This value function is independent of t since the environment is stationary after $t = 2$. Since other agents' strategies satisfy S3, the endowment process of a deviating agent who starts at (x, δ) satisfies the following property: if the state is s_1, any endowment $\tilde{x}$ reached with positive

[11] The value function is defined at the beginning of the period, before knowing whether the game ends or there is another round of trading.

probability at future dates must satisfy

$$\varphi\tilde{x}^1 + (1-\varphi)\,\tilde{x}^2 \leq \varphi x^1 + (1-\varphi)\,x^2.$$

This property holds because in s_1 neither informed nor uninformed agents will accept trades that increase the expected value of the proposer's endowment computed using the probabilities φ and $1-\varphi$. A similar property holds in s_2, reversing the roles of the probabilities φ and $1-\varphi$. This properties, together with concavity of the utility function, imply that the continuation utility $V(x,\delta)$ is bounded as follows:

$$V(x,\delta) \leq \delta u\left(\varphi x^1 + (1-\varphi)\,x^2\right) + (1-\delta)\,u\left((1-\varphi)\,x^1 + \varphi x^2\right). \tag{72}$$

In the continuation equilibrium, all agents start from a perfectly diversified endowment with $x^1 = x^2$. So an agent can achieve the upper bound in (72) by not trading. This rules out any deviation by proposers on the equilibrium path. A similar argument shows that the off-the-equilibrium-path responses in S3 are optimal.[12] So we have an equilibrium for $t \geq 2$. The argument for no trade in periods $t \geq 2$ is closely related to the no-trade theorem of Milgrom and Stokey (1982).

Turning to period $t = 1$, consider a rich informed proposer with endowment $x_{1,0}$ in state s_1. If he deviates and offers $z \neq z_{1,E}$ the uninformed responder's belief goes to $\delta = 0$. Then the offer will only be accepted if $V(x_{1,0} + z, 0) \geq V(x_{1,0}, 0)$ and the payoff of the proposer, if the offer is accepted, would be $V(x_{1,0} - z, 1)$. In Section 7.3, we derive an upper bound on this payoff. We can then find parametric examples and choose η so that offering $z_{1,E}$ yields a higher payoff. The reason why this is possible is that, if the proposer offers $z_{1,E}$, the uninformed responder's belief remains at $\delta = 1/2$, so the informed proposer is able to trade at better terms in period 1.

Consider next a poor informed proposer with endowment $x_{2,0}$ in state s_1. If he deviates, the uninformed responder adjusts his belief to $\delta = 1$. Since the poor informed proposer also holds belief $\delta = 1$, we can show that the best deviation by a poor informed is to make an offer that reaches the 45 degree line for both. However, the responder's outside option is higher at $\delta = 1$ than at $\delta = 1/2$, because he holds a larger endowment of asset 1. This makes the participation constraint of the responder tighter than at the equilibrium offer $z_{2,E}$ and allows us to construct parametric examples in which the poor informed proposer prefers not to deviate.

Having shown that both the rich informed proposer and the poor informed proposer prefer not to deviate, we can then show that an uninformed proposer prefers not to deviate either, using the argument that the payoff of an uninformed agent under a deviation is weakly dominated by the average of the payoffs of an informed agent with the same endowment. In Section 7.3, we derive sufficient conditions that rule out deviations in period $t = 1$ and show

[12] See Proposition 6 in Section 7.3.

how to construct parametric examples that satisfy these conditions. The following proposition contains such an example.

Proposition 4 *Suppose the utility function is $u(c) = c^{1-\sigma}/(1-\sigma)$ and the parameters $(\sigma, \omega, \varphi)$ are in a neighborhood of $\left(4, 9/10, 9^4/(9^4+1)\right)$. There is an $\eta \in (0,1)$ and a cutoff $\bar{\alpha} > 0$ for the fraction of informed agents in the game,such that S1-S3 and B1-B3 form an equilibrium if $0 \leq \alpha < \bar{\alpha}$.*

An important ingredient in the construction of the example in the proposition is to assume that the fraction of informed agents α is sufficiently small. This puts a bound on the utility from trading in periods $t \geq 2$, because it implies that an agent only gets a chance to trade with informed agents with a small probability. In particular, property S3 means that an agent who starts at x at $t = 2$ and only trades with uninformed agents before the end of the game can only reach endowments $\tilde{x}$ that satisfy both

$$\varphi \tilde{x}^1 + (1-\varphi)\tilde{x}^2 \leq \varphi x^1 + (1-\varphi) x^2,$$

and

$$(1-\varphi)\tilde{x}^1 + \varphi \tilde{x}^2 \leq (1-\varphi) x^1 + \varphi x^2.$$

This restriction is crucial in constructing upper bounds on the continuation utility of deviating agents at date $t = 1$. The intuition is that the trades $z_{i,E}$ at date $t = 1$ are proposed and accepted in equilibrium because the outside option is to trade with fully diversified, mostly uninformed agents in periods $t = 2, 3, ...$ In period $t = 1$ there are large gains from trade coming from the fact that agents are not diversified, but all these gains from trade are exhausted in the first round of trading. From then on, the presence of asymmetric information limits the gains from trade for a deviating agent who is still undiversified at the end of $t = 1$.

Somewhat surprisingly, under the assumptions in Proposition 4 an equilibrium can be constructed for any value of γ. This is because what bounds the continuation utility in period $t = 2$ is the small probability of trading with informed agents in future periods. So for any value of γ, we can choose the cutoff $\bar{\alpha}$ sufficiently small to make the probability of trading with informed agents approach zero and sustain our equilibrium. Larger values of γ correspond to economies in which agents trade more frequently before the game ends. Then the observation above can be interpreted as follows. There can be economies close to the frictionless limit—i.e., with γ close to 1—in which no information is revealed in equilibrium. This happens because more frequent trade implies that the diversification motive for trade is exhausted more quickly. However, once the diversification motive is exhausted, the no-trade theorem implies that no further trade occurs and so no further information is revealed.

When the mass of informed agents α is zero, the equilibrium holds for all γ so we can take $\gamma \to 1$. We then have an economy in which the limit equilibrium allocation is an allocation in which uninformed agents with the *same* endowments get different consumption levels depending on whether they were selected as proposers or responders in the very first period of the game. So we have an example of an economy in which the presence of more frequent rounds of trading does not lead, in the limit, to a perfectly competitive outcome, unlike in the economies with perfect information analyzed by Gale (1986, a). This shows that the presence of asymmetric information can have powerful effects in decentralized economies. Again, the underlying idea is that the no-trade theorem limits the agents' ability to trade in the long run, and this induces agents to accept trades in the early stages of the game, when diversification motives are stronger. This undermines the ability of future rounds of trading to act as a check on the monopoly power of proposers in the early stages of the game.

7.2 Example 2: an equilibrium with learning

We now turn to an example in which uninformed agents acquire perfect information in the first round of trading. As in Example 1, we assume that in the first round of trading each agent is matched with an agent with complementary endowments. Moreover, we assume that almost all agents are informed, so there is only a zero mass of uninformed agents. We also assume that preferences display constant absolute risk aversion:

$$u(c) = -e^{-\rho a}.$$

This assumption allows us to characterize analytically the value function $V(x, \delta)$ for $\delta = 0$ and $\delta = 1$. Notice that CARA preferences do not satisfy the property $\lim_{c \to 0} u(c) = -\infty$, which was assumed in Section 2 (Assumption 2). However, the only purpose of that property was to ensure that endowments stay in a compact set with probability close to 1 in equilibrium. Here, we can check directly that endowments remain in a compact set in equilibrium. So all our general results still apply.

The analysis of this example proceeds in two steps. First, we characterize the equilibrium focusing on the behavior of informed agents—which can be done, given that uninformed agents are in zero mass. Second, we look at the uninformed agent's problem at date $t = 1$ and derive conditions that ensure that his optimal strategy is to experiment, making an offer that perfectly reveals the state s.

For the first step, we need to derive four equilibrium offers $z_{i,E}(s)$ which depend on the proposer type i and on the state s. In equilibrium, all proposers of type i make offer $z_{i,E}(s)$ in state s, all responders accept, and both proposer and responder reach a point on the 45 degree line. The offers $z_{i,E}(s)$ are found maximizing $V(x_{i,0} - z, \delta)$ subject to

$$V(x_{-i,0} + z, \delta) \geq V(x_{-i,0}, \delta),$$

with $\delta = 1$ if s_1 and $\delta = 0$ if s_2. Since the two agents share the same beliefs, it is not difficult to show that the solution to this problem yields an allocation on the 45 degree line for both agents and that they stop trading from period $t = 2$ onwards. For this argument it is sufficient to use the upper bound (72) which was used for our Example 1 and also holds here. We then obtain the following proposition.[13]

Proposition 5 *If all agents are informed, there is an equilibrium in which all agents reach the 45 degree line in the first round of trading and stop trading from then on.*

Before turning to our second step, however, we need to derive explicitly the form of the V function and the offers $z_{i,E}(s)$. These steps are more technical and are presented in Section 7.4. The assumption of CARA utility helps greatly in these derivations, as it allows us to show that the value function takes the form $V(x) = -\exp\left\{-\rho x^1\right\} f\left(x^2 - x^1\right)$ for some decreasing function f and that the function f can be obtained as the solution of an appropriate functional equation.

We can then turn to our second step and consider uninformed agents in period $t = 1$. The case of uninformed responders is easy. Since they meet informed proposers with probability 1 and these proposers make different offers in the two states, they acquire perfect information on s and respond like informed agents, accepting the offer and reaching the 45 degree line.

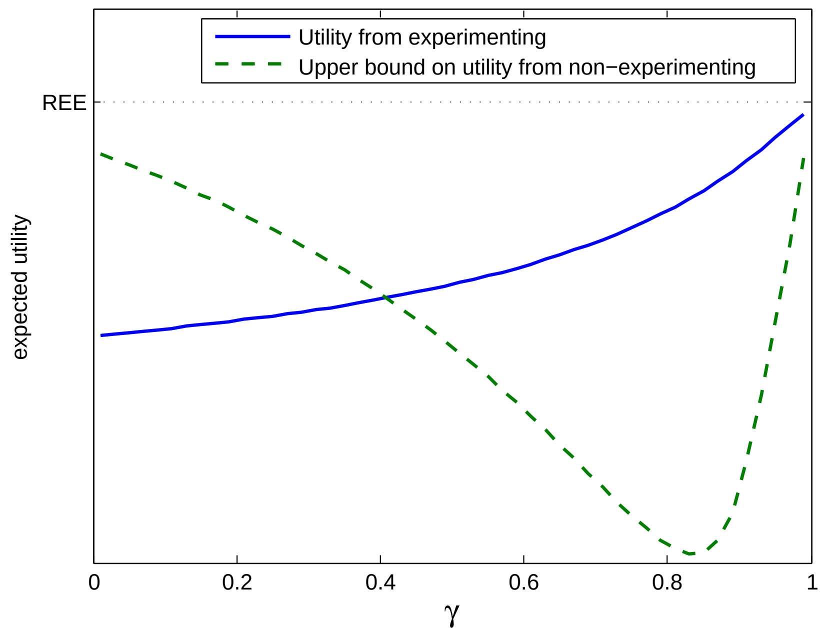
The case of uninformed proposers is harder. We want to show that an uninformed proposer with endowment i makes offer $z_{i,E}(s_1)$ if $i = 1$ and offer $z_{i,E}(s_2)$ if $i = 2$, where $z_{i,E}(s)$ are the offers derived above for informed agents. In other words, uninformed proposers mimic the behavior of *rich* informed proposers. By making these offers, uninformed proposers get to learn exactly the state s, because their offer is accepted with probability 1 in one state and rejected with probability 1 in the other. While this implies that they become informed from period $t = 2$ onwards, it also implies that with probability $1/2$ they do not reach the 45 degree line in the first round of trading. Our characterization of the V function in Section 7.4 allows us to compute their payoff in this case and to characterize their trading in periods $t = 2, 3, \ldots.$ In particular, the optimal behavior of uninformed agents who fail to trade in period $t = 1$ is to make an infinite sequence of trades in all periods $t \geq 2$ in which they are selected as proposers. The difference $x^1 - x^2$ is reduced by a factor of $1/2$ each time they get to trade, so that they asymptotically reach the 45 degree line.

Let us show that the offers described above are optimal for the uninformed proposer. We focus on type $i = 1$ as the case of $i = 2$ is symmetric. To prove that offering $z_{1,E}(s_1)$ is optimal at time $t = 1$ we need to check that:

1. offer $z_{1,E}(s_1)$ is rejected by informed agents in s_2;
2. offer $z_{1,E}(s_1)$ dominates any other offer accepted by informed agents only in s_1;

[13] A detailed proof is in Section 7.4.

Figure 1: Example 2



3. offer $z_{1,E}(s_1)$ dominates any offer accepted by informed agents only in state s_2;

4. offer $z_{1,E}(s_1)$ dominates any offer accepted by informed agents in both states.

Conditions 1 to 3 are proved in Section 7.4 and hold for any choice of parameters. The most interesting condition is the last one, which ensures that the uninformed agent prefers to learn even though it entails not trading with probability 1/2 in the first period. In Section 7.4, we derive an upper bound for the expected utility from any offer accepted by informed responders in both s_1 and s_2. Making the following assumptions on parameters

$$\rho = 1, \quad \varphi = 2/3, \quad \omega = 1,$$

we can compute the expected utility from offering $z_{1,E}(s_1)$ and the upper bound just discussed. The values we obtain are plotted in Figure 1 for different values of γ.

As the figure shows, there is a range of γ for which experimenting dominates non-experimenting, so uninformed agents offer $z_{1,E}(s_1)$. Notice that as γ goes to 1 the expected utility of the uninformed agent converges to the expected utility of the informed agent and both converge to the expected utility in a Walrasian rational expectations equilibrium. So unlike in Example 1, in this case, as the frequency of trading increases, the equilibrium payoffs converge to those of a perfectly competitive rational expectations equilibrium.

As a final remark, notice that in this example some agents—the uninformed proposers whose offer is rejected in the first round—only reach an efficient allocation asymptotically. However, we can characterize the speed at which this convergence occurs. To measure distance from efficiency let us use the distance $d_t \equiv |x_t^1 - x_t^2|$. Since this distance is reduced by 1/2 every time the agents gets to make an offer and trade, after n trades in which the agent is selected as the proposer the distance if reduced to $2^{-n} d_0$. It is then possible to show that for every ϵ there is a γ large enough that the probability of $d_t < \epsilon$ is larger than $1 - \epsilon$. That is, with γ close to 1 the allocation approaches efficiency also for uninformed first-round proposers.

7.3 Example 1: Proofs

Proposition 6 *In the economy of Example 1, individual strategies are optimal for $t \geq 2$.*

Proof. Consider first a proposer, informed or uninformed, who has not deviated up to time t, so he holds an endowment x on the 45 degree line, either (η, η) or $(1 - \eta, 1 - \eta)$, and beliefs$\delta \in [0, 1]$. From inequality (72) the expected payoff from any deviating strategy is bounded above by

$$\delta u \left(\varphi x^1 + (1 - \varphi) x^2\right) + (1 - \delta) u \left((1 - \varphi) x^1 + \varphi x^2\right) = u \left(x^1\right),$$

given that $x^1 = x^2$. Making a zero offer today and not trading in all future periods achieves the upper bound $u \left(x^1\right)$, so the agent cannot gain by deviating. Turning to a responder, consider first an uninformed responder. Suppose he receives an off-the-equilibrium path offer $z \neq 0$ at $t \geq 2$, with

$$\min \left\{\varphi z^1 + (1 - \varphi) z^2, (1 - \varphi) z^1 + \varphi z^2\right\} < 0.$$

Suppose, without loss of generality, that $\varphi z^1 + (1 - \varphi) z^2$ is smaller than $(1 - \varphi) z^1 + \varphi z^2$, and thus smaller than zero. From B3, his belief after receiving offer z is $\delta = 1$. Using (72), his expected utility after accepting the offer is bounded above by

$$u \left(\varphi \left(x^1 + z^1\right) + (1 - \varphi) \left(x^2 + z^2\right)\right) < u \left(\varphi x^1 + (1 - \varphi) x^2\right) = u \left(x^1\right).$$

Rejecting the offer yields the payoff $u \left(x^1\right)$ and strictly dominates accepting the offer. A similar argument shows optimality for informed responders. ■

The following lemma provides an upper bound on the continuation utility which will be used below. Define the function

$$\begin{aligned} W(x, \delta) \equiv \max_z \quad & U(x - z, \delta) \\ s.t. \quad & \varphi z^1 + (1 - \varphi) z^2 \geq 0, \\ & (1 - \varphi) z^1 + \varphi z^2 \geq 0, \end{aligned} \tag{73}$$

and the quantity

$$\Xi(\alpha) \equiv \frac{1-\gamma}{1-\gamma+\alpha\gamma/2},$$

which is the probability of trading only with uninformed agents between any period $t \geq 2$ and the end of the game.

Lemma 10 *The following is an upper bound on the value function $V(x,\delta)$:*

$$V(x,\delta) \leq \Xi(\alpha)\left[\delta W(x,1) + (1-\delta) W(x,0)\right] + (1-\Xi(\alpha))\,\overline{u} \text{ for } \delta \in \{0,1\}.$$

Proof. Consider first an informed agent with $\delta = 1$. Consider a deviating strategy starting at $(x,1)$ at time t, and consider any history h^{t+j} along which the agent only meets uninformed agents. Let $x\left(h^{t+j}\right) = x + \sum z_n$ be his endowment at that history, where z_n are all the successful trades made along the history. Each trade z_n must satisfy $\varphi z_n^1 + (1-\varphi) z_n^2 \geq 0$ and $(1-\varphi) z_n^1 + \varphi z_n^2 \geq 0$, and so $z = \sum z_n$ must satisfy the same inequalities. By the definition of W the expected utility at history h^{t+j} then satisfies

$$U\left(x\left(h^{t+j}\right),1\right) \leq W(x,1).$$

Following any history in which the agent meets informed agent the utility is bounded by the upper bound $\overline{u}$. Taking expectation over all future histories yields the bound $\Xi W(x,1) + (1-\Xi)\,\overline{u}$. An uninformed agent cannot do better than receiving perfect information on the state s and then re-optimize, which yields the bound in the lemma. ■

For the following result we define

$$\begin{aligned} w_R^* \equiv \max_z \quad & W(x_{1,0} - z, 1) \\ s.t. \quad & W(x_{2,0} + z, 0) \geq U(x_{2,0}, 0). \end{aligned} \tag{74}$$

Proposition 7 *If η satisfies*

$$u(1-\eta) > w_R^*, \tag{75}$$

$$u(\eta) < U(x_{2,0}, 0), \tag{76}$$

$$u(\eta) > \frac{1}{2} W(x_{2,0}, 0) + \frac{1}{2} W(x_{2,0}, 1). \tag{77}$$

there is an $\hat{\alpha} \in (0,1)$ such that if $\alpha < \hat{\alpha}$, the strategies in S1-S3 are individually optimal.

Proof. Proposition 6 shows optimality in all periods $t \geq 2$, so we can restrict attention to time $t = 1$. Consider first the behavior of responders. We focus on responders with $x_{2,0}$, the case of responders with $x_{1,0}$ is symmetric. All responders are uninformed and they accept the equilibrium offer if

$$u(\eta) \geq \frac{1}{2}V(x_{2,0},1) + \frac{1}{2}V(x_{2,0},0).$$

Using Lemma 10 a sufficient condition for (7.3) is

$$u(\eta) \leq \Xi(\alpha)\left[\frac{1}{2}W(x_{2,0},0) + \frac{1}{2}W(x_{2,0},1)\right] + (1-\Xi(\alpha))\,\overline{u}.$$

Assumption (77) together with $\lim_{\alpha\to 0}\Xi(\alpha) = 1$, ensure that this condition holds for $\alpha \to 0$.

Consider next informed proposers. We focus on proposers with endowment $x_{1,0}$, the case of proposers with $x_{2,0}$ is symmetric. Suppose the proposer deviates by offering $z \neq z_{1,E}$ and the responder's belief goes to $\delta = 0$. The z offer will be rejected if

$$\Xi(\alpha)\,W(x_{2,0}+z,0) + (1-\Xi(\alpha))\,\overline{u} < U(x_{2,0},0),$$

given that the left-hand side is an upper bound on the continuation utility after accepting the offer and the right-hand side is a lower bound on the continuation utility after rejecting the offer. If the state is s_1, the proposer is a rich informed agent and his continuation utility is bounded above by $W(x_{1,0}-z,1)$. The utility from deviating is then bounded above by

$$\begin{aligned} w_R(\alpha) = \max_z \quad & \Xi(\alpha)\,W(x_{1,0}-z,1) + (1-\Xi(\alpha))\,\overline{u} \\ s.t. \quad & \Xi(\alpha)\,W(x_{2,0}+z,0) + (1-\Xi(\alpha))\,\overline{u} \geq U(x_{2,0},0). \end{aligned}$$

The function $w_R(\alpha)$ is continuous in α and $w_R(0) = w_R^*$ so assumption (75) ensures that $u(1-\eta) > w_R(\alpha)$ as $\alpha \to 0$. If the state is s_2, the proposer is a poor informed agent and the utility from deviating is bounded above by

$$\begin{aligned} w_P(\alpha) = \max_z \quad & \Xi(\alpha)\,W(x_{1,0}-z,0) + (1-\Xi(\alpha))\,\overline{u} \\ s.t. \quad & \Xi(\alpha)\,W(x_{2,0}+z,0) + (1-\Xi(\alpha))\,\overline{u} \geq U(x_{2,0},0). \end{aligned}$$

When $\alpha = 0$ the solution to this problem is given by perfect risk sharing with $x_{2,0}^1 + z^1 = x_{2,0}^2 + z^2 = u^{-1}(U(x_{2,0},0))$. The proposer's payoff is then

$$w_P(0) = u\left(1 - u^{-1}(U(x_{2,0},0))\right).$$

This payoff is strictly dominated by offering $z_{1,E}$ if

$$1-\eta > 1 - u^{-1}(U(x_{2,0},0)),$$

which is equivalent to assumption (76). A continuity argument ensures that $u(1-\eta) > w_P(\alpha)$ for $\alpha \to 0$.

Finally, consider uninformed proposers. Since informed proposers can condition their strategy on the realization of s, an uninformed proposer cannot do better than the expected gain of the informed proposer's deviations. Since this gain is negative under both values of s, the expected gain is negative and the uninformed proposer strictly prefers not to deviate. ■

7.3.1 Proof of Proposition 4

Given Proposition 7 we need to find parameters that satisfy conditions (75)-(77). The following lemma simplifies this construction.

Lemma 11 *The function W satisfies the following properties:*

1. $W(x_{2,0}, 1) = u(\varphi(1-\omega) + (1-\varphi)\omega)$,
2. *If*

$$\frac{\varphi u'(\omega)}{(1-\varphi)\, u'(1-\omega)} = 1 \tag{78}$$

then

$$w_R^* = W(x_{1,0}, 1) = \varphi u(\omega) + (1-\varphi)\, u(1-\omega)\,. \tag{79}$$

Proof. Consider problem (73), which defines W. The problem is concave, so the following first-order conditions, together with the constraints, are sufficient for an optimum

$$\begin{aligned} \pi(\delta)\, u'\left(x^1 - z^1\right) &= \varphi\lambda + (1-\varphi)\,\mu, \\ (1-\pi(\delta))\, u'\left(x^2 - z^2\right) &= (1-\varphi)\,\lambda + \varphi\mu. \end{aligned}$$

At $(x, \delta) = (x_{2,0}, 1)$, we have a solution with $\lambda > 0$, $\mu = 0$, and $z^2 > 0 > z^1$, which gives us property 1.

At $(x, \delta) = (x_{2,0}, 0)$ we have a solution at $z = 0$ with

$$\begin{aligned} \lambda &= \frac{u'(1-\omega) - u'(\omega)}{\frac{\varphi}{1-\varphi} - \frac{1-\varphi}{\varphi}} > 0, \\ \mu &= \frac{\frac{\varphi}{1-\varphi} u'(\omega) - \frac{1-\varphi}{\varphi} u'(1-\omega)}{\frac{\varphi}{1-\varphi} - \frac{1-\varphi}{\varphi}} > 0, \end{aligned} \tag{80}$$

where the second inequality follows from (78) and $\varphi > 1/2$. This implies $W(x_{2,0}, 0) = U(x_{2,0}, 0)$ and the envelope theorem implies

$$\frac{\partial W(x_{2,0}, 0)}{\partial x^1} = (1-\varphi) u'(1-\omega)\,, \tag{81}$$

$$\frac{\partial W(x_{2,0}, 0)}{\partial x^2} = \varphi u'(\omega)\,. \tag{82}$$

A symmetric argument applies to the case $(x, \delta) = (x_{1,0}, 1)$, leading to $W(x_{1,0}, 1) = U(x_{1,0}, 1)$ and

$$\begin{aligned}
\frac{\partial W(x_{1,0}, 1)}{\partial x^1} &= \varphi u'(\omega) \\
\frac{\partial W(x_{1,0}, 1)}{\partial x^2} &= (1-\varphi) u'(1-\omega).
\end{aligned}$$

Consider problem (74), which defines w_R^*. The first order conditions for this problem are

$$\begin{aligned}
\frac{\partial W(x_{1,0} - z, 1)}{\partial x^1} &= \lambda \frac{\partial W(x_{2,0} + z, 0)}{\partial x^1}, \\
\frac{\partial W(x_{1,0} - z, 1)}{\partial x^2} &= \lambda \frac{\partial W(x_{2,0} + z, 0)}{\partial x^2}.
\end{aligned}$$

These conditions are satisfied by setting $z = 0$ and $\lambda = 1$, so we have $w_R^* = W(x_{1,0}, 1) = U(x_{1,0}, 1)$. ■

Given Proposition 7 and Lemma 11, to construct an example it is sufficient to find a utility function, probabilities and endowments that satisfy the four conditions:

$$\begin{aligned}
\varphi u'(\omega) &= (1-\varphi) u'(1-\omega) \\
u(1-\eta) &> \varphi u(\omega) + (1-\varphi) u(1-\omega), \\
u(\eta) &< \varphi u(\omega) + (1-\varphi) u(1-\omega), \\
u(\eta) &> \frac{1}{2} u(\varphi(1-\omega) + (1-\varphi)\omega) + \frac{1}{2}(\varphi u(\omega) + (1-\varphi) u(1-\omega)).
\end{aligned}$$

With CRRA utility the first condition boils down to

$$\frac{\varphi}{1-\varphi} = \left(\frac{\omega}{1-\omega}\right)^\sigma,$$

and the remaining conditions are satisfied for $\sigma = 4$, $\omega = 0.9$, $\varphi = 0.9^4 / (1 + 0.9^4)$ and $\eta = 0.1265$.

7.4 Example 2: Proofs

7.4.1 Characterization of the value function V

Throughout this section, we fix the value of δ at either at 0 or 1. Our objective is to prove Proposition 8 below, which shows that the maximum continuation utility $V(x, \delta)$ is well defined for all $x \in R^2$ and shows how to compute it. We exploit the fact that with exponential utility, the per-period utility U takes the form

$$U(x, \delta) = -e^{-\rho x_1} f_0(x_2 - x_1), \tag{83}$$

where

$$f_0(\xi) \equiv \pi(\delta) + (1 - \pi(\delta)) e^{-\rho\xi}.$$

Proposition 8 *If the utility function is exponential and all agents are on the 45 degree line at $t = 2$, then there exists a continuation equilibrium in which the maximum continuation utility $V(x, \delta)$ is well defined for any $x \in R^2$, agents accept any offer that satisfies $V(z, \delta) \geq V(0, \delta)$, and V takes the form*

$$V(x, \delta) = -e^{-\rho x_1} f(x_2 - x_1), \tag{84}$$

where f solves the functional equation

$$f(\xi) = (1 - \gamma) f_0(\xi) + \gamma \left(\frac{1}{2} f(\xi) + \frac{1}{2} \left[f\left(\frac{\xi}{2} \right) \right]^2 \right).$$

The proof of this proposition is split in a number of lemmas.

Lemma 12 *Suppose the continuation utility $V(x, \delta)$ is well defined for all $x \in R^2$. Suppose all agents are on the 45 degree line and an agent with endowment x accepts offer z if it satisfies $V(x + z, \delta) \geq V(x, \delta)$. Then V satisfies two properties:*

(i) it takes the form (84) for some function f;

(ii) it satisfies the Bellman equation

$$V(x, \delta) = (1 - \gamma) U(x, \delta) + \gamma \left(\frac{1}{2} V(x, \delta) + \frac{1}{2} \max_{z: V(z, \delta) \geq V(0, \delta)} V(x - z, \delta) \right). \tag{85}$$

Proof. Consider two agents with endowments x and $x + a$, where a is any scalar. The second agent can follow the trading strategy of the first agent and get, in every period, the same utility level scaled by a factor $e^{-\rho a}$. This implies $V(x + a, \delta) \geq e^{-\rho a} V(x, \delta)$. A symmetric argument for the first agent implies $V(x, \delta) \geq e^{\rho a} V(x + a, \delta)$. Combining these two inequalities, we have

$$V(x + a, \delta) = e^{-\rho a} V(x, \delta) \text{ for all } a. \tag{86}$$

Defining $f(\xi) = -V((0, \xi), \delta)$, this implies (84). If x is on the 45 degree line, (86) implies that $V(x + z, \delta) \geq V(x, \delta)$ is equivalent to $V(z, \delta) \geq V(0, \delta)$. So if all agents are on the 45 degree line, an agent with endowment x who gets selected to make an offer, chooses z to maximize $V(x - z, \delta)$ subject to $V(z, \delta) \geq V(0, \delta)$. A standard dynamic programming argument implies that the value function $V(x, \delta)$ satisfies the functional equation (85). ■

Our next step is to show that there is a function V that solves the functional equation (85). Let T denote the mapping that, given a function $v : R^2 \to R$, yields

$$Tv(x) = (1 - \gamma) U(x, \delta) + \gamma \left(\frac{1}{2} v(x) + \frac{1}{2} \max_{z: v(z) \geq v(0)} v(x + z) \right).$$

A fixed point of T is a solution to the functional equation (85). To establish existence of a fixed point our strategy is the following: restrict attention to v functions that are generated by functions f in some set $\mathcal{B}$; define a self-map $\hat{T}$ on the space $\mathcal{B}$; find a fixed point of $\hat{T}$ in $\mathcal{B}$ and use it to construct a function v that is a fixed point of the original operator T.

Definition 4 *$\mathcal{A}$ is the set of continuous, non-increasing functions $f : R \to R_+$ that satisfy $f(0) = 1$ and*

$$f\left(\lambda\xi' + (1-\lambda)\xi''\right) \leq \left[f\left(\xi'\right)\right]^{\lambda}\left[f\left(\xi''\right)\right]^{1-\lambda} \text{ for all } \xi' \neq \xi'' \text{and all } \lambda \in (0,1). \tag{87}$$

Condition (87) is essentially a property of log-convexity of f. The next lemma shows that (87) is equivalent to concavity of v.

Lemma 13 *Let $v(x) = -e^{-\rho x_1} f(x_2 - x_1)$ for some continuous, non-increasing function f : $R \to R_+$. The function v is concave iff f satisfies (87).*

Proof. v is concave iff the set $\{(x_1, \xi) : v(x_1, x_1 + \xi) \geq -\kappa\}$ is convex for any $\kappa > 0$ (for $\kappa \leq 0$ the set is empty). But $v(x_1, x_1 + \xi) \geq -\kappa$ is equivalent to $e^{-\rho x_1} f(\xi) \leq \kappa$. Take two values ξ' and ξ'' and choose x_1' and x_1'' so that

$$e^{-\rho x_1'} f\left(\xi'\right) = e^{-\rho x_1''} f\left(\xi''\right) = 1.$$

The convexity of $\{(x_1, \xi) : e^{-\rho x_1} f(\xi) \leq 1\}$ implies that

$$e^{-\rho\left[\lambda x_1' + (1-\lambda)x_1''\right]} f\left(\lambda\xi' + (1-\lambda)\xi''\right) \leq 1 = \left[e^{-\rho x_1'} f\left(\xi'\right)\right]^{\lambda}\left[e^{-\rho x_1''} f\left(\xi''\right)\right]^{(1-\lambda)},$$

which yields property (87). The converse is easy and is omitted. ■

Lemma 14 *Take any function $v(x) = -e^{-\rho x_1} f(x_2 - x_1)$ for some $f \in \mathcal{A}$. Then, the following offer solves the maximization problem in (85)*

$$\begin{aligned} z_1 &= \frac{1}{\rho}\log f\left(\frac{x_2 - x_1}{2}\right), \\ z_2 &= z_1 + \frac{x_2 - x_1}{2}, \end{aligned}$$

and

$$Tv(x) = -e^{\rho x_1} h(x_2 - x_1),$$

where h is in $\mathcal{A}$ and satisfies

$$h(\xi) = (1-\gamma) f_0(\xi) + \gamma\left(\frac{1}{2} f(\xi) + \frac{1}{2}\left[f\left(\frac{\xi}{2}\right)\right]^2\right). \tag{88}$$

Proof. Using $\xi = x_2 - z_2 - (x_1 - z_1)$, we can rewrite the optimization problem as

$$\begin{aligned} \max_{z_1, z_2} \quad & -e^{-\rho(x_1-z_1)} f(\xi), \\ \text{s.t.} \quad & -e^{-\rho z_1} f(x_2 - x_1 - \xi) \geq -1. \end{aligned} \tag{89}$$

Substituting for z_1 in the constraint, we have

$$\max_{\xi} -e^{-\rho x_1} f(\xi) f(x_2 - x_1 - \xi). \tag{90}$$

From (87) and

$$\frac{1}{2}\xi + \frac{1}{2}(x_2 - x_1 - \xi) = \frac{x_2 - x_1}{2},$$

we have

$$f(\xi) f(x_2 - x_1 - \xi) \geq \left[f\left(\frac{x_2 - x_1}{2} \right) \right]^2. \tag{91}$$

The last inequality implies that setting $\xi = (x_2 - x_1)/2$ is optimal. This gives us the optimal value for $z_2 - z_1$. To get the optimal level of z_1 we use the constraint (89). The expression for h follows from substituting the optimal choices of z_1 and ξ in the objective function (90), substituting in (85) and using (83). Continuity and monotonicity of h and the fact that $h(0) = 1$ follow immediately from (88) and the definition of f_0. It remains to establish the log-convexity of h. The maximization problem in (85) yields a concave function of x, because v is concave. Then Tv is a convex combination of concave functions and so is concave. Lemma 13 implies that h satisfies (87). ■

Lemma 14 suggests that, to prove existence of a fixed point for T we define the mapping $\hat{T}$ as

$$\hat{T} f(\xi) = (1 - \gamma) f_0(\xi) + \gamma \left(\frac{1}{2} f(\xi) + \frac{1}{2} \left[f\left(\frac{\xi}{2} \right) \right]^2 \right),$$

and look for a fixed point of this mapping. The advantage is that we can choose any positive scalar M and take as the domain of $\hat{T}$ the space of continuous functions on the interval $[0, M]$, since $\xi \in [0, M]$ implies $\xi/2 \in [0, M]$.

Definition 5 *$\mathcal{B}_M$ is the set of continuous, non-increasing functions $f : [0, M] \to [0, 1]$ that satisfy $f(0) = 1$, (87), and:*

$$f(\xi') - f(\xi) \geq -\rho(\xi' - \xi) \ \text{ if } \xi' \geq \xi. \tag{92}$$

The additional property (92) is useful as it ensures the equicontinuity of the functions in $\mathcal{B}_M$. We can show that $\hat{T}$ is a self-map on $\mathcal{B}_M$.

Lemma 15 *$\hat{T}$ is a mapping from $\mathcal{B}_M$ to $\mathcal{B}_M$.*

Proof. That $\hat{T}$ preserves continuity, monotonicity, $f(\xi) \geq 0$, $f(0) = 1$, and (87) is an immediate corollary of Proposition 14. It remains to prove that $\hat{T}$ preserves the bound $f(\xi) \leq 1$ and that it preserves (92). Property (87) implies $f(\xi/2)^2 \leq f(\xi)$. Substituting in the definition of $\hat{T}f$, we then have

$$\hat{T}f(\xi) \leq (1-\gamma) f_0(\xi) + \gamma f(\xi) \leq (1-\gamma) f_0(0) + \gamma f(0) = 1,$$

since both f_0 and f are non-increasing. To prove (92), take two values $\xi' \geq \xi$. By convexity, f_0 satisfies

$$f_0(\xi') - f_0(\xi) \geq f_0'(\xi)(\xi' - \xi) = -\rho(1-\pi) e^{-\rho\xi}(\xi' - \xi) \geq -\rho(\xi' - \xi).$$

Moreover, since f satisfies

$$f(\xi') - f(\xi) \geq -\rho(\xi' - \xi)$$

and is bounded above by 1 we have

$$\left[f\left(\frac{1}{2}\xi'\right)\right]^2 - \left[f\left(\frac{1}{2}\xi\right)\right]^2 = \left[f\left(\frac{1}{2}\xi'\right) + f\left(\frac{1}{2}\xi\right)\right]\left[f\left(\frac{1}{2}\xi'\right) - f\left(\frac{1}{2}\xi\right)\right] \geq -2\rho\left(\frac{1}{2}\xi' - \frac{1}{2}\xi\right).$$

Combining the last three inequalities we have

$$\begin{gathered}\hat{T}f(\xi') - \hat{T}f(\xi) = \\ (1-\gamma)[f_0(\xi') - f_0(\xi)] + \tfrac{\gamma}{2}[f(\xi') - f(\xi)] + \tfrac{\gamma}{2}\left\{\left[f\left(\tfrac{1}{2}\xi'\right)\right]^2 - \left[f\left(\tfrac{1}{2}\xi\right)\right]^2\right\} \\ \geq -\rho(\xi' - \xi),\end{gathered}$$

which completes the argument.

We can now state our existence result for f. ■

Lemma 16 *The mapping $\hat{T}$ has a fixed point f in $\mathcal{B}_M$.*

Proof. Define the sequence of functions $\{f_n\}_{n=0}^{\infty}$ starting at f_0 and letting $f_n = \hat{T}f_{n-1}$ for $n = 1, 2, 3....$ First, we want to show that the sequence $\{f_n\}$ is monotone. We prove it by induction. Notice that

$$f_1(\xi) = \left(1 - \frac{\gamma}{2}\right) f_0(\xi) + \frac{\gamma}{2}\left[f_0\left(\frac{1}{2}\xi\right)\right]^2 \leq f_0(\xi), \tag{93}$$

since the log-convexity of f_0 implies $f_0(\xi/2)^2 \leq f_0(\xi)$. The definition of the sequence means that

$$f_n(\xi) = (1-\gamma) f_0(\xi) + \frac{\gamma}{2} f_{n-1}(\xi) + \frac{\gamma}{2} f_{n-1}\left(\frac{1}{2}\xi\right).$$

Writing the same equation at $n+1$ and taking differences side by side we have

$$f_{n+1}(\xi) - f_n(\xi) = \frac{\gamma}{2}\left[f_n(\xi) - f_{n-1}(\xi)\right] + \frac{\gamma}{2}\left[f_n\left(\frac{1}{2}\xi\right) - f_{n-1}\left(\frac{1}{2}\xi\right)\right].$$

This means that $f_n \leq f_{n-1}$ implies $f_{n+1} \leq f_n$. Since $f_1 \leq f_0$ from(93), by induction we have $f_n \leq f_{n-1}$ for all n.

$$f_1(\xi) - f_0(\xi) = \frac{\gamma}{2}\left[f_n(\xi) - f_{n-1}(\xi)\right] + \frac{\gamma}{2}\left[f_n\left(\frac{1}{2}\xi\right) - f_{n-1}\left(\frac{1}{2}\xi\right)\right].$$

Since f_0 is in $\mathcal{B}_M$, all the functions in the sequence are in $\mathcal{B}_M$ and so they all satisfy

$$-\rho\left(\xi' - \xi\right) \leq f_n\left(\xi'\right) - f_n(\xi) \leq 0$$

for any pair $\xi' \geq \xi$. This implies that the sequence $\{f_n\}_{n=0}^{\infty}$ is uniformly bounded and equicontinuous. Notice that $\mathcal{B}_M$ is closed in the sup-norm topology. Then by the Arzelà-Ascoli theorem, the sequence $\{f_n\}_{n=0}^{\infty}$ admits a subsequence $\{f_{n_k}\}_{k=0}^{\infty}$ that converges uniformly to a function f in $\mathcal{B}_M$. Moreover, the fact that the original sequence $\{f_n\}_{n=0}^{\infty}$ is monotone implies that it also converges uniformly to f. It is easy to show that the mapping $\hat{T}$ is continuous on $\mathcal{B}_M$. Therefore,

$$f = \lim_{n\to\infty} f_n = \lim_{n\to\infty} \hat{T} f_{n-1} = \hat{T} \lim_{n\to\infty} f_{n-1} = \hat{T} f,$$

which completes the argument. ■

To complete the proof of Proposition 8, we need to go back to the function $V(x,\delta)$. If $x_2 \geq x_1$ we can choose any $M > x_2 - x_1$, find the function f that is a fixed point of $\hat{T}$ on $\mathcal{B}_M$ and set $V(x,\delta) = -e^{-\rho x_1} f(x_2 - x_1)$. If $x_2 < x_1$, we can proceed in a symmetric fashion, and prove the existence of a function g such that $V(x,\delta) = -e^{-\rho x_2} g(x_1 - x_2)$. The function f is then found setting

$$f(\xi) = e^{-\rho\xi} g(-\xi),$$

completing the proof.

7.4.2 Equilibrium in periods $t \geq 2$

We need to show that if an agent is already on 45 degree line, his optimal strategy is never to trade. That is we need to show that the trade $z = (0,0)$ is a solution to the maximization problem

$$\max_z V(x - z, \delta),\ \text{s.t.} V(z,\delta) \geq V(0,\delta),$$

if $x_1 = x_2$. To do so we argue that V satisfies the property

$$V(x,\delta) \leq u\left(\pi(\delta) x_1 + (1 - \pi(\delta)) x_2\right) \tag{94}$$

with equality if $x_1 = x_2$.

Proposition 9 *The function V satisfies (94).*

Proof. First, we prove that the mapping T preserves this property. If v satisfies (94), then we want to prove that the function J defined as

$$J(x) \quad = \quad \max_z v(x-z), \text{ s.t. } v(z) \geq v(0),$$

also satisfies (94). Suppose, by contradiction, that $J(x) > u(\pi x_1 + (1-\pi)x_2)$. Then there exists a z such that $v(x-z) > u(\pi x_1 + (1-\pi)x_2)$. Since v satisfies (94), we have

$$u(\pi x_1 + (1-\pi)x_2) < u(\pi(x_1 - z_1) + (1-\pi)(x_2 - z_2))$$

which implies

$$\pi x_1 + (1-\pi)x_2 < \pi(x_1 - z_1) + (1-\pi)(x_2 - z_2).$$

Similarly, $v(z) \geq v(0)$ and (94) imply

$$0 \leq \pi z_1 + (1-\pi)z_2.$$

Summing side by side the last two inequalities yields a contradiction. The function Tv is equal to a convex combination of U and J. It is easy to show that U satisfies (94) and the property is preserved under convex combination. So Tv satisfies (94). It remains to prove that the condition holds as an equality if $x_1 = x_2$, and this follows from the fact that $z = 0$ is always feasible. Since the function $V(x, \delta)$ can be derived as the limit of sequence of functions v_n, starting at $v_0 = U$ (see Lemma 16), U satisfies (94), and the set of functions that satisfy (94) is closed, the result follows. ■

The following is an immediate corollary.

Corollary 1 *For $t \geq 2$, zero trade is optimal for all agents with endowment on the 45 degree line.*

Proof. In the proof of Proposition 9, we show that

$$\max_z \{V(x-z,\delta), \text{ s.t. } V(z,\delta) \geq V(0,\delta)\} \leq u(\pi x_1 + (1-\pi)x_2).$$

If $x_1 = x_2$, setting $z = 0$ achieves the upper bound $u(\pi x_1 + (1-\pi)x_2)$, hence zero trade is optimal. ■

7.4.3 Strategies at $t = 1$

Let us first consider equilibrium offers of informed agents. We focus on rich informed in s_1, but analogous results hold for the other cases.

Proposition 10 *In period $t = 1$, if informed agent of type 1 meets type 2 and s_1, the proposer offers a trade $z_{1,E}(s_1)$ such that*

$$x_{1,0} - z_{1,E}(s_1) = (1 - \eta, 1 - \eta)$$
$$x_{2,0} + z_{1,E}(s_1) = (\eta, \eta)$$

where

$$\eta = 1 - \omega - [\log f(2\omega - 1)] / \rho.$$

Proof. Given the value function V derived above, it is easy to check that η satisfies $V((\eta, \eta), 1) = V((1 - \omega, \omega), 1)$, since $V((\eta, \eta), 1) = -e^{-\rho\eta}$ and $V((1 - \omega, \omega), 1) = -e^{-\rho(1-\omega)} f(2\omega - 1)$. We need to show that $z_{1,E}(s_1)$ is optimal for the proposer. To do so, notice that, given the definition of η, the inequality $V((1 - \omega, \omega) + z) \geq V((1 - \omega, \omega))$ can be rewritten as $V((\eta, \eta) + z - z_{1,E}(s_1)) \geq V((\eta, \eta))$ or, given the properties of V, as $V(z - z_{1,E}(s_1), \delta) \geq V(0, \delta)$. So the maximization problem of the proposer can be rewritten as

$$\max_{\tilde{z}} \{ V((1 - \eta, 1 - \eta) - \tilde{z}, \delta) \text{ s.t. } V(\tilde{z}, \delta) \geq V(0, \delta) \}.$$

The argument for Corollary 1 shows that it is optimal to choose $\tilde{z} = 0$, i.e., $z = z_{1,E}(s_1)$. ■

7.4.4 Uninformed agents experiment

Without loss of generality, consider an uninformed proposer of type 1, with endowment $(\omega, 1 - \omega)$. We want to show that at $t = 1$ he finds it optimal to experiment, by making an offer that is only accepted by the informed agents in one state of the world. In particular, we want to show that it is optimal for him to offer $z_{1,E}(s_1)$. If the offer is accepted he stops trading, if it is rejected he trades in all following periods, whenever he is the proposer, offering the trades described in Lemma 14. Given that the agent learns the state s from the fact that his offer is rejected and given that from $t = 2$ onwards he will meet with probability 1 informed agents with endowments on the 45 degree line, his behavior for $t \geq 2$ is optimal by the results of the previous subsections.

To prove optimality at $t = 1$ we need to check that:

1. offer $z_{1,E}(s_1)$ is rejected by informed agents in s_2;

2. offering $z_{1,E}(s_1)$ is better than any other offer accepted by the informed agent only in s_1;

3. offering $z_{1,E}(s_1)$ is better than any offer accepted by the informed agent only in state s_2;

4. offering $z_{1,E}(s_1)$ is better than any offer accepted by the informed agent in both states.

To check part 1, we need to show that

$$V((1-\omega,\omega)+z_{1,E}(s_1),0) < V((1-\omega,\omega),0). \tag{95}$$

But since $(1-\omega,\omega)+z_{1,E}(s_1)=(\eta,\eta)$ (where η is defined in Proposition 10) and $V((\eta,\eta),0)=V((\eta,\eta),1)=V((1-\omega,\omega),1)$ this condition boils down to

$$V((1-\omega,\omega),1) < V((1-\omega,\omega),0).$$

This inequality follows from the fact that, given that $\omega > 1-\omega$, $V((1-\omega,\omega),\delta)$ is monotone decreasing in δ.

Part 2 can be proved as follows. To ensure that the offer is accepted by the informed in state s_1 the offer must satisfy

$$V((1-\omega,\omega)+z,1) \geq V((1-\omega,\omega),1), \tag{96}$$

$$V((1-\omega,\omega)+z,0) < V((1-\omega,\omega),0) \tag{97}$$

The payoff of the uninformed agent if he makes an offer accepted only in state 1 is

$$\frac{1}{2}V((\omega,1-\omega)-z,1)+\frac{1}{2}V((\omega,1-\omega),0). \tag{98}$$

Consider the problem of maximizing (98) subject to (96) and (97). If we relax the problem by omitting constraint (97), the optimal offer is $z_{1,E}(s_1)$, because it maximizes the first term of (98) and the second term is a constant independent of z. But since $z_{1,E}(s_1)$ satisfies (95), the second constraint is also satisfied and so $z_{1,E}(s_1)$ solves the original problem. The payoff from this offer is

$$V_E=\frac{1}{2}V((\omega,1-\omega)-z_{1,E}(s_1),1)+\frac{1}{2}V((\omega,1-\omega),0).$$

To check part 3, we need to check that there is an upper bound on the utility the proposer can get by making an offer accepted only in state 2 and that this upper bound correspond to the payoff from no trade $(1/2)V((\omega,1-\omega),1)+(1/2)V((\omega,1-\omega),0)$. In particular, inspecting indifference curves for our numerical examples, we see that all the offers that induce acceptance only in state 2 involve that the proposer buys asset 1 in exchange for asset 2 and that these offers are dominated by no trade.

To check part 4 notice that the payoff of the uninformed agent if he makes an offer accepted in both states is bounded above by the solution to the following problem

$$\overline{V}_{NE} = \max_{z} \frac{1}{2} V\left((\omega, 1-\omega) - z, 1\right) + \frac{1}{2} V\left((\omega, 1-\omega) - z, 0\right) \tag{99}$$

subject to

$$V\left((1-\omega, \omega) + z, \delta\right) \geq V\left((1-\omega, \omega), \delta\right)$$

for $\delta = 0, 1$. This is because the continuation utility of the uninformed is lower than or equal to the objective function in (99). So we need to check the inequality

$$\overline{V}_{NE} < V_E.$$

This is the condition discussed in the main text and represented graphically in Figure 1.