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By

Peter C. B. Phillips and Ying Wang

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YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.yale.edu/

# Limit Theory for Locally Flat Functional Coefficient Regression*

Peter C. B. Phillips[a,b,c,d] and Ying Wang[e]

[a]University of Auckland, [b]Yale University,
[c]Singapore Management University, [d]University of Southampton
[e]Renmin University of China

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**Abstract**

Functional coefficient (FC) regressions allow for systematic flexibility in the responsiveness of a dependent variable to movements in the regressors, making them attractive in applications where marginal effects may depend on covariates. Such models are commonly estimated by local kernel regression methods. This paper explores situations where responsiveness to covariates is locally flat or fixed. In such cases, the limit theory of FC kernel regression is shown to depend intimately on functional shape in ways that affect rates of convergence, optimal bandwidth selection, estimation, and inference. The paper develops new asymptotics that take account of shape characteristics of the function in the locality of the point of estimation. Both stationary and integrated regressor cases are examined. Locally flat behavior in the coefficient function has, as expected, a major effect on bias and thereby on the trade-off between bias and variance, and on optimal bandwidth choice. In FC cointegrating regression, flat behavior materially changes the limit distribution by introducing the shape characteristics of the function into the limiting distribution through variance as well as centering. Both bias and variance depend on the number of zero derivatives in the coefficient function. In the boundary case where the number of zero derivatives tends to infinity, near parametric rates of convergence apply for both stationary and nonstationary cases. Implications for inference are discussed and simulations characterizing finite sample behavior are reported.

*JEL classification:* C14; C22.

*Keywords*: Boundary asymptotics; Functional coefficient regression; Limit theory; Locally flat regression coefficient; Near-parametric rate.

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# 1 Introduction

Kernel approaches to nonparametric regression use localized versions of standard statistical methods to fit shape characteristics of nonlinear functions in statistical models. These methods have been extensively used in applied research across the social, business, and natural sciences. The methods are particularly useful in assessing the role of nonlinearities and parameter instabilities and are used in modeling cross section, time series, and panel data. An especially useful model for which these methods have been developed is functional coefficient (FC) regression. Such regressions allow the responses of a dependent variable to depend locally in a systematic way on movements in other variables.

This paper demonstrates that the limit theory in FC regression depends on the functional shape of the regression coefficient in ways that involve rates of convergence, asymptotic variance, bandwidth selection, and inference. Standard limit theory for kernel regression shows clearly how functional shape affects bias, which is well known to depend on the local first two derivatives of the regression function and the first derivative of the density of the covariate. The limit theory changes in material ways when these and possibly higher derivatives are zero at the point of estimation. Recent work on FC cointegrating regression (Phillips and Wang, 2020) pointed out dependence of the asymptotic variance on the first derivative of the functional coefficient in estimating cointegrating equations. But the effects of flat functional shape on the limit distribution, including the bias function and limiting variance, have not been explored in earlier nonparametric literature on FC regression in either stationary or nonstationary cases. There also appears to be no former research on the implications of flat functional shape on the limit theory for standard kernel density estimation or kernel regression.

The present paper develops new asymptotics that involve these shape characteristics of the function in the locality of the point of estimation. In particular, locally flat behavior in the coefficient function is shown to have a major effect on the form of the asymptotic distribution as well as the rate of convergence, with important differences between stationary and nonstationary regressions. Local flatness in the coefficient function at some point in the covariate space may be regarded as an intermediate case between the usual FC model and regression with a fixed coefficient, allowing for responses of the dependent variable to be unresponsive to movements in other variables at this point in their support. The primary focus in this paper is to develop asymptotics for FC regression under such flatness conditions. Related effects to those described here may be expected to apply in other nonparametric regression models where flatness occurs in nonlinear nonparametric regressions.

The paper is organized as follows. The new limit theory is given in Section 2, which covers both stationary and nonstationary FC regression. Section 3 discusses the implications of the limit theory for inference. Section 4 provides simulation evidence corroborating the asymptotics. Section 5 concludes. Proofs of the main results, several subsidiary lemmas, and computation details are given in the Appendix. Additional technical details are provided in the Online

Supplement to this paper. Throughout the paper we use the notation $\equiv_d$ to signify equivalence in distribution, $\sim_a$ to signify asymptotic equivalence, $\rightsquigarrow$ to denote weak convergence on the relevant probability space, $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ to denote floor and ceiling functions, $[\cdot]$ to signify the rounded part of a real number, and $\mu_j(K) = \int_{\mathcal{K}} u^j K(u)du$, $\nu_j(K) = \int_{\mathcal{K}} u^j K^2(u)du$ for kernel moment functions, where $\mathcal{K}$ is the support of the kernel function $K$. According to the context, we use := and =: to signify definitional equality. Unless otherwise indicated $\int$ denotes $\int_0^1$.

## 2 Asymptotic theory for locally flat FC estimation

The standard FC regression model is a simple extension of linear regression, taking the following form

$$y_t = x_t'\beta(z_t) + u_t \tag{2.1}$$

in which the covariate $z_t$ determines the strength or weakness of the response of $y_t$ to the regressor $x_t$. The regressor $x_t$ is a $p \times 1$ time series, which may be stationary or nonstationary. The covariate $z_t$ is a $q \times 1$ time series and is commonly, although not always, assumed to be stationary. The error term $u_t$ is a scalar stationary process, often taken to be a martingale difference. In view of its flexibility as a convenient extension of fixed parameter regression, the model has been extensively studied and applied in econometrics. A popular textbook reference is by Li and Racine (2007, Chapter 9.3). Many papers have studied estimation and inference in this model under various assumptions, including early work by Cai et al. (2000) on stationary regression and much subsequent work on nonstationary regressions covering both cointegrated and noncointegrated models (Juhl, 2005; Xiao, 2009; Cai et al., 2009; Sun et al., 2011; Wang et al., 2019).

Kernel weighted local least squares regression is a standard approach to estimating the functional coefficient $\beta(\cdot)$ in (2.1). The local level least squares estimate of $\beta(z)$ is $\hat{\beta}(z) = \left(\sum_{t=1}^n x_t x_t' K_{tz}\right)^{-1}\left(\sum_{t=1}^n x_t y_t K_{tz}\right)$ with kernel function $K_{tz} = K((z_t - z)/h)$ and bandwidth $h$. The estimate $\hat{\beta}(z)$ may be decomposed in the usual manner into 'bias' and 'variance' terms as

$$\left(\sum_{t=1}^n x_t x_t' K_{tz}\right)\left(\hat{\beta}(z) - \beta(z)\right) = \sum_{t=1}^n x_t x_t' [\beta(z_t) - \beta(z)] K_{tz} + \sum_{t=1}^n x_t u_t K_{tz}. \tag{2.2}$$

Under suitable regularity conditions the limit theory for $\hat{\beta}(z)$ is normal or mixed normal after standard corrections are employed for bias and suitable recentering or undersmoothing is employed (Phillips and Wang, 2020). These asymptotics lead to a theory of estimation and inference for both stationary, cointegrating, and mixed regressor cases. Our treatment extends the existing limit theory to address the impact of locally flat behavior in the regression coefficient function $\beta(\cdot)$. We start with the stationary case.

### 2.1 The FC stationary model

It is convenient for exposition to use a prototypical version of the model (2.1) in which the following conditions are assumed.

**Assumption 1.**

*(i)* *The observable time series* $\{x_t, z_t\}$ *are strictly stationary* $\alpha$*-mixing processes with mixing numbers* $\alpha(j)$ *that satisfy* $\sum_{j\geq 1} j^c[\alpha(j)]^{1-2/\delta} < \infty$ *for some* $\delta > 2$, $c > \tau(1-2/\delta)$ *and* $\tau > 1$ *with finite moments of order* $p > 2\delta > 4$ *and* $\mathbb{E}x_t x_t' = \Sigma_{xx} > 0$. *The density* $f(z)$ *of the scalar process* $z_t$ *and the joint density* $f_{0,j}(s_0, s_j)$ *of* $(z_t, z_{t+j})$ *are bounded above and away from zero over their supports with uniformly bounded and continuous derivatives to the second order.*

*(ii)* $\{u_t\}$ *is a martingale difference sequence (mds) with respect to the filtration* $\mathcal{F}_t = \sigma\{\{x_s\}_{s=1}^{\infty}; \{z_s\}_{s=1}^{\infty}; u_t, u_{t-1}, \cdots\}$, $\mathbb{E}(u_t^2|\mathcal{F}_{t-1}) = \sigma_u^2$ *a.s., and* $\mathbb{E}(u_t^4) < \infty$.

*(iii)* $\{x_t\}, \{z_t\}$ *and* $\{u_t\}$ *are mutually independent.*

*(iv)* *The kernel function* $K(\cdot)$ *is a bounded probability density function symmetric about zero with* $\mu_j(K) = \int_{\mathcal{K}} u^j K(u)du$, $\nu_j(K) = \int_{\mathcal{K}} u^j K^2(u)du$, *and support* $\mathcal{K}$ *either* $[-1, 1]$ *or* $\mathbb{R} = (-\infty, \infty)$.

*(v)* $\beta(z)$ *is a smooth function with uniformly bounded continuous derivatives to order* $L + 1$ *for some integer* $L \geq 1$.

*(vi)* $n \to \infty$ *and* $h \to 0$.

The stationarity conditions in Assumption (i) accord with earlier work on nonparametric and functional coefficient kernel regression for which the mixing requirements are commonly used to enable development of asymptotic theory in time series FC regression (e.g., Fan and Yao, 2008; Cai et al., 2000). A stronger mixing decay rate condition $c > \tau(1 - 2/\delta)$ some $\delta > 2$ and $\tau > 1$ in (i) is used in place of the more usual condition $c > 1 - 2/\delta$ to assist in the nonparametric limit distribution theory under dependence. The mds condition in (ii) and exogeneity and independence conditions in (iii) are convenient for the limit theory. Relaxation of those conditions requires alternative methods such as FC instrumental variable methods and additional technical complications that are not within the goals of the present work to address. The kernel assumptions in (iv) are commonly employed but when bandwidths are very small, as they are in some of the results herein, kernels with support $\mathcal{K}$ on the entire real line $\mathbb{R}$ are better suited, or other methods used to avoid finite sample failure in the kernel-weighted signal in the regression.

The smoothness conditions (v) on $\beta(z_t)$ and its derivatives are needed for the theory developed here because the limiting bias expressions rely on higher order derivatives of $\beta(z_t)$.

When the smoothness degree parameter $L$ is unknown and estimated a stronger condition may be required to allow for potential overestimation of $L$ in practice. Condition (vi) is standard in nonparametric work and specific rate conditions involving $(n, h)$ are given as needed in the results below. However, as shown in the analysis of limit behavior when $L \to \infty$, the optimal bandwidth may no longer satisfy the contraction condition $h \to 0$ in (vi).

Our first result details the limit theory for the FC regression estimator $\hat{\beta}(z)$ in model (2.1) for the stationary case under locally flat conditions on the coefficient function.

**Theorem 2.1.** *If Assumption 1 holds, if $\beta(z)$ has derivatives $\beta^{(\ell)}(z) = 0$ at $z$ for all $\ell = 1, 2, ..., L-1$ and some integer $L \geq 1$ for which $\beta^{(L)}(z) \neq 0$, then the following limit theory holds when $nh \to \infty$*

$$\sqrt{nh}\left\{\hat{\beta}(z) - \beta(z) - h^{L^*}\mathcal{B}_L(z)\right\} \rightsquigarrow \mathcal{N}\left(0, \Omega_S(z)\right), \tag{2.3}$$

*where* $L^* = (L+1)\mathbf{1}_{\{L=odd\}} + L\mathbf{1}_{\{L=even\}}$, $\Omega_S(z) = \frac{\nu_0(K)\sigma_u^2}{f(z)}\Sigma_{xx}^{-1}$,

$$\mathcal{B}_L(z) = \frac{\mu_{L^*}(K)}{f(z)} C_L(z) = \frac{\mathcal{G}_L(z)}{f(z)} \tag{2.4}$$

$\mu_{L^*}(K) = \mu_L(K) \times \mathbf{1}_{\{L=even\}} + \mu_{L+1}(K)\mathbf{1}_{\{L=odd\}}$, $\mathcal{G}_L(z) = \mu_{L^*}(K)C_L(z)$ *and*

$$C_L(z) = \frac{f(z)\beta^{(L)}(z)}{L!}\mathbf{1}_{\{L=even\}} + \left[\frac{\beta^{(L)}(z)}{L!}f^{(1)}(z) + \frac{\beta^{(L+1)}(z)}{(L+1)!}f(z)\right]\mathbf{1}_{\{L=odd\}}. \tag{2.5}$$

Theorem 2.1 shows that flatness in the functional coefficient $\beta(\cdot)$ at $z$ affects the limit theory of $\hat{\beta}(z)$ in the stationary $x_t$ regressor case only through the bias function $h^{L^*}\mathcal{B}_L(z) = h^{L^*}\frac{\mathcal{G}_L(z)}{f(z)}$ in (2.3). The bias order $O(h^{L^*})$ and the functional form $\mathcal{G}_L(z)$ are affected. The bias function $\mathcal{G}_L(z)$ depends on the first two non-zero derivatives $\{\beta^{\ell}(z);\ \ell = L, L+1\}$ of $\beta(z)$, as well as the density $f(z)$ and its first derivative $f^{(1)}(z)$, the latter appearing as is usual in nonparametric regression. When $L$ is even the dependence is confined to the derivative $\beta^L(z)$ and the density $f(z)$. The limiting variance formula $\Omega_S(z) = \frac{\nu_0(K)\sigma_u^2}{f(z)}\Sigma_{xx}^{-1}$ is unchanged from the standard case without flatness and the convergence rate remains $\sqrt{nh}$. So, the effect of local flatness in $\beta(z)$ affects the limit theory of FC regression only via the bias function.

As $L$ rises with an increasing degree of flatness in the regression coefficient at $z$, the bias function in (2.3), which is of order $O(h^{L^*})$, falls when $h \to 0$ as $n \to \infty$. When estimation bias falls it is natural to select a wider bandwidth to reduce variance. Correspondingly, the usual plug-in optimal bandwidth formula changes, with resulting adjustment to the convergence rate. This can be conveniently shown in the scalar coefficient function $\beta(z)$ case, for which the optimal bandwidth formula for minimizing asymptotic mean squared error can be deduced from

(2.3) in the usual way, giving (using the scalar $x_t$ case to illustrate)

$$h_{\text{opt}^*} = \left( \frac{\Omega_S(z)}{2L^* \mathcal{B}_L(z)^2} \right)^{\frac{1}{2L^*+1}} \frac{1}{n^{1/(2L^*+1)}}. \tag{2.6}$$

In the conventional case where $L = 1$ and $L^* = 2$, we have the usual optimal bandwidth rate $h_{\text{opt}^*} = O(n^{-\frac{1}{5}})$. More generally, and taking $L$ to be even for convenience so that $L^* = L$ and $\mathcal{B}_L(z) = \left( \frac{\beta^{(L)}(z)}{L!} \right) f(z) \mu_L(K)$, we have

$$h_{\text{opt}^*} = \left( \frac{L!(L-1)!\Omega_S(z)}{2[\mu_L(K) f(z) \beta^{(L)}(z)]^2)} \right)^{1/(2L+1)} \frac{1}{n^{1/(2L+1)}} = \frac{c_L(z)}{n^{1/(2L+1)}} \tag{2.7}$$

where $c_L(z) = d_L(z) [L!(L-1)!]^{1/(2L+1)}$ with $d_L(z) = \left( \frac{\Omega_S(z)}{2[\mu_L(K) f(z) \beta^{(L)}(z)]^2)} \right)^{1/(2L+1)}$. For instance, when the functional coefficient has the polynomial form $\beta(z) = \sum_{j=0}^{q} a_j z^{L+j}$ which is locally flat to order $L-1$ at $z = 0$ when $a_0 \neq 0$, we have $\beta^{(L)}(z) = L! \sum_{j=0}^{q} \frac{(L+j)!}{L!j!} a_j z^j$ and $\beta^{(L)}(0) = a_0 L! = O(L!)$. The same applies when the locally flat coefficient function $\beta(z)$ has the asymptotically regular form $\beta(z) \sim_a a_0 z^L$ as $L \to \infty$. In such cases, it is evident that $c_L(z) = O([L!/\beta^{(L)}(z)]^{1/L}) = O(1)$ as $L \to \infty$ and the optimal bandwidth $h_{\text{opt}^*}$ in (2.7) approaches the non-shrinking rate $O(1/n^0) = O(1)$. Hence, for large $L$ the associated optimal convergence rate is $\sqrt{n h_{opt^*}}$ which approaches $\sqrt{n}$, giving a near-parametric convergence rate for extremely flat functions.

This behavior matches the heuristic that when a functional coefficient is nearly flat and bias is small from neighboring observation points, averaging over those observations by using a wider (or asymptotically non-shrinking) bandwidth is useful in reducing variance and thereby mean squared error. Note, however, that for this optimal choice of bandwidth as $L \to \infty$, in such cases we have $\mathcal{B}_L(z) = O(\beta^{(L)}(z)/L!) = O(1)$ as $L \to \infty$ so that $\sqrt{nh} h^{L^*} \mathcal{B}_L(z) = O(n^{1/2} h^{L+1/2}) = O(n^{1/2} n^{-\frac{L+1/2}{2L+1}}) = O(1)$ following (2.7) for the case that $L$ is even. So the bias term is, as usual, not negligible for the optimal choice of bandwidth. What Theorem 2.1, formula (2.7), and this asymptotic bias analysis show is that when the coefficient function is nearly flat in the neighborhood of the point of estimation, near parametric convergence rates are possible with the same limit normal distribution and variance as in other cases.

## 2.2 The FC cointegrating regression model

For exposition we use a cointegrating regression equation with full rank $I(1)$ exogenous regressors and functional coefficients. The model is a prototype of more complex systems and provides results that show the impact of flat behavioral characteristics in the functional coefficients on rates of convergence, estimation, inference, and bandwidth selection in a nonstationary framework. These simplifying conditions enable the use of standard kernel-weighted least squares regression. Similar analyses to those given here will be needed in more complex modeling environments under endogeneity and cointegrated equations with possibly cointegrated or even functionally cointegrated regressors. Extensions to address such complexities would involve procedures such as 'fully modified' FCC kernel regression. Some related FM methods have been designed for the time varying parameter framework of cointegration (Phillips et al., 2017; Li et al., 2016; Gao and Phillips, 2013) and may be developed for FC cointegrating models. But they are not the subject of the present work and are left for future research.

The following assumption modifies the conditions of Assumption 1 and provides for a simple cointegrating regression analogue of model (2.1).

**Assumption 2.**

*(i)* $\{x_t\}$ *is a full rank unit root process satisfying the functional law* $\frac{1}{\sqrt{n}}x_{\lfloor n\cdot\rfloor} \rightsquigarrow B_x(\cdot)$, *where* $B_x$ *is vector Brownian motion with variance matrix* $\Sigma_{xx} > 0$.

*(ii)* $\{z_t\}$ *is a strictly stationary* $\alpha$*-mixing scalar process with mixing numbers* $\alpha(j)$ *that satisfy* $\sum_{j\geq 1} j^c[\alpha(j)]^{1-2/\delta} < \infty$ *for some* $\delta > 2$ *and* $c > \tau(1-2/\delta)$ *and* $\tau > 1$ *with finite moments of order* $p > 2\delta > 4$. *The density* $f(z)$ *of* $z_t$ *and joint density* $f_{0,j}(s_0, s_j)$ *of* $(z_t, z_{t+j})$ *are bounded above and away from zero over their supports with uniformly bounded and continuous derivatives to the second order.*

*(iii) Assumptions 1(ii) - (vi) hold.*

The high level assumption (i) on the functional limit behavior of the regressor $x_t$ is convenient, commonly used, and justified by standard primitive conditions (e.g., Phillips and Solo, 1992). Assumption 2(ii) mirrors Assumption 1(i) for the covariate $z_t$. The independence conditions in Assumption 1 (iii) are restrictive, particularly in cointegrating regressions. They may be partially relaxed, as for example in Li et al. (2016, 2020) in time-varying parameter cointegrating regression. Such extensions require different methods of estimation, as indicated earlier. In further extensions of this type to FCC regression models, many of the findings of the present work on the effects of local flatness of the functional coefficient will be relevant and can be explored in future work. The moment conditions in (v2) on $\beta(z_t)$ and its derivatives are needed in the nonstationary case because they figure in the development and appear in the asymptotic variance formula. The remaining conditions are as in Assumption 1.

**Theorem 2.2.** *If Assumption 2 holds, if* $\beta(z)$ *has derivatives* $\beta^{(\ell)}(z) = 0$ *at* $z$ *for all* $\ell = 1, 2, ..., L-1$ *and some integer* $L \geq 1$ *for which* $\beta^{(L)}(z) \neq 0$, *and* $\mathbb{E}||\beta^{(L)}(z_t)||^2 < \infty$, *then the following limit theory holds under the respective rate conditions indicated:*

$$(i)\ n\sqrt{h}\left\{\hat{\beta}(z) - \beta(z) - h^{L^*}\mathcal{B}_L(z)\right\} \rightsquigarrow \mathcal{MN}\left(0, \Omega_{NS}(z)\right), \qquad \text{if } nh^{2L} \to 0, nh \to \infty, \quad (2.8)$$

$$(ii)\ \sqrt{\frac{n}{h^{2L-1}}}\left\{\hat{\beta}(z) - \beta(z) - h^{L^*}\mathcal{B}_L(z)\right\} \rightsquigarrow \mathcal{MN}\left(0, \Omega_L(z)\right), \qquad \text{if } nh^{2L} \to \infty, \quad (2.9)$$

*(iii)* $n^{1-\frac{1}{4L}}\left(\hat{\beta}(z)-\beta(z)-h^{L^*}\mathcal{B}_L(z)\right)$

$$\rightsquigarrow c^{\frac{1}{2}-\frac{1}{4L}}\times\mathcal{MN}\left(0,\Omega_L(z)\right)+c^{-\frac{1}{4L}}\mathcal{MN}\left(0,\Omega_{NS}(z)\right)$$

$$\equiv_d \mathcal{MN}\left(0,c^{1-\frac{1}{2L}}\Omega_L(z)+c^{-\frac{1}{2L}}\Omega_{NS}(z)\right),\qquad \text{if } nh^{2L}\to c\in(0,\infty),\quad (2.10)$$

*where* $L^*=(L+1)\mathbf{1}_{\{L=odd\}}+L\mathbf{1}_{\{L=even\}}$*,*

$$\Omega_{NS}(z)=\frac{\nu_0(K)\sigma_u^2}{f(z)}\left(\int B_xB_x'\right)^{-1},\tag{2.11}$$

$$\Omega_L(z)=\frac{\nu_{2L}(K)}{(L!)^2f(z)}\left(\int B_xB_x'\right)^{-1}\left(\int B_xB_x'\left(B_x'\beta^{(L)}(z)\right)^2\right)\left(\int B_xB_x'\right)^{-1},\tag{2.12}$$

*and where the bias function* $\mathcal{B}_L(z)=\frac{\mu_{L^*}(K)}{f(z)}C_L(z)=\frac{\mathcal{G}_L(z)}{f(z)}$*, just as in Theorem 2.1.*

The division of the limit theory of FC cointegrating regression into three categories was discovered in Phillips and Wang (2020) for the case where $L=1$. Theorem 2.2 extends those results to the general case and reveals the effect on both the limit theory and the convergence rate of local flatness in the coefficient function at the point of estimation. As shown in Phillips and Wang (2020) and, as is evident in the proof of Theorem 2.2, the presence of multiple categories to the limit theory arises because two different sources of variability occur in the asymptotics – one from the random elements of the bias function and one from the sample covariance of the regressor and the equation error. Correspondingly, the form of the limit theory itself changes, according to the behavior of $nh^{2L}$.

Category (i) where $nh^{2L}\to 0$ is comparable to the stationary case, but with convergence rate $n\sqrt{h}$ that embodies the $O(\sqrt{n})$ order of the $I(1)$ regressor $x_t$ and a limit variance matrix that replaces the stationary sample moment matrix limit $\Sigma_{xx}$ with the corresponding quadratic functional $\int B_xB_x'$ for the nonstationary case in the limit matrix $\Omega_{NS}$ in (2.11). The bias function $h^{L^*}\mathcal{B}_L(z)$ in the centering of $\hat{\beta}(z)$ is identical to the stationary case and has the same order $O(h^{L^*})$. Mixed normal limit theory, but with different rates of convergence and different variance matrices, applies in cases (i), (ii) and the intermediate case (iii).

**Remark 2.1.** ***(Convergence-rate optimal bandwidth order)*** *In case (iii) of Theorem 2.2 where* $nh^{2L}\to c$ *for some constant* $c\in(0,\infty)$*, the bandwidth* $h\sim_a (c/n)^{\frac{1}{2L}}$ *and then the convergence rate in case (ii) becomes* $\sqrt{n/h^{2L-1}}=O(\sqrt{n^{1+\frac{2L-1}{2L}}})=O(n^{1-\frac{1}{4L}})$*. Similarly, the convergence rate in case (i) becomes* $n\sqrt{h}=O(n^{1-\frac{1}{4L}})$ *when* $h\sim_a (c/n)^{\frac{1}{2L}}$*. This duality between the two cases implies that the convergence rates in cases (i) and (ii) merge to the same* $O(n^{1-\frac{1}{4L}})$ *rate for the intermediate situation where the bandwidth satisfies* $nh^{2L}\to c$*. In fact, the case* $nh^{2L}\to c\in(0,\infty)$ *yields the maximum convergence rate outcome for FCC regression because, for the boundary cases where* $nh^{2L}\to 0$ *or* $nh^{2L}\to\infty$*, we find that the respective convergence rates are* $n\sqrt{h}=o(n^{1-\frac{1}{4L}})$ *and* $\sqrt{n/h^{2L-1}}=o(n^{1-\frac{1}{4L}})$*. Thus, the FCC kernel regression* *convergence rate is optimal in the intermediate case where* $nh^{2L} \to c \in (0, \infty)$. *The associated convergence-rate optimal bandwidth, denoted* $h_{opt}$, *is* $h_{opt} \sim_a (c/n)^{\frac{1}{2L}} = O(n^{-\frac{1}{2L}})$. *The limit distribution is a mixture of the mixed normal component* $\mathcal{MN}(0, \Omega_L(z))$ *(which comes from the random element of the bias function) and the mixed normal component* $\mathcal{MN}(0, \Omega_{NS}(z))$ *(which comes from the usual equation error term). The coefficients in this mixture are* $c^{\frac{1}{2}-\frac{1}{4L}}$ *and* $c^{-\frac{1}{4L}}$. *For instance, if the locally flat function* $\beta(z)$ *has the asymptotically regular form* $\beta(z) \sim_a a_0 z^L$ *as* $L \to \infty$, *the variance matrix* $\Omega_L(z) = O((\beta^{(L)}(z)/L!)^2) = O(1)$. *Then the* $c^{1-\frac{1}{2L}}\Omega_L(z)$ *component in the asymptotic variance in* (2.10) *remains stable when* $L$ *is large just like the* $c^{-\frac{1}{2L}}\Omega_{NS}(z)$ *component. Therefore the random element coming from the bias function cannot be ignored even when the functional coefficient is sufficiently flat at the point of estimation.*

*However, with* $h = O(n^{-\frac{1}{2L}})$, *based on case (iii) of Theorem 2.2, the bias term cannot be neglected because it is of order* $O(n^{1-\frac{1}{4L}} \times n^{-\frac{L^*}{2L}}) = O(n^{\frac{4L-2L^*-1}{4L}}) \to \infty$ *when* $L \geq 2$. *Therefore when discussing the optimal bandwidth order, we need to take the bias effect into consideration, not only the convergence rate. This requires examination of the Mean Squared Error (MSE) optimal bandwidth order, as given next.*

**Remark 2.2.** ***(Optimal bandwidth order)*** *We explore the optimal bandwidth order with respect to Root Mean Squared Error (RMSE). Let* $h = O(n^\gamma)$, $-1 < \gamma < 0$, *and* $\hat{\beta}(z) - \beta(z) = O(n^{g_L(\gamma)})$. *The exponent function* $g_L(\gamma)$ *in the latter expression represents the order of the RMSE, which is determined by the maximum of the bias order and the standard deviation order. The subindex* $L$ *in* $g_L(\gamma)$ *indicates that the RMSE order function varies with parameter* $L$.

*First consider the case where* $L$ *is odd in which case* $L^* = L + 1$. *Based on result (i) of Theorem 2.2, when* $nh^{2L} \to 0$ *or equivalently* $\gamma < -\frac{1}{2L}$, *we have* $\hat{\beta}(z) - \beta(z) = O_p(\frac{1}{n\sqrt{h}} + h^{L^*}) = O_p(n^{-1-\gamma/2} + n^{(L+1)\gamma})$. *Then we have* $g_L(\gamma) = \max\{-1 - \gamma/2, (L+1)\gamma\}$ *when* $\gamma < -\frac{1}{2L}$. *Similarly, based on result (ii) of Theorem 2.2, we have* $g_L(\gamma) = \max\{(L-1/2)\gamma - 1/2, (L+1)\gamma\}$ *when* $\gamma > -\frac{1}{2L}$. *Following result (iii) of Theorem 2.2, we have* $g_L(\gamma) = \max\{-1 + \frac{1}{4L}, (L+1)\gamma\}$ *when* $\gamma = -\frac{1}{2L}$. *Straightforward analysis yields*

$$g_1(\gamma) = \begin{cases} 2\gamma, & -1/3 \leq \gamma < 0 \\ -\frac{1-\gamma}{2}, & -1/2 < \gamma < -1/3 \\ -(1+\gamma/2), & -1 < \gamma \leq -1/2 \end{cases} \tag{2.13}$$

*and*

$$g_L(\gamma) = \begin{cases} (L+1)\gamma, & -\frac{2}{2L+3} \leq \gamma < 0 \\ -1 - \frac{\gamma}{2}, & -1 < \gamma < -\frac{2}{2L+3} \end{cases} \tag{2.14}$$

*for* $L \geq 3$.

*Similarly, suppose* $L$ *is even, in which case* $L^* = L$, *and* $g_L(\gamma)$ *can be derived based on Theorem 2.2. Thus, if* $nh^{2L} \to 0$ *or equivalently* $\gamma < -\frac{1}{2L}$, *we have* $\hat{\beta}(z) - \beta(z) = O_p(\frac{1}{n\sqrt{h}} +$

$h^{L^*}) = O_p(n^{-1-\gamma/2} + n^{L\gamma})$. *When* $\gamma = -\frac{1}{2L}$, *we have* $g_L(\gamma) = \max\{-1 + \frac{1}{4L}, L\gamma\} = \max\{-1 + \frac{1}{4L}, -1/2\} = -1/2$; *and when* $\gamma > -\frac{1}{2L}$, *we have* $g_L(\gamma) = \max\{(L-1/2)\gamma - 1/2, L\gamma\}$. *Standard calculations yield*

$$g_L(\gamma) = \begin{cases} L\gamma, & -\frac{2}{2L+1} \le \gamma < 0 \\ -1 - \frac{\gamma}{2}, & -1 < \gamma < -\frac{2}{2L+1} \end{cases} \tag{2.15}$$

*when* $L$ *is even. Note that* (2.14) *and* (2.15) *can be combined as*

$$g_L(\gamma) = \begin{cases} L^*\gamma, & -\frac{2}{2L^*+1} \le \gamma < 0 \\ -1 - \frac{\gamma}{2}, & -1 < \gamma < -\frac{2}{2L^*+1} \end{cases} \tag{2.16}$$

*for* $L \ge 2$.

*The* $g_L(\gamma)$ *functions are plotted in Figure 1. Evidently, when* $L = 1$, *the RMSE optimal bandwidth order is* $h_{opt^*} = O(n^{-1/2})$, *which equals the convergence-rate optimal bandwidth order* $h_{opt}$. *For* $L \ge 2$, *the optimal bandwidth is* $h_{opt^*} = O(n^{-\frac{2}{2L^*+1}})$, *which is smaller than the convergence-rate optimal bandwidth order* $h_{opt} = O(n^{-\frac{1}{2L}})$ *when* $L \ge 2$. *The discrepancy between these two optimal bandwidth rates is due to the fact that when* $L \ge 2$ *bias dominates variance in result (iii) of Theorem 2.2. To reduce bias, the RMSE optimal bandwidth prefers to select a smaller order. When* $L$ *is large, we can see the order of* $h_{opt^*}$, *viz.,* $-\frac{2}{2L^*+1}$, *is close to zero and then* $h_{opt^*}$ *diminishes to zero at a very slow rate as* $n \to \infty$. *This outcome is consistent with heuristics as* $\beta(z)$ *is close to a constant function at the estimation point* $z$ *when* $L$ *is large; and estimation of an almost constant function requires only a very low degree of localization.*

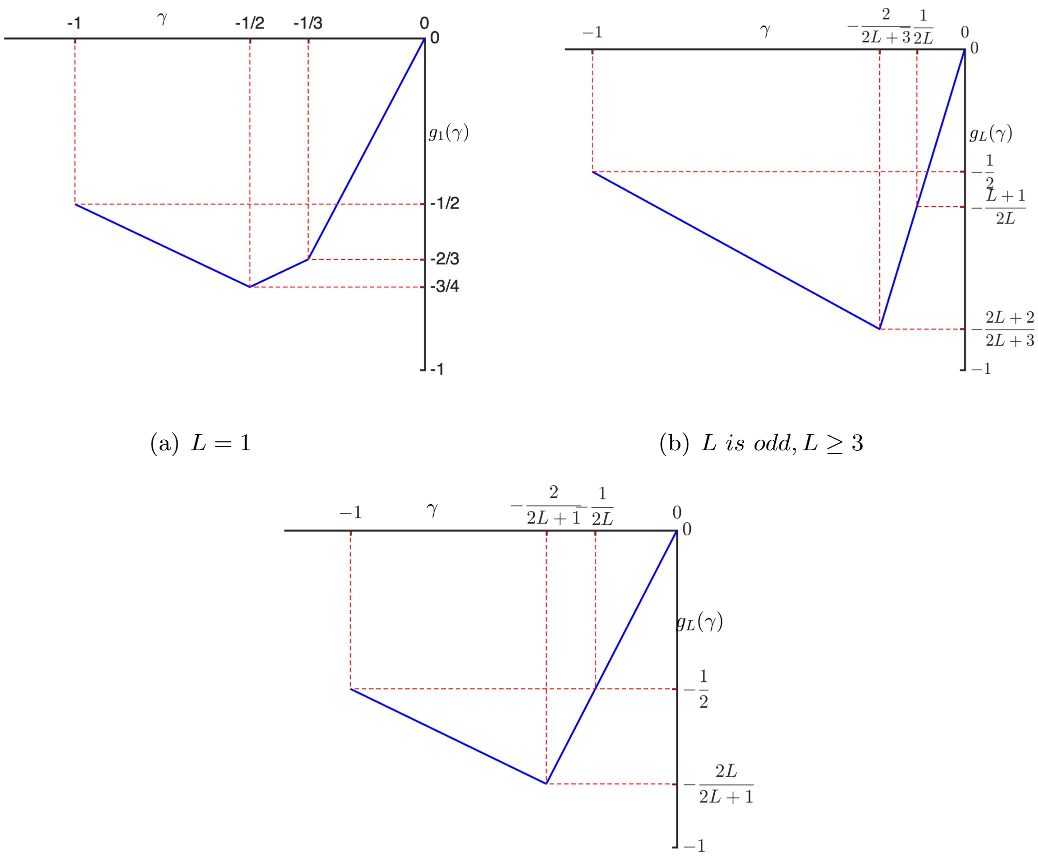
**Remark 2.3.** ***(MSE optimal bandwidth formula)*** *The above analysis tells us that the RMSE optimal bandwidth order, or equivalently, the MSE optimal bandwidth order, is achieved within case (i) of Theorem 2.2. Taking the standard approach to optimal bandwidth selection that balances bias and variance (and using the scalar* $x_t$ *case for convenience) leads to the following formula compared with the stationary regressor case given in* (2.6)

$$h_{opt^*} = \left(\frac{\Omega_{NS}(z)}{2L^*\mathcal{B}_L(z)^2}\right)^{\frac{1}{2L^*+1}} \frac{1}{n^{2/(2L^*+1)}} \equiv c_L(z) n^{-\frac{2}{2L^*+1}} \tag{2.17}$$

*where* $c_L(z) = \left(\frac{\Omega_{NS}(z)}{2L^*\mathcal{B}_L(z)^2}\right)^{\frac{1}{2L^*+1}}$. *To illustrate, suppose* $L$ *is even in which case*

$$c_L(z) = \left(\frac{\Omega_{NS}(z)}{2L\mathcal{B}_L(z)^2}\right)^{\frac{1}{2L+1}} = \left(\frac{\nu_0(K)\sigma_u^2 L!(L-1)!}{2\mu_L(K)^2 f(z)(\int B_x^2)\beta^{(L)}(z)^2}\right)^{\frac{1}{2L+1}}. \tag{2.18}$$

*If, as before, the functional coefficient is flat at* $z = 0$ *with polynomial form* $\beta(z) = \sum_{j=0}^q a_j z^{L+j}$ *and* $a_0 \ne 0$, *then* $\beta^{(L)}(z) = L! \sum_{j=0}^q \frac{(L+j)!}{L!j!} a_j z^j = O(L!)$ *and* $c_L(z) = O_p(1)$ *as* $L \to \infty$, *the randomness of* $c_L(z)$ *arising from the presence of the quadratic functional* $\int B_x^2$ *in* (2.18).




(a) $L = 1$

(b) $L$ *is odd*, $L \geq 3$

(c) $L$ *is even*

Figure 1: Plots of $g_L(\gamma)$ for $L \geq 1$.

*Again, the optimal bandwidth $h_{opt^*} = O_p(n^{-2/(2L+1)})$ and for large $L$ the optimal bandwidth shrinks at a very slow rate and the associated optimal convergence rate $n\sqrt{h_{opt^*}}$ approaches $n$, giving a near-parametric convergence rate for extremely flat functions. This suggests that larger bandwidth is needed for large $L$. In practice in both stationary and nonstationary cases, $L$ is typically unknown, so $c_L(z)$ and the optimal bandwidth order are also unknown in the absence of information about $\beta(z)$ and its derivatives. But while estimation of optimal bandwidths by cross validation or by the use of derivative function estimates is possible, these methods typically lead to very slow convergence rates in optimal bandwidth formulae even in the simplest cases (Hall and Marron, 1987; Hall et al., 1991). So the above findings are likely to be mainly of importance and use for theoretical work.*

**Remark 2.4.** ***(Asymptotics with MSE optimal bandwidth)*** *When $L = 1$, choice of the MSE optimal bandwidth order $h_{opt^*}$ leads to asymptotics that are determined according to case (iii) of Theorem 2.2 since $nh_{opt^*}^{2L} = O(1)$ when $L = 1$. More specifically, the limit theory for $\hat{\beta}(z)$ is given by*

$$n^{3/4}\left(\hat{\beta}(z) - \beta(z) - h_{opt^*}^2\mathcal{B}_1(z)\right) \rightsquigarrow \mathcal{MN}\left(0, c^{1/2}\Omega_L(z) + c^{-\frac{1}{2}}\Omega_{NS}(z)\right), \tag{2.19}$$

*which matches the result in Phillips and Wang (2020, Theorem 2.1(c)) for the standard case of no flatness in $\beta(z)$. In this case, the bias can be neglected because $n^{3/4}\times h_{opt^*}^2 = O(n^{-1/4}) = o(1)$. When $L \geq 2$, we have $nh_{opt^*}^{2L} \to 0$ and the limit theory is determined by case (i) of Theorem 2.2. Specifically with $h = h_{opt^*} = O(n^{-\frac{2}{2L^*+1}})$, we have*

$$n^{\frac{2L^*}{2L^*+1}}\left\{\hat{\beta}(z) - \beta(z) - h_{opt^*}^{L^*}\mathcal{B}_L(z)\right\} \rightsquigarrow \mathcal{MN}\left(0, \Omega_{NS}(z)\right). \tag{2.20}$$

*In this case, the random bias component involving $\Omega_L(z)$ can be ignored asymptotically but the deterministic bias term cannot be neglected because $n^{\frac{2L^*}{2L^*+1}} \times h_{opt^*}^{L^*} = O(n^{\frac{2L^*}{2L^*+1}} \times n^{-\frac{2L^*}{2L^*+1}}) = O(1)$. Using the MSE optimal bandwidth $h_{opt^*} = O(n^{-\frac{2}{2L^*+1}})$, the fastest convergence rate that $\hat{\beta}(z)$ can achieve is $O_p(n^{-\frac{2L^*}{2L^*+1}})$ when $L \geq 2$. As $L \to \infty$, the fastest convergence rate approaches $O_p(n^{-1})$, leading to the parametric cointegrating regression convergence rate $\hat{\beta}(z) - \beta(z) = O_p(n^{-1})$ as $L \to \infty$. As in the stationary case, this matches heuristic arguments because $\beta(z)$ approaches a constant function at the estimation point $z$ when $L \to \infty$.*

# 3 Implications for Inference

## 3.1 Procedures for inference

When $L$ is known or is correctly hypothesized standard test statistics for inference about the functional coefficient $\beta(z)$ can be constructed in a standard way. Following Phillips and Wang (2020), but allowing now for local flatness in the coefficient function, we start with the matrix normalization

$$\hat{T}(z;L) = \hat{V}_n(z;L)^{-1/2}[\hat{\beta}(z) - \beta(z) - h^{L^*}\hat{\mathcal{B}}_L(z)], \tag{3.1}$$

where $\hat{V}_n(z;L) = A_n(z)^{-1}\hat{\Omega}_n(z;L)A_n(z)^{-1}$ with $A_n(z) = \sum_{t=1}^n x_t x_t' K_{tz}$,

$$\hat{\Omega}_n(z;L) = \nu_0(K)\hat{\sigma}_u^2 \sum_{t=1}^n x_t x_t' K_{tz} + \sum_{t=1}^n x_t x_t' \{x_t' \frac{1}{L!}\hat{\beta}^{(L)}(z)(z_t - z)^L K_{tz}\}^2 \tag{3.2}$$

and

$$\hat{\mathcal{B}}_L(z) = \mu_{L^*}(K)\left\{\frac{\hat{\beta}^{(L)}(z)}{L!}1_{\{L=even\}} + \left[\frac{\hat{\beta}^{(L)}(z)}{L!}\frac{\hat{f}^{(1)}(z)}{\hat{f}(z)} + \frac{\hat{\beta}^{(L+1)}(z)}{(L+1)!}\right]1_{\{L=odd\}}\right\}. \tag{3.3}$$

The statistic $\hat{T}(z;L)$ follows the same design as the robust $t$-test statistic developed in Phillips and Wang (2020) for the non-flat case with $L = 1$.

The bias component $h^{L^*}\hat{\mathcal{B}}_L(z)$ in (3.1) and the second term of $\hat{\Omega}_n(z;L)$ in (3.2) are both infeasible in practical work unless $L$ is known or is stated as part of a null hypothesis such as

$$\mathbb{H}_0 : \beta(z) = \beta_0, L = L_0. \tag{3.4}$$

One way to determine $L$ empirically is to test whether successive derivatives of $\beta(z)$ are zero at the point of estimation using consistent kernel estimates,[1] $\hat{\beta}^{(\ell)}(z)$, of the derivative functions $\beta^{(\ell)}(z)$ and conducting inference to detect zero derivatives at the point of interest. In most practical cases this procedure would involve examination of only the first derivative or first two derivatives ($\ell = 1, 2$). Nonetheless, empirical determination of the correct degree of flatness is inevitably subject to pre-test bias from testing (and sequential testing) whether the derivatives are zero. Finding a feasible pivotal test statistic that incorporates such information has proved challenging. Section 3.3.1 below discusses some of the difficulties involved in the direct estimation of the derivative order parameter $L$. Fortunately, simulation evidence presented below indicates that the naive approach of assuming there is no flatness in the function (i.e., $L = 1$) works well in terms of coverage compared with the infeasible test procedure that employs correct information about $L$.

Under $\mathbb{H}_0$ the statistic $\hat{T}(z;L_0)$ may be used to construct a robust Hotelling's $T^2$ type statistic based on the quadratic form $\hat{T}_2(z;L_0) = \hat{T}(z;L_0)'\hat{T}(z;L_0)$, so that

$$\hat{T}_2(z;L_0) = [\hat{\beta}(z) - \beta_0 - h^{L_0^*}\hat{\mathcal{B}}_{L_0}(z)]'\hat{V}_n(z;L_0)^{-1}[\hat{\beta}(z) - \beta_0 - h^{L_0^*}\hat{\mathcal{B}}_{L_0}(z)].$$

The following result shows that under the null $\mathbb{H}_0$ with use of the correct value of $L$ the statistics $\hat{T}(z;L_0)$ and $\hat{T}_2(z;L_0)$ satisfy $\hat{T}(z;L_0) \rightsquigarrow \mathcal{N}(0, I_p)$ and $\hat{T}_2(z;L_0) \rightsquigarrow \chi_p^2$ as $n \to \infty$. This pivotal

[1] Derivative estimates can be obtained in the usual way by differentation of the kernel estimate $\hat{\beta}(z)$.

limit theory provides a basis for performing inference about $\beta(z)$ when the functional coefficient is locally flat and the flatness parameter $L$ is correctly hypothesized. This approach covers both stationary and nonstationary regressor cases.

**Theorem 3.1.** *Under either Assumption 1 or 2, when the null hypothesis $\mathbb{H}_0$ holds and $nh \to \infty$, $\hat{T}(z;L_0) \rightsquigarrow \mathcal{N}(0,I_p)$ and $\hat{T}_2(z;L_0) \rightsquigarrow \chi_p^2$.*

## 3.2 Test power

When the null hypothesis is false and the true value of the functional coefficient $\beta(z) \neq \beta_0$ but the maintained hypothesis $L = L_0$ is correct, asymptotic power can be explored under local alternatives of the form

$$\mathbb{H}_{1,\beta} : \beta(z) = \beta_0 + \rho_n m(z),$$

where $m(z)$ is a $p$-vector function whose modulus is bounded away from the origin and $\rho_n$ is a real sequence for which $\rho_n \to 0$. Let $\chi_p^2(\alpha)$ be the $1-\alpha$ right tail critical value of the $\chi_p^2$ distribution. Then, under $\mathbb{H}_{1,\beta}$ we have

$$\lim_{n\to\infty} P\left(\hat{T}_2(z;L_0) > \chi_p^2(\alpha)\right) = 1, \tag{3.5}$$

for any $\rho_n$ satisfying $\rho_n^2 nh \to \infty$ if $x_t$ is stationary and Assumption 1 holds or for $\rho_n$ satisfying $\rho_n^2 n^2 h \to \infty$ if $x_t$ is nonstationary, $nh^{2L_0} \to 0$ and Assumption 2 holds. To prove (3.5) first consider the stationary case. In view of Theorem 2.1 we have, under $\mathbb{H}_{1,\beta}$,

$$\sqrt{nh}[\hat{\beta}(z) - \beta_0 - \rho_n m(z) - h^{L_0^*}\hat{\mathcal{B}}_{L_0}(z)] \rightsquigarrow \mathcal{N}(0, \Omega_S(z)).$$

Since $\frac{1}{nh}A_n(z) \to_p \Sigma_{xx} f(z)$ and

$$\frac{1}{nh}\hat{\Omega}_n(z;L_0) = \nu_0(K)\hat{\sigma}_u^2 \frac{1}{nh}\sum_{t=1}^n x_t x_t' K_{tz} + \frac{1}{nh}\sum_{t=1}^n x_t x_t' \{x_t' \frac{\hat{\beta}^{(L_0)}(z)}{L_0!}(z_t - z)^{L_0} K_{tz}\}^2 \to_p \nu_0(K)\sigma_u^2 f(z)\Sigma_{xx}, \tag{3.6}$$

just as in the proof of (A.24), we obtain

$$nh\hat{V}_n(z;L_0) = \left(\frac{1}{nh}A_n(z)\right)^{-1}\frac{1}{nh}\hat{\Omega}_n(z;L_0)\left(\frac{1}{nh}A_n(z)\right)^{-1} \to_p \Omega_S(z). \tag{3.7}$$

It follows that under $\mathbb{H}_{1,\beta}$ and Assumption 1

$$\begin{aligned}\hat{T}(z;L_0) &= \hat{V}_n(z;L_0)^{-1/2}[\hat{\beta}(z) - \beta_0 - h^{L^*}\hat{\mathcal{B}}_{L_0}(z)] \\ &= \hat{V}_n(z;L_0)^{-1/2}[\hat{\beta}(z) - \beta_0 - \rho_n m(z) - h^{L^*}\hat{\mathcal{B}}_{L_0}(z)] + [nh\hat{V}_n(z;L_0)]^{-1/2}\sqrt{nh}\rho_n m(z) \\ &\sim_a \mathcal{N}(\Omega_S(z)^{-1/2}\sqrt{nh}\rho_n m(z), I_p),\end{aligned} \tag{3.8}$$

so that (3.5) holds when $\rho_n^2 nh \to \infty$ and $m(z) \neq 0$ in the stationary case.

In the nonstationary case, the analysis can be carried out separately depending on the rate of $nh^{2L_0}$. We take $nh^{2L_0} \to 0$ as an example. When $nh^{2L_0} \to 0$ and $nh \to \infty$, from Theorem 2.2 (i) under $\mathbb{H}_{1,\beta}$, we have

$$n\sqrt{h}\left\{\hat{\beta}(z) - \beta_0 - \rho_n m(z) - h^{L_0^*}\hat{\mathcal{B}}_{L_0}(z)\right\} \rightsquigarrow \mathcal{MN}\left(0, \Omega_{NS}(z)\right), \tag{3.9}$$

where $\Omega_{NS}(z) = \frac{\nu_0(K)\sigma_u^2}{f(z)}\left(\int_0^1 B_x B_x'\right)^{-1}$. Now $\frac{1}{n^2h}A_n(z) \rightsquigarrow \int_0^1 B_x B_x' f(z)$ and $\frac{1}{n^2h}\hat{\Omega}_n(z;L_0) \rightsquigarrow \nu_0(K)\sigma_u^2 f(z)\int_0^1 B_x B_x'$, so that

$$n^2h\hat{V}_n(z;L_0) = \left(\frac{1}{n^2h}A_n(z)\right)^{-1}\frac{1}{n^2h}\hat{\Omega}_n(z;L_0)\left(\frac{1}{n^2h}A_n(z)\right)^{-1} \rightsquigarrow \Omega_{NS}(z). \tag{3.10}$$

Hence, under $\mathbb{H}_{1,\beta}$, Assumption 2 and with $nh^{2L_0} \to 0$ and $nh \to \infty$ we have

$$\begin{aligned}
&\hat{T}(z;L_0) = \hat{V}_n(z;L_0)^{-1/2}[\hat{\beta}(z) - \beta_0 - h^{L_0^*}\hat{\mathcal{B}}_{L_0}(z)] \\
&= \hat{V}_n(z;L_0)^{-1/2}[\hat{\beta}(z) - \beta_0 - \rho_n m(z) - h^{L_0^*}\hat{\mathcal{B}}_{L_0}(z)] + [n^2h\hat{V}_n(z;L_0)]^{-1/2}\sqrt{n^2h}\rho_n m(z) \\
&\sim_a \mathcal{MN}_m(\Omega_{NS}(z)^{-1/2}\sqrt{n^2h}\rho_n m(z), I_p),
\end{aligned} \tag{3.11}$$

where $\mathcal{MN}_m(\cdot,\cdot)$ signifies a mean mixture normal distribution.[2] The test statistic $\hat{T}_2(z;L_0)$ then diverges when $\rho_n^2 n^2 h \to \infty$ because it is asymptotically distributed as a mixture noncentral chi-squared variate with the divergent noncentrality parameter $n^2h\rho_n^2 m(z)'\Omega_{NS}(z)^{-1}m(z) \to \infty$. It follows that (3.5) holds in the nonstationary case when $\rho_n^2 n^2 h \to \infty$ and $m(z) \neq 0$.

Results for $nh^{2L_0} \to \infty$ and $nh^{2L_0} \to c \in (0,\infty)$ can be obtained in similar ways and the details are omitted. In the case where $nh^{2L_0} \to \infty$, the condition $n\rho_n^2/h^{2L_0-1} \to \infty$ is needed for the test to be consistent. When $nh^{2L_0} \to c \in (0,\infty)$ the test is consistent if $n^{1-\frac{1}{4L_0}}\rho_n \to \infty$ holds.

Before closing this section we point out that this test is not designed to detect alternatives specifically about $L$ in either stationary or nonstationary cases. To illustrate the difficulties involved in such alternatives, we take the stationarity case with $nh^{2L_0} \to 0$ and consider the alternative $\mathbb{H}_{1,L} : \beta(z) = \beta_0,\ L_0 < L$ where the flatness degree $L$ exceeds the hypothesized $L_0$. To examine test power observe that

$$\sqrt{nh}\left\{\hat{\beta}(z) - \beta(z) - h^{L_0^*}\hat{\mathcal{B}}_{L_0}(z)\right\} = \sqrt{nh}\left\{\hat{\beta}(z) - \beta(z) - h^{L^*}\hat{\mathcal{B}}_L(z)\right\} + \sqrt{nh}\left\{h^{L^*}\hat{\mathcal{B}}_L(z) - h^{L_0^*}\hat{\mathcal{B}}_{L_0}(z)\right\}.$$

The order of the second term depends on both $L$ and $L_0$, noting that the order of $\hat{\mathcal{B}}_{L_0}(z)$

[2] A random $p$-vector $\xi$ has a mean mixture normal density with covariance matrix $I_p$ if the density of $\xi$ is the mixture density $\frac{1}{(2\pi)^{p/2}}\int_{\vartheta\in\Theta} e^{-\frac{1}{2}(x-\vartheta)'(x-\vartheta)}dP(\vartheta)$ where $P(\vartheta)$ is the probability measure of the mean-mixing variate vector $\vartheta$.

depends on $L_0$ and $L$ through the empirical estimates $\hat{\beta}^{(L_0)}(z)$ and $\hat{\beta}^{(L_0+1)}(z)$ that are used in the construction of the test statistic. Due to the fact that $L$ is unknown under $\mathbb{H}_{1,L}$, the order of magnitude of the component $\sqrt{nh}\left\{h^{L^*}\hat{\mathcal{B}}_L(z) - h^{L_0^*}\hat{\mathcal{B}}_{L_0}(z)\right\}$ cannot be precisely determined and the power characteristics of the test are not known. Since these properties of the test in the case of departures $L$ from the hypothesized $L_0$ are unknown, the statistic is not designed to test hypotheses concerning the flatness order $L$.

## 3.3 Challenges in test construction when $L$ is unknown

### 3.3.1 Direct estimation of $L$

The statistic $\hat{T}(z;L)$ cannot be used in practical work if $L$ is unknown or is not part of the null or an explicit maintained hypothesis. A natural approach if this were not the case but if $L$ were directly estimable (by $\hat{L}$, say) would be to employ plug-in estimates $\hat{\beta}^{(\hat{L})}(z)$ and $\hat{\beta}^{(\hat{L}+1)}(z)$ of the required derivatives of $\beta(z)$ in the bias and variance matrix components of $\hat{T}(z;L)$. However, as the analysis below reveals, in the general case of unknown $L$ such a plug-in approach encounters difficulties because of the challenge of direct consistent estimation of $L$. Further, as earlier analysis reveals, the optimal bandwidth order in functional coefficient regression depends on the flatness degree parameter $L$. Since $L$ is a higher order property of an unknown nonparametric function $\beta(z)$, this dependence poses a subtle question of how to determine the bandwidth $h$ in estimation and inference.

In this respect, noting that $\beta\left(z_t\right)-\beta\left(z\right) = \frac{\beta^{(L)}(\tilde{z}_t)}{L!}\left(z_t - z\right)^L$ where $\tilde{z}_t$ lies on the line segment between $z_t$ and $z$ and $\beta^{(L)}\left(z\right) \neq 0$ by assumption, it follows that as $n \to \infty$ and $h \to 0$

$$L_n^{\dagger} = \frac{1}{\log\left(h\right)} \log\left(\frac{\frac{1}{n}\sum_{t=1}^{n}\left|\beta(z_t) - \beta(z)\right| K_{tz}}{\frac{1}{n}\sum_{t=1}^{n} K_{tz}}\right) \to_p L,$$

as shown in the Online Supplement - see (**??**). Of course, $L_n^{\dagger}$ is an infeasible rate estimator reliant on the unknown function $\beta(\cdot)$ in a neighbourhood of $z$. It has a slow logarithmic convergence rate with $L_n^{\dagger} - L = O_p(1/\log(h))$, so that when $h = n^{-\delta}$ for some $\delta > 0$ we have $L_n^{\dagger} - L = O_p(1/\log(n))$. More specifically, $\log(h)(L_n^{\dagger} - L) = \log\left(\left|\frac{\beta^{(L)}(z)}{L!}\right|\right) + \log\left(\int |s|^L K(s)ds\right) + o_p(1)$, as shown in (**??**).

Setting $w_{tz} = K_{tz}/\sum_{t=1}^{n} K_{tz}$, this limit behavior suggests the following 'plausible' practical estimate of $L$

$$\hat{L} = \frac{1}{\log(h)} \log\left(\sum_{t=1}^{n}\left|\hat{\beta}(z_t) - \hat{\beta}(z)\right| w_{tz}\right), \tag{3.12}$$

which can be computed using a preliminary bandwidth $h$ satisfying $h \to 0$ and $nh \to \infty$. However, when $L > 1$ the estimator $\hat{L}$ is not consistent. Intuitively, this is because the nonparametric estimator $\hat{\beta}(\cdot)$ does not necessarily satisfy $\hat{\beta}^{(\ell)}(w) = 0$ for $\ell < L$ and $w$ in a neighborhood of

$z$. Hence the approximation $\hat{\beta}(z_t) - \hat{\beta}(z) \sim_a \frac{\hat{\beta}^{(L)}(z)}{L!}(z_t - z)^L$ no longer holds for $z_t$ in the neighborhood of $z$. Instead, for $z_t = z + ph$ we have $\hat{\beta}(z_t) - \hat{\beta}(z) = O_p(h + \frac{1}{\sqrt{nh}})$ when $x_t$ is stationary and $\hat{\beta}(z_t) - \hat{\beta}(z) = O_p(h + \frac{1}{n\sqrt{h}})$ when $x_t$ is nonstationary, as demonstrated in Section 2 in the Online Supplement, and these error orders are not sufficient to ensure that $\hat{L}$ in (3.12) is consistent. Therefore, the estimator $\hat{\beta}(z)$ does not retain the higher order flat property of $\beta(z)$ and therefore cannot be used to recover the flatness parameter $L$. The feasibility of direct consistent estimation of $L$ requires further study and is left for future research.[3]

### 3.3.2 Adaptive statistic design

This section comments briefly on the possibility of constructing an adaptive test statistic that does not require knowledge of $L$. The idea stems from Remarks 3.2 and 3.3 in Phillips and Wang (2020) where a statistic is developed that incorporates bias and variance matrix estimators that do not involve $L$ but instead rely on local information about the function obtained by kernel estimation. In principle, it is straightforward to extend this idea to the case where $L > 1$. Take the stationary case as an example. The adaptive bias estimator is defined as

$$\hat{B}(z) = A_n(z)^{-1} \left( \sum_{t=1}^{n} x_t x_t' \right) \frac{1}{n} \sum_{t=1}^{n} [\hat{\beta}(z_t) - \hat{\beta}(z)] K \left( \frac{z_t - z}{h} \right),$$

where the sample average $\frac{1}{n} \sum_{t=1}^{n} [\hat{\beta}(z_t) - \hat{\beta}(z)] K \left( \frac{z_t - z}{h} \right)$ is introduced to approximate $\mathbb{E}[\beta(z_t) - \beta(z)] K_{tz}$. Unfortunately this adaptive bias estimator $\hat{B}(z)$ is not consistent for the true bias when $L > 1$ because local kernel estimation in $\hat{\beta}(z_t) - \hat{\beta}(z)$ is insufficiently precise to capture the required derivative components. In consequence, the limit of $\hat{B}(z)$ has many additional terms when $L > 1$. Moreover, direct (bias correction) adjustment to achieve consistent bias estimation is not possible because the limit of $\hat{B}(z)$ depends on the unknown value of $L$. More details are provided in the Online Supplement showing how the adaptive bias estimator fails in flat regions of the function where $L > 1$ in both stationary and nonstationary cases. There are

[3] A further complication that should be mentioned is that even if a consistent estimator of $L$ were available, bias correction requires specification of the bandwidth factor $h(L) = h^L$ in $h^L \mathcal{B}_L(z)$, which presents additional difficulties. For example, whereas the infeasible estimator $L_n^\dagger \to_p L$, the consistency of $L_n^\dagger$ does not mean that $h^{L_n^\dagger} \sim_a h^L$. Indeed, by Taylor expansion

$$h(L^\dagger) - h(L) = h^{(1)}(\tilde{L})(L^\dagger - L) = h^{\tilde{L}} \log(h)(L^\dagger - L) = h^L \log \left( \left| \frac{\beta^{(L)}(z)}{L!} \right| \int |s|^L K(s) ds \right) + o_p(h^L) \tag{3.13}$$

since $\tilde{L}$ is on the line segment between $L^\dagger$ and $L$ and $L^\dagger \to_p L$. Then

$$h(L^\dagger) = h(L) \times \left( 1 + \log \left( \left| \frac{\beta^{(L)}(z)}{L!} \right| \int |s|^L K(s) ds \right) \right) + o_p(h^L),$$

and $h(L^\dagger) = h^{L^\dagger} \sim_a d_L h^L$, where $d_L = \left( 1 + \log \left( \left| \frac{\beta^{(L)}(z)}{L!} \right| \int |s|^L K(s) ds \right) \right)$, so that $h^{\hat{L}}$ is inconsistent. Thus, the slow rate of convergence of $L_n^\dagger$ interferes with the consistent estimation of the factor $h^L$ needed for bias correction.

further obstacles to inference in the adaptive bias estimator $\hat{B}(z)$ due to additional variation that affects the limit distribution of the bias centered term $\hat{\beta}(z) - \beta(z) - \hat{B}(z)$. In the nonstationary case, the variance of this term depends on $L$ and $\beta^{(L)}(z)$, making it difficult to estimate the limit variance adaptively without introducing further bias effects. These complications combine to make it difficult to design an adaptive statistic in cases where the flatness degree is unknown, leaving this pursuit as a challenge for future research.

Section 4.2 below studies the finite sample performance of the infeasible oracle statistic $\hat{T}(z; L)$ where the unknown true value $L$ is used in construction of the test. For comparison, the naive $t$-ratio $\hat{T}(z; L = 1)$ which sets $L = 1$ is implemented to reveal the consequences of ignoring potential local flatness in the coefficient function and using the base statistic $\hat{T}(z; L = 1)$. The findings shed light on the empirical relevance for inference of failing to utilize local flatness information about $\beta(z)$ when flatness is unanticipated.

# 4 Simulations

The simulation experiments that follow employ a simple prototypical framework for evaluating the adequacy of the asymptotic theory. We explore the behavior of the functional coefficient estimators and the adequacy of the limit theory in locally flat and non-flat regions of the function. The following sections consider estimation and inference in stationary and nonstationary cases, separately.

## 4.1 Estimation

**Nonstationary $x_t$**

In the first experiment the model (2.1) is used with a single $I(1)$ exogenous regressor $x_t$ generated as a random walk with *iid* $\mathcal{N}(0, \sigma_x^2)$ innovations $\epsilon_{xt}$ and zero initialization $x_0$, *iid* $\mathcal{N}(0, \sigma_u^2)$ equation errors $u_t$, and $iidU[-1, 2]$ covariates $z_t$. We set $\sigma_x^2 = 1$ and $\sigma_u^2 = 1$. Throughout the simulations, the number of replications used is $10,000$ and the coefficient function is the quartic $\beta(z) = z^4$, for which the first three derivatives at $z_1 = 0$ are zero, $\beta^{(4)}(z_1) = 4!$ and $\beta^{(1)}(z_2) = 4z_2^3 = 4$ at $z_2 = 1$.

Figure 2 shows the mean bias (plotted in the left panel), standard deviation (plotted in the middle panel) and RMSE (plotted in the right panel) for $\hat{\beta}(z)$ calculated at the points $\{z = 0, 1\}$ using samples of size $n = 100, 400$ and $800$, based on $10,000$ replications. In estimation we employ a Gaussian kernel and the bandwidth formula $h = \hat{\sigma}_z \times n^{\gamma}$. The range $-0.90 \leq \gamma \leq -0.05$ is used to meet the condition $nh \to \infty$ and to avoid extremely small bandwidths for which there is considerable imprecision in the simulation estimates, as is evident in the plotted curves for the standard deviation and RMSE near the left limit of the domain of definition.[4] The plots show significant differences in estimator behavior between the two points

[4]This imprecision is related to the fact that when $nh \to c < \infty$ the asymptotic theory changes and no

of estimation $\{z = 0, 1\}$, which we summarize as follows.

(i) Bias increases as the bandwidth widens and the bandwidth power $\gamma \to 0$. For very wide bandwidths, estimates at both $z_1 = 0$ and $z_2 = 1$ suffer large bias. However, bias is smaller and usually much smaller at the point $z_1 = 0$ of locally flat functional form than at point $z_2 = 1$. These findings all match the asymptotic theory in Theorem 2.2, which shows that bias has order $h^{L^*}$, which is $h^4$ when $z_1 = 0$ where $L^* = L = 4$, compared with $h^2$ when $z_2 = 1$ where $L^* = L + 1 = 2$ with $L = 1$.

(ii) Standard deviation rises in estimation at both points of estimation as the bandwidth becomes very small when $\gamma \to -1$ or as bandwidth becomes very large when $\gamma \to 0$. This outcome corresponds to asymptotic theory where there are three convergence rates for the cases given in Theorem 2.2, where it is shown that the highest convergence rate (or minimum standard deviation) occurs in the intermediate bandwidth contraction case with $h = O(n^{-\frac{1}{2L}})$. When the bandwidth is very small ($\gamma$ close to -1), considerable volatility in the standard deviation estimates was found even with a large number of replications, particularly for smaller sample sizes. We therefore only report results for $\gamma \geq -0.90$ and some volatility in the estimates is evident in the graphics close to this lower limit. The standard deviation of $\hat{\beta}(z)$ at $z_2 = 1$ is seen to be substantially greater than that at $z_1 = 0$ except for small bandwidths, again matching the limit theory.

(iii) The RMSE curves demonstrate similar U-shaped patterns to those of the standard deviation curves. This simulation evidence corroborates the analysis in Remark 2.2, where it is shown that the RMSE order $g_L(\gamma)$ has a check function shape with $L = 4$. Further, the RMSE is considerably lower when $\beta(z)$ is flat at $z_1 = 0$ than when the coefficient function is rising at $z_2 = 1$. These gains hold throughout a wide range of bandwidth powers except for smaller bandwidths.

(iv) Across panels (a) (b) and (c) in Figure 2, the main impact of larger sample sizes is the anticipated reduction in the bias, standard deviation, and RMSE, which applies to both $z_1 = 0$ and $z_2 = 1$ cases and across all bandwidth powers.

Table 1: Finite Sample Optimal Bandwidth Order Estimates

| | $x_t$ is nonstationary | | | | $x_t$ is stationary | |
|---|---|---|---|---|---|---|
| | StDev optimal | | RMSE optimal | | RMSE optimal | |
| | $z_1 = 0$ | $z_2 = 1$ | $z_1 = 0$ | $z_2 = 1$ | $z_1 = 0$ | $z_2 = 1$ |
| $n = 100$ | -0.19 | -0.56 | -0.27 | -0.56 | -0.20 | -0.40 |
| $n = 400$ | -0.18 | -0.56 | -0.26 | -0.56 | -0.17 | -0.38 |
| $n = 800$ | -0.18 | -0.55 | -0.25 | -0.55 | -0.17 | -0.36 |
| $n = \infty$[5] | -0.13 | -0.50 | -0.22 | -0.50 | -0.11 | -0.20 |

invariance principle applies. Readers are referred to Phillips and Wang (2020) for further analysis and discussion of this phenonomenon.

[5]The numbers in this row are the optimal bandwidth orders based on the asymptotic theory as $n \to \infty$.

To better illustrate the optimal bandwidth order discussed in Remarks 2.1 and 2.2, we report the bandwidth power values corresponding to the minimum points of the standard deviation and RMSE curves from the simulations in Figure 2. Results are collected in Table 1 under the panel headed "$x_t$ is nonstationary". According to Remark 2.1, the convergence-rate, or equivalently, the standard-deviation optimal bandwidth order is achieved at $-\frac{1}{2L}$, which is $-\frac{1}{8} \approx -0.13$ for $z_1 = 0$ ($L = 4$) and $-\frac{1}{2}$ for $z_2 = 1$ ($L = 1$). Following Remark 2.2, the RMSE optimal bandwidth order is $-\frac{2}{2L+1} = -\frac{2}{9} \approx -0.22$ for $z_1 = 0$ ($L = 4$) and $-\frac{1}{2}$ for $z_2 = 1$ ($L = 1$). These are the figures reported in the last row of Table 1 for $n = \infty$. Only when $L = 1$ are these two optimal bandwidth orders the same both here and for $z_2 = 1$ in Table 1. When $L = 4$, the convergence-rate optimal bandwidth power is larger than the RMSE optimal bandwidth power. In Table 1 it is evident that for $z_1 = 0$, the standard-deviation optimal bandwidth order estimates are larger than the RMSE optimal bandwidth order estimates. Moreover, as the sample size $n$ increases, the optimal bandwidth order estimates approach the corresponding limit values reported in the last row for $n = \infty$. These results again corroborate the analysis in Remarks 2.1 and 2.1 showing that the RMSE optimal bandwidth rate equals the convergence-rate optimal bandwidth order when $L = 1$ or is less than the convergence-rate optimal bandwidth order when $L \geq 2$.

**Stationary $x_t$**

In the second experiment the same model (2.1) is used but with a stationary exogenous regressor $x_t$ generated by the autoregression $x_t = \theta x_{t-1} + \epsilon_{xt}$ with $iid\, \mathcal{N}(0, \sigma_x^2)$ innovations $\epsilon_{xt}$ and zero initialization $x_0$, $iid\, \mathcal{N}(0, \sigma_u^2)$ equation errors $u_t$, and $iid U[-1, 2]$ covariates $z_t$. We set $\sigma_x^2 = 1$, $\sigma_u^2 = 1$, and $\theta = 0.5$. Again 10,000 replications are employed. The results for bias, standard deviation and RMSE are shown in Figure 3. The plots for the stationary case mirror those in Figure 2 for the FCC case. The imprecision in the simulation estimates at small bandwidths is more severe than in the nonstationary case and results are accordingly reported here for the reduced bandwidth power region $-0.8 \leq \gamma \leq -0.05$. The findings for the stationary case are summarized below.

(v) The main difference with the nonstationary model occurs in the standard deviation curves. Different from the nonstationary case, Theorem 2.1 shows that the convergence rate on the left hand side is unaffected by the local flatness parameter $L$ or the bandwidth rate condition $nh^{2L}$. We therefore expect to see monotonously decreasing standard deviation curves for both points of estimation $\{z = 0, 1\}$ as the bandwidth power $\gamma$ increases. From the middle panel of Figure 3, we observe that the standard deviation curve for $z_1 = 0$ indeed shows a decreasing pattern as $\gamma$ increases to 0, but that for $z_2 = 1$ the curve starts to rise slightly when $\gamma$ is close

For the case that $x_t$ is nonstationary, the standard-deviation optimal bandwidth order is the convergence-rate optimal bandwidth order analyzed in Remarks 2.1. It is given as $-\frac{1}{2L}$, which is $-\frac{1}{8} \approx -0.13$ for $z_1 = 0$ ($L = 4$) and $-\frac{1}{2}$ for $z_2 = 1$ ($L = 1$). The RMSE optimal bandwidth follows Remark 2.2, which is $-\frac{2}{2L^*+1} = -\frac{2}{9} \approx -0.22$ for $z_1 = 0$ ($L = L^* = 4$) and $-\frac{1}{2}$ for $z_2 = 1$ ($L = 1$). For the case that $x_t$ is stationary, the RMSE optimal bandwidth order is $-\frac{1}{2L^*+1}$ as given in (2.6). Then the true value for $z_1 = 0$ ($L = L^* = 4$) is $-\frac{1}{9} \approx -0.11$ and that for $z_2 = 1$ ($L = 1$, $L^* = 2$) is $-\frac{1}{5}$.

to 0. This is explained by the randomness that is present in the bias function in finite samples. Although the randomness in the bias function is of smaller order than that of the usual error term asymptotically and therefore does not figure in the limit theory, it can still affect finite sample performance. Moreover, the finite sample effects are less severe when the functional coefficient is locally flatter (with larger $L$) because the bias is smaller when $L$ is larger. This explains why a marked rise in the curve is only observed towards the right limit near $\gamma = 0$ of the domain of definition of the standard deviation curves for $z_2 = 1$ but not for the curves for $z_1 = 0$.

(vi) For both RMSE curves, there is also a clear minimum RMSE bandwidth choice as in the nonstationary case. Furthermore, the curves indicate that the minimum RMSE bandwidth power $\gamma$ is larger for estimation at $z_1 = 0$ than at $z_2 = 1$. Direct evidence of this difference is given by the estimates of the RMSE optimal bandwidth power reported in Table 1 under the panel '$x_t$ is stationary'. These findings corroborate the analysis concerning the optimal RMSE bandwidth order following Theorem 2.1.

(vii) The plots in Figure 3 show the finite sample gains in estimation that occur from local flatness of the functional coefficient. These gains occur for bandwidths large enough to be well beyond the region where there is imprecision in the simulation estimates of the standard deviation and RMSE.

## 4.2 Inference

This Section reports findings on the finite sample performance of the $t$-ratios discussed in Section 3. Three statistics are considered: (i) the infeasible statistic $\hat{T}(z; true\ L)$ in which the true value of $L$ is used; (ii) the 'naive' statistic $\hat{T}(z; L = 1)$ where $L = 1$ is used as the simplest case without any attention to potential flatness; and (iii) the oracle $t$-ratio $T(z; true\ L)$, which assumes $L$, the derivatives $(\beta^{(L)}(z), \beta^{(L+1)}(z))$, and other components $\sigma_u^2$, $f(z)$ and $f^{(1)}(z)$ are known. The oracle and infeasible statistics provide two baselines to assess the relative performance of the naive statistic for comparative purposes. Other details concerning computation are given in the Appendix. The same generating mechanism is used as in the previous section and we again consider the two evaluation points $z_1 = 0$ (with $L = 4$) and $z_2 = 1$ (with $L = 1$). A second order Epanechnikov kernel is used in the computations.

The empirical densities of these $t$-ratio statistics are shown in Figure 4 for stationary $x_t$ and in Figure 5 for nonstationary $x_t$. From Figure 4, at the flat point $z_1 = 0$ the densities of the oracle statistic are evidently extremely close to the standard normal. Densities of the naive and infeasible statistics show some discrepancy from the standard normal, but the distribution of the infeasible statistic is closer to standard normal when the sample size is large. The improvement of the infeasible over the naive statistic reveals the gains from knowledge of $L$, or equivalently, the consequences of ignoring local flatness in the coefficient function at the point of flatness. At the non-flat point, the oracle statistic is the one closest to standard normal. The naive and infeasible statistics are identical in this case because the true value of $L$ is 1. Compared to the performance at the flat point $z_1 = 0$, the densities are closer to standard normal at the non-flat point. In the nonstationary case in Figure 5, the oracle statistic is again extremely close to standard normal at the flat point. The naive and infeasible statistics are competitive in performance, although both are too densely distributed at the origin. At the non-flat point, the naive and the infeasible distributions are again identical and their performance is competitive to that of the oracle statistic. In conclusion, ignoring local flatness in the coefficient function seems to cause some efficiency loss at the flat point. To see this feature more clearly and examine its implications for inference we examine coverage rates and confidence interval lengths of the associated test statistics.

Table 2 reports coverage rates and lengths of the confidence intervals constructed at the two points $z_1 = 0$ and $z_2 = 1$ using the three statistics $\hat{T}(z; true\ L)$, $\hat{T}(z; L = 1)$, and $T(z; true\ L)$. In the stationary case, the oracle statistic has the best coverage rates and these are close to the nominal level. The naive and infeasible statistics have similar coverage rates but these are lower than those of the oracle statistic. At the flat point, the infeasible statistic has narrower confidence bands than the naive statistic. This finding reflects the efficiency gain of knowing $L$, or the efficiency loss of ignoring local flatness. In the nonstationary case, the efficiency loss of the naive versus the infeasible statistic is evident from the much wider confidence bands produced by the naive statistic at the flat point. The naive choice and the infeasible statistic suffer a mild over-coverage problem in the nonstationary case. In sum, the naive and infeasible statistics share similar coverage rates but with wider confidence bands at the flat region in the naive statistic case. So ignoring local flatness has greater consequences in terms of wider confidence bands than lower coverage rates.

The coverage rate curves and lengths of the confidence intervals are plotted in Figure 6 over the support $[-1, 2]$ of $z_t$ for sample sizes $n = 200$ and $n = 800$. The oracle statistic shows an evident advantage in coverage over the other two methods. The naive and the infeasible curves are identical except at the flat point $z = 0$. The efficiency gain in the infeasible statistic compared to the naive statistic manifests in the narrower confidence bands at $z = 0$, especially in the nonstationary case. This finding is consistent with what is observed in Table 2. The conclusion is, again, that naive inference using $L = 1$ leads to wider confidence bands in the flat region as a consequence of ignoring the local flatness in the coefficient function.

Thus, at the cost of some efficiency loss in terms of wider confidence interval in the flat region compared to the infeasible statistic which uses the true value of $L$, the naive statistic that employs the simple setting $L = 1$ remains an adequate option for inference when straightforward implementation is a priority.

Table 2: Coverage rates and confidence interval length (in brackets) at points $(z_1, z_2)$ based on the $t$-ratios $\hat{T}(z; true\ L)$, $\hat{T}(z; L=1)$, and $T(z; true\ L)$

| | $x_t$ is nonstationary | | | $x_t$ is stationary | | |
|---|---|---|---|---|---|---|
| $n$ | $\hat{T}(z; true\ L)$ | $\hat{T}(z; L=1)$ | $T(z; true\ L)$ | $\hat{T}(z; true\ L)$ | $\hat{T}(z; L=1)$ | $T(z; true\ L)$ |
| | | $z_1 = 0(L=4)$ | | | $z_1 = 0(L=4)$ | |
| 100 | 0.96 | 0.97 | 0.95 | 0.85 | 0.85 | 0.95 |
| | (0.24) | (0.67) | (0.21) | (0.62) | (0.74) | (0.66) |
| 200 | 0.97 | 0.97 | 0.95 | 0.88 | 0.87 | 0.95 |
| | (0.12) | (0.31) | (0.11) | (0.44) | (0.56) | (0.48) |
| 800 | 0.98 | 0.97 | 0.95 | 0.91 | 0.90 | 0.95 |
| | (0.04) | (0.10) | (0.03) | (0.24) | (0.32) | (0.25) |
| | | $z_2 = 1(L=1)$ | | | $z_2 = 1(L=1)$ | |
| 100 | 0.96 | 0.96 | 0.95 | 0.91 | 0.91 | 0.94 |
| | (0.78) | (0.78) | (0.71) | (1.14) | (1.14) | (1.15) |
| 200 | 0.97 | 0.97 | 0.96 | 0.91 | 0.91 | 0.95 |
| | (0.39) | (0.39) | (0.36) | (0.79) | (0.79) | (0.80) |
| 800 | 0.97 | 0.97 | 0.95 | 0.92 | 0.92 | 0.95 |
| | (0.12) | (0.12) | (0.12) | (0.40) | (0.40) | (0.41) |

## 5 Conclusion

This paper extends existing limit theory in functional coefficient regression to accommodate locally constant coefficients in the regression model (2.1), allowing for both stationary and nonstationary regressors $x_t$. The findings show that, in the stationary case, the primary effects on the limit theory involve estimation bias, which in turn affects optimal bandwidth choice and optimal convergence rates. In the nonstationary case, both bias and dispersion are affected in the limit theory. As a result, the conditions that separate the limit theory into three different categories are affected by the flatness degree parameter. In particular, both bias and variance depend on the number $(L-1)$ of zero derivatives in the coefficient function, with consequential effects on optimal bandwidth choice and rates of convergence. In the boundary case where $L \to \infty$ near parametric rates of convergence apply for both stationary and nonstationary cases. In both cases, locally flat functional coefficients make wider bandwidth choices beneficial compared with those implied by standard limit theory. But optimal bandwidth choice is complicated by the fact that bias-variance trade-offs may not correspond to optimal convergence rates and bias correction is more complex due to the locally flat behavior of the coefficient function.

In closing it is worth mentioning that extensions of the type given here are relevant to existing asymptotic theory for nonparametric estimation whenever locally flat functional behavior is present in other models such as probability densities, models with nonstationary regressors that are more complex than $I(1)$ processes and models with time varying parameters. Common practice in the latter models, for instance, is to use weak trend formulations of the parameters, leading to time dependent coefficients of the form $\beta(\frac{t}{n})$. Trend formulations of this type in both stationary and nonstationary systems will lead to asymptotics that involve extensions similar to those developed here, particularly in the stationary regressor case where bias expressions, bias order, and optimal bandwidth choice will all be influenced by flatness in the function. Similarly, in time varying parameter cointegrated systems of the type studied in Phillips et al. (2017), the limit theory will be affected by locally flat regions of the coefficient function. An important simplification in both these cases is that the coefficient function $\beta(\cdot)$ is deterministic, which means that the bias component affects centering but will not contribute directly to variability and the form of the limit distribution, as it can do in models with nonstationary regressors. These are some extensions of the present theory that seem worthy of full investigation in future research.

In all of the above models, any regions of flatness in the function being estimated are typically unknown *a priori*, including the degree of local flatness, just as the function itself is unknown. Our analysis shows that in such cases the formulae based on standard asymptotics that are used to measure bias and variance in nonparametric estimation are only approximate and rates of convergence may be wrong, especially in cases of nonstationary regressors. After extensive attempts we have found it extremely challenging to devise a feasible inference procedure that accommodates empirical information about unknown locally flat characteristics of a functional coefficient. Fortunately, simulation results reveal that use of the naive test statistic that simply ignores the possibility of local flatness is an acceptable approach to inference in practice. In fact, the naive procedure achieves similar coverage rates to those of the infeasible statistic that uses additional correct information about the degree of local flatness at the cost of slightly wider confidence bands. Empirical estimation of the degree of local flatness and improved inferential procedures that take account of potential flatness both merit further research.

# Appendix

## A Proof of the Theorems

**Proof of Theorem 2.1** We analyze the components in the following normalized decomposition of the estimation error

$$\left(\sum_{t=1}^{n} x_t x_t' K_{tz}\right)\left(\hat{\beta}\left(z\right)-\beta\left(z\right)\right)=\sum_{t=1}^{n} x_t x_t'[\beta(z_t)-\beta(z)]K_{tz}+\sum_{t=1}^{n} x_t u_t K_{tz}.$$

$$=\sum_{t=1}^{n} x_t x_t' \mathbb{E}\xi_{\beta t}+\sum_{t=1}^{n} x_t x_t' \eta_t+\sum_{t=1}^{n} x_t u_t K_{tz}, \tag{A.1}$$

with $\xi_{\beta t} = [\beta(z_t) - \beta(z)]K_{tz}$ and $\eta_t = \xi_{\beta t} - \mathbb{E}\xi_{\beta t}$. Starting with the kernel-weighted signal matrix, we have

$$\frac{1}{nh}\sum_{t=1}^{n} x_t x_t' K_{tz} = \frac{1}{nh}\sum_{t=1}^{n} x_t x_t' \mathbb{E}\left(K_{tz}\right) + \frac{1}{nh}\sum_{t=1}^{n} x_t x_t' \zeta_{tK} \tag{A.2}$$

where $\zeta_{tK} = K_{tz} - \mathbb{E}\left(K_{tz}\right)$ and $\mathbb{E}K_{tz} = h\int K\left(r\right)f\left(z+rh\right)dr = hf(z) + O(h^3)$. Since $\mathbb{E}K_{tz}^2 = h\int K^2\left(r\right)f\left(z+rh\right)dr = hf(z)\int K^2(r)dr + o(h) = hf(z)\nu_0(K) + o(h)$, where $\nu_j(K) = \int u^j K^2(u)du$, it follows that $\mathbb{V}ar(\zeta_{tK}) = \mathbb{E}K_{tz}^2 - (\mathbb{E}K_{tz})^2 = O(h)$ and so $\zeta_{tK} = O_p(\sqrt{h})$. We deduce that when $nh \to \infty$

$$\frac{1}{nh}\sum_{t=1}^{n} x_t x_t' K_{tz} = \frac{1}{n}\sum_{t=1}^{n} x_t x_t' \left\{f(z) + o(1)\right\} + \frac{1}{\sqrt{nh}}\sum_{t=1}^{n} x_t x_t' \frac{\zeta_{tK}}{\sqrt{nh}} \to_p \Sigma_{xx} f(z), \tag{A.3}$$

since $\frac{1}{\sqrt{nh}}\sum_{t=1}^{n} x_t \otimes x_t \zeta_{tK} \rightsquigarrow \mathcal{N}\left(0, \nu_0(K)f(z)\mathbb{E}\left[x_t x_t' \otimes x_t x_t'\right]\right)$ from Lemma B.1(d)(i).

Next, from the proof of Lemmas B.2(c) and (B.6), we have

$$\mathbb{E}\xi_{\beta t} = h^{L^*+1}C_L(z) + o(h^{L^*+1}),$$

where $L^* = L \times \mathbf{1}_{\{L=\text{even}\}} + (L+1)\,\mathbf{1}_{\{L=\text{odd}\}}$, $\mu_{L^*}(K) = \mu_L(K) \times \mathbf{1}_{\{L=\text{even}\}} + \mu_{L+1}(K)\mathbf{1}_{\{L=\text{odd}\}}$ and $C_L(z)$ defined in (2.5). Upon normalization and using Lemma B.2(c), the first term in (A.1) is then

$$\frac{1}{nh^{L^*+1}}\sum_{t=1}^{n} x_t x_t' \mathbb{E}\left(\xi_{\beta t}\right) \to_p \Sigma_{xx} C_L(z). \tag{A.4}$$

The second term of (A.1) is, upon normalization and using Lemma B.2(b)(i),

$$\frac{1}{\sqrt{nh^{2L+1}}}\sum_{t=1}^{n} x_t x_t' \eta_t \rightsquigarrow \mathcal{N}\left(0, \frac{\nu_{2L}(K)f\left(z\right)}{\left(L!\right)^2}\mathbb{E}\left[(x_t'\beta^{(L)}(z))^2 x_t x_t'\right]\right) = \mathcal{N}\left(0, \mathbb{E}\left[(x_t' V_{\eta\eta,L} x_t) x_t x_t'\right]\right), \tag{A.5}$$

provided $nh \to \infty$. Otherwise $\frac{1}{\sqrt{nh^{2L+1}}}\sum_{t=1}^{n} x_t x_t' \eta_t = O_p(1)$ but no central limit theorem holds, as shown in Lemma B.2(b)(ii). The final term of (A.1) is, after suitable normalization and using Lemma B.1(c)(i),

$$\frac{1}{\sqrt{nh}}\sum_{t=1}^{n} x_t u_t K_{tz} \rightsquigarrow \mathcal{N}\left(0, \nu_0(K)f(z)\sigma_u^2 \Sigma_{xx}\right), \tag{A.6}$$

provided $nh \to \infty$. Otherwise from Lemma B.1(c)(ii), $\frac{1}{\sqrt{nh}}\sum_{t=1}^{n} x_t u_t K_{tz} = O_p(1)$ but no invariance principle applies.

Standardizing by the weighted signal matrix and recentering (A.1) we have the estimation error decomposition

$$\hat{\beta}(z)-\beta(z)-\left(\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\sum_{t=1}^{n}x_tx_t'\mathbb{E}\xi_{\beta t}=\left(\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\sum_{t=1}^{n}x_tx_t'\eta_t+\left(\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\sum_{t=1}^{n}x_tu_tK_{tz}, \tag{A.7}$$

or, with each component appropriately standardized, as

$$\hat{\beta}(z)-\beta(z)-h^{L^*}\left(\frac{1}{nh}\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\frac{1}{nh^{L^*+1}}\sum_{t=1}^{n}x_tx_t'\mathbb{E}\xi_{\beta t}$$
$$=\sqrt{\frac{h^{2L-1}}{n}}\left(\frac{1}{nh}\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\frac{1}{\sqrt{nh^{2L+1}}}\sum_{t=1}^{n}x_tx_t'\eta_t+\frac{1}{\sqrt{nh}}\left(\frac{1}{nh}\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\frac{1}{\sqrt{nh}}\sum_{t=1}^{n}x_tu_tK_{tz}. \tag{A.8}$$

Using (A.4), (A.5) and (A.6) in (A.8), we have

$$\sqrt{nh}\left(\hat{\beta}(z)-\beta(z)-h^{L^*}\left(\frac{1}{nh}\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\frac{1}{nh^{L^*+1}}\sum_{t=1}^{n}x_tx_t'\mathbb{E}\xi_{\beta t}\right)$$
$$=h^{L}\left(\frac{1}{nh}\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\frac{1}{\sqrt{nh^{2L+1}}}\sum_{t=1}^{n}x_tx_t'\eta_t+\left(\frac{1}{nh}\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\frac{1}{\sqrt{nh}}\sum_{t=1}^{n}x_tu_tK_{tz}$$
$$=O_p(h^L)+\left(\frac{1}{nh}\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\frac{1}{\sqrt{nh}}\sum_{t=1}^{n}x_tu_tK_{tz} \tag{A.9}$$
$$\rightsquigarrow\mathcal{N}\left(0,\frac{\nu_0(K)\sigma_u^2}{f(z)}\Sigma_{xx}^{-1}\right). \tag{A.10}$$

Using (A.3) and (A.4) we have

$$\left(\frac{1}{nh}\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\frac{1}{nh^{L^*+1}}\sum_{t=1}^{n}x_tx_t'\mathbb{E}\xi_{\beta t}\to_p\frac{\mathcal{G}_L(z)}{f(z)}, \tag{A.11}$$

which leads to

$$\sqrt{nh}\left(\hat{\beta}(z)-\beta(z)-h^{L^*}\mathcal{B}_L(z)\right)\rightsquigarrow\mathcal{N}\left(0,\frac{\nu_0(K)\sigma_u^2}{f(z)}\Sigma_{xx}^{-1}\right) \tag{A.12}$$

where $\mathcal{B}_L(z)=\frac{\mathcal{G}_L(z)}{f(z)}$, giving the stated result for the first part, which holds whenever $nh\to\infty$ ensuring the central limit theorem (A.12).

In cases where $nh\to c\in[0,\infty)$, in view of Lemma B.1(c)(ii) and Lemma B.2(b)(ii), we still have $\frac{1}{\sqrt{nh}}\sum_{t=1}^{n}x_tu_tK_{tz}=O_p(1)$ and $\frac{1}{\sqrt{nh^{2L+1}}}\sum_{t=1}^{n}x_tx_t'\eta_t=O_p(1)$ although no invariance principle holds. Further, in view of Lemma B.2(c) we have $\sum_{t=1}^{n}x_tx_t'\mathbb{E}\xi_{\beta t}=O_p(nh^{L^*+1})$. But the signal matrix $\sum_{t=1}^{n}x_tx_t'K_{tz}=O_p(\sqrt{nh})$ and so the last term in (A.7) is $O_p(1)$. Therefore $\hat{\beta}(z)$ is inconsistent when $nh\to c\in[0,\infty)$. ■

**Proof of Theorem 2.2**

**Case (i)** We start again with the decomposition (A.7) and rescale the components according to their asymptotic behavior, as determined in Lemma B.3, so that

$$\begin{aligned}
&\hat{\beta}(z)-\beta(z)-h^{L^*}\left(\frac{1}{n^2h}\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\frac{1}{n^2h^{L^*+1}}\sum_{t=1}^{n}x_tx_t'\mathbb{E}\xi_{\beta t}\\
&=\sqrt{\frac{h^{2L-1}}{n}}\left(\frac{1}{n^2h}\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\frac{1}{\sqrt{n^3h^{2L+1}}}\sum_{t=1}^{n}x_tx_t'\eta_t+\frac{1}{n\sqrt{h}}\left(\frac{1}{n^2h}\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\frac{1}{n\sqrt{h}}\sum_{t=1}^{n}x_tu_tK_{tz}.
\end{aligned}\tag{A.13}$$

Then, since $nh^{2L}\to 0$ in this case, we rescale the equation by $n\sqrt{h}$, giving

$$\begin{aligned}
&n\sqrt{h}\left(\hat{\beta}(z)-\beta(z)-h^{L^*}\left(\frac{1}{n^2h}\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\frac{1}{n^2h^{L^*+1}}\sum_{t=1}^{n}x_tx_t'\mathbb{E}\xi_{\beta t}\right)\\
&=\sqrt{nh^{2L}}\left(\frac{1}{n^2h}\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\frac{1}{\sqrt{n^3h^{2L+1}}}\sum_{t=1}^{n}x_tx_t'\eta_t+\left(\frac{1}{n^2h}\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\frac{1}{n\sqrt{h}}\sum_{t=1}^{n}x_tu_tK_{tz}\\
&=o_p(1)+\left(\frac{1}{n^2h}\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\frac{1}{n\sqrt{h}}\sum_{t=1}^{n}x_tu_tK_{tz}\\
&\rightsquigarrow\left(f(z)\int B_xB_x'\right)^{-1}\left(\int B_xdB_{uK}\right)\equiv_d\mathcal{MN}\left(0,\frac{\nu_0(K)\sigma_u^2}{f(z)}\left(\int B_xB_x'\right)^{-1}\right),
\end{aligned}\tag{A.14}$$

the mixed normality following from Lemma B.3 (d)(i). In view of Lemma B.3(b) and Lemma B.3(c)(i) we have

$$\left(\frac{1}{n^2h}\sum_{t=1}^{n}x_tx_t'K_{tz}\right)^{-1}\frac{1}{n^2h^{L^*+1}}\sum_{t=1}^{n}x_tx_t'\mathbb{E}\xi_{\beta t}\to_p\frac{\mathcal{G}_L(z)}{f(z)},\tag{A.15}$$

for the bias function. Hence,

$$n\sqrt{h}\left(\hat{\beta}(z)-\beta(z)-h^{L^*}\mathcal{B}_L(z)\right)\rightsquigarrow\mathcal{MN}\left(0,\Omega_{NS}(z)\right),\tag{A.16}$$

with $\Omega_{NS}(z)=\frac{\nu_0(K)\sigma_u^2}{f(z)}\left(\int B_xB_x'\right)^{-1}$, as given in the stated result (2.8) for case (i). ■

**Case (ii)**

When $nh^{2L} \to \infty$ the bandwidth goes to zero slower than $O(\frac{1}{\sqrt{n^{1/2L}}})$. To derive the limit theory in this case, rescale (A.13) by $\sqrt{n/h^{2L-1}}$, giving

$$\begin{aligned}
&\sqrt{\frac{n}{h^{2L-1}}}\left(\hat{\beta}(z)-\beta(z)-h^{L^*}\left(\frac{1}{n^2h}\sum_{t=1}^n x_tx_t'K_{tz}\right)^{-1}\frac{1}{n^2h^{L^*+1}}\sum_{t=1}^n x_tx_t'\mathbb{E}\xi_{\beta t}\right)\\
&=\left(\frac{1}{n^2h}\sum_{t=1}^n x_tx_t'K_{tz}\right)^{-1}\frac{1}{\sqrt{n^3h^{2L+1}}}\sum_{t=1}^n x_tx_t'\eta_t+\frac{1}{\sqrt{nh^{2L}}}\left(\frac{1}{n^2h}\sum_{t=1}^n x_tx_t'K_{tz}\right)^{-1}\frac{1}{n\sqrt{h}}\sum_{t=1}^n x_tu_tK_{tz}\\
&=\left(\frac{1}{n^2h}\sum_{t=1}^n x_tx_t'K_{tz}\right)^{-1}\frac{1}{\sqrt{n^3h^{2L+1}}}\sum_{t=1}^n x_tx_t'\eta_t+o_p(1)\\
&\rightsquigarrow\left(f(z)\int B_xB_x'\right)^{-1}\left(\int B_xB_x'dB_{\eta,L}\right) &\text{(A.17)}\\
&\equiv_d \mathcal{MN}\left(0,\frac{\nu_{2L}(K)}{f(z)(L!)^2}\left(\int B_xB_x'\right)^{-1}\int B_xB_x'\left(B_x'\beta^{(L)}(z)\right)^2\left(\int B_xB_x'\right)^{-1}\right), &\text{(A.18)}
\end{aligned}$$

using Lemma B.3(a) and (A.3), where $B_{\eta,L}$ is Brownian motion with variance matrix $V_{\eta\eta,L} = \frac{\nu_{2L}(K)f(z)}{(L!)^2}\beta^{(L)}(z)\beta^{(L)}(z)'$.

Since $B_{\eta,L}$ is singular Brownian motion whenever $p > 1$ we may write the inner product $B_x(r)'B_{\eta,L}(r)$ in the equivalent form $B_x(r)'B_{\eta,L}(r) = \left(B_x(r)'\beta^{(L)}(z)\right)B_{f,L}(r)$, where $B_{f,L}$ is scalar Brownian motion with variance $\frac{\nu_{2L}(K)f(z)}{(L!)^2}$. Then in view of the independence of $B_x$ and $B_{\eta,L}$ we have

$$\int B_xB_x'dB_{\eta,L} \equiv_d \mathcal{MN}\left(0,\frac{\nu_{2L}(K)f(z)}{(L!)^2}\int B_xB_x'\left(B_x'\beta^{(L)}(z)\right)^2\right), \tag{A.19}$$

which leads to the mixed normal limit distribution given in (A.18). Combining this result with the bias function evaluation obtained earlier in (A.15) yields

$$\sqrt{\frac{n}{h^{2L-1}}}\left(\hat{\beta}(z)-\beta(z)-h^{L^*}\mathcal{B}_L(z)\right)\rightsquigarrow\left(f(z)\int B_xB_x'\right)^{-1}\left(\int B_xB_x'dB_{\eta,L}\right)\equiv_d \mathcal{MN}\left(0,\Omega_L(z)\right), \tag{A.20}$$

where $\Omega_L(z) = \frac{\nu_{2L}(K)f(z)}{(L!)^2}\left(\int B_xB_x'\right)^{-1}\int B_xB_x'\left(B_x'\beta^{(L)}(z)\right)^2\left(\int B_xB_x'\right)^{-1}$, giving the stated result (ii) of Theorem 2.2. ■

**Case (iii)**

Since $nh^{2L} \to c$ for some constant $c \in (0,\infty)$, $h \sim_a (c/n)^{\frac{1}{2L}}$ and then $\sqrt{n/h^{2L-1}} = O(\sqrt{n^{1+\frac{2L-1}{2L}}}) = O(n^{1-\frac{1}{4L}}) = n\sqrt{h}$. It follows that the first and second terms on the right side of (A.13) have the same order and both therefore appear to contribute to the asymptotics. So, upon rescaling

(A.13) by $n^{1-\frac{1}{4L}}$ we find that

$$\begin{aligned}
&n^{1-\frac{1}{4L}}\left(\hat{\beta}(z)-\beta(z)-h^{L^*}\left(\frac{1}{n^2h}\sum_{t=1}^n x_tx_t'K_{tz}\right)^{-1}\frac{1}{n^2h^{L^*+1}}\sum_{t=1}^n x_tx_t'\mathbb{E}\xi_{\beta t}\right)\\
&=(nh^{2L})^{\frac{1}{2}-\frac{1}{4L}}\left(\frac{1}{n^2h}\sum_{t=1}^n x_tx_t'K_{tz}\right)^{-1}\frac{1}{\sqrt{n^3h^{2L+1}}}\sum_{t=1}^n x_tx_t'\eta_t+\frac{1}{(nh^{2L})^{\frac{1}{4L}}}\left(\frac{1}{n^2h}\sum_{t=1}^n x_tx_t'K_{tz}\right)^{-1}\frac{1}{n\sqrt{h}}\sum_{t=1}^n x_tu_tK_{tz},\\
&=c^{\frac{1}{2}-\frac{1}{4L}}\times\left(\frac{1}{n^2h}\sum_{t=1}^n x_tx_t'K_{tz}\right)^{-1}\frac{1}{\sqrt{n^3h^{2L+1}}}\sum_{t=1}^n x_tx_t'\eta_t+c^{-\frac{1}{4L}}\left(\frac{1}{n^2h}\sum_{t=1}^n x_tx_t'K_{tz}\right)^{-1}\frac{1}{n\sqrt{h}}\sum_{t=1}^n x_tu_tK_{tz}.
\end{aligned}\tag{A.21}$$

The asympototics are then jointly determined by the two terms of (A.21). Conditional on $\mathcal{F}_x$, these terms are uncorrelated as the conditional covariance involves the matrix

$$\mathbb{E}\left(\frac{1}{\sqrt{n^3h^{2L+1}}}\sum_{t=1}^n x_tx_t'\eta_t\right)\left(\frac{1}{n\sqrt{h}}\sum_{t=1}^n x_tu_tK_{tz}\right)'=\frac{1}{\sqrt{n^5h^{L+1}}}\sum_{t,s=1}^n\mathbb{E}\left(x_tx_s'(x_t'\eta_tu_sK_{sz})\right)=0.\tag{A.22}$$

From (A.21) and the bias function calculation (A.15) which continues to hold, it follows that when $nh^{2L}\to c>0$

$$\begin{aligned}
&n^{1-\frac{1}{4L}}\left(\hat{\beta}(z)-\beta(z)-h^{L^*}\mathcal{B}_L(z)\right)\\
&\sim_a c^{\frac{1}{2}-\frac{1}{4L}}\times\left(f(z)\int B_xB_x'\right)^{-1}\left(\int B_xB_x'dB_{\eta,L}\right)+c^{-\frac{1}{4L}}\left(f(z)\int B_xB_x'\right)^{-1}\left(\int B_xdB_{uK}\right)\\
&\equiv_d c^{\frac{1}{2}-\frac{1}{4L}}\times\mathcal{MN}\left(0,\Omega_L(z)\right)+c^{-\frac{1}{4L}}\mathcal{MN}\left(0,\Omega_{NS}(z)\right)\\
&=\mathcal{MN}\left(0,c^{1-\frac{1}{2L}}\Omega_L(z)+c^{-\frac{1}{2L}}\Omega_{NS}(z)\right).
\end{aligned}\tag{A.23}$$

This proves result (iii) of Theorem 2.2. ■

**Proof of Theorem 3.1**

**(i) Stationary** $x_t$ We assume $L$ is known. Using Lemma B.1, $\hat{\sigma}_u^2\to_p\sigma_u^2$, and any consistent derivative estimator $\hat{\beta}^{(L)}(z)$ of $\beta^{(L)}(z)$, we have

$$\begin{aligned}
&\hat{\Omega}_n(z;L)=\nu_0(K)\hat{\sigma}_u^2\sum_{t=1}^n x_tx_t'K_{tz}+\sum_{t=1}^n x_tx_t'\left\{x_t'\frac{1}{L!}\hat{\beta}^{(L)}(z)(z_t-z)^LK_{tz}\right\}^2\\
&\sim_a nh\left(\frac{\nu_0(K)\sigma_u^2}{nh}\sum_{t=1}^n x_tx_t'K_{tz}+\frac{1}{nh}\sum_{t=1}^n x_tx_t'\left\{x_t'\frac{\beta^{(L)}(z)}{L!}(z_t-z)^LK_{tz}\right\}^2\right)
\end{aligned}$$

$$
\begin{aligned}
&\sim_a nh\left(\nu_0(K)\sigma_u^2 f(z)\,\Sigma_{xx}+\frac{1}{h}\mathbb{E}\left[x_t x_t'\left\{x_t'\frac{\beta^{(L)}(z)}{L!}(z_t-z)^L K_{tz}\right\}^2\right]\right)\\
&\sim_a nh\left(\nu_0(K)\sigma_u^2 f(z)\,\Sigma_{xx}+\frac{1}{h}\mathbb{E}\left[x_t x_t'\left(x_t'\frac{\beta^{(L)}(z)}{L!}\right)^2\right]\int\left\{(z_t-z)^L K\left(\frac{z_t-z}{h}\right)\right\}^2 f(z_t)\,dz_t\right)\\
&\sim_a nh\left(\nu_0(K)\sigma_u^2 f(z)\,\Sigma_{xx}+\mathbb{E}\left[x_t x_t'\left(x_t'\frac{\beta^{(L)}(z)}{L!}\right)^2\right]\int\left\{(sh)^L K(s)\right\}^2 f(z+sh)\,ds\right)\\
&\sim_a nh\left(\nu_0(K)\sigma_u^2 f(z)\,\Sigma_{xx}+h^{2L}\frac{\nu_{2L}(K)\,f(z)}{(L!)^2}\mathbb{E}\left[x_t x_t'\left(x_t'\beta^{(L)}(z)\right)^2\right]\right)\\
&\sim_a nh\nu_0(K)\sigma_u^2 f(z)\,\Sigma_{xx}. \qquad (A.24)
\end{aligned}
$$

Then

$$
\begin{aligned}
nh\hat{V}_n(z;L)&=\left[\frac{1}{nh}A_n(z)\right]^{-1}\left[\frac{1}{nh}\hat{\Omega}_n(z;L)\right]\left[\frac{1}{nh}A_n(z)\right]^{-1}\\
&\to_p [f(z)\,\Sigma_{xx}]^{-1}\left[\nu_0(K)\sigma_u^2 f(z)\,\Sigma_{xx}\right][f(z)\,\Sigma_{xx}]^{-1}=\frac{\nu_0(K)\sigma_u^2}{f(z)}\Sigma_{xx}^{-1}=\Omega_S(z). \qquad (A.25)
\end{aligned}
$$

Combining (A.25) and Theorem 2.1 gives

$$
\begin{aligned}
\hat{T}(z;L)&=\hat{V}_n(z;L)^{-1/2}\left(\hat{\beta}(z)-\beta(z)-h^{L^*}\mathcal{B}_L(z)\right)\\
&\sim_a [\Omega_S(z)]^{-1/2}\sqrt{nh}\left(\hat{\beta}(z)-\beta(z)-h^{L^*}\mathcal{B}_L(z)\right)\rightsquigarrow\mathcal{N}(0,I_p),
\end{aligned}
$$

and $\hat{T}_2(z;L)\rightsquigarrow\chi_p^2$ follows. Further, in view of (A.24) in the stationary case, the simpler estimate $\tilde{\Omega}_n(z;L)=\nu_0(K)\hat{\sigma}_u^2\sum_{t=1}^n x_t x_t' K_{tz}$, which is based solely on the variance term, can be employed and the same limit theory applies.

**(ii) Nonstationary** $x_t$ We again assume that $L$ is known. We analyze each case of the Theorem in turn.

**Case (a)** Using Lemma B.3(c) we have $\frac{1}{n^2h}\sum_{t=1}^n x_t x_t' K_{tz}\rightsquigarrow f(z)\int B_x B_x'$. In place of (A.24) and again using a consistent derivative estimator $\hat{\beta}^{(L)}(z)\to_p\beta^{(L)}(z)$ and $\hat{\sigma}_u^2\to_p\sigma_u^2$, we now have

$$
\begin{aligned}
\hat{\Omega}_n(z;L)&=\nu_0(K)\hat{\sigma}_u^2\sum_{t=1}^n x_t x_t' K_{tz}+\sum_{t=1}^n x_t x_t'\left\{x_t'\frac{1}{L!}\hat{\beta}^{(L)}(z)(z_t-z)^L K_{tz}\right\}^2\\
&\sim_a n^2 h\left(\frac{\nu_0(K)\sigma_u^2}{n^2h}\sum_{t=1}^n x_t x_t' K\left(\frac{z_t-z}{h}\right)+\frac{h^{2L}}{n^2h}\sum_{t=1}^n x_t x_t'\left\{x_t'\frac{\beta^{(L)}(z)}{L!}\left(\frac{z_t-z}{h}\right)^L K\left(\frac{z_t-z}{h}\right)\right\}^2\right)
\end{aligned}
$$

$$\sim_a n^2h\left(\nu_0(K)\sigma_u^2 f(z)\int B_xB_x' + nh^{2L-1}\left[\frac{1}{n}\sum_{t=1}^{n}\frac{x_t}{\sqrt{n}}\frac{x_t'}{\sqrt{n}}\left\{\frac{x_t'}{\sqrt{n}}\frac{\beta^{(L)}(z)}{L!}\left(\frac{z_t-z}{h}\right)^L K\left(\frac{z_t-z}{h}\right)\right\}^2\right]\right)$$

$$\sim_a n^2h\left(\nu_0(K)\sigma_u^2 f(z)\int B_xB_x' + nh^{2L-1}\left[\int B_xB_x\left\{B_x'\frac{\beta^{(L)}(z)}{L!}\right\}^2\mathbb{E}\left\{\left(\frac{z_t-z}{h}\right)^{2L}K\left(\frac{z_t-z}{h}\right)^2\right\}\right]\right)$$

$$\sim_a n^2h\left(\nu_0(K)\sigma_u^2 f(z)\int B_xB_x' + nh^{2L-1}\left[\int B_xB_x\left\{B_x'\frac{\beta^{(L)}(z)}{L!}\right\}^2\int p^{2L}K(p)^2 f(z+ph)\,dph\right]\right)$$

$$\sim_a n^2h\left(\nu_0(K)\sigma_u^2 f(z)\int B_xB_x' + nh^{2L}\frac{\nu_{2L}(K)f(z)}{(L!)^2}\int B_xB_x\left\{B_x'\beta^{(L)}(z)\right\}^2\right) \tag{A.26}$$

$$\sim_a n^2h\nu_0(K)\sigma_u^2 f(z)\int B_xB_x' \quad \text{when } nh^{2L}\to 0. \tag{A.27}$$

It follows that in this case

$$\begin{aligned}
n^2h\hat{V}_n(z;L) &= \left[\frac{1}{n^2h}A_n(z)\right]^{-1}\left[\frac{1}{n^2h}\hat{\Omega}_n(z;L)\right]\left[\frac{1}{n^2h}A_n(z)\right]^{-1}\\
&\rightsquigarrow \left[f(z)\int B_xB_x'\right]^{-1}\left[\nu_0(K)\sigma_u^2 f(z)\int B_xB_x'\right]\left[f(z)\int B_xB_x'\right]^{-1}\\
&= \frac{\nu_0(K)\sigma_u^2}{f(z)}\left(\int B_xB_x'\right)^{-1} = \Omega_{NS}(z),
\end{aligned} \tag{A.28}$$

from which we deduce from Theorem 2.2(i) that

$$\begin{aligned}
\hat{T}(z;L) &= \hat{V}_n(z;L)^{-1/2}\left(\hat{\beta}(z)-\beta(z)-h^{L^*}\hat{\mathcal{B}}_L(z)\right)\\
&\sim_a \left[\Omega_{NS}(z)\right]^{-1/2} n\sqrt{h}\left(\hat{\beta}(z)-\beta(z)-h^{L^*}\mathcal{B}_L(z)\right) \rightsquigarrow \mathcal{N}(0,I_p).
\end{aligned}$$

Then $\hat{T}_2(z;L)\rightsquigarrow\chi_p^2$, as required.

**Case (b)** When $nh^{2L}\to\infty$ we have by calculations similar to those leading to (A.26)

$$\begin{aligned}
\hat{\Omega}_n(z;L) &= \nu_0(K)\hat{\sigma}_u^2\sum_{t=1}^{n}x_tx_t'K_{tz} + \sum_{t=1}^{n}x_tx_t'\left\{x_t'\frac{1}{L!}\hat{\beta}^{(L)}(z)(z_t-z)^LK_{tz}\right\}^2\\
&\sim_a n^2h\left(\nu_0(K)\sigma_u^2 f(z)\int B_xB_x' + nh^{2L}\frac{\nu_{2L}(K)f(z)}{(L!)^2}\int B_xB_x\left\{B_x'\beta^{(L)}(z)\right\}^2\right)\\
&= n^3h^{2L+1}\left(\frac{1}{nh^{2L}}\nu_0(K)\sigma_u^2 f(z)\int B_xB_x' + \frac{\nu_{2L}(K)f(z)}{(L!)^2}\int B_xB_x\left\{B_x'\beta^{(L)}(z)\right\}^2\right)\\
&\sim_a n^3h^{2L+1}\frac{\nu_{2L}(K)f(z)}{(L!)^2}\int B_xB_x\left\{B_x'\beta^{(L)}(z)\right\}^2.
\end{aligned}$$

Hence, in this case

$$\frac{n}{h^{2L-1}}\hat{V}_n(z;L) = \left[\frac{1}{n^2h}A_n(z)\right]^{-1}\left[\frac{1}{n^3h^{2L+1}}\hat{\Omega}_n(z;L)\right]\left[\frac{1}{n^2h}A_n(z)\right]^{-1}$$
$$\sim_a \left[f(z)\int B_xB_x'\right]^{-1}\left[\frac{v_{2L}(K)f(z)}{(L!)^2}\int B_xB_x\left\{B_x'\beta^{(L)}(z)\right\}^2\right]\left[f(z)\int B_xB_x'\right]^{-1}$$
$$= \frac{\nu_{2L}(K)}{(L!)^2 f(z)}\left(\int B_xB_x'\right)^{-1}\left(\int B_xB_x\left\{B_x'\beta^{(L)}(z)\right\}^2\right)\left(\int B_xB_x'\right)^{-1} = \Omega_L(z), \qquad \text{(A.29)}$$

and from Theorem 2.2(ii) we deduce that

$$\hat{T}(z;L) = \hat{V}_n(z;L)^{-1/2}\left(\hat{\beta}(z) - \beta(z) - h^{L^*}\hat{\mathcal{B}}_L(z)\right)$$
$$= \left[\frac{n}{h^{2L-1}}\hat{V}_n(z;L)\right]^{-1/2}\sqrt{\frac{n}{h^{2L-1}}}\left(\hat{\beta}(z) - \beta(z) - h^{L^*}\hat{\mathcal{B}}_L(z;L)\right)$$
$$\sim_a [\Omega_L(z)]^{-1/2}\sqrt{\frac{n}{h^{2L-1}}}\left(\hat{\beta}(z) - \beta(z) - h^{L^*}\mathcal{B}_L(z)\right) \rightsquigarrow \mathcal{N}(0, I_p),$$

and $\hat{T}_2(z;L) \rightsquigarrow \chi^2_p$, as required.

**Case (c)** When $nh^{2L} \to c \in (0,\infty)$ we have by calculations similar to those leading to (A.26)

$$\hat{\Omega}_n(z;L) = \nu_0(K)\hat{\sigma}_u^2\sum_{t=1}^n x_tx_t'K_{tz} + \sum_{t=1}^n x_tx_t'\left\{x_t'\frac{1}{L!}\hat{\beta}^{(L)}(z)(z_t - z)^L K_{tz}\right\}^2$$
$$\sim_a n^2h\left(\nu_0(K)\sigma_u^2 f(z)\int B_xB_x' + nh^{2L}\frac{\nu_{2L}(K)f(z)}{(L!)^2}\int B_xB_x\left\{B_x'\beta^{(L)}(z)\right\}^2\right)$$
$$\sim_a n^2h\left(\nu_0(K)\sigma_u^2 f(z)\int B_xB_x' + c\frac{\nu_{2L}(K)f(z)}{(L!)^2}\int B_xB_x\left\{B_x'\beta^{(L)}(z)\right\}^2\right).$$

Then

$$n^2h\hat{V}_n(z;L) = \left[\frac{1}{n^2h}A_n(z)\right]^{-1}\left[\frac{1}{n^2h}\hat{\Omega}_n(z;L)\right]\left[\frac{1}{n^2h}A_n(z)\right]^{-1}$$
$$\sim_a \left[f(z)\int B_xB_x'\right]^{-1}\left[\nu_0(K)\sigma_u^2 f(z)\int B_xB_x' + c\frac{\nu_{2L}(K)f(z)}{(L!)^2}\int B_xB_x\left\{B_x'\beta^{(L)}(z)\right\}^2\right]\left[f(z)\int B_xB_x'\right]^{-1}$$
$$= \Omega_{NS}(z) + c\Omega_L(z). \qquad \text{(A.30)}$$

Hence, when $nh^{2L} \to c$ or $h \sim_a (c/n)^{\frac{1}{2L}}$ and $n^2h \sim_a c^{\frac{1}{2L}}n^{2-\frac{1}{2L}}$ as $n \to \infty$ we have

$$n^{2-\frac{1}{2L}}\hat{V}_n(z;L) \rightsquigarrow c^{-\frac{1}{2L}}\Omega_{NS}(z) + c^{1-\frac{1}{2L}}\Omega_L(z).$$

From Theorem 2.2(iii) it now follows that when $nh^{2L} \to c$ we have

$$\hat{T}(z;L) = \hat{V}_n(z;L)^{-1/2}\left(\hat{\beta}(z) - \beta(z) - h^{L^*}\hat{\mathcal{B}}_L(z)\right)$$
$$= \left[n^{2-\frac{1}{2L}}\hat{V}_n(z;L)\right]^{-1/2} n^{1-\frac{1}{4L}}\left(\hat{\beta}(z) - \beta(z) - h^{L^*}\hat{\mathcal{B}}_L(z)\right)$$
$$\sim_a [c^{-\frac{1}{2L}}\Omega_{NS}(z) + c^{1-\frac{1}{2L}}\Omega_L(z)]^{-1/2} n^{1-\frac{1}{4L}}\left(\hat{\beta}(z) - \beta(z) - h^{L^*}\mathcal{B}_L(z)\right) \rightsquigarrow \mathcal{N}(0, I_p),$$

and again $\hat{T}_2(z;L) \rightsquigarrow \chi^2_p$, as required. ■

# B Useful Lemmas

**Lemma B.1.** *Under Assumption 1, the following hold as $n \to \infty$:*

*(a) (i) If $nh \to \infty$, $\{\frac{1}{\sqrt{nh}}\sum_{t=1}^{\lfloor n\cdot\rfloor}\zeta_{tK}, \frac{1}{\sqrt{nh}}\sum_{t=1}^{\lfloor n\cdot\rfloor}u_t K_{tz}\} \rightsquigarrow \{B_{\zeta K}(\cdot), B_{uK}(\cdot)\}$, where $\{B_{\zeta K}, B_{uK}\}$ are independent Brownian motions with respective variances $\nu_0(K)f(z)$, and $\nu_0(K)\sigma_u^2 f(z)$, with $\zeta_{tK} = K_{tz} - \mathbb{E}K_{tz}$ and $K_{tz} = K(\frac{z_t - z}{h})$;*
*(ii) If $nh \to c \in [0,\infty)$, then $\{\frac{1}{\sqrt{nh}}\sum_{t=1}^{\lfloor n\cdot\rfloor}\zeta_{tK}, \frac{1}{\sqrt{nh}}\sum_{t=1}^{\lfloor n\cdot\rfloor}u_t K_{tz}\} = O_p(1)$ but no invariance principle holds.*

*(b) (i) If $nh \to \infty$, $\frac{1}{nh}\sum_{t=1}^n x_t x_t' K_{tz} \to_p \Sigma_{xx} f(z)$;*
*(ii) If $nh \to c \in [0,\infty)$, $\frac{1}{\sqrt{nh}}\sum_{t=1}^n x_t x_t' K_{tz} = O_p(1)$ but no invariance principle holds.*

*(c) (i) If $nh \to \infty$, $\frac{1}{\sqrt{nh}}\sum_{t=1}^n x_t u_t K_{tz} \rightsquigarrow \mathcal{N}\left(0, \nu_0(K)\sigma_u^2 f(z)\Sigma_{xx}\right)$;*
*(ii) If $nh \to c \in [0,\infty)$, $\frac{1}{\sqrt{nh}}\sum_{t=1}^n x_t u_t K_{tz} = O_p(1)$ but no invariance principle holds.*

*(d) (i) If $nh \to \infty$, $\frac{1}{\sqrt{nh}}\sum_{t=1}^n x_t \otimes x_t \zeta_{tK} \rightsquigarrow \mathcal{N}(0, \nu_0(K)f(z)\mathbb{E}[x_t x_t' \otimes x_t x_t'])$;*
*(ii) If $nh \to c \in [0,\infty)$, $\frac{1}{\sqrt{nh}}\sum_{t=1}^n x_t x_t' \zeta_{tK} = O_p(1)$ but no invariance principle holds.*

**Proof of Lemma B.1**

**Part (a) (i) and (ii)** See Lemma B.1(a) of Phillips and Wang (2020). For later use, note that $\mathbb{E}K_{tz} = hf(z) + o(h)$, and $\mathbb{E}K_{tz}^2 = hf(z)\nu_0(K) + o(h)$, so that $\mathbb{V}ar(\zeta_{tK}) = hf(z)\nu_0(K) + o(h)$ and $\zeta_{tK} = K_{tz} - \mathbb{E}(K_{tz}) = O_p(\sqrt{h})$. Further, $\mathbb{V}ar(u_t K_{tz}) = h\nu_0(K)\sigma_u^2 f(z) + o(h)$ and $\mathbb{E}u_t K_{tz}^2 = 0$, so that the limit processes $(B_{\zeta K}(r), B_{uK}(r))$ are independent. The functional laws follow by standard weak convergence methods when $nh \to \infty$ and (ii) follows by showing the $O_p(1)$ property directly whereas the CLT does not hold because of failure of the Lindeberg condition, just as in the proof of Phillips and Wang (2020, Lemma B.1(a)(ii)). ■

**Part (b)** (i) We have

$$\frac{1}{nh}\sum_{t=1}^n x_t x_t' K_{tz} = \frac{1}{n}\sum_{t=1}^n x_t x_t' \frac{\mathbb{E}K_{tz}}{h} + \frac{1}{nh}\sum_{t=1}^n x_t x_t' \zeta_{tK}$$
$$= \frac{1}{n}\sum_{t=1}^n x_t x_t'\{f(z) + o(1)\} + O_p\left(\frac{1}{\sqrt{nh}}\right) \to_p \Sigma_{xx} f(z), \tag{B.1}$$

as in (A.3) by virtue of Lemma B.1(d)(i) and the law of large numbers. For Part (ii), when $nh \to c \in [0, \infty)$ we have $\frac{1}{\sqrt{nh}} \sum_{t=1}^{n} x_t x_t' \zeta_{tK} = O_p(1)$ as in Lemma B.1(d)(ii) with no invariance principle holding. Then, as in (B.1),

$$\frac{1}{\sqrt{nh}} \sum_{t=1}^{n} x_t x_t' K_{tz} = \sqrt{nh} \frac{1}{n} \sum_{t=1}^{n} x_t x_t' \{f(z) + o(1)\} + \frac{1}{\sqrt{nh}} \sum_{t=1}^{n} x_t x_t' \zeta_{tK} = O_p(1), \tag{B.2}$$

as stated. ■

**Part (c)** Result (i) follows by the martingale central limit theorem. Stability holds because the martingale conditional variance matrix is

$$\left\langle \frac{1}{\sqrt{nh}} \sum_{t=1}^{n} x_t u_t K_{tz} \right\rangle = \sigma_u^2 f(z) \nu_0(K) \frac{1}{nh} \sum_{t=1}^{n} x_t x_t' \to_p \sigma_u^2 \nu_0(K) f(z) \Sigma_{xx}. \tag{B.3}$$

Setting $w_{tK} = x_t u_t K_{tz}$ and noting that $\mathbb{E}\left\| \frac{w_{tK}}{\sqrt{h}} \right\|^2 < \infty$ we have, given $\epsilon > 0$,

$$\frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left\{ \left\| \frac{w_{tK}}{\sqrt{h}} \right\|^2 1_{[\|w_{tK}\| > \epsilon\sqrt{nh}]} \right\} \to 0 \tag{B.4}$$

and the Lindeberg condition holds when $nh \to \infty$, giving the stated result. Part (ii) follows because, although the stability condition continues to hold as in (B.3), the Lindeberg condition fails when $nh \to c \in [0, \infty)$ as (B.4) no longer tends to zero. In the present case, with scalar $x_t$, *iid* $\{(u_t, z_t)\}$ and independent, strictly stationary components with respective densities $\{f_x(x), f_u(u), f(s)\}$ we have, given $\epsilon > 0$ and $nh \to c \in [0, \infty)$

$$\frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left\{ \left( \frac{w_{tK}}{\sqrt{h}} \right)^2 1_{[|w_{tK}| > \epsilon\sqrt{nh}]} \right\} = \frac{1}{h} \int x^2 u^2 K\left( \frac{s-z}{h} \right)^2 f_x(x) f_u(u) f(s) 1_{[|xuK(\frac{s-z}{h})| > \epsilon\sqrt{nh}]} dxduds$$

$$= \int x^2 u^2 K(p)^2 f_x(x) f_u(u) f(z+ph) 1_{[|xuK(p)| > \epsilon\sqrt{nh}]} dxdudp$$

$$\to \begin{cases} \nu_0(K) f(z) \mathbb{E}(x_t^2 u_t^2) > 0 & \text{if } nh \to 0 \\ \int x^2 u^2 K(p)^2 f_x(x) f_u(u) f(z) 1_{[|xuK(p)| > \epsilon\sqrt{c}]} dxdudp > 0 & \text{if } nh \to c \in (0, \infty) \end{cases},$$

leading to failure in the Lindeberg condition. ■

**Part (d)** Parts (i) and (ii) follow in the same way as Parts (c)(i) and (ii). ■

**Lemma B.2.** *Under Assumption 1, if $\beta^{(\ell)}(z) = 0$ for $\ell = 0, 1, \cdots, L-1$ and $\beta^{(L)}(z) \neq 0$, then the following hold as $n \to \infty$ and $h \to 0$:*

*(a) (i) If $nh^{2L+1} \to \infty$, $\frac{1}{\sqrt{nh^{2L+1}}} \sum_{t=1}^{\lfloor n\cdot \rfloor} \eta_t \rightsquigarrow B_{\eta,L}(\cdot)$, where $B_{\eta,L}(\cdot)$ is Brownian motion with variance matrix $V_{\eta\eta,L} = \frac{\nu_{2L}(K) f(z)}{(L!)^2} \beta^{(L)}(z) \beta^{(L)}(z)'$, with $\eta_t = \xi_{\beta t} - \mathbb{E}\xi_{\beta t}$ and $\xi_{\beta t} = [\beta(z_t) -$*

$\beta(z)]K_{tz}$;

*(ii) If* $nh \to c \in [0,\infty)$, *then* $\frac{1}{\sqrt{nh^{2L+1}}}\sum_{t=1}^{\lfloor n\cdot\rfloor}\eta_t = O_p(1)$, *but no invariance principle holds.*

*(b)* *(i) If* $nh^{2L+1}\to\infty$, $\frac{1}{\sqrt{nh^{2L+1}}}\sum_{t=1}^{n}x_tx_t'\eta_t \rightsquigarrow \mathcal{N}\left(0,\frac{\nu_{2L}(K)f(z)}{(L!)^2}\mathbb{E}\left[(x_t'\beta^{(L)}(z))^2x_tx_t'\right]\right)$;

*(ii) If* $nh \to c\in[0,\infty)$, $\frac{1}{\sqrt{nh^{2L+1}}}\sum_{t=1}^{n}x_tx_t'\eta_t = O_p(1)$, *but no invariance principle holds.*

*(c)* $\frac{1}{nh^{L^*+1}}\sum_{t=1}^{n}x_tx_t'\mathbb{E}\xi_{\beta t}\to_p \mathcal{G}_L(z)\Sigma_{xx}$, *where* $L^* = L\times\mathbf{1}_{\{L=even\}}+(L+1)\,\mathbf{1}_{\{L=odd\}}$, $\mathcal{G}_L(z) = \mu_{L^*}(K)C_L(z)$ *with* $\mu_{L^*}(K) = \mu_L(K)\mathbf{1}_{\{L=even\}}+\mu_{L+1}(K)\mathbf{1}_{\{L=odd\}}$, *and*

$$C_L(z) = \frac{f(z)\beta^{(L)}(z)}{L!}\mathbf{1}_{\{L=even\}} + \left[\frac{\beta^{(L)}(z)}{L!}f^{(1)}(z)+\frac{\beta^{(L+1)}(z)}{(L+1)!}f(z)\right]\mathbf{1}_{\{L=odd\}}. \tag{B.5}$$

**Proof of Lemma B.2**

**Part (a) (i)** Phillips and Wang (2020) proved a related result when $L = 1$ in their Lemma B.1(b). A similar argument is employed here. But for general $L$ we need to compute the first and second moments of $\eta_t = \xi_{\beta t} - \mathbb{E}\xi_{\beta t}$ and deal with the precise form of the local behavior of the coefficient function $\beta(\cdot)$ in the neighborhood of the point of estimation $z$. To this end, the proof uses the following Taylor representations

$$\begin{aligned}\beta(z+ph)-\beta(z) =& \beta^{(L)}(z)\frac{p^Lh^L}{L!}+\beta^{(L+1)}(\tilde{z}_p)\frac{p^{L+1}h^{L+1}}{(L+1)!}\\ f(z+ph) =& f(z)+f^{(1)}(\breve{z}_p)\,ph,\end{aligned}$$

where $\left\{\beta^{(j)}(z)=0; j=1,...,L-1\right\}$ and with $\tilde{z}_p$ and $\breve{z}_p$ on the line segment connecting $z_t$ and $z$. The first and second moments of $\eta_t$ may now be deduced. Specifically, using the symmetry of $K$, we have

$$\begin{aligned}&\mathbb{E}\xi_{\beta t} = \mathbb{E}[\beta(z_t)-\beta(z)]K_{tz} = \int_{-1}^{1}[\beta(s)-\beta(z)]K((s-z)/h)f(s)ds\\ &= h\int_{-1}^{1}[\beta(z+ph)-\beta(z)]K(p)f(z+ph)dp\\ &= h\int_{-1}^{1}\left[\frac{h^Lp^L}{L!}\beta^{(L)}(z)+\frac{h^{L+1}p^{L+1}}{(L+1)!}\beta^{(L+1)}(\tilde{z}_p)\right]K(p)\left[f(z)+f^{(1)}(\breve{z}_p)\,ph\right]dp\\ &= \left\{h\int_{-1}^{1}\frac{h^L}{L!}\beta^{(L)}(z)f(z)\,p^LK(p)dp+o(h^{L+1})\right\}\times\mathbf{1}_{\{L=\text{even}\}}\\ &+\left\{h\int_{-1}^{1}\left[\frac{h^{L+1}}{L!}\beta^{(L)}(z)f^{(1)}(\breve{z}_p)+\frac{h^{L+1}}{(L+1)!}\beta^{(L+1)}(\tilde{z}_p)f(z)\right]p^{L+1}K(p)dp+o(h^{L+2})\right\}\times\mathbf{1}_{\{L=\text{odd}\}}\\ &= \left\{h^{L+1}\mu_L(K)\frac{\beta^{(L)}(z)}{L!}f(z)+o(h^{L+1})\right\}\times\mathbf{1}_{\{L=\text{even}\}}\end{aligned}$$

$$+\left\{h^{L+2}\mu_{L+1}(K)\left[\frac{\beta^{(L)}(z)}{L!}f^{(1)}(z)+\frac{\beta^{(L+1)}(z)}{(L+1)!}f(z)+o(1)\right]+o(h^{L+2})\right\}\times\mathbf{1}_{\{L=\text{odd}\}}$$
$$=h^{L^*+1}\mu_{L^*}(K)C_L(z)+o(h^{L^*+1})=:h^{L^*+1}\mathcal{G}_L(z)+o(h^{L^*+1}),\tag{B.6}$$

where $L^*$, $\mu_{L^*}(K)$, $\mathcal{G}_L(z)=\mu_{L^*}(K)C_L(z)$, and $C_L(z)$ are defined in the statement of the Lemma. Next

$$\begin{aligned}\mathbb{E}\xi_{\beta t}\xi_{\beta t}' &= \mathbb{E}[(\beta(z_t)-\beta(z))(\beta(z_t)-\beta(z))'K((z_t-z)/h)^2]\\
&=h\int_{-1}^{1}(\beta(z+hs)-\beta(z))(\beta(z+hs)-\beta(z))'K(s)^2f(z+hs)ds\\
&=h\int_{-1}^{1}\left[\frac{h^Ls^L}{L!}\beta^{(L)}(z)+o\left(h^L\right)\right]\left[\frac{h^Ls^L}{L!}\beta^{(L)}(z)+o\left(h^L\right)\right]'K(s)^2\left[f(z)+o(1)\right]ds\\
&=\frac{h^{2L+1}}{(L!)^2}f(z)\,\beta^{(L)}(z)\beta^{(L)}(z)'\int_{-1}^{1}s^{2L}K^2(s)ds+o(h^{2L+1})\\
&=\frac{h^{2L+1}}{(L!)^2}\nu_{2L}(K)f(z)\,\beta^{(L)}(z)\beta^{(L)}(z)'+o(h^{2L+1}).\end{aligned}$$

It follows that

$$\mathbb{V}ar(\eta_t)=\mathbb{E}\xi_{\beta t}\xi_{\beta t}'-(\mathbb{E}\xi_{\beta t})(\mathbb{E}\xi_{\beta t})'=\frac{h^{2L+1}}{(L!)^2}\nu_{2L}(K)f(z)\,\beta^{(L)}(z)\beta^{(L)}(z)'+o(h^{2L+1}),\tag{B.7}$$

and $\eta_t=O_p\left(h^{L+\frac{1}{2}}\right)$. Next, in view of (B.6) the serial covariances satisfy

$$\mathbb{C}ov(\xi_{\beta t},\xi_{\beta t+j})=\mathbb{E}\xi_{\beta t}\xi_{\beta t+j}'-(\mathbb{E}\xi_{\beta t})(\mathbb{E}\xi_{\beta t})'=\mathbb{E}\xi_{\beta t}\xi_{\beta t+j}'+O(h^{2L^*+2})$$

and by virtue of the strong mixing of $z_t$, measurability of $\beta(\cdot)$, and Davydov's lemma the covariances satisfy the bound

$$|\mathbb{C}ov(\xi_{\beta t},\xi_{\beta t+j})|\leq 8\left(\mathbb{E}\left|\xi_{\beta t}\right|^{\delta}\right)^{2/\delta}|\alpha(j)|^{1-2/\delta}=A_\beta h^{2L+2/\delta}|\alpha(j)|^{1-2/\delta}+o(h^{2L+2/\delta}),\tag{B.8}$$

where $A_\beta=8(\int\left|\frac{\beta^{(L)}(\tilde{z}_p)}{L!}\right|^{\delta}|p|^{\delta L}K(p)^{\delta}dpf(z))^{2/\delta}$, since $\mathbb{E}\left|\xi_{\beta t}\right|^{\delta}=h^{1+L\delta}\int\left|\frac{\beta^{(L)}(\tilde{z}_p)}{L!}\right|^{\delta}|p|^{\delta L}K(p)^{\delta}dpf(z)+o(h^{1+L\delta})$ in a similar way to (B.6), and where $\tilde{z}_p$ is on the line segment connecting $z$ and $z+hp$. Further, for $j\neq 0$ and using the joint density $f_{0,j}(s_0,s_j)$ of $(z_t,z_{t+j})$ we have

$$\begin{aligned}\mathbb{E}\xi_{\beta t}\xi_{\beta t+j}' &= \mathbb{E}[(\beta(z_t)-\beta(z))\left(\beta(z_{t+j})-\beta(z)\right)'K_{tz}K_{t+j,z}]\\
&=\int\int\left(\beta(s_0)-\beta(z)\right)\left(\beta(s_j)-\beta(z)\right)'K\left(\frac{s_0-z}{h}\right)K\left(\frac{s_j-z}{h}\right)f_{0,j}(s_1,s_j)ds_0ds_j\\
&=h^2\int\int(\beta(z+hp_0)-\beta(z))(\beta(z+hp_j)-\beta(z))'K(p_0)K(p_j)f_{0,j}(z+hp_0,z+hp_j)dp_0dp_j\end{aligned}$$

$$= \left\{ h^{2L+2} \frac{\beta^{(L)}(z)}{L!} \frac{\beta^{(L)}(z)'}{L!} f_{0,j}(z,z) \int p_1^L K(p_1) dp_1 \int p_2^L K(p_2) dp_2 + o(h^{2L+2}) \right\} \times \mathbf{1}_{\{L=\text{even}\}}$$
$$+ \left\{ h^{2L+4} \left[ \frac{\beta^{(L+1)}(z)}{(L+1)!} \frac{\beta^{(L+1)}(z)'}{(L+1)!} f_{0,j}(z,z) + \frac{\beta^{(L+1)}(z)}{(L+1)!} \frac{\beta^{(L)}(z)'}{L!} \frac{\partial f_{0,j}}{\partial s_j}(z,z) \right. \right.$$
$$\left. + \frac{\beta^{(L)}(z)}{L!} \frac{\beta^{(L+1)}(z)'}{(L+1)!} \frac{\partial f_{0,j}}{\partial s_0}(z,z) + \frac{\beta^{(L)}(z)}{L!} \frac{\beta^{(L)}(z)'}{L!} \frac{\partial^2 f_{0,j}}{\partial s_0 \partial s_j}(z,z) \right] \int p_1^{L+1} K(p_1) dp_1 \int p_2^{L+1} K(p_2) dp_2$$
$$\left. + o(h^{2L+4}) \right\} \times \mathbf{1}_{\{L=\text{odd}\}}$$
$$= O(h^{2L^*+2}) \leq O(h^6) \text{ for } L \geq 1. \tag{B.9}$$

We now deduce that the long run variance matrix of $\eta_t$, or variance matrix of the standardized partial sum $\frac{1}{\sqrt{nh^{2L+1}}} \sum_{t=1}^n \eta_t$, is

$$\mathbb{V}^{LR}(\eta_t) = \mathbb{E} \left[ \frac{1}{\sqrt{nh^{2L+1}}} \sum_{t=1}^n \eta_t \right] \left[ \frac{1}{\sqrt{nh^{2L+1}}} \sum_{t=1}^n \eta_t \right]' = \frac{1}{nh^{2L+1}} \sum_{t=1}^n \mathbb{E} \eta_t \eta_t' + \frac{1}{nh^{2L+1}} \sum_{t \neq s} \mathbb{E} \eta_t \eta_s'$$
$$= \frac{1}{h^{2L+1}} \mathbb{E} \eta_t \eta_t' + o(1) \to \frac{\nu_{2L}(K) f(z)}{(L!)^2} \beta^{(L)}(z) \beta^{(L)}(z)' =: V_{\eta\eta,L}, \tag{B.10}$$

which follows from (B.7) and standard arguments concerning the $o(1)$ magnitude of the sum of the autocovariances of kernel-weighted stationary processes. In particular, from the $\alpha$ mixing property of $z_t$ and using a sum splitting argument and results (B.6), (B.8) and (B.9) above, we have

$$\frac{1}{nh^{2L+1}} \sum_{t \neq s} \mathbb{E} \eta_t \eta_s' = \frac{1}{h^{2L+1}} \sum_{j=-n+1, j \neq 0}^{n-1} \left[ 1 - \frac{|j|}{n} \right] \left[ \mathbb{E} \xi_{\beta t} \xi_{\beta t+j}' - (\mathbb{E} \xi_{\beta t}) (\mathbb{E} \xi_{\beta t})' \right]$$
$$= \frac{1}{h^{2L+1}} \sum_{j=-M, j \neq 0}^{M} \left[ 1 - \frac{|j|}{n} \right] \left[ \mathbb{E} \xi_{\beta t} \xi_{\beta t+j}' - (\mathbb{E} \xi_{\beta t}) (\mathbb{E} \xi_{\beta t})' \right]$$
$$+ \frac{1}{h^{2L+1}} \sum_{M < |j| < n} \left( 1 - \frac{|j|}{n} \right) \left[ \mathbb{E} \xi_{\beta t} \xi_{\beta t+j}' - (\mathbb{E} \xi_{\beta t}) (\mathbb{E} \xi_{\beta t})' \right]$$
$$= O\left( \frac{M h^{2L^*+2}}{h^{2L+1}} \right) + O\left( \frac{1}{h^{2L+1}} \left( \mathbb{E} |\xi_{\beta t}|^\delta \right)^{2/\delta} \sum_{M < |j| < n} \alpha_j^{1-2/\delta} \right)$$
$$= O\left( Mh \times \mathbf{1}_{\{L=\text{even}\}} + Mh^3 \times \mathbf{1}_{\{L=\text{odd}\}} \right) + O\left( \frac{h^{2\frac{1+L\delta}{\delta}}}{h^{2L+1} M^c} \sum_{M < |j| < \infty} j^c \alpha_j^{1-2/\delta} \right)$$
$$= O\left( Mh \times \mathbf{1}_{\{L=\text{even}\}} + Mh^3 \times \mathbf{1}_{\{L=\text{odd}\}} \right) + O\left( \frac{1}{h^{1-2/\delta} M^c} \sum_{M < |j| < \infty} j^c \alpha_j^{1-2/\delta} \right)$$

$$= O\left(Mh \times \mathbf{1}_{\{L=\text{even}\}} + Mh^3 \times \mathbf{1}_{\{L=\text{odd}\}}\right) + o\left(\frac{1}{(M^{\frac{c}{1-2/\delta}} h)^{1-2/\delta}}\right) = o\left(1\right), \tag{B.11}$$

for a suitable choice of $M \to \infty$ such that $Mh \to 0$, $M^\tau h \to \infty$, and $\frac{M}{n} \to 0$, with $\tau > 1$, $c > \tau(1-2/\delta)$ and $\delta > 2$. It then follows by arguments similar to the central limit theory for weakly dependent kernel regression in Robinson (1983), Masry and Fan (1997), and Fan and Yao (2003, theorem 6.5) that the standardized partial sum process of $\eta_t$ satisfies a triangular array functional law giving

$$\frac{1}{\sqrt{nh^{2L+1}}} \sum_{t=1}^{\lfloor n\cdot \rfloor} \eta_t \rightsquigarrow B_{\eta,L}(\cdot), \tag{B.12}$$

where $B_{\eta,L}$ is vector Brownian motion with variance matrix $V_{\eta\eta,L} = \frac{\nu_{2L}(K)f(z)}{(L!)^2}\beta^{(L)}(z)\beta^{(L)}(z)'$. The effective sample size condition $nh \to \infty$ is required for this result. ■

**Part (a) (ii)** Otherwise, when $nh \to c \in [0, \infty)$ we have

$$\frac{1}{\sqrt{nh^{2L+1}}} \sum_{t=1}^{\lfloor n\cdot \rfloor} \eta_t = O_p(1), \tag{B.13}$$

but no invariance principle applies. Taking scalar $x_t$ and *iid* $\{z_t\}$ for ease of notation, the stability condition

$$\mathbb{E}\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{\eta_t}{\sqrt{h^{2L+1}}}\right)^2 = \frac{1}{n}\sum_{t=1}^{n}\mathbb{E}\left(\frac{\eta_t}{\sqrt{h^{2L+1}}}\right)^2 = \nu_2(K)f(z)\left(\frac{\beta^{(L)}(z)}{L!}\right)^2 + O(h),$$

is satisfied so that $\frac{1}{\sqrt{nh^{2L+1}}}\sum_{t=1}^{n}\eta_t = O_p(1)$ but the Lindeberg condition fails. To show this, note that $\eta_t = \xi_{\beta t} - \mathbb{E}\xi_{\beta t} = \xi_{\beta t} + O(h^{L^*+1})$. Given $\epsilon > 0$, $nh \not\to \infty$ and $\beta^{(L)}(z) \neq 0$ imply

$$\begin{aligned}
&\frac{1}{n}\sum_{t=1}^{n}\mathbb{E}\left\{\left(\frac{\eta_t}{\sqrt{h^{2L+1}}}\right)^2 1_{\left[|\eta_t|>\epsilon\sqrt{nh^{2L+1}}\right]}\right\} \\
&= \int \frac{[[\beta(z_t)-\beta(z)]K_{tz} + O(h^{L^*+1})]^2}{h^{2L+1}} 1_{\left[|[\beta(z_t)-\beta(z)]K_{tz}+O(h^{L^*+1})|>\epsilon\sqrt{nh^{2L+1}}\right]} f(z_t)dz_t \\
&= \int \frac{[\frac{\beta^{(L)}(z)}{L!}h^L p^L K(p) + O(h^{L^*+1})]^2}{h^{2L}} 1_{\left[|\frac{\beta^{(L)}(z)}{L!}h^L p^L K(p)+O(h^{L^*+1})|>\epsilon\sqrt{nh^{2L+1}}\right]} f(z+ph)dp \\
&= \left(\beta^{(L)}(z)/L!\right)^2 f(z)\int p^{2L}K^2(p) 1_{\left[|\frac{\beta^{(L)}(z)}{L!}p^L K(p)|>\epsilon\sqrt{nh}\right]}dp + O(h) \\
&\to \begin{cases} \left(\frac{\beta^{(L)}(z)}{L!}\right)^2 f(z)\nu_{2L}(K) > 0 & \text{if } nh \to 0, \\ \left(\frac{\beta^{(L)}(z)}{L!}\right)^2 f(z)\int p^{2L}K(p)^2 1_{\left[\left|\frac{\beta^{(L)}(z)}{L!}p^L K(p)\right|>\epsilon\sqrt{c}\right]}dp > 0 & \text{if } nh \to c. \end{cases}
\end{aligned}$$

■

**Part (b)** This part follows in essentially the same way as Part (a) and the proof is omitted. ■

**Part (c)** From (B.6) we have

$$\mathbb{E}\xi_{\beta t} = h^{L^*+1}\mu_{L^*}(K)C_L(z) + o(h^{L^*+1}) = h^{L^*+1}\mathcal{G}_L(z) + o(h^{L^*+1}) \tag{B.14}$$

from which it follows directly that $\frac{1}{nh^{L^*+1}}\sum_{t=1}^n x_t x_t' \mathbb{E}\xi_{\beta t} \to_p \mathcal{G}_L(z)\Sigma_{xx}$, as required. ■

**Lemma B.3.** *Under Assumption 2 and if $\beta^{(\ell)}(z) = 0$ for $\ell = 1, \cdots, L-1$ and $\beta^{(L)}(z) \neq 0$, the following hold as $n \to \infty$ and $h \to 0$:*

*(a) (i) If $nh \to \infty$ , $\frac{1}{\sqrt{n^3 h}}\sum_{t=1}^n x_t x_t' \zeta_{tK} \rightsquigarrow \int B_x B_x' dB_{\zeta K} \equiv_d \mathcal{MN}\left(0, \nu_0(K) f(z) \int B_x B_x' \otimes B_x B_x'\right)$, and $\frac{1}{\sqrt{n^3 h^{2L+1}}}\sum_{t=1}^n x_t x_t' \eta_t \rightsquigarrow \int B_x B_x' dB_{\eta,L} \equiv_d \mathcal{MN}\left(0, \frac{\nu_{2L}(K) f(z)}{(L!)^2} \int B_x B_x' \left[B_x'\beta^{(L)}(z)\right]^2\right)$;*
*(ii) If $nh \to c \in [0,\infty]$, $\frac{1}{\sqrt{n^3 h^{2L+1}}}\sum_{t=1}^n x_t x_t' \eta_t = O_p(1)$, and $\frac{1}{\sqrt{n^3 h}}\sum_{t=1}^n x_t x_t' \zeta_{tK} = O_p(1)$, both without invariance principles holding.*

*(b) With $C_L(z)$ defined as in (B.5) and $\mathcal{G}_L(z) = \mu_{L^*}(K)C_L(z)$,*

$$\frac{1}{n^2 h^{L^*+1}}\sum_{t=1}^n x_t x_t' \mathbb{E}\xi_{\beta t} \rightsquigarrow \mu_{L^*}(K)C_L(z)\int B_x B_x' = \mathcal{G}_L(z)\int B_x B_x', \tag{B.15}$$

*where $L^* = L \times \mathbf{1}_{\{L=even\}} + (L+1)\,\mathbf{1}_{\{L=odd\}}$.*

*(c) (i) If $nh \to \infty$, $\frac{1}{n^2 h}\sum_{t=1}^n x_t x_t' K_{tz} \rightsquigarrow \int B_x B_x' f(z)$;*
*(ii) If $nh \to c \in [0,\infty)$, $\frac{1}{\sqrt{n^3 h}}\sum_{t=1}^n x_t x_t' K_{tz} = O_p(1)$ but no invariance principle holds.*

*(d) (i) If $nh \to \infty$, $\frac{1}{\sqrt{n^2 h}}\sum_{t=1}^n x_t u_t K_{tz} \rightsquigarrow \int B_x dB_{uK} \equiv_d \mathcal{MN}\left(0, \nu_0(K)\sigma_u^2 f(z)\int B_x B_x'\right)$;*
*(ii) If $nh \to c \in [0,\infty)$, $\frac{1}{\sqrt{n^2 h}}\sum_{t=1}^n x_t u_t K_{tz} = O_p(1)$ but no invariance principle holds.*

**Proof of Lemma B.3**

**Part (a)(i)** First observe that when $nh \to \infty$ we have the joint convergence

$$\left(\frac{1}{\sqrt{n}}x_{\lfloor n\cdot\rfloor},\quad \frac{1}{\sqrt{nh}}\sum_{t=1}^{\lfloor n\cdot\rfloor}\zeta_{tK},\quad \frac{1}{\sqrt{nh^{2L+1}}}\sum_{t=1}^{\lfloor n\cdot\rfloor}\eta_t\right) \rightsquigarrow \left(B_x(\cdot),\quad B_{\zeta K}(\cdot),\quad B_{\eta,L}(\cdot)\right), \tag{B.16}$$

by virtue of Assumption 1, Lemma B.1(a)(i) and Lemma B.2(a)(i), where the Brownian motion $\{B_x\}$ is independent with $\{B_{\zeta K}, B_{\eta,L}\}$ by virtue of the exogeneity of $x_t$. The covariance between $B_{\zeta K}$ and $B_{\eta,L}$ is $\frac{\beta^{(L)}(z)}{L!}\nu_L(K)f(z)$. This follows from the fact that the contemporaneous covariance $\mathbb{E}\zeta_{tK}\eta_t = \mathbb{E}K_{tz}^2(\beta(z_t) - \beta(z)) - \mathbb{E}K_{tz}\mathbb{E}K_{tz}(\beta(z_t) - \beta(z)) = h^{L+1}\frac{\beta^{(L)}(z)}{L!}\nu_L(K)f(z) + O(h^{L^*+2}) = O(h^{L+1})$ and the cross serial covariance $\mathbb{E}\zeta_{tK}\eta_{t+j} = O(h^{L^*+2})$ for $j \neq 0$, so that combined with the weak dependence of $z_t$ and an argument along the same lines as that leading to (B.10) we have

$$\mathbb{E}\left(\frac{1}{\sqrt{nh}}\sum_{t=1}^{\lfloor n\cdot\rfloor}\zeta_t \times \frac{1}{\sqrt{nh^{2L+1}}}\sum_{t=1}^{\lfloor n\cdot\rfloor}\eta_t\right) = \frac{1}{h^{L+1}}\mathbb{E}\left(\zeta_{tK}\eta_t\right)\{1+o(1)\} = \frac{\beta^{(L)}(z)}{L!}\nu_L(K)f(z)\{1+o(1)\}.$$

Weak convergence to the stochastic integral limits

$$\frac{1}{\sqrt{n^3h}}\sum_{t=1}^{n}x_tx_t'\zeta_{tK} = \sum_{t=1}^{n}\left(\frac{x_t}{\sqrt{n}}\frac{x_t'}{\sqrt{n}}\right)\frac{\zeta_{tK}}{\sqrt{nh}} \rightsquigarrow \int B_xB_x'dB_{\zeta K}, \tag{B.17}$$

$$\frac{1}{\sqrt{n^3h^{2L+1}}}\sum_{t=1}^{n}x_tx_t'\eta_t = \sum_{t=1}^{n}\left(\frac{x_t}{\sqrt{n}}\frac{x_t'}{\sqrt{n}}\right)\frac{\eta_t}{\sqrt{nh^{2L+1}}} \rightsquigarrow \int B_xB_x'dB_{\eta,L} \tag{B.18}$$

then follows directly from Ibragimov and Phillips (2008, theorem 4.3) when $nh \to \infty$, respectively. Both stochastic integrals have mixed normal distributions, viz.,

$$\int B_x \otimes B_x dB_{\zeta K} \equiv_d \mathcal{MN}\left(0, \nu_0(K)f(z)\int B_xB_x' \otimes B_xB_x'\right), \tag{B.19}$$

$$\int B_xB_x'dB_{\eta,L} \equiv_d \mathcal{MN}\left(0, \frac{\nu_{2L}(K)f(z)}{(L!)^2}\int B_xB_x'\left(B_x(r)'\beta^{(L)}(z)\right)^2\right), \tag{B.20}$$

and the stated result holds. ■

**Part (a) (ii)** When the rate conditions $nh \to \infty$ fails and, instead $nh \to c \in [0,\infty)$ applies, it follows from Lemma B.1(a)(ii) and Lemma B.2(a)(ii) that $\frac{1}{\sqrt{nh}}\sum_{t=1}^{n}\zeta_{tK} = O_p(1)$ and $\frac{1}{\sqrt{nh^{2L+1}}}\sum_{t=1}^{n}\eta_t = O_p(1)$, respectively, but with no invariance principles holding. Correspondingly, in place of (B.17) and (B.18), we have

$$\frac{1}{\sqrt{n^3h}}\sum_{t=1}^{n}x_tx_t'\zeta_{tK} = \sum_{t=1}^{n}\left(\frac{x_t}{\sqrt{n}}\frac{x_t'}{\sqrt{n}}\right)\frac{\zeta_{tK}}{\sqrt{nh}} = O_p(1), \tag{B.21}$$

$$\frac{1}{\sqrt{n^3h^{2L+1}}}\sum_{t=1}^{n}x_tx_t'\eta_t = \sum_{t=1}^{n}\left(\frac{x_t}{\sqrt{n}}\frac{x_t'}{\sqrt{n}}\right)\frac{\eta_t}{\sqrt{nh^{2L+1}}} = O_p(1), \tag{B.22}$$

again without invariance principles. ■

**Part (b)**. Using (B.14) and standard weak convergence methods we have

$$\frac{1}{n^3h^{L^*+1}}\sum_{t=1}^{n}x_tx_t'\mathbb{E}\xi_{\beta t} = \frac{1}{n}\sum_{t=1}^{n}\frac{x_t}{\sqrt{n}}\frac{x_t'}{\sqrt{n}}\frac{\mathbb{E}\xi_{\beta t}}{h^{L^*+1}} \rightsquigarrow \mathcal{G}_L(z)\int B_xB_x',$$

giving the stated result. ■

**Part (c)**. Using (B.21), Part (i) follows as in Phillips and Wang (2020, Lemma B.1(c)(i)), giving

$$\frac{1}{n^2 h}\sum_{t=1}^{n} x_t x_t' K_{tz} = \frac{1}{n}\sum_{t=1}^{n} \frac{x_t}{\sqrt{n}}\frac{x_t'}{\sqrt{n}}\frac{\mathbb{E}K_{tz}}{h} + o_p(1) \rightsquigarrow \int B_x B_x' f(z). \tag{B.23}$$

The proof of (ii) follows as in Phillips and Wang (2020, Lemma B.1(c)(ii)). ■

**Part (d)**. As in Part (a)(i), when $nh \to \infty$ we have $\frac{1}{\sqrt{nh}}\sum_{t=1}^{n\lfloor\cdot\rfloor} u_t K_{tz} \rightsquigarrow B_{uK}(\cdot)$ as in Lemma B.1(a) and then weak convergence to the stochastic integral $\frac{1}{\sqrt{n^2 h}}\sum_{t=1}^{n} x_t u_t K_{tz} \rightsquigarrow \int B_x dB_{uK}$ follows directly from Ibragimov and Phillips (2008, theorem 4.3). Due to the independence between $B_x$ and $B_{uK}$, we have $\int B_x dB_{uK} \equiv_d \mathcal{MN}\left(0, \nu_0(K)\sigma_u^2 f(z)\int B_x B_x'\right)$.
(ii) If $nh \to c \in [0,\infty)$, then $\frac{1}{\sqrt{nh}}\sum_{t=1}^{n\lfloor\cdot\rfloor} u_t K_{tz} = O_p(1)$ as in Lemma B.1(a)(ii), and then $\frac{1}{\sqrt{n^2 h}}\sum_{t=1}^{n} x_t u_t K_{tz} = O_p(1)$ but no invariance principle holds. ■

# C Additional Computational Details

The following paragraphs provide further details of how the three statistics in Figures 4, 5, 6 and Table 2 were computed.

(i) Computation of the naive $t$-ratio $\hat{T}(z; L=1)$ follows the definition (3.1). Since the use of $\hat{T}(z; L=1)$ implies belief that $L=1$, the optimal bandwidth order for that case is employed in the computation. For the computation of $\hat{\beta}(z)$ and $K_{tz}$ the bandwidth $h = \hat{\sigma}_z n^{\gamma}$ was used with $\gamma = -1/2$ for nonstationary $x_t$ and $\gamma = -1/5$ for stationary $x_t$. For the other unknown components $\beta^{(1)}$, $\beta^{(2)}(z)$, $f(z)$ and $f^{(1)}(z)$ involved in (3.2) and (3.3), different bandwidth orders were used. Specifically, $\hat{f}(z)$ used $\gamma = -1/5$, $\hat{f}^{(1)}(z)$ used $\gamma = -1/7$, $\hat{\beta}^{(1)}$ was estimated using local linear estimation with $\gamma = -1/7$ for stationary $x_t$ and $\gamma = -2/7$ for nonstationary $x_t$, and $\hat{\beta}^{(2)}$ was estimated by local quadratic estimation with $\gamma = -1/9$ for stationary $x_t$ and $\gamma = -2/9$ for nonstationary $x_t$. Those orders were selected based on optimal bandwidth order rules in the case of local $p$-th order polynomial estimation to estimate $\beta^{(p)}(z)$ for $p = 1, 2$ and rely on ongoing work by the authors for local $p$-th order polynomial estimation in functional coefficient regression.

(ii) For the infeasible statistic $\hat{T}(z; true\ L)$, the true $L$ is used in the computation. For $L=1$, it is identical to the naive choice. For $L=4$, we need to estimate $\beta^{(4)}(z)$. We use local 4-th order estimation with bandwidth order $\gamma = -1/13$ for stationary $x_t$ and $\gamma = -2/13$ for nonstationary $x_t$. For the computation of $\hat{\beta}(z)$ and $K_{tz}$ the optimal order $\gamma = -1/(2L^*+1)$ for stationary $x_t$ and $\gamma = -2/(2L^*+1)$ for nonstationary $x_t$ were used.

(iii) For computation of the oracle $t$-ratio $T(z; true\ L)$, the quantities $L$, $\beta^{(L)}(z)$, $\beta^{(L+1)}(z)$, $f(z)$, $f^{(1)}(z)$ and $\sigma_u^2$ were assumed known. Given a known $L$, the optimal order was used in the estimation of $\hat{\beta}(z)$ and computation of $K_{tz}$. More specifically, for $L$ greater than 1, the optimal order used is $\gamma = -1/(2L^*+1)$ for stationary $x_t$ and $\gamma = -2/(2L^*+1)$ for nonstationary $x_t$. For $L=1$, the optimal order is $\gamma = -1/5$ for stationary $x_t$ and $\gamma = -1/2$ for nonstationary $x_t$.

(v) For the residual variance $\sigma_u^2$ the same estimate was used in the naive choice and the infeasible statistics. We used $\gamma = -1/2$ to compute $\hat{\beta}(z_t)$ and hence the residual estimates $\hat{u}_t = y_t - x_t\hat{\beta}(z_t)$. Then $\sigma_u^2$ was estimated by $\hat{\sigma}_u^2 = \frac{1}{n}\sum_{t=1}^{n}\hat{u}_t^2$.

**Supplementary Material**

Peter C. B. Phillips and Ying Wang (2021). Online Supplement to "Limit Theory for Locally Flat Functional Coefficient Regressions."

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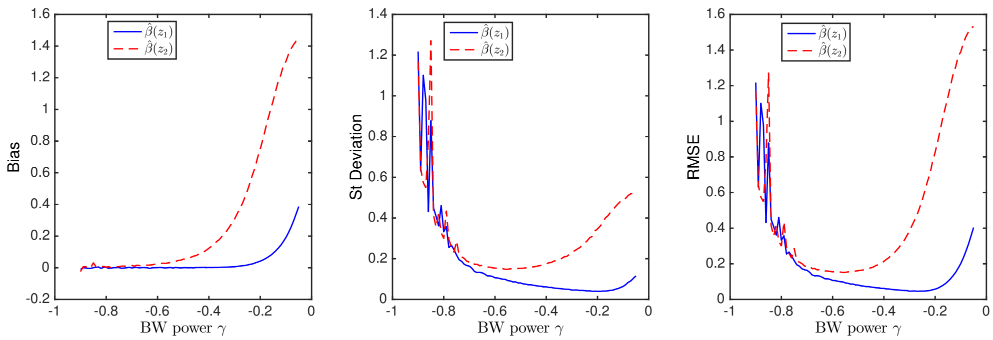
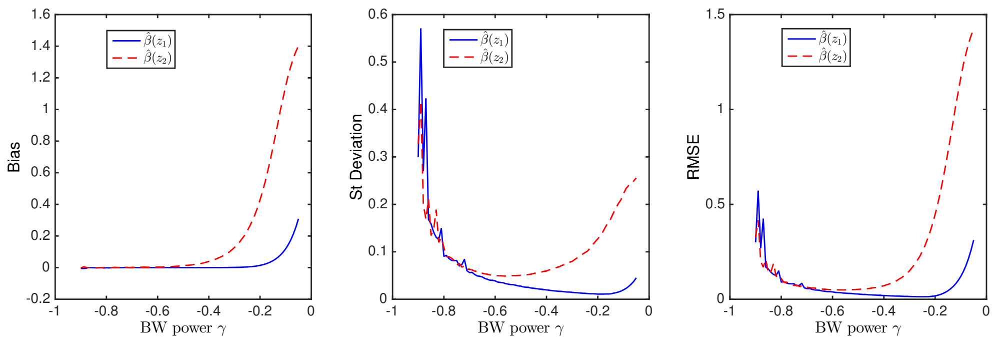
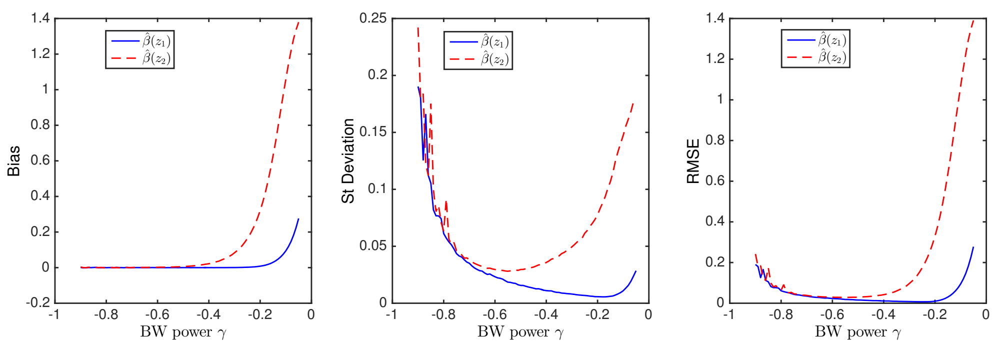



(a) $n = 100$

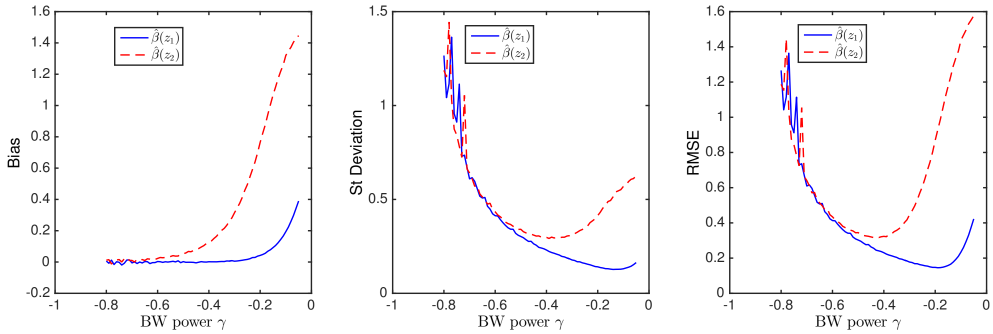



(b) $n = 400$

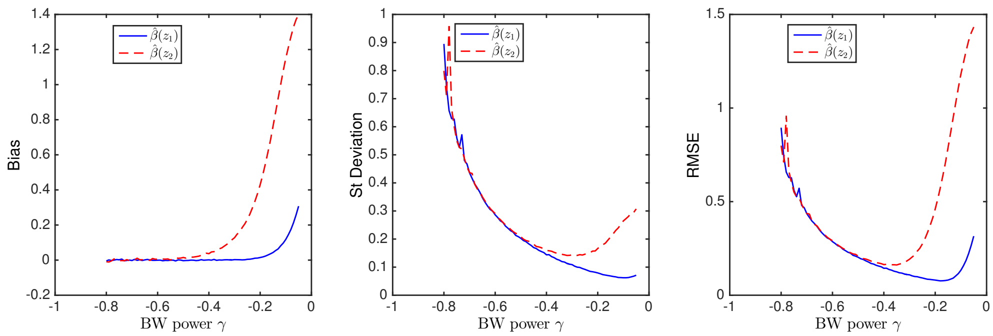



(c) $n = 800$

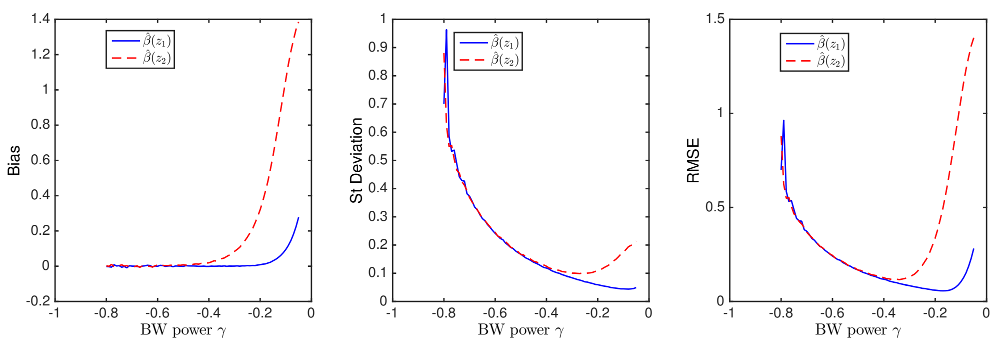
Figure 2: Nonstationary case: bias, standard deviation and RMSE plots for FCC estimator $\hat{\beta}(z)$ at points $z_1 = 0$ and $z_2 = 1$ for the quartic coefficient function $\beta(z) = z^4$. The figures show bias, standard deviation, and RMSE in the left, middle and right panels as functions of bandwidth power $\gamma$ ($-0.9 \leq \gamma \leq -0.05$) in $h = \hat{\sigma}_z \times n^{\gamma}$ for Model (2.1) with $I(1)$ regressor $x_t$ and sample size $n = 100, 400$ and 800.




(a) $n = 100$

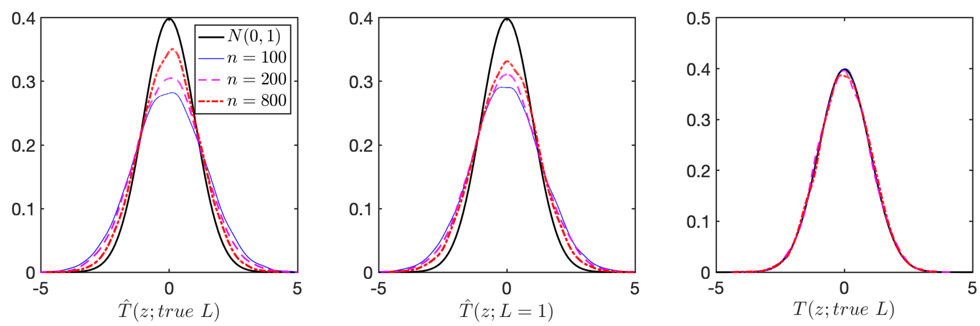



(b) $n = 400$

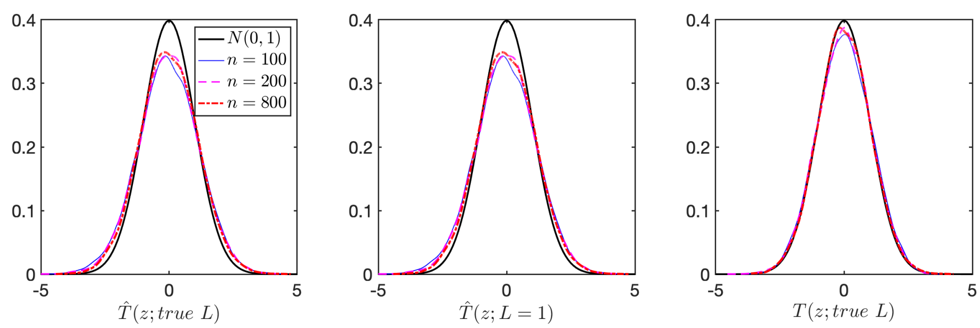



(c) $n = 800$

Figure 3: Stationary case: bias, standard deviation and RMSE plots for the FC estimator $\hat{\beta}(z)$ at points $z_1 = 0$ and $z_2 = 1$ for the quartic coefficient function $\beta(z) = z^4$. The figures show bias, standard deviation, and RMSE in the left, middle and right panels as functions of bandwidth power $\gamma$ $(-0.80 \leq \gamma \leq -0.05)$ in $h = \hat{\sigma}_z \times n^{\gamma}$ for Model (2.1) with a stationary autoregressive regressor $x_t$ with autoregressive coefficient $\theta = 0.5$ and sample size $n = 100, 400$ and $800$.

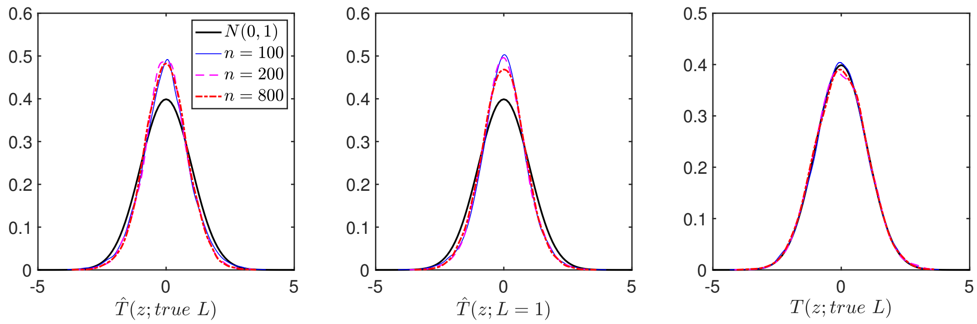



(a) $z_1 = 0, L = 4$

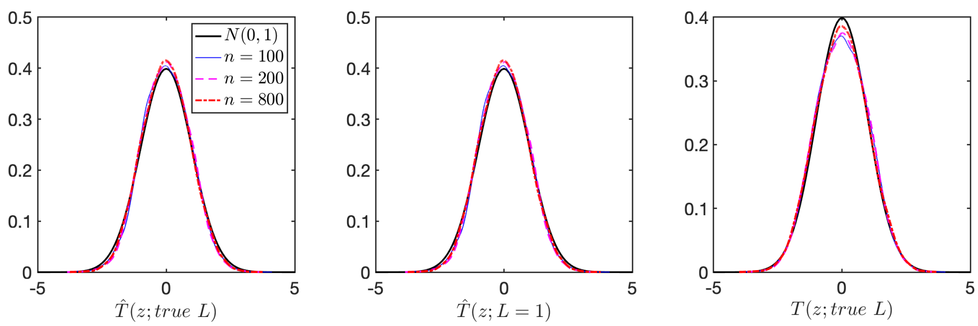



(b) $z_2 = 1, L = 1$

Figure 4: Empirical densities of the self-normalized $t$-ratio $\hat{T}(z; true\ L)$, $\hat{T}(z; L = 1)$ and $T(z; true\ L)$ when $x_t$ is stationary.

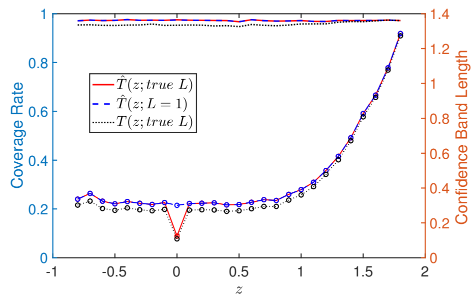



(a) $z_1 = 0, L = 4$

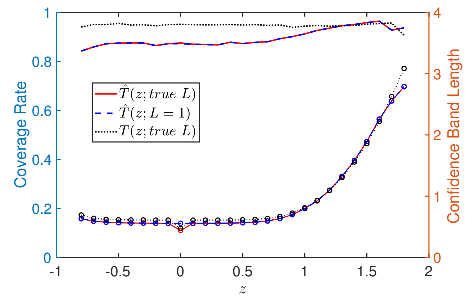



(b) $z_2 = 1, L = 1$

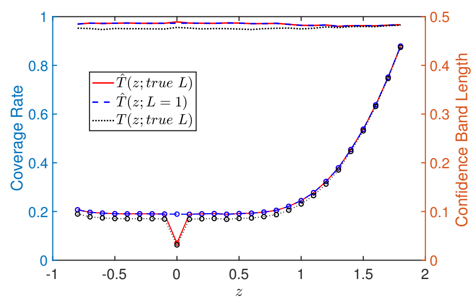
Figure 5: Empirical densities of the self-normalized $t$-ratio $\hat{T}(z; true\ L)$, $\hat{T}(z; L = 1)$ and $T(z; true\ L)$ when $x_t$ is nonstationary.



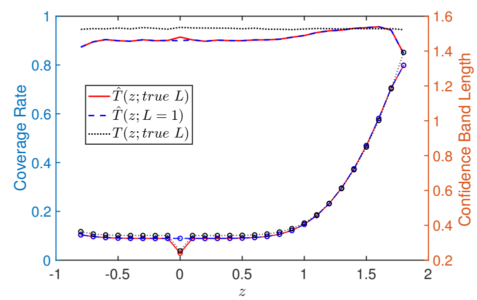

(a) $x_t$ is nonstationary, $n = 200$




(b) $x_t$ is stationary, $n = 200$




(c) $x_t$ is nonstationary, $n = 800$




(d) $x_t$ is stationary, $n = 800$

Figure 6: Coverage rate (left scale) and length (right scale, with lines marked by circles) of the 95% confidence bands over the support of $z_t$ for $n = 200$ and $n = 800$, from 10,000 replications.