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Water Storage Policy in a Simplified Hydroelectric System *

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SUMMARY: This papers deals with an electricity generating system that combines one hydroelectric reservoir and generating plant with unlimited thermal generating capacity operating at increasing incremental cost. It constructs a water storage policy that minimizes the operating cost of thermal generation over the planning period while meeting a prescribed demand for power given as a function of time. Associated with this policy are valuations ("efficiency prices") imputed to the power generated and the water used and ("efficiency rents") to the use of reservoir and turbine capacities. These valuations, which vary over time, can be used in coordinating the operation of different generating systems, in electricity rate making, and in designing new systems or modifying existing ones. It is hoped that the methods developed can be extended beyond present simplifying assumptions: that future power demand and water inflow to the reservoir are known with certainty, and that the effect of variation in the head of water can be ignored.

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** I am indebted to Dr. John D.C. Little and to Professor Grant Robley for valuable information about hydroelectric systems.

1. Introduction.

1.1 Purpose of the Study. In a recent article, John D.C. Little [1955] has presented a procedure for computing an optimal water storage policy for an electricity generating system consisting of one hydroelectric unit (reservoir and generating station) and one or more supplementary thermal stations. The latter stations are characterized by an incremental cost that increases with the rate of thermal generation. The main objective of Little's study is to develop and try out a computational procedure that recognizes uncertainty about future water inflow into the reservoir, and to assess the average saving obtainable by a policy of mininizing the mathematical expectation of cost in one particular instance - an approximate model of the Grand Coulee plant on the Columbia River.

While the present study borrows most of its assumptions from Little's work, its intent is somewhat different. Our first purpose is to study the characteristics of an optimal storage policy and to study its dependence on the future paths of water inflow and power demand. Our second purpose is to associate with an optimal storage policy imputed values or "efficiency prices," of the power generated and of the water used or in storage, and imputed "efficiency rents" for the use of the hydroelectric generating plant and of the reservoir. These prices (and rents) are functions of time, and are useful in coordinating the operation of hydroelectric and thermal plants, and of different hydroelectric plants on the same or on different rivers or rivers systems. They can also serve as a basis for rate making according to incremental cost. If the conditions of water inflow and of demand for power assumed in deriving the valuation functions are likely to be repeated seasonally over an extended period of time, these valuations can also help in making decisions about the sizes of reservoirs, about generating capacities, and about watershed improvements to influence runoff.

The mathematical analysis needed to attain these objectives is helped forward decisively by treating time as a continuous variable, even though its computational implementation unless completely graphical would be likely to fall back on a discrete time concept. On the other hand, we shall make this first exploration of our problem on the basis of an assumption of complete certainty about future water inflow and power demand. Little's finding that the explicit recognition of uncertainty in the Grand Coulee situation would save about one percent of supplementary generation cost shows that in some climates policies based on suitable assumptions about water inflow and power demand treated as completely certain are not too far off the mark.* However, our main purpose is the development of concepts

* Actually, the assumptions made with respect to water inflow in this case are based on the worst year on record rather than some average, reflecting a high weight given to the avoidance of interruption of power supply in dry years.

and tools of analysis, rather than immediate applicability. It is believed that our results have value also as a starting point for the systematic analysis of cases involving uncertainty.

The technological facts will therefore be simplified and approximated also in other respects. If in spite of this we aim for mathematical precision in reasoning, the motivation is similar to that which makes a computer add several spurious zero decimals to inaccurate data when embarking on a lengthy computation. Strict consistency of conclusions with premises will help in intuitively assessing the importance of discrepancies between premises and reality.

1.2 The Model. The following list of variables and notations suggests the main features of the model.

Class of Variables	Variable	Notation*	Dimension**
Time	time	t	t
	planning period	τ	t
Given constants	reservoir capacity	Ω	ℓ^3
	turbine capacity	φ	ℓ^3/t
	initial, and minimum final, store of water	Ω_o, Ω_τ	ℓ^3
Given technological functions	cost of thermal generation at rate s	$\psi(s)$	$\$/t$
	conversion function (power output if $v=1$ and store of water $= W$)	$\lambda(W)$	e/ℓ^3
Given functions of time	rate of inflow of water into the reservoir	$\xi(t)$	ℓ^3/t
	demand for power	$\zeta(t)$	e/t
Policy functions	rate of discharge of water through turbines	$v(t)$	ℓ^3/t
	spillage of water	$u(t)$	ℓ^3/t

Class of Variables	Variable	Notation*	Dimension**
Derived functions	net increase in the store of water	$w(t)$	ℓ^3/t
	maximum rate of turbine discharge	$\eta(t)$ or $y(t)$	ℓ^3/t
	rate of supplementary thermal generation	$s(t)$	e/t
Prices and rents	price of power	$p(t)$	$\$/e$
	price of water	$q(t)$	$\$/\ell^3$
	rent of reservoir	$Q(t)$	$\$/t$
	rent of turbines	$R(t)$	$\$/t$

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* Greek symbols denote given quantities, latin symbols (other than t) denote policy variables and variables dependent thereon. Capital letters refer to stocks, small letters (other than t, τ and μ) refer to flows.

** In this column, t = time, ℓ = length, ℓ^3 = volume, $\$$ = money and e = electric energy.

The determination of water storage policy is assumed to be made for a finite period ahead, $0 \leqq t \leqq \tau$, to be called the planning period, and starts from a given initial store of water,*

$$W(0) = \Omega_0 , \quad \text{where} \quad 0 \leqq \Omega_0 \leqq \Omega . \tag{1.1}$$

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* The store of water is defined to exclude "dead storage," that is, water at levels below the lowest turbine gates.

At time t the store of water then is

$$W(t) = W(0) + \int_0^t w(t')dt' , \qquad 0 \leqq t \leqq \tau , \tag{1.2}$$

where the instantaneous net rate of increase $w(t)$ in the store of water depends on the policy functions through the inventory identity

$$w(t) = \xi(t) - u(t) - v(t) . \tag{1.3}$$

This identity says that the net excess of inflow $\xi(t)$ over outflow through turbines $v(t)$ or spillways $u(t)$ becomes a net increase to the store of water.*

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* Evaporation and leakage, if recognized at all, must be regarded as a deduction from inflow in defining $\xi(t)$. The dependence of these water losses on the store of water must then be ignored.

We shall restrict the end-of-planning-period store of water by a prescribed lower bound

$$W(\tau) \geqq \Omega_\tau \quad \text{where} \quad 0 \leqq \Omega_\tau \leqq \Omega . \tag{1.4}$$

The special case where the final store of water is of no concern is then obtained by setting $\Omega_o = 0$.

A characteristic feature of the problem is the simultaneous presence of restrictions on stores and on flows. The given size of the reservoir places the store of water between two bounds

$$0 \leqq W(t) \leqq \Omega , \qquad 0 \leqq t \leqq \tau , \tag{1.5}$$

if we ignore minor fluctuations in reservoir capacity due to wind, flow or temperature. On the other hand, turbine capacity places similar bounds on the rate of discharge,

$$0 \leqq v(t) \leqq \varphi , \qquad 0 \leqq t \leqq \tau , \tag{1.6}$$

if we ignore possible harmful downstream effects of low rates of discharge implicitly recognized by Little,* and if we disregard the possibility of

* ℓ.c. Table I on p. 192, and correspondence.

using thermal power to pump water back into the reservoir.

The power output from a given rate of discharge v through the turbines will be treated as proportional to v, although this is only approximately true. The factor of proportionality λ in turn depends primarily on the "head" of water, that is, on the difference h in surface level between the water in the reservoir and the "tailwater" back of the dam. The precise form of this dependence is determined by turbine design, and in some plants the conversion factor λ decreases for heads larger than a designed most favorable level. We shall however assume constant turbine efficiency, which makes λ proportional to h. The head h in turn depends primarily on the store W of water, if we ignore the possible effect of total rate of outflow $u(t) + v(t)$ on tailwater level. While the precise form of this dependence is determined by the shape of the reservoir, h is necessarily an increasing function of W. We shall therefore write

$$(1.7) \qquad v \cdot \lambda(W)$$

for the power output of a discharge v, and choose the units of water flow and of electric power in such a way that $\lambda(W)$ satisfies

$$(1.8) \qquad 0 \leq \lambda(0) \leq \lambda(W) < \lambda(W') \leq \lambda(\Omega) = 1 \quad \text{for} \quad 0 \leq W < W' \leq \Omega .$$

We shall refer to $\lambda(W)$ as the conversion function.

In some cases the range of reservoir surface levels is small compared with the drop in elevation from turbines gates to the tailwater pool. In this case a good approximation is obtained by ignoring all variation in the conversion factor,

$$(1.9) \qquad \lambda(W) = 1 \quad \text{for} \quad 0 \leqq W \leqq \Omega ,$$

and a considerable simplification of the analysis results. The present paper is entirely concerned with this special case.

Since the problem to be studied is in the first instance one of operation rather than design, we take into account only operating cost, but not long run commitments such as service on capital invested in reservoir and generating stations, or labor and maintenance cost independent of fluctuations in power output. On this basis, the operating cost of hydroelectric generation can be neglected as compared with that of thermal generation, in which fuel input is the dominant operating cost.

In arranging for the necessary thermal generation, one will naturally first utilize the most efficient units of available capacity, and draw on additional units or stations in order of increasing operating cost, possibly including transmission losses. It is therefore justified - and extremely helpful to the analysis - to assume that the incremental operating cost of thermal generation, starting at a positive level γ at the rate $s=0$, can never decrease but can increase when going to higher and higher rates of generation. We shall express these conditions in terms of the total cost function $\psi(s)$ rather than the incremental cost, to avoid the unrealistic assumption of differentiability of $\psi(s)$ for all values of s. We thus specify that $\psi(s)$ is increasing with s (1.10a), has a positive slope at $s=0$ (1.10b), and is convex (1.10c), that is, the curve representing $\psi(s)$ remains below any of its chords.

$$(1.10)\quad \begin{cases} (1.10a) & 0 = \psi(0) \leqq \psi(s) < \psi(s') \quad \text{for} \quad 0 \leqq s < s' , \\ (1.10b) & \psi(s) \geqq \gamma \cdot s \quad \text{for} \quad 0 \leqq s , \quad \text{where} \quad \gamma > 0 , \\ (1.10c) & \mu\psi(s) + \mu^*\psi(s^*) \geqq \psi(\mu s + \mu^* s^*) \quad \text{for} \quad 0 < \mu = 1-\mu^* < 1 \end{cases}$$

These requirements are satisfied in the realistic case where $\psi(s)$ follows a broken line (see Figure 1) with successive segments increasing in slope. It so happens that the presence of linear segments in $\psi(s)$ introduces considerable indeterminacy in the optimal storage policy. We shall therefore in most of this

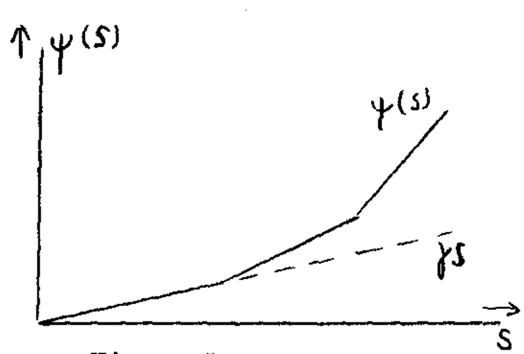

Figure 1. Convex and piece-wise linear cost function of thermal generation.

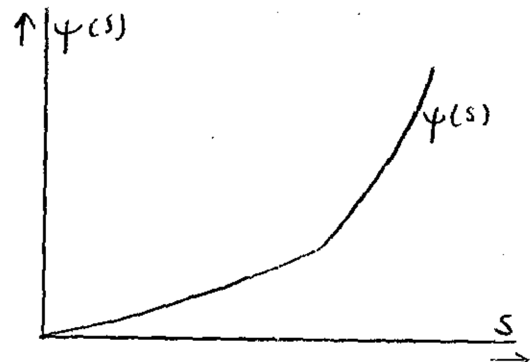

Figure 2. Strictly convex cost function of thermal generation.

paper use an assumption of strict convexity,

$$(1.10c') \qquad \mu\psi(s) + \mu^*\psi(s^*) > \psi(\mu s + \mu^* s^*) \quad \text{for} \quad 0 < \mu = 1-\mu^* < 1 \quad \text{and} \quad s \neq s^* ,$$

which introduces some degree of upward curvature throughout the curve (see Figure 2) without precluding a finite number of upward kinks. Conditions (1.10a), (1.10b) and (1.10c') will together be referred to as (1.10').

The total operating cost of thermal generation over the planning period can now be written as

(1.11) $$C = \int_0^\tau \psi \left\{ s(t) \right\} dt,$$

where the rate of such generation,

(1.12) $$s(t) = \zeta(t) - v(t) \cdot \lambda \left\{ W(t) \right\}$$

is that part of the given demand for power $\zeta(t)$ not met by hydroelectric generation. Our problem is to minimize the quantity C by proper choice of the policy functions $u(t)$, $v(t)$, subject to the restraints stated above, and subject to the nonnegativity of the variables $s(t)$, $u(t)$, $v(t)$ measuring the levels of the irreversible processes of thermal generation, spillage and discharge, respectively.

In order to have a meaningful problem, we specify further that water inflow and power demand over the planning period are nonnegative* bounded

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* This implies an assumption that evaporation and leakage does not exceed gross water inflow. While this assumption is not indispensable, its removal would somewhat complicate the analysis.

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functions of time,

(1.13) $$0 \leqq \xi(t) \leqq \bar{\xi} , \quad 0 \leqq \zeta(t) \leqq \bar{\zeta} , \quad 0 \leqq t \leqq \tau .$$

If the prescribed minimum end-of-period store Ω_τ exceeds the initial store Ω_o we must also require the stronger condition

$$(1.14) \qquad \int_o^\tau \xi(t)\, dt \geqq \Omega_\tau - \Omega_o$$

for the prescribed minimum store Ω_τ to be attainable.

1.3 Remarks on mathematical tools and notation. The problem of this paper belongs to the category of inventory or production smoothing problems on which fundamental work has been done in the past few years.* The price imputation to be developed in Section 3 is made possible by special traits of the present problem. As long as the conversion function is treated as a constant, the restraints on the policy functions indicated above define a convex set,** and the minimand is a convex functional of the policy functions.*** Hence

* See in particular Arrow, Harris and Marschak [1951], Dvoretzky, Kiefer and Wolfowitz [1952], Modigliani and Hohn [1955] and Morin [1955].

** That is, if $(u(t),v(t))$ and $(u^*(t),v^*(t))$ are two feasible policies, then $(\mu u(t) + \mu^* u^*(t), \mu v(t) + \mu^* v^*(t))$ is a feasible policy whenever $0 < \mu = 1 - \mu^* < 1$. Here a feasible policy is one that satisfies the restraints specified.

*** That is, if $C(u,v)$ denotes the minimand, and (u,v) and (u^*,v^*) are feasible policy functions, then $\mu\, C(u,v) + \mu^*\, C(u^* v^*) \geqq C(\mu u + \mu^* u^*, \mu v + \mu^* v^*)$ whenever $0 < \mu = 1 - \mu^* < 1$.

in this case we have a problem of nonlinear but convex programming, the main complication being that the policies to be determined are functions of a continuous time variable rather than vectors with only a finite number of components. While the latter case has been rather fully explored,* the more difficult former case is still the

* See, for instance, Kuhn and Tucker [1951], and Tucker [1957].

subject of continuing mathematical research.** In the present study

** See Ky Fan [1956] and unpublished studies by Hurwicz [1954] and by Bratton [1955].

we shall use the results of this research only heuristically. We shall first obtain the unique optimal policy by direct construction. Instead of proving its optimality directly (which is entirely possible), we shall then specify the efficiency price functions, and give a direct proof of that one of the two so-called "saddle-point inequalities" of convex programming which is relevant to our present purpose. This inequality will yield us a proof of the optimality of the constructed policy as well as an interpretation of the efficiency prices themselves which clarifies their uses already indicated.

Knowledge of convex programming theory is therefore not presupposed in our proofs, but underlies the choice of propositions to be proved.

The more technical parts of the proofs are set off in starred sections, and can be passed over by readers interested mainly in results.

The obvious restriction $0 \leqq t, t', \ldots \leqq \tau$ on the time symbols $t, t', \ldots$ will henceforth be omitted except where needed for clarity. Likewise the identity connecting rates of flow and cumulative functions will be assumed throughout without restatement. With one exception, all cumulative functions will start at $t = 0$ at the value zero,

$$(1.15) \qquad U(t) \equiv \int_0^t u(t)dt', \ldots, \quad \Xi(t) \equiv \int_0^t \xi(t')dt', \ldots .$$

The exception is formed by the various store-of-water functions $W(t)$, $W_v(t)$, $\Omega(t)$, $W(t,s)$, $\hat{W}(t)$, all of which start at Ω_0 as shown in (1.1), (1.2). We use the symbol $\equiv$ for equality be definition. The integration used is Riemann or in some cases Riemann-Stieltjes integration. The following notation defines a function of v important in connection with the turbine restraints,

$$(1.16) \qquad [0 \mid v \mid \varphi] = \begin{cases} 0 & \text{if } v < 0 \\ v & \text{if } 0 \leqq v \leqq \varphi \\ \varphi & \text{if } \varphi < v . \end{cases}$$

1.3* We shall make certain regularity assumptions concerning the given functions $\xi(t)$, $\zeta(t)$, which, while not essential for the propositions proved, simplify the proofs and appear to be unobjectionable from the point of view of applications. We shall call an interval $t_1 \leqq t < t_2$ <u>uniform</u> with regard to a function $f(t)$ if $f(t)$ is either increasing, or constant, or decreasing throughout that interval. We shall call a function $f(t)$ <u>regular</u> on $0 \leqq t < \tau$

if it is continuous to the right for all such t, continuous to the left for all but a finite number of values of t, and if the interval of definition can be partitioned into a finite number of intervals uniform with regard to $f(t)$. In what follows, we shall assume that $\xi(t)$, $\zeta(t)$ and $\xi(t) - \zeta(t)$ are regular on $0 \leq t < \tau$. If $f(t)$ is discontinuous to the left at $t = t_o$, we write $f(t_o - 0) \equiv \lim_{\substack{t \to t_o \\ t < t_o}} f(t)$.

We shall denote the set of elements x with the property P by $\{x \mid P\}$. Closed and half open intervals are denoted by

$$(1.17) \qquad [t_1,t_2] \equiv \left\{ t \mid t_1 \leqq t \leqq t_2 \right\}, \quad [t_1,t_2) \equiv \left\{ t \mid t_1 \leqq t < t_2 \right\}.$$

Since we shall use no open intervals, the notation (t_1,t_2) remains available for the vector (ordered pair) with elements t_1, t_2. On the other hand, $\{t_1,t_2\}$ denotes the set consisting of two elements, t_1 and t_2, without regard to order. $T \cup T'$ and $T \cap T'$ denote the union and the intersection, respectively, of the sets T and T', and $T \subset T'$ and $T' \supset T$ express equivalently that T is contained in T'. Finally, $t \in T$ denotes that t is an element of T.

2. Construction of an optimal storage policy.

2.1 Automatic overflow policy. In this section 2 we shall construct an optimal water storage policy under the assumption (1.9) that the power equivalent of one unit of water discharge is independent of the water level in the reservoir. The problem to be solved then is that of minimizing

$$C(v) \equiv \int_0^{\tau} \psi\,(\,\zeta(t) - v(t)\,)dt \tag{2.1}$$

subject to the restraints enumerated, which we shall now group under two headings, those of <u>turbine feasibility</u>* (2.2) and those of <u>reservoir feasibility</u> (2.3),

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* (2.2b) arises from the nonnegativity of the **eliminated** variable $s(t)$.

(2.2)

$$\begin{cases} (2.2a) & 0 \leqq v(t) \leqq \varphi \\ (2.2b) & v(t) \leqq \zeta(t), \end{cases}$$

(2.3)

$$\begin{cases} (2.3a) & u(t) + v(t) + w(t) = \xi(t) \\ (2.3b) & u(t) \geqq 0 \\ (2.3c) & 0 \leqq W(t) \leqq \Omega \\ (2.3d) & W(0) = \Omega_o\,, \qquad\qquad W(\tau) \geqq \Omega_\tau\,. \end{cases}$$

A policy satisfying both groups of constraints is simply called <u>feasible</u>.

The fact that we do not recognize flood control or navigation as objectives suggests that, for a given feasible discharge policy $v(t)$, one can never go wrong if one postpones all spilling of water until the reservoir is incapable of holding the further net additions arising from that discharge policy. To explore the idea of such an <u>automatic overflow policy</u>, we consider a <u>guide function</u>

$$W_v(t) \equiv \Omega_o + \Xi(t) - V(t)\,, \tag{2.4}$$

which represents what would be the store of water at time t under the discharge policy v(t) if reservoir capacity were unbounded from above and below, and if no spillage were ever permitted.

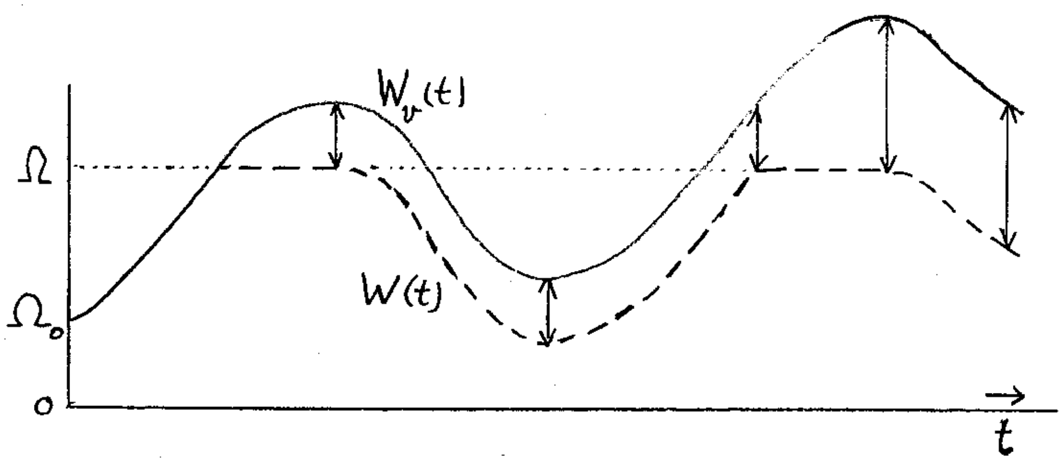

Figure 3. Illustration of an automatic overflow policy

As illustrated by Figure 3, under an automatic overflow policy the store of water $W(t)$ follows the guide function $W_v(t)$ from $t = 0$ up to the point beyond which the latter exceeds Ω. Thereafter the reservoir remains full as long as $W_v(t)$ continues to increase, but as soon as $W_v(t)$ reaches a temporary maximum overflow stops and thereafter $W(t)$ preserves a constant difference with $W_v(t)$ for as long as this is feasible, and so on. We shall show in section 2.1* that, if the given discharge policy $v(t)$ is feasible in combination with some spillage policy $u(t)$, it is feasible also in combination with the automatic overflow policy $u_v(t)$ determined by it. Since $u(t)$ does not directly enter the minimand, it follows that we can in all later discussion of the present case assume that spillage only takes the form of automatic overflow.

2.1* The automatic overflow policy may be defined through its cumulative function

$$(2.5) \qquad U_v(t) \equiv \max \left\{ 0 \; , \; \max_{0 \leqq t' \leqq t} W_v(t') - \Omega \right\} ,$$

and the resulting store of water is derived from the guide function by

$$(2.6) \qquad W(t) = W_v(t) - U_v(t) = \begin{cases} W_v(t) & \text{if } \max\limits_{0 \leqq t' \leqq t} W_v(t') \leqq \Omega , \\ \Omega + W_v(t) - \max\limits_{0 \leqq t' \leqq t} W_v(t') & \text{otherwise.} \end{cases}$$

Now let $v(t)$ be a discharge policy feasible in combination with some spillage policy $u(t)$, automatic or not. We translate the reservoir restraints (2.3) in terms of $u(t)$ and the guide function $W_v(t)$ derived from $v(t)$, where it is understood that $W_v(0) = \Omega_o$ throughout.

$$(2.7) \quad \begin{cases} (2.7b) & U(0) = 0 \text{ , } U(t) \text{ is nondecreasing,} \\ (2.7c) & 0 \leqq W_v(t) - U(t) \leqq \Omega , \\ (2.7d) & W_v(\tau) - U(\tau) \geqq \Omega_\tau . \end{cases}$$

From (2.7) we have in turn, for $t' \leqq t$

$$(2.8) \quad \begin{cases} (2.8b) & W_v(t) \geqq U(t) \geqq 0 , \\ (2.8c) & W_v(t) \geqq U(t) \geqq U(t') \geqq W_v(t') - \Omega , \\ (2.8d) & W_v(\tau) \geqq U(\tau) + \Omega_\tau \geqq U(t') + \Omega_\tau \geqq W_v(t') - \Omega + \Omega_\tau , \end{cases}$$

which, in terms of $W_v(t)$ only, requires

$$
(2.9)\quad \begin{cases} (2.9c) \quad W_v(t) \geqq \max \left\{ 0 \, , \, \max\limits_{0 \leqq t' \leqq t} W_v(t') - \Omega \right\} \, , \\[2ex] (2.9d) \quad W_v(\tau) \geqq \max\limits_{0 \leqq t \leqq \tau} W_v(t) - \Omega + \Omega_\tau \, . \end{cases}
$$

But whenever (2.9) is met, the automatic overflow policy $U_v(t)$ together with $v(t)$ meets the reservoir restraint (2.7), because (2.5) implies (2.7b), (2.5) and (2.9c) imply the first inequality in (2.7c), (2.6) implies the second, and (2.5) and (2.9d) imply (2.7d). Our contention is thereby established. In addition we have found (2.9) to be the expression of the reservoir restraints in terms of the discharge policy $v(t)$ alone.

2.2 Demarcation of periods of abundance and of scarcity of water. It is obvious that, if it were reservoir-feasible to make the discharge throughout the planning period equal to its turbine-feasible upper bound (turbine capacity or power demand whichever smaller) this would be optimal. It is therefore a logical next step to study the guide function associated with this particular maximal discharge policy. If for brevity we write the turbine restraint (2.2) in the form

$$
(2.10)\qquad v(t) \leqq \eta(t) \equiv \min \left\{ \varphi \, , \, \zeta(t) \right\} \, ,
$$

and denote the cumulative function of the maximal discharge $\eta(t)$ by $H(t)$, this guide function is defined by

$$
(2.11)\qquad W_\eta(t) \equiv \Omega_0 + \Xi(t) - H(t) \equiv \Omega(t) \, , \text{ say.}
$$

Obviously, the feasibility of the maximal discharge policy is not affected by any rises in the guide curve $\Omega(t)$, indicating a temporary excess of water inflow over the maximal discharge. Inspection of Figure 3 suggests, however, and our analysis in section 2.1* confirms (see (2.9c)), that any dip in $\Omega(t)$ extending by more than Ω below any previously assumed value, would signal infeasibility of the maximal discharge policy in the intervening part of the period. The swings of the guide curve $\Omega(t)$ therefore hold the clue to the character of the solution in each part of the planning period. To bring this out more fully, we shall now construct a subdivision of the planning period into alternating periods of <u>scarcity</u> and of a<u>bundance</u> (or at least <u>sufficiency</u>) of water, with the following characteristics. Within a typical period of abundance, the guide function rises by at least Ω from a minimum at the beginning of that period to a maximum at its end, while any downward swings within the period cannot exceed Ω in range.

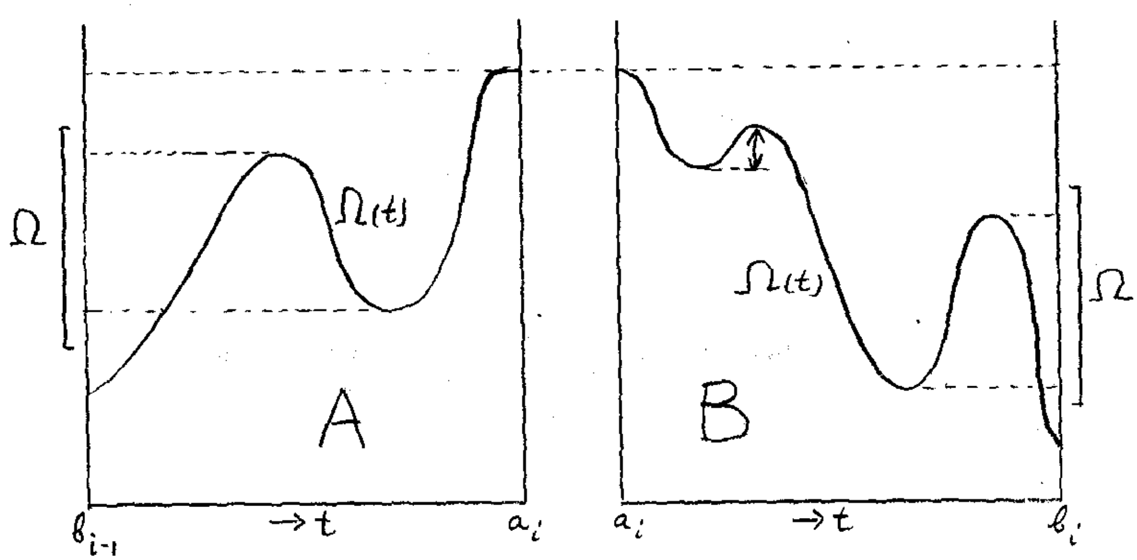

Figure 4. Guide curves in periods of water abundance (A) and scarcity (B).

To be precise, if $b_{i-1} \leqq t \leqq a_i$ is the period of abundance,

$$(2.12)\begin{cases}(2.12a) & \Omega(b_{i-1}) + \Omega \leqq \Omega(a_i) \ , \\ (2.12b) & \Omega(b_{i-1}) \leqq \Omega(t) \leqq \Omega(a_i) \quad \text{for} \quad b_{i-1} < t < a_i \ , \\ (2.12c) & \Omega(t) - \Omega(t') \leqq \Omega \qquad \text{for} \quad b_{i-1} < t < t' < a_i \ .\end{cases}$$

Within a typical period of scarcity, $a_i \leqq t \leqq b_i$ say, the guide function falls by more than Ω from a unique maximum at the beginning to a unique minimum at the end, and within the period never rises by as much as Ω ,

$$(2.13)\begin{cases}(2.13a) & \Omega(a_i) - \Omega > \Omega(b_i) \ , \\ (2.13b) & \Omega(a_i) > \Omega(t) > \Omega(b_i) \quad \text{for} \quad a_i < t < b_i \ , \\ (2.13c) & \Omega(t') - \Omega(t) < \Omega \qquad \text{for} \quad a_i < t < t' < b_i \ .\end{cases}$$

These statements, illustrated in Figure 4, need modification for <u>terminal</u> periods, that is, for the first and last period demarcated, or for the entire planning period if it should have only one character, scarcity or abundance.

2.2* Let us call a point t_o such that $0 < t_o < \tau$ a (local) <u>maximizer</u> of $\Omega(t)$ if there exist numbers $\gamma \geqq 0$ and δ , $\epsilon > 0$ such that

$$(2.14)\quad \left.\begin{cases}\Omega(t) < \Omega(t_o) & \text{for} \quad t_o - \gamma - \delta \leqq t < t_o - \gamma \\ \Omega(t) = \Omega(t_o) & \text{for} \quad t_o - \gamma \leqq t \leqq t_o \\ \Omega(t) < \Omega(t_o) & \text{for} \quad t_o < t \leqq t_o + \epsilon\end{cases}\right\} \quad \begin{matrix}\text{and} \\ 0 \leqq t \leqq \tau \ ,\end{matrix}$$

and a <u>minimizer</u> if for some such numbers

$$(2.15)\qquad \left.\begin{bmatrix} \Omega(t) > \Omega(t_o) & \text{for} & t_o - \delta \leqq t < t_o \\ \Omega(t) = \Omega(t_o) & \text{for} & t_o \leqq t \leqq t_o + \gamma \\ \Omega(t) > \Omega(t_o) & \text{for} & t_o + \gamma < t \leqq t_o + \gamma + \epsilon \end{bmatrix}\right\} \begin{matrix}\text{and} \\ 0 \leq t \leq \tau\ .\end{matrix}$$

This means that a flat maximum is represented by the right end point of the interval on which the local extremum is attained, a flat minimum by the left end point.

Because of the regularity conditions assumed in section 1.3* for $\xi(t)$ and $\zeta(t)$, there exists only a finite number, $n'' - 1$ say, of extremizers, which we denote in increasing order by t_i, $i = 1, \ldots, n'' - 1$. To these we add $t_o \equiv 0$ and $t_{n''} \equiv \tau$ as nominal "extremizers," even though the definitions (2.14), (2.15) are not applied to these points. With each t_i except $t_{n''} = \tau$ we associate a half-open interval

$$(2.16)\qquad I_i \equiv \begin{cases} \{W \mid 0 \leqq W < \Omega\} & \text{for } i = 0\ , \\ \{W \mid \Omega(t_i) - \Omega \leqq W < \Omega(t_i)\} & \text{if } t_i \text{ is a maximizer,} \\ \{W \mid \Omega(t_i) \leqq W < \Omega(t_i) + \Omega\} & \text{if } t_i \text{ is a minimizer,} \end{cases}$$

as illustrated by horizontal bands in Figure 5. Write $t_o = 0 \equiv t^{(o)}$ and delete all extremizers $t_1, \ldots, t_{i_1 - 1}$ between t_o and the first $t_{i_1} = t^{(1)}$ for which $\Omega(t_i)$ falls outside I_1. Next delete all extremizers

$t_{i_1}+1, \ldots, t_{i_2}-1$ until the first $t_{i_2} \equiv t^{(2)}$ for which $\Omega(t_i)$ falls outside I_{i_1}. Carrying this process through the entire sequence completes the first round of deletions. Restore $t_{n''} = \tau \equiv t^{(n')}$ if it was deleted, and name $t^{(n')}$ by the following rule, which supplements the definitions (2.14), (2.15) nominally for the case that $t_0 = \tau$,

$$\text{.17)}\left\{\begin{array}{l} \text{if } t^{(n'-1)} \text{ is a minimizer, } t^{(n')} = \tau \text{ is a } \begin{cases} \text{"minimizer" if } \Omega(\tau) < \Omega(t^{(n'-1)}) + \Omega_\tau \\ \text{"maximizer" if } \Omega(t^{(n'-1)}) + \Omega_\tau \leqq \Omega(\tau), \end{cases} \\ \text{if } t^{(n'-1)} \text{ is a maximizer, } t^{(n')} = \tau \text{ is a } \begin{cases} \text{"minimizer" if } \Omega(\tau) < \Omega(t^{(n'-1)}) - \Omega + \Omega_\tau \\ \text{"maximizer" if } \Omega(t^{(n'-1)}) - \Omega + \Omega_\tau \leqq \Omega(\tau). \end{cases} \end{array}\right.$$

In the second round, we delete from any run (uninterrupted subsequence) of maximizers contained in the sequence $t^{(j)}$ all but the last maximizer of the run, and likewise from any run of minimizers all but the last. The first element $t^{(0)} = 0$ is retained.

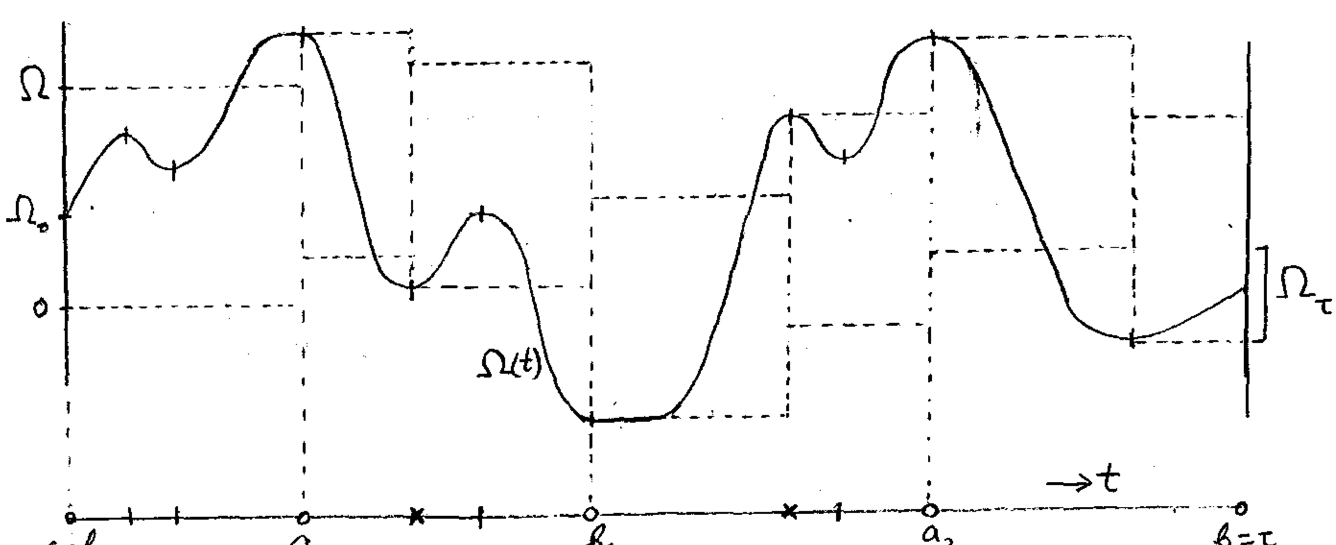

Figure 5. Subdivision of the planning period.

Figure 5 illustrates the construction. Extremizers deleted in the first and second round are marked on the time axis by a dash and a cross, respectively, while those ultimately retained are marked by a circle. The latter form an alternating sequence

$$(2.18)\qquad (b_o)\ ,\ a_1, b_1, a_2\ , \ldots\ b_{i-1}, a_i, b_i\ , \ldots,\ a_n, (b_n)$$

of maximizers a_i and minimizers b_i, provided we supplement the definitions (2.14), (2.15) also for $t_o = 0$, in such a way as to preserve the alternating pattern: If the sequence has at least three elements, and if the second element is a maximizer, we call it a_1 and write $b_o = 0$ for the first. If the second is a minimizer, it is called b_1 and we write $a_1 = 0$. If the next to last element is a maximizer, call it a_n and write $b_n = \tau$; if it is a minimizer, call it b_{n-1} and write $a_n = \tau$. If only two elements occur, we write $a_1 = 0$, $b_1 = \tau$ if $\Omega(\tau) < \Omega_\tau$ and $b_o = 0$, $a_1 = \tau$ if $\Omega(\tau) \geq \Omega_\tau$.

The sequence (2.18) gives the desired demarcation of periods of abundance $(b_{i-1} \leqq t < a_i)$ and of scarcity $(a_i \leqq t < b_i)$. In order to avoid separate formulations for terminal periods, we shall use the upper and lower bounding functions* $\bar{\Omega}_t$ and $\underline{\Omega}_t$ for $W(t)$ defined in (2.19). Then for any period $b_{i-1} \leqq t < a_i$

---

* We ignore that the term "bounding function" is inaccurate for $\bar{\Omega}_t$ in the point $t = \tau$.

		for $t = 0$	$0 < t < \tau$	$t = \tau$
	values of $\bar{\Omega}_t$	Ω_o	Ω	Ω_τ
(2.19)	$\underline{\Omega}_t$	Ω_o	0	Ω_τ

of abundance the following inequalities are implied in the construction of the sequence (2.18)

(2.20)

$$\text{(2.20a)} \quad \Omega(b_{i-1}) - \Omega(a_i) \leqq \underline{\Omega}_{b_{i-1}} - \bar{\Omega}_{a_i} \,,$$

$$\text{(2.20b)} \quad \Omega(b_{i-1}) - \Omega(t) \leqq \underline{\Omega}(b_{i-1}) - \underline{\Omega}_t \quad \text{and} \quad \Omega(t) - \Omega(a_i) \leqq \bar{\Omega}_t - \bar{\Omega}_{a_i} \quad \text{for} \quad b_{i-1} < t < a_i \,,$$

$$\text{(2.20c)} \quad \Omega(t) - \Omega(t') \leqq \bar{\Omega}_t - \underline{\Omega}_{t'} \quad \text{for} \quad b_{i-1} < t < t' < a_i \,.$$

The implications (2.20a and b) are straightforward. (2.20c) is also implied because, if it were violated for some (t,t'), we could define a maximizer a'_i with $b_{i-1} < a'_i < t'$ and a minimizer b'_i with $a'_i < b'_i < a_i$, by

$$\text{(2.21)} \quad \Omega(a'_i) - \bar{\Omega}_{a'_i} \equiv \max_{b_{i-1} \leqq t'' \leqq t'} (\Omega(t'') - \bar{\Omega}_{t''}) \,, \; \Omega(b'_i) - \underline{\Omega}_{b'_i} \equiv \min_{a'_i \leqq t'' \leqq a_i} (\Omega(t'') - \underline{\Omega}_{t''})$$

(where the possible discontinuities of $\bar{\Omega}_t$ and $\underline{\Omega}_t$ do not prevent the extrema from being attained). But these extremizers would have satisfied

$$(\Omega(a'_i) - \bar{\Omega}_{a'_i}) - (\Omega(b'_i) - \underline{\Omega}_{b'_i}) \geqq (\Omega(a'_i) - \bar{\Omega}_{a'_i}) - (\Omega(t') - \underline{\Omega}_{t'})$$

$$\text{(2.22)} \qquad \geqq (\Omega(t) - \bar{\Omega}_t) - (\Omega(t') - \underline{\Omega}_{t'}) > 0 \,,$$

and hence could not have been deleted in either of the two rounds of deletions.

Similarly, for any period $a_i \leq t < b_i$ of scarcity,

(2.23)

$$\text{(2.23a)} \quad \Omega(a_i) - \Omega(b_i) > \overline{\Omega}_{a_i} - \underline{\Omega}_{b_i} ,$$

$$\text{(2.23b)} \quad \Omega(a_i) - \Omega(t) > \overline{\Omega}_{a_i} - \overline{\Omega}_t \quad \text{and} \quad \Omega(t) - \Omega(b_i) > \underline{\Omega}_t - \underline{\Omega}_{b_i} \quad \text{for} \quad a_i < t < b_i ,$$

$$\text{(2.23c)} \quad \Omega(t) - \Omega(t') > \underline{\Omega}_t - \overline{\Omega}_{t'} \quad \text{for} \quad a_i < t < t' < b_i .$$

The fact that in this case all inequalities are of the strict type results from the treatment of flat extrema and from the exclusion of the right end point from each interval I_i .

The statements (2.12), (2.13) for nonterminal periods are obtained from (2.20)and (2.23), respectively, by substituting

$$\text{(2.24)} \qquad \underline{\Omega}_{b_{i-1}} = 0 \ , \ \overline{\Omega}_{a_i} = \Omega \ , \ \underline{\Omega}_{b_i} = 0 \ .$$

2.3 Discharge policy in periods of abundance of water. We have now completed the subdivision of the planning period into alternating periods $b_{i-1} \leqq t < a_i$ of abundance and periods $a_i \leqq t < b_i$ of scarcity of water. In the light of the detailed analysis of section 2.2*, a nonterminal period of abundance can now be defined as a period for which (2.12) holds such that its beginning b_{i-1} cannot be advanced, its end a_i cannot be delayed without violating (2.12b) or (2.12c). A nonterminal period of scarcity is one for which (2.13) holds while a_i cannot be advanced or b_i delayed without violating (2.13b) or (2.13c). Similar definitions for terminal periods can be based on (2.20) and (2.23).

Since our minimand (2.1) is an integral of cost over the entire planning period, it can be written as a sum

$$(2.25) \qquad C(v) \equiv \sum_i \int_{b_{i-1}}^{a_i} \psi(\zeta(t) - v(t))dt + \sum_i \int_{a_i}^{b_i} \psi(\zeta(t) - v(t))dt$$

of the costs incurred in each particular period. We can therefore look on our problem as composed of a number of cost minimization problems, one for each period, linked only by the fact that the store of water at the end of one period, $W(a_i)$ or $W(b_i)$, equals the store of water at the beginning of the next period.

The particular subdivision we have chosen is such that these links present no difficulty. Periods of abundance not ending in τ necessarily end up with a full reservoir $W(a_i) = \Omega$, regardless of the store of water $W(b_{i-1})$ at the beginning of such a period, and regardless of the discharge policy within the period (but provided, as we assume throughout, an automatic overflow policy is followed). The reasoning only needs to be sketched. In the first place, for a given initial store $W(b_i)$, the store at any later time in the period is at least as high under any turbine-feasible discharge policy as it is under the maximal discharge policy. Now consider the relations between the store of water $W(t)$ under a maximal discharge policy and the swings of the guide function $\Omega(t)$, as indicated by Figure 3. $W(t)$ moves parallel to $\Omega(t)$ (which now takes the place of $W_\psi(t)$) except where prevented from doing so by the upper reservoir bound (Ω). If under this rule $W(t)$ never crosses the lower reservoir bound (0), the maximal discharge policy is thereby found feasible. Now, as suggested by Figure 4, the first inequality in (2.12b) ensures that $W(t)$ does not cross the lower reservoir bound before reaching the upper bound, and (2.12c) ensures that $W(t)$ does not cross the lower bound after reaching the upper bound. Furthermore, (2.12a) ensures that, regardless of the nonnegative beginning-of-period store of water $W(b_{i-1})$, $W(t)$ will reach the upper bound at the end $t = a_i$ of the period of abundance or earlier. Finally, the second inequality in (2.12b) guarantees that if the upper bound is reached earlier, it will be attained also at $t = a_i$. This reasoning can be verified either graphically, or by explicit mathematical analysis similar to that of Section 2.1*. Its extension to terminal periods on the basis of (2.20) is straightforward: A period of abundance with $b_i = 0$ but $a_i < \tau$ also ends with a full reservoir, and a period of abundance with $a_i = \tau$ ends with a store of water $W(\tau)$ not less than the prescribed lower bound Ω_τ.

To sum up, the initial store $W(b_{i-1})$ of a period of abundance (if not given as $W(0) = \Omega_0$) does not limit the choice of a discharge policy, and the final store $W(a_i)$ (or, if $a_i = \tau$, its conformity with the specified lower bound) is independent of the discharge policy. In all these cases the maximal discharge policy is feasible. Since any discharge rate below the turbine-feasible maximum would involve a higher cost of supplementary generation, it follows that the maximal discharge policy is uniquely optimal in each period of abundance. That is, that the policy

$$(2.26) \qquad v(t) = \hat{v}(t) \equiv \eta(t) , \qquad b_{i-1} \leqq t < a_i ,$$

minimizes the cost integral in (2.25) for the period in question within the given restraints, regardless of the policies applied in other periods.

2.4 <u>Discharge policy in periods of scarcity of water</u>. The problem for each period of scarcity of water $a_i \leqq t < b_i$, which in this section we shall simply write as $a \leqq t < b$, is now that of minimizing the cost integral

$$(2.27) \qquad C_{a,b}(v) \equiv \int_a^b \psi\left\{\zeta(t) - v(t)\right\} dt ,$$

subject to turbine (2.2) and reservoir (2.3) restraints for $a \leqq t \leqq b$, with the initial and final stores of water restrained by

$$(2.28) \qquad W(a) = \begin{cases} \Omega_o & \text{if } a = 0 \\ \Omega & \text{if } a > 0 , \end{cases}$$

and

$$(2.29) \qquad \left.\begin{matrix} 0 \\ \Omega_\tau \end{matrix}\right\} \leqq W(b) \leqq \Omega \quad \text{if } b \begin{cases} < \tau \\ = \tau . \end{cases}$$

In this section 2.4 we shall through heuristic discussion explore the nature of the solution. In section 2.4* we shall establish its existence and its salient characteristics. It will save reasoning effort if we postpone the proof of its optimality until Section 3.4 after we have imputed prices to the resources occurring in our problem.

Periods of scarcity are characterized by the circumstance - see (2.13a) - that a maximal discharge policy throughout such a period would imply a negative $(b < \tau)$ or otherwise too low $(b = \tau)$ store of water at the end of the period. The problem therefore is one of apportioning the use of the available water to the various parts of the period in such a way that the now unavoidable supplementary thermal generation is performed at minimum cost. This suggests in the first place that it will be uneconomical to end a period of scarcity with more than the required minimum store or water because as we have already seen any excess over that minimum is of no value in the remaining part of the planning period. We therefore prescribe instead of (2.29),

$$(2.30) \qquad \hat{W}(b) = \begin{cases} 0 \\ \Omega_\tau \end{cases} \text{ for } b \begin{cases} < \tau \\ = \tau . \end{cases}$$

The total amount of thermal generation for the period of scarcity is thereby (in our units) set equal to the water deficit

$$(2.31) \quad \hat{S}(b) - \hat{S}(a) = [Z(b) - \Xi(b) - \hat{W}(b)] - [Z(a) - \Xi(a) - \hat{W}(a)]$$

of the period, where $\hat{W}(a)$ and $\hat{W}(b)$ are prescribed by (2.28) and (2.30).

The increasing character of incremental cost further suggests that if a constant rate

$$(2.32) \qquad \hat{s}(t) = \hat{s} \equiv (\hat{S}(b) - \hat{S}(a)) / (b - a) \qquad \text{for } a \leq t < b$$

is feasible throughout the period, this is uniquely optimal -- a statement contained as a special case in the results of Section 3.4 below. This simple solution may be precluded by turbine restraints, by reservoir restraints, or by both. The problem then becomes one of equalizing the rate of thermal generation as much as the restraints permit.

Considering first the turbine restraints, we shall say that a discharge policy $v(t)$ is derived from a target rate or target function $\tilde{s}(t)$ of thermal generation if (see (1.16) for notation)

$$v(t) = [0 \mid \zeta(t) - \tilde{s}(t) \mid \varphi] . \tag{2.33}$$

This means, as illustrated in Figure 6, that the actual rate $\zeta(t) - v(t)$ of thermal generation is equal to the target rate whenever that is turbine-feasible, and otherwise is as close to the target rate as the turbine restrictions permit.

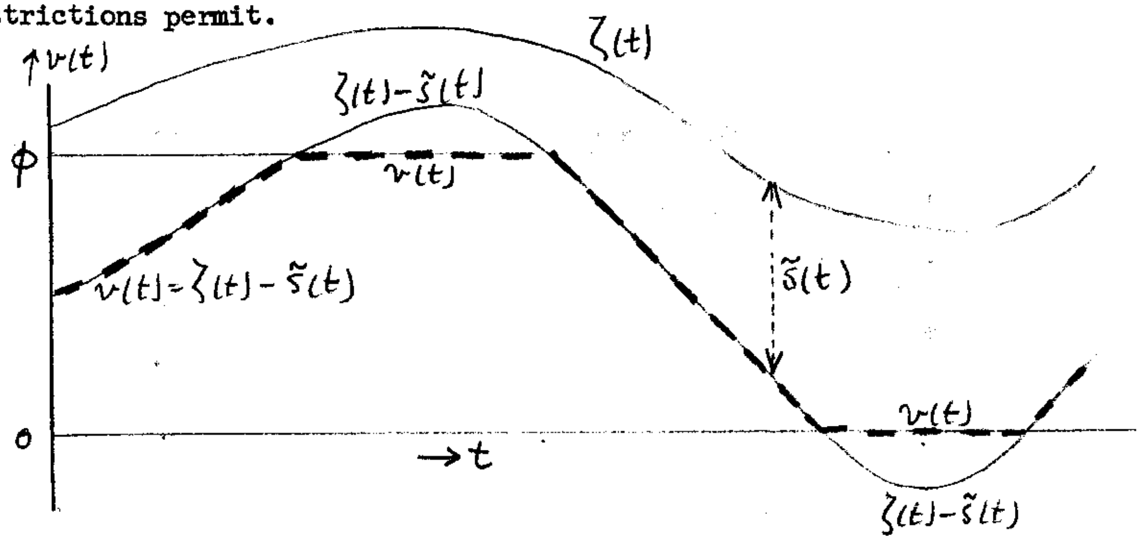

Figure 6. Discharge policy $v(t)$ derived from a target rate $\tilde{s}(t)$ of supplementary thermal generation.

Again, our results in Section 3.4 imply that, if there is a target rate $\hat{s}$ such that the discharge policy $\hat{v}(t)$ derived from that constant rate $\hat{s}(t) = \hat{s}$ is reservoir-feasible and ends up the period with the required store of water, then that discharge policy is uniquely optimal. This in turn shifts the problem to one of equalizing the target rate as much as reservoir restraints permit.

That the reservoir restraints can preclude a constant target rate can be seen from a simple example in which we assume that the turbine restraints by themselves permit a constant actual rate of thermal generation. Figure 7 shows a possible guide function $\Omega(t)$ (solid line) in a period of scarcity $a \leq t \leq b$ (with $0 < a < b < \tau$), referred to the scale at the left. This function satisfies

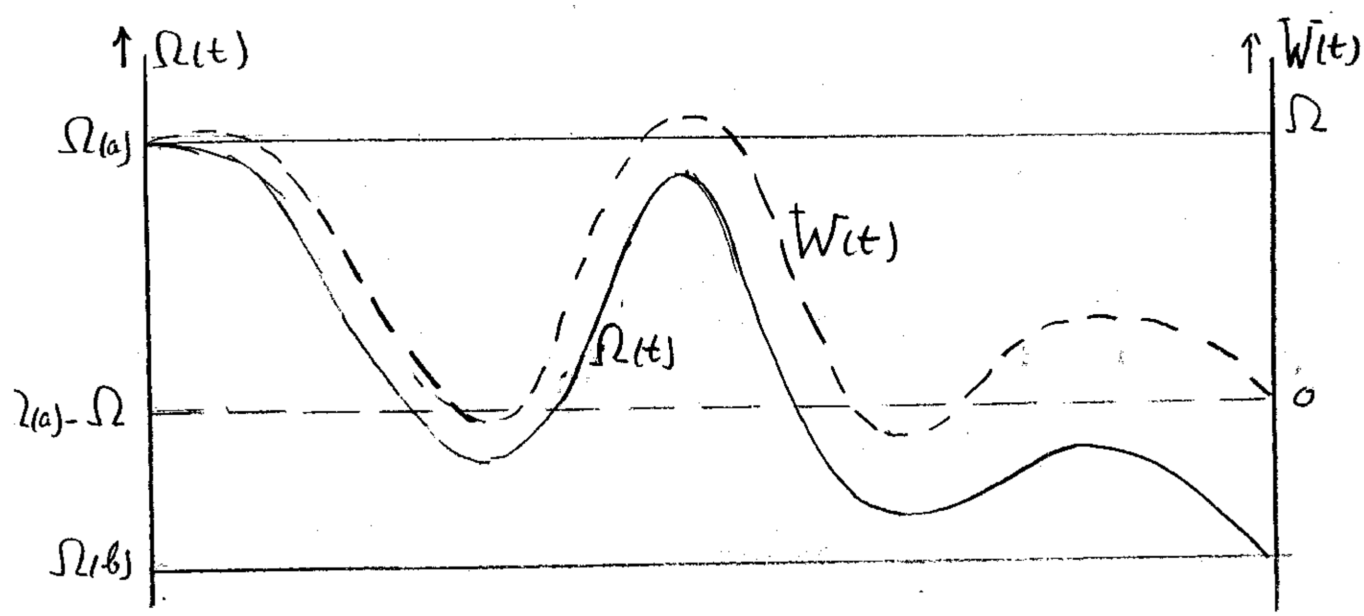

Figure 7. A guide function that precludes a constant target rate of thermal generation.

the conditions (2.13) for a nonterminal period of scarcity (2.34). In contrast, the corresponding store of water function (dotted line, scale at right)

$$W(t) = \Omega + \Omega(t) - \Omega(a) + \sigma_o (t - a) , \tag{2.34}$$

derived from a constant rate of thermal generation

$$\sigma_o \equiv (\Omega(a) - \Omega(b) - \Omega) / (b - a) \tag{2.35}$$

chosen so as to make $W(b) = 0 = W(a) - \Omega$, is seen to violate the upper and lower reservoir restraints in several places.

A graphical procedure illustrated by Figure 8 now suggests itself. We draw in Figure 8A a family of guide functions $W(t,s)$, all starting at the full reservoir level Ω . The curve for $W(t,s)$ represents what would be the store of water at any time t if a constant target rate s of thermal generation were followed throughout, within turbine restraints, and if no reservoir restraints existed. Increasing s raises the curve $W(t,s)$ by an amount increasing with time.

Starting with the curve labeled $s = 0$, we find it violating a reservoir bound for the first time for $t > t_o$, and the violated restraint is (necessarily) the lower bound. This suggests increasing s until such a value s_1 , is reached that a further increase in s would violate the upper bound, at time a_1 , even before the lower bound is violated. The segment of the curve $W(t,s)$ for which $a \leq t \leq a_1$ is now moved down to Figure 8B as the solution $\hat{W}(t)$ for that time interval. Since the continuation of that curve would still leave the lower bound

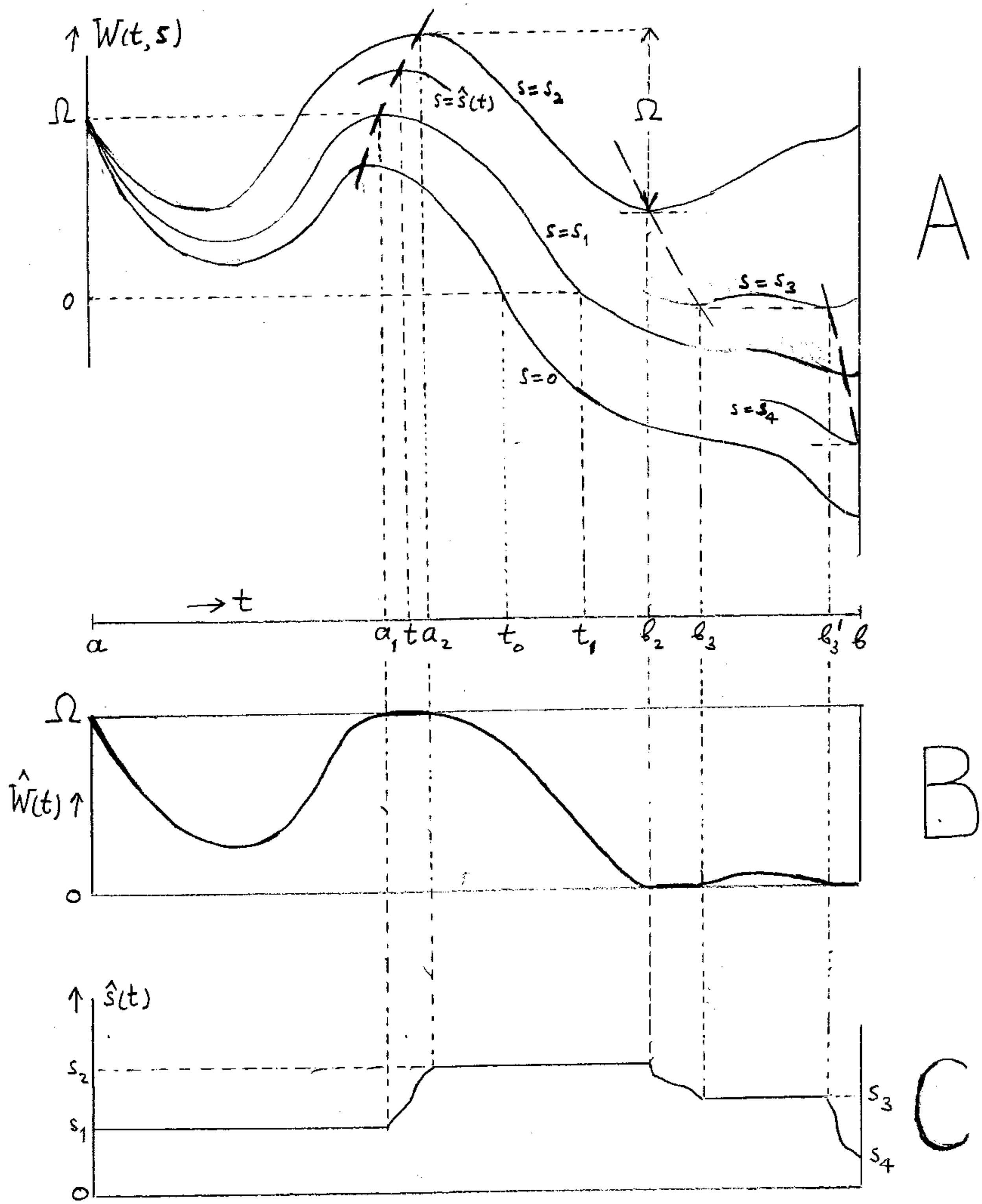

Figure 8. Construction of the optimal storage policy in a period of scarcity.

violated at t_1 , we search the diagram for curves with still higher s until we meet one, labeled $s = s_2$, of which the decline from its highest peak after a_1 does not exceed Ω . If this decline runs from a_2 to b_2 the segment $a_2 \leqq t \leqq b_2$ of the curve $W(t,s)$ is now reduced by a suitable constant to form another piece

$$(2.36) \qquad \hat{W}(t) \equiv W(t,s_2) - W(a_2,s_2) + \Omega \qquad \text{for} \quad a_2 \leqq t \leqq b_2$$

of the solution which just fits between the reservoir bounds. In the intervening interval $a_1 \leqq t \leqq a_2$ the target rate is equated to an increasing function $\hat{s}(t)$ of time determined in such a way that the reservoir remains full $(\hat{W}(t) = \Omega)$ throughout this interval. This function is traced out by the locus of maxima of $W(t,s)$ (dotted line) if s increases from s_1 (maximum reached at $t = a_1$) to s_2 (maximum reached at $t = a_2$). Beyond b_2 the curve $W(t,s_2) - W(b_2,s_2)$ rises again to a value at b exceeding the value prescribed by (2.30), 0 in this case. Therefore, $\hat{s}(t)$ is gradually decreased so as to keep the reservoir empty $(\hat{W}(t) = 0)$ from b_2 to b_3 , where a curve labeled $s = s_3$ with a horizontal double tangent (or double supporting line) is reached. This curve is adjusted by a constant to serve as solution $\hat{W}(t)$ between the two points of contact, b_3 and b_3' , whereafter the reservoir is again kept empty by further decrease of $\hat{s}(t)$ until the value s_4 is reached at b . The course of $\hat{s}(t)$ is shown in Figure 8C.

This description follows the particular shape of the family of curves $W(t,s)$ represented in Figure 8, and should not be taken as a general statement of the rules for construction an optimal policy. In particular, Figure 8 contains no case of a return from an empty to a full reservoir within the period of scarcity. While (2.13) assures us that the curve labeled $s = 0$ exhibits no rise by Ω , for higher values of s , as suggested by Figure 7, such a rise is quite possible, and may become incorporated in the optimal storage policy.

A complete statement of the construction of the unique optimal storage policy $\hat{W}(t)$ in a period of scarcity is given in Section 2.4*. In particular, it will be shown that this policy can be derived from a target function $\hat{s}(t)$ of thermal generation which is constant whenever the reservoir is partly filled, which can increase only when the reservoir is full, and can decrease only when it is empty. The proof of the optimality of the policy so constructed is given in Section 3.4.

2.4* The family of guide functions used in constructing the optimal storage policy is defined by

$$(2.36) \qquad W(t,s) \equiv \Omega_a + \int_a^t (\xi(t') - [0 \mid \zeta(t') - s \mid \varphi])dt' , \qquad 0 \leqq s \leqq \zeta .$$

Each of these functions satisfies,

$$(2.37) \qquad | W(t',s) - W(t,s) | \leqq \omega \cdot | t' - t | \quad \text{for} \quad 0 \leqq s \leqq \zeta ,$$

where $\omega \equiv \max \left\{ \xi , \varphi \right\}$, and each pair satisfies

(2.38a and b)

$$\leqq W(t',s') - W(t,s') - W(t',s) + W(t,s) \leqq (t' - t)(s' - s) , \quad \text{if} \quad t \leqq t' \quad \text{and} \quad 0 \leqq s \leqq s' \leqq \zeta,$$

which implies continuity of $W(t,s)$ with regard to t and s .
Finally, from (2.10), (2.11), and (1.13),

(2.39a and b)
$$W(t,0) = \Omega_a + \Omega(t) - \Omega(a),$$
$$W(t,\zeta) = \Omega_a + \Xi(t) - \Xi(a),$$

where $s = \zeta$ represents a policy of meeting all power demand by thermal generation alone.

In order to include terminal periods in our formulations, we define further

(2.40)
$$W^*(t,s) \equiv \begin{cases} \Omega & \text{if } t = a \\ W(t,s) & \text{if } a < t < b \\ W(b,s) - \Omega_b & \text{if } t = b \end{cases} .$$

This function is continuous in t and s except possibly for $t=a$ and $t=b$, where discontinuities in t (but not in s) may occur such that

(2.41)
$$W^*(a,s) \geq W^*(a+0,s), \qquad W^*(b-0,s) \geq W^*(b,s)$$

Asterisks can be ignored whenever $\Omega_a = \Omega$ and $\Omega_b = 0$

We define and study two functionals of $W^*(t,s)$ of which the relation to the reservoir restraints is obvious. These are the <u>maximum rise</u> (I for <u>i</u>ncrease)

(2.42)
$$I(s) \equiv \underset{a \leq t \leq t' \leq b}{\text{lub}} \left[W^*(t',s) - W^*(t,s) \right], \qquad 0 \leq s \leq \zeta ,$$

and the range (D for decrease, the case that interests us)

(2.43a and b) $$D(s) \equiv \underset{\substack{a \leq t \leq b \\ a \leq t' \leq b}}{\text{lub}} \left[W^*(t',s) - W^*(t,s) \right] = \underset{a \leq t \leq b}{\text{lub}} W^*(t,s) - \underset{a \leq t \leq b}{\text{glb}} W^*(t,s)$$

Figure 9 illustrates the definitions in a case where $\Omega_a < \Omega$ but $\Omega_b = 0$.

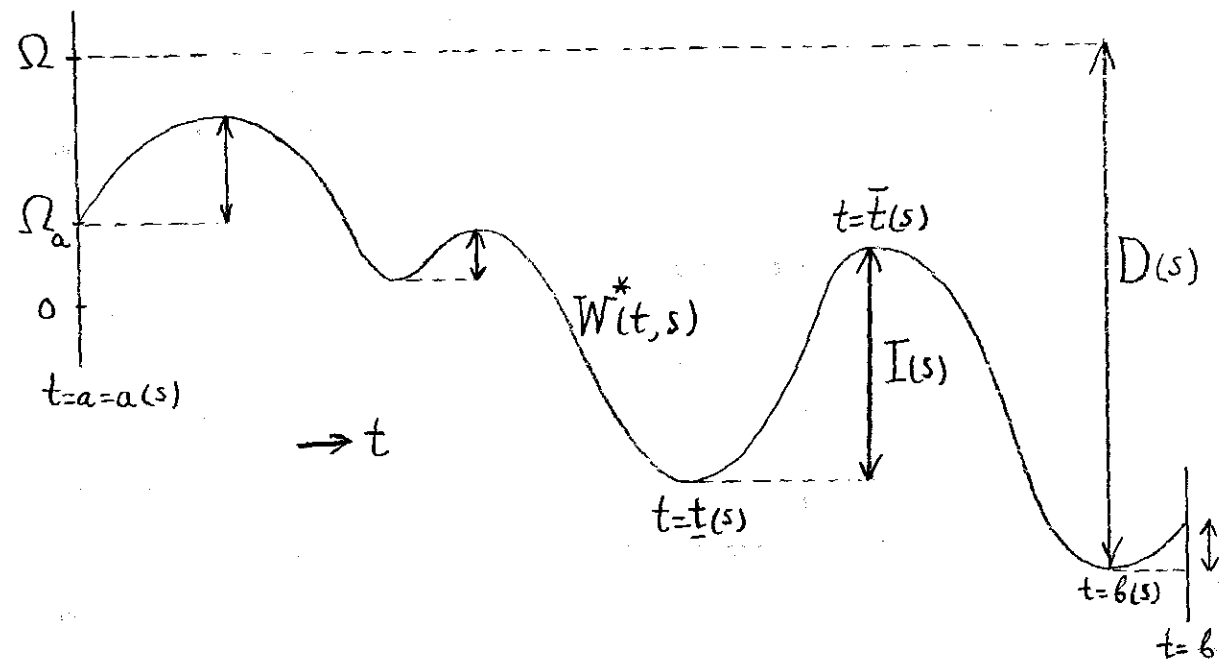

Figure 9. The maximum rise $I(s)$ and the range $D(s)$ of $W^*(t,s)$

Since the set of permitted values of (t, t') defining $D(s)$ includes that defining $I(s)$, and since the latter set includes points where $t = t'$,

(2.44) $$0 \leq I(s) \leq D(s) \quad .$$

In (2.42) the asterisks can be omitted because of (2.41), and if they are omitted $I(s)$ is necessarily an attained maximum. From this, (2.39a) and (2.23) it follows that

(2.45) $$I(0) < \Omega < D(0)$$

In order to show that $I(s)$ is continuous and nondecreasing for $0 \leqq s \leqq \zeta$ we write $(\underline{t}(s), \bar{t}(s))$ for some pair of points (t, t') where the maximum in (2.42) (asterisks omitted) is attained, and define

(2.46) $$I(s, s') \equiv W(\bar{t}(s), s') - W(\underline{t}(s), s') .$$

Then we have from (2.42)

(2.47a and b) $$I(s') = I(s', s') \geqq I(s, s')$$

and from this and (2.38), if $s < s'$,

(2.48) $$\begin{aligned} 0 \leqq I(s, s') - I(s, s) &\leqq I(s', s') - I(s, s) \leqq I(s', s') - I(s', s) \\ &\leqq (\bar{t}(s') - \underline{t}(s'))(s' - s) \leqq (b-a)(s'-s) , \end{aligned}$$

which together with (2.47a) proves the contention.

With regard to $D(s)$, we shall define a specific pair of points $(t, t') = (a(s), b(s))$, say, where the least upper bound in (2.43) is attained (if anywhere), by specifying

(2.49)
(2.49a) $$W^*(t', s) \leqq W^*(a(s)-0, s) > W^*(t, s) \quad \text{for } a \leqq t' < a(s) < t \leqq b ,$$
(2.49b) $$W^*(t, s) > W^*(b(s)+0, s) \leqq W^*(t'', s) \quad \text{for } a \leqq t < b(s) < t'' \leqq b ,$$

where the additions of -0 and $+0$ to the time arguments are without effect except in the extreme cases $W^*(b-0,s) = W(b,s)$ and $W^*(a+0,s) = W(a,s)$, where they remove the asterisks. Obviously there cannot be more than one such pair $(a(s), b(s))$. It follows from (2.43b) and the continuity of $W(t,s)$ that there exists one such pair, and

(2.50) $$D(s) = W^*(a(s)-0, s) - W^*(b(s)+0, s) .$$

Finally, from (2.49), (2.40) and (2.23), we have

(2.51a and b) $$a(0) = a, \qquad b(0) = b \quad .$$

In order to show that $a(s)$ is nondecreasing for $0 \leqq s \leqq \zeta$, we derive from (2.49a) and (2.40) that

(2.52) $$W^*(a(s')-0,s) - W^*(a(s)-0,s) \leqq 0 \leqq W^*(a(s')-0,s') - W^*(a(s)-0,s')$$

Now if we had $a(s') < a(s)$ and $0 \leqq s < s' \leqq \zeta$, (2.38a) with $a(s')$ inserted for t, $a(s)$ for t', would imply that the inequality between the first and last member of (2.52) could also be reversed, hence that both equality signs in (2.52) would apply. But then we would have $W^*(a(s'),s) = W^*(a(s)-0,s)$, and $a(s') < a(s)$ would contradict the definition (2.49a) of $a(s')$

It follows further from (2.52) that $a(s)$ is continuous to the right for $0 \leqq s < \zeta$. To see this, let s_0 by a point where $a(s_0-0) < a(s_0+0)$, and

and let $0 \leqq s < s_0 < s' \leqq \zeta$. Then, by inserting $t = a(s)$ and $t' = a(s')$ in (2.38b), we now find that the last member of (2.52) exceeds the first by at most $(b-a)(s'-s)$. It follows that

$$W^*(a(s_0-0), s_0) = W^*(a(s_0+0)-0, s_0), \tag{2.53}$$

and hence by (2.49a) that $a(s_0) = a(s_0+0)$. The case in which $s_0 = 0$ is similarly discussed by replacing $s_0 - 0$ by s_0 . One proves in the same way that $b(s)$ is nonincreasing and continuous to the left for $0 \leqq s \leqq \zeta$.

Since, in view of (2.39b) and (1.13), $W(t, \zeta)$ is nondecreasing we have from (2.49)

$$a(\zeta) = a \text{ or } b, \qquad b(\zeta) = a \text{ or } b. \tag{2.54}$$

This leads to the following two cases.

Case I; $a(\zeta) = a$, $b(\zeta) = b$. This can occur only if

$$\Xi(b) - \Xi(a) < \min\{\Omega - \Omega_a, \Omega_b\} \tag{2.55}$$

which, since the left hand member is nonnegative, can occur only if $a = 0$, $b = \tau$, that is, if the entire planning period is one period of scarcity. In that case (1.14) assures us that

$$D(\zeta) = \Omega - (W(b, \zeta) - \Omega_b) \leqq \Omega, \tag{2.56}$$

using (2.50), (2.40) and (2.39b). Hence, because of (2.45) and the continuity of $D(\zeta)$ there exists a number $\tilde{\sigma}$ such that

(2.57a and b) $0 < \tilde{\sigma} \equiv \min\{s \mid D(s) = \Omega\} \leqq \zeta$, so $D(\tilde{\sigma}) = \Omega$.

We now define as the optimal storage policy

(2.58) $$\hat{W}(t) \equiv W(t, \tilde{\sigma}) \quad \text{for } a \leqq t \leqq b \quad .$$

This policy achieves the prescribed end-of-period store of water $\hat{W}(b) = \Omega_b$ because of (2.57b), (2.50) and (2.40). $\hat{W}(t)$ stays between but never reaches the reservoir bounds 0 and Ω because of (2.49). The target rate of thermal generation is the constant $\tilde{\sigma}$ throughout the planning period.

Case II; $a(\zeta) \geqq b(\zeta)$. We now define

(2.59) $$\sigma \equiv \min\{s \mid 0 \leqq s \leqq \zeta \text{ and } a(s) \geqq b(s)\} .$$

Then $0 \leqq \sigma \leqq \zeta$, and from a comparison of (2.42) and (2.43)

(2.60) $$D(s) = I(s) \quad \text{for} \quad \sigma \leqq s \leqq \zeta$$

because the point $(t, t') = (a(s), b(s))$ in which $D(s)$ is attained now lies in the set defining $I(s)$.

On the other hand,

(2.61) $$D(s) = \underset{a \leqq t' \leqq t \leqq b}{\text{lub}} [W^*(t', s) - W^*(t, s)] \quad \text{for} \quad 0 \leqq s \leqq \sigma .$$

For $0 \leqq s < \sigma$ this follows from the fact that then $a(s) < b(s)$ by (2.59), whereas for $s = \sigma$ it follows from (2.53) if we take $t' = a(\sigma - 0)$ and $t = b(\sigma + 0)$, which by (2.59) implies $t' \leqq t$. For $0 \leqq s \leqq \sigma$, therefore, $D(s)$ can be defined in terms of $-W^*(t,s)$ in the same way in which $I(s)$ is defined by (2.42) in terms of $W^*(t,s)$.

Since the inequalities (2.38) on which our reasoning has been based are reversed if $W^*(t,s)$ is replaced by $-W^*(t,s)$, it follows that $D(s)$ is continuous and nonincreasing for $0 \leqq s \leqq \sigma$. We thus have

(2.62) $$I(0) \leqq I(s) \leqq I(\sigma) = D(\sigma) \leqq D(s) \leqq D(0) \quad \text{for } 0 \leqq s \leqq \sigma.$$

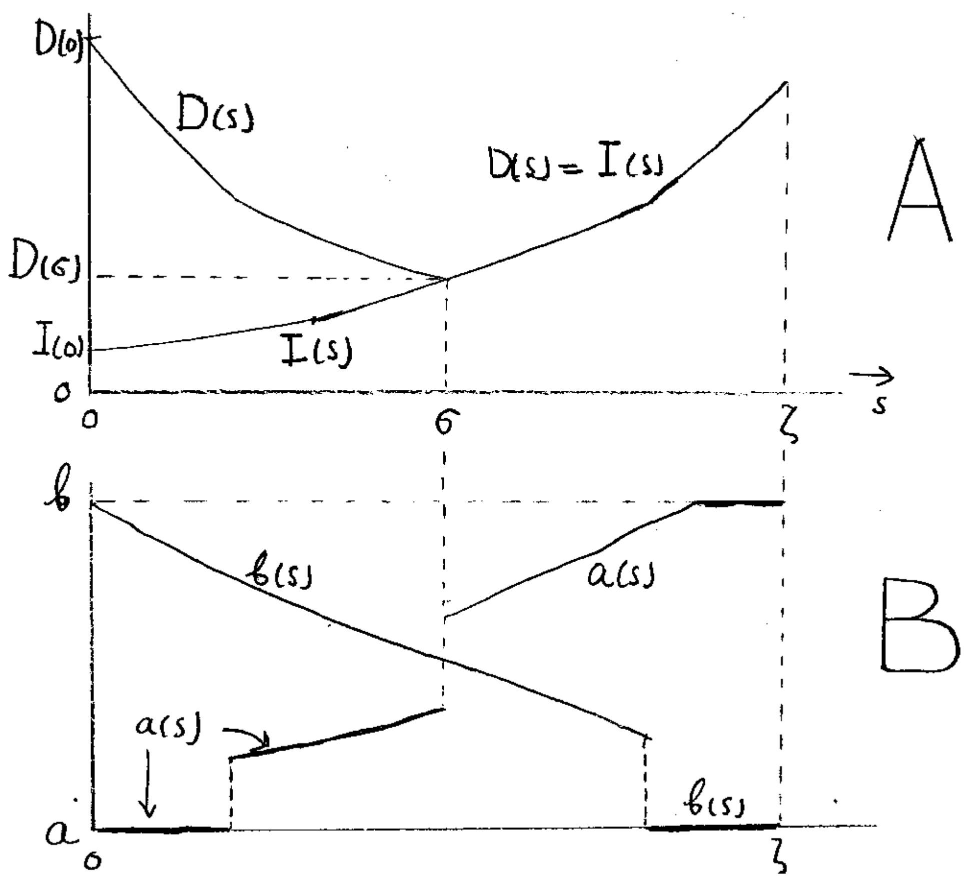

Figure 10. Illustration of I(s) and D(s)

Figure 10A illustrates the properties of $D(s)$ and $I(s)$ that we have found, with possible related graphs of $a(s)$ and $b(s)$ shown in Figure 10B.

We now proceed to the construction of the optimal storage policy by specifying a target rate of thermal generation $\tilde{s}(t)$ which is a function of time, in

$$\hat{W}(t) \equiv \Omega_a + \int_0^t \left(\xi(t') - [\,0 \mid \zeta(t') - \tilde{s}(t') \mid \phi\,] \right) dt'. \tag{2.63}$$

$\tilde{s}(t)$ will in turn be defined for $a \leqq t < b$ by

$$\tilde{s}(t) \equiv glb \{ s \mid t \in T(s) \}, \tag{2.64}$$

where $T(s)$ is a set function which associates with each point s of the interval $0 \leqq s \leqq \zeta$ a subset $T(s)$ of the interval $a \leqq t < b$ in such a way that*

* The construction of $\hat{W}(t)$ for the already completed Case I can be subsumed under this construction by specifying that $T(s)$ is empty for $0 \leqq s < \hat{\sigma}$ and that $T(s) = [a, b)$ for $\hat{\sigma} \leq s \leqq \zeta$.

(2.65)

(2.65a and b) $T(0)$ is empty, $T(\zeta) = [a, b)$,

(2.65c) $T(s) \subset T(s')$ if $0 \leqq s < s' \leqq \zeta$.

Because of (2.65c), $t \in T(s_0)$ implies $t \in T(s)$ for $s_0 \leqq s \leqq \zeta$. Hence for each t the set in (2.64) is an interval which contains its right end point ζ because of (2.65b), and of which the left end point, defined to be $\tilde{s}(t)$, is nonnegative because of (2.65a), so

(2.66) $$0 \leqq \tilde{s}(t) \leqq \zeta \quad \text{for} \quad a \leqq t < b .$$

Comparison of (2.45) and (2.62) leads us to distinguish two subcases.

Subcase II.1; $D(\sigma) \leqq \Omega$. In this case, since $D(s)$ is continuous and $D(0) > \Omega$, there is a number $\tilde{\sigma}$ such that

(2.67a and b) $$0 \leqq \tilde{\sigma} \equiv \min\{s \mid D(s) = \Omega\} \leqq \sigma, \quad \text{so} \quad D(\tilde{\sigma}) = \Omega .$$

We now define $T(s)$ by

(2.68)
(2.68a) $$T(s) \equiv [a, a(s)) \cup [b(s), b) \quad \text{for} \quad 0 \leqq s < \tilde{\sigma} ,$$
(2.68b) $$T(s) = [a, b) \quad \text{for} \quad \tilde{\sigma} \leqq s \leqq \zeta .$$

This definition accords with (2.66) because of (2.51) and the monotonicity properties of $a(s)$ and $b(s)$. It implies through (2.64) that, for $0 \leqq s < \tilde{\sigma}$,

(2.69)
(a) if $a(s-0) < a(s)$, $\tilde{s}(t) = s$ for $a(s-0) \leqq t < a(s)$,

(b) if $a(s) < a(s')$ for all $s' > s$, $\tilde{s}(a(s)) = s$,

(c) if $b(s) < b(s-0)$, $\tilde{s}(t) = s$ for $b(s) \leqq t < b(s-0)$,

(d) if $b(s) < b(s')$ for all $s' < s$, $\tilde{s}(b(s)) = s$,

(e) and $\tilde{s}(t) = \tilde{\sigma}$ for $\tilde{\alpha} \equiv a(\tilde{\sigma}-0) \leqq t < b(\tilde{\sigma}-0) \equiv \tilde{b}$,

where $\tilde{\alpha} \leqq \tilde{b}$ because of (2.59) and (2.67a). It is seen from (2.69) that $\tilde{s}(t)$ is continuous to the right for $a \leqq t < b$.

We shall study the implications of (2.69) with regard to $\hat{W}(t)$ as defined by (2.63). We begin with the case assumed in (2.69b), where we now write $s = s_0$ for a point where $a(s)$ is continuous and increasing to the right, and $s' = s_0'$ for the smallest value of s' such that $a(s') = a(s_0)$. Then $s_0' \leqq s_0$. We note that if in particular $a(s_0) = a$, (2.49a), (2.37) and the assumed continuity of $a(s)$ for $s = s_0$ imply that $\Omega_a = \Omega$.

Writing for brevity $a_0 \equiv a(s_0)$ we will study

(2.70) $$W(a_0, t, s) \equiv W(t,s) - W(a,s) = \int_{a_0}^{t} \left(\xi(t') - [0 \,|\, \zeta(t') - s \,|\, \varphi] \right) dt'$$

in the neighborhoods of $t = a_0$ and $s = s_0$ or $s = s_0'$. From (2.49a) we have that, for some $\delta > 0$,

(2.71)
$$\begin{cases} \text{(a)} & W(a_0, t, s) > 0 \quad \text{for } s_0'' \leqq s < s_0' \text{ and } t = a(s), \text{ where } a(s) < a_0, \\ \text{(b)} & W(a_0, t, s) < 0 \quad \text{for } 0 \leqq s \leqq s_0 \text{ and } a_0 < t \leqq b, \\ \text{(c)} & W(a_0, t, s) \geqq 0 \quad \text{for } s_0 < s \leqq s_0 + \delta \text{ and } t = a(s), \text{ where } a(s) > a_0, \end{cases}$$

where $s_0'' = 0$ if $\Omega_a = \Omega$ and $s_0'' = \min\{s \,|\, a(s) > a\}$ otherwise.

Because of the regularity assumptions there exist positive functions $\varepsilon(t')$ and $\varepsilon'(t')$ of t' such that $\xi(t)$, $\zeta(t)$ and $\xi(t) - \zeta(t)$ are continuous and monotonic for $t' - \varepsilon'(t') \leqq t < t'$ and for $t' \leqq t \leqq t' + \varepsilon(t')$. Now, if $\varepsilon_0 \equiv \varepsilon(a_0)$, (2.72) $\begin{cases} \text{(2.72a)} & \text{either} \quad 0 < \zeta(a_0) - s_0 \leqq \phi \ , \\ \text{(2.72b)} & \text{or} \quad 0 = \zeta(a_0) - s_0 \ \text{ and } \zeta(t) \text{ increases on } a_0 \leqq t \leqq a_0 + \varepsilon_0 \ , \end{cases}$

because otherwise one could find positive numbers $\delta' < \delta$ and $\varepsilon' < \varepsilon_0$ such that, for any t in $a_0 < t \leqq a_0 + \varepsilon'$, $W(a_0, t, s)$ is independent of s for $s_0 \leqq s \leqq s_0 + \delta'$, in which case (2.71b) and (2.71c) cannot both be true. We begin with assuming (2.72a) and

(2.73) if $\zeta(a_0) - s_0 = \phi$ then $\zeta(t)$ is constant or decreases on $a_0 \leqq t \leqq a_0 + \varepsilon_0$.

In that case $[\,0 \mid \zeta(t) - s \mid \phi\,] = \zeta(t) - s$ and hence from (2.69)

$$\text{(2.74)} \qquad \frac{W(a_0, t, s)}{t - a_0} = \xi(t') - \zeta(t') + s \quad \text{where } a_0 \leqq t' \leqq t$$

for all t, s such that $a_0 < t \leqq a_0 + \varepsilon''$ and $s_0 \leqq s \leqq s_0 + \delta''$, where $0 < \varepsilon'' \leqq \varepsilon_0$ and $0 < \delta'' \leqq \delta$. This is compatible with (2.71b) and (2.71c) only if

$$\text{(2.75)} \qquad \xi(a_0) - \zeta(a_0) + s_0 = 0 \ , \ \text{and } \xi(t) - \zeta(t) \text{ decreases on } a_0 \leqq t \leqq a_0 + \varepsilon_0 \ .$$

The same conclusion is obtained in each of the other two cases, (2.72b) and the obvious alternative to (2.73), by examining an extra term arising in (2.73) when $\zeta(t) - s < 0$ or $> \phi$, respectively.

Similarly, if $a(s)$ is also continuous in s_0', we can show from a study of (2.71a) and (2.71b) in left neighborhoods of $t = a_o$ and $s = s_o'$ that, if $\varepsilon'(a_o) \equiv \varepsilon_o'$,

(2.76) $\xi(a_o - 0) - \zeta(a_o - 0) + s_o' = 0$ and $\xi(t) - \zeta(t)$ decreases on $a_o - \varepsilon_o' \leqq t < a_o$.

In particular, if $s_o' = s_o$, $\xi(t) - \zeta(t)$ is continuous and decreasing in a neighborhood of $t = a_o = a(s_o)$, and conversely.

It follows from this analysis that if $a(s)$ is continuous on an interval $s' \leqq s < s''$ and if $a' \equiv a(s') < a(s'' - 0) \equiv a''$ then by (2.69b), (2.75) and (2.76),

(2.77) $\tilde{s}(t) = \xi(t) - \zeta(t) \leqq \tilde{s}(t') \leqq \tilde{s}(a'')$ for $a' \leqq t \leqq t' < a''$,

and, by (2.63) and (2.72),

(2.78) $$\hat{W}(t) = \hat{W}(a') \quad \text{for} \quad a' \leqq t \leqq a'' .$$

By similar reasoning one finds that, if $b(s)$ is continuous on an interval $s' \leqq s < s'' \leqq \tilde{\sigma}$ and if $b'' \equiv b(s'' - 0) < b(s') \equiv b'$, then

(2.79a and b) $\tilde{s}(b') \leqq \tilde{s}(t) = \xi(t) - \zeta(t) \leqq \tilde{s}(t')$ and $\hat{W}(t) = \hat{W}(b')$ for $b'' < t' \leqq t \leqq b'$.

We can now indicate the nature of the optimal storage policy $\hat{W}(t)$ in the simplest case in which both $a(s)$ and $b(s)$ are continuous for $0 \leqq s < \tilde{\sigma}$.

⟵ We have from (2.78), (2.63), (2.69b,e,d), (2.36) and (2.79b), respectively,

(2.80)

$$\text{(2.80a)}\quad \hat{W}(t) = \Omega_a = \Omega \quad \text{for } a \leqq t \leqq \tilde{a} \quad \text{provided } a < \tilde{a},$$

$$\text{(2.80b)}\quad \hat{W}(t) = \hat{W}(\tilde{a}) + W(t, \tilde{\sigma}) - W(\tilde{a}, \tilde{\sigma}) \quad \text{for } \tilde{a} \leqq t \leqq \tilde{b},$$

$$\text{(2.80c)}\quad \hat{W}(t) = \Omega_b = 0 \quad \text{for } \tilde{b} \leqq t \leqq b \quad \text{provided } \tilde{b} < b,$$

where $\hat{W}(\tilde{a}) = \Omega_a$ if $\tilde{a} = a$, in which case $\Omega_a < \Omega$ is possible. (2.80b) indeed gives $\hat{W}(\tilde{b}) = 0$ if $\tilde{b} < b$ and $\hat{W}(b) = \Omega_b$ if $\tilde{b} = b$, because of (2.67b), (2.50) and (2.53). Moreover, (2.80b) satisfies the reservoir restraints because of (2.49), (2.53) and (2.69e).

In the case in which $a(s)$ has discontinuities for $0 \leqq s < \tilde{\sigma}$, let $s_0 \equiv \min\{s \mid a(s-0) < a(s)\}$. Then, by (2.69a), (2.63) and (2.36),

$$\text{(2.81)}\quad \hat{W}(t) = \hat{W}(a(s_0-0)) + W(t, s_0) - W(a(s_0-0), s_0) \quad \text{for } a(s_0-0) \leqq t \leqq a(s_0).$$

Here $\hat{W}(a(s_0-0)) = \Omega_a$ if $a(s_0-0) = a$, and $\hat{W}(a(s_0-0)) = \Omega$ otherwise. In either case $\hat{W}(a(s_0)) = \Omega$ from (2.53) since $a(s_0) < b$ because $s_0 < \tilde{\sigma}$. Moreover, (2.81) satisfies the upper reservoir restraint because of (2.49a) and (2.53), and the lower reservoir restraint because $I(s_0) \leqq I(\tilde{\sigma}) \leqq D(\tilde{\sigma}) = \Omega$ and hence $\hat{W}(t) = \Omega - [W(a(s_0), s_0) - W(t, s_0)] \geqq \Omega - \Omega = 0$.

If $a(s_0-0) > a$ and hence $\hat{W}(a(s_0-0)) = \Omega = \hat{W}(a(s_0))$, it follows from $\hat{W}(t) \leqq \Omega$ that $\xi(t) - [0 \mid \zeta(t) - s_0 \mid \phi]$ is either identically equal to zero, or else first negative and ultimately positive, for $a(s_0-0) \leqq t < a(s_0)$. It then follows from the regularity assumptions that there are at most a finite number of discontinuities of $a(s)$ for $0 \leqq s < \tilde{\sigma}$. The optimum policy is therefore given by (2.80) modified by (2.81) for each discontinuity s_0 of $a(s)$ with $0 \leqq s_0 < \tilde{\sigma}$, and by similar interpolations for any discontinuities s_0 of $b(s)$ with $0 \leqq s_0 < \tilde{\sigma}$.

Subcase II.2; $D(\sigma) = I(\sigma) > \Omega$. In this case, the continuity of $I(s)$ and $I(0) < \Omega$ imply that there is a number $\tilde{\sigma}_1$, such that

(2.82a and b) $$0 \leqq \tilde{\sigma}_1 \equiv \min\{s \mid I(s) = \Omega\} < \sigma, \quad \text{so} \quad I(\tilde{\sigma}_1) = \Omega,$$

and, by (2.42) with asterisks omitted, there exists a sequence of $n \geqq 1$ pairs of points $(\underline{t}_i, \bar{t}_i)$ such that $a \leqq \underline{t}_1 < \bar{t}_1 < \underline{t}_2 < \bar{t}_2 < \dots < \bar{t}_n \leqq b$ and, for $i = 1, \dots, n$,

(2.83)
$$\text{(a)} \quad W(\underline{t}_i, \tilde{\sigma}_1) + \Omega = W(\bar{t}_i, \tilde{\sigma}_1),$$
$$\text{(b)} \quad W(t', \tilde{\sigma}_1) > W(\underline{t}_i, \tilde{\sigma}_1) \leqq W(t, \tilde{\sigma}_1) \leqq W(\bar{t}_i, \tilde{\sigma}_1) > W(t'', \tilde{\sigma}_1) \quad \text{for} \quad (a \leqq t' < \underline{t}_i \leqq t \leqq \bar{t}_i < t'' \leqq b),$$

and, for $i = 1, \dots, n-1$,

(2.84)
$$\text{(a)} \quad W(\bar{t}_i, \tilde{\sigma}_1) - \Omega > W(\underline{t}_{i+1}, \tilde{\sigma}_1),$$
$$\text{(b)} \quad W^*(a, \tilde{\sigma}_1) - \Omega > W(\underline{t}_1, \tilde{\sigma}_1) \quad \text{if} \quad a < \underline{t}_1,$$
$$\text{(c)} \quad W(\bar{t}_n, \tilde{\sigma}_1) - \Omega > W^*(b, \tilde{\sigma}_1) \quad \text{if} \quad \bar{t}_n < b,$$
$$\text{(d)} \quad W(t', \tilde{\sigma}_1) - W(t, \tilde{\sigma}_1) < \Omega \quad \text{for} \quad \bar{t}_i \leqq t \leqq t' \leqq \underline{t}_{i+1},$$

where (2.84d) also holds for $i = 0$ with $\bar{t}_0 \equiv a$ if* $a < \underline{t}_1$, and for $i = n$ with $\underline{t}_{n+1} = b$ if $\bar{t}_n < b$.

---

* We can have $a = \underline{t}_1$ or $\bar{t}_n = b$ only in the extreme cases that $\Omega_a = 0$ or $\Omega_b = \Omega$.

Here (2.83a) and the second and third inequality in (2.83b) follow from (2.82b). The first and last inequalities in (2.83b) serve to define $\underline{t}_i$ and $\bar{t}_i$ uniquely. If (2.84a,b,c) were violated, two successive pairs $(\underline{t}_i, \bar{t}_i)$ and $(\underline{t}_{i+1}, \bar{t}_{i+1})$ could be replaced by a single pair $(\underline{t}_i, \bar{t}_{i+1})$, or (2.82b) would be contradicted, and if (2.84d) were violated, an additional pair $(\underline{t}_i', \bar{t}_i')$ would exist with $\bar{t}_i < \underline{t}_i' < \bar{t}_i' < \underline{t}_{i+1}$, or (2.82b) would again be contradicted. Since by (2.37) and (2.83a) $\bar{t}_i - \underline{t}_i \geqq \Omega/\omega > 0$, n is finite.

We now maintain the definition (2.68a) of $T(s)$ for $0 \leqq s < \tilde{\sigma}_1$ and modify (2.68b) in part to

$$T(\tilde{\sigma}_1) = [a, a(\tilde{\sigma}_1)) \bigcup_{i=1}^{n} [\underline{t}_i, \bar{t}_i) \bigcup [b(\tilde{\sigma}_1), b) \quad . \tag{2.85}$$

If $\underline{t}_1 < a(\tilde{\sigma}_1)$ we have from (2.49a) that $\bar{t}_1 = a(\tilde{\sigma}_1)$, and the inclusion of $[\underline{t}_1, \bar{t}_1)$ in (2.85) has no effect. Similarly, if $\bar{t}_n > b(\tilde{\sigma}_1)$, $\underline{t}_n = b(\tilde{\sigma}_1)$ and the inclusion of $[\underline{t}_n, \bar{t}_n)$ has no effect. In any case, the previous solution (2.80) modified by (2.81) remains valid for $a \leqq t \leqq a(\tilde{\sigma}_1)$ and for $b(\tilde{\sigma}_1) \leqq t \leqq b$, where $a(\tilde{\sigma}_1) < b(\tilde{\sigma}_1)$ because $\tilde{\sigma}_1 < \sigma$. On any interval $[\underline{t}_i, \bar{t}_i)$ with*

$a(\tilde{\sigma}_1) < \underline{t}_i < \bar{t}_i < b(\tilde{\sigma}_1)$ that occurs in (2.85) we have $\tilde{s}(t) = \tilde{\sigma}_1$ and hence

$$\hat{W}(t) = \hat{W}(\underline{t}_i) + W(t, \tilde{\sigma}_1) - W(\underline{t}_i, \tilde{\sigma}_1) \text{ for } \underline{t}_i \leqq t \leqq \bar{t}_i \,, \tag{2.86}$$

---

* $\underline{t}_i = a(\tilde{\sigma}_1)$ cannot occur for $i > 1$ because $\bar{t}_1 < \underline{t}_2$, and cannot occur for $i = 1$ because it could imply $a(\tilde{\sigma}_1) = a$, $\Omega_a = 0$, which would again imply $\bar{t}_1 = a(\tilde{\sigma}_1) > a$, a contradiction.

which by (2.85) meets the reservoir restraints if and only if

$$(2.87) \qquad \hat{W}(\underline{t}_i) = 0 \quad \text{and hence} \quad \hat{W}(\bar{t}_i) = \Omega \quad .$$

The set $[a, b) - T(\tilde{\epsilon}_1)$ on which $\tilde{s}(t)$ has not yet been defined is a union of intervals $[\bar{t}_i, \underline{t}_{i+1})$. On each such interval, $W(t, \tilde{\epsilon}_1)$ satisfies precisely the same conditions as $\Omega(t) = W(t, 0)$ satisfies on $[a, b)$. This can be verified by comparing (2.23) with (2.84) and with the first and last inequalities in (2.83b). The same analysis can therefore be repeated for each such interval with new functionals $D_i(s)$, $I_i(s)$ defined for $\tilde{\epsilon}_1 \leqq s \leqq \zeta$ analogously to (2.42) and (2.43). If on such an interval Subcase II.1 applies, the analogue of (2.80) qualified by (2.81) defines the optimal policy on that interval. If Subcase II.2 applies, a further subdivision occurs for $s = \tilde{\epsilon}_2 > \tilde{\epsilon}_1$ which incorporates at least one subinterval of length not less than Ω/ω into $T(\tilde{\epsilon}_2)$. Hence Subcase II.2 can occur only a finite number of times in all intervals to be examined. Since case I cannot occur on any interval but the entire planning period, the construction of the optimal storage policy has now been completed, and the resulting policy $\hat{W}(t)$ has the properties enumerated at the end of Section 2.4.

3. Price imputations for power, water, and reservoir and turbine capacities.

3.1 Purposes of efficiency prices. Once an optimal storage policy has been specified, it is possible to associate with it imputed valuations of the resources that occur in the system, also known as "shadow," "accounting," "intrinsic" or "efficiency" prices. These efficiency prices have an incremental or "marginal" interpretation. We want the price of power at any time to represent the incremental cost of one extra unit of electrical energy to be produced in a short period containing that time. Likewise, the price of water is to express the highest possible saving in the operating cost of thermal generation if it is known in advance that one extra unit of water will be added to the reservoir at a certain time. Because of the non-linearity of the cost function $\psi(s)$, this interpretation is approximately valid only if the added units are small. Likewise, the efficiency prices of reservoir and turbine capacity are to represent the derivatives of the minimum $C(\hat{v})$ of operating cost for the planning period with respect to each capacity, Ω or ϕ , whenever such a derivative exists.

There is, of course, no necessary connection between the efficiency price of power and the power rates charged to customers. The first and foremost use of these prices is in the distribution of power loads over interconnected generating systems of which the managements pursue a policy of minimizing aggregate cost of the power supplied by all systems. It has been correctly argued by economic theorists that determination of power rates (and of all other prices) according to marginal cost would make it possible for users of electric energy to make allocative decisions that minimize aggregate cost of production of whatever is produced — or, even more broadly, that enable consumers to distribute their expenditures over various commodities in such a way as to minimize the aggregate social cost of the satisfactions received.* Since neither the practical

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* For an excellent summary of the literature on this issue, see Nancy Ruggles [1949]. For a statement of the general theory see Koopmans [1957] and for an example of an application to another industry see Vickrey [1955].

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difficulties of, nor some theoretical objections to, marginal cost pricing have been fully explored, it is contended here only that if marginal cost pricing of electric power rates is desired, the present study contributes to the analytical tools for determining such prices.

The efficiency price of water has an obvious application when additional investments in the watershed can increase the flow of water by an approximately known function of time. Comparison of the efficiency prices of reservoir and turbine capacities with the incremental costs of constructing these capacities will help in designing such capacities to fit anticipated seasonally recurrent conditions of watershed runoff and power demand.

3.2 Definitions of the efficiency prices. The data underlying these definitions are the cost function $\psi(s)$ of thermal generation, and the characteristics of the optimal storage policy $\hat{W}(t)$, which for this purpose is best represented by the target rate $s(t)$ of thermal generation from which it is derived. Because of the convexity* (1.10c) of

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* See, for instance, W. Fenchel, "Convex cones, sets and functions," Princeton University, Logistics Research Project, 1953.

$\psi(s)$, there exists for every $s \geqq 0$ a number $\psi'(s)$ such that

(3.1) $$\psi(s') \geqq \psi(s) + \psi'(s)\cdot(s'-s) \quad \text{for all} \quad s' \geqq 0 .$$

In any point s where $\psi(s)$ is differentiable, the number $\psi'(s)$ is uniquely determined by (3.1) and equals the first derivative of $\psi(s)$. If $\psi(s)$ has different right and left derivatives $\psi'(s+0)$ and $\psi'(s-0)$, we have $\psi'(s-0) < \psi'(s+0)$, and we retain freedom to give to $\psi'(s)$ any value such that

(3.2) $$\psi'(s-0) \leqq \psi'(s) \leqq \psi'(s+0) ,$$

with the understanding that $\psi'(0-0) \equiv 0$. No matter how this freedom is used, we have*

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* Fenchel, l. c., p. 71

(3.3) $$\psi'(s) < \psi'(s') \quad \text{if} \quad 0 \leqq s < s' .$$

We now define as the efficiency price of power

$$p(t) \equiv \psi'(\hat{s}(t)), \quad \text{where} \quad \hat{s}(t) \equiv \zeta(t) - \hat{v}(t). \tag{3.4}$$

Here $\hat{s}(t)$ is the actual rather than the target rate of thermal generation, occurring at time t in the optimal policy. We allow the indeterminacy (3.2) to persist for such times t at which $\hat{s}(t)$ indicates a kink in $\psi(s)$, where we specify

$$\underline{p}(t) \equiv \psi'(\hat{s}(t)-0) \leqq p(t) \leqq \psi'(\hat{s}(t)+0) \equiv \bar{p}(t). \tag{3.5}$$

$\bar{p}(t)$ is the extra cost of one additional unit of power, $\underline{p}(t)$ the saving from generating one unit less.

In contrast, the price of water $q(t)$ is similarly derived from the target rate $\tilde{s}(t)$ of thermal generation as defined by (2.64),

$$q(t) \equiv \psi'(\tilde{s}(t)), \tag{3.6}$$

where in case of indeterminacy

$$\underline{q}(t) \equiv \psi'(\tilde{s}(t)-0) \leqq q(t) \leqq \psi'(\tilde{s}(t)+0) \equiv \bar{q}(t). \tag{3.7}$$

Since availability of more water means a need for less thermal generation, $\underline{q}(t)$ now is the value of an added unit of water, $\bar{q}(t)$ the cost of diverting one unit.

Since by (2.33)

$$\zeta(t) - \hat{s}(t) = [0 \mid \zeta(t) - \tilde{s}(t) \mid \phi], \tag{3.8}$$

our choice of units of power and of water equates their prices, $p(t) = q(t)$, unless turbine restraints are operative. If the upper turbine bound is reached, $\hat{s}(t) \geqq \tilde{s}(t)$ and hence by (3.3) $p(t) \geqq q(t)$. If the lower bound is reached, $\hat{s}(t) \leqq \tilde{s}(t)$ and $p(t) \leqq q(t)$. Hence if we write

$$(3.9) \qquad r(t) \equiv p(t) - q(t) \equiv r^{+}(t) - r^{-}(t) \equiv \begin{cases} r^{+}(t) & \text{if } r(t) \geqq 0 \\ -r^{-}(t) & \text{if } r(t) < 0 , \end{cases}$$

we can look upon $r^{+}(t)$, the excess (when positive) of the price of power over that of the water used to generate the power, as a scarcity rent (per unit of time) on turbine capacity, which is positive only when that capacity is in full use, and otherwise zero.* In case of indeterminacy of either

* The negative component $r^{-}(t)$ of $r(t)$ may be used in deciding whether or not to provide capacity for pumping water back into the reservoir.

$p(t)$ or $q(t)$ or both, we set

$$(3.10) \qquad \underline{r}(t) \equiv \underline{p}(t) - \bar{q}(t) \leqq r(t) \leqq \bar{p}(t) - \underline{q}(t) \equiv \bar{r}(t) .$$

Since higher turbine capacity permits the use of more water to reduce the rate of thermal generation, $\underline{r}(t)$ is the rental for increases in capacity, $\bar{r}(t)$ that for decreases.

Finally, the integral

$$(3.11)\qquad R \equiv \int_0^{\tau} r(t)\,dt\,, \quad \text{where } \underline{R} \equiv \int_0^{\tau} \underline{r}(t)\,dt \leqq R \leqq \int_0^{\tau} \bar{r}(t)\,dt \equiv \bar{R}$$

defines the efficiency price of turbine capacity for the planning period.

The efficiency price Q of reservoir capacity is related to the price of water $q(t)$ differently from (3.11) because the reservoir is a means of shifting the use of water in time, whereas the turbines convert water into power. By (3.3) and (3.6) $q(t)$, or in case of indeterminacy either one of its bounds $\underline{q}(t)$ or $\bar{q}(t)$, shares with $\tilde{s}(t)$ the property of being constant over time except when the reservoir is full, when $q(t)$ can increase, or when it is empty, when $q(t)$ can decrease. This reflects the fact that a situation where a shift in the use of water if feasible would produce a saving can arise in connection with an optimal storage policy only if that shift is in fact not feasible. The only circumstance that can prevent a forward shift is a full reservoir at some intermediate point in time, the only obstacle to a backward shift an intervening empty reservoir.

This suggests that the rental on reservoir capacity is related to the increase in $q(t)$ while the reservoir is full. We therefore write

$$(3.12)\qquad q(t) \equiv q^{\uparrow}(t) - q^{\downarrow}(t)\,,$$

where $q^{\uparrow}(t)$ and $q^{\downarrow}(t)$ are nondecreasing functions such that $q^{\uparrow}(t)$ absorbs all increases, $q^{\downarrow}(t)$ all decreases of $q(t)$ — which are known to occur on disjoint intervals. Now $q^{\uparrow}(t'+0) - q^{\uparrow}(t-0)$ is defined as the rental on reservoir capacity during any interval (or instant if $t'=t$)

in which the reservoir is full, and

$$Q = q^{\uparrow}(\tau - 0) - q^{\uparrow}(0+0) \tag{3.13}$$

as the efficiency price of reservoir capacity for the entire planning period.*

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* In case of indeterminacy, the lower bound $\underline{Q}$ applicable to increases in capacity is obtained by replacing the defining function $q(t)$ in (3.12) by a function $q^*(t)$ which equals $\underline{q}(t)$ on all intervals on which the store of water falls from Ω (or Ω_0) to 0 (or Ω_τ), $\bar{q}(t)$ on all intervals during which the store of water rises from 0 (or Ω_0) to Ω (or Ω_τ), and a permissible but constant value on all intervals during which the store of water departs from either Ω or 0 without reaching the other of these values. The upper bound $\bar{Q}$ applicable to decreases in capacity is similarly defined with the roles of $\bar{q}(t)$ and $\underline{q}(t)$ reversed.

3.3 <u>Profit maximization in a fictitious market</u>. We shall now derive an inequality which will yield us a proof of the optimality of the storage policy $\hat{W}(t)$ constructed in Section 2, as well as important properties of the efficiency prices. We consider an alternative hydroelectric system, in which the net inflow of water $x(t)$ may be different from $\xi(t)$ as a result of improvements to the watershed, or of different rates of diversion of water to competing purposes such as irrigation. Likewise, the power demand $z(t)$ may differ from $\zeta(t)$ but as before $x(t)$ and $z(t)$ are nonnegative. We consider further an alternative policy, restrained through

(3.14a and b) $\quad z(t) = s(t) + v(t), \qquad x(t) - u(t) - v(t) = w(t),$

where the nonnegative functions $s(t)$, $v(t)$, $u(t)$, represent policies with regard to thermal generation, turbine discharge, and spillage, each of which may differ from the corresponding functions $\hat{s}(t)$, $\hat{v}(t)$, $\hat{u}(t)$ in the policy constructed in Section 2. The initial store of water $W(0)$, and the final store $W(\tau)$ it implies through (3.14b), are required only to be nonnegative. No upper capacity restraints are imposed, for the moment, on $v(t)$ or $W(t)$, but the latter function is required to be nonnegative for $0 \leqq t \leqq \tau$ as well as the former. Finally, the cost function $\psi(s)$ of thermal generation remains the same as before. For brevity, the alternative policy is denoted $X \equiv (x(t), z(t), s(t), u(t), W(0))$.

We now consider a fictitious situation where the manager of the hydroelectric system chooses his policy in the following manner. He must buy or rent all resources he uses at the efficiency prices defined in the preceding Section 3.2. He must pay the total cost of such thermal generation as he requires. He sells the power generated at its efficiency price, and will receive an allowance for the value of the water in storage at the end of the planning period. He then chooses a policy that maximizes his profit. To be precise, the "profit" is defined as

$$(3.15) \quad \Pi \equiv \int_0^\tau \left[z(t)p(t) - x(t)q(t) - v(t)r^+(t) - \psi(s(t)) \right] dt - \int_0^\tau W(t)\, dq^\uparrow(t) - W(0)q(0) + W(\tau)q(\tau-0),$$

The negative terms not yet itemized are payments for water received (an initial stock $W(0)$ and a flow $x(t)$ throughout the period) and rent payments for use of turbine and reservoir capacity proportional to the amount of use, $v(t)$ or $W(t)$.

We can simplify (3.15) through the identity

$$W(\tau)q(\tau-0)-W(0)q(0)=\int_0^\tau W(t)\,dq(t)+\int_0^\tau w(t)q(t)\,dt, \tag{3.16}$$

which by (3.14b), (3.12) and (3.9) leads to

(3.17)
(a and b) $\Pi=\Pi_1+\Pi_2+\Pi_3$, $\quad \Pi_1\equiv\int_0^\tau[s(t)p(t)-\psi(s(t))]\,dt$,

(c and d) $\Pi_2\equiv-\int_0^\tau[v(t)r^-(t)+u(t)q(t)]\,dt$, $\quad \Pi_3\equiv-\int_0^\tau W(t)\,dq^{\downarrow}(t)$.

Now let us compare any policy X open to our manager with the policy $\hat{X}\equiv(\xi(t), \zeta(t), \hat{s}(t), \hat{u}(t), \Omega_0)$ constructed in Section 2 to fit the data $\xi(t)$, $\zeta(t)$, Ω_0, Ω, Ω_τ, ϕ underlying the restraints of that construction. Since $\hat{v}(t)=0$ whenever $r^-(t)>0$, $u(t)=0$ whenever $q(t)>0$, and $W(t)=0$ whenever $q(t)$ decreases, and since $v(t)$, $r^-(t)$, $u(t)$, $q(t)$, $W(t)$ and $dq^{\downarrow}(t)$ are all nonnegative, we have

$$\Pi_2\leqq\hat{\Pi}_2=0, \qquad \Pi_3\leqq\hat{\Pi}_3=0, \tag{3.18}$$

if $\hat{\Pi}_i$ corresponds to the policy of Section 2. Finally, from (3.1), (3.4) and (3.18),

$$\Pi\leqq\Pi_1\leqq\hat{\Pi}_1=\int_0^\tau[\hat{s}(t)p(t)-\psi(\hat{s}(t))]\,dt=\hat{\Pi}. \tag{3.19}$$

It follows that our manager cannot do better than he does by adopting the policy $\hat{X}$ of Section 2. The price system that we have associated with that policy "sustains" it, in that in calculations based on that price system the policy in question is as profitable as any. It is to be noted that the alternative policies include policies that involve different water and power flows, and that violate the upper turbine and reservoir restraints and the initial and final store-of-water specifications. The price system therefore gives indirect expression to these restraints by making their violation unprofitable.

The maximum attainable "profit" is positive whenever $\hat{s}(t) > 0$ on some interval, and is found to accrue entirely from the fact that whenever $\hat{s}(t) > 0$ the marginal cost $p(t) = \psi'(\hat{s}(t))$ at which power is sold exceeds the average cost $\psi(\hat{s}(t))/\hat{s}(t)$ of its generation, as is seen by insertin[g] $s = \hat{s}(t)$, $s' = 0$ in (3.1).

3.4 Proof of optimality of the policy construction in Section 2.

We now specify that the alternative policy X be a policy $X_0 \equiv (\xi(t), \zeta(t), s(t), u(t), \Omega_0)$ that satisfies all the restraints of Section 2, without necessarily being the "optimal" policy $\hat{X}$. Denoting all quantities relating to the policy X_0 by a subscript 0 , we have from (3.15) and (3.19),

$$0 \leq \hat{\Pi} - \Pi_0 = \int_0^T \left[(v_0(t) - \hat{v}(t))\, i^+(t) + \psi(s_0(t)) - \psi(\hat{s}(t)) \right] dt \tag{3.20}$$

$$- \int_0^T (\hat{W}(t) - W_0(t))\, dq^\uparrow(t) + (\hat{W}(T) - W_0(T))\, q(T-0).$$

However, $\hat{v}(t) = \phi$ if $i^+(t) > 0$, and $v_0(t) \leq \phi$ throughout, $\hat{W}(t) = \Omega$ if $q^\uparrow(t)$ increases and $W_0(t) \leq \Omega$ throughout, and

$\hat{W}(\tau)=\Omega_\tau$ if $q(\tau-0)>0$ and $W_o(\tau) \geqq \Omega_\tau$. It follows that

(3.21) $$C_o \equiv \int_0^\tau \psi(s_o(t))\,dt \geqq \int_0^\tau \psi(\hat{s}(t))\,dt \equiv \hat{C},$$

which proves that the policy $\hat{X}$ constructed in Section 2 is indeed an optimal one. Moreover, equality in (3.19) would, in view of (3.1), require $s_o(t) = \hat{s}(t)$ for $0 \leqq t < \tau$, if we specify that both $s_o(t)$ and $\hat{s}(t)$ are continuous to the right. Hence the policy $\hat{X}$ is the unique optimal policy involving automatic overflow.

3.5 <u>A further property of the efficiency prices</u>. We shall indicate without full proof one further important property of the efficiency prices, which appears when we study how the minimum attainable cost $\hat{C}$ depends on the data of the system studied. Using the brief notation $\Xi \equiv (\xi(t), \zeta(t), \Omega_o, \Omega, \Omega_\tau, \phi)$ for these data, we find by inserting the policy $\hat{X}$ in (3.15) and using (3.9) and (3.13) that

(3.22) $$\begin{aligned}\hat{\Pi} \equiv \hat{\Pi}(\Xi) = \int_0^\tau [\zeta(t)\,p(t) - \xi(t)\,q(t)]\,dt - \hat{C}(\Xi) \\ -\phi R - \Omega Q - \Omega_o q(0) + \Omega_\tau q(\tau-0).\end{aligned}$$

Consider a neighboring system with data $\Xi^{(\mu)} \equiv \Xi + \mu \Xi'$, where $\Xi' \equiv (\xi'(t), \dots)$ is a set of "incremental" data restricted only the requirement that, for some $\mu = \mu_o > 0$, and hence for all $0 \leqq \mu \leqq \mu_o$, the data be consistent in the sense that all components of $\Xi^{(\mu)}$ be nonnegative and that $\Omega_o^{(\mu)} \leqq \Omega^{(\mu)}$ and $\Omega_\tau^{(\mu)} \leqq \Omega^{(\mu)}$

We now subtract (3.22) from the corresponding inequality for the neighboring system, solve for $\hat{C}(\Xi^{(\mu)}) - \hat{C}(\Xi)$, divide both members by μ and take the limit for $\mu \to +0$. Then if the data Ξ imply determinate efficiency prices $p(t)$, $q(t)$ for $0 \leqq t < \tau$, we obtain

$$\left(\frac{d\hat{C}(\Xi^{(\mu)})}{d\mu}\right)_{\mu=+0} = \int_0^\tau \left[\zeta'(t)p(t) - \xi'(t)q(t)\right] dt - \phi' R - \Omega' Q - \Omega'_0 q(0) + \Omega'_\tau q(\tau - 0) . \tag{3.23}$$

Furthermore, if the data $\Xi^{(\mu)}$ are compatible also for some $\mu'_0 < 0$, then (3.23) also gives the left derivative of the cost minimum at $\mu = 0$. Finally, if $p(t)$ and $q(t)$ are not determinate, one can make permitted* choices

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* See the footnote on p. below for the precise meaning of "permitted."

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so that (3.23) indicates the right derivative of the cost minimum, and other permitted choices so that (3.23) represents the left derivative. It follows that if the data can be modified at given incremental costs which are different from the corresponding efficiency prices (if unique) or fall outside their permitted ranges (if not unique), another system can be realized of which the minimum cost (algebraically) falls below that of the given system by more than the cost of the modification. In this way, the efficiency prices bear on design problems of hydroelectric systems as well as on operation problems.

The proof of (3.23) depends on continuity of the efficiency prices $p^{(\mu)}(t)$, $q^{(\mu)}(t)$ at the point $\mu = 0$, or upper semicontinuity if indeterminacies exist at that point.* In particular, one needs to show that, with suitable permitted choices

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* Besides the indeterminacies already indicated in (3.5), (3.7) and (3.10), on further type of indeterminacy needs to be recognized in this proof, which can arise when an interval $[t_1, t_2]$ on which the upper turbine restraint is operative $(\hat{v}(t) = \phi$ for $t_1 \leq t < t_2)$ contains a point $\bar{t} > t_1$ at which the reservoir is full. Whereas (3.6) holds $q(t)$ constant until such a point $\bar{t}$ is reached, thus giving a lower bound to the price of water on $[t_1', \bar{t})$, an alternative definition allows $q(t)$ to increase as soon and as much as is compatible with $\hat{v}(t) = \phi$ for $t_1 \leq t < \bar{t}$ and maintains the nondecreasing character of $q(t)$ for $t_1 \leq t \leq \bar{t}$, thus providing an upper bound to the range of the price of water. A similar difficulty cannot arise when the lower turbine restraint is operative $(\hat{v}(t) = 0)$, because $\xi(t) \geqq 0$ prevents the reservoir from becoming empty in this case.

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of the prices, $\left[\hat{\Pi}(\Xi^{(\mu)}) - \hat{\Pi}(\Xi)\right]/\mu \to 0$ if $\mu \to +0$. This is indeed suggested by the last equality in (3.19), where in the case of determinacy $p(t) = \psi'(\hat{s}(t))$. Hence $\hat{\Pi}(\Xi^{(\mu)}) - \hat{\Pi}(\Xi)$ depends on $\Xi^{(\mu)}$ and Ξ only through $\hat{s}^{(\mu)}(t) - \hat{s}(t)$, and is of second order in this quantity, as one would expect since $\hat{\Pi}(\Xi)$ is the value of a minimum attained for $s(t) = \hat{s}(t)$. The burden of the proof is then to show that $\left[\hat{s}^{(\mu)}(t) - \hat{s}(t)\right]/\mu$ has a bounded limit for $0 \leqq t \leqq \tau$.

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