ADDENDUM TO
A MULTIVARIATE STOCHASTIC UNIT ROOT MODEL
WITH AN APPLICATION TO DERIVATIVE PRICING

by

Offer Lieberman and Peter C. B. Phillips

COWLES FOUNDATION PAPER NO. 1552a

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

2021

http://cowles.yale.edu/

# A Multivariate Stochastic Unit Root Model with an Application to Derivative Pricing - An Addendum to Lieberman and Phillips, *Journal of Econometrics* 196 (2017), 99-110. *

Offer Lieberman† and Peter C. B. Phillips‡

June 14, 2021

In Theorem 7 of Lieberman and Phillips (*Journal of Econometrics* 196 (2017), 99-110), it is stated that $\hat{\mu}_n \Rightarrow \mu B(a, \Sigma_u)$, where

$$B(a, \Sigma_u) = H_a^*(1) - a' \int_0^1 H_a^*(r)\, dB_u(r) - \frac{1}{2} a' \Sigma_u a \int_0^1 H_a^*(r)\, dr,$$

and

$$H_a^*(r) = e^{a' B_u(r)} \int_0^r e^{-a' B_u(p)} dp.$$

In fact, stochastic differentiation of $H_a^*(r)$ reveals that

$$dH_a^*(r) = dr + a' H_a^*(r)\, dB_u(r) + \frac{1}{2} a' \Sigma_u a \int_0^1 H_a^*(r)\, dr,$$

from which it follows that

$$H_a^*(1) = \int_0^1 dH_a^*(r) = 1 + a' \int_0^1 H_a^*(r)\, dB_u(r) + \frac{1}{2} a' \Sigma_u a \int_0^1 H_a^*(r)\, dr,$$

*We thank Tim Ginker for his research assistance during the time the original paper was written and for recently drawing our attention to some of the points in this Addendum.

†Department of Economics, Bar-Ilan University, Ramat Gan 52900, Israel. E-mail: offer.lieberman@gmail.com

‡Yale University, University of Auckland, Southampton University, and Singapore Management University.

so that $B(a, \Sigma_u) = 1$. In other words, $\hat{\mu}_n$ is consistent and as a result Remarks 9-10, which provide an alternative consistent estimator of $\mu$, are superflous. In view of the above

$$\left(\hat{\sigma}^{\mu}_{\varepsilon,n}\right)^2 := \frac{1}{n}\sum_{t=2}^{n}\left(Y_t - \hat{\mu}_n - e^{\hat{a}_n' u_t/\sqrt{n}} Y_{t-1}\right)^2 \Rightarrow \sigma_\varepsilon^2 - \frac{\left(\int_0^1 H_a(r)\,dr\right)^2}{\int_0^1 H_a^2(r)\,dr} \Sigma_{u\varepsilon}' \Sigma_u^{-1} \Sigma_{u\varepsilon},$$

$$\hat{\Sigma}^{\mu}_{u\varepsilon,n} := \frac{1}{n}\sum_{t=2}^{n}\left(Y_t - \hat{\mu}_n - e^{\hat{a}_n' u_t/\sqrt{n}} Y_{t-1}\right) u_t \Rightarrow \left(1 - \frac{\left(\int_0^1 H_a(r)\,dr\right)^2}{\int_0^1 H_a^2(r)\,dr}\right) \Sigma_{u\varepsilon}$$

and consistent estimators of $\sigma_\varepsilon^2$ and $\Sigma_{u\varepsilon}$ can be obtained in a straightforward manner by replacing $\int_0^1 H_a(r)\,dr$ and $\int_0^1 H_a^2(r)\,dr$ by $n^{-2}\sum Y_t$ and $n^{-3}\sum Y_t^2$, respectively, and plugging

$$\hat{\Sigma}^{*}_{\mu\varepsilon,n} := \left(1 - \frac{\left(\sum Y_t\right)^2}{n\sum Y_t^2}\right)^{-1} \hat{\Sigma}_{\mu\varepsilon,n} \Rightarrow \Sigma_{u\varepsilon}$$

into

$$\left(\hat{\sigma}^{\mu *}_{\varepsilon,n}\right)^2 := \left(\hat{\sigma}^{\mu}_{\varepsilon,n}\right)^2 + \frac{\left(\sum Y_t\right)^2}{n\sum Y_t^2} \hat{\Sigma}^{*\prime}_{\mu\varepsilon,n} \hat{\Sigma}^{-1}_{u,n} \hat{\Sigma}^{*}_{\mu\varepsilon,n} \Rightarrow \sigma_\varepsilon^2.$$