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NASH EQUILIBRIA OF MARKET GAMES: II. FINITENESS

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by

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1. Introduction

In this paper we continue the investigation of the Nash Equilibria of market games, begun in [3]. Our aim here is to show that if traders' utilities are C^2 then for "almost all" market games the set of Nash Equilibria is finite. Needless to say we take our cue from the analogous results obtained for the Walras Equilibria of markets by Debreu [2] and Smale [5]. However, despite our focus on market games, we can cull out of our proof a finiteness result for various classes of noncooperative games which need have nothing to do with markets. This will be amplified in Remark 4 in Section 4.

2. The Market Game

For concreteness we will focus our attention on the "sell-all model" of a trading game in strategic form ([3], [4], [7]).

For any integer r, let $I_r = \{1, 2, \ldots, r\}$, and let Ω^r be the nonnegative orthant of the Euclidean space of dimension r. Let I_n be the set of traders and I_{k+1} the set of commodities in which they trade. Each trader $i \in I_n$ is characterized by an initial endowment $a^i \in \Omega^{k+1}$ and a utility function $u^i : \Omega^{k+1} \to R$. (Here a^i_j is the quantity of

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commodity j held by trader i .) We assume that $a^i > 0$ for all $i \in I_n$.

To recast the market as a game in strategic form, we single out the $(k+1)^{st}$ commodity as a money. There are k trading posts, one for each of the other commodities. Traders are required to put up all of their first k commodities for sale in these trading posts, and use their endowment of commodity money for bidding on them. The strategy set S^i of trader i consists of bids on the k trading posts, but he is constrained to bid within a^i_{k+1} :

$$S^i = \{b^i \in \Omega^k : \sum_{j \in I_k} b^i_j \leq a^i_{k+1}\} .$$

Given a choice of strategies $(b^1, \ldots, b^n)$, $b^i \in S^i$, prices $p(b) \in \Omega^k$ are formed in the trading posts and the markets cleared, with the final bundle $x^i(b)$ accruing to i , according to the rules:

$$p_j(b) = \frac{\overline{b}_j}{\overline{a}_j} \quad (\text{where } \overline{b}_j = \sum_{i \in I_n} b^i_j \, , \text{ etc.})$$

$$x^i_j(b) = \begin{cases} \dfrac{b^i_j}{p_j(b)} & \text{if } p_j(b) > 0 \\ 0 & \text{if } p_j(b) = 0 \end{cases}$$

for $j = 1, \ldots, k$; and

$$x^i_{k+1}(b) = a^i_{k+1} - \sum_{j=1}^{k} b^i_j + \sum_{j=1}^{k} p_j(b) a^i_j .$$

(We may interpret $x^i_j(b) = 0$ to be a confiscation of goods in the absence of any bid.) A Nash Equilibrium (N.E.) of this game is a choice of strategies $(\hat{b}^1, \ldots, \hat{b}^n)$, $\hat{b}^i \varepsilon S^i$, such that for all $i \varepsilon I_n$,

$$u^i(x^i(\hat{b})) \geq u^i(x^i(\hat{b}|b^i)) , \quad b^i \varepsilon S^i ,$$

where $(\hat{b}|b^i)$ is the same as $\hat{b}$ but with $\hat{b}^i$ replaced by b^i .

3. Finiteness of Nash Equilibria

Let U denote the linear space of all C^2 functions from Ω^{k+1} to R , endowed with the Whitney topology (see [6] for definition). And let Ω_+^k denote the strictly positive orthant of R^k , i.e., $\Omega_+^k = \{x \varepsilon \Omega^k : x > 0\}$. In our analysis, the initial money endowments of the traders , $a_{k+1}^1, \ldots, a_{k+1}^n$, will--for convenience (see Remark 1)--be held fixed, and their utilities and other endowments will vary. Thus we may think of a market game as given by a point $z = (a^1, \ldots, a^n, u^1, \ldots, u^n) \varepsilon (\Omega_+^k)^n \times (U)^n = Z$. The set of all the N.E. of this game we will denote by $\eta(z)$. We are now ready to state our result.

Theorem. There exists a dense and open subset Y of Z such that, for any $z \varepsilon Y$, $\eta(z)$ is finite.

Proof. It will help to first break up $\eta(z)$ into certain subsets. Consider mappings $\alpha : I_n \times I_k \to \{0,1\}$, and denote the set of all such mappings by A . We will interpret $\alpha(i,j) = 1$ (= 0) to mean that--at the N.E. in question--trader i bids positively (zero) on the j^{th} trading post, and partition $\eta(z)$ by all choices of $\alpha \varepsilon A$. Let $I(\alpha) = \{i \varepsilon I_n : \alpha(i,j) = 1$ for some $j \varepsilon I_k\}$, and for $T \subset I(\alpha)$ define

$$\eta(z,\alpha,T) = \{(b^1, \ldots, b^n) \in \eta(z) : b_j^i > 0 \text{ if, and only if, } \alpha(i,j) = 1 \text{ ; and } \sum_{j\in I_k} b_j^i = a_{k+1}^i \text{ if, and only if, } i \in T\} .$$

Then clearly $\{\eta(z,\alpha,T) : \alpha \in A, T \subset I(\alpha)\}$ yields a partition of $\eta(z)$. Now fix an $\alpha \in A$, $\alpha \neq 0$, and a $T \subset I(\alpha)$. (Here T may be the empty set.) For each $i \in I(\alpha)$ define $S^i(\alpha,T)$ as follows:

if $i \in T$,

$$S^i(\alpha,T) = \{b^i \in \Omega^k : b_j^i > 0 \text{ if, and only if, } \alpha(i,j) = 1 \text{ ; } \sum_{j\in I_k} b_j^i = a_{k+1}^i\} \text{ ;}$$

if $i \in I(\alpha)\setminus T$,

$$S^i(\alpha,T) = \{b^i \in \Omega^k : b_j^i > 0 \text{ if, and only if, } \alpha(i,j) = 1 \text{ ; } \sum_{j\in I_k} b_j^i < a_{k+1}^i\} .$$

Let $S(\alpha,T) = \bigtimes_{i\in I(\alpha)} S^i(\alpha,T)$. Denote $\sum_{j\in I_k} \alpha(i,j)$ by $\bar{\alpha}_i$, $\sum_{i\in I(\alpha)} \bar{\alpha}_i$ by $\bar{\alpha}$, and $|T|$ by t. Observe that the dimension of $S^i(\alpha,T) = \bar{\alpha} - t$.

Consider $R^{\bar{\alpha}}$ with its axes indexed by "admissible" pairs (i,j), i.e. those for which $\alpha(i,j) = 1$. For any $v \in R^{\bar{\alpha}}$, v_j^i will be its $(i,j)^{th}$ component. We construct a mapping $\phi(z,\alpha,T) : S(\alpha,T) \to R^{\bar{\alpha}}$. Take any b in $S(\alpha,T)$. View it as an element of $S = \bigtimes_{i\in I_n} S^i$ by putting $b^i = 0$ for $i \in I_n\setminus I(\alpha)$. Then define--for an admissible (i,j) --

$$\phi_j^i(z,\alpha,T)(b) = \frac{\partial u^i(x^i(b))}{\partial x_j^i} \cdot \frac{\bar{a}_j}{\bar{b}_j} + \frac{\partial u^i(x^i(b))}{\partial x_{k+1}^i} \left[\frac{a_j^i}{\bar{a}_j} - 1 \right] ,$$

where the u^i and a^i are, of course, according to z .

Now suppose $b \in \eta(z,\alpha,T)$. Then obviously $b \in S(\alpha,T)$. Furthermore a moment's reflection reveals that we must have, with $I(i,\alpha) = \{j \in I_k : \alpha(i,j) > 0\}$,

$$\phi_j^i(z,\alpha,T)(b) = \phi_\ell^i(z,\alpha,T)(b)$$

for all j and ℓ in $I(i,\alpha)$, and $i \in T$; and also

$$\phi_j^i(z,\alpha,T)(b) = 0$$

for all $j \in I(i,\alpha)$ and $i \in I(\alpha)\setminus T$.

Hence if we consider $\Delta(\alpha,T) \subset R^{\bar{\alpha}}$ given by

$$\Delta(\alpha,T) = \{y \in R^{\bar{\alpha}} : y_j^i = y_\ell^i \quad \text{if } i \in T , \ j \in I(i,\alpha) , \ \ell \in I(i,\alpha) ; \quad y_j^i = 0 \text{ otherwise}\} ,$$

we get $\eta(z,\alpha,T) \subset (\phi^{-1}(z,\alpha,T))(\Delta(\alpha,T)) \subset S(\alpha,T)$. But now note:

(1) the mapping $z \to \phi(z,\alpha,T)$ is a C' representation (see [1] for definition).

(2) $\Delta(\alpha,T)$ is a submanifold of $R^{\bar{\alpha}}$ of codimension $\bar{\alpha} - t$.

(3) $S(\alpha,T)$ is a manifold of dimension $\bar{\alpha} - t$.

(4) The evaluation map

$$e_\phi : Z \times S(\alpha,T) \to R^{\bar{\alpha}}$$

given by

$$e_\phi(z,b) = (\phi(z,\alpha,T))(b)$$

is transversal to $\Delta(\alpha,T)$. (See [1] for definition of transversal.)

(1), (2) and (3) are clear. We will check (4) below.

By Thom's Transversal Density Theory (see [1]) there is a dense and open set $Z(\alpha,T)$ of Z such that for any $z \in Z(\alpha,T)$, $\phi(z,\alpha,T)$ is transversal to $\Delta(\alpha,T)$. This implies that $(\phi^{-1}(z,\alpha,T))(\Delta(\alpha,T))$ has codimension $\bar{\alpha} - t$, i.e. dimension 0 . Consequently it is a discrete set and, being bounded, must consist of a finite number of points. A fortiori $\eta(z,\alpha,T)$ is finite for $z \in Z(\alpha,T)$. Disregarding the unique N.E. with all bids equal to zero, we have

$$\eta(z) = \cup\{\eta(z,\alpha,T) : \alpha \in A, \alpha \neq 0, T \subset I(\alpha)\} .$$

Therefore if $z \in Y = \cap\{Z(\alpha,T) : \alpha \in A, \alpha \neq 0, T \subset I(\alpha)\}$, $\eta(z)$ is finite. But Y , being a finite intersection of dense open sets in Z , is itself dense and open in Z .

To complete the proof we need to check (4). Take any $(z,b) \in Z \times S(\alpha,T)$ and suppose $e_\phi(z,b) = w \in R^{\bar{\alpha}}$. Let v be an arbitrary vector in $R^{\bar{\alpha}}$. First we will show that there is a smooth path $\{(z_t, b_t)\}_{t=0}^{1}$ in $Z \times S(\alpha,T)$ such that:

$$e_\phi(z_0, b_0) = w , \quad (z_0, b_0) = (z, b)$$

$$\frac{d}{dt}[e_\phi(z_t, b_t)]\Big|_{t=0} = v .$$

Letting $z = (a^1, \ldots, a^n, u^1, \ldots, u^n)$ define (z_t, b_t) by:

$$b_t = b$$

$$z_t = a^1, \ldots, a^n, u_t^1, \ldots, u_t^n$$

where

$$u_t^i(x) = u^i(x) + \sum_{j=1}^{k} (v_j^i \bar{b}_j / \bar{a}_j) t x_j$$

for $x \in \Omega^{k+1}$.

Then

$$[e_\phi(z_t, b_t)]_j^i = \phi_j^i(z,\alpha,T) + tv_j^i$$

and thus the path constructed has the requisite properties.

To complete the check of (4) we must prove that for any $(z,b) \in Z \times S(\alpha,T)$ and $w \in \Delta(\alpha,T)$ such that $e(z,b) = w$, the inverse image $\left(T_{(z,b)}e_\emptyset\right)^{-1}\left(T_w\Delta(\alpha,T)\right)$ splits. But this is obvious from the finite-dimensionality of $R^{\overline{\alpha}}$.

Q.E.D.

4. Remarks and Extensions

(1) The requirement that the money endowment be positive is easily waived. Indeed suppose $Z = (\Omega_+^{k+1})^n \times (U)^n$. Now the domain $S(\alpha,T)$ of the mapping $\phi(z,\alpha,T)$ depends upon $z \in Z$. However define $\hat{S}(\alpha,T)$ exactly like $S(\alpha,T)$ but with a_{k+1}^i put equal to 1 for all $i \in T$. Note that $S(\alpha,T)$ is diffeomorphic to $\hat{S}(\alpha,T)$ in the obvious way. Via this diffeomorphism we can consider each $\phi(z,\alpha,T)$ to be defined on $\hat{S}(\alpha,T)$, which is invariant of z , and go through the same proof with straightforward changes.

(2) Also we could have kept the endowments (or utilities) fixed with $Z = (U)^n$ (or $Z = (\Omega_+^{k+1})^n$) and clearly arrived at a similar finiteness result.

(3) The existence of N.E. for such market games has been explored in [3], [7]. For instance let

$$U' = \{u \in U : Du > 0\}$$

and $Z' = (\Omega_+^{k+1})^n \times (U')^n$. Then we know that for any $z \in Z'$, $\eta(z) \neq \phi$. But Z' is an open set in Z . Hence our theorem enables us to say that there is a dense and open subset Y' of Z' (to wit: $Y \cap Z'$) such that for any $z \in Y'$, $\eta(z)$ is nonempty and finite. Of course bigger open sets than Z' exist which guarantee existence of N.E. (see [3], [7]). We took Z' only for ease of description.

(4) From our proof we can extract a finiteness result for Nash Equilibria of noncooperative games in general. Let $S^1, \ldots, S^n$ be the strategy sets of the players in I_n . Assume that each S^i is a bounded subset of R^{k_i} for some integer k_i and that it is "smoothly triangulable" with each of the simplices diffeomorphic to the standard simplex. Denote by Π the

set of all C^2 functions from $\bigtimes_{i \in I_n} S^i$ to the reals. For each $\pi = (\pi^1, \ldots, \pi^n) \in (\Pi)^n$ consider the noncooperative game which has π^i as the payoff function of player i. Let $\eta(\pi)$ be the set of all its N.E. Then imitating our proof—and indeed with more ease—we can show that there exists a dense and open subset Π' of Π such that $\eta(\pi)$ is finite whenever $\pi \in \Pi'$. In fact there is no real need to even keep the S^i fixed. Let M^i be a C^1-manifold (for $i \in I_n$) and suppose that the strategy sets $S^i(m)$ vary with $m \in M^i$. Suppose however that (a) each $S^i(m)$ is diffeomorphic to some smoothly triangulable set $S^i \subset R^{k_i}$ (as before) via $d^i(m) : S^i \to S^i(m)$, (b) the mapping $\Psi : S^i \times M^i \to R^{k_i}$ given by $\Psi(s,m) = (d^i(m))(s)$ is C^1. Let the space of games be $W = (\bigtimes_{i \in I_n} M^i) \times (\Pi)^n$, where Π now consists of C^2 functions defined on $R^{k_1 + \ldots + k_n}$. Again, by imitating our proof we can exhibit a dense and open set W' in W for which the Nash Equilibria are finite.

(5) A related result has been obtained by Harsanyi [5] and Wilson [8]. However, they dealt with "finite" games in which the "pure strategy sets" of the players are finite. Where finiteness (<u>only</u>) of the NE is concerned our method yields a simplification of their proof. On the other hand, it is very likely that oddness of the NE can be proved in our context as well, but it would require a more delicate analysis.

(6) Finiteness of the NE can be established even for non-atomic market games (see [4] for a detailed description of these) under appropriate differentiability assumptions.

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