DEFAULT, EFFICIENCY AND UNIQUENESS

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Abstract

An adequate description of economic dynamics requires the introduction of a monetary system including default penalties and expectations in a society whose economy utilizes money and credit. This essay notes and discusses several of the factors involved in the use of money and credit in a process oriented economy. It links these observations with the general equilibrium treatment of the same underlying economy and formulates a government guidance game where the government sets several key parameters in a monetary economy sufficient to select a unique equilibrium. Low information and error correction are noted. The links to the first and second welfare theorems of GE are also considered as is the setting of the price level.

Keywords: General equilibrium, strategic market games, uniqueness, aggregation, information, disequilibrium, minimal institutions, playable games.

JEL Classifications: C7, D50, E4

1 Government Guidance

This essay lays out the connection between the static, non-strategic, general equilibrium (GE) model of an economy and game theoretic, strategic market game (SMG) models of the same economy. It notes that in the latter price formation mechanisms and default or bankruptcy conditions must be defined.explicitly if any credit is created; and that the specification of these conditions may have a considerable influence on the outcomes. A basic observation is that no matter how simple a fully defined process model may be it is fundamentally institutional,the mechanism that supports process itself may be interpreted as such.

In essence the overall argument can be sketched in several simple points

1. Classical GE exchange economies have no institutional money, but rather just a unit of account that sums to zero. The price level is of no concern. It is defined on $(0, \infty)$. Default is of no concern.

2. The competitive equilibria (CE) of a GE model are not necessarily unique. In fact, in this paper we reconsider an example with 3 CEs.

3. There are two ways uniqueness fails (a) there is more than one fixed point implying more than one way to distribute all physical resources. (b) the price level is not linked to an asset money. In a dynamic economy both of these may be avoided.

4. As soon as we try to build a process model, we are forced to consider cash flow constraints. There are three fairly natural ways to handle these constraints. (a) we can consider that a referee hands out an accounting money line of credit where each individual must return his credit line at the end or be penalized for failing to do so. (b) Trade uses a commodity money where in simpler subcases the money could be separable and could be linear in utility. (c) an intrinsically worthless fiat is issued whose future value is determined by expectations.[1] All cases are feasible. If borrowing is permitted, a default penalty may be needed if an individual is unable to pay back her debts.[2]

[1]Its value lies with the worth of liquidity, not in consumption.

[2]In all of the discussion in this essay we limit our concern to one round of trade in a single period.

5. As soon as we build a model using any feasible trading method even for the GE model, we can calculate liquidity needs at any CE for any trading method. This gives us a way to differentiate among CEs – if total nonmonetary wealth is to sum to the same monetary amount W, then each CE requires a different level of liquidity relative to wealth normalized to W.

6. Case 4 (b) above is a straightforward but mathematically messy way to develop the conditions of "enough money" when the commodity selected as a money enters into the utility functions generally. Dubey and Shapley [4] and others have considered this possibility, including some examples where it is not feasible to satisfy the conditions for there to be enough of a commodity money if it enters the preferences of individuals in an inappropriate manner.

7. Case 4 (c) above, with fiat, may be considered with and without borrowing and these are discussed below.

8. All cases require a counting of the degrees of freedom (DOF) and discussing what is needed to fully nail down uniqueness and a price level. Fiat money introduces n degrees of freedom into an economy with n trader types. A candidate for minimal government might be a commodity money such as gold with 100% reserve lending; but this requires a separate specific discussion as to why it may or may not be economically and politically feasible (see [18] Chapter 5).

Government as a player Government may be treated in two ways. First, it can be an active goal oriented agent with some strategic abilities such as setting an interest rate, bankruptcy penalties and a bound on borrowing, and possibly taxes and subsidies. Second, its role may be just regulatory, so its control variables may be regarded as a given set of parameters. This avoids being explicit about government goals. The former stresses the macro-economic connection while the latter emphasizes the, micro-economic connection. For simplicity we choose the latter.

2 General Equilibrium (GE) Model

A general equilibrium exchange economy (or "GE economy" for short) Γ can be described by the set of the trader types, their utility functions and their endowments

The variability in the velocity of money is an empirical fact; but even without this complicating feature many of the basic properties of the monetary system we study can be investigated.

$\Gamma = \{R^m_+, u^i, a^i\}^n_{i=1}$. where

R^m_+ = the commodity space for m commodities,

u^i = the utility function for individual i, $i = 1, ..., n$.

a^i = the initial endowment to individual i of the m commodities

For any GE economy, a *competitive equilibrium* (CE) as a pair $\{(x^1, ..., x^n), p\}$ (here x^i is an m-dimensional consumption bundle for player i, and p is an m-dimensional price vector) satisfying a) for each player i, x^i maximizes $u^i(x)$ over all x satisfying $\sum p_j x_j \leq \sum p_j a^i_j$; and b) $\sum x^i = \sum a^i$. This is the standard concept from microeconomics, in which players are each maximizing their own utility and supply equals demand.

An observation here (relevant when we discuss strategic market game equivalents) is that if (x, p) is a CE in a GE model, then so is $(x, \alpha p)$ for any positive α. Thus there is complete degree of freedom in the equilibrium prices of the model.

In general, there is no guarantee that there is a unique equilibrium allocation x. However, there are known conditions which do guarantee the uniqueness:

- Uniqueness will exist if all traders have the same utility function;
- if all have a commodity in common that enters the utility functions in a linear separable manner and all have a sufficient supply;
- if there is gross substitutability.

Shapley and Shubik [16] give an $m = n = 2$ example satisfying none of the above conditions, which contains three equilibria. In the latter part of this paper, we consider a "strategic market game version" of their model, and compare results.

We remind the reader that as described in the previous section, there is no process described in the GE model formulation, no explicit accounting of money. For that we need strategic market games.

3 Strategic Market Games (SMGs)

Associated with any GE economy there are many ways to define a strategic market game. These have been discussed elsewhere [17][15][18]. Here we enlarge the GE model explicitly, by not merely constructing a process model, but also by introducing government as an extra strategic dummy atomic player. Its powers over the economy are specified by the financial instruments it controls, such as fiat money supply, taxation and subsidies.

3.1 The Most General SMG

The most general strategic market game may be described by: $\Psi = \{R^m_+ \times R, , \tilde{p}_j, \Pi, \Lambda, \rho, \beta, \theta, B, \{U^i, a^i, s^i, \tau^i\}^n_{i=1}, k\}$ where the extra dimension R in the commodity set includes the amount of fiat held by the players The i's index player types, of which there are n in all. Each individual of type i is assumed to be identical.

The government control set is $\Pi, \Lambda, \rho, \theta, B, \tau^i$. In our formulation here, the government is not a formal player in the game, but instead supplies the exogenous parameters listed. There are other models in which the government IS a formal player in the game, with its own objectives. See Quint and Shubik [15].

- The parameter Π may be described as official per unit expectations of the purchasing power, or cost of living index, for any money held at settlement (in a finite dynamic program it is part of the salvage value) [3].
- Λ is the common penalty, per unit, for any money owed (or negative money) appearing in the second period or day of settlement; after the game is over measured in terms of the 'disutility or negative worth' of the penaty relative to the size of the bankruptcy measured in fiat [4].
- The B is a bound placed on the central bank's issue of fiat. The bound may not exist formally, but history has seen unconstrained printing of fiat eventually destroying the functioning of the economy.
- τ^i are the lump sum taxes $(-\Delta)$ and subsidies $(+\Delta)$ on each individual of type i.
- $\tilde{p}_j$ are the previous market prices [5]

[3] If we consruct a playable game with only nondurable commodities, no history beyond previous prices and only one period of play there is little leaning feasible so that any valuation of worth at settlement must depend at most on given parameters and the outcomes of a single move by each agent.

[4] Philosophically it is difficult to be precise about a bankruptcy penalty and its indvidual worth. By concentrating our thought on an experimental game we can at least be reasonably specific i our assmptions.

[5] In a "rational expectations world" the behavoral mystery of how future expectations are formed is solved by assuming that the coordination probelm that matches the ex ante expectation is identical with the ex post result. This, though intellectually satisfyng to some, sidesteps the dynamics by the assumption of an error proof foresight. Here, as we do consider an equilibrium solution we permit it to be conditioned on a parametrically given set of expectations. How the expectations are formed and modified in a many period economy is, essentially, one of the key open questions in economic dynamics; and, in our opinion, does not have a pure economic answer free from considerable context.

- ρ is the money rate of interest,
- β is the natural discount
- θ is a lending parameter

We also define two derived variables where

- Π^i is individual type i's expectation of the per unit worth of money in the future. It is calculated as $\Pi^i = \beta\mu^i\Pi$. The discount factor β denotes that settlement comes one period after the action and the μ^i is the Lagrangian that personalizes the valuation for type i.[6]

- Λ^i is individual type i's per unit worth of the government imposed bankruptcy penalty. It is calculated as $\Lambda^i = \beta\tilde{\mu}^i\Lambda$. Again β denotes that settlement comes one period after the action and the μ^i is the Lagrangian that personalizes the valuation.

There is a conceptual problem in specifying the magnitude of the Λ. From the viewpoint of the government it is a generalized "marginal disutility" of debt at the point of bankruptcy. If we wanted to be physically specific it could be the disutiltiy of going to debtor's prison, or of other actual penalties. If this has been selected large enough,[7] then no one will wish to go bankrupt strategically.

At the point separating bankruptcy from nonbankruptcy, there will be left and right hand derivatives that may differ.

Finally, the parameter k signifies the number of individuals of type $i.(i = 1, ..., n)$. The two important cases are where $k = 1$ and where $k = \infty$. In the $k = \infty$ case, the interpretation is that there is a continuum of individuals of each type. More about this in Section 9.

Individuals of type i wish to optimize, over all nonnegative $((b_1^i, ..., b_m^i, q_1^i, ..., q_m^i, d^i)$, the following

[6]Mathematically, μ^i is the optimal value of the Lagrangian multiplier of a "no bankruptcy allowed" constraint for type i, which is equivalent to our case below with $\Lambda = \infty$. See Quint-Shubik [15] for details.

[7]If there is an upper bound to the amount of money and credit that can be issued, for a bounded velocity, there is an upper bound to price, hence a high enough penalty to avoid strategic bankruptcy near quilibrium is well-defined.

$$
\begin{aligned}
\max U^i &= u^i\left(\frac{b_1^i}{p_1}+a_1^i-q_1^i,\ldots,\frac{b_m^i}{p_m}+a_m^i-q_m^i\right) & (1)\\
&+\Pi^i\left(s^i+\tau^i+\sum_{j=1}^m p_j q_j^i-\sum_{j=1}^m b_j^i-\rho d^i\right)^+ & (2)\\
&+\Lambda^i\left(s^i+\tau^i+\sum_{j=1}^m p_j q_j^i-\sum_{j=1}^m b_j^i-\rho d^i\right)^- & (3)
\end{aligned}
$$

subject to

$$
s^i+\tau^i-\sum_{j=1}^m b_j^i+d^i\geq 0 \qquad (\lambda^i)
$$

where

$a^i =$ the initial goods resources of i. For all goods $a_j > 0$ where $a_j = \sum_{i=1}^n a_j^i$.

s^i = the initial amount of fiat money held by i before taxes or subsidies, where $s^i \geq 0$ for all i. We also assume $s > 0$ where $s = \sum_{i=1}^n s^i$.

τ^i = the initial amount of money lump sum taxes on or subsidies for i, where we assume $\tau^i \geq -s^i$ and is bounded above where $\tau = \sum_{i=1}^n \tau^i$.

$\rho =$ the money rate of interest fixed on the government's outside bank

$\beta =$ the natural discount rate

$b_j^i =$ the bid of type i for good j

$q_j^i =$ the offers for sale of good j by individuals of type i

$\theta =$ the leverage factor in borrowing

$d^i =$ the amount of fiat borrowed by i

Here the notation $(x)^+$ means $\max(x,0)$ and $(x)^-$ means $\min(x,0)$.

We select the SMG with the minimally simple price formation mechanism, where price is the ratio of the sum of money bid to the quantity of goods offered. If k is finite, this is

$$
p_j=\frac{\sum_{i=1}^n\sum b_j^i}{\sum_{i=1}^n\sum q_j^i} \qquad \text{for } j=1,\ldots,m. \tag{4}
$$

where the second summation sign in numerator and denominator is the summation over the k individuals of each type. [If $k=\infty$, these "second summations" are replaced by integrals – see Section 9.] Any SMG utilizing this price formation is called a *Cournot SMG*.

Here we note another distimction between the $k = 1$ and $k = \infty$ case. In the former, a single individual can affect the price via the selection of his bids and offers. In the latter, he cannot because the size of his individual bid is miniscule compared the magnitude of both numerator and denominator. Hence in their own optimization the individuals take prices as a given constant. This simplifies the analysis compared with the $k = 1$ case.

We look for noncooperative equilibria (NE) for the above game. More specifically, we look for NEs in which all individuals of the same type are doing the same thing; these are called "type specific noncooperative equilibria", or TSNEs. In symbols, a TSNE is a set of n vectors $\{(b^i, q^i, d^i)\}_{i=1}^n$ (each of which optimizes the individual utility maximization problem above), plus a price vector which satisfies (4).

3.2 Where does negative money come from?

Debts are financial instruments different from fiat money. The debts are usually denominated in fiat money to be paid. Thus at settlement the debt is expunged by converting it into fiat money (in the United States fiat is good for all debts public or private). When an individual cannot pay and goes bankrupt, no fiat is destroyed but the bankruptcy procedure expunges the debt and also may impose a negative reward for the debt not paid. As history indicates the punishment is partially or completely societal and not necessarily economic. It has included death, slavery, deportation, indentured servitude, debtors' prison, garnishing of future income and other penalties.

3.3 An aside on taxes and transfers

Macroeconomic models, unless they are very low dimensional growth models, often do not specify the number of control variables and the degrees of freedom in the system. This may, in part, be encouraged by the concern for dynamic equilibrium paths and a desire to avoid the complexity of boundary conditions. Our belief is that financial control is heavily involved with boundary conditions and inequalities and, at best, transient paths to an equilibrium which may not always exist. Among the most important devices in influencing adjustment are taxation and subsidies. In the models here they can be treated explicitly. In particular the influence of a lump sum transfer is dealt with explicitly in considering the welfare theorems of GE.

The construction here converts the general equilibrium model into a process model. Although it is ridiculously simple behavioristically, it can actually be played as a one period game. The full Pandora's box concerning learning and expectations opens with the multistage game and Jurgen Huber, Martin Shubik and Shyam Sunder [6] have experimented with this model.

Comment: *In essence the simple SMG model selected here is a closed one period economy in terms of agents; but is open in time and it is connected to the other periods by the one durable asset, fiat money.*

4 Control, Dimensionality, and Other Modeling Issues

When we compare the set of GE exchange economies against the set of strategic market games (SMG) presented here, we observe that there is a vast profusion of games that map down to any GE model. The GE models in concentrating on the existence of equilibria can dispense with the institutional detail needed to describe the carriers of process in a SMG. Even with a relatively parsimonious description of the financial and control structure the proliferation of models is considerable, but this is to be expected when one attempts to model process. Many of the control variables such as Λ are set equal for all. This is usually reasonable when information requirements are considered [8]. Even though the physical penalty may, in law, be the same for all, like default it may be valued differently by each agent.

4.1 Error correction

A dynamic system that acknowledges the possibility of error and disequilibrium requires the appropriate forms of redundancy that help to correct errors. The availability of lending and the variation of the money supply provides a system where in some situations the ability to borrow may help to correct error, This is illustrated by Walter Bagehot's masterful discussion of Lender of Last Resort [1].

4.2 The use of money and the leaky conduit

At best clean abstract models pick up essences, not the details of reality. We utilize a stark simplistic view of both fiat money and taxation. We imagine strict rules of the game that require all individuals to use fiat money in their transactions, to borrow only from a government bank, and to fully pay their taxes. In reality the monetary and

[8]Although, in fact, the legal process does some (expensive) hand-tailoring for individual cases.

taxation system is not unlike a leaky irrigation conduit system where many mountain streams pour inefficiently into the conduit and many individuals tap into the system legally or illegally, while others dig wells for themselves.

4.3 Dimensional analysis and the redundancy of control instruments

In the pure GE model there is no explicit government control. It is proved that given the assumptions made, efficient prices exist and they are not necessarily unique. No selection mechanism is provided. We suggest that this mathematical problem may be resolved in two considerably different ways. First, one may either search for restricting conditions within the GE models that guarantee uniqueness of equilibrium (cutting down the generality of the models). Alternatively, we may embed the GE economy into a large class of monetary economies where government selection of certain 'rules of the game' may still leave competitive markets to form price, but the extra rules guide the outcome and possibly select or, at least maintain a unique equilibrium.

4.4 Degrees of freedom and control

In any macroeconomic textbook different government control instruments are discussed, but there is rarely if ever any discussion of the number of degrees of freedom in the private sector and the number of control variables available to the government. In many of the standard low dimensional growth models it is easy to construct economic models where the dimensions of the control variable exceed the degrees of freedom in the market structure. This appears as a paradox; but is easily explained with the observation that a number of control variables cannot be utilized for political reasons. For example, in the U.S. economy a marginal income tax rate of 80% for those with incomes of over $100 million a year is economically feasible but may not be politically feasible.

The information requirements for control variables need to be kept relatively simple; trying to hand-tailor a bankruptcy penalty to fit each individual would require considerable detail. Thus Λ (rather than many parametersis) is far more reasonable in practise. Another example that is discussed below calls for simple rules that are information parsimonious for partially secured lending.

We may compare the GE models and the SMG games noted above. Setting aside government control, the GE and SMG models share many features Each representation has n agent types and m goods. The distribution of goods a_j^i is bounded by the conditions that for each $j = 1, ..., m$, $\sum_{i=1}^n a_j^i = a_j > 0$. There are apparently m prices, but as only ratios matter there is an extra price to be fixed by imposing a condition such as

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_j^i p_j = W$$

The GE model is described mathematically in such a manner that all details concerning the payment structure are irrelevant. All that matters is that the books balance at the end in accounting units (we could call this accounting money but apart from balancing the books it has no operational meaning). Money as an asset, credit and default and bankruptcy are not relevant.

We switch to an associated set of SMG because a full mechanism description of a playable game must be presented. We must be explicit about many items that are at best implicit in the GE model. A discussion has been given elsewhere of the concept of "minimal institution" [18]; we argue that even with trying to minimize the description of an exchange economy with a transactions technology there is an enormous proliferation of models.

In the construction of a SMG we may add a synthetic good whose supply is controlled by government or by an extra atomic, large "outside player". We call the synthetic good **outside money** or more simply **fiat money**. Its properties must be specified. It is a durable that does not depreciate physically and is valid in the discharge of all debts. Furthermore all debts are denominated in units of this good. The existence of a fiat money requires the extra dimension R denoted in the description of an SMG given above. The initial conditions require the specification of money holdings $s^i$of all individuals and the government bank B. This requires $n+1$ dimensions. As the bank is capable of creating its fiat money the B could be regarded as a fiction, but otherwise it may be regarded as a limit placed by the bank on the amount of central bank money that it can or will create, given that the monetary unit has been fixed.

Although touched upon in Section 2, we reemphasize here the **control variables available to the government** are $\tau^i, \Lambda, \Pi, \rho, \theta, B$.

The strategic choices of the individuals in the SMG are (b^i, q^i, d^i) the (b^i, q^i) are bids and offers forming prices as contrasted with the (x^i) for the GE where the x^i are purchases or sales at a given set of prices. The d^i are loans.

The Π^i in the SMG are the expectations of individuals who can believe what they like about what money will be worth at the day of settlement. One can invoke a rational expectations or noncooperative equilibrium solution as a way to produce their actions; but in a one period game there is no evidence on which to base one's learning. In an actual experimental game the referee can supply the terminal conditions and it is for the individuals to decide whether or not to believe the referee. This is tantamount to adding the variable Π to the government strategy. In an actual economy this is having the government supply an estimation of inflation or deflation and each agent utilizing it to form her estimate of $\Pi^i = \beta\mu^i\Pi$ this reflects the government forecast modified by each agent's individual valuation of the marginal worth of a unit of money to her at settlement.

4.5 A comment on expectations and hoarding

The expected value of terminal claims to money is a behavioral parameter associated with learning. This appears when we have a large number T of active periods followed by a settlement period. Here we have only a $T = 1$ period followed by settlement [9]. As a first approximation we may assume that all use the same expectation Π concerning the final or salvage value of fiat.

In the broad class of models described here it is possible, but unlikely that Π is high enough, (as in a deep deflation) hoarding becomes an optimal strategy. Unless supervision were sufficient, an individual might even borrow, hoard, conceal (such as a foreign unregistered account) and declare bankruptcy. We omit an analysis of this interesting possibility and confine our discussion to where the expected value of money is "small enough" to rule this out.

Generally the expected value of money to an individual is a function of history. Here the only history available is reflected in the initial conditions.

4.6 Some Conditions on the SMG

In the SMGs considered here we impose an elementary financial structure including a means of payment that enters the utility function only in the terminal conditions, when economic default occurs, or when a salvage value is attached to a terminal stock of money.

[9]A critical physical feature in modeling economic actvity using dynamic programming models is how long a time is a period meant to represent? When dealing with commodity trade a day may be a reasonable unit; but it is commodity and agent specific.

4.6.1 The default penalty and fiat money

The existence of a default penalty Λ imposed by government whose dimensions are (disutility of punishment)/Fiat, even though the penalty is by no means purely economic but is determined by an intermix of law, politics and economics this links fiat directly to preferences and hence to utility via the individual Λ^i.

4.6.2 An aside on uncertainty

Although we do not deal with exogenous uncertainty here, it must be noted that even with any random variable without complete markets the importance of the default mechanism is as a public good, that measures the joint willingness of a society to absorb losses from individual behavior (see [18]).

4.6.3 Money as a measure

Although a unit of money to one individual is not identical in utility terms to that of another individual it provides a unit that has the same purchasing power for all and can be transferred with ease and provides a monotonic measure of wealth.

4.7 Ownership and initial conditions

A modeling problem is faced when initial conditions specify the individual ownership of fiat money s^i. In the GE economy this money does not exist. In strategic dynamic economies it exists. Intuitively it appears to be some form of 'free energy' or liquidity. At an interior equilibrium it should be irrelevant, or at least invisible as the books always balance. But in a GE model there is either no future, or it is taken care of by rational expectations applied to an infinite horizon. There are two somewhat different attractive ways we can construct playable games of a finite economy specifying how to evaluate any fiat left over at the end:

1. We may assume that each agent i at the end of the game is required to return s^i units of fiat to the referee. It is as though government has given each individual i an interest free loan of the fiat for the duration of the game. If less is returned the agent is bankrupt. Any excess money held has a terminal value. The final

wealth condition is:

$$
\begin{aligned}
&u^i + \Pi^i \left(\sum_{j=1}^{m} p_j q_j^i - \sum_{j=1}^{m} b_j^i - \rho d^i \right)^+ \\
&+\Lambda^i \left(\sum_{j=1}^{m} p_j q_j^i - \sum_{j=1}^{m} b_j^i - \rho d^i \right)^-
\end{aligned} \tag{5}
$$

2. or we could consider that the s^i are owned unencumbered. This leads to a final wealth condition of

$$
\begin{aligned}
&u^i + \Pi^i \left(s^i + \sum_{j=1}^{m} p_j q_j^i - \sum_{j=1}^{m} b_j^i - \rho d^i \right)^+ \\
&+\Lambda^i \left(s^i + \sum_{j=1}^{m} p_j q_j^i - \sum_{j=1}^{m} b_j^i - \rho d^i \right)^-
\end{aligned} \tag{6}
$$

We limit our analysis to the second model as operationally fiat money is an asset owned by the individual who has it. Thus we are embedding the model into an infinite horizon world where history is given by the initial conditions, including the fact that individuals may own some fiat money and the future is implicit in their expectations.

4.8 The CE and TSNE solutions

Although we deal with a process model and a well defined game we look mathematically only at equilibrium properties in order to compare model solutions. We consider TSNEs and consider where they may or may not coincide with the relevant CEs of the general equilibrium model associated with the SMG.

4.9 A disclaimer on learning

Any experimentation also calls for a consideration of learning dynamics. There is a large, important and growing behavioral and game theory literature on learning [11], [12] that we do not discuss as it requires more informational complexity than is given here and is not directly relevant to the central theme of uniqueness studied here. In our analysis because we deal only with a single period in which all agents make a single simultaneous move, expectations can be treated as a parameter attached to a linear separable utility term. A justification for this is given below in warning about a common mistake concerning the role of a linear separable term in the utility function.

4.10 Two basic misinterpretations to be avoided

In proceeding from a static GE model to a multistage SMG model it is easy to fall into two basic conceptual traps. The first deals with the concept of a dynamic solution and equations of motion; the second concerns the existence of a linearly separable utility term in the representation of preferences. Both are dealt with independently here.

4.10.1 Solutions: Structure, behavior and equations of motion

In an evolving system it is easy to intermix structure and behavior as, for example, political behavior on one time scale can influence economic structure and hence economic behavior proceeding on a different time scale. Our construct for a SMG assumes the existence of individuals with preferences; but beyond that makes no behavioral assumptions whatsoever concerning how they will behave. The structure of an individual's payoff function for an economy that lasts actively for T time periods and then has a settlement date at $t = T + 1$ is given by:

$$U^i = \sum_{t=1}^{T+1} U_t^i = \tag{7}$$

$$\sum_{t=1}^{T} \beta^{t-1} \left\{ u_t^i + \beta \Lambda^i \left(s_t^i + \tau_t^i + \sum_{j=1}^{m} p_{jt} q_{jt}^i - \sum_{j=1}^{m} b_{jt}^i - \rho d_t^i \right)^- \right\} \tag{8}$$

$$+\beta^{T+1} \left\{ \begin{array}{c} \Pi_T^i \left(s_T^i + \tau_T^i + d_T^i - \sum_{j=1}^{m} b_{jT}^i \right)^+ \\ +\Lambda^i \left(s_T^i + \tau_T^i + \sum_{j=1}^{m} p_{jT} q_{jT}^i - \sum_{j=1}^{m} b_{jT}^i - \rho d_T^i \right)^- \end{array} \right\} \tag{9}$$

$$\text{where } s_{t+1}^i = s_t^i + \tau_t^i + d_t^i - \sum_{j=1}^{m} b_{jt}^i \text{ if positive,} \tag{10}$$

$$\text{or } s_{t+1}^i = G > 0 \text{ if negative} \tag{11}$$

$$\text{subject to } \left(s_t^i + \tau_t^i + d_t^i - \sum_{j=1}^{m} b_j^i \right) \qquad \mu_t^i \text{ for } t = 1, ..., T \tag{12}$$

$$\text{subject to} \left(s_t^i + \tau_t^i + \sum_{j=1}^{m} p_{jt} q_{jt}^i - \sum_{j=1}^{m} b_{jt}^i - \rho d_t^i \right) \qquad \mu_t \text{ for } t = 1, ..., T \tag{13}$$

$$\text{where } 0 \leq q_{jt}^i \leq a_{jt}^i, b_{jt}^i \geq 0, 0 \leq d_t^i \leq \theta \left(s_t^i + \tau_t^i \right) \tag{14}$$

The number $G > 0$ may be regarded as some small sum left to or given a bankrupt so that she is not totally destitute and is able to get back into the game. In actual bankruptcy rules where long run physical assets are involved these involve conventions such as leaving craftsmen or farmers some of their elemental tools of trade. In the one decision period model G is not relevant.

$\beta =$ the time preference discount, where we assume $0 \leq \beta \leq 1$.

We note in the equations above that as fiat had no intrinsic value, except as a constraint it yields no consumption worth in any active market period. At the end of active market time a salvage value for fiat is given.

4.10.2 Terminal conditions and separable utility

A basic misinterpretation of this work is to believe that it does nothing more than state that with the existence of a durable commodity money that enters the utility function as a linearly separable term automatically gives the conditions for the uniqueness of the CE. It looks as if we do that, but there is a subtle key difference between what we do and that interpretation that is present when we use only one active period and a settlement day. This difference is made far clearer if we consider several periods in a finite model, as is described immediately above. In particular the only use of separability until the terminal conditions is in the assumptions of the separability of the utility function from period to period. The links between the periods appear to be linear only when the cash flow constraints are all interior. Bankruptcy can occur any time after the first period and when it does it is evaluated at that period.

5 Solutions with the SMG

We utilize the TSNE solution, not because we believe in its sufficiency as a solution concept, but because if nothing else, it shows the problems that exist even at the simplest level of process model where there is only one decision period, i.e. $T = 1$

5.1 Government Lending Fiat

In order to introduce a money other than an accounting money we construct a strategic market game (SMG) model with an initial distribution of fiat s^i and borrowing of more fiat at an interest rate $\rho > 0$ set by the government. There are many institutional ways to do so. We suggest two. The first stresses credit evaluation, and control. We note it in an example. The second is simpler more laissez faire and utilized in our general treatment.

Partially secured lending: It is assumed that the government is required to lend fiat money to any individual who wishes to borrow with a credit limit for each such that up to a multiple θ of an individual's cash s^i,or $d^i = \theta s^i$ can be borrowed where

$$\sum_{i=1}^{n} \theta s^i \leq B$$

is the limit on the total and $\theta > 0$. This is a stylized partially secured lending.

The information and calculation requirements of this rule are minimal. All the bank needs to know is the total amount of cash outstanding. When a request to borrow arrives it requires that evidence is supplied by any requesting borrower i that she has s^i.

Unlimited lending: The bank stands to lend any amount. Neither θ nor B are relevant the naturally limiting factors are that the bank is passive and hence not pushing loans,and that loans cost a positive interest rate ρ [10].

A comment on the Lagrangians The GE and SMG models are mathematically highly different, but under certain special circumstances they yield the same resource distribution and relative prices. The equilibrium set for many of the SMG models will tend to involve some boundary, and will be reflected by the Lagrangians differing from those given by the CE.

5.2 Two Problems

We consider two problems, the first is relatively straightforward and is hardly surprising but still requires careful formal proof as is given in the next section. It shows that when we specialize from general equilibrium models to strategic market games with appropriate rational expectations initial and terminal conditions the welfare theorems hold. The second item deals implicitly with disequilibrium, even a simple example breaks into a morass of special cases as is illustrated in the appendix by direct computation of examples for the many different boundary conditions involved.

Comment: *With rational expectations settlement if* $\Pi = 0$ *prices are selected to match market prices such that all fiat is removed from the system.*

If the $\Pi > 0$ the SMG is still well defined but will generally have a boundary solution and fiat money left over at settlement.

6 Existence and Equivalence

In this section's analysis we are concerned with the case where there are an infinite number of individuals of each of the n types (the "$k = \infty$ case" from Section 2). We use x^i_{m+1} to denote agent type i's final holding of the fiat money, which can be either positive or negative. We assume that with the fiat money, type i's consumption set is extended from $\Re^m_+$ to $\Re^m_+ \times \Re$. Then, the extension of u^i to U^i as given in (6) can be written as

$$U^i(x^i, x^i_{m+1}) = u^i(x^i) + \Pi^i \max\{0, x^i_{m+1}\} + \Lambda^i \min\{0, x^i_{m+1}\} \tag{15}$$

for $(x^i, x^i_{m+1}) \in \Re^m_+ \times \Re$.

[10] This does not rule out the unlikely possibility of completely correlated massive borrowing that could introduce knife edge equilibria.

To prevent agents from purposely incurring debt by borrowing, we make the following assumption which also guarantees that U^i is concave whenever u^i is.

Assumption 1: $\Lambda^i \geq \Pi^i$ for $i = 1, 2, \cdots, n$.

As mentioned before,, a strategy for agent i is a list $(q_1^i, \cdots, q_m^i, b_1^i, \cdots, b_m^i, d^i)$ of quantities q_j^i of commodity j agent i offers to sell, b_j^i of the fiat money he bids for commodity j, $j = 1, \cdots, m$, and loan amount d^i. The strategy is feasible if

$$0 \leq d^i \leq \theta(s^i + \tau^i),\ 0 \leq q_j^i \leq a_j^i,\ b_j^i \geq 0,\ j = 1, \cdots, m, \tag{16}$$

$$s^i + \tau^i + d^i - \sum_{j=1}^{m} b_j^i \geq 0 \tag{17}$$

For each commodity $1 \leq j \leq m$, let $Q_j > 0$ denote the total offer for sale and $B_j > 0$ the total bid in units of the fiat money to buy commodity j by all consumers other than agent i. The bundle (x^i, x_{m+1}^i) that agent i receives with strategy (q^i, b^i, d^i) is given by

$$x_j^i = a_j^i - q_j^i + \frac{b_j^i}{p_j},\ 1 \leq j \leq m,\ x_{m+1}^i = s^i + \tau^i + \sum_{j=1}^{m} p_j q_j^i - \sum_{j=1}^{m} b_j^i - \rho d^i, \tag{18}$$

where

$$p_j = \frac{B_j + b_j^i}{Q_j + q_j^i},\ 1 \leq j \leq m. \tag{19}$$

Lemma 1 *Let $(Q_1, \cdots, Q_m, B_1, \cdots, B_m)$ be the profile of total offers and bids by agents other than agent i. Assume $Q_j > 0$ and $B_j > 0$ for all j. Then, the set of bundles agent i can achieve via his feasible strategies is a concave curve.*

Proof. Let (x^i, x_{m+1}^i) be an achievable bundle. Then, there exists a feasible strategy (q^i, b^i, d^i) that together with (x^i, x_{m+1}^i) satisfies (18). Thus, for $1 \leq j \leq m$,

$$p_j q_j^i - b_j^i = p_j(a_j^i - x_j^i) \tag{20}$$

and by (19),

$$p_j q_j^i - b_j^i = B_j - p_j Q_j. \tag{21}$$

Combining (20) and (21),

$$p_j = \frac{B_j}{a_j^i + Q_j - x_j^i}. \tag{22}$$

Now, by (20), (22), and the last equation (determination of x^i_{m+1}) in (18), we obtain

$$x^i_{m+1} = s^i + \tau^i + \sum_{j=1}^{m}(p_j q^i_j - b^i_j) - \rho d^i = s^i + \tau^i + \sum_{j=1}^{m} \frac{B_j(a^i_j - x^i_j)}{a^i_j + Q_j - x^i_j} - \rho d^i. \qquad (23)$$

The concavity of the curve composed of agent is achievable bundles follows from (23).
■

Observe that the strategy profile at which no one bids nor offers is always a NE. We consider active NEs in which there are non-zero offers and bids for some commodity. A common approach for proving the existence of such NEs is to consider ϵ-modified SMGs for $\epsilon > 0$, in which an outside agency places a fixed offer of $\epsilon > 0$ and a fixed bid of $\epsilon > 0$ for each of the m commodities. The existence of an active NE can be established by first showing the existence of active NEs for ϵ-modified SMGs and the considering the limits of such NEs as ϵ approaches to zero. This approach was adopted in Dubey and Shubik [5]. By (16)-(18), the strategy set of each consumer is compact and convex. With the concavity of agents' utility functions and Lemma 1, the same proof of existence of an active NE in Dubey and Shubik [5] can be applied to the present setting. For this reason, we summarize the existence result in Theorem 1 below without proof.

By (16), $s^i + \tau^i + d^i \leq s^i + \tau^i + \theta(s^i + \tau^i) = (1 + \theta)(s^i + \tau^i)$. Thus, agent i is moneyed (i.e., endowed with the fiat money) if and only if $s^i + \tau^i > 0$. We say that commodity j is desirable to agent i if u^i is increasing in x^i_j for any given quantities of the other goods.

Theorem 1 *Suppose (i) $s^i + \tau^i > 0$ for all i, (ii) each commodity is desirable for at least two agents, and (iii) each commodity is endowed by at least two agents. Then, there exists an active NE.*

Dubey and Shapley [4] considered the equivalence between the Walrasian equilibrium allocations of the general equilibrium model and the Nash equilibrium allocations of the strategic market game, when there is neither borrowing nor salvage value of the fiat money. They showed that the equilibrium allocations are equivalent if the per unit penalty is large enough relative to the marginal utilities of income, which are implied by the Walrasian equilibrium allocations, are Nash equilibrium allocations. We show that the equivalence continues to hold when the marginal utility of the salvage value is low but the per unit of penalty is high relative to the marginal utilities of income implied by the Walrasian equilibrium allocations. That is, when Assumption 1 is satisfied.

The following assumption is adapted from Assumption A' in Dubey and Shapley [4] for our present setting.

Assumption 2: For any $\delta > 0$, there exists a number $\kappa(\delta) > 0$ such that for all $j = 1, 2, \cdots, m$,

$$\frac{u^i(x^i + \Delta e_j) - u^i(x^i)}{\Delta} < \kappa(\delta)\Pi^i.$$

for all sufficiently small $\Delta > 0$ and $x \in \Re^m_+$ with $x_j \geq \delta$.

Essentially, Assumption 2 requires that for each type, the marginal utility of salvage money relative to the marginal utilities of other goods be bounded from below, unless consumptions of other goods are near zero levels.

For the following assumption, fix $\delta > 0$ such that $0 < \delta < \min_j \int a_j$ and let $T_1, T_2, \cdots, T_n$ denote the partition of T such that traders of T_i are identically characterized by $(\Re^m, u^i, a^i, s^i, \tau^i)$.

Assumption 3: For each type i,

$$\theta(s^i + \tau^i) \geq \frac{m^2 \kappa(\delta) \max_{1 \leq j \leq m} \int a_j}{\min_{1 \leq i \leq n} \mu(T_t)}.$$

Assumption 3 means that each type has enough money from initial endowment and borrowing. With Assumptions 1-3 in place, all the conditions required in Theorem 6 of Dubey and Shapley [4] are satisfied. Consequently, the following equivalence theorem holds.

Theorem 2 *Assume Assumptions 1-3 are satisfied. Then, (i) the prices and allocations of any CE are achieved at some NEs; (ii) the CE prices and allocations of $\mathcal{E}$ coincide with those achieved at open NEs.*

Using Theorem 2, we can establish the following equivalence between PO and NE allocations.

Theorem 3 *Assume Assumptions 1-3 are satisfied. Then, any given PO allocation of $\mathcal{E}$ can be a NE allocation for an expansion of $\mathcal{E}$ with suitable fiat money and taxes.*

Proof. Let $x = (x_1, x_2, \cdots, x_n)$ be a type-symmetric PO allocation for $\mathcal{E}$ with no fiat money endowments nor taxes. Then, by the second fundament theorem of welfare economics, there exists wealth transfers $\tau = (\tau^1, \tau^2, \cdots, \tau^n)$ such that x is a CE allocation with these wealth transfers (see Mas-Colell et. al [13]). Next, set $s^i = p \cdot a^i$ for all i, where p is the CE price vector with wealth transfers τ. Then, $((x^i, x^i_{m+1})^n_{i=1}, (p, p_{m+1}))$ with $p_{m+1} = 1$ is also a CE with agent i's utility function in (15), his commodity endowment a^i, fiat money endowment s^i, lump-sum tax τ^i, and with both Λ^i and Π^i given by the agent i's marginal utility of income (the Lagrangian multiplier) associated with the CE. Notice that $s^i + \tau^i > 0$ whenever $p \cdot x^i > 0$ for all i. By Theorem 2, x is a commodity allocation of an open SE. Conversely, an open SE allocation is a competitive equilibrium allocation from Theorem 2. Thus, it follows from the First Fundamental Theorem of welfare economics that the open SE allocation is Pareto optimal. ■

7 Uniqueness and borrowing

We turn to the problem of uniqueness in GE and SMG models. Suppose that the CEs are all normalized in such a way that the monetary wealth measure at each CE is the same. Then, we conjecture that the vectors of the Lagrange multipliers associated with the agents' utility maximization problems in the CEs do not Pareto dominate each other.

If the conjecture holds, then as shown in Qin and Shubik [14], for each CE of $\mathcal{E}$, a credit money can be instituted such that the CE is the unique CE for the extension of $\mathcal{E}$ that includes the credit money with the marginal utility of agent i associated with the CE of the original economy as default penalties. By Theorem 2, the open SEs of the SMG model with the same credit money and default penalties are also unique, as long as agents' salvage values of the credit money does not exceed their default penalties. We thus have:

Theorem 4 *Let (x, p) be a CE of $\mathcal{E}$ and for $i = 1, 2, \cdots, n$, let Π^i and s^i be the marginal utility of income and value of endowment of agent i associated with this CE. Assume Π^i satisfies Assumption 2 for i. Then, for $(\Lambda^i, \tau^i)^n_{i=1}$ and θ that satisfy Assumptions 1 and 3, the open NE of the SMG is unique and coincides with the CE.*

The three CE example in [16] is used in the examples below, as well as in an experimental game. The solution illustrates the selection problem. In this one shot game the referee announces its control variables, including its expectation on the future value of money. In the experimental game run by Huber, Shubik and Sunder [7] this helped to determine what was actually paid out to the subjects. The players knew the promised settlement conditions before they moved [11].

[11] In actuality the government may not know much about individual utilities or endowments but at

8 Volume of trade, cash flow and wealth

The financial volume of trade depends on the initial conditions involving the distribution of goods and services, the amount of money and credit, and the technology of trade. Here we have considered payments in fiat where the value of the fiat is supported by a given valuation in the next period. Given this there are natural extreme upper and lower bounds.to trade.

The absolute lower bound is 0, when the initial distribution is Pareto optimal and no trade is needed. The upper bound is 100%, when preferences are such that each sells all of her assets and buys other goods.

The **volume of trade is highly institutional** it depends heavily on the **current trading technology** and the speed of clearance. Were trade all instantaneous all goods would be liquid and the amount of government or bank money needed for trade would be zero.[12]

9 An SMG version of Shapley-Shubik's 3-NE Market

Here we consider an example of a GE economy with three CEs together with an associated SMG, and show how they are related.

9.1 The Original GE Model

The GE economy is that of Shapley and Shubik[16], a two-player, two-good economy which has three equilibria. Player 1 ("Ivan") is endowed with 40 units of Good #1 ("rubles"). Player 2 ("John") is endowed with 50 units of Good #2 ("dollars"). Their utility functions are $u^1(x,y) = x + 100(1 - e^{-y/10})$ and $u^2(x,y) = y + 110(1 - e^{-x/10})$ respectively, where x represents the consumption of Good #1and y the consumption of Good #2. The authors calculate Edgeworth's contract curve as essentially the set of Pareto optimal allocations (x,y) for Player 2 ("John")[13], for which $y = x+50-10\ln 110$,

any point in time it may estimate the future purchasing power of money.

[12]The formal study of liquidity requirements calls for the introduction of the details concerning the bridge financing of the Δt between when A has paid B, but B has not received the payment for use. There is a large literature on bridging fnance and the risks in transactions from Adam Smith on many references are given in [18],[15].

[13]It would seem more natural here to be defining x and y in terms of consumption for Player 1 (not Player 2), but we stick with the convention from the Shapley-Shubik paper.

$4.83 \leq x \leq 40$, plus some allocations where $x = 40$ (for any such allocation Player 1 receives $(40 - x, 50 - y)$). They also calculate the three equilibria for the economy, given in the following table:

	x	y	40-x	50-y	Ex Ratio	Payoff 1	Payoff 2
Initial	0	50	40	0	-	40.00	50.00
CE1	7.74	10.74	32.26	39.26	5.07:1	133.69	70.01
CE2	26.83	29.82	13.17	20.18	0.75:1	199.88	132.30
CE3	36.78	39.77	3.22	10.23	0.28:1	167.27	146.99

Table 1: The three equilibria for the GE economy

The first two columns with numbers are the initial endowments/equilibrium consumptions for Player 2, while the next two columns give those for Player 1..The "Ex Ratio" column gives the exchange ratio between the two goods at the three CEs. Finally, the last two columns give the utility payoffs to the players initially and at each of the three equilibria.

9.2 The SMG

Our "related SMG" is a special case of the general model from Section 2, in which there are two types of player and a continuum of each type. So $k = \infty$ and $n = 2$. We also assume that ρ is arbitrarily high, so that no individual ever wants to borrow, so that essentially there is no loan market. The utility functions and initial endowments are those of Shapley and Shubik.

To formally define our SMG we may start with the above GE economy. But now there are two *types* of players, a continuum of each type. The players are each infinitesimally small; hence no individual's actions can affect price (they are all "price takers). Within each type, all the players are identical. Hence players of the first type have the same utility function for the two goods, namely the above u^1, while all the Type 2 players have u^2. Also, the initial endowment of 40 units of Good #1is spread equally over the continuum of players of Type 1, and the 50 units of Good #2 is spread equally over the continuum of players of Type 2. Finally, each trader type has an aggregate initial endowment of 1000 in fiat money, again spread equally among the individual traders.

The strategies for the players are to make bids and put goods up for sale. Specifically, if i is a Type 1 player, his strategy is (b^i, q^i), where b^i is the amount of money he bids for Type 2 good and q^i is the amount of his own endowment of Type 1 good that he puts up for sale. Similarly, if j is a Type 2 player her strategy is a vector $(\overline{b}^j, \overline{q}^j)$, where this time $\overline{b}^j$is the amount she bids for Type 1 good and $\overline{q}^j$ the amount of her own Type 2 good put up for sale. Prices are then formed by $p = \frac{\int \overline{b}^j}{\int q^i}$for Good #1 and $\overline{p} = \frac{\int b^i}{\int \overline{q}^j}$ for Good #2. Each player's individual consumption (of the "opposite" good) can then be recovered by dividing his/her bid by the appropriate price.

Since the players in each type are identical, we restrict our search to "type specific noncooperative equilibrium (TSNE) outcomes", i.e. equilibria in which all players of the same type do the same thing and receive the same outcome. Hence we may identify a strategy for the Type 1 players by (b, q), where $b = \int b^i$and $q = \int q^i$. Similarly, we may write a strategy for the Type 2 traders by $(\bar{b}, \bar{q})$. Again, see [15] for details.

Initial holdings are $(40, 0, 1000)$ and $(0, 50, 1000)$. We are now ready to formulate the SMG. Traders of Type 1 solve the following optimization problem:

$$
\begin{array}{ll}
\max\limits_{b,q} & 40 - q + 100\left(1 - e^{-b/10\overline{p}}\right) + \Pi^1(1000 + pq - b) \\
s.t. & 1000 - b \geq 0 \qquad (\lambda) \\
 & b \geq 0, 0 \leq q \leq 40
\end{array}
$$

The objective function here is for Type 1 Traders' aggregate utility, as the sum of utility from the two traded goods plus the salvage value of the fiat at the end of the game (Π^1 is the per unit salvage value for fiat). Since the bids for the traders cannot exceed the initial cash on hand, we have a constraint (λ). Similarly, there is a constraint that the Type 1 Traders cannot put more up for sale than their initial endowment. This is the $q \leq 40$.

Similarly, the Type 2 Traders solve the following

$$
\begin{array}{ll}
\max\limits_{\bar{b},\bar{q}} & 50 - \bar{q} + 110\left(1 - e^{-\bar{b}/10p}\right) + \Pi^2(1000 + \overline{p}\overline{q} - \bar{b}) \\
s.t. & 1000 - \bar{b} \geq 0 \qquad (\overline{\lambda}) \\
 & \bar{b} \geq 0, 0 \leq \bar{q} \leq 50
\end{array}
$$

Finally, the prices, bids, and quantities sold must satisfy the balance conditions

$$
p = \frac{\overline{b}}{q} \text{ and } \overline{p} = \frac{b}{\overline{q}}.
$$

This model is solved, for all values of $\Pi^1$and Π^2, in the Appendix. Note that there are four cases

Case 1.1: This is where both player types have a lot of money. Thus their "cash flow constraints" $(\lambda)$and $(\overline{\lambda})$ are both loose. The meaning of 'a lot of money" is defined precisely in the Appendix.

Case 1.2: Trader Type 2 has a lot of money, but Trader Type 1does not. Here (λ) holds tightly while $(\overline{\lambda})$holds loosely. This holds if Π^2 is relatively large and Π^1small.

Case 1.3: Trader Type 1has a lot of money but Trader Type 2does not. Here $(\overline{\lambda})$holds tightly while (λ)holds loosely. This tends to hold if $\Pi^1$is relatively large and Π^2small.

Case 1.4: Now both trader types have little money. Here both $(\lambda)$and $(\overline{\lambda})$are both tight. It holds if both $\Pi^1$and Π^2 are small.

We are now able to point out the correspondence between this model and the original GE model. The contract curve in the GE Edgeworth box is the set of Pareto optimal allocations at which both players could be optimizing their utilities, but where supply does not necessarily equal demand. For the 3 CE economy the main part of the contract curve is given as $y = x + 50 - 10\ln 110$, where $x =$ Player 2's consumption of Good 1 and $y =$ Player 2's consumption of Good 2. In our SMG, to find the corresponding outcomes, we need to look where there are no binding monetary constraints (Case 1.1). Here Type 1 and Type 2 consume $(40 - 10\ln\left(\frac{11\Pi^1}{\Pi^2}\right), 10\ln\left(\frac{10\Pi^2}{\Pi^1}\right))$ and $(10\ln\left(\frac{11\Pi^1}{\Pi^2}\right), 50 - 10\ln\left(\frac{10\Pi^2}{\Pi^1}\right))$ respectively. So, if we let $x =$Type $2\prime s$consumption of Good 1and $y =$Type $2\prime s$consumption of Good 2,this is $y = 50 - 10\ln\left(\frac{10\Pi^2}{\Pi^1}\right)$and $x = 10\ln\left(\frac{11\Pi^1}{\Pi^2}\right)$. But then $y = 50 - 10\ln\left(\frac{10\Pi^2}{\Pi^1}\right) = 50 + 10\ln\left(\frac{\Pi^1}{10\Pi^2}\right) = 50 + 10\ln\left(\frac{11\Pi^1}{\Pi^2}\right) - 10\ln 11 - 10\ln 10 = 50 + x - 10\ln 110$. This is exactly the above Shapley-Shubik formula for the contract curve.

Next, the set of equilibria for the economy is the set of points on the contract curve for which supply equals demand, i.e. $pq = b$.[14] In the Appendix, for Case 1.1 we calculate the equilibrium formulas $p = \frac{1}{\Pi^1}$, $q = 10\ln\left(\frac{11\Pi^1}{\Pi^2}\right)$, and $b = \frac{10}{\Pi^2}\ln\left(\frac{10\Pi^2}{\Pi^1}\right)$. Hence $pq = b$ becomes $\frac{10}{\Pi^1}\ln\left(\frac{11\Pi^1}{\Pi^2}\right) = \frac{10}{\Pi^2}\ln\left(\frac{10\Pi^2}{\Pi^1}\right)$. Canceling the 10's and rearranging gives $\frac{\Pi^2}{\Pi^1}\ln\left(\frac{11\Pi^1}{\Pi^2}\right) = \ln\left(\frac{10\Pi^2}{\Pi^1}\right)$. Now let $\gamma = \frac{\Pi^2}{\Pi^1}$be the exchange ratio. Then this last equation is $\gamma\ln\left(\frac{11}{\gamma}\right) = \ln(10\gamma)$. As in [16] this has precisely three solutions, where $\gamma \cong 5.07$, $\gamma \cong .75$, and $\gamma \cong .28$.

For each of these equilibria, we observe that (unlike in the GE model, see Section

[14]This is "supply = demand" for Good #1; equivalently we could write $\overline{pq} = \overline{b}$ for Good #2.

2), it is NOT true that $p$an equilibrium vector implies that αp is also an equilibrium vector, for any positive α. In fact there is an upper bound on the α, due to the traders'cash flow constraints. For instance, for the second equilibrium above, where $\gamma = \frac{\Pi^2}{\Pi^1} = .75$, the cash flow constraint requires $b \leq 1000$, which is $\frac{10}{\Pi^2} \ln\left(\frac{10\Pi^2}{\Pi^1}\right) \leq 1000$, which is $\frac{1}{\Pi^2} \ln(7.5) \leq 100$. But $\overline{p}$ is equal to $\frac{1}{\Pi^2}$, so this gives $\overline{p} \leq 49.6$. And since $\frac{\Pi^2}{\Pi^1} = \frac{p_1}{p_2} = .75$, we have $p \leq 37.2$. These values of 37.2 and 49.6 we call the "top feasible prices" for the equilibrium; any nonnegative $(p, \overline{p})$ that is less than or equal to (37.2, 49.6) can serve as the equilibrium prices, so long as $\frac{p}{\overline{p}} = .75$.[15]

By similar calculations, we can calculate the top feasible prices for the first equilibrium as (129.1,25.5) and those for the third equilibrium as (27.2, 97.1).

Next, let us suppose Π^2 is equal to 1. Then our three equilibria require $p = 5.07$, $p = .75$, and $p = .28$ respectively, each with $\overline{p} = 1$. The value of Π^1must be $\frac{1}{p}$, which is .20, 1.33, and 3.57 respectively for the three NEs. Next, $q = 10 \ln\left(\frac{11\Pi^1}{\Pi^2}\right) = 7.73$, 26.86, and 36.71 respectively, and $\overline{q} = 10 \ln\left(\frac{10\Pi^2}{\Pi^1}\right) = 39.26$, 20.15, and 10.30. The volume of trade in this model is given by pq (for the Type 1 traders) plus $\overline{pq}$ (for the Type 2 traders), for a total of $pq + \overline{pq}$. Under the three equilibria, this turns out to be 78.48, 40.30, and 20.58 respectively.

The optimal values of the Lagrangian multipliers λ and $\overline{\lambda}$ are zero.

9.3 Cash Flow and Wealth

Before continuing with further SMG examples we return to the GE model to consider if there is any Mechanism in the CE formulation alone to select among equilibria. In order to consider this we enlarge our above calculations as follows

	p	$\bar{p}$	q	$\overline{q}$	pq	$b = \overline{pq}$	Net	% Vol
NE1	5.07	1	7.73	39.26	39.26	39.26	0	.31
NE2	.75	1	26.86	20.15	20.15	20.15	0	.50
NE3	.28	1	36.71	10.30	10.30	10.30	0	.17
1-TO-1	1	1	23.03	23.98	23.03	23.98	.96	.52

Table 2

The four rows above correspond to our three NEs, and to a case where $\Pi^1 = \Pi^2 = p = \overline{p} = 1$. This is not an equilibrium, but rather a case where $pq \neq b$. The column

[15]We could also have started with the Type 2 traders' cash flow constraint $1000 - \overline{b} \geq 0$, which would have given us the same bounds on p and $\overline{p}$.

marked "Net" is the inbalance, i.e the difference between pq and b. The last column "% Vol" is the "ratio of the money value of trade to the money value of all assets", which is the volume of trade divided by $40p + 50\overline{p}$. This is the percentage of liquidity required to stay in Case 1.1. The fifth and sixth columns contain the goods volume of trade for the two agent types. At each equilibrium, the money volumes are the same size for the two types. However we can distinguish among the equilibria by noting that generically there will be only one equilibrium with a **minimum volume of trade relative to the total worth of the resources** at those prices.[16] All that this says is that if the government is ingenious enough to pick the right money supply and consumer price forecast[17] it can pick out a single price level with minimal liquidity needs. In the example if $s = \bar{s} = 39.26$, there will be enough liquidity to handle all instances, and trade relative to nonmonetary wealth is minimal.

Before adding more structure we observe that by definition the volume of trade is less than or equal to total wealth [18]. The immediate interpretation of the volume of trade in a single period model of a monetary economy is as a cash flow constraint. The CE model is too abstract to pick this up. Although one can select an arbitrary price level of trade in accounting money the initial ownership of fiat is a datum and is connected with the utility function by both the expectation of positive value at settlement and by the disutility of a default penalty, but this does not hold here as borrowing is not considerd.

9.4 The Experimental Game

Huber, Shubik, and Sunder [7] conducted (an approximation of) the above game as an experiment in the "lab"[19] Players' moves were recorded in the four cases from Table 2, namely a) where $\Pi^1 = 1$ and $\Pi^2 = 5.07$, b) where $\Pi^1 = 1$ and $\Pi^2 = .75$, c) where $\Pi^1 = 1$ and $\Pi^2 = .28$, and d) where $\Pi^1 = 1$ and $\Pi^2 = 1$. In case a) and case c), the traders essentially behaved as they "should", playing their NE strategies. In case d), which is not an NE, the "Net" was about 1% of the volume of trade and so could not

[16]This does not provide any guidance to that equilibrium. In order to consider guidance, at least some form of process model is required, and this must be provided by equations of motion applied to the SMG.

[17]If lending is permitted, then a bankruptcy penalty is also required.

[18]In dynamics one has take care of wash sales (see [18]).

[19]The actual game played was slightly different from the game solved here inasmuch as the 1000 was deducted from payoffs made, but as this is a constant it does not influence the optimization if the solution does not involve bankruptcy. Furthermore, there were only five players of each tiype in the game, rather than the continuum in our model. .

be rejected by our theory. However, in case b) this percentage was about 3% (instead of the theory-predicting 0%). This seems to indicate that in this case, with $\Pi^1$and Π^2 so close to each other, that players might have acted as if they were the same, and this dominated CE or NE considerations [7]

In the experiment noted [7] even without permitting individual borrowing we were able to inroduce the possibility of default by reqiring that the initial supply of fiat be returned to the government at settlement and one would be penalized for failing to do so, but be given credit if one had more money. If Π and Λ were set equal there would be no discontinuity in the marginal worth of fiat at the origin. If the marginal penalty were larger than the marginal value of an extra purchase the lower bound on the price level without bankruptcy would sink below the upper bound established by Π.

We now have a playable game with the government having set the total money supply at s, named a default penalty Λ and supplied a prediction Π of the future purchasing power of money.

The rules are such as if government had supplied (possibly too little or too much) liquidity to all interest free, but requiring that it be returned at the end of the game. Its distribution could be done randomly (like a Milton Friedman helicopter).

It follows there is large class of games where all CEs are feasible without borrowing, all that is required is that the distribution of money and the expected terminal value for money are such that all cash flow constraints are met.

9.5 Basins of Attraction and Excess Demand

An open question is: can we offer a measure of a finite domain of influence or basin of attraction around each CE? Candidates for doing so are the maxima and minima of the excess demands. They can be calculated from $pq - b$ which may be expressed as

$$\frac{10}{\Pi^1} \ln\left(\frac{11\Pi^1}{\Pi^2}\right) - \frac{10}{\Pi^2} \ln\left(\frac{10\Pi^2}{\Pi^1}\right)$$

or this is the equivalent of finding the extrema of

$$\frac{\Pi^2}{\Pi^1} \ln\left(\frac{11\Pi^1}{\Pi^2}\right) - \ln\left(\frac{10\Pi^2}{\Pi^1}\right)$$

Set $\gamma = \Pi^2/\Pi^1$ taking the derivative wrt γ we obtain an approximate value in each interval of $\gamma = .46$ and $\gamma = 2.85$

The consumptions on the Pareto optimal surfaces are $(8.26, 15.26)$ for Type 1 and $(31.74, 34.74)$for Type 2 and $(26.49, 33.50)$ for Type 1 and $(13.51, 16.50)$for Type 2 [20].

As we do not address dynamics explicitly we cannot assume that the center CE which is unstable under a simple first order price dynamics is also unstable under other plausible dynamics (see [9]).

9.6 Examples with Lending

The variation of the money supply provides an obvious source of further control with the need to introduce conditions on the quantity and price of money.

In the Appendix (Section 11.2) we lay down an extension of the above SMG, in which banking is present..This parallels the banking models from [15], for a model with different utility functions. We have solved this model, and plan to present our results in a future paper..Here we remark that the solution with the Shapley-Shubik utility functions is a lot more complicated (with more cases), owing to the nature of the utility functions.

We can then extend this model to include strategic bankruptcy, much in the spirit of the model in Chapter 8 of [15].

9.7 A Comment on the Price Level

With fiat money, the question of the price level becomes complex. In the usual CE model the price level is homogeneous of order 0 or $(\cdot)$; with the SMG this is now no longer has to be the case. A finite bankruptcy penalty puts a lower bound on the price level and connects the monetary unit to utility and the amount of fiat plus credit sets an upper bound on price at equilibrium $[\cdot]$. If the expectations of the future value of money $\gamma^*\Pi$ are high enough for at least one individual there will be some individual whose policy involves touching a financial transaction constraint.

Even at this level of simplicity the absolute price level is not always determined but a continuum of knife edge equilibria may be obtained that do not survive perturbations[21].

[20] A highly different way is by investigating the point at which the core splits into seperate points. Kumar and Shubik [10] investigate the splitting of the core of this economy regarded as a cooperative game.

[21] Furthermore in an actual considerations such as the relative prices among many durables with different lifetimes appear to cut down on zones of knife edge equilibria.

10 Concluding Remarks

The GE model is extremely parsimonious in the number of parameters it utilizes. The SMG, even when presented with minimal mechanisms [22] requires many mechanisms to channel economic process. These mechanisms call forth many parameters. It is a natural property of the dynamics of trade to promote the existence of a money (for proof see [2] and [3]).The presence of a money and a powerful government enable the government to erect a control system over a fiat (or commodity) money supply. It however, raises questions concerning government knowledge of the system it controls. Government may easily set parameters that do not guide the system appropriately, but may need error correction.

This power in particular enables a government to resolve the possibility of the multiplicity of equilibria by changing constraints in the game thus resolving the deep problem of the presence of multiple fixed points in a GE system not by a mathematical analysis of the GE system, but by enlarging the problem to accommodate process in a larger context. The presence of a loan market and flexible money supply influence the adjustment [23].

A fundamental mathematical distinction between general equilibrium analysis with stress on equilibrium and the SMG models here is that while the former stresses statics and equilibrium, and deals with equalities; the latter stresses the role of money, inequalities and dynamics.

If the use of money involves any cost whatsoever there is an incentive for a society to minimize to that cost and this is sufficient to select among multiple outcomes. The ideal cash flow constraint is zero in the perfect clearinghouse in the sky. As technology improves the transactions need for an outside money decreases.

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11 APPENDIX

In constructing the SMG associated with a 2 player type, 2 commodity model there are 4 minimal model cases without banking and 16 cases with simple banking we illustrate the simplest no banking first. We utilize the specific model used above in Section 5.2.

11.1 Case 1: No Borrowing

There are two types of trader, with a continuum of agents of each type. Traders of the first type solve the following utility maximization problem

$$\begin{aligned} \max_{b,q} \quad & 40 - q + 100\left(1 - e^{-b/10\bar{p}}\right) + \Pi^1(1000 + pq - b) \\ s.t. \quad & 1000 - b \geq 0 \qquad (\lambda) \\ & b \geq 0, 0 \leq q \leq 40 \end{aligned} \tag{24}$$

The notation here is as follows. The decision variables for the Type 1 traders are $b =$ the amount of fiat bid per unit for Good #2 and $q =$ the amount of their own Good #1 put up for sale. The other quantities present are $p =$ the price per unit of Good #1, $\bar{p} =$ the price per unit of Good #2, and $\Pi_1 =$ the per unit salvage value ("worth") of fiat at the end of the game for Type 1 players.

The first order condition (FOC) with respect to b is

$$100\frac{1}{10\bar{p}}e^{\frac{-b}{10\bar{p}}} - \Pi^1 - \lambda = 0 \Rightarrow 10e^{-b/10\bar{p}} = \bar{p}(\lambda + \Pi^1) \tag{25}$$

where λ is the multiplier associated with the constraint (λ). With respect to q we have

$$-1 + \Pi^1 p = 0 \Rightarrow p = \frac{1}{\Pi^1} \tag{26}$$

Meanwhile, the optimization problem for the Type 2 traders is as follows:

$$\begin{aligned} \max_{\bar{b},\bar{q}} \quad & 50 - \bar{q} + 110\left(1 - e^{-\bar{b}/10p}\right) + \Pi^2(1000 + \bar{p}\bar{q} - \bar{b}) \\ s.t. \quad & 1000 - \bar{b} \geq 0 \qquad (\bar{\lambda}) \\ & \bar{b} \geq 0, 0 \leq \bar{q} \leq 50 \end{aligned}$$

New notations above are $\bar{b}$ = the amount of fiat bid by Type 2 players for Good #1, $\bar{q}$ = the amount of Good #2 put up for sale by Type 2 players, and Π^2 = the per unit salvage value of fiat at the end of the game for Type 2 players. The first-order conditions, with respect to $\bar{b}$ and $\bar{q}$ respectively, are

$$11e^{-\bar{b}/10p} = p(\bar{\lambda} + \Pi^2) \tag{27}$$

and

$$\bar{p} = \frac{1}{\Pi^2} \tag{28}$$

Finally, the markets for Good #1 and Good #2 must "balance", i.e. the amounts bid, put up for sale, and the price must be related by the relations

$$p = \frac{\bar{b}}{q} \text{ and } \bar{p} = \frac{b}{\bar{q}}. \tag{29}$$

There are four cases to consider

Case 1.1: Both player types have a lot of money. Another way to say that "both player types have a lot of money" is to assume that both Π^1 and Π^2 are large. This ensures that they are not motivated to bid a lot (because the fiat itself is valuable to them at the end of the game), and hence their cash flow constraints are loose. So the multipliers $\lambda = \bar{\lambda} = 0$,

Then (25) $\Rightarrow 10e^{-b/10\bar{p}} = \bar{p}\Pi^1 = \frac{\Pi^1}{\Pi^2} \Rightarrow e^{-\Pi^2 b/10} = \frac{\Pi^1}{10\Pi^2} \Rightarrow b = \frac{10}{\Pi^2}\ln\left(\frac{10\Pi^2}{\Pi^1}\right)$. And

$\overline{q} = \frac{b}{p} = 10 \ln \left(\frac{10\Pi^2}{\Pi^1} \right)$.

Similarly, (27) $\Rightarrow 11e^{-\overline{b}/10p} = p\Pi^2 = \frac{\Pi^2}{\Pi^1}$. So $e^{-\Pi^1\overline{b}/10} = \frac{\Pi^2}{11\Pi^1} \Rightarrow \overline{b} = \frac{10}{\Pi^1} \ln \left(\frac{11\Pi^1}{\Pi^2} \right)$. And $q = \frac{\overline{b}}{p} = 10 \ln \left(\frac{11\Pi^1}{\Pi^2} \right)$.

In general, the final consumption of the two goods for the two types are given by $(40 - q, \overline{q})$ for the Type 1 traders, and $(q, 50 - \overline{q})$ for the Type 2 traders. Here, this works out to be $(40 - 10 \ln \left(\frac{11\Pi^1}{\Pi^2} \right), 10 \ln \left(\frac{10\Pi^2}{\Pi^1} \right))$ and $(10 \ln \left(\frac{11\Pi^1}{\Pi^2} \right), 50 - 10 \ln \left(\frac{10\Pi^2}{\Pi^1} \right))$ respectively.

This case holds if

1) $b \leq 1000 \Rightarrow \frac{1}{\Pi^2} \ln \left(\frac{10\Pi^2}{\Pi^1} \right) \leq 100$.

2) $\overline{b} \leq 1000 \Rightarrow \frac{1}{\Pi^1} \ln \left(\frac{11\Pi^1}{\Pi^2} \right) \leq 100$.

3) $b, \overline{q} \geq 0 \Rightarrow \frac{\Pi^2}{\Pi^1} \geq \frac{1}{10}$

4) $\overline{b}, q \geq 0 \Rightarrow \frac{\Pi^1}{\Pi^2} \geq \frac{1}{11}$

5) $q \leq 40 \Rightarrow \ln \left(\frac{11\Pi^1}{\Pi^2} \right) \leq 4 \Rightarrow \frac{\Pi^1}{\Pi^2} \leq \frac{e^4}{11} \cong 4.963$

6) $\overline{q} \leq 50 \Rightarrow \ln \left(\frac{10\Pi^2}{\Pi^1} \right) \leq 5 \Rightarrow \frac{\Pi^2}{\Pi^1} \leq \frac{e^5}{10} \cong 14.841$

Example: $\Pi^1 = \Pi^2 = 1$. Then $b = 10 \ln 10$, $\overline{b} = 10 \ln 11$, $q = 10 \ln 11$, $\overline{q} = 10 \ln 10$, $p = \overline{p} = 1$.

*Comment:*This is the case where Π^1 and Π^2 are large, but roughly equal. The "largeness" guarantees that conditions 1) and 2) above hold. If they are large but NOT roughly equal, we have

Case 1.1A: Π^1, Π^2 large, $\frac{\Pi^1}{\Pi^2} \leq \frac{10}{e^5} \cong \frac{1}{14.841}$. Then $\overline{b}$, $q = 0$, $\overline{q} = 50$, $\overline{p} = \frac{1}{\Pi^2}$, and $b = \frac{50}{\Pi^2}$ The market for Good #1 breaks down because Type 2 traders have no incentive to bid anything (they value the fiat too much). Meanwhile, the Type 2 traders put all of their goods up for sale (in an effort to get even more money). Consumptions for the two trader types are $(40, 50)$ and $(0, 0)$.

Case 1.1B: Π^1, Π^2 large, $\frac{10}{e^5} \leq \frac{\Pi^1}{\Pi^2} \leq \frac{1}{11}$. Then $\overline{b}$, $q = 0$, $\overline{q} = 10 \ln \left(\frac{10\Pi^2}{\Pi^1} \right)$, $\overline{p} = \frac{1}{\Pi^2}$, and $b = \frac{10}{\Pi^2} \ln \left(\frac{10\Pi^2}{\Pi^1} \right)$. The market for Good #1 is still out, but now the Type 2 traders no longer put all of their goods up for sale. Consumptions are $(40, 10 \ln \left(\frac{10\Pi^2}{\Pi^1} \right))$ and $(0, 50 - 10 \ln \left(\frac{10\Pi^2}{\Pi^1} \right))$

Case 1.1C: Π^1, Π^2 large, $\frac{1}{11} \leq \frac{\Pi^1}{\Pi^2} \leq \frac{e^4}{11} \cong 4.963.$. This is the "$\Pi^1$ and Π^2 roughly equal" case from above.

Case 1.1D: Π^1, Π^2 large, $\frac{e^4}{11} \leq \frac{\Pi^1}{\Pi^2} \leq 10$. Then $q = 40$, $\overline{b} = \frac{40}{\Pi^1}$, $p = \frac{1}{\Pi^1}$, $\overline{p} = \frac{1}{\Pi^2}$, $\overline{q} = 10 \ln \left(\frac{10\Pi^2}{\Pi^1} \right)$and $b = \frac{10}{\Pi^2} \ln \left(\frac{10\Pi^2}{\Pi^1} \right)$. Consumptions are $(0, 10 \ln \left(\frac{10\Pi^2}{\Pi^1} \right))$ and

$(40, 50 - 10\ln\left(\frac{10\Pi^2}{\Pi^1}\right))$.

Case 1.1E: Π^1, Π^2 large, $\frac{\Pi^1}{\Pi^2} \geq 10$. Then $b, \overline{q} = 0$, $q = 40$, $p = \frac{1}{\Pi^1}$, and $\overline{b} = \frac{40}{\Pi^1}$. Now the market for Good #2 breaks down because the Type 1 traders have no incentive to bid. Consumptions are $(0, 0)$ and $(40, 50)$.

Case 1.2: Type 2 traders have "a lot of money"but Type 1 traders do not. From the same logic as the introduction to Case 1.1, this means the multiplier λ is positive while the multiplier $\overline{\lambda}$ is equal to zero. For this to happen typically Π^1must be much smaller than Π^2.

Note that we again have conditions (26) and (28) which are $p = \frac{1}{\Pi^1}$ and $\overline{p} = \frac{1}{\Pi^2}$. And then λ positive means constraint (λ) is tight, which is $b = 1000$. Hence $\overline{q} = \frac{b}{\overline{p}} = 1000\Pi^2$. Next, we again have (27) $\Rightarrow 11e^{-\overline{b}/10p} = p\Pi^2 = \frac{\Pi^2}{\Pi^1}$. So $e^{-\Pi^1\overline{b}/10} = \frac{\Pi^2}{11\Pi^1} \Rightarrow \overline{b} = \frac{10}{\Pi^1}\ln\left(\frac{11\Pi^1}{\Pi^2}\right)$, with $q = \frac{\overline{b}}{p} = 10\ln\left(\frac{11\Pi^1}{\Pi^2}\right)$. Then, using (25), we calculate $\lambda = \frac{10e^{-b/10\overline{p}}}{\overline{p}} - \Pi^1 = 10\Pi^2 e^{-100\Pi^2} - \Pi^1$. Consumptions for the two trader types are again given by the formulas $(40 - q, \overline{q})$ and $(q, 50 - \overline{q})$, which here work out to be $(40 - 10\ln\left(\frac{11\Pi^1}{\Pi^2}\right), 1000\Pi^2)$ and $(10\ln\left(\frac{11\Pi^1}{\Pi^2}\right), 50 - 1000\Pi^2)$.

The calculations above hold under the following conditions:

1) First and foremost, we need λ to be nonnegative, i.e $\Pi^1 \leq 10\Pi^2 e^{-100\Pi^2}$. This is the defining requirement for Case 1.2. For the rest of the conditions below, if they don't hold one must use a "subcase" as in Cases 1.1A - Case 1.1E above, where either one type of agent puts all of their goods for sale, or else one of the goods markets breaks down and nothing is sold.

2) $\overline{b} \leq 1000 \Rightarrow \frac{1}{\Pi^1}\ln\left(\frac{11\Pi^1}{\Pi^2}\right) \leq 100$.

3) $\overline{b}$, $q \geq 0 \Rightarrow \frac{\Pi^1}{\Pi^2} \geq \frac{1}{11}$.

4) $q \leq 40 \Rightarrow \ln\left(\frac{11\Pi^1}{\Pi^2}\right) \leq 4 \Rightarrow \frac{\Pi^1}{\Pi^2} \leq \frac{e^4}{11} \cong 4.963$

+5) $\overline{q} \leq 50 \Rightarrow 1000\Pi^2 \leq 50 \Rightarrow \Pi^2 \leq .05$

Example: $\Pi^1 = .001$ and $\Pi^2 = .01$. Then $p = 1000, \overline{p} = 100$, $b = 1000$, $\overline{b} = 10000\ln 1.1 \cong 953.1$, $q \cong .953$, $\overline{q} = 10$, and $\lambda \cong .0357$.

Case 1.3 Type 1 traders have "a lot of money" but Type 2 traders do not. Now $\lambda = 0$ and $\overline{\lambda} > 0$. Typically Π^2 is a lot smaller than Π^1.

We still have $p = \frac{1}{\Pi^1}$ and $\overline{p} = \frac{1}{\Pi^2}$. Next, $\overline{\lambda} > 0$ means that constraint $(\overline{\lambda})$ is tight, which is $\overline{b} = 1000$. Thus $q = \frac{\overline{b}}{p} = 1000\Pi^1$. Next, (25) $\Rightarrow 10e^{-b/10\overline{p}} = \overline{p}\Pi^1 = \frac{\Pi^1}{\Pi^2}$. So $e^{-\Pi^2 b/10} = \frac{\Pi^1}{10\Pi^2} \Rightarrow b = \frac{10}{\Pi^2}\ln\left(\frac{10\Pi^2}{\Pi^1}\right)$. And $\overline{q} = \frac{b}{\overline{p}} = 10\ln\left(\frac{10\Pi^2}{\Pi^1}\right)$. Finally, using (27), we have $\overline{\lambda} = \frac{11e^{-\overline{b}/10p}}{p} - \Pi^2 = 11\Pi^1 e^{-100\Pi^1} - \Pi^2$. Consumptions for the trader types are $(40 - 1000\Pi^1, 10\ln\left(\frac{10\Pi^2}{\Pi^1}\right))$ and $(1000\Pi^1, 50 - 10\ln\left(\frac{10\Pi^2}{\Pi^1}\right))$.

The calculations above hold under the following conditions:

1) First and foremost, we need $\overline{\lambda}$ to be nonnegative, i.e $\Pi^2 \leq 11\Pi^1 e^{-100\Pi^1}$. This is the defining requirement for Case 1.3. For the rest of the conditions below, if they don't hold one must use a "subcase" as in Cases 1.1A - Case 1.1E above, where either one type of agent puts all of their goods for sale, or else one of the goods markets breaks down and nothing is sold.

2) $b \leq 1000 \Rightarrow \frac{1}{\Pi^2} \ln\left(\frac{11\Pi^2}{\Pi^1}\right) \leq 100$.

3) b, $\overline{q} \geq 0 \Rightarrow \frac{\Pi^2}{\Pi^1} \geq \frac{1}{10}$.

4) $\overline{q} \leq 50 \Rightarrow \ln\left(\frac{10\Pi^2}{\Pi^1}\right) \leq 5 \Rightarrow \frac{\Pi^2}{\Pi^1} \leq \frac{e^5}{10} \cong 14.841$

5) $q \leq 40 \Rightarrow 1000\Pi^1 \leq 40 \Rightarrow \Pi^1 \leq .04$

Example: $\Pi^1 = .01$ and $\Pi^2 = \frac{1}{900} = .00\overline{11}$. Then $p = 100$, $\overline{p} = 900$, $b = 9000 \ln\left(\frac{10}{9}\right) \cong 948.24$, $b = 1000$, $q = 10$, $\overline{q} \cong 1.054$, and $\overline{\lambda} \cong .029$.

Case 1.4: Both trader types have little money. This is $\lambda > 0$ and $\overline{\lambda} > 0$. This corresponds to both $\Pi^1$and Π^2 being small (and roughly equal), because in that case traders have incentive to bid all of their money (because it won't be worth much at the end of the game).

Again we have $p = \frac{1}{\Pi^1}$ and $\overline{p} = \frac{1}{\Pi^2}$. The multipliers λ and $\overline{\lambda}$. being positive means that constraints (λ) and ($\overline{\lambda}$) hold tightly; hence $b = \overline{b} = 1000$. The balance conditions (29) then give $q = 1000\Pi^1$ and $\overline{q} = 1000\Pi^2$. Finally, we have $\lambda = \frac{10e^{-b/10\overline{p}}}{\overline{p}} - \Pi^1 = 10\Pi^2 e^{-100\Pi^2} - \Pi^1$ and $\overline{\lambda} = \frac{11e^{-\overline{b}/10p}}{p} - \Pi^2 = 11\Pi^1 e^{-100\Pi^1} - \Pi^2$. Consumptions for the two trader types are $(40 - 1000\Pi^1, 1000\Pi^2)$ and $(1000\Pi^2, 50 - 1000\Pi^1)$ respectively. The conditions in order for this case to hold are only that $\lambda \geq 0$ and $\overline{\lambda} \geq 0$, i.e. that $\Pi^1 \leq 10\Pi^2 e^{-100\Pi^2}$ and $\Pi^2 \leq 11\Pi^1 e^{-100\Pi^1}$. But note that these conditions can both hold only if $\Pi^1$and Π^2 are both small.

11.2 Case 2: A Model with Banking

An inkling of disequilibrium dynamics can be obtained in this one period model when the possibility of borrowing is introduced. Among the reasons for borrowing is strategic bankruptcy. It can take place to take advantage of regulation that sets bankruptcy penalties too low.

Case 2.1 Borrowing without Bankruptcy

In our first model with banking, we consider a model in which the traders are not allowed to go bankrupt. We define ρ as the exogenous rate of interest for loans in the economy. New decision variables for the trader types d and $\overline{d}$,, where $d =$ the amount of loan to the Type 1 traders, to be paid back at the end of the game, and $\overline{d}$ = the amount of loan to the Type 2 traders, to be paid back at the end of the game. The utility maximization problem for Trader Type 1 is

$$\begin{aligned}
\max_{b,q,d} \quad & 40 - q + 100\left(1 - e^{-b/10\bar{p}}\right) + \Pi^1(1000 - b + \frac{d}{1+\rho} + pq - d) \\
s.t. \quad & 1000 + \frac{d}{1+\rho} - b \geq 0 \qquad (\lambda) \\
& 1000 + \frac{d}{1+\rho} - b + pq - d \geq 0 \ (\mu) \\
& b, d \geq 0, 0 \leq q \leq 40
\end{aligned}$$

while for Trader Type 2 it is

$$\begin{aligned}
\max_{\overline{b},\overline{q},\overline{d}} \quad & 40 - \overline{q} + 110\left(1 - e^{-\overline{b}/10p}\right) + \Pi^2(1000 - \overline{b} + \frac{\overline{d}}{1+\rho} + \overline{pq} - \overline{d}) \\
s.t. \quad & 1000 + \frac{\overline{d}}{1+\rho} - \overline{b} \geq 0 \qquad (\overline{\lambda}) \\
& 1000 + \frac{\overline{d}}{1+\rho} - \overline{b} + \overline{pq} - \overline{d} \geq 0 \ (\overline{\mu}) \\
& \overline{b}, \overline{d} \geq 0, 0 \leq \overline{q} \leq 50
\end{aligned}$$

....

Note that each trader type now has an extra constraint ((μ) and ($\overline{\mu}$)) respectively) in their optimizations. These reflect the "no bankruptcy" requirement, i.e that when accounts are settled at the end of the game, the traders must have a nonnegative amount of money to their name.

With two constraints for each of the two trader types, there are now 4 multipliers which could each be positive or zero. Hence there are 16 cases to consider. We have done the analysis for all sixteen cases, and will present this in a future paper.