THE BENEFIT OF COLLECTIVE REPUTATION

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Abstract

We study a model of reputation with two long-lived firms that sell their products under a collective brand or under two different individual brands. Firms face a moral hazard problem because their quality investments are not observed. Investments can only be sustained due to reputational concerns. In a collective brand, consumers cannot distinguish between the two firms. We show that in the long run, this makes it harder to establish a good reputation because of the incentives to free-ride on the other firm's investments. But in the short run it mitigates the temptation to milk good reputation. Consequently, a collective brand can provide stronger incentives to invest in quality if firms are sufficiently impatient. We explain the connection between incentives and the type of industry in which the firms operate as captured by the underlying signal structure and consumers' prior beliefs. We discuss the relation to country-of-origin labelling, agricultural cooperatives, and other collective brands.

1 Introduction

Firms make substantial investments to build strong brands. The American Marketing Association defines a brand as "a name ... that identifies one seller's good ... as *distinct* from those of other sellers."[1] Sometimes, a number of firms sell their products under a shared name or a collective brand that carries a collective reputation shaped by the firms who use

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[1] https://www.ama.org/resources/pages/dictionary.aspx

the name. For example, a bottle of wine carries an appellation, such as Bordeaux or Champagne, which applies to many producers in the same region. Many lay consumers cannot distinguish among the names of individual producers and rely on appellations to make their purchase decisions. Country of origin labelling serves a similar function. For example, Volkswagen advertises "the power of German engineering" and Swiss watchmakers, even the ones with strong individual brands, emphasize that their watches are "Swiss made."

Both individual and collective brands are means to build a good reputation. When building reputation, a firm faces a moral hazard problem; its investment in quality is unobservable to current consumers, and the reputational return on its investment can only be collected in the future. In this paper, we study how sustaining reputation in a collective brand is different from that of an individual brands.[2]

At first glance, collective brands may seem like a bad idea. If several firms operate under one brand name, each firm has an incentive to free-ride on other firms' investments. Moreover, a firm's investment in its own quality has a weaker effect on the brand value of a collective brand because consumers are uncertain about the relationship between the collective brand's reputation and the specific firm they interact with. In other words, the "precision" of the signal that is generated by a firm's investment in quality is lower in a collective brand, which weakens the incentive to invest in quality.

Nevertheless, under some circumstances, a collective reputation can serve as a commitment device for investment in high quality. If a brand is very successful (possibly as a result of previous large investments), then a firm might be discouraged from additional investment because the returns from it become small. Such a firm might become complacent or rest on its laurels, so to speak. Analogously, if a brand develops a bad reputation (possibly as a result of no investment), then returns on investment are also low, and the firm might give up investment altogether. Collective brands can mitigate these "discouragement effects" faced by individual firms after very good or very bad histories by making extreme beliefs about the value of the brand less likely. We describe circumstances where this benefit of a collective brand outweighs the benefits of individual brands.

To compare the two branding regimes, we extend a model of reputation in the vein of Mailath and Samuelson (2001).[3] In our model, two (or more) long lived firms make investment decision over time. Firms' investments are unobservable to consumers. There

[2]Of course, in practical situations firms are endowed with features of both individual and collective brands. For example, Volkswagen has a strong individual brand, and at the same time belongs to the group of German auto makers. For simplicity, we abstract away from such hybrid situations and focus on pure collective and individual brands in order the present the difference between individual and collective brands in the starkest possible terms.

[3]Mailath and Samuelson (2001) consider the case of a single firm (and individual reputation). We consider two or more firms that may sell their products under an individual or a collective brand.

are two types of firms, competent and incompetent. Only the competent type has the option to make a costly investment. Consumers observe past quality levels, which are noisy signals of past investment decisions.

The key distinction between an individual and a collective brand lies in consumers' observation of past quality realizations. Consumers observe a firm-specific record under an individual brand, and a group-specific record under a collective brand. This has two implications. First, each signal produced by a collective brand is a noisier signal about each firm than an individual brand's signal. In particular, consumers remain uncertain about each firm's types because they are unaware of which firm produced the signal. Second, a collective brand generates more signals than an individual brand because each one of its members can produce a signal.

We focus on the most efficient equilibrium in which a competent type always makes an investment. We call this equilibrium the Reputational Equilibrium. We examine whether it is easier for an individual or a collective brand to sustain this reputational equilibrium. In this equilibrium, a firm's reputation is given by consumers' posterior beliefs about the competence of the brand's members. Consumers are willing to pay more for a good that is produced by a brand with better past outcomes. Therefore, the reputational equilibrium exists if and only if a competent type's expected return from investment is larger than the investment cost after each possible history.

An investment generates both short-run and long-run benefits. In the *short-run*, a firm may want to exploit its current reputation if it has already reached a very good reputation. In such a case, additional investment yields only a modest improvement to reputation, which may be insufficient to justify the cost. A collective brand can improve investment incentives in the short-run because its noisier signals prevent its reputation from becoming so good. Hence, a member of a collective brand is more motivated to contribute to its reputation. However, in the *long-run*, a member of a collective brand may be tempted to free-ride on efforts by other members of the brand. So, in the long-run, a collective brand provides less investment incentives than an individual brand. It follows that when short-run incentives are more important, then a collective brand provides stronger incentives to invest than an individual brand.[4]

Moreover, we show that a collective brand is more likely to thrive in markets that require specialized knowledge in order to produce high quality products, such as markets for expensive wine, watches, and cars. In such markets, an individual firm can quickly attain an excellent reputation as an individual brand, which is bad for incentives. This effect is especially pronounced if firms are very likely to possess the required specialized knowl-

[4]As shown in Section 4, in some circumstances this result holds even if firms are infinitely patient.

edge ex-ante, which is more likely in developed economies. If, however, the market consists mostly of incompetent firms who lack this specialized knowledge, as would be the case in a developing economy, then we show that individual brands provide stronger incentives for investment.

We show that the benefits from the additional commitment power that is provided by a collective brand can be large enough so as to induce a competent firm to form a collective brand with an incompetent firm. In such a case, the socially optimal branding regime coincides with firms' optimal choice, so no regulation is required. However, it is also possible that a competent firm would prefer an individual to a collective brand, even though the latter induces incentives to invest while the former does not. In such cases, regulation that promotes collective brands improves overall efficiency.

Collective reputation building is also relevant in other domains. Any good that is purchased online is essentially an "experience good" whose quality cannot be ascertained by consumers at the time of purchase (Nelson (1970)).[5] Nosko and Tadelis (2015) show that a consumer who has a bad experience with one seller in an online platform such as eBay or Amazon, is less likely to buy through that platform again, which is evidence of a "reputational externality" that sellers in the platform exert over one another. Such a reputational externality is characteristic of a collective brand. Yet another example for the reputational externality that is produced by collective reputation is provided by organizations that require their members to wear uniforms, such as the Police, military forces, Girl and Boy Scouts, etc. Uniforms foster the creation of a reputational externality among their wearers because they blur individual identities.[6]

The paper is structured as follows. The next section discusses the related literature. In Section 3 we present the model, define the equilibrium concept, and introduces the key distinction between an individual and a collective brand in terms of consumers' beliefs. In Section 4 we describe circumstances under which an individual or a collective brand provides stronger incentives for investments. In Section 5, we examine a competent type's brand formation decision, and consider whether it would want to form a collective brand with an incompetent firm. In Section 6 we present extensions of the basic model that allow for longer memory and more than two firms, respectively. All proofs are relegated to Appendices.

[5]Experience goods also include nondurables such as wine, durables such as appliances and cars, and many service providers such as lawyers, doctors, and mechanics.

[6]Stereotypes provide yet another example.

2 Related Literature

Our work is related to the theoretical economics literature on reputation in markets for experience goods, as well as to the literature on umbrella branding, country-of-origin and career concerns.

The idea that reputational concerns can help a firm to produce high quality even though consumers are unable to verify the quality of an experience good (Nelson (1970)) goes back to Klein and Leffler (1981). The subsequent literature has explored the implications of this argument and has argued that it must contend with two major difficulties: the first is that for it to be sustainable, reputation must generate profits, and it is not clear how this is possible in a competitive environments where profits are driven down to zero.[7] The second difficulty, which has been famously noted by Holmström (1999), is that in the long run, the firm would develop an excellent reputation for quality. Any observation of low quality would thus be attributed to bad luck and would therefore not affect prices, with the consequence that the firm's incentives to continue to exert the costly effort necessary to produce high quality would be destroyed.

Several elegant solutions to these difficulties were offered. Hörner (2002) noted that if consumers can observe the consumer bases of firms in the market, then a firm may be discouraged from producing low quality for fear of losing its consumer base (see also Fedele and Tedeschi (2014)). Mailath and Samuelson (2001) formulated the insight that individual reputation can be sustained if consumers' beliefs about the type of the firm are bounded away from one, as would be the case if the firm's type is drawn afresh in every period, in a way that is unobservable to consumers. In this paper, we assume instead that consumers have finite memories as in Moav and Neeman (2010) and Liu and Skrzypacz (2014). This allows us to solve for the threshold cost below which firms invest in quality, which is intractable in Mailath and Samuelson (2001)'s model.

Using this framework we are able to show that acting as a collective can help to sustain high reputation. The mechanism is related to the one studied by Bar-Isaac (2007) who considers an overlapping generations model in the moral hazard-in-teams (career concerns) framework developed by Holmström (1999). He shows that senior entrepreneurs who sell the firm in the next period have an incentive to exert effort and work with young juniors who themselves also need to build a good reputation.

Research that identifies the benefits of collective reputation is scarce. Tirole (1993) is probably the first who formalized an analytical model of collective reputation in context of

[7] See, e.g., Kranton (2003). In our model we abstract away from this difficulty by assuming that firms make take-it-or-leave-it price offers to consumers, but our results would continue to hold as long as firms capture at least some of the surplus that is generated by their transactions with consumers.

a large organization. In Tirole's model, a group's reputation is an aggregate of the reputations of the individual members of the group. As is the case in models with statistical discrimination, there can be different steady states equilibria, and in particular one with "low corruption" and another with "high corruption".[8] As in his model, we do not require a common trait of group members, but unlike in Tirole (1993) our focus is on the moral hazard problem and the "brand management" rather than on statistical steady state inferences. Thus, our model is more relevant for long-lived firms as decision makers rather than individuals being assigned different tasks.

More recently, Fishman et al. (2014) consider a two-period model in which an individual firm can only generate one signal per period. A collective brand that includes many firms can send many signals and so provide better information to consumers. This informational benefit outweighs firms' incentive to free-ride on other firms' investment efforts provided the number of brand members is not too large.[9] This model abstracts away from issues of commitment and dynamic trade-offs, which are the focus of our analysis. [10] Notably, unlike Bar-Isaac (2007), Tirole (1993), and Fishman et al. (2014), our focus is to compare collective to individual reputation building.

Collective reputation has also been studied in the context of umbrella branding in which an existing brand name is extended to a new product line, and thus the brand reputation is formed by performance of its multiple products. Wernerfelt (1988), Choi (1998), Cabral (2000), Miklós-Thal (2012), and Moorthy (2012) have examined the incentives that a monopolist has to signal quality by pooling reputation for different products. In a setting of moral hazard with consumers' perfect monitoring of product quality Andersson (2002) and Hakenes and Peitz (2008) show that umbrella branding always provides stronger investment incentives as one deviation puts both markets at risk. Yu (2017) examines the extent of risk sharing across product markets as a function of relatedness between markets and shows that independent branding can be a disciplinary device if the relatedness is too high. Others have considered settings where free-riding incentives are more pronounced. Zhang (2015) examines country-of-origin labeling. He shows that the ability to free-ride on other firms' quality investments implies that high quality firms have an incentive to dissociate themselves from the country-of-origin label, which in turn mitigates free-riding and can improve the reputation for the group.

[8] Levin (2009) extends Tirole (1993) by considering the case where the cost of high effort evolves stochastically over time.

[9] Indeed, Winfree and McCluskey (2005) claim that the large number of apple growers in Washington, contributed to the decline in the quality of Washington apples during the 1990s.

[10] Fleckinger (2016) considers collective reputation under Cournot competition where consumers only learn the average quality in the market. He studies the effect of the number of firms on welfare, and shows that quality is decreasing in the number of firms whereas quantity increases.

3 Model and Definitions

3.1 Model

Our model captures the following type of scenario:[11]

> Two drivers, Adam and Brian, work for New Haven Limo Services, a company that provides limousine services. Every day, a customer who needs the service calls the company and is (randomly) assigned to an available driver. The customer observes the reviews posted by the previous customers and pays to the driver before the service is provided. After the ride, the customer posts a review on the quality of the service on the company's website. A competent driver can improve the ride experience through the exertion of costly effort; an incompetent driver cannot.
>
> The limo service can decide whether to reveal or conceal the names of the drivers in the posted reviews. In the former case, customers can check the past records of individual drivers. So, each driver establishes an individual reputation. In the latter case, customers cannot distinguish between the two drivers' records. So, the drivers establish a collective reputation.

More formally, we consider a market with two firms that produce a vertically differentiated experience good, that can be of either good (G) or bad (B) quality, at zero cost. In every period, $t \in \{\dots, -1, 0, 1, \dots\}$, one short-lived consumer with unit demand arrives and is randomly matched with one of the firms.

Firms. Each firm is competent (C) with probability $\mu \in (0, 1)$, or incompetent (I) with probability $1 - \mu$. The two firms' types $\theta \in \{C, I\}$ are independently drawn from the same publicly known distribution. The two firms' realized types become known to the firms, but not to consumers. A competent firm that is matched with a consumer can make an investment at cost $c > 0$ to improve the quality of the good it produces in that period: investment yields a good quality (G) with probability π_H while non investment yields good quality with probability $\pi_L < \pi_H$. An incompetent firm cannot invest and produces good quality with probability π_L.

Consumers. Consumers do not observe the firms' investment decisions, but they do observe the quality of goods produced in the last two periods.[12] Consumers update their beliefs about the type of the firm they are matched with. After its investment decision,

[11] We thank Robert Zeithammer for suggesting this example.

[12] In Section 6.1, we extend the model to any finite number of periods of observation larger than two.

the matched firm makes a take-it-or-leave-it offer to the consumer.[13] The consumer either accepts or rejects the firm's offer and then leaves the market.

Payoffs. We normalize the payoff of a consumer who does not buy the good to 0. A consumer who buys the good at a price p receives a payoff of $1 - p$ if the good is of good quality, and $-p$ otherwise. A firm that sells in period t at price p_t enjoys a payoff of $v_t = p_t - c$ at t if it invests at t, and $v_t = p_t$ if it does not. A firm that does not sell in any given period obtains a payoff of 0 in that period. Firms discount their future payoffs by $\delta \in [0, 1)$.

Branding Regimes. In a collective brand, or if the two firms sustain their reputation collectively, consumers cannot distinguish between the identities of the two firms. This means that consumers obtain a signal about the collective in every period, regardless of which firm they are matched with. Thus, the set of relevant histories for a collective brand is $\mathcal{H}^{\text{col}} = \{G, B\}^2$. In contrast, if firms maintain an individual reputation or form individual brands, then consumers can distinguish between them. Consequently, consumers observe the quality produced in the last two periods by the firm they have been matched. Thus, the set of relevant histories for an individual brand is $\mathcal{H}^{\text{ind}} = \{G, B, \varnothing\}^2$ where $\varnothing$ represents a failure to match.

We denote a history at time t by $\mathbf{h}_t \equiv h_{t-2}h_{t-1} \in \mathcal{H}^b$ ($b \in \{\text{ind}, \text{col}\}$) where h_{t-n} denotes the quality of the good produced in period $t - n$. Notice that the matching process ensures that the two firms sell the same expected quantity under the two regimes, but at possibly different prices .

Equilibrium. We focus on *stationary equilibria* in which strategies depend only on the relevant histories specified above. A stationary equilibrium is defined by an investment and pricing strategy of firms, a purchasing strategy of consumers, and consumers' beliefs over the set of firm's types. For simplicity, we assume that consumers purchase the good when indifferent.

3.2 Beliefs and Signal Structure

Posterior beliefs. In the case of an individual brand, posterior beliefs given a history $\mathbf{h}_t \in \mathcal{H}^{\text{ind}}$ are given by the probability $\Pr^{\text{ind}}(C|\mathbf{h}_t)$ that the firm the consumer is matched with is competent.

In the case of a collective reputation or brand, posterior beliefs are given by a probability distribution over the pairs of types of the two firms. We denote the posterior belief that the two firms' types are $s \in \{C, I\}^2$ given history $\mathbf{h}_t \in \mathcal{H}^{\text{col}}$ by $\eta_s(\mathbf{h}_t)$. The posterior belief that

[13] This assumption implies that all surplus goes to the firm in equilibrium, which simplifies the analysis. Firms must receive some surplus for reputation to be desirable.

the matched firm is competent given a history $\mathbf{h}_t$ is

$$\Pr^{\text{col}}(C|\mathbf{h}_t) = \eta_{CC}(\mathbf{h}_t) + \frac{1}{2}(\eta_{CI}(\mathbf{h}_t) + \eta_{IC}(\mathbf{h}_t)). \tag{1}$$

The *reputation* of a brand – both individual and collective – corresponds to the two posterior beliefs $\Pr^{\text{ind}}(C|\mathbf{h}_t)$ and $\Pr^{\text{col}}(C|\mathbf{h}_t)$, respectively. In equilibrium, each player's strategy maximizes its payoffs given other players' strategies and beliefs. Posterior beliefs are derived from the realized histories and the firms' strategies by Bayes' rule whenever possible.

For most of the paper we focus on the stationary equilibrium in which competent firms invest in quality whenever they are matched with a consumer, after each and every history and independently of the other firm's type. We call this the **reputational equilibrium**. In such an equlibrium, upon observing a history $\mathbf{h}_t$, a consumer is willing to pay a price

$$p^b(\mathbf{h}_t) = \Pr^b(C|\mathbf{h}_t) \cdot \pi_H + (1 - \Pr^b(C|\mathbf{h}_t)) \cdot \pi_L, \tag{2}$$

where $b \in \{\text{ind}, \text{col}\}$. Thus, this is also the reputational equilibrium price.

The reputational equilibrium is *socially optimal* if (and only if)

$$\Delta\pi \equiv \pi_H - \pi_L \geq c, \tag{3}$$

which we assume to be satisfied throughout the paper.

The game also has other stationary equilibria. For example, a "no investment" equilibrium, in which a competent firm never invests in quality, always exists. We discuss other stationary equilibria in Section 5 where we discuss endogenous brand formation.

When we compare the incentives induced by collective versus individual brands, it is useful to focus on two types of signal structures that are easy to interpret and that highlight the benefit of collective reputation:

1. *"Exclusive knowledge" ($\pi_L = 0$, $\pi_H \in (0,1)$)*: In this case, a firm cannot produce a good outcome without making an investment. Consequently, the observation of good quality reveals competence. Such a signal structure fits industries in which some special technology or expertise is required in order to produce high quality products, such as in watches, automobiles, electronics, etc.

2. *"Quality control" ($\pi_H = 1$, $\pi_L \in (0,1)$)*: In this case, a competent firm that invests is guaranteed a good outcome. Consequently, observation of bad quality (in the reputational equilibrium) reveals incompetence. Such a signal structure fits industries in

which consistency is required in order to produce high quality products, such as in manufacturing or service industries.

Throughout the paper, we formulate all of our results for the case of exclusive knowledge, and mention the analogous results for the case of quality control in remarks. Note that by continuity, all of our results hold also in the cases where π_L is sufficiently close to zero and the value of π_H is held fixed (exclusive knowledge) and π_H is sufficiently close to one and π_L is held fixed (quality control), respectively.

4 Reputational Equilibrium

In this section, we derive necessary and sufficient conditions for the existence of the reputational equilibrium, which is also the most efficient equilibrium, under the two branding regimes. We show that a reputational equilibrium exists if and only if the investment cost c is smaller than or equal to a threshold cost $\bar{c}^b$, $b \in \{\text{ind}, \text{col}\}$, that depends on the branding regime. Then, we identify which branding regime sustains the reputational equilibrium for a larger set of costs c by comparing these two threshold costs. The branding regime with the higher threshold cost $\bar{c}^b$ is said to induce stronger incentives to invest.

4.1 Individual Brand

In a reputational equilibrium, a competent firm invests after every possible history. Therefore, in such an equilibrium, after every history $\mathbf{h}_t$ the consumer that is matched with the firm updates her posterior belief about the firm's type and computes her willingness to pay $p^{ind}(\mathbf{h}_t)$. The firm invests in quality after a history $\mathbf{h}_t$ if its expected return from investment, taking into account the effect of this investment on the consumers' future willingness to pay, is greater than its cost. The next proposition characterizes the threshold cost of investment above which investment is not worthwhile.

Proposition 1. *The reputational equilibrium exists for an individual brand if and only if the cost of investment c is such that*

$$c \leq \bar{c}^{ind} \equiv \min_{h_{t-1} \in \{G,B,\varnothing\}} \bar{c}^{ind}(h_{t-1})$$

where $\bar{c}^{ind}(h_{t-1})$ denotes the expected benefit from investment after history $\mathbf{h}_t = h_{t-2}h_{t-1}$.

The function $\bar{c}^{ind}(h_{t-1})$ is given by

$$\begin{aligned}\bar{c}^{ind}(h_{t-1}) \equiv\ & \tfrac{\Delta\pi}{2}\cdot\delta\cdot\Big[\big(p^{ind}(h_{t-1}G) - p^{ind}(h_{t-1}B)\big) + \\ & \delta\cdot\textstyle\sum_{h_{t+1}\in\{G,B,\varnothing\}} Pr(h_{t+1})\cdot(p^{ind}(Gh_{t+1}) - p^{ind}(Bh_{t+1}))\Big]\end{aligned} \tag{4}$$

where $Pr(h_{t+1})$ denotes the probability distribution of the outcome realized in period $t+1$ ($Pr(G) = \frac{\pi_H}{2}$, $Pr(B) = \frac{1-\pi_H}{2}$, and $Pr(\varnothing) = \frac{1}{2}$).

Notice that the threshold cost $\bar{c}^{\text{ind}}$ is the sum of expected short-run and long-run price premia that arise from investment, as explained below. It is important to note that all these price premia can be explicitly expressed as a function of the parameters of the model, which include the prior belief μ, and the probabilities π_H and π_L.[14] We relegate the explicit expression to the Appendix A because it is lengthy and not insightful in itself.

The firm's investment in period t increases the probability of producing a good outcome at t and this will be observed by the consumer that is matched with the firm in the next two periods $t+1$ and $t+2$. Upon observing $h_t = G$, such a consumer would be willing to pay more than if it observed $h_t = B$. So, the threshold cost is given by the sum of expected price premiums in the following two periods. The differences in expected price premiums in periods $t+1$ and $t+2$ induce *short-run* and *long-run* incentives to invest, respectively.

In the short-run, a consumer that is matched with the firm in period $t+1$ observes a history $\mathbf{h}_{t+1} = h_{t-1}h_t$. So, by investing in period t, the firm enjoys a price premium $p^{ind}(h_{t-1}G) - p^{ind}(h_{t-1}B)$. This premium is small if the firm has a very good or very bad reputation following the history h_{t-1}. For example, in the exclusive knowledge environment ($\pi_L = 0$, $\pi_H \in (0,1)$, $\mu \in (0,1)$), following history $h_{t-1} = G$, a consumer's posterior belief is updated to $\Pr^{ind}(C|h_{t-1}h_t) = 1$. T hus, for $h_{t-1} = G$, the short-run price premium vanishes, or $p^{ind}(h_{t-1}G) - p^{ind}(h_{t-1}B) = 0$.[15] This illustrates the difficulty of inducing a commitment to invest through short-run incentives for an individual brand. An individual brand can develop a very good reputation through investment, but it is then tempted to exploit its reputation.

In the long-run, the consumer that is matched with the firm in period $t+2$ no longer observes the original history ($\mathbf{h}_t$). Instead, she observes $\mathbf{h}_{t+2} = h_t h_{t+1}$. In the reputational equilibrium a competent type invests in all periods following t if matched with a consumer. So, an investment at t also generates a long-run price premium $p^{\text{ind}}(Gh_{t+1}) - p^{\text{ind}}(Bh_{t+1})$. If h_{t+1} is equal to G with a high probability, then the firm would be tempted to rely on

[14]This is because reputational equilibrium probabilities and prices can be explicitly calculated for every history. For example, $\Pr^{\text{ind}}(C|GG) = \frac{\mu\pi_H^2}{\mu\pi_H^2+(1-\mu)\pi_L^2}$ and then $p^{\text{ind}}(GG)$ can be calculated using (2). The other probabilities and prices can be similarly calculated.

[15]More specifically, by equation (2), $p^{ind}(GG) - p^{ind}(GB) = \pi_H \cdot (\Pr^{ind}(C|GG) - \Pr^{ind}(C|GB)) = 0$.

its future equilibrium investments, which would hurt its incentives to invest. However, in the case of an individual brand, the firm would not be matched with a consumer in period $t+1$ with probability $\frac{1}{2}$, and in this case $h_{t+1} = \varnothing$. This long-run consideration may provide sufficient *discipline* for an individual brand to invest at t.

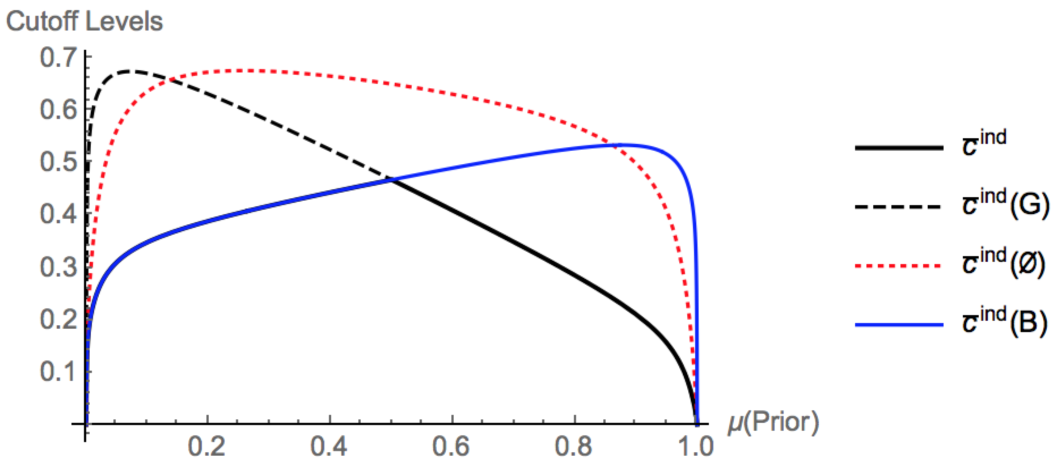


Figure 1: The Cutoff Levels for Independent Branding for $\pi_H = 0.975,\ \pi_L = 0.025,\ \delta = 0.9$

To completely characterize the threshold $\bar{c}^{ind}$, we need to take the minimum of $\bar{c}^{ind}(h_{t-1})$ over all histories $h_{t-1} \in \mathcal{H}^{ind}$. Figure 1 depicts $\bar{c}^{ind}(h_{t-1})$ as a function of the prior probability that a firm is competent, μ. As expected, the threshold vanishes at $\mu = 0$ and $\mu = 1$ because in these cases consumers' beliefs are unaffected by observed history so the price premiums associated with investment are zero. Obviously, in these cases the firm cannot be induced to invest.

Figure 1 also depicts the history on which $\bar{c}^{ind}$ is attained. It shows that for a large μ, $\bar{c}^{ind} = \bar{c}^{ind}(G)$, that is, the firm faces the weakest incentive to invest after a good outcome. This is because observation of a good outcome pushes posterior beliefs further up, which tempts the firm to milk its good reputation. For a low value of μ, $\bar{c}^{ind} = \bar{c}^{ind}(B)$ because given a pessimistic prior, the observation of a bad outcome pushes posterior beliefs further down, which discourages the firm from investment because it doesn't change beliefs sufficiently anyway.

The next Lemma shows that this observation is true also more generally.

Lemma 1. *If μ is sufficiently large, or*

$$\mu \geq \frac{\pi_L(1-\pi_L)}{\pi_H(1-\pi_H) + \pi_L(1-\pi_L)}$$

then $\bar{c}^{ind} = \bar{c}^{ind}(G)$*. Otherwise,* $\bar{c}^{ind} = \bar{c}^{ind}(B)$*.*

4.2 Collective Brand

If the two firms form a collective brand, then consumers observe the performance history of the collective brand $\mathbf{h}_t \in \mathcal{H}^{col} = \{G, B\}^2$ without being able to distinguish whether it was produced by the particular firm with which they have been matched, or the other firm in the collective brand. Compared to an individual brand, the history of a collective brand provides a noisier signal about firms' types, but unlike in the case of an individual brand in which a firm may fail to match and produce an outcome in any given period, a collective brand produces an outcome or signal is produced in every period. Specifically, the consumer forms beliefs over the types of the two firms, $s \in \{C, I\}^2$, denoted $\eta_s(\mathbf{h}_t)$, which allow her to compute beliefs about the competence of the firm that are given by:

$$\Pr^{col}(C|\mathbf{h}_t) = \hat{\eta}_{CC}(\mathbf{h}_t) + \frac{1}{2}(\hat{\eta}_{CI}(\mathbf{h}_t) + \hat{\eta}_{IC}(\mathbf{h}_t)).$$

Then consumer then determines her willingness to pay, $p^{\text{col}}(\mathbf{h}_t)$, according to equation (2).

The general approach for the characterization of the reputational equilibrium under a collective brand is similar to the approach for individual brand. We state the result for a collective brand in the following proposition.

Proposition 2. *A reputational equilibrium exists for a collective brand if and only if the cost of investment* c *is such that*

$$c \leq \bar{c}^{col} \quad \equiv \quad \min_{h_{t-1} \in \{G,B\},\, \theta \in \{C,I\}} \bar{c}^{col}(h_{t-1}, \theta)$$

where $\bar{c}^{col}(h_{t-1}, \theta)$ *denotes the expected benefit from investment after history* $\mathbf{h}_t = h_{t-2}h_{t-1}$ *if the other firm is of type* $\theta \in \{C, I\}$ *and is given by*

$$\begin{aligned} \bar{c}^{col}(h_{t-1}, \theta) \equiv \; & \tfrac{\Delta\pi}{2} \cdot \delta \cdot \big[\big(p^{col}(h_{t-1}G) - p^{col}(h_{t-1}B) \big) + \\ & \delta \cdot \textstyle\sum_{h_{t+1} \in \{H,G\}} Pr(h_{t+1}|\theta) \cdot (p^{col}(Gh_{t+1}) - p^{col}(Bh_{t+1})) \big], \end{aligned} \tag{5}$$

where $Pr(h_{t+1}|\theta)$ *denotes the probability distribution over the realized outcome in period* $t+1$*. If the other firm is competent, then* $Pr(G|C) = \pi_H$ *and* $Pr(B|C) = 1 - Pr(G|C)$*. If it is incompetent, then* $Pr(G|I) = \frac{\pi_H + \pi_H}{2}$ *and* $Pr(B|I) = 1 - Pr(G|C)$*.*

As with an individual brand, the existence of a reputational equilibrium for a collective brand is characterized by a threshold rule. The difference between the two cases stems from the fact that in the case of a collective brand the investment incentives of a competent firm depend on the other firm's investment. Accordingly, the threshold cost $\bar{c}^{col}$ is the minimum of $\bar{c}^{col}(h_{t-1}, \theta)$ over the history $h_{t-1} \in \{G, B\}$ and the other firm's type, $\theta \in \{C, I\}$, which is unobserved by consumers. As in the case of an individual brand, the function $\bar{c}^{col}(h_{t-1}, \theta)$ can be expressed in terms of the primitives of the model, but since the resulting expression is long, we relegate it to the Appendix A.

In the short-run, a competent firm in a collective brand expects a price premium of $p^{col}(h_{t-1}G) - p^{col}(h_{t-1}B)$ from investment that depends on the consumers' prior belief μ, and probabilities π_H and π_L. For example, in the case of exclusive knowledge described above ($\pi_L = 0$, $\pi_H \in (0, 1)$, and $\mu \in (0, 1)$), upon observation of an outcome $h_{t-1} = G$, the consumer learns that one firm in the collective is competent, but the type of the other firm remains unknown. This implies that the firm has an incentive to invest even after a good outcome in order to improve its reputation. In other words, $p^{col}(h_{t-1}G) - p^{col}(h_{t-1}B) > 0$.[16] That is, consumers' limited information about individual firms within the collective brand mitigates each firm's short-run Moral Hazard problem.

In the long-run, the firm's investment in period t can contribute to its reputation in period $t+2$. The price premium that is generated by investment is given by $p^{col}(Gh_{t+1}) - p^{col}(Bh_{t+1})$. A collective brand produces an outcome in every period, regardless of which firm a consumer visits. So, a firm may *free-ride* on its own as well as on the other firm's future investment. This results in weaker long-run incentives to invest in a collective brand compared to an individual brand.

Notice that an individual brand faces a more severe commitment problem in the short-run, while a collective brand faces a bigger problem in the long-run. This tradeoff plays a central role in the comparison presented in the next subsection.

Figure 2 depicts the expected return from investment $\bar{c}^{col}(h_{t-1}, \theta)$ for each history $h_{t-1} \in \{G, B\}$ and type $\theta \in \{C, I\}$ of the other firm. The solid line represents the threshold cost $\bar{c}^{col}$, which is given by the minimum of $\bar{c}^{col}(h_{t-1}, \theta)$ over $h_{t-1} \in \{G, B\}$ and $\theta \in \{C, I\}$, as a function of μ.

If consumers' prior beliefs are very optimistic, then a competent member firm in a collective brand faces a commitment problem because it has a reputation that is good enough to exploit. This commitment problem is more severe after a good history $h_{t-1} = G$ when the firm expects the other firm to invest in the future ($\theta = C$). Thus, for a large value of μ, $\bar{c}^{col} = \bar{c}^{col}(G, C)$.

[16] Specifically, $p^{col}(GG) - p^{col}(GB) = \frac{3\mu+1}{2\mu+2} - \frac{1}{2} > 0$ and $p^{col}(BG) - p^{col}(BB) = \frac{1}{2} - \frac{\mu(1-\mu)}{2\mu^2-6\mu+4} > 0$.

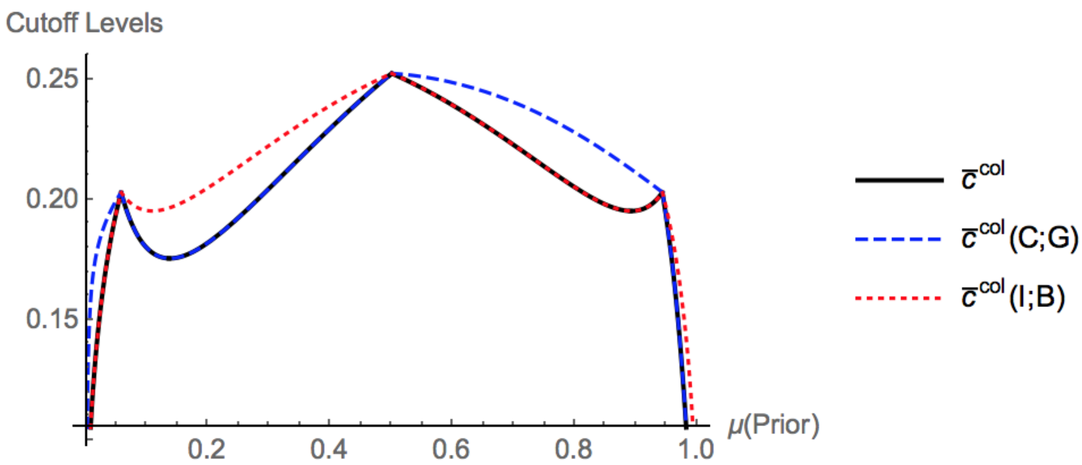


Figure 2: The Cutoff Levels for Collective Branding for $\pi_H = 0.975,\ \pi_L = 0.025,\ \delta = 0.9$

If μ is small, then the firm does not invest because it becomes discouraged. This discouragement becomes more severe after a bad history $h_{t-1} = B$ if the firm expects the other firm no not invest in the future ($\theta = I$).

Lemma 2. *For μ close to 1, $\bar{c}^{col} = \bar{c}^{col}(G, C)$, and for μ close to 0, $\bar{c}^{col} = \bar{c}^{col}(B, I)$.*

4.3 Comparing Individual and Collective Brands

In this section, we examine the conditions under which a collective brand sustains the reputational equilibrium on a larger set of investment costs than an individual brand. Other equilibria are discussed in the next section. For simplicity, we focus on the case of exclusive knowledge where $\pi_L = 0$ and $\pi_H \in (0, 1)$. In this case, a good outcome G reveals competence, which allows us to derive results that are easy to interpret.

The next proposition shows that in the case of exclusive knowledge, a collective brand sustains the reputational equilibrium for a larger set of costs than an individual brand if the discount factor δ is not too large. This is because as explained in the previous subsection, a collective brand provides stronger short-run incentives than individual reputation.

Proposition 3. *Suppose that $\pi_L = 0$ and $\pi_H \in (0, 1)$. There exists a threshold discount factor $\overline{\delta} \in [0, 1]$ such that $\bar{c}^{col} > \bar{c}^{ind}$ if and only if $\delta < \overline{\delta}$. Moreover, if π_H is sufficiently large, then $\overline{\delta} = 1$, or $\bar{c}^{col} > \bar{c}^{ind}$ for all $\delta \in [0, 1)$.*

The fact that a collective brand induces stronger incentives to invest for all discount factors $\delta < 1$ if π_H is sufficiently large is due to the fact that a higher π_H implies that signals are more accurate, which strengthens the incentive to milk reputation. We verify in simulations that the critical threshold $\overline{\delta}$ is increasing in π_H.

The magnitude of the short-run benefit that is provided by a collective brand depends critically on the prior beliefs μ. In particular, in the case of exclusive knowledge, this magnitude is larger if μ is large, or consumers are optimistic about firms' competence. This is formalized in the following proposition.

Proposition 4. *Suppose that $\pi_L = 0$, $\pi_H \in (0, 1)$, and δ is not too large. Then, if μ is sufficiently close to 1, then $\bar{c}^{col} > \bar{c}^{ind}$; if μ is close to 0, then $\bar{c}^{col} < \bar{c}^{ind}$.*

The intuition for this result is as follows. Recall that for a large μ, consumers have optimistic prior beliefs about each firm. This implies that an individual brand is more tempted to exploit its reputation. However, consumers remain relatively more uncertain about a collective brand because even after observation of a good outcome it is still possible that the other firm in the collective is incompetent. Therefore, a firm in a collective brand is relatively more motivated to invest.

If μ is small, then firms are concerned with building up reputation. This takes place over time, which implies that firms' long-run incentives become more important. Consequently, individual reputation induces stronger incentives to invest.

Remark 1. *(Quality control) In the case of quality control ($\pi_H = 1$ and $\pi_L \in (0, 1)$), a competent always produces a good outcome as long as it makes an investment. So, given the reputational equilibrium, a bad outcome reveals the firm's incompetence. In this setting, we can show that 1) $\bar{c}^{col} > \bar{c}^{ind}$ if δ is not too large and μ is sufficiently close to 0, and 2) $\bar{c}^{col} < \bar{c}^{ind}$ if μ is sufficiently close to 1. In this environment, after producing bad outcomes, an individual brand's reputation plunges, which discouraged the firm from further investing. However, a bad outcome for a collective brand is noisier information, as it proves incompetence of one firm but the type of the other firm remains uncertain. As a result, a collective brand's reputation can be somewhat recovered by additional investments. In fact, the insight that individual reputation provides stronger long-run incentives as well as numerical simulations suggest this result holds for any signal structure.*[17]

Figure 3 depicts the threshold costs for individual and collective reputation $\bar{c}^{ind}$ and $\bar{c}^{col}$, respectively. The higher is the threshold cost, the stronger is the incentive to invest. Figure 3 shows that for μ close to 1, a collective brand dominates individual brands, while the opposite is true for μ close to 0. Proposition 3 implies that a collective brand is less attractive when δ is large, but the discount factor used in Figure 3 is $\delta = 0.9$, which shows that a collective brand dominates individual brands for a rather large set of parameter values.

[17] See Online Appendix B.2. for a formal result for quality control case and its proof.

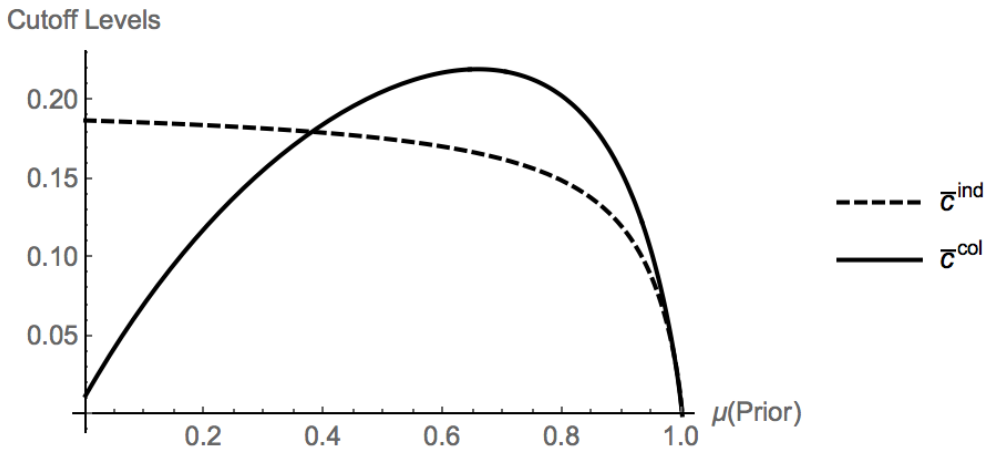


Figure 3: Comparison of Cutoff Levels for $\pi_L = 0$, $\pi_H = 0.93$, $\delta = 0.9$

Our observations seem to be consistent with observed practice. The parameter μ may be interpreted as the *baseline reputation* of firms in the market, industry, or country. It is reasonable to assume that firms in developed economies would have a better baseline reputation than those in developing economies. Furthermore, an exclusive knowledge environment describes industries in which production requires an advanced technology or expertise, such as advanced electronics, automobiles, watches, etc. In such industries, Proposition 4 implies that collective brands would thrive in developed economies, but less so in developing ones. Indeed, car manufacturers in Germany often emphasize their country of origin. In contrast, in countries with lower baseline reputations, such as China and South Korea, firms try to develop their individual brands, and sometimes even detach these individual brands from their country of origin (see also Zhang (2015)).

5 Brand Formation

So far, we have examined each brand regime as exogenously given. This may be realistic in some applications. For example, a country might require each local manufacturer to label its country of origin. And producers of wine, cheese, and coffee may become part of an appellation that is determined by their geographical location. However, in other examples it is a firm's strategic decision whether develop an individual or a collective brand. In this section, we examine this decision.

The reason we cannot simply apply the results of the previous section is that the choice between an individual and a collective brand provides information about firms' types. For example, a competent firm may form a collective brand if and only if the other firm is also competent. Moreover, from a welfare perspective, this analysis may generate implications for regulation policy. Collective reputation is socially optimal if cost c is such that $\bar{c}^{col} > c > \bar{c}^{ind}$. If, in this case, competent firms prefer to establish an individual rather than a collective reputation brand, then regulation that requires firms to label their country-of-origin or appellation more prominently may promote social welfare. However, we show that at least in the case of exclusive knowledge where in addition the prior beliefs μ are sufficiently strong, collective reputation induces a stronger commitment to invest exactly when it is also more efficient, so no regulation is needed.

To analyze this question, we assume that firms make their branding decisions at the beginning of the entire game that is described in Section 3. Specifically, after the types of the two firms are determined, firms learn each other's type and decide whether to operate as an individual or collective brand.[18] We analyze the best stationary equilibrium for each branding decision, and compare firm's profits in both.

We focus our attention on the more interesting case where a collective brand induces stronger incentives to invest than an individual brand, or where cost c is such that $\bar{c}^{\text{ind}} < c < \bar{c}^{\text{col}}$. In all other cases, individual reputation (weakly) dominates a collective reputation.

5.1 Stationary Equilibria

We first investigate which stationary equilibria exist when investment costs c are too high for the reputational equilibrium to exist. Recall that the set of relevant histories for a brand is given by $\mathcal{H}^b$ for $b \in \{\text{ind}, \text{col}\}$. A stationary equilibrium strategy specifies a mapping from the set of relevant histories into investment decisions. It can therefore be characterized by a subset $\mathcal{S} \subset \mathcal{H}^b$ of histories after which a competent firm invests.

As noted in Section 4, the expected return from an investment depends on the outcome produced in the previous period only. So, for an individual brand, any equilibrium strategy is a subset of the set $\{G, \varnothing, B\}$. We have already discussed two of the $2^3 = 8$ stationary equilibria: the *no investment equilibrium* that corresponds to the subset $\mathcal{S} = \varnothing$, which always exists, and the reputational equilibrium, that corresponds to the subset $\mathcal{S} = \{G, \varnothing, B\}$. There are six other candidates for a stationary equilibria $\mathcal{S} = \{G, \varnothing\}, \{G\}, \{\varnothing, B\}, \{B\}, \{G, B\}$, and $\{\varnothing\}$.[19] Similarly, for collective brands, stationary equilibria can be described by the sets $\mathcal{S} = \{G, B\}, \{G\}, \{B\}$, and $\varnothing$. In the next proposition, we identify which stationary equilib-

[18] For simplicity, we assume that while a competent firm's decision whether or not to brand with another firm may depend on the other firm's type, after a collective brand is formed, firm's decisions are independent of the other firm's type.

[19] $\mathcal{S} = \varnothing$ represents the no investment equilibrium; $\mathcal{S} = \{\varnothing\}$ represents the stationary equilibrium in which a competent type invests if and only if it failed to match in the previous period and no outcome was generated.

ria exist in the case of exclusive knowledge where in addition μ is close to one. Proposition 4 ensures that in this case $\bar{c}^{\text{ind}} < \bar{c}^{\text{col}}$.

Proposition 5. *Suppose that $\pi_L = 0$, $\pi_H \in (0,1)$ and μ is sufficiently large (to ensure that $\bar{c}^{ind} < \bar{c}^{col}$).*

1. *If the cost c is such that $c > \bar{c}^{ind}$, then the "no investment" equilibrium is the unique equilibrium for an individual brand.*[20]

2. *If the cost c is such that $c > \bar{c}^{col}$, then the "no investment" equilibrium is the unique equilibrium for a collective brand.*

In general, each equilibrium exists under a different set of conditions. However, in the parameter region that we focus on, where $\pi_L = 0$, $\pi_H \in (0,1)$, and μ is sufficiently large, conditions for the existence are very stringent. To understand the intuition for this result, consider the two equilibria $\mathcal{S} = \{G, \varnothing\}$ and $\mathcal{S} = \{G\}$. The arguments for other equilibria are similar. For both equilibria, the firm's optimal decision following a bad outcome is not to invest. Knowing this, consumers pay a low price (equal to π_L) to a firm that has produced a bad quality in the previous period. At the same time, a firm that just produced a good outcome is maximally rewarded with a price equal to π_H because it reveals the firm's competence. The large difference between the firm's payoff after bad and good outcomes implies that a firm would benefit from deviating and investing after a bad history because it would generate a higher payoff than non investment. Deterring this deviation requires that the cost of investment c is larger than $\bar{c}^{ind}$, which is precluded by assumption.

5.2 Profits and Endogenous Brand Formation

Next, we compare the firm's expected payoff in each equilibrium. In any stationary equilibrium, the per-period profit is determined by the payoff-relevant history, $\mathbf{h} \in \mathcal{H}^b$. The expected profit is the mean per-period profit obtained by averaging over all possible histories. The probability weight for each outcome is determined by the stationary distribution that is induced by the equilibrium strategies and beliefs.

Conditional on the history $\mathbf{h}$, consumers facing an individual or collective brand $b \in \{ind, col\}$ update their beliefs that the firm is competent to $\Pr^b(C|\mathbf{h})$. Then, the price the firm receives depends on the equilibrium strategy of the competent type, which is given by

[20] If π_H is close to 1 and $\pi_L \in (0,1)$ (quality control) and μ is close to 0, then the equilibrium $\mathcal{S} = \{G, \varnothing\}$ exists if $\bar{c}^{\text{ind}} < c < \bar{c}^{\text{col}}$, which is the case if and only if $\delta < \frac{2\pi_L}{3+\pi_L}$, and the equilibrium $\mathcal{S} = \{G\}$ exists if and only if $\delta > \frac{2\pi_L^2}{1+2\pi_L^2}$. This makes analysis of this case more involved than the case of exclusive knowledge, but it can nevertheless be analyzed along similar lines.

the probability that a competent firm invests after history $\mathbf{h}$, denoted $\sigma^{\mathcal{S}}(\mathbf{h}) \in \{0,1\}$. As a result, the per-period profit conditional on the observed history $\mathbf{h}$ is given by:

$$\Pi^b_{\mathcal{S}}(\mathbf{h}) = \Pr^b(C|\mathbf{h}) \cdot \sigma^{\mathcal{S}}(\mathbf{h}) \cdot \pi_H + (1 - \Pr^{\mathrm{b}}(C|\mathbf{h}) \cdot \sigma^{\mathcal{S}}(\mathbf{h})) \cdot \pi_L - c \cdot \sigma^{\mathcal{S}}(\mathbf{h}),$$

which is equal to the equilibrium probability of producing high quality. So, the *mean expected per-period profit* of a competent firm is given by:

$$\Pi^{\mathrm{b}}_{\mathcal{S}} = \sum_{\mathbf{h}\in\mathcal{H}^b} \Big(\Pr^b_s(\mathbf{h}) \cdot \Pi^b(\mathbf{h})$$

where $\Pr^b_s(\mathbf{h})$ denotes the stationary probability distribution over histories, which is determined by the brand's unobserved type $s \in \{C, I\}^2$ and the equilibrium strategy $\sigma^{\mathcal{S}}$. For example, under the reputational equilibrium, a collective brand produces a history $\mathbf{h} = GG$ with probability $\Pr^{col}_s(GG)$ for $s \in \{C, I\}^2$ that is given by:

$$\Pr^{col}_{CC}(GG) = \pi_H^2, \quad \Pr^{col}_{CI}(GG) = \Pr^{col}_{IC}(GG) = \left(\tfrac{\pi_H+\pi_L}{2}\right)^2, \quad \Pr^{col}_{II}(GG) = \pi_L^2$$

To identify the optimal branding strategy, we compare the expected per-period profits under the best feasible equilibrium across individual and collective brand.

Proposition 6. *Suppose that $\pi_L = 0$, $\pi_H \in (0,1)$, and μ is sufficiently large to ensure that $\bar{c}^{ind} < \bar{c}^{col}$.*

1. *If the cost c is such that $\bar{c}^{ind} < c < \bar{c}^{col}$, then a competent firm prefers to form a collective brand with another firm to establishing its own individual brand, regardless of the type the other firm.*
2. *If the cost c is such that $0 < c < \bar{c}^{ind}$, then a competent firm prefers to form an individual brand.*
3. *If the cost c is such that $c > \bar{c}^{col}$, then a competent firm is indifferent between an individual and a collective brand.*

Proposition 6 shows that for industries that require exclusive knowledge and have many competent firms (μ close to 1), the commitment value of a collective brand can induce competent firms to brand with another firm regardless of its competence. This is the case for an intermediate level of the investment cost $\bar{c}^{ind} < c < \bar{c}^{col}$. If $c < \bar{c}^{ind}$ then a competent firm prefers an individual to a collective brand because in this case, the fact that the former induces more extreme beliefs turns into an advantage because it also implies higher prices.

Remark 2. *(Quality control) When π_H is close to 1 and μ is close to 0, a competent firm does not want to form a collective brand with an incompetent firm, even if μ is sufficiently small to guarantee $\bar{c}^{ind} < \bar{c}^{col}$ and $c \in (\bar{c}^{ind}, \bar{c}^{col})$. The reason is that for such costs c, an individual brand can sustain intermediate equilibria with investment after some histories, but not all. Such an equilibrium yields higher profits for an individual brand than the more efficient reputational equilibrium for a collective brand because the firm expects the other (incompetent) firm to not invest, which will adversely affect consumers' beliefs and future prices. Furthermore, for small μ an adverse selection problem arises: since the probability of a firm being competent is small, consumers' willingness to pay in a reputational equilibrium is low because even after good histories their beliefs remain relatively low (more so for a collective brand). Thus, from an ex-ante perspective, the benefits captured by a competent firm do not outweigh the investment costs. The formal analysis is relegated to Online Appendix B.2.*

Propositions 6 and Remark 2 imply that in countries with a high baseline reputation (a large μ) country of origin labeling contributes to social welfare by improving firms' ability to commit to invest in quality in industries with exclusive knowledge such as French wine, Swiss watches, German automobiles, Japanese electronics, US software, etc. In contrast, producers of generic products such as screws, basic clothes, etc., in such countries should advertise their own brand only. The exact opposite conclusion applies in countries with a low baseline reputation (a small μ). In such countries, social welfare is maximized when manufacturers of generic goods label their country of origin while manufacturers of specialized goods avoid it.

These theoretical results are consistent with anecdotal evidence. For example the collective brand "Made in China" is advertised by sub-suppliers on platforms such as 'Made-in-China.com'," while successful high-tech companies such as Huawei try to build their own brand names. On the other hand, German sub-suppliers of generics such as ThyssenKrupp count on their own brand reputation.

Because firms in "quality control" industries may be reluctant to form a collective brand, the implementation of the optimal branding strategy might require some government intervention if the baseline reputation of firms is low. Indeed, the regulation of the labeling of country of origin is an important issue in many countries. The standard argument is that firms should be required to label their product with certain information in order to provide better consumer protection. The insights developed here suggest that the type of labeling, in particular the inclusion of country of origin, may also affect the incentives of firms to invest in quality.

6 Extensions

6.1 T-period Memory

The intuition for why a collective brand may induce stronger incentives to invest does not depend on the length of consumers' memory. In this section, we extend consumers' memory to T periods and show that our main results still hold in the following sense: the range of discount factors δ for which a collective brand provides a stronger incentive to invest than an individual brand ($\bar{c}^{\text{ind}} < \bar{c}^{\text{col}}$) is non-empty for all T. Moreover, as T tends to infinity, it becomes larger than in the case of a 2−period memory.

In general, with a longer memory, each single investment becomes less important. Thus, the benefit of a single investment decreases in T both in the case of individual and collective brands. However, the benefit of investment is more adversely affected for individual brands. The intuition is identical to that for the 2-period memory. With a longer memory, an individual brand can reach more extreme reputations following a sequence of good or bad outcomes, which worsens the associated moral hazard problem.

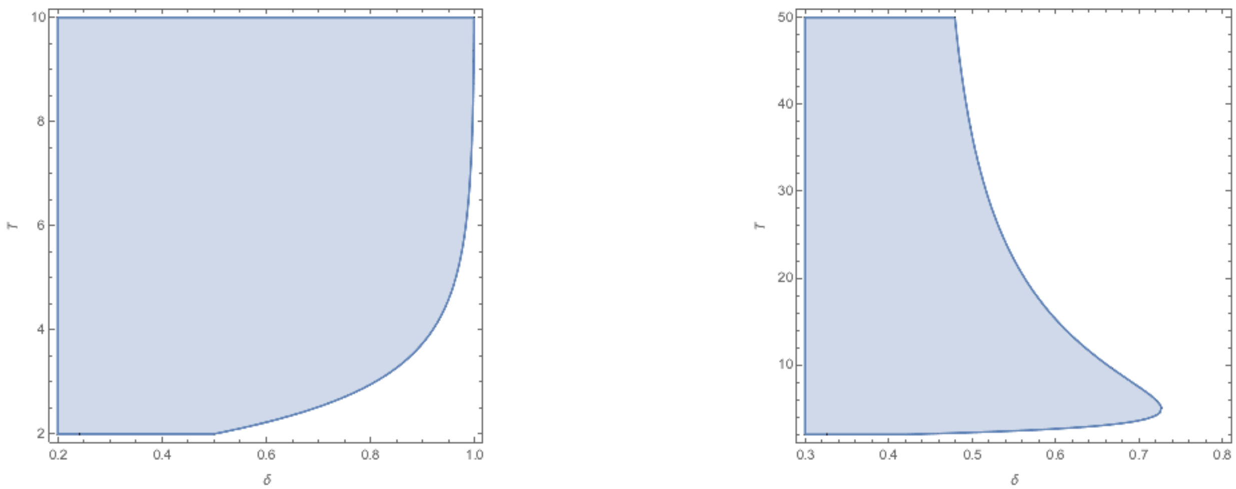
The following proposition generalizes Proposition 3 to T periods. A detailed analysis and proofs of the T−period case is relegated to Appendix B.1.

Proposition 7. *If $\pi_L = 0$, $\pi_H \in (0, 1)$, and μ is sufficiently close to 1, then a collective brand sustains a reputational equilibrium for higher investment costs than an individual brand ($\bar{c}^{ind} < \bar{c}^{col}$) if the discount factor δ is small enough. Moreover, the region of δ for which $\bar{c}^{ind} < \bar{c}^{col}$ increases monotonically in T and converges to $[0, 1]$.*

Figure 4 exhibits the range of parameters for which a collective brand induces stronger incentives to invest than an individual brand. For the case of exclusive knowledge, a larger δ requires a correspondingly larger memory T for a collective brand to outperform an individual brand. Panel (b) exhibits the analogous result for the case of quality control. While the cutoff $\bar{\delta}$ is non-monotonic in T, we show in Online Appendix B.1. that it converges to $\frac{1}{2}$.

6.2 Many Firms

In this subsection we generalize the model by allowing for an arbitrary number of firms $n \geq 3$. We maintain the assumptions of a 2-period memory and that the consumer that arrives in each period is randomly matched with one firm. This implies that the sets of possible histories are still given by $\mathcal{H}^{\text{ind}}$ and $\mathcal{H}^{\text{col}}$, respectively.



(a) $\pi_L = 0$ and $\mu = 1$ (b) $\pi_H = 1$, $\mu = 0$ and $\pi_L = 0.8$

Figure 4: Region in the T-δ space where $\bar{c}^{\text{col}} > \bar{c}^{\text{ind}}$ (dark) and $\bar{c}^{\text{ind}} > \bar{c}^{\text{col}}$ (light)

The analysis of an individual brand is hence very similar to the analysis with $n = 2$. In the collective case, consumers facing a collective brand cannot distinguish between the identities of individual firms. They care about the expected quality of a randomly matched firm. Thus, the updating depends on the number of firms and the signal is weaker with more firms because the consumer knows that she is less likely to have observed the history of the firm she is matched with. As is the case with $n = 2$, the reputational equilibrium exists for a collective brand if and only if the cost of investment is smaller than or equal to a minimum threshold that describes the expected benefit from investment given histories and other firms' types.

Because the benefit of free-riding increases with the number of firms, the benefit of collective reputation as a commitment device for investment decreases with n. However, as shown by the next proposition, it is still the case that collective reputation induces stronger incentives for effort than individual reputation under conditions that are similar to those described in Proposition 5.

We consider the case of exclusive knowledge where $\pi_L = 0$ so that one good outcome G almost fully reveals that the firm is competent. Thus, $p^{\text{ind}}(\mathbf{h}) = \pi_H$ as long as $\mathbf{h}$ contains one G. The reputational equilibrium exists for an individual brand if and only if the cost of investment c satisfies

$$c \leq \bar{c}_n^{\text{ind}} \quad \equiv \min_{h_{t-1} \in \{G,B,\varnothing\}} \bar{c}_n^{\text{ind}}(h_{t-1})$$

where $\bar{c}_n^{\text{ind}}(h_{t-1})$ denotes the expected benefit from investment given history $\mathbf{h}_t = h_{t-2}h_{t-1}$.

All in all, Proposition 3 can be generalized as follows:

Proposition 8. *Suppose that π_L is sufficiently close to* 0*. A collective brand sustains a reputational equilibrium for higher investment costs than an individual brand ($\bar{c}^{col} > \bar{c}^{ind}$) if consumers' prior belief μ about the firm's type is sufficiently high and the discount factor $\delta \leq \bar{\delta}$ is smaller than or equal to some threshold discount factor $\bar{\delta} < 1$. The threshold discount factor $\bar{\delta}$ is decreasing in the number of firms n.*

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A Appendix: Proofs

A.1 Proofs of Section 4

Proof. [Proposition 1] The posterior beliefs $\hat{\mu}^{\text{ind}}$ about the quality of the product after observing history $\mathbf{h}_t = h_{t-2}h_{t-1}$ are given by Bayes' rule:

$$\hat{\mu}^{\text{ind}}(GG) = \frac{\mu\pi_H^2}{\mu\pi_H^2 + (1-\mu)\pi_L^2}, \qquad \hat{\mu}^{\text{ind}}(GB) = \hat{\mu}^{\text{ind}}(BG) = \frac{\mu\pi_H(1-\pi_H)}{\mu\pi_H(1-\pi_H) + (1-\mu)\pi_L(1-\pi_L)},$$
$$\hat{\mu}^{\text{ind}}(BB) = \frac{\mu(1-\pi_H)^2}{\mu(1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2}, \qquad \hat{\mu}^{\text{ind}}(G\varnothing) = \hat{\mu}(\varnothing G) = \frac{\mu\pi_H}{\mu\pi_H + (1-\mu)\pi_L}$$
$$\hat{\mu}^{\text{ind}}(\varnothing\varnothing) = \mu, \qquad \hat{\mu}^{\text{ind}}(B\varnothing) = \hat{\mu}^{\text{ind}}(\varnothing B) = \frac{\mu(1-\pi_H)}{\mu(1-\pi_H) + (1-\mu)(1-\pi_L)}.$$

The reputational equilibrium exists if and only if a competent firm invests whenever visited following all histories, i.e.,

$$p^{\text{ind}}(h_{t-2}h_{t-1}) - c + \delta \cdot \big(\pi_H V(h_{t-1}G) + (1-\pi_H)V(h_{t-1}B)\big) \geq$$
$$p^{\text{ind}}(h_{t-2}h_{t-1}) + \delta \cdot \big(\pi_L V(h_{t-1}G) + (1-\pi_L)V(h_{t-1}B)\big)$$

which is equivalent to

$$c \leq \overline{c}^{\text{ind}}(h_{t-1}) := \delta \cdot (\pi_H - \pi_L) \cdot (V(h_{t-1}G) - V(h_{t-1}B)).$$

Then, $V(h_{t-1}G)$ and $V(h_{t-1}B)$ can be written as

$$V(h_{t-1}G) = \frac{p^{\text{ind}}(h_{t-1}G) - c}{2} + \frac{\delta}{2} \cdot \big(\pi_H V(GG) + (1-\pi_H)V(GB) + V(G\varnothing)\big)$$
$$V(h_{t-1}B) = \frac{p^{\text{ind}}(h_{t-1}B) - c}{2} + \frac{\delta}{2} \cdot \big(\pi_H V(BG) + (1-\pi_H)V(BB) + V(B\varnothing)\big).$$

Then, the difference is

$$V(h_{t-1}G) - V(h_{t-1}B) = \frac{p^{\text{ind}}(h_{t-1}G) - p^{\text{ind}}(h_{t-1}B)}{2} + \frac{\delta}{2} \cdot \Big(\pi_H \underbrace{(V(GG) - V(BG))}_{\frac{p^{\text{ind}}(GG) - p^{\text{ind}}(GB)}{2}}$$
$$+ (1-\pi_H) \underbrace{(V(GB) - V(BB))}_{\frac{p^{\text{ind}}(GB) - p^{\text{ind}}(BB)}{2}} + \underbrace{V(G\varnothing) - V(B\varnothing)}_{\frac{p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing)}{2}}\Big).$$

□

Proof. [Lemma 1] First, note that

$$
\begin{aligned}
p^{\text{ind}}(GG) - p^{\text{ind}}(GB) &= (\pi_H - \pi_L) \cdot \left(\frac{\mu \pi_H^2}{\mu \pi_H^2 + (1-\mu)\pi_L^2} - \frac{\mu \pi_H (1-\pi_H)}{\mu \pi_H (1-\pi_H) + (1-\mu)\pi_L(1-\pi_L)} \right) \\
&= \frac{\mu(1-\mu)\pi_H \pi_L (\pi_H - \pi_L)^2}{\Pr(GG) \cdot \Pr(GB)}, \\
p^{\text{ind}}(GB) - p^{\text{ind}}(BB) &= (\pi_H - \pi_L) \cdot \left(\frac{\mu \pi_H (1-\pi_H)}{\mu \pi_H (1-\pi_H) + (1-\mu)\pi_L(1-\pi_L)} - \frac{\mu (1-\pi_H)^2}{\mu (1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2} \right) \\
&= \frac{\mu(1-\mu)(1-\pi_H)(1-\pi_L)(\pi_H - \pi_L)^2}{\Pr(GB) \cdot \Pr(BB)}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing) &= (\pi_H - \pi_L) \cdot \left(\frac{\mu \pi_H}{\mu \pi_H + (1-\mu)\pi_L} - \frac{\mu(1-\pi_H)}{\mu(1-\pi_H) + (1-\mu)(1-\pi_L)} \right) \\
&= \frac{\mu(1-\mu)(\pi_H - \pi_L)^2}{\Pr(G) \cdot \Pr(B)} \\
&\geq \min\{p^{\text{ind}}(GG) - p^{\text{ind}}(GB),\ p^{\text{ind}}(GB) - p^{\text{ind}}(BB)\}.
\end{aligned}
$$

Hence, the minimum is attained at $h_{-1} = G$ if and only if

$$
\begin{aligned}
& \tfrac{\pi_H \pi_L}{\Pr(GG) \cdot \Pr(GB)} \leq \tfrac{(1-\pi_H)(1-\pi_L)}{\Pr(GB) \cdot \Pr(BB)} \\
\Leftrightarrow \quad & \Pr(BB) \cdot \pi_H \pi_L \leq \Pr(GG) \cdot (1-\pi_H)(1-\pi_L) \\
\Leftrightarrow \quad & \pi_H \pi_L (\mu(1-\pi_H)^2 + (1-\mu)(1-\pi_L)^2) \leq (1-\pi_H)(1-\pi_L)(\mu \pi_H^2 + (1-\mu)\pi_L^2) \\
\Leftrightarrow \quad & \mu \pi_H (1-\pi_H) \geq (1-\mu)\pi_L(1-\pi_L)
\end{aligned}
$$

This inequality holds if and only if $\mu \geq \bar{\mu} \equiv \frac{\pi_L(1-\pi_L)}{\pi_H(1-\pi_H)+\pi_L(1-\pi_L)}$. □

Proof. [Proposition 2] Let us denote by $V(h;\theta)$ the present discounted expected equilibrium profit of a competent firm when branding with a θ-type firm after history $\mathbf{h}_t \in \mathcal{H}^{\text{col}}$ at the beginning of the period before the consumer is assigned to either firm.

Then, a reputational equilibrium exists if and only if for all $\mathbf{h}_t, \theta$

$$
\begin{aligned}
p^{\text{col}}(h_{t-2}h_{t-1}) - c + \delta \cdot \big(\pi_H V(h_{t-1}G;\theta) + (1-\pi_H)V(h_{t-1}B;\theta)\big) &\geq \\
p^{\text{col}}(h_{t-2}h_{t-1}) + \delta \cdot \big(\pi_L V(h_{t-1}G;\theta) + (1-\pi_L)V(h_{t-1}B;\theta)\big)&.
\end{aligned}
$$

This is equivalent to

$$
c \leq \overline{c}^{\text{col}}(h_{t-1}) \equiv \delta \cdot (\pi_H - \pi_L) \cdot (V(h_{t-1}G;\theta) - V(h_{t-1}B;\theta)).
$$

First, note that for all $q_1, q_2, x \in \{G, B\}$, we have that $V(q_1 x, \theta) - V(q_2 x, \theta) = \frac{p^{\text{col}}(q_1 x) - p^{\text{col}}(q_2 x)}{2}$. Using this, we can calculate

$$\begin{aligned} V(h_{t-1}G;\theta) - V(h_{t-1}B;\theta) &= \frac{p^{\text{col}}(h_{t-1}G) - p^{\text{col}}(h_{t-1}B)}{2} \\ &+ \frac{\delta\pi_H}{2}\underbrace{\left(V(GG,\theta) - V(BG,\theta)\right)}_{\frac{p^{\text{col}}(GG)-p^{\text{col}}(BG)}{2}} + \frac{\delta(1-\pi_H)}{2}\underbrace{\left(V(GB,\theta) - V(BB,\theta)\right)}_{\frac{p^{\text{col}}(GB)-p^{\text{col}}(BB)}{2}} \\ &+ \frac{\delta\pi(\theta)}{2}\left(V(GG,\theta) - V(BG,\theta)\right) + \frac{\delta(1-\pi(\theta))}{2}\left(V(GB,\theta) - V(BB,\theta)\right) \end{aligned}$$

where $\pi(\theta) = \pi_L$ if $\theta = I$ and π_H if $\theta = C$. □

Proof. [Lemma 2] As noted in Section 3, upon observing a history $\mathbf{h}_t \in \mathcal{H}^{\text{col}}$, a consumer places a probability $\eta_s(\mathbf{h}_t)$ on the group's type $s \in \{CC, CI, IC, II\}$. These beliefs are given by:

$$\begin{aligned} \eta_{CC}(GG) &= \frac{\mu^2\pi_H^2}{\mu^2\pi_H^2 + 2\mu(1-\mu)\left(\frac{1}{4}\pi_H^2 + \frac{1}{2}\pi_H\pi_L + \frac{1}{4}\pi_L^2\right) + (1-\mu)^2\pi_L^2}, \\ \eta_{CI}(GG) &= \eta_{IC}(GG) \\ &= \frac{\mu(1-\mu)\left(\frac{1}{4}\pi_H^2 + \frac{1}{2}\pi_H\pi_L + \frac{1}{4}\pi_L^2\right)}{\mu^2\pi_H^2 + 2\mu(1-\mu)\left(\frac{1}{4}\pi_H^2 + \frac{1}{2}\pi_H\pi_L + \frac{1}{4}\pi_L^2\right) + (1-\mu)^2\pi_L^2}, \\ \eta_{II}(GG) &= 1 - \eta_{CC}(GG) - 2\eta_{CI}(GG), \\ \eta_{CC}(GB) &= \frac{\mu^2\pi_H(1-\pi_H)}{\mu^2\pi_H(1-\pi_H) + 2\mu(1-\mu)\frac{1}{4}\left(\pi_H(1-\pi_H) + \pi_H(1-\pi_L) + \pi_L(1-\pi_H) + \pi_L(1-\pi_L)\right) + (1-\mu)^2\pi_L(1-\pi_L)}, \\ \eta_{CI}(GB) &= \eta_{IC}(GB) \\ &= \frac{\mu(1-\mu)\frac{1}{4}\left(\pi_H(1-\pi_H) + \pi_H(1-\pi_L) + \pi_L(1-\pi_H) + \pi_L(1-\pi_L)\right)}{\mu^2\pi_H(1-\pi_H) + 2\mu(1-\mu)\frac{1}{4}\left(\pi_H(1-\pi_H) + \pi_H(1-\pi_L) + \pi_L(1-\pi_H) + \pi_L(1-\pi_L)\right) + (1-\mu)^2\pi_L(1-\pi_L)}, \\ \eta_{II}(GB) &= 1 - \eta_{CC}(GB) - 2\eta_{CI}(GB), \\ \eta_{CC}(BB) &= \frac{\mu^2(1-\pi_H)^2}{\mu^2(1-\pi_H)^2 + 2\mu(1-\mu)\left(\frac{1}{4}(1-\pi_H)^2 + \frac{1}{2}(1-\pi_H)(1-\pi_L) + \frac{1}{4}(1-\pi_L)^2\right) + (1-\mu)^2(1-\pi_L)^2}, \\ \eta_{CI}(BB) &= \eta_{IC}(BB) \\ &= \frac{\mu(1-\mu)\left(\frac{1}{4}(1-\pi_H)^2 + \frac{1}{2}(1-\pi_H)(1-\pi_L) + \frac{1}{4}(1-\pi_L)^2\right)}{\mu^2(1-\pi_H)^2 + 2\mu(1-\mu)\left(\frac{1}{4}(1-\pi_H)^2 + \frac{1}{2}(1-\pi_H)(1-\pi_L) + \frac{1}{4}(1-\pi_L)^2\right) + (1-\mu)^2(1-\pi_L)^2}, \\ \eta_{II}(BB) &= 1 - \eta_{CC}(BB) - 2\eta_{CI}(BB). \end{aligned}$$

Then, the consumer's posterior belief is about the firm being competent is given by

$$\Pr(\mathbf{h}_t) = \eta_{CC}(\mathbf{h}_t) + \frac{1}{2}(\eta_{CI}(\mathbf{h}_t) + \eta_{IC}(\mathbf{h}_t))$$

and $p^{\text{col}}(\mathbf{h}_t) = (\pi_H - \pi_L)\Pr(\mathbf{h}_t) + \pi_L$. Thus, the price differentials are given by:

$$\begin{aligned} p^{\text{col}}(GG) - p^{\text{col}}(GB) &= \frac{\mu(1-\mu)(\pi_H-\pi_L)^2\left(\mu^2(\pi_H-\pi_L)^2 + 2\mu(\pi_H-\pi_L)\pi_L + \pi_L(\pi_H+\pi_L)\right)}{4\cdot\Pr(GG)\cdot\Pr(GB)} \\ p^{\text{col}}(GB) - p^{\text{col}}(BB) &= \frac{\mu(1-\mu)(\pi_H-\pi_L)^2\left(\mu^2(\pi_H-\pi_L)^2 - 2\mu(\pi_H-\pi_L)(1-\pi_L) + (1-\pi_L)(2-\pi_H-\pi_L)\right)}{4\cdot\Pr(GB)\cdot\Pr(BB)} \end{aligned}$$

Thus, $p^{\text{col}}(GG) - p^{\text{col}}(GB) < p^{\text{col}}(GB) - p^{\text{col}}(BB)$ if and only if

$$\frac{\mu^2(\pi_H - \pi_L)^2 + 2\mu(\pi_H - \pi_L)\pi_L + \pi_L(\pi_H + \pi_L)}{\Pr(GG)} < \frac{\mu^2(\pi_H - \pi_L)^2 - 2\mu(\pi_H - \pi_L)(1 - \pi_L) + (1 - \pi_L)(2 - \pi_H - \pi_L)}{\Pr(BB)}$$

Taking the limit $\mu \to 1$ on both sides, the inequality becomes

$$\frac{\pi_H(\pi_H + \pi_L)}{\pi_H^2} < \frac{(\pi_H - \pi_L)^2 - 2(\pi_H - \pi_L)(1 - \pi_L) + (1 - \pi_L)(2 - \pi_H - \pi_L)}{(1 - \pi_H)^2}.$$

This is equivalent to $\pi_L(1 - \pi_H) < \pi_H(1 - \pi_H)$, i.e., it is always satisfied. Similarly, for $\mu \to 0$, the inequality is equivalent to

$$\frac{\pi_L(\pi_H + \pi_L)}{\pi_L^2} < \frac{(1 - \pi_L)(2 - \pi_H - \pi_L)}{(1 - \pi_L)^2}$$

which simplifies to $\pi_H < \pi_L$ which is never satisfied. Thus, by continuity $p^{\text{col}}(GG) - p^{\text{col}}(GB) < p^{\text{col}}(GB) - p^{\text{col}}(BB)$ for sufficiently large μ and $p^{\text{col}}(GG) - p^{\text{col}}(GB) > p^{\text{col}}(GB) - p^{\text{col}}(BB)$ for sufficiently small μ. The statement of the proposition follows from the definition of $\bar{c}^{\text{col}}(h_{t-1}, \theta)$ in (5).

One can show that unlike in the independent branding case, as μ increases, the binding history changes from B to G, back to B and then to G, but it does not yield additional insights, so we omit the proof and statement. □

Proof. [Proposition 3] Let us assume that π_L is fixed and sufficiently small (or equal to 0). It follows from Proposition 1 that for μ sufficiently large $\bar{c}^{\text{ind}}(G)$ determines the cutoff cost. Also, by Proposition 2, for sufficiently large μ, $\bar{c}^{\text{col}}(G; C)$ determines $\bar{c}^{\text{col}}$. Thus, it suffices to compare $\bar{c}^{\text{ind}} = \bar{c}^{\text{ind}}(G)$ and $\bar{c}^{\text{col}} = \bar{c}^{\text{col}}(G; C)$.

First, for an individual brand,

$$\begin{aligned}
\lim_{\pi_L \to 0} \bar{c}^{\text{ind}}(G) &= \delta \cdot \frac{\pi_H}{2} \lim_{\pi_L \to 0} \left(1 + \frac{\delta \pi_H}{2}\right) \Bigg(\underbrace{p^{\text{ind}}(GG) - p^{\text{ind}}(GB)}_{\to 0} \Bigg) \\
&\quad + \frac{\delta(1 - \pi_H)}{2} \left(p^{\text{ind}}(GB) - p^{\text{ind}}(BB)\right) + \frac{\delta}{2}\left(p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing)\right) \\
&= \delta^2 \cdot \frac{\pi_H^2}{2}(1 - \mu) \left(\frac{1 - \pi_H}{2} \cdot \frac{1}{1 - \mu\pi_H(2 - \pi_H)} + \frac{1}{2} \cdot \frac{1}{1 - \mu\pi_H} \right)
\end{aligned}$$

Then, $\lim_{\mu \to 1} \frac{1}{1-\mu} \lim_{\pi_L \to L} \bar{c}^{\text{ind}}(G) = \frac{\delta^2 \pi_H^2}{2(1 - \pi_H)}$.

For a collective brand,

$$\begin{aligned}
\lim_{\pi_L \to 0} \bar{c}^{\text{col}}(G;C) &= \delta \cdot \frac{\pi_H}{2} \lim_{\pi_L \to 0} (1+\delta\pi_H)(p^{\text{col}}(GG) - p^{\text{col}}(GB)) + \delta(1-\pi_H)(p^{\text{col}}(GB) - p^{\text{col}}(BB)) \\
&= \delta \cdot \frac{\pi_H}{2}(1+\delta\pi_H) \cdot \frac{(1-\mu)\mu\pi_H}{(1+\mu)(2-(1+\mu)\pi_H)} \\
&\quad + \delta^2 \cdot \frac{\pi_H}{2} \cdot (1-\pi_H) \cdot \frac{(1-\mu)\pi_H\left(2-\pi_H(1+\mu(2-\mu\pi_H))\right)}{\left((1-\mu\pi_H)^2 + \mu(1-\pi_H)^2 + 1 - \mu\right)\left(2-(1+\mu)\pi_H\right)}
\end{aligned}$$

Thus, $\lim_{\mu \to 1} \frac{1}{1-\mu} \lim_{\pi_L \to 0} \bar{c}^{\text{col}}(G;C) = \frac{\delta}{2} \frac{\pi_H^2(1+2\delta)}{4(1-\pi_H)}$.

So, $\lim_{\mu \to 1} \frac{1}{1-\mu} \lim_{\pi_L \to 0} \bar{c}^{\text{col}}(G;C) > \lim_{\mu \to 1} \frac{1}{1-\mu} \lim_{\pi_L \to 0} \bar{c}^{\text{ind}}(G)$ if and only if $\frac{1}{2} > \delta$. Thus, as long as $\delta \leq \frac{1}{2}$ for sufficiently small π_L and μ sufficiently close to 1, $\bar{c}^{\text{col}} \geq \bar{c}^{\text{ind}}$. Moreover, $\lim_{\pi_H \to 1}(1-\pi_H) \lim_{\pi_L \to 0} \bar{c}^{\text{col}}(G;C) > \lim_{\pi_H \to 1}(1-\pi_H) \lim_{\pi_L \to 0} \bar{c}^{\text{ind}}(G)$ for all $\delta \in [0,1]$ if and only if $\mu > \frac{1}{3}$.

Note for general μ, π_H, a collective brand can be better for higher values of δ, namely as long as

$$\delta < \frac{\mu}{(\mu+1)\left(-\frac{\mu\pi_H}{(\mu+1)} + \frac{(2-\pi_H(1+\mu))(\pi_H(\mu(2\pi_H-3)-1)+2)}{2(1-\mu\pi_H)(\mu(\pi_H-2)\pi_H+1)} - \frac{(1-\pi_H)(\pi_H(\mu(\mu\pi_H-2)-1)+2)}{(\mu\pi_H(\mu\pi_H+\pi_H-4)+2)}\right)} \equiv \bar{\delta}$$

which is for example satisfied for any $\delta \leq 1$ for sufficient large π_H. □

Proof. [Proposition 4]

If μ is close to 0, we get

$$\begin{aligned}
\lim_{\mu \to 0} \frac{1}{1-\mu} \lim_{\pi_L \to 0} \bar{c}^{\text{ind}}(G) &= \pi_H \frac{\delta(2-\pi_H)}{2}, \text{ and} \\
\lim_{\mu \to 0} \frac{1}{1-\mu} \lim_{\pi_L \to 0} \bar{c}^{\text{col}}(G) &= \pi_H \frac{\delta(2-\pi_H)(1-\pi_H)}{2(2-\pi_H)}.
\end{aligned}$$

Clearly, $\delta \cdot \frac{2-\pi_H}{2} > \delta \cdot \frac{1-\pi_H}{2}$ for all $\pi_H \in (0,1)$. So, for a μ close 0, there is a π_L close to 0 such that $\bar{c}^{\text{ind}} \geq \bar{c}^{\text{col}}$. □

A.2 Proofs of Section 5

Proof. [Proposition 5] This proposition states that, for each branding regime, if the reputational equilibrium does not exist ($c > \bar{c}^{\text{b}}$), the unique equilibrium is the no investment equilibrium for b $\in$ {ind, col}. This requires of a proof that no other alternative equilibria exist. For this, we proceed with the following proof strategy. First, we characterize the conditions under which alternative equilibria (besdies the reputational equilibrium and no investment equilibrium) exist for each branding regime. Second, we identify which equilibria exist if for different values of c, the investment cost.

I. Individual brand: For all equilibria other than the reputational equilibrium and no investment equilibrium, the competent type sometimes invests and other times not. This implies the cost of investment cannot be too large or too small. In other words, for an equilibrium specified by a subset $\mathcal{S}$ of $\mathcal{H}^{\text{ind}} = \{G, B, \varnothing\}$, there exist $\overline{C}_{\mathcal{S}}^{\text{ind}}$ and $\underline{C}_{\mathcal{S}}^{\text{ind}}$ such that the equilibrium exists if and only if $c \in (\underline{C}_{\mathcal{S}}^{\text{ind}}, \overline{C}_{\mathcal{S}}^{\text{ind}})$. These cutoff levels are characterized for each of the six remaining equilibria.

I–1. $\{G, \emptyset\}-$ **and** $\{G\}-$**Equilibrium**

In these equilibria, following a good history, the firm finds it optimal to invest in quality. On the other hand, following a bad history, it is optimal not to make an investment. Each condition corresponds to the following equtions:

$$p^{\text{ind}}(xG) - c + \delta\big(\pi_H V^{\text{ind}}(GG) + (1-\pi_H)V^{\text{ind}}(GB)\big) > p^{\text{ind}}(xG) + \delta\big(\pi_L V^{\text{ind}}(GG) + (1-\pi_L)V^{\text{ind}}(GB)\big), \quad (6)$$

$$p^{\text{ind}}(xB) - c + \delta\big(\pi_H V^{\text{ind}}(BG) + (1-\pi_H)V^{\text{ind}}(BB)\big) < p^{\text{ind}}(xB) + \delta\big(\pi_L V^{\text{ind}}(BG) + (1-\pi_L)V^{\text{ind}}(BB)\big).^{21} \quad (7)$$

Following the notation introduced in previous sections, $V^{\text{ind}}(h)$ for a history $h \in \{G, B, \varnothing\}^2$ is the equilibrium payoff at the beginning of the period *before* the consumer is assigned to either firm. The conditions above are equivalent to

$$c < \delta \cdot \Delta\pi \cdot (V^{\text{ind}}(GG) - V^{\text{ind}}(GB)). \quad (8)$$

$$c > \delta \cdot \Delta\pi \cdot (V^{\text{ind}}(BG) - V^{\text{ind}}(BB)). \quad (9)$$

(a) Computing Conditions (8) **and** (9) We express each of $V^{\text{ind}}(GG)$ and $V^{\text{ind}}(GB)$, and take their difference, which we denote by $A \equiv V^{\text{ind}}(GG) - V^{\text{ind}}(GB)$. A competent makes an investment following GG, but not GB. Therefore,

$$V^{\text{ind}}(GG) = \frac{p^{\text{ind}}(GG) - c}{2} + \frac{\delta}{2} \cdot \big(\pi_H V^{\text{ind}}(GG) + (1-\pi_H)V^{\text{ind}}(GB) + V^{\text{ind}}(G\varnothing)\big)$$

$$V^{\text{ind}}(GB) = \frac{p^{\text{ind}}(GB)}{2} + \frac{\delta}{2} \cdot \big(\pi_L V^{\text{ind}}(BG) + (1-\pi_L)V^{\text{ind}}(BB) + V^{\text{ind}}(B\varnothing)\big).$$

Then, the difference is equivalent to:

$$A = \frac{p^{\text{ind}}(GG) - p^{\text{ind}}(GB) - c}{2} + \frac{\delta}{2} \cdot \big(\pi_H(\underbrace{V^{\text{ind}}(GG) - V^{\text{ind}}(GB)}_{=A}) + \pi_L(\underbrace{V^{\text{ind}}(GB) - V^{\text{ind}}(BG)}_{=-B})$$
$$+ (1-\pi_L)(\underbrace{V^{\text{ind}}(GB) - V^{\text{ind}}(BB)}_{=0}) + \underbrace{V^{\text{ind}}(G\varnothing) - V^{\text{ind}}(B\varnothing)}_{\frac{p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing)}{2}}\big).$$

The underbraces show simplification of expressions in the equation above. Let us denote $V^{\text{ind}}(GB) - V^{\text{ind}}(BG) = -B$. The difference $V^{\text{ind}}(GB) - V^{\text{ind}}(BB)$ vanishes because both the period payoff and continuation payoffs are the same.[22] $V^{\text{ind}}(G\varnothing) - V^{\text{ind}}(B\varnothing) = \frac{1}{2} \cdot (p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing))$ because the continuation payoff is the same.

The expression B can be obtained similarly by computing $V^{\text{ind}}(BG)$ and $V^{\text{ind}}(GB)$ and then subtracting. A competent type would invest following BG, so the payoff function is

$$V^{\text{ind}}(BG) = \frac{p^{\text{ind}}(BG) - c}{2} + \frac{\delta}{2} \cdot \big(\pi_H V^{\text{ind}}(GG) + (1 - \pi_H) V^{\text{ind}}(GB) + V^{\text{ind}}(G\varnothing)\big).$$

So, the difference B is

$$\begin{aligned} B = {} & \frac{p^{\text{ind}}(BG) - p^{\text{ind}}(GB) - c}{2} + \frac{\delta}{2} \cdot \big(\pi_H \underbrace{(V^{\text{ind}}(GG) - V^{\text{ind}}(GB))}_{=A} + \pi_L \underbrace{(V^{\text{ind}}(GB) - V^{\text{ind}}(BG))}_{=-B} \\ & + (1 - \pi_L) \underbrace{(V^{\text{ind}}(GB) - V^{\text{ind}}(BB))}_{=0} + \underbrace{V^{\text{ind}}(G\varnothing) - V^{\text{ind}}(B\varnothing)}_{\frac{p(G\varnothing) - p(B\varnothing)}{2}}\big). \end{aligned}$$

So, we can solve for simultaneous equations:

$$\begin{aligned} A &= \frac{p^{\text{ind}}(GG) - p^{\text{ind}}(GB) - c}{2} + \frac{\delta}{2} \cdot \Big(\pi_H \cdot A - \pi_L \cdot B + \frac{p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing)}{2}\Big) \\ B &= \frac{p^{\text{ind}}(BG) - p^{\text{ind}}(GB) - c}{2} + \frac{\delta}{2} \cdot \Big(\pi_H \cdot A - \pi_L \cdot B + \frac{p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing)}{2}\Big), \end{aligned}$$

which gives

$$A = \frac{-2c + 2(p^{\text{ind}}(GG) - p^{\text{ind}}(GB)) + \delta(\pi_L \cdot (p^{\text{ind}}(GG) - p^{\text{ind}}(BG)) + p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing))}{2(2 - \delta \cdot \Delta\pi)}.$$

Finally, we plug this into the initial condition for incentive compatibility in equation (8) and collect c on the same side:

$$c < \overline{C}^{\text{ind}}_{\mathcal{S}} := \frac{\delta \cdot \Delta\pi}{2} \cdot \Big(p^{\text{ind}}(GG) - p^{\text{ind}}(GB) + \frac{\delta}{2}(\pi_L(p^{\text{ind}}(GG) - p^{\text{ind}}(BG)) + p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing))\Big) \tag{10}$$

We can repeat a similar exercise and find that Condition (9) corresponds to

$$c > \underline{C}^{\text{ind}}_{\mathcal{S}} := \frac{\delta \cdot \Delta\pi}{2} \cdot \Big(p^{\text{ind}}(BG) - p^{\text{ind}}(BB) + \frac{\delta}{2}(\pi_H \cdot (p^{\text{ind}}(GG) - p^{\text{ind}}(BG)) + p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing))\Big) \tag{11}$$

[22] Consumers pay π_L upon observing both GB and BB, so the difference in per-period profits vanishes, too.

(b) Computing $\overline{C}^{\mathbf{ind}}_{\mathcal{S}}$ and $\underline{C}^{\mathbf{ind}}_{\mathcal{S}}$

For a given equilibrium strategy, we identify consumers' posterior beliefs and prices, and plug into $\overline{C}^{\text{ind}}_{\mathcal{S}}$ and $\underline{C}^{\text{ind}}_{\mathcal{S}}$.

If a firm plays a $\{G, \emptyset\}$-equilibrium, then for a competent type the transition matrix of the Markov chain from the previous outcome $h_{t-1} \in \{G, \varnothing, B\}$ to $h_t \in \{G, \varnothing, B\}$ is given by

$$\begin{pmatrix} \frac{\pi_H}{2} & \frac{1}{2} & \frac{1-\pi_H}{2} \\ \frac{\pi_H}{2} & \frac{1}{2} & \frac{1-\pi_H}{2} \\ \frac{\pi_L}{2} & \frac{1}{2} & \frac{1-\pi_L}{2} \end{pmatrix}$$

Thus, the stationary distribution of outcomes for a competent firm is given by $\Pr_C(G) = \frac{\pi_H+\pi_L}{2(2-\pi_H+\pi_L)}$, $\Pr_C(B) = \frac{1-\pi_H}{2-\pi_H+\pi_L}$, $\Pr_C(\emptyset) = \frac{1}{2}$. For an incompetent type, the stationary distribution is $\Pr_I(G) = \frac{\pi_L}{2}$, $\Pr_I(B) = \frac{1-\pi_L}{2}$, and $\Pr_I(\emptyset) = \frac{1}{2}$ because it never makes an investment.

Then, the posterior beliefs in this equilibrium, denoted by $\hat{\mu}^{\text{ind}}(h)$ is obtained by Bayes' rule. For example, $\mu(BG) = \frac{\mu\frac{1-\pi_H}{2-\pi_H+\pi_L}}{\mu\frac{1-\pi_H}{2-\pi_H+\pi_L}+(1-\mu)\frac{1-\pi_L}{2}}$ and $\mu(GG) = \frac{\mu\frac{\pi_H+\pi_L}{2(2-\pi_H+\pi_L)}\pi_H}{\mu\frac{\pi_H+\pi_L}{2(2-\pi_H+\pi_L)}\pi_H+(1-\mu)\frac{\pi_L}{2}\pi_L}$. Then,

$$\lim_{\mu\to 1}\lim_{\pi_L\to 0}\underline{C}^{\text{ind}}_{\{G,\emptyset\}} = \lim_{\mu\to 1}\lim_{\pi_L\to 0}\overline{C}^{\text{ind}}_{\{G,\emptyset\}} = \frac{\delta\pi_H^2}{2} > 0.$$

Since the lower and upper bound coincide in the limit, c has to be arbitrarily close to $\frac{\delta\pi_H^2}{2}$ for $\mathcal{S} = \{G, \varnothing\}$ to be an equilibrium.

We can show similarly that for a $G-$equilibrium, we have exactly the same cutoff levels in the limit:

$$\lim_{\mu\to 1}\lim_{\pi_L\to 0}\underline{C}^{\text{ind}}_{\{G\}} = \lim_{\mu\to 1}\lim_{\pi_L\to 0}\overline{C}^{\text{ind}}_{\{G\}} = \frac{\delta\pi_H^2}{2} > 0.$$

(c) Proof that $\{G, \varnothing\}-$ and $\{G\}-$equilibrium do not exist almost surely. If $c \in (\overline{c}^{\text{ind}}, \overline{c}^{\text{col}})$, then neither of these equilibria exists for an individual brand. This is because for $\pi_L = 0$ and μ close to 1, $\overline{c}^{\text{ind}}$ and $\overline{c}^{\text{col}}$ converge to 0. On the other hand, $\{G, \varnothing\}-$ and $\{G\}-$equilibrium exist only for c arbitrarily close to $\frac{\delta\pi_H^2}{2}$. In other words, these equilibria require a higher investment cost to exist. This is because an investment is more appealing in this equilibrium, and only with higher investment cost, the firm can be discouraged from making an investment following a bad outcome. A good outcome leads to an investment, which in turn leads to a good outcome. This forms a virtuous circle, which the firm would want to be part of even if it has produced a bad outcome.

If $c > \overline{c}^{\text{ind}}$, as long as c is bounded away from $\frac{\delta\pi_H^2}{2}$, neither of $\{G, \varnothing\}-$ and $\{G\}-$equilibrium would exist. So, we can say these equilibria do not exist almost surely.

I–2. Proof that $\{B,\varnothing\}-$ and $\{B\}-$equilibrium do not exist.

In these equilibria, a competent firm invests following a bad outcome, but not after a good outcome. Then, an investment does not yield sufficient future benefits because consumers punish the firm for having produced a good outcome. This implies that the cutoff levels for these equilibria would be very small, if not negative.

We can repeat the same analysis to characterize the upper and lower bounds. We find that for $\mathcal{S} = \{B, \varnothing\}$ and $\{B\}$, in the limit

$$\lim_{\mu\to 1}\lim_{\pi_L\to 0}\underline{C}^{\mathcal{S}} = \lim_{\mu\to 1}\lim_{\pi_L\to 0}\overline{C}^{\mathcal{S}} = -\frac{\delta\pi_H^2}{2} < 0.$$

Therefore, these equilibria does not exist for any positive investment cost.

I–3. Proof that $\{G,B\}-$ and $\{\varnothing\}-$equilibrium do not exist.

It remains to examine two more equilibria: $\mathcal{S} = \{G, B\}$ and $\{\varnothing\}$. These equilibria demonstrate strategies non-monotonic in the firm's reputation in the sense that the firm takes the same action following a good and bad outcome, but a different one following an empty outcome.

First, for $\mathcal{S} = \{G, B\}$, the firm must find it optimal to invest following a good and bad outcome, but not after an $\varnothing$-outcome. Then, the equilibrium exists if and only if

$$\delta\Delta\pi\cdot(V_{\{G,B\}^{\text{ind}}}(\varnothing G) - V^{\text{ind}}_{\{G,B\}}(\varnothing B)) \le c \le \min_{x\in\{G,B\}}\left(\delta\Delta\pi\cdot(V^{\text{ind}}_{\{G,B\}}(xG) - V^{\text{ind}}_{\{G,B\}}(xB))\right).\Bigg($$

In this equilibrium, the firm behaves the same following a good and a bad outcome. Therefore, the future payoffs of $V^{\text{ind}}_{\{G,B\}}(yG)$ and $V^{\text{ind}}_{\{G,B\}}(yB)$ are exactly the same for any $y \in \{G, B, \varnothing\}$. So, the difference in these payoff functions is equivalent to the difference in immediate per-period profit. Therefore, the equilibrium can exist for some $c > 0$ if and only if

$$p^{\text{ind}}_{\{G,B\}}(\varnothing G) - p^{\text{ind}}_{\{G,B\}}(\varnothing B) < \min_{x\in\{G,B\}}\left(p^{\text{ind}}_{\{G,B\}}(xG) - p^{\text{ind}}_{\{G,B\}}(xB)\right).\Bigg($$

If $\pi_L = 0$, the right-hand side is zero for $x = G$, as one good outcome reveals the firm to be competent. On the other hand, the left-hand side is positive. Therefore, the inequality cannot hold.

We can similarly show that the $\mathcal{S} = \{\varnothing\}-$equilibrium does not exist. If it did, the competent type would invest following an $\varnothing$ outcome, but not after a good or bad one. So, the following condition must hold:

$$\max_{x\in\{G,B\}}\frac{\Delta\pi}{2}(V^{\text{ind}}_{\{\varnothing\}}(xG) - V^{\text{ind}}_{\{\varnothing\}}(xB)) \le c \le \frac{\Delta\pi}{2}(V^{\text{ind}}_{\{\varnothing\}}(\varnothing G) - V^{\text{ind}}_{\{\varnothing\}}(\varnothing B)).$$

Similarly, comparing the difference in payoff functions is equivalent to comparing that in prices: $\max_{x\in\{G,B\}} p^{\text{ind}}_{\{\varnothing\}}(xG) - p^{\text{ind}}_{\{\varnothing\}}(xB) \leq p^{\text{ind}}_{\{\varnothing\}}(\varnothing G) - p^{\text{ind}}_{\{\varnothing\}}(\varnothing B)$. In this equilibrium, the firm does not invest following a good or bad equilibrium. So, consumers pay the minimal price, i.e. $p^{\text{ind}}_{\{\varnothing\}}(yG) = p^{\text{ind}}_{\{\varnothing\}}(yB) = \pi_L = 0$ for all $y \in \{G, B, \varnothing\}$. Therefore, both the left-hand and the right-hand side vanish, and the equilibrium does not exist.

I–4. Statement for Individual Branding

Summing up the previous analysis from **I–1** to **I–3** above, we conclude that if $\pi_L = 0$ and μ is sufficiently large, and if $c > \overline{c}^{\text{ind}}$, the no investment equilibrium is the unique equilibrium for an individual brand.

This implies that whenever the reputational equilibrium does not exist for an individual brand, the competent type never makes investment, and consumers pay the minimal price. This suggests that if $c > \overline{c}^{\text{ind}}$, collective branding would be an attractive option for the firm as long as it provides more commitment power, i.e., it admits a more profitable equilibrium, such as the reputational equilibrium.

Next, we move on to collective branding and prove the statement regarding a collective brand in Proposition 6.

II. Collective brand:

If $c > \overline{c}^{\text{col}}$, the reputational equilibrium does not exist. For a collective brand, an equilibrium is prescribed by $\mathcal{S}$, a subset of $\mathcal{H}^{\text{col}} = \{G, B\}$. So, besides the reputational and no investment equilibrium, there are two alternative equilibria: $\mathcal{S} = \{G\}$, or $\{B\}$. We identify conditions under which a competent type finds it optimal to follow a given equilibrium strategy. The payoff of a firm in a collective brand depends on the type of the other firm, $\theta \in \{C, I\}$. We denote a payoff function at a two-period history h by $V^{\text{col}}_{\mathcal{S}}(h;\theta)$. Whenever we focus on one specific equilibrium, we omit the notation $\mathcal{S}$.

II–1. $\{G\}$–equilibrium

In an equilibrium with $\mathcal{S} = \{G\}$, following a good history, a firm finds it optimal to invest in quality, but not following a bad history. Similar to equations (8) and (9), the condition for existence is

$$\underline{C}^{\text{col}}(\theta) \leq c \leq \overline{C}^{\text{col}}(\theta),$$

where $\underline{C}^{\text{col}}(\theta) := \delta \cdot \Delta\pi \cdot (V^{\text{col}}(BG;\theta) - V^{\text{col}}(BB;\theta))$ and $\overline{C}^{\text{col}}(\theta) := \delta \cdot \Delta\pi \cdot (V^{\text{col}}(GG;\theta) - V^{\text{col}}(GB;\theta))$

First, suppose $\theta = C$ so that the other firm is also competent. For any $x \in \{G, B\}$

$$V^{\text{col}}(xG; C) = \frac{p^{\text{col}}(xG) - c}{2} + \delta \cdot \big(\pi_H(V^{\text{col}}(GG; C) - V^{\text{col}}(GB; C)) + V^{\text{col}}(GB; C))\big)$$
$$V^{\text{col}}(xB; C) = \frac{\pi_L}{2} + \delta \cdot \big(\pi_L V^{\text{col}}(BG; C) + (1 - \pi_L)V^{\text{col}}(BB; C)\big).$$

and hence, for any $x, y \in \{G, B\}$

$$V^{\text{col}}(xG; C) - V^{\text{col}}(yB; C) = \frac{p^{\text{col}}(xG) - \pi_L - c}{2} + \delta \cdot \big(\pi_H(V^{\text{col}}(GG; C) - V^{\text{col}}(GB; C))$$
$$+ \pi_L(V^{\text{col}}(GB; C) - V^{\text{col}}(BG; C)) + (1 - \pi_L)\underbrace{(V^{\text{col}}(GB; C) - V^{\text{col}}(BB; C))}_{=0}\big).$$

Thus, we can calculate $V(GG) - V(GB)$ and $V(GB) - V(BG)$ to be

$$V^{\text{col}}(GG) - V^{\text{col}}(GB) = \frac{p^{\text{col}}(GG) - \pi_L - c + \delta\pi_L(p^{\text{col}}(GG) - p^{\text{col}}(BG))}{2(1 - \delta(\pi_H - \pi_L))}$$
$$V^{\text{col}}(BG) - V^{\text{col}}(GB) = \frac{p^{\text{col}}(BG) - \pi_L - c + \delta\pi_H(p^{\text{col}}(GG) - p^{\text{col}}(BG))}{2(1 - \delta(\pi_H - \pi_L))} = V(BG) - V(BB).$$

All in all, the equilibrium exists if and only if

$$\frac{\frac{\delta}{2}(\pi_H - \pi_L)}{1 - \frac{\delta}{2}(\pi_H - \pi_L)}\big(p^{\text{col}}(BG) - \pi_L + \delta\pi_H(p^{\text{col}}(GG) - p^{\text{col}}(BG))\big) \leq c \leq$$
$$\frac{\frac{\delta}{2}(\pi_H - \pi_L)}{1 - \frac{\delta}{2}(\pi_H - \pi_L)}\big((p^{\text{col}}(GG) - \pi_L) + \delta\pi_L(p^{\text{col}}(GG) - p^{\text{col}}(BG))\big)$$

Let us next assume that the other firm is incompetent. In that case we have for any $x \in \{G, B\}$

$$V^{\text{col}}(xG) = \frac{p^{\text{col}}(xG) - c}{2} + \delta \cdot \big(\frac{\pi_H + \pi_L}{2}(V^{\text{col}}(GG) - V^{\text{col}}(GB)) + V^{\text{col}}(GB)\big)$$
$$V^{\text{col}}(xB) = \frac{\pi_L}{2} + \delta \cdot \big(\pi_L V^{\text{col}}(BG) + (1 - \pi_L)V^{\text{col}}(BB)\big).$$

and hence, for any $x, y \in \{G, B\}$

$$V^{\text{col}}(xG) - V^{\text{col}}(yB) = \frac{p^{\text{col}}(xG) - \pi_L - c}{2} + \delta \cdot \big(\frac{\pi_H + \pi_L}{2}(V^{\text{col}}(GG) - V^{\text{col}}(GB))$$
$$+ \pi_L(V(GB) - V(BG))\big).$$

All in all, the equilibrium exists if and only if

$$\frac{\frac{\delta}{2}\frac{\pi_H-\pi_L}{2}}{1-\frac{\delta}{2}\frac{\pi_H-\pi_L}{2}}\left(p^{\text{col}}(BG)-\pi_L+\delta\frac{\pi_H+\pi_L}{2}(p^{\text{col}}(GG)-p^{\text{col}}(BG))\right)\leq c\leq$$
$$\frac{\frac{\delta}{2}\frac{\pi_H-\pi_L}{2}}{1-\frac{\delta}{2}\frac{\pi_H-\pi_L}{2}}\left((p^{\text{col}}(GG)-\pi_L)+\delta\pi_L(p^{\text{col}}(GG)-p^{\text{col}}(BG))\right)$$

All in all, a $\{G\}$-equilibrium can be sustained if and only if

$$\underline{C}^{\text{col}}\equiv\frac{\frac{\delta}{2}(\pi_H-\pi_L)}{1-\frac{\delta}{2}(\pi_H-\pi_L)}\left(p^{\text{col}}(BG)-\pi_L+\delta\pi_H(p^{\text{col}}(GG)-p^{\text{col}}(BG))\right)\leq c\leq$$
$$\frac{\frac{\delta}{2}\frac{\pi_H-\pi_L}{2}}{1-\frac{\delta}{2}\frac{\pi_H-\pi_L}{2}}\left((p^{\text{col}}(GG)-\pi_L)+\delta\pi_L(p^{\text{col}}(GG)-p^{\text{col}}(BG))\right)\equiv\overline{C}^{\{G\}}_{\text{col}}$$

Taking $\pi_L\to 0$ yields

$$\frac{\frac{\delta}{2}\pi_H}{1-\frac{\delta}{2}\pi_H}\left(p^{\text{col}}(BG)+\delta\pi_H(p^{\text{col}}(GG)-p^{\text{col}}(BG))\right)\leq c\leq\frac{\frac{\delta}{2}\frac{\pi_H}{2}}{1-\frac{\delta}{2}\frac{\pi_H}{2}}p^{\text{col}}(GG) \tag{12}$$

In order to calculate the prices we need to calculate the stationary distribution of states given the transition matrix from the consumer's perspective. If both firms are competent it is given by

$$\begin{pmatrix}\pi_H & 1-\pi_H\\ \pi_L & 1-\pi_L\end{pmatrix}$$

Thus, the stationary probability of being in state G is $\Pr_{CC}(G)=\frac{\pi_L}{1-(\pi_H-\pi_L)}$ and the probability of being in state B is $\Pr_{CC}(B)=\frac{1-\pi_H}{1-(\pi_H-\pi_L)}$. If one is competent and the other is incompetent, then it is given by

$$\begin{pmatrix}\frac{\pi_H+\pi_L}{2} & 1-\frac{\pi_H+\pi_L}{2}\\ \pi_L & 1-\pi_L\end{pmatrix}$$

Thus, the stationary probability of being in state G is $\Pr_{CI}(G)=\frac{\pi_L}{1-\frac{\pi_H-\pi_L}{2}}$ and the probability of being in state B is $\Pr_{CI}(B)=\frac{1-\frac{\pi_H+\pi_L}{2}}{1-\frac{\pi_H-\pi_L}{2}}$. If both are incompetent, then $\Pr_{II}(G)=\pi_L$ and the probability of being in state B is $\Pr_{II}(B)=1-\pi_L$. Note that as $\pi_L\to 0$, B always becomes an absorbing state. Hence, the after observing a history GG, a consumer updates his belief about facing a competent firm is

$$\Pr(C|GG)=\frac{\mu^2\frac{\pi_L}{1-(\pi_H-\pi_L)}\pi_H+\mu(1-\mu)\frac{\pi_L}{1-\frac{\pi_H-\pi_L}{2}}\frac{\pi_H+\pi_L}{2}}{\mu^2\frac{\pi_L}{1-(\pi_H-\pi_L)}\pi_H+2\mu(1-\mu)\frac{\pi_L}{1-\frac{\pi_H-\pi_L}{2}}\frac{\pi_H+\pi_L}{2}+(1-\mu)^2\pi_L^2}$$

and after observing a history BG, a consumer updates his belief about facing a competent firm is

$$\Pr(C|BG) = \frac{\mu^2 \frac{1-\pi_H}{1-(\pi_H-\pi_L)}\pi_L + \mu(1-\mu)\frac{1-\frac{\pi_H+\pi_L}{2}}{1-\frac{\pi_H-\pi_L}{2}}\pi_L}{\mu^2 \frac{1-\pi_H}{1-(\pi_H-\pi_L)}\pi_L + 2\mu(1-\mu)\frac{1-\frac{\pi_H+\pi_L}{2}}{1-\frac{\pi_H-\pi_L}{2}}\pi_L + (1-\mu)^2\pi_L^2}$$

Thus, as $\pi_L \to 0$ (12) converges to

$$\lim_{\pi_L\to 0} \underline{C}^{\text{col}} = \frac{\frac{\delta}{2}\pi_H}{1-\frac{\delta}{2}\pi_H}\pi_H\mu \left(1 + \delta\pi_H \left(\frac{\mu\frac{\pi_H}{1-\pi_H} + (1-\mu)\frac{1}{1-\frac{\pi_H}{2}}\frac{\pi_H}{2}}{\mu\frac{\pi_H}{1-\pi_H} + 2(1-\mu)\frac{1}{1-\frac{\pi_H}{2}}\frac{\pi_H}{2}} - \frac{1}{2-\mu} \right) \right) \le c$$
$$\le \lim_{\pi_L\to 0} \overline{C}^{\text{col}} = \frac{\frac{\delta}{2}\frac{\pi_H}{2}}{1-\frac{\delta}{2}\frac{\pi_H}{2}}\pi_H\mu \frac{\mu\frac{\pi_H}{1-\pi_H} + (1-\mu)\frac{1}{1-\frac{\pi_H}{2}}\frac{\pi_H}{2}}{\mu\frac{\pi_H}{1-\pi_H} + 2(1-\mu)\frac{1}{1-\frac{\pi_H}{2}}\frac{\pi_H}{2}}.$$

Then, as $\mu \to 1$ we can write

$$\lim_{\mu\to 1}\frac{1}{\mu}\lim_{\pi_L\to 0}\underline{C}^{\text{col}} = \frac{\frac{\delta}{2}\pi_H}{1-\frac{\delta}{2}\pi_H}\pi_H \quad > \quad \frac{\frac{\delta}{2}\frac{\pi_H}{2}}{1-\frac{\delta}{2}\frac{\pi_H}{2}}\pi_H = \lim_{\mu\to 1}\frac{1}{\mu}\lim_{\pi_L\to 0}\overline{C}^{\text{col}}.$$

Thus, as for large μ, there is no $c > 0$ such that a $\{G\}$-equilibrium exists.

II–2. $\{B\}-$equilibrium

Similarly, one can show that no $\{B\}-$equilibrium can exist for sufficiently large μ and π_L close to zero.

Therefore, for $\pi_L = 0$ and μ close to 1, if $c > \overline{c}^{\text{col}}$, the no investment equilibrium is the unique equilibrium. This proves the proposition. □

Proof. [Proposition 6] For $c \in (\overline{c}^{\text{ind}}, \overline{c}^{\text{col}})$, the only equilibrium for an individual brand is the "no investment" equilibrium by Proposition 5. In this equilibrium, its average profits are given by $\lim_{\mu\to 1}\lim_{\pi_L\to 0}\Pi^{\text{ind}} \approx \pi_L \approx 0$. In a collective brand, regardless of the other firm's competency, the firm's average profit in a reputation equlibrium is given by $\lim_{\mu\to 1}\lim_{\pi_L\to 0}\Pi^{\text{col}} = \pi_H - c$. Therefore, the firm always prefers branding with another firm to staying alone as long as $c < \pi_H$ with is the case by assumption.

For $c \in (0, \min\{\overline{c}^{\text{ind}}, \overline{c}^{\text{col}}\})$, the reputational equilibrium exists for an individual and collective brand. Thus, after any history, consumers expect competent firms to invest, but the belief updating after a particular history is different. An individual firm makes an average profit of

$$\Pi^{\text{ind}} = 0.25 \cdot \Big(\pi_H^2 p^{\text{ind}}(GG) + 2 \cdot \pi_H(1-\pi_H)p^{\text{ind}}(GB) + (1-\pi_H)^2 p^{\text{ind}}(BB) + 2 \cdot \pi_H p^{\text{ind}}(G\emptyset) + 2 \cdot (1-\pi_H)p^{\text{ind}}(B\emptyset) + p^{\text{ind}}(\emptyset\emptyset)\Big) - c$$

A competent firm forms a brand with another competent firm makes an average profit of

$$\Pi^{\text{col}} = \pi_H^2 p^{\text{col}}(GG) + \pi_H(1-\pi_H)p^{\text{col}}(GB) + \pi_H(1-\pi_H)p^{\text{col}}(BG) + (1-\pi_H)^2 p^{\text{col}}(BB) - c.$$

Then, as $\mu \to 1$ and $\pi_L \to 0$ the difference in average profits satisfies

$$\lim_{\mu \to 1} \lim_{\pi_L \to 0} \frac{\Pi^{\text{ind}} - \Pi^{\text{col}}}{(1-\mu)^2} = \frac{\pi_H^3(\pi_H(\pi_H((0.125\pi_H - 0.5)\pi_H + 0.75) - 0.5) + 0.125)}{(1-\pi_H)^6} > 0.$$

Thus, for large μ, a firm always prefers to stay alone to branding with another firm. Note that branding with an incompetent firm is always less attractive than branding with a competent firm.

When $c > \bar{c}^{\text{col}} > \bar{c}^{\text{ind}}$, only the no-investment equilibrium exist for a collective brand, as well as an individual brand. Thus, the average profit in both scenarios is $\pi_L = 0$. □

A.3 Proof of Section 6.1

Please see the online appendix for proof of Proposition 7.

A.4 Proof of Section 6.2

Proof. [Proposition 2] Note that the history that must minimizes the benefit from investment is $h_{t-1} = GG$ and

$$\bar{c}_N^{\text{ind}}(GG) \equiv \pi_H^2 \frac{\delta^2}{N^2}\left[(1-\pi_H)\frac{1-\mu}{\mu(1-\pi_H)^2 + 1 - \mu} + (N-1) \cdot \frac{1-\mu}{\mu(1-\pi_H) + 1 - \mu}\right]$$

Thus, we can write

$$\lim_{\mu \to 1} \lim_{\pi_L \to 0} \frac{\bar{c}_N^{\text{ind}}(GG)}{1-\mu} = \frac{\pi_H^2 \cdot \delta^2}{N} \frac{1}{(1-\pi_H)}$$

In the collective case, for $\pi_L \approx 0$ the probability of facing a C-firm after a history h with u G-observations and $2-u$ B-observations simplifies to

$$\lim_{\pi_L\to 0}\Pr^{\text{col}}(C|h) \;=\; \frac{\sum_{i=1}^{N}\binom{N}{i}\mu^i(1-\mu)^{N-i}\sum_{v=u}^{2}\binom{2-u}{v-u}\left(\frac{i}{N}\right)^{v+1}\pi_H^u(1-\pi_H)^{v-u}\left(\frac{N-i}{N}\right)^{2-v}}{\sum_{i=0}^{N}\binom{N}{i}\mu^i(1-\mu)^{N-i}\sum_{v=u}^{2}\binom{2-u}{v-u}\left(\frac{i}{N}\right)^{v}\pi_H^u(1-\pi_H)^{v-u}\left(\frac{N-i}{N}\right)^{2-v}}$$

This equation is Bayes' rule. The denominator is the total probability that a history h is produced. Provided that i of N firms are competent, the second summation is the probability that u good and $2-u$ bad outcomes are produced. With $\pi_L = 0$, only competent type can produce a good outcome. $\left(\frac{i}{N}\right)^v \cdot \pi_H^u \cdot (1-\pi_H)^{v-u}$ is the probability that a competent type is drawn $v \geq u$ times and produce u good and $v-u$ bad outcomes. The remaining $2-v$ bad outcomes are generated if an incompetent type is drawn, which happens with probability $\left(\frac{N-i}{N}\right)^{2-v}$. Summing this over i gives the total probability. On the numerator is simply a joint probability that the collective brand produces h and a randomly drawn firm is competent. Therefore, there has to exist at least one competent type, which is represented in the lower bound $i=1$ in the first sum. An additional factor of $\frac{i}{N}$ in the second summation completes the expression.

Now we can calculate the price differences:

$$\lim_{\pi_L\to 0}\Pr^{\text{col}}(C|GG) - \Pr^{\text{col}}(C|GB) =$$

$$\sum_{i=1}^{N}\binom{N}{i}\mu^i(1-\mu)^{N-i}\sum_{j=1}^{N}\binom{N}{j}\mu^j(1-\mu)^{N-j}$$

$$\frac{\pi_H^2\left(\frac{i}{N}\right)^3\pi_H\left(\frac{j}{N}\left(\frac{N-j}{N}\right)+\left(\frac{j}{N}\right)^2(1-\pi_H)\right)-\pi_H\left(\left(\frac{i}{N}\right)^2\left(\frac{N-i}{N}\right)+\left(\frac{i}{N}\right)^3(1-\pi_H)\right)\pi_H^2\left(\frac{j}{N}\right)^2}{\left(\sum_{j=1}^{N}\binom{N}{j}\mu^j(1-\mu)^{N-j}\pi_H^2\left(\frac{j}{N}\right)^2\right)\left(\sum_{j=1}^{N}\binom{N}{j}\mu^j(1-\mu)^{N-j}\pi_H\left(\frac{j}{N}\left(\frac{N-j}{N}\right)+\left(\frac{j}{N}\right)^2(1-\pi_H)\right)\right)} =$$

$$\sum_{i=1}^{N}\binom{N}{i}\mu^i(1-\mu)^{N-i}\sum_{j=1}^{N}\binom{N}{j}\mu^j(1-\mu)^{N-j}\pi_H^3\left(\frac{i}{N}\right)^2\frac{j}{N}$$

$$\frac{\left(\frac{i}{N}\left(\frac{N-j}{N}\right)+\frac{i}{N}\frac{j}{N}(1-\pi_H)\right)-\left(\frac{j}{N}\left(\frac{N-i}{N}\right)+\frac{i}{N}\frac{j}{N}(1-\pi_H)\right)}{\left(\sum_{j=1}^{N}\binom{N}{j}\mu^j(1-\mu)^{N-j}\pi_H^2\left(\frac{j}{N}\right)^2\right)\left(\sum_{j=1}^{N}\binom{N}{j}\mu^j(1-\mu)^{N-j}\pi_H\left(\frac{j}{N}\left(\frac{N-j}{N}\right)+\left(\frac{j}{N}\right)^2(1-\pi_H)\right)\right)} =$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\binom{N}{j}\binom{N}{i}\mu^{i+j}(1-\mu)^{2N-i-j}\pi_H^3\left(\frac{i}{N}\right)^2\frac{j}{N}\frac{\frac{i}{N}-\frac{j}{N}}{Pr^{\text{col}}(GG)Pr^{\text{col}}(GB)}$$

and consequently

$$\begin{aligned}&\lim_{\mu\to1}\frac{1}{1-\mu}\lim_{\pi_L\to0}(\Pr^{\text{col}}(C|GG)-\Pr^{\text{col}}(C|GB)) = \\ &\pi_H^3\frac{N-1}{N^2}\frac{1}{\pi_H^3(1-\pi_H)} = \frac{N-1}{N^2(1-\pi_H)}\end{aligned}$$

Similarly,

$$\begin{aligned}&\lim_{\pi_L\to0}(\Pr^{\text{col}}(C|GB)-\Pr^{\text{col}}(C|BB)) = \\ &\sum_{i=1}^{N}\binom{N}{i}\sum_{j=0}^{N}\binom{N}{j}\mu^{i+j}(1-\mu)^{2N-i-j}\pi_H\frac{i}{N} \\ &\frac{\left(\left(\frac{i}{N}\right)\frac{N-i}{N}+\left(\frac{i}{N}\right)^2(1-\pi_H)\right)\left(\left(\frac{N-j}{N}\right)^2+2\left(\frac{j}{N}\right)\frac{N-j}{N}(1-\pi_H)+\left(\frac{j}{N}\right)^2(1-\pi_H)^2\right)}{\Pr^{\text{col}}(GB)\Pr^{\text{col}}(BB)} \\ &-\frac{\left(\left(\frac{j}{N}\right)\frac{N-j}{N}+\left(\frac{j}{N}\right)^2(1-\pi_H)\right)\left(\left(\frac{N-i}{N}\right)^2+2\left(\frac{i}{N}\right)\frac{N-i}{N}(1-\pi_H)+\left(\frac{i}{N}\right)^2(1-\pi_H)^2\right)}{\Pr^{\text{col}}(GB)\Pr^{\text{col}}(BB)}\end{aligned}$$

and consequently

$$\lim_{\mu\to1}\frac{1}{1-\mu}\lim_{\pi_L\to0}(\Pr^{\text{col}}(C|GB)-\Pr^{\text{col}}(C|BB)) = \frac{N+\pi_H-N\pi_H}{N^2(1-\pi_H)^2}$$

Thus, the cutoff for $\theta = C$ and $h = GG$ can be written as

$$\begin{aligned}\lim_{\mu\to1}\frac{1}{1-\mu}\bar{c}^{\text{col}}(GG,C) &= \pi_H^2\left[\left(\frac{\delta}{N}+\frac{\delta^2}{N}\pi_H\right)\left(\frac{N-1}{N^2(1-\pi_H)}+\frac{\delta^2}{N}(1-\pi_H)\frac{N-(N-1)\pi_H}{N^2\pi_H(1-\pi_H)^2}\right]\right. \\ &= \frac{\delta(N-1+\delta N)\pi_H^2}{N^3(1-\pi_H)}\end{aligned}$$

Thus, we can write

$$\begin{aligned}&\lim_{\mu\to1}\lim_{\pi_L\to0}\frac{1}{1-\mu}(\bar{c}^{\text{col}}(GG,C)-\bar{c}^{\text{ind}}(G)) = \\ &\pi_H^2\frac{\delta}{N}\left[\frac{N-1+\delta N}{N^2(1-\pi_H)}-\frac{\delta}{1-\pi_H}\right] = \\ &\frac{\delta\pi_H^2\left(2(N-1)-\delta N(N^2-2)\right)}{2N^3(1-\pi_H)}\end{aligned}$$

Which is positive for $\delta < \frac{2(N-1)}{N(N^2-2)}$.

□

B On-line Appendix

B.1 Appendix: $T-$Period Memory Analysis and Proofs

In this section, we extend our analysis to a T-period memory for $T > 2$. With a $T-$period memory, a relevant history at period t is of the form $\mathbf{h}_t \in \mathcal{H}^{\text{ind}} := \{G, \varnothing, B\}_T^T$ for an individual brand and $\mathbf{h}_t \in \mathcal{H}_T^{\text{col}} := \{G, B\}^T$ for a collective brand. The history consists of outcomes produced in the previous T periods, $\mathbf{h}_t = h_{t-T}h_{t-T+1}\cdots h_{t-1}$. As time proceeds, consumers' new history consists of the most recent outcomes from $\mathbf{h}_t$ and new outcomes. Let us denote the n most recent outcomes by $\mathbf{h}_t^n = h_{t-n}\cdots h_{t-1}$ for any $1 \leq n \leq T$.

As in Section 4, we start by finding conditions under which the reputational equilibrium exists for an individual and a collective brand. Then, we compare the respective parameter regions to find where the equilibrium exists under a collective, but not under an individual brand. The analysis is similar to that in Section 4, so to avoid redundancy, we omit details.

B.1.1 Individual brand

In a reputational equilibrium, a competent firm must find it optimal to invest after any history. To rule out profitable deviations, we consider the firm's investment decision at period t (also often referred as today) given that the firm will invest whenever visited in the future. By investing, it can add $h_t = G$ to the history $\mathbf{h}_t$ with a greater probability, which will be remembered in the next T periods. $k+1$ periods after period t, consumers would have forgotten the $k+1$ oldest outcomes, and $k+1$ new outcomes are added to the relevant history

$$\mathbf{h}_{t+k+1} = \mathbf{h}_t^{T-k-1}\underbrace{h_t h_{t+1}\cdots h_{t+k}}_{\text{new outcomes}} = \mathbf{h}_t^{T-k} h_t \mathbf{r}_{t+k+1}^{k-1}.$$

The new outcomes are denoted by $h_t \mathbf{r}_{t+k+1}^k$, where h_t is the result of today's investment decision. To simplify the notation and to distinguish the known (old) outcomes and those to be realized, we denote future outcomes $\mathbf{r}_{t+k+1}^k$. Then, conditional on realizing the future outcomes $\mathbf{f}$, the benefit of investing in period t comes from a probabilistic improvement in the history from $\mathbf{h}_t^{T-k-1}B\mathbf{r}_{t+k+1}^k$ to $\mathbf{h}_t^{T-k-1}G\mathbf{r}_{t+k+1}^k$. This allows the firm to receive a higher price $p^{\text{ind}}(\mathbf{h}_t^{T-k-1}G\mathbf{r}_{t+k+1}^k) - p^{\text{ind}}(\mathbf{h}_t^{T-k-1}B\mathbf{r}_{t+k+1}^k)$. The total expected benefit from a decision to invest today then is a sum of such price differences, weighted according to the probability of realizing $\mathbf{r}_{t+k+1}^k$ and accounting for an appropriate discounting.

So, we can compute the benefit of an investment for each history. Then, the reputational equilibrium exists if and only if the cost of investment is less than the minimum of benefits over all histories. We summarize this in the next lemma, which is a general statement of

Lemma 1.

Lemma 3. *For an individual brand, there exists a constant $\overline{c}^{ind} > 0$ such that the reputatoinal equilibrium exists if and only if $c \leq \overline{c}^{ind}$ where*

$$\overline{c}^{ind} = \min_{\mathbf{h}_t^{T-1}} \overline{c}^{ind}(\boldsymbol{h}_t^{T-1}) := \frac{\delta \Delta \pi}{2} \cdot \sum_{k=0}^{T-1} \delta^k \left(\sum_{\mathbf{f} \in \{G, \varnothing, B\}^k} Pr(\mathbf{f})(p(\mathbf{h}_t^{T-k-1} G\mathbf{f}) - p(\mathbf{h}_t^{T-k-1} B\mathbf{f})) \right). \tag{13}$$

Proof. As in Lemma 1, we obtain an expression for the cutoff in terms of price differences. Let $V(\mathbf{h}_t)$ be the expected payoff to the firm in equilibrium:

$$V^{\text{ind}}(\mathbf{h}_t) \equiv \tfrac{1}{2}(p(\mathbf{h}_t) - c) + \delta \left(\frac{\pi_H}{2} \cdot V^{\text{ind}}(\mathbf{h}_t^{T-1} G) + \frac{1-\pi_H}{2} \cdot V^{\text{ind}}(\mathbf{h}_t^{T-1} B) + \frac{1}{2} \cdot V^{\text{ind}}(\mathbf{h}_t^{T-1} \varnothing) \right)$$

As the consumer visits the firm with probability $\frac{1}{2}$, the firm's expected period-t profit is $\frac{1}{2}(p(\mathbf{h}_t) - c)$. The expected future payoff depends on the realized outcome in the current period. The firm produces outcomes G, B, $\varnothing$ with probabilities $\frac{\pi_H}{2}$, $\frac{1-\pi_H}{2}$, $\frac{1}{2}$, respectively.

Once the firm is visited, it should be optimal for the firm to invest always. Given a history $\mathbf{h}_t$ and a consumer's visit, the expected payoff from following the equilibrium strategy is

$$p(\mathbf{h}_t) - c + \delta(\pi_H \cdot V^{\text{ind}}(\mathbf{h}_t^{T-1} G) + (1 - \pi_H) \cdot V^{\text{ind}}(\mathbf{h}_t^{T-1} B)). \tag{14}$$

By deviating and not investing today, the firm expects to obtain the following payoff

$$p(\mathbf{h}_t) + \delta(\pi_L \cdot V^{\text{ind}}(\mathbf{h}_t^{T-1} G) + (1 - \pi_L) \cdot V^{\text{ind}}(\mathbf{h}_t^{T-1} B)).$$

By investing in quality, the firm is able to produce a good outcome with a greater probability π_H, which improves the future payoffs. Then, the condition for the existence of the reputational equilibrium can be expressed as a cutoff-rule; the invest cost is always less than its benefit. So,

$$c \leq \overline{c}^{\text{ind}} := \delta \cdot \Delta \pi \cdot \min_{\mathbf{h}_t^{T-1} \in \{G, \varnothing, B\}^{T-1}} \Delta V^{\text{ind}}(\mathbf{h}_t^{T-1}), \tag{15}$$

where $\Delta V^{\text{ind}}(\mathbf{h}_t^{T-1}) := V^{\text{ind}}(\mathbf{h}_t^{T-1} G) - V^{\text{ind}}(\mathbf{h}_t^{T-1} B)$. The firm is able to receive a higher price in the next T periods due to the good outcome produced today. For this reason, $\Delta V(\mathbf{h}_t^{T-1})$ is a present-discounted weighted-sum of price premiums, as we saw in the analysis for two-period memory:

The future payoff, conditional on producing a good outcome, is

$$V^{\text{ind}}(\mathbf{h}_t^{T-1}G) = \underbrace{\frac{1}{2}\sum_{k=0}^{T-1}\delta^k \sum_{\mathbf{f}\in\{G,\varnothing,B\}^k} \Pr(\mathbf{f})(p(\mathbf{h}^{T-k-1}G\mathbf{f})-c)}_{\text{first } T \text{ periods}} + \underbrace{\frac{1}{2}\sum_{j=0}^{\infty}\delta^{T+j} \sum_{\mathbf{g}\in\{G,\varnothing,B\}^T} \Pr(\mathbf{g})(p(\mathbf{g})-c)}_{\text{after } T \text{ periods}}.$$

$$= \frac{1}{2}\sum_{k=0}^{T-1}\delta^k \left(\left(\sum_{i+j+l=k} (\frac{\pi_H}{2})^i (\frac{1-\pi_H}{2})^j (\frac{1}{2})^l \left(\sum_{N_G(\mathbf{f})=i, N_B(\mathbf{f})=j} p(\mathbf{h}^{T-1-k}G\mathbf{f})\right) - c\right)\right)$$

$$+ \frac{1}{2}\delta^T \sum_{k=0}^{\infty}\delta^k \left(\left(\sum_{i+j+l=T} (\frac{\pi_H}{2})^i (\frac{1-\pi_H}{2})^j (\frac{1}{2})^l \left(\sum_{N_G(\mathbf{g})=i, N_B(\mathbf{g})=j} p(\mathbf{g})\right) - c\right)\right)$$

Given a history $\mathbf{h}_t^{T-1}G$, the relevant history k periods later becomes $\mathbf{h}^{T-k-1}G\mathbf{f}$. That is, consumers replace oldest k memories with a new memory realized throughout k periods, i.e., $\mathbf{f} \in \mathcal{H}^k$. Conditional on the realization of $\mathbf{f}$, the firm's period-profit is $p(\mathbf{h}^{T-k-1}G\mathbf{f}) - c$. This realization occurs with a probability denoted by $\Pr(\mathbf{f})$. Accouting for these probabilities and discounting, we obtain the first double sum in the equation. Once T periods have passed and consumers no longer remember the good outcome of the investment made in period t, the firm's relevant history can be any $T-$period history, $\mathbf{g} \in \mathcal{H}_T^{\text{ind}}$. So, we obtain the second double sum by weighting and discounting each period-profit appropriately. The firm receives a period-profit if and only if the consumer visits, and therefore we divide the whole expression by 2.

To compute $\Pr(\mathbf{f})$, counting the number of good, bad and empty histories is just enough, as the order of each outcome does not matter. Let $N_h(\mathbf{h}_t)$ for $h \in \{G, B, \varnothing\}$ and $\mathbf{h}_t \in \mathcal{H}_T^{\text{ind}}$ be the count of an outcome of type h in the T-period history $\mathbf{h}_t$. For example, $N_G(G\varnothing G) = 2$, $N_B(G\varnothing G) = 0$ and $N_\varnothing(G\varnothing G) = 1$. Suppose $N_G(\mathbf{f}) = i$, $N_B(\mathbf{f}) = j$, and $N_\varnothing(\mathbf{f}) = l$, respectively, such that $i + j + l = k$. Then, $\Pr(\mathbf{f}) = (\frac{\pi_H}{2})^i \cdot (\frac{1-\pi_H}{2})^j (\frac{1}{2})^l$. The next two lines in the equation are results of simply plugging in these probabilities.

Likewise, the future payoff to the firm if it produced a bad outcome would be

$$V^{\text{ind}}(\mathbf{h}_t^{T-1}B) = \frac{1}{2}\sum_{k=0}^{T-1}\delta^k \left(\left(\sum_{i+j+l=k} (\frac{\pi_H}{2})^i (\frac{1-\pi_H}{2})^j (\frac{1}{2})^l \left(\sum_{N_G(\mathbf{f})=i, N_B(\mathbf{f})=j} p(\mathbf{h}^{T-1-k}B\mathbf{f})\right) - c\right)\right)$$

$$+\frac{1}{2}\delta^T \sum_{k=0}^{\infty}\delta^k \left(\left(\sum_{i+j+l=T} (\frac{\pi_H}{2})^i (\frac{1-\pi_H}{2})^j (\frac{1}{2})^l \left(\sum_{N_G(\mathbf{g})=i, N_B(\mathbf{g})=j} p(\mathbf{g})\right) - c\right)\right)$$

Therefore, subtracting the two gives

$$
\begin{aligned}
\Delta V^{\text{ind}}(\mathbf{h}_t^{T-1}) &= \tfrac{1}{2} \cdot \sum_{k=0}^{T-1} \delta^k \sum_{\mathbf{f} \in \{G,\varnothing,B\}^k} \Pr(\mathbf{f})(p(\mathbf{h}^{T-k-1}G\mathbf{f}) - p(\mathbf{h}^{T-k-1}B\mathbf{f})) \\
&= \tfrac{1}{2} \cdot \sum_{k=0}^{T-1} \delta^k \left(\sum_{i+j+l=k} (\tfrac{\pi_H}{2})^i (\tfrac{1-\pi_H}{2})^j (\tfrac{1}{2})^l \sum_{N_G(\mathbf{f})=i, N_B(\mathbf{f})=j} (p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})) \right)
\end{aligned}
$$

Plugging this into (15) completes the proof. □

To obtain an explicit expression for $\overline{c}^{\text{ind}}$, we need to uncover the minimum operator by identifying the binding history for different parameter regions. As in the two-period memory case, we focus on two special signal structures: exclusive knowledge ($\pi_L = 0$) and quality control ($\pi_H = 1$). The former provides an environment where building an extremely high level of reputation is easy for a competent firm, as one good outcome completely reveals its type. Therefore, we can attain the minimum by choosing a history that has a lasting damage to the firm's incentives. This implies that any history $\mathbf{h}_t^{T-1}$ with $h_{t-1} = G$ does the job. Since the most recent outcome in the history is good, consumers know perfectly the firm's type to be good until $t = T - 1$. This eliminates all the benefits to be realized until period $t + T - 1$. The only expression that survives in equation (13) is the very last period $(t + T)$ when $h_{t-1} = G$ will have been forgotten. As this benefit is discounted by δ^T, a longer history clearly hurts investment incentives for an individual brand.

Under the structure of quality control ($\pi_H = 1$), one bad outcome completely reveals a firm to be an incompetent type. Then, similarly, any history with $h_{t-1} = B$ attains the minimum because it puts a bad stamp on the brand for until period $t + T - 1$. Then, all benefits other than ones to be realized in the very last period $(t + T)$, again discounted by δ^T.

Therefore, $\lim_{\pi_L \to 0} \overline{c}^{\text{ind}} = \lim_{\pi_L \to 0} \overline{c}^{\text{ind}}(\mathbf{h}_t)$ where $h_{t-1} = G$, and $\lim_{\pi_H \to 1} \overline{c}^{\text{ind}} = \lim_{\pi_H \to 1} \overline{c}^{\text{ind}}(\mathbf{g}_t)$ where $g_{t-1} = B$. We state next lemma with characterization of the cutoff once we take limits for μ.

Lemma 4. *(i) In an the environment with exclusive knowledge ($\pi_L = 0$), a history in which the most recent outcome is G attains $\overline{c}^{ind}$. If μ is close to 0,*

$$\lim_{\mu \to 1} \lim_{\pi_L \to 0} \frac{\overline{c}^{ind}}{1 - \mu} = \frac{\delta^T \pi_H^2}{2(1 - \pi_H)} \tag{16}$$

(ii). In an environment with quality control ($\pi_H = 1$), a history in which the most recent outcome is B attains $\overline{c}^{ind}$. If μ is close to 0,

$$\lim_{\mu \to 0} \lim_{\pi_H \to 1} \frac{\overline{c}^{ind}}{\mu} = \frac{\delta^T (1 - \pi_L)^2}{2^T \pi_L} \cdot (\frac{1 + \pi_L}{\pi_L})^{T-1}. \tag{17}$$

Proof. First, the binding constraints are identified. Second, the cutoff-level is computed. As the exact cutoff level involves a minimum operator, we need to compare $\Delta V(\mathbf{h}_t^{T-1})$ for all $\mathbf{h}_t^{T-1} \in \{G, B, \varnothing\}$.

First, suppose $\pi_L = 0$, $\pi_H \in (0,1)$. This is the case of exclusive technology where a good outcome reveals the firm to be competent. So, $\mu(\mathbf{h}) = 1$ if and only if $N_G(\mathbf{h}) \geq 1$. Here, the price $p(\mathbf{h}) = \pi_H \cdot \mu(\mathbf{h})$. So,

$$\begin{aligned} p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f}) &= \pi_H \cdot (\mu(\mathbf{h}^{T-1-k}G\mathbf{f}) - \mu(\mathbf{h}^{T-1-k}B\mathbf{f})) \\ &= \pi_H \cdot (1 - \mu(\mathbf{h}^{T-1-k}B\mathbf{f})). \end{aligned}$$

This vanishes if and only if $N_G(\mathbf{h}^{T-1-k}B\mathbf{f}) \geq 1$, i.e. there is at least one good outcome in this history. To find a history that minimizes $\Delta V(\cdot)$, we want as many of the price difference as possible to vanish. For this purpose, it suffices to have $h_{t-1} = G$. Recall h_{t-1} is the outcome produced a period before the focal investment decision. So, the good outcome reveals the firm's competence until it is forgotten T periods later. So, with $h_{t-1} = G$, $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f}) = 0$ for all $\mathbf{f} \in \{G, B, \varnothing\}^k$ for $0 \leq k \leq T-2$. For $k = T-1$, h_{t-1} is forgotten and the relevant price premium is $p(G\mathbf{f}) - p(B\mathbf{f})$. So, for $h_{t-1} = G$,

$$\Delta V^{\text{ind}}(\mathbf{h}) \to_{\pi_L \to 0} \frac{1}{2} \cdot \delta^{T-1} \sum_{\mathbf{f} \in \mathcal{H}^{T-1}} \Big(\Pr(\mathbf{f})(p(G\mathbf{f}) - p(B\mathbf{f}))$$

That is, all benefits other than the one realized in the last period vanish. And, this part is independent of $\mathbf{h}$, the history at the time of investment decision. Therefore, $h_{-1} = G$ indeed attains the minimum for $\Delta V(\cdot)$.

Clearly, $p(G\mathbf{f}) - p(B\mathbf{f})$ vanishes for any $N_G(\mathbf{f}) \geq 1$. Therefore, terms that survive in the equation above are $\mathbf{f}$ of length $T-1$ that only consists of B and/or $\varnothing$. Therefore,

$$\begin{aligned} \lim_{\pi_L \to 0} \Delta V^{\text{ind}}(\mathbf{h}) &= \frac{\delta^{T-1}}{2} \cdot \left(\sum_{j=0}^{T-1} \left(\binom{T-1}{j} \left(\frac{1-\pi_H}{2} \right)^j \left(\frac{1}{2}\right)^{T-1-j} \cdot \pi_H \left(\hat{\mu}(GB^j \varnothing^{T-1-j}) - \hat{\mu}(B^{j+1}\varnothing^{T-1-j}) \right) \right) \right) \\ &= \frac{\delta^{T-1}}{2} \cdot \left(\sum_{j=0}^{T-1} \left(\binom{T-1}{j} \left(\frac{1-\pi_H}{2}\right)^j \left(\frac{1}{2}\right)^{T-1-j} \cdot \pi_H \left(1 - \frac{\mu(1-\pi_H)^{j+1}}{\mu(1-\pi_H)^{j+1} + 1 - \mu} \right) \right) \right) \\ &= \frac{\pi_H(1-\mu)}{2^T} \cdot \delta^{T-1} \left(\sum_{j=0}^{T-1} \left(\binom{T-1}{j} \frac{(1-\pi_H)^j}{\mu(1-\pi_H)^{j+1} + (1-\mu)} \right) \right) \end{aligned}$$

The first equality holds because $\hat{\mu}(GB^j\varnothing^{T-1-j}) = 1$ because a good history causes a full revelation, and $\hat{\mu}(B^{j+1}\varnothing^{T-1-j}) = \frac{\mu(1-\pi_H)^{j+1}}{\mu(1-\pi_H)^{j+1}+1-\mu}$. Simply plugging into (15) proves the lemma for $\pi_L = 0$ and $\pi_H \in (0,1)$. In particular, $\lim_{\mu \to 1} \lim_{\pi_L \to 0} \frac{\Delta V^{\text{ind}}(\mathbf{h})}{1-\mu} = \frac{\pi_H}{2(1-\pi_H)} \cdot \delta^{T-1}$.

Now, consider the case where $\pi_H = 1$ and $\pi_L \in (0,1)$. Here, a bad outcome is revealing of a firm's incompetence. Therefore, $\mu(\mathbf{h}) = 0$ if and only if $N_B(\mathbf{h}) \geq 1$, and $p(\mathbf{h}) = \pi_L$. We omit details for this case, as it is very similar to the previous case.

From equation (16), $h_{t-1} = B$ attains the minimum for $\Delta V^{\text{ind}}(\cdot)$. Then, all price premiums other than the ones to be realized in the last period vanish. Therefore,

$$\begin{aligned}
\Delta V^{\text{ind}}(\mathbf{h}) \quad &\to_{\pi_H \to 1} \quad \frac{1}{2} \cdot \delta^{T-1} \sum_{\mathbf{f} \in \mathcal{H}^{T-1}} \Big(\Pr(\mathbf{f})(p(G\mathbf{f}) - p(B\mathbf{f})) \Big) \\
&= \quad \frac{\delta^{T-1}(1-\pi_L)}{2} \sum_{j=0}^{T-1} \binom{T-1}{j} (\frac{1}{2})^j (\frac{1}{2})^{T-1-j} \Big(\hat{\mu}(G^{j+1} \varnothing^{T-1-j}) - \hat{\mu}(BG^j \varnothing^{T-1-j}) \Big) \\
&= \quad \frac{\delta^{T-1}(1-\pi_L)\mu}{2^T} \sum_{j=0}^{T-1} \binom{T-1}{j} \Big(\frac{1}{\mu + (1-\mu)\pi_L^{j+1}} \Big)
\end{aligned}$$

Plugging this into (15) completes the proof. In particular,

$$\lim_{\mu \to 0} \lim_{\pi_H \to 1} \frac{\Delta V^{\text{ind}}(\mathbf{h})}{\mu} = \frac{\delta^{T-1}(1-\pi_L)}{2^T \pi_L} \cdot (\frac{1+\pi_L}{\pi_L})^{T-1}. \tag{18}$$

□

As we see in equations (16) and (17), the expected benefit to be realized in the last period is a weighted sum, depending on realization of $\mathbf{f}$, the future outcomes following the focal investment decision at period t. The price differences are of the form $p^{\text{ind}}(G\mathbf{f}) - p^{\text{ind}}(B\mathbf{f})$, where $\mathbf{f} \in \{G, \varnothing, B\}^{T-1}$. Under $\pi_L = 0$, if any outcome in $\mathbf{f}$ is G, the difference vanishes, as one good outcome reveals the firm to be competent. So, the summation accounts for the cases where $\mathbf{f} \in \{\varnothing, B\}^{T-1}$, i.e. only bad or empty outcomes constitute $\mathbf{f}$. Likewise, under $\pi_H = 1$, the price difference vanishes if and only if there is a B in $\mathbf{f}$. So, (17) sums over the cases $\mathbf{f} \in \{G, \varnothing\}^{T-1}$.

B.1.2 Collective brand

A longer memory also makes a collective brand to reach a higher level of reputation by producing good outcomes, which makes it hard for firms in the group to further exert a costly investment. However, as we saw in the analysis of the main model with a two-period memory, consumers' limited observability for a collective brand alleviates this problem; as consumers cannot observe history at firm-level, they can never learn perfectly about the types of two firms in the group. Therefore, a competent firm can always improve the brand reputation by investing in quality.

The relevant history for a collective brand with $T-$period memory is $\mathbf{h}_t \in \mathcal{H}_T^{\text{col}} =$

$\{G,B\}^T$. The next lemma establishes the necessary and sufficient condition for the existence of reputational equilibrium. Let $\theta \in \{C,I\}$ denote the other firm's type. $\Pr(\mathbf{f};\theta)$ for $\mathbf{f} \in \{G,B\}^k$ and $\theta \in \{C,I\}$ with $0 \le k \le T$ is the the probability that the brand produces a sequence of outcomes $\mathbf{f}$ in k periods if a competent firm always invests.

Lemma 5. *For a competent firm within a collective brand, there exists a constant $\bar{c}^{col} > 0$ such that the reputational equilibrium exists if and only if $c \le \bar{c}^{col}$ where*

$$\bar{c}^{col} = \min_{\boldsymbol{h}_t^{T-1},\theta} \bar{c}^{col}(\boldsymbol{h}_t^{T-1},\theta) := \frac{\delta\Delta\pi}{2} \cdot \sum_{k=0}^{T-1} \delta^k \Big(\sum_{\mathbf{f}\in\{G,B\}^k} Pr(\mathbf{f};\theta)(p(\mathbf{h}_t^{T-k-1}G\mathbf{f}) - p(\mathbf{h}_t^{T-k-1}B\mathbf{f}))\Big), \tag{19}$$

where $\mathbf{h}_t^{T-1} \in \{G,B\}^{T-1}$ and $\theta \in \{C,I\}$.

Proof. As this lemma is a straightforward generalization of lemma 2, we omit many details. Also, we adopt notation from the proof for 3. Let $V_\theta^{\text{col}}(\mathbf{h}_t)$ denote the payoff to a competent firm of a collective brand before the customer's visit. The brand can be one of types $s \in \{CC, CI\}$.

$$V_\theta^{\text{col}}(\mathbf{h}_t) \equiv \underbrace{\frac{1}{2}(p(\mathbf{h}_t) - c)}_{\text{current period profit}} + \delta \left(\underbrace{\frac{\pi_H + \pi(\theta)}{2} \cdot V_\theta^{\text{col}}(\mathbf{h}_t^{T-1}G) + (1 - \frac{\pi_H + \pi(\theta)}{2}) \cdot V_\theta^{\text{col}}(\mathbf{h}_t^{T-1}B)}_{\text{continuation payoff}} \right).$$

In the current period the firm makes $p(\mathbf{h}_t) - c$ if visited and 0 otherwise. In the next period, the brand will face a history $\mathbf{h}_t^{T-1}G$ or $\mathbf{h}_t^{T-1}B$ depending on today's investment outcome, which also depends on the type of the other firm. So, on average, the firm produces a G with a probability $\frac{\pi_H+\pi(\theta)}{2}$ and a B otherwise.

Once the firm is visited, it should be optimal for the firm to invest always. After a history $\mathbf{h}_t$, by following the equilibrium strategy, the firm expects to receive

$$p(\mathbf{h}_t) - c + \delta(\pi_H \cdot V_\theta^{\text{col}}(\mathbf{h}_t^{T-1}G) + (1-\pi_H) \cdot V_\theta^{\text{col}}(\mathbf{h}_t^{T-1}B))$$

The firm's expected payoff from a deviation is

$$p(\mathbf{h}_t) + \delta(\pi_L \cdot V_\theta^{\text{col}}(\mathbf{h}_t^{T-1}G) + (1-\pi_L) \cdot V_\theta^{\text{col}}(\mathbf{h}_t^{T-1}B)).$$

This is equivalent to

$$c \le \bar{\mathbf{c}}^{\text{col}} := \delta \cdot \Delta\pi \cdot \min_{\mathbf{h}_t^{T-1}\in\{G,B\}^{T-1}} \Delta V_\theta^{\text{col}}(\mathbf{h}_t^{T-1}), \tag{20}$$

where $\Delta V_\theta^{\mathrm{col}}(\mathbf{h}_t^{T-1}) := V_\theta^{\mathrm{col}}(\mathbf{h}_t^{T-1}G) - V_\theta^{\mathrm{col}}(\mathbf{h}_t^{T-1}B)$.

The future payoff, conditional on producing an outcome of either G or B, is

$$V_\theta^{\mathrm{col}}(\mathbf{h}_t^{T-1}G) = \underbrace{\frac{1}{2}\sum_{k=0}^{T-1}\delta^k \sum_{\mathbf{f}\in\mathcal{H}^k}\Pr(\mathbf{f};\theta)(p(\mathbf{h}^{T-k-1}G\mathbf{f}) - c)}_{\text{First } T \text{ Periods}} + \underbrace{\frac{1}{2}\sum_{j=0}^{\infty}\delta^{T+j}\sum_{\mathbf{g}\in\mathcal{H}^T}\Pr(\mathbf{g};\theta)(p(\mathbf{g}) - c)}_{\text{After } T \text{ Periods}},$$

$$V_\theta^{\mathrm{col}}(\mathbf{h}_t^{T-1}B) = \underbrace{\frac{1}{2}\sum_{k=0}^{T-1}\delta^k \sum_{\mathbf{f}\in\mathcal{H}^k}\Pr(\mathbf{f};\theta)(p(\mathbf{h}^{T-k-1}B\mathbf{f}) - c)}_{\text{First } T \text{ Periods}} + \underbrace{\frac{1}{2}\sum_{j=0}^{\infty}\delta^{T+j}\sum_{\mathbf{g}\in\mathcal{H}^T}\Pr(\mathbf{g};\theta)(p(\mathbf{g}) - c)}_{\text{After } T \text{ Periods}}$$

In each period, the brand produces a G with a probability $\frac{\pi_H+\pi(\theta)}{2}$ and a B with the complementary probility. Therefore, for any $\mathbf{h}_t \in \mathcal{H}^{\mathrm{col}}$, if $N_G(\mathbf{h}_t) = i$ and $N_B(\mathbf{h}_t) = j = t - i$, $\Pr(\mathbf{f};\theta) = (\frac{\pi_H+\pi(\theta)}{2})^i(1 - \frac{\pi_H+\pi(\theta)}{2})^j$.

Therefore, subtracting the two gives

$$\begin{aligned}\Delta V_\theta^{\mathrm{col}}(\mathbf{h}_t^{T-1}) &= \frac{1}{2}\cdot\sum_{k=0}^{T-1}\delta^k\sum_{\mathbf{f}\in\mathcal{H}^k}\Pr(\mathbf{f};\theta)(p(\mathbf{h}^{T-k-1}G\mathbf{f}) - p(\mathbf{h}^{T-k-1}B\mathbf{f})) \qquad (21)\\ &= \frac{1}{2}\cdot\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j=k}(\frac{\pi_H+\pi(\theta)}{2})^i(1-\frac{\pi_H+\pi(\theta)}{2})^j\left(\sum_{N_G(\mathbf{f})=i}(p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f}))\right)\right)\end{aligned}$$

Plugging this into (20) completes the proof. □

This lemma generalizes lemma 2. The cutoff now depends on the type of the other firm, as it affects realization of future outcomes **f** through $\Pr(\mathbf{f};\theta)$. Also, prices here are different from those in the individual brand because conditional on a history, posterior beliefs are different.

First, in the exclusive knowledge case, $\pi_L = 0$ and μ close to 1. Then, a good outcome is informative. However, the informativeness of each additional good outcome must be decreasing. For example, having one good outcome compared to none is quite desirable, as it reveals the existence of at least one competent firm. But, having a fifth good outcome in the history in addition to existing four is not as valuable, as consumers already believe with a high probability that both firms are competent. So, in this parameter region, the binding constraint would be provided by an environment that produces as many good outcomes as possible. Naturally, $\mathbf{h}_t^{T-1} = G^{T-1}$ and $\theta = C$ would do the job.

Second, in quality control, $\pi_H = 1$ and μ close to 0. Then, while a bad outcome is informative, it's informativeness decreases as there are more bad outcomes in the history. So, the binding condition would be provided by $\mathbf{h}_t^{T-1} = B^{T-1}$ and $\theta = I$, as together they produce as many bad outcomes as possible in the brand's history.

Then, we can compute the cutoff levels explicitly:

Lemma 6. *(i) Under the environment of exclusive technology ($\pi_L = 0$), if μ is close to 1, $\overline{c}^{col} = \overline{c}^{col}(G^{T-1}, C)$ and*

$$\lim_{\mu \to 1} \lim_{\pi_L \to 0} \frac{\overline{c}^{col}}{1-\mu} = \frac{\delta \pi_H^2}{2^{T+1}(1-\pi_H)} \cdot \frac{1-(2\delta)^T}{1-2\delta} \tag{22}$$

(ii) Under the quality control ($\pi_H = 1$), if μ is close to 0, $\overline{c}^{col} = \overline{c}^{col}(B^{T-1}, I)$ and

$$\lim_{\mu \to 0} \lim_{\pi_H \to 1} \frac{\overline{c}^{col}}{\mu} = \frac{\delta(1-\pi_L)^2}{2^{T+1}\pi_L} \cdot \frac{1-(\frac{\delta}{2}\frac{1+3\pi_L}{\pi_L})^T}{1-\frac{\delta}{2}\frac{1+3\pi_L}{\pi_L}} \tag{23}$$

Proof. The exact cutoff levels in lemma 5 is a discounted sum of price premiums over T periods. It is not feasible to obtain an explicit expression for general parameter regions. We find it useful to understand posterior beliefs denoted by $\eta(\cdot)$. Facing a collective brand, consumers update beliefs over types of the brand, $s \in \{CC, CI, IC, II\}$, and use this to compute the probability of visiting a competent firm: $\eta(\cdot) = \eta_{CC}(\cdot) + \frac{1}{2}(\eta_{CI}(\cdot) + \eta_{IC}(\cdot))$. So, $\eta(\mathbf{h}_t)$, if $N_G(\mathbf{h}_t) = i$, is

$$\eta(\mathbf{h}_t) = \frac{\mu^2 \cdot \pi_H^i (1-\pi_H)^{T-i} + \mu(1-\mu) \cdot (\frac{\pi_H+\pi_L}{2})^i (1-\frac{\pi_H+\pi_L}{2})^{T-i}}{\mu^2 \cdot \pi_H^i (1-\pi_H)^{T-i} + 2\mu(1-\mu) \cdot (\frac{\pi_H+\pi_L}{2})^i (1-\frac{\pi_H+\pi_L}{2})^{T-i} + (1-\mu)^2 \cdot \pi_L^i (1-\pi_L)^{T-i}} \tag{24}$$

It is infeasible to obtain an explicit expression for $\Delta V_\theta(\cdot)$, not to mention the overall cutoff, $\overline{c}^{\text{col}}$. As we did in previous analyses, we i) focus on two signal structures ($\pi_L = 0$ vs. $\pi_H = 1$), ii) identify the binding history and the brand type, and iii) obtain the cutoff level.

First, consider the case $\pi_L = 0$. Then, after a history $\mathbf{h}_t$, the consumer pays $p(\mathbf{h}_t) = \eta(\mathbf{h}_t) \cdot \pi_H$. The reputational benefit realized in each period is the price difference made available by one more good outcome in the history, and thus is of a form $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})$, where $N_G(\mathbf{h}^{T-1-k}G\mathbf{f}) = N_G(\mathbf{h}^{T-1-k}B\mathbf{f}) + 1$. And, here we claim that this difference is decreasing in i for a large enough μ. That is, when $\pi_L = 0$ and μ is large, the price premium reduces as the number of good outcomes becomes large. If this were true, $\mathbf{h}_t^{T-1} = G^{T-1}$ and $\theta = C$ would provide the minimum for $\Delta V_\theta(\mathbf{h}_t^{T-1})$, as these two conditions both places the brand under histories with more good outcomes. We formally state this and prove:

Claim 1. Suppose $\pi_L = 0$ and μ is close to 1. Let $\boldsymbol{r}_1, \boldsymbol{r}_2 \in \mathcal{H}_T^{col}$ such that $N_G(\boldsymbol{r}_1) = i+1$ and $N_G(\boldsymbol{r}_2) = i$. Then, $p(\mathbf{r}_1) - p(\mathbf{r}_2)$ is decreasing in i. So, $\boldsymbol{h}_t^{T-1} = G^{T-1}$ and $\theta = C$ attains the minimum for $\Delta V_\theta(\boldsymbol{h}_t^{T-1})$, and hence are the binding condition for the cutoff level, $\overline{c}^{col}$.

The intuition is the following. As long as there is a good outcome in the history, consumers believe the brand has either one or two competent firms. But, as they see more good outcomes, they become more convinced that both firms are competent. As more good outcomes resolve consumers' uncertainty, the price difference becomes small. Mathematically, from equation (24),

$$
\begin{aligned}
\lim_{\pi_L \to 0} (\eta(\mathbf{r}_1) - \eta(\mathbf{r}_2)) &= \frac{\mu^2 \cdot \pi_H^{i+1}(1-\pi_H)^{T-i-1} + \mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}}{\mu^2 \cdot \pi_H^{i+1}(1-\pi_H)^{T-i-1} + 2\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}} \\
&- \frac{\mu^2 \cdot \pi_H^{i}(1-\pi_H)^{T-i} + \mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}}{\mu^2 \cdot \pi_H^{i}(1-\pi_H)^{T-i} + 2\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}} \\
&= \frac{\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}}{\mu^2 \cdot \pi_H^{i}(1-\pi_H)^{T-i} + 2\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}} \\
&- \frac{\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}}{\mu^2 \cdot \pi_H^{i+1}(1-\pi_H)^{T-i-1} + 2\mu(1-\mu)\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}}
\end{aligned}
$$

Then, taking $\frac{\eta(\mathbf{r}_1)-\eta(\mathbf{r}_2)}{1-\mu}$ to a limit as $\mu \to 1$,

$$
\begin{aligned}
\lim_{\mu \to 1} \lim_{\pi_L \to 0} \frac{\eta(\mathbf{r}_1) - \eta(\mathbf{r}_2)}{1-\mu} &= \frac{\cdot(\frac{\pi_H}{2})^{i}(1-\frac{\pi_H}{2})^{T-i}}{\pi_H^{i}(1-\pi_H)^{T-i}} - \frac{\cdot(\frac{\pi_H}{2})^{i+1}(1-\frac{\pi_H}{2})^{T-i-1}}{\pi_H^{i+1}(1-\pi_H)^{T-i-1}} \\
&= \frac{1}{(1-\pi_H)2^{i+1}}(\frac{1-\frac{\pi_H}{2}}{1-\pi_H})^{T-i-1},
\end{aligned}
$$

which is clearly decreasing in i. Therefore, for any positive integer T, there is a $\bar{\mu}$ close enough to 1 so that, for any $\mu > \overline{\mu}$, the difference in beliefs (and thus prices) is decreasing in i, the number of good outcomes in the history. This completes the proof for the claim.

Then, we plug in $\mathbf{h}^{T-1} = G^{T-1}$ and $\theta = C$ to compute:

$$
\begin{aligned}
\lim_{\mu \to 1} \lim_{\pi_L \to 0} \frac{\Delta V_C^{\text{col}}(G^{T-1})}{1-\mu} &= \frac{\pi_H}{2} \cdot \sum_{k=0}^{T-1} \delta^k \left(\sum_{i+j=k} \pi_H^i (1-\pi_H)^j \left(\sum_{N_G(\mathbf{f})=i} (\eta(\mathbf{h}^{T-1-k}G\mathbf{f}) - \eta(\mathbf{h}^{T-1-k}B\mathbf{f})) \right) \right) \\
&= \frac{\pi_H}{2(1-\pi_H)} \cdot \sum_{k=0}^{T-1} \delta^k \sum_{i+j=k} \left(\pi_H^i (1-\pi_H)^j \binom{k}{i} \frac{1}{2^{T-k+i}} (\frac{1-\frac{\pi_H}{2}}{1-\pi_H})^j \right) \\
&= \frac{\pi_H}{2^{T+1}(1-\pi_H)} \cdot \sum_{k=0}^{T-1} (2\delta)^k \\
&= \frac{\pi_H}{2^{T+1}(1-\pi_H)} \cdot \frac{1-(2\delta)^T}{1-2\delta}
\end{aligned}
$$

Next, we consider the case $\pi_H = 1$. Then, the price consumer pays after a history $\mathbf{h}_t$

is $p(\mathbf{h}_t) = \eta(\mathbf{h}_t) + (1 - \eta(\mathbf{h}_t))\pi_L$. In this setting, a bad outcome is very informative, as it reveals existence of an incompetent firm in the brand. And, intuitively as there are more bad outcomes in the history, informativeness of each bad outcome decrease. Therefore, the price premium to be realized k period after the focal investment decision conditional on the new outcomes $\mathbf{f}$ is $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})$, and this decreases in i, where $i = N_G(\mathbf{h}^{T-1-k}B\mathbf{f})$. We state it formally in the next claim.

Claim 2. Suppose $\pi_H = 1$ and μ is close to 0. And let $N_G(\mathbf{h}^{T-1-k}B\mathbf{f}) = i$. Then, $p(\mathbf{h}^{T-1-k}G\mathbf{f}) - p(\mathbf{h}^{T-1-k}B\mathbf{f})$ is increasing in i. Then, $\boldsymbol{h}_t^{T-1} = B^{T-1}$ and $\theta = I$ attains the minimum for $\Delta V_\theta(\boldsymbol{h}_t^{T-1})$, and hence are the binding condition for the cutoff level, $\bar{c}^{col}$.

From equation (24),

$$\begin{aligned}
\lim_{\pi_H \to 1} \eta(G^T) &= \frac{\mu^2 + \mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^T}{\mu^2 + 2\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^T + (1-\mu)^2\cdot\pi_L^T} \\
\lim_{\pi_H \to 1} \eta(\mathbf{h}) &= \frac{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^{T-i}}{2\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^{T-i} + (1-\mu)^2\cdot\pi_L^i(1-\pi_L)^{T-i}}
\end{aligned}$$

Then, $\lim_{\pi_H \to 1}(\eta(\mathbf{r}_1) - \eta(\mathbf{r}_2)) =$

$$\begin{aligned}
&\frac{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^{i+1}(\frac{1-\pi_L}{2})^{T-i-1}}{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^{i+1}(\frac{1-\pi_L}{2})^{T-i-1} + (1-\mu)^2\cdot\pi_L^{i+1}(1-\pi_L)^{T-i-1}} \\
-\ &\frac{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^{T-i}}{\mu(1-\mu)\cdot(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^{T-i} + (1-\mu)^2\cdot\pi_L^i(1-\pi_L)^{T-i}}
\end{aligned}$$

Then, taking $\frac{\eta(\mathbf{r}_1)-\eta(\mathbf{r}_2)}{\mu}$ to a limit as $\mu \to 0$,

$$\begin{aligned}
\lim_{\mu \to 0}\lim_{\pi_H \to 1} \frac{\eta(\mathbf{r}_1) - \eta(\mathbf{r}_2)}{\mu} &= (\frac{1+\pi_L}{2\pi_L})^{i+1}\frac{1}{2^{T-i-1}} - (\frac{1+\pi_L}{2\pi_L})^i\frac{1}{2^{T-i}} \\
&= \frac{1}{2^T}\frac{(1+\pi_L)^i}{\pi_L^{i+1}}.
\end{aligned}$$

This is clearly increasing in i. Therefore, there is a $\bar{\mu}_{\pi_H=1}$ close enough to 0 so that the difference in beliefs (and thus prices) is increasing in i, the number of good outcomes in the history. This completes the proof for the claim.

Then, we plug in $\mathbf{h}^{T-1} = B^{T-1}$ and $\theta = I$ to compute:

$$\begin{aligned}
\lim_{\mu\to 0}\lim_{\pi_H\to 1}\frac{\Delta V_I^{\text{col}}(B^{T-1})}{\mu} &= \frac{1-\pi_L}{2}\cdot\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j=k}(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^j\left(\sum_{N_G(\mathbf{f})=i}(\frac{1}{2^T}\frac{(1+\pi_L)^i}{\pi_L^{i+1}})\right)\right)\\
&= \frac{1-\pi_L}{2^{T+1}}\cdot\sum_{k=0}^{T-1}\delta^k\left(\sum_{i+j=k}(\frac{1+\pi_L}{2})^i(\frac{1-\pi_L}{2})^j\binom{k}{i}\frac{(1+\pi_L)^i}{\pi_L^{i+1}}\right)\\
&= \frac{1-\pi_L}{2^{T+1}\pi_L}\cdot\sum_{k=0}^{T-1}\frac{\delta^k}{2^k}\left(\frac{1+3\pi_L}{\pi_L}\right)^k\\
&= \frac{1-\pi_L}{2^{T+1}\pi_L}\cdot\frac{1-\frac{\delta^T}{2^T}(\frac{1+3\pi_L}{\pi_L})^T}{1-\frac{\delta}{2}(\frac{1+3\pi_L}{\pi_L})} \qquad (25)
\end{aligned}$$

□

Even in the limits, benefits of investment for a collective brand do not vanish, and the cutoff turns out to be a sum of what turns out to be a finite geometric sequence. Unlike the cutoff for an individual brand, the cutoff is not discounted by δ^T, so it decreases in T as a much slower rate. This highlights the advantage of collective brands over individual ones.

B.1.3 Comparing Individual and Collective Brands

It remains to prove the statement in Proposition 7, in particular the conditions under which $\bar{c}^{\text{col}}$ is greater than $\bar{c}^{\text{ind}}$. We compare the cutoff levels obtained in euqations (16) and (22), and (17) and (23).

Proof. [Proposition 7] For the good news case with $\pi_L = 0$, we compare the cutoff levels we obtained by taking limit of μ to 1. $\bar{c}^{\text{col}} \geq \bar{c}^{\text{ind}}$ in this region if $\lim_{\mu\to 1}\lim_{\pi_L\to 0}\frac{\Delta V_C^{\text{col}}(G^{T-1})}{1-\mu} > \lim_{\mu\to 1}\lim_{\pi_L\to 0}\frac{\Delta V^{\text{ind}}(G^{T-1})}{1-\mu}$.

$$\begin{aligned}
&\frac{\pi_H}{2^{T+1}(1-\pi_H)}\cdot\frac{1-(2\delta)^T}{1-2\delta} > \frac{\pi_H}{2(1-\pi_H)}\cdot\delta^{T-1}\\
\Leftrightarrow \quad &\delta\cdot\frac{1-(2\delta)^T}{1-2\delta} > (2\delta)^T. \qquad (26)
\end{aligned}$$

If $\delta < \frac{1}{2}$, this condition is equivalent to $\frac{\delta}{1-2\delta} > \frac{(2\delta)^T}{1-(2\delta)^T}$. This holds true for every $T \geq 2$. This is because when $T = 2$, $\frac{\delta}{1-2\delta} > \frac{(2\delta)^2}{1-(2\delta)^2}$ if and only if $\delta < 1/2$. Also, the right-hand side is decreasing in T for $\delta < 1/2$.

If $\delta > \frac{1}{2}$, the condition is equivalent to $\frac{1}{2}\cdot\frac{2\delta}{2\delta-1} > \frac{(2\delta)^T}{(2\delta)^T-1}$. We can define $f(x,T) = \frac{x^T}{x^T-1}$, which is decreasing in x^T. For $x > 1$, x^T is increasing both in x and T. The condition above can be re-written as $\frac{1}{2} \cdot f(2\delta, 1) > f(2\delta, T)$. Therefore, the inequality is more likely to hold for a larger T. Also, $\frac{1}{2} \cdot f(2\delta, 1) - f(2\delta, T)$ is decreasing in δ. Therefore, there exists $\overline{\delta}(T)$ such that the condition holds if and only if $\delta < \overline{\delta}(T)$. Greater T expands the scope of this inequality, and therefore $\overline{\delta}(T)$ increases in T.

For the case with $\pi_H = 1$, we compare $\lim_{\mu\to 0} \lim_{\pi_H \to 1} \frac{\Delta V^{\text{ind}}(B^{T-1})}{\mu}$ from equation (18) and $\lim_{\mu\to 0} \lim_{\pi_H \to 1} \frac{\Delta V_I^{\text{col}}(B^{T-1})}{\mu}$ (25), and a collective brand sustains the reputational equilibrium better if and only if

$$\frac{1-\pi_L}{2^{T+1} \cdot \pi_L} \cdot \frac{1 - \frac{\delta^T}{2^T}(\frac{1+3\pi_L}{\pi_L})^T}{1 - \frac{\delta}{2}(\frac{1+3\pi_L}{\pi_L})} > \frac{\delta^{T-1}(1-\pi_L)}{2^T \cdot \pi_L} \cdot (\frac{1+\pi_L}{\pi_L})^{T-1}$$

$$\Leftrightarrow \quad \frac{1}{2} \cdot \frac{1 - \frac{\delta^T}{2^T}(\frac{1+3\pi_L}{\pi_L})^T}{1 - \frac{\delta}{2}(\frac{1+3\pi_L}{\pi_L})} > (\delta \cdot \frac{1+\pi_L}{\pi_L})^{T-1} \tag{27}$$

Because the left-hand side is always increasing in T, (27) is more likely to hold if $\frac{\delta(1+\pi_L)}{\pi_L} \leq 1$. Otherwise, if $\frac{\delta(1+\pi_L)}{\pi_L} > 1$, the right-hand side diverges as T goes to infinity. So, in order for the condition to hold, the left-hand side must diverge at a faster rate. The left-hand side converges if and only if $\delta < \frac{2\pi_L}{1+3\pi_L}$. So, if $\frac{\pi_L}{1+\pi_L} < \delta < \frac{2\pi_L}{1+3\pi_L}$, the condition holds only for a small enough T. If $\delta > \frac{2\pi_L}{1+3\pi_L}$, we can show that the condition cannot hold for T too large.

□

B.2 Appendix: Proofs for the Quality Control Case

This section proves claims made in the paper regarding the quality control case, i.e. $\pi_H = 1$. In this environment, a competent type always produces a good outcome if it exerts investment efforts. An incompetent type can sometimes can sometimes produce a good outcome, and other times a bad outcome, i.e. $\pi_L \in (0, 1)$. So, upon producing a bad outcome, the firm's type to be incompetent.

We prove the following statement that corresponds to Remark 1:

Proposition 9. *(Remark 1) Suppose $\pi_H = 1$ so that a bad outcome reveals a firm's incompetence. If μ is sufficiently close to 0, then $\overline{c}^{col} > \overline{c}^{ind}$ if and only if δ is not too large. If μ is sufficiently close to 1, then $\overline{c}^{col} < \overline{c}^{ind}$.*

Proof. Here, set $\pi_H = 1$ and $0 < \pi_L < 1$. It follows from Lemma 1 that $\overline{c}^{\text{ind}} = \overline{c}^{\text{ind}}(B)$. Similarly, $\overline{c}^{\text{col}} = \overline{c}^{\text{col}}(B; I)$ for high and low values of μ.

With an individual brand, $\lim_{\pi_H \to 1} \bar{c}^{\text{ind}}(B) =$

$$
\begin{aligned}
& \frac{\delta(1-\pi_L)}{2} \cdot \lim_{\pi_H \to 1} (p^{\text{ind}}(GB) - p^{\text{ind}}(BB)) + \frac{\delta}{2}(p^{\text{ind}}(GG) - p^{\text{ind}}(GB) + p^{\text{ind}}(G\varnothing) - p^{\text{ind}}(B\varnothing)) \\
= \quad & \frac{\delta^2(1-\pi_L)^2 \cdot \mu}{4} \left(\frac{1}{\mu + (1-\mu)\pi_L^2} + \frac{1}{\mu + (1-\mu)\pi_L} \right) \Bigg(\\
:= \quad & \frac{\delta(1-\pi_L)^2 \cdot \mu}{4} \cdot Y^{\text{ind}}(\mu, \pi_L).
\end{aligned}
$$

Under a collective brand, $\lim_{\pi_H \to 1} \bar{c}^{\text{col}}(B; I) =$

$$
\begin{aligned}
& \frac{\delta(1-\pi_L)}{2} \cdot \lim_{\pi_H \to 1} p^{\text{col}}(BG) - p^{\text{col}}(BB) + \\
= \quad & \frac{\delta}{2} \cdot ((1+\pi_L) \cdot (p^{\text{col}}(GG) - p^{\text{col}}(GB)) + (1-\pi_L)(p^{\text{col}}(GB) - p^{\text{col}}(BB))) \\
= \quad & \frac{\delta(1-\pi_L)^2 \cdot \mu}{4} \cdot Y^{\text{col}}(\mu, \pi_L)
\end{aligned}
$$

where $Y^{\text{col}}(\mu, \pi_L) = \frac{-2(1+\delta)\mu^3(1-\pi_L)^2 + 2\pi_L(\delta + 2\pi_L + 3\delta\pi_L) + \mu(2+\delta+4(1+\delta)\pi_L - (10+9\delta)\pi_L^2) - 2\mu^2(1-\pi_L)(4\pi_L + \delta(-1+3\pi_L))}{(2-\mu)(\mu(1-\pi_L)+2\pi_L)(\mu(1+\mu)+2(1-\mu)\mu\pi_L+(2-\mu)(1-\mu)\pi_L^2)}$. To make a comparison for μ close to 0, it is sufficient to compare Y^{ind} and Y^{col} in that region:

$$
\begin{aligned}
\lim_{\mu \to 0} Y^{\text{ind}}(\mu, \pi_L) &= \frac{\delta(1+\pi_L)}{\pi_L^2} \\
\lim_{\mu \to 0} Y^{\text{col}}(\mu, \pi_L) &= \frac{2\pi_L + \delta(1+3\pi_L)}{4\pi_L^2}
\end{aligned}
$$

So, $\lim_{\mu \to 0} Y^{\text{col}} > \lim_{\mu \to 0} Y^{\text{ind}}$ if and only if $\delta < \frac{2\pi_L}{3+\pi_L}$. Thus, by continuity, if $\pi_H = 1$ and μ is close to 0, $\bar{c}^{\text{col}} \geq \bar{c}^{\text{ind}}$ for $\delta < \frac{2\pi_L}{3+\pi_L}$.

On the other hand, if μ is close to 0, $\bar{c}^{\text{ind}} \geq \bar{c}^{\text{col}}$ holds always because

$$
\lim_{\mu \to 1} Y^{\text{ind}}(\mu, \pi_L) = 2\delta > \lim_{\mu \to 1} Y^{\text{col}}(\mu, \pi_L) = \frac{1}{2}\delta(1+\pi_L)
$$

for all values of π_L. Therefore, for $\pi_H = 1$ and μ close to 0, an individual brand sustains the reputational equilibrium better. □

Proposition 10. *(Remark 2) Suppose* $\pi_H = 1$. *Then,* $\bar{c}^{ind} < \bar{c}^{col}$ *for all sufficiently small* μ *if and only if* $\delta < \frac{2\pi_L}{3+\pi_L}$. *In that case, for* μ *close to* 0 *and* $c \in (\bar{c}^{ind}, \bar{c}^{col})$, *the* $\{G, \varnothing\}-$*equilibrium always exists for an individual brand. The* $\{G\}-$*equilibrium exists for all sufficiently small* μ *if and only if* $\frac{2\pi_L^2}{1+2\pi_L} < \delta < \frac{2\pi_L}{3+\pi_L}$.

Proof. Now consider the region where $\pi_H = 1$ and μ is close to 0. We already ruled out existence of two equilibria: $\mathcal{S} = \{G, B\}$, and $\{\varnothing\}$. We now show that, for $\overline{c}^{\text{ind}} < c < \overline{c}^{\text{col}}$, $\{B, \varnothing\}-$ and $\{B\}-$equilibrium do not exist, by verifying $\overline{C}^{\text{ind}}_{\mathcal{S}} < \overline{c}^{\text{ind}}$. (Recall an equilibrium $\mathcal{S}$ exists if and only if $\underline{C}^{\text{ind}}_{\mathcal{S}} < c < \overline{C}^{\text{ind}}_{\mathcal{S}}$.)

From equations 11 and 10, for an equilibrium $\mathcal{S} = \{B\}$ and $\mathcal{S} = \{B, \varnothing\}$

$$\begin{aligned}
\lim_{\pi_H \to 1} \overline{C}^{\text{ind}}_{\{B\}} &= \frac{\delta^2 \mu (1-\pi_L)^3 (1+\pi_L)}{4(\mu(1-\pi_L)^2 + (3-\pi_L)\pi_L} > 0 \\
\lim_{\pi_H \to 1} \overline{C}^{\text{ind}}_{\{B,\varnothing\}} &= \frac{\delta^2 \mu (1-\pi_L)^3}{2\mu(2-\pi_L)(1-\pi_L) + 2(3-\pi_L)\pi_L} > 0
\end{aligned}$$

Note that

$$\begin{aligned}
\lim_{\pi_H \to 1} \overline{c}^{\text{ind}} &= \delta\mu(1-\pi_L)^2 \cdot \frac{\delta(\pi_L(1+\pi_L) + \mu(2-\pi_L-\pi_L^2))}{4(\pi_L - \mu(1-\pi_L))(\pi_L^2 + \mu(1-\pi_L^2))}, \\
\lim_{\pi_H \to 1} \overline{C}^{\text{ind}}_{\{B,\varnothing\}} &= \delta\mu(1-\pi_L)^2 \cdot \frac{\delta(1-\pi_L)}{2\mu(2-\pi_L)(1-\pi_L) + 2(3-\pi_L)\pi_L}, \\
\lim_{\pi_H \to 1} \overline{C}^{\text{ind}}_{\{B\}} &= \delta\mu(1-\pi_L)^2 \cdot \frac{\delta(1-\pi_L)(1+\pi_L)}{4(\mu(1-\pi_L)^2 + (3-\pi_L)\pi_L}.
\end{aligned}$$

To compare these values close for μ close to 0, each of these expressions is divided by μ and taken to limit for $\mu \to 0$. Then, the limit is:

$$\frac{\overline{c}^{\text{ind}}}{\mu} \to \frac{\delta(1+\pi_L)}{4\pi_L^2}, \quad \frac{\overline{C}^{\text{ind}}_{\{B,\varnothing\}}}{\mu} \to \frac{\delta(1-\pi_L)}{2(3-\pi_L)\pi_L}, \quad \frac{\overline{C}^{\text{ind}}_{\{B\}}}{\mu} \to \frac{\delta(1-\pi_L^2)}{4(3-\pi_L)\pi_L}.$$

For all values of π_L, $\frac{\delta(1+\pi_L)}{4\pi_L^2} > \frac{\delta(1-\pi_L)}{2(3-\pi_L)\pi_L}$ and $\frac{\delta(1+\pi_L)}{4\pi_L^2} > \frac{\delta(1-\pi_L^2)}{4(3-\pi_L)\pi_L}$, which proves the non-existence of $\{B, \varnothing\}-$ and $\{B\}-$equilibria.

Next, we show the existence of two equilibria, $\mathcal{S} = \{G\}$ and $\{G, \varnothing\}$ in the relevant parameter region. We show this by showing that, in that region, the interval $(\underline{C}_{\mathcal{S}}, \overline{C}_{\mathcal{S}})$ contains $(\overline{c}^{\text{ind}}, \overline{c}^{\text{col}})$, i.e. $\underline{C}^{\text{ind}}_{\mathcal{S}} < \overline{c}^{\text{ind}}$ and $\overline{C}^{\text{ind}}_{\mathcal{S}} > \overline{c}^{\text{col}}$. If $\mathcal{S} = \{G\}$,

$$\begin{aligned}
\lim_{\mu \to 0} \frac{\lim_{\pi_H \to 1} \underline{C}^{\text{ind}}_{\{G\}}}{\delta\mu(1-\pi_L)^2} &= \lim_{\mu \to 0} \frac{\delta(1-\mu)(2-\pi_L)(1+\pi_L) + 2(\pi_L + \pi_L^2 + \mu(2-\pi_L-\pi_L^2))}{4(1-(1-\mu)\pi_L)(\pi_L(1+\pi_L) + \mu(2-\pi_L-\pi_L^2))} \\
&= \frac{2\pi_L + \delta(2-\pi_L)}{4(1+\pi_L)\pi_L} \\
\lim_{\mu \to 0} \frac{\lim_{\pi_H \to 1} \overline{C}^{\text{ind}}_{\{G\}}}{\delta\mu(1-\pi_L)^2} &= \lim_{\mu \to 0} \frac{(4 + (1-\mu)\pi_L(4 + \delta(2-\pi_L)(1+\pi_L)))}{4(1-(1-\mu)\pi_L)(\pi_L(1+\pi_L) + \mu(2-\pi_L-\pi_L^2))} \\
&= \frac{4 + \delta(2-\pi_L)\pi_L}{4(1+\pi_L)\pi_L}.
\end{aligned}$$

If $\mathcal{S} = \{G, \varnothing\}$,

$$\lim_{\mu\to 0} \frac{\lim_{\pi_H\to 1} \underline{C}^{\text{ind}}_{\{G,\varnothing\}}}{\delta\mu(1-\pi_L)^2} = \frac{\delta}{4(\mu + (1-\mu)\pi_L^2)} = \frac{\delta}{4\pi_L^2}$$
$$\lim_{\mu\to 0} \frac{\lim_{\pi_H\to 1} \overline{C}^{\text{ind}}_{\{G,\varnothing\}}}{\delta\mu(1-\pi_L)^2} = \frac{2+\delta\pi_L}{4(\mu + (1-\mu)\pi_L^2)} = \frac{\delta\pi_L + 2}{4\pi_L^2}.$$

From the cutoff levels $\overline{c}^{\text{ind}}$ and $\overline{c}^{\text{col}}$ obtained in proofs for Section 4,

$$\lim_{\mu\to 0} \lim_{\pi_H\to 1} \frac{\overline{c}^{\text{ind}}}{\delta\mu(\pi_H - \pi_L)^2} = \frac{\delta(1+\pi_L)}{4\pi_L^2},$$
$$\lim_{\mu\to 0} \lim_{\pi_H\to 1} \frac{\overline{c}^{\text{col}}}{\delta\mu(\pi_H - \pi_L)^2} = \frac{2\pi_L + \delta(1+3\pi_L)}{16\pi_L^2}.$$

Recall that we are focusing on the case that $\overline{c}^{\text{col}} > \overline{c}^{\text{ind}}$, i.e., $\delta < \frac{2\pi_L}{3+\pi_L}$. Now we check the existence of equilibrium $\mathcal{S} = \{G, \varnothing\}$ and $\{G\}$ by comparing $(\underline{C}_{\mathcal{S}}, \overline{C}_{\mathcal{S}})$ and $(\overline{c}^{\text{ind}}, \overline{c}^{\text{col}})$.

First, $\overline{c}^{\text{col}} < \overline{C}^{\text{ind}}_{\{G,\varnothing\}}$ and $\overline{c}^{\text{ind}} > \underline{C}^{\text{ind}}_{\{G,\varnothing\}}$ for all values of δ and π_L, so $(\overline{c}^{\text{ind}}, \overline{c}^{\text{col}}) \subset (\underline{C}_{\{G,\varnothing\}}, \overline{C}_{\{G,\varnothing\}})$. This implies that whenever $c \in (\overline{c}^{\text{ind}}, \overline{c}^{\text{col}})$, the $\{G, \varnothing\}-$equilibrium exists whenever $\delta < \frac{2\pi_L}{3+\pi_L}$.

Furthermore, $\overline{c}^{\text{col}} < \overline{C}^{\text{ind}}_{\{G\}}$ holds whenever $\delta < \frac{2\pi_L}{3+\pi_L}$. In contrast, $\overline{c}^{\text{ind}} > \overline{C}^{\text{ind}}_{\{G\}}$ holds if and only if $\delta > \frac{2\pi_L^2}{1+2\pi_L}$. Therefore, the $\{G\}-$equilibrium exists whenever $\frac{2\pi_L^2}{1+2\pi_L} < \delta < \frac{2\pi_L}{3+\pi_L}$. $\square$

We have shown that a $\mathcal{S} = \{G, \varnothing\}$-equilibrium always exists for sufficiently small μ as long as $\delta < \frac{2\pi_L}{3+\pi_L}$. If a firm plays a $\{G, \varnothing\}$-equilibrium, then for a competent type the transition matrix of the Markov chain from the previous outcome $h_{t-1} \in \{G, \varnothing, B\}$ to $h_t \in \{G, \varnothing, B\}$ as $\pi_H \to 1$ is given by

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{\pi_L}{2} & \frac{1}{2} & \frac{1-\pi_L}{2} \end{pmatrix}$$

Thus, the stationary distribution of outcomes for a competent firm is given by $\Pr_C(G) = \frac{1}{2}$, $\Pr_C(B) = 0$, $\Pr_C(\varnothing) = \frac{1}{2}$. For an incompetent type, the stationary distribution is $\Pr_I(G) = \frac{\pi_L}{2}$, $\Pr_I(B) = \frac{1-\pi_L}{2}$, and $\Pr_I(\varnothing) = \frac{1}{2}$ because it never makes an investment.

Then, the posterior beliefs in this equilibrium, denoted by $\hat{\mu}^{\text{ind}}(h)$ is obtained by Bayes' rule, i.e., $\mu(\varnothing B) = \mu(B\varnothing) = \mu(BG) = \mu(GB) = \mu(BB) = 0$, $\mu(G\varnothing) = \frac{\mu}{\mu+(1-\mu)\pi_L}$, $\mu(\varnothing G) = \frac{\mu}{\mu+(1-\mu)\pi_L}$, $\mu(\varnothing\varnothing) = \mu$ and $\mu(GG) = \frac{\mu}{\mu+(1-\mu)\pi_L^2}$. We can write the profit of a competent firm in an individual brand as

$$\lim_{\pi_H\to 1} \Pi^{\text{ind}}_{\{G,\varnothing\}} = \frac{1}{4}\left(2 \cdot \frac{\mu + (1-\mu)\pi_L^2}{\mu + (1-\mu)\pi_L} + \mu + (1-\mu)\pi_L + \frac{\mu + (1-\mu)\pi_L^3}{\mu + (1-\mu)\pi_L^2}\right) - c$$

and in the collective brand's reputational equilibrium the profits of a firm in a CI-brand are given by

$$\lim_{\pi_H \to 1} \Pi^{\mathrm{col}} = \left(\frac{1+\pi_L}{2}\right)^2 p^{\mathrm{col}}(GG) + 2 \cdot \frac{(1+\pi_L)(1-\pi_L)}{4} p^{\mathrm{col}}(GB) + \left(\frac{1-\pi_L}{2}\right)^2 p^{\mathrm{col}}(BB) - c.$$

Then, one can show that

$$\lim_{\mu \to 0} \lim_{\pi_H \to 1} \frac{\Pi^{\mathrm{col}} - \Pi^{\mathrm{ind}}_{\{G,\varnothing\}}}{\mu} = \frac{0.125\pi_L^4(\pi_L(\pi_L(3.5-2.5\pi_L)+0.5)-1.5)}{\pi_L^6} < 0.$$

Thus, as π_H is close to 1 a collective firm prefers to play an $\{G, \varnothing\}$-equilibrium to a reputation equilibrium even though it is socially optimal to form a collective brand. Note that for $\pi_H = 1$ individual and collective brands are equally efficient from a welfare perspective.