AFFECTIVE PORTFOLIO ANALYSIS: RISK, AMBIGUITY AND (IR)RATIONALITY

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Abstract

Ambiguous assets are characterized as assets where objective and subjective probabilities of tomorrow's asset-returns are ill- defined or may not exist, e.g., bitcoin, volatility indices or any IPO. Investors may choose to diversify their portfolios of fiat money, stocks and bonds by investing in ambiguous assets, a fourth asset class, to hedge the uncertainties of future returns that are not risks.

(IR)rational probabilities are computable alternative descriptions of the distribution of returns for ambiguous assets. (IR)rational probabilities can be used to define an investor's (IR)rational expected utility function in the class of non-expected utilities.

Investment advisors use revealed preference analysis to elicit the investor's composite preferences for risk tolerance, ambiguity aversion and optimism.

Investors rationalize (IR)rational expected utilities over portfolios of fiat money, stocks, bonds and ambiguous assets by choosing their optimal portfolio investments with (IR)rational expected utilities. Subsequently, investors can hedge future losses of their optimal portfolios by purchasing minimum-cost portfolio insurance.

Keywords: Behavioral Finance, Prospect Theory, Afriat Inequalities

JEL CLassification B31, C91, D9

1 Introduction

In the theory of decision-making under uncertainty, ambiguous assets are assets where objective and subjective probabilities of tomorrow's asset-returns are ill- defined or may not exist. If so, then tomorrow's uncertain payoffs are characterized by (IR)rational state probabilities which depend on the investor's (IR)rational state of mind.

(IR)rational probabilities are computable moments of the distribution of returns for ambiguous assets. (IR)rational probabilities are computable alternative descriptions of the distribution of returns for ambiguous assets. (IR)rational probabilities may be used to define an investor's (IR)rational expected utility function in the class of non-expected utilities. Investors may choose to diversify portfolios of fiat money, stocks and bonds by investing in ambiguous assets to hedge the uncertainties of future returns that are not risks. Investors select optimal portfolios of fiat money, stocks, bonds and ambiguous assets by rationalyzing recent portfolio investments with (IR)rational expected utilities and hedging forecasts of future losses of the chosen optimal portfolios by purchasing minimum-cost portfolio insurance. The theory of (IR)rational portfolio analysis differs significantly from the mean-variance analysis of the efficient trade-off between risk and return in diversified portfolios of risky assets. See Chapter 1 in Lam (2016), where investment advisors implement the elicitation of investor's risk tolerance and loss aversion with questionnaires, framed as a series of hypothetical investing scenarios, often lacking demographic controls.

This is an instance of stated preference analysis.The method of elicitation proposed in this paper is revealed preference analysis which is predicated on the history of investor's portfolio choices in asset markets.As is now well known, the refutable implications of market equilibria can be derived from revealed preference analysis.

The origin of (IR)rational portfolio analysis is the Keynesian notion of (IR)rational equilibrium in asset markets. Keynes viewed equilibrium prices in asset markets as a balance of the sales of bears, the pessimists, and the purchases of bulls, the optimists.

Subjective expected utility theory, originally proposed by Savage as the foundation of Bayesian statistics, is a theory of decision-making under uncertainty that "... does not leave room for optimism or pessimism to play any role in the person's judgment" (Savage, 1954, p. 68).

This viewpoint is not the perspective of Keynes.That is, "equilibrium prices in asset markets will be fixed at the point at which the sales of the bears and the purchases of the bulls are balanced" (Keynes, 1930). In Keynes, equilibrium in asset markets is an (IR)rational notion. Keynes argued that It is the optimism and pessimism of investors not the risk and return of assets that determine future asset-returns.

The equilibration of optimistic and pessimistic beliefs of investors is rationalized by investors maximizing (IR)rational expected utility functions subject to budget constraints defined by asset-prices and expenditures of investors.The family of (IR)rational expected utilities is a subclass of non-expected utility functions in the theory of decision-making under uncertainty.

(IR)rational expected utility functions represent the preferences of investors for optimism defined as the composition of the investor's preferences for risk and preferences for ambiguity.That is, an investor may be risk averse or risk seeking and ambiguity averse or ambiguity seeking and optimistic or pessimistic.

If U(x) is a representation of the investor's preferences for risk, and J(y) is a representation of the investor's preferences for ambiguity, where the state-utility vector y= U(x) for some limited liability state-contingent claim x, then V(x) = J(U(x)), the composition of U(x) and J(y), represents the investor's preferences for optimism.

In the decision-theoretic literature, averse preferences are represented by strictly concave utility representations; and seeking preferences are represented by strictly convex utility representations.

This convention is followed in this manuscript to describe Keynes's notion of how bulls and bears invest in asset-markets. Talking heads on cable TV often summarize today's financial news as a " bear market" or a "bull market".

If (IR)rational utility functions are smooth, then the (IR)rational Afriat inequalities are defined as the first order conditions for maximizing the composite utility function, V(x), subject to a budget constraint, where the gradient of V is computed using the chain rule. Solving the (IR)rational

Afriat inequalities for smooth (IR)rational utility functions is, in general, NP-hard.That is, in the worst case the (IR)rational Afriat inequalities are exponential in the number of inequalities and unknowns.

Suppose V(x)=J(U(x)), where U:X→Y, J: Y→R. X is the family of limited liability assets or state-contingent claims, and Y is a family of state-utility vectors, where X and Y are N dimensional linear vector spaces.

If U is a diagonal NxN matrix, then DV(x) = DU(x) [ΔJ(y)] is the pointwise product of DU(x) and [ΔJ(y)]. That is, in general, DV(x) is bilinear,

hence the ensuing NP-hard computational complexity.

The family of positive linear functions is a family of utility functions that are closed under composition. L(x) is a positive linear function if L(x) = d·x, for some fixed d > 0 and all x > 0 in $(R)^N$.

If the utility functions for risk and ambiguity are positive linear functions, then their composition, the utility function for optimism, is also a positive linear function.

Suppose U(x) = b· x and J(k) = a· k, where a and b are positive, then V(x)=J(U(x)) is also a positive linear utility function, where V(x) = c· x and c is the pointwise product of a and b. Hence the marginal utility of expenditures in the affective Afriat inequalities for V can be normalized to one for all elicited optimal choices of the investor.

Arbitrary systems of linear inequalities can be solved in polynomial time as a function of the number of inequalities and unknowns, using interior-point algorithms.

2 Approximation Theorems

This observation suggests approximation theorems, where NP-hard systems of (IR)rational Afriat inequalities are approximated by linear systems of inequalities

The family of smooth (IR)rational expected utilities are derived from smooth (IR)rational utilities using the Legendre duality theorem for smooth convex functions, assuming that the gradient of V(x) is 1 to 1 on the interior of X, the positive orthant of R^N

In the nonsmooth case, the Legendre-Fenchel duality theorem can be used in lieu of Fenchel's duality theorem to derive an equivalent family of representations of nonsmooth (IR)rational preferences as a family of (IR)rational expected utility functions, without invoking the chain rule. For any function V(x), the bi-conjugate, denoted $V^{**}(x)$,is the sup of all the convex functions majorized by $V^{**}(x)$, hence convex, and the bi-conjugate of - V (x) is the inf of all the concave functions minorized by - $V^{**}(x)$, hence concave.

Theorem (1) If $V_{LB}(x) := V^{**}(x)$ and $V_{UB}(x):=- V^{**}(x)$, then

$V_{LB}(x) < V^{**}(x) < V_{UB}(x)\ V_{LB}(x)$.

To derive an approximation theorem for testing the feasibility of the convex (IR)rational Afriat inequalities, we define the family of relaxed linear (IR)rational Afriat inequalities, indexed by the scalar $t > 0$.The relaxed (IR)rational linear Afriat inequalities are feasible for sufficiently large t. Minimizing t with respect to the observations defines the optimal linear approximation, where the shadow prices for the dual linear program are proxies for the degree of approximation. A proxy for the investor's unobservable true preferences over assets is the piece-wise, linear Afriat function that approximately rationalizes the optimal observed individual asset-demands. Note, it is not assumed that the investor's true preferences are represented by (IR)rational utility functions.

To test the feasibility of convex (IR)rational Afriat inequalities for $V_{LB}(x)$, consider the relaxed convex, (IR)rational Afriat Inequalities and solve the following linear program:

(P)

$$t^* = [\text{Max } t_j : \text{s.t. } 0 \le t_j]$$

$$V_{LB}(x_i)-V_{LB}(x_j) \le ß_j\, p_j \cdot (x_i - x_j) + t_j$$

Theorem (2) t*=0 iff the convex IR(rational) Afriat inequalities are feasible.

To test the feasibility of concave (IR)rational Afriat inequalities for $V_{LB}(x)$, consider the relaxed convex/concave (IR)rational Afriat Inequalities and solve the following linear program:

(Q)

$$s^* = [\text{Max } s_j : \text{s.t. } 0 \le s_j]$$

$$ß_j p_j \cdot (x_i - x_j) + s_j \le V_{LB}(x_i)-V_{LB}(x_j)$$

Theorem (3) s*=0 iff the concave (IR)rational Afriat inequalities are feasible.

(P) and (Q) are linear systems of inequalities that can be solved in polynomial time.

Using Afriat's construction we construct the piecewise linear convex functions:

$$V^{\#}_{LB}(x) = \max \{1<j : V_{LB}(x_j) + ß_j\, p_j \cdot (x - x_j) + t_j$$

Using Afriat's construction we construct the piecewise linear concave functions:

$$V^{\#}_{UB}(x) = \min \{1<j : s_j + ß_j p_j \cdot (x - x_j) + V_{LB}(x_j)$$

Theorem 4:There exists functions that bound the unobserved V_{LB} (x),the biconjugate of the (IR)rational utility function V(x).These functions are computable in polynomial time.

3 Prospect Theory

The fourfold pattern of preferences discussed in chapter 29 of Thinking Fast and Slow (2011) by Daniel Kahneman is described as "one of the core achievements of prospect theory". In a 2x2 contingency table, where the columns are high probability. (certainty effect) and low probability (possibility effect).and the rows are gains and losses from the status quo. The entries in the four cells are illustrative prospects. One cell is a surprise, where in the high probability/losses cell. Kahneman and Tversky observe risk seeking with negative prospects, commonly referred to as loss aversion. In his insightful monograph, Kahneman identifies “three cognitive principles at the core of prospect theory. They play an essential role in the evaluation of financial outcomes…. The third principle is loss aversion.” Prospect theory and its generalization cumulative prospect theory are empirical, psychological theories of decision making under risk, inspired by the Allais paradox. (IR)rational portfolio analysis, theory, extends the fourfold pattern of decision-making under risk to a fourfold pattern of decision-making under risk and ambiguity. (IR)rational portfolio analysis is an empirical, psychological theory of decision making under risk and ambiguity, inspired by the Ellsberg’s paradox

The fourfold pattern of (IR)rational decision-making under risk and ambiguity is also a 2x2 contingency table, where the columns are Risk Averse and Risk Seeking and the rows are Ambiguity Averse and Ambiguity Seeking. Entries in the cells are preferences for optimism derived from sufficient conditions for the composition of convex and concave functions as specified in the theory of disciplined convex programming. See Lemma 1.in Grant, et al (2006)

Composition Theorem for Convex/Concave Functions

If f*:* R→R is *convex and nondecreasing and*

g: R^N→R is convex, then h = fog is convex.

If f: R→R is convex and nonincreasing and

g*:* R^N→R is concave, then fog is convex.

If f: R→R is concave and nondecreasing and

g: R^N→R^N is concave, then f o g is concave.

If: R→R is concave and nonincreasing and

g: R^N→R^N is convex, then f o g is concave. f

For (IR)rational utilities the Composition theorem implies:

If J is concave and nonincreasing and U is convex, then the investor is pessimistic.

If J is convex and nondecreasing and U is convex, then the investor is optimistic.

If J is concave and decreasing and U is concave,then the investor is pessimistic

If J is convex and nonincreasing and U is concave, then the investor is optimistic

The Fourfold Pattern of (IR)rational decision-making under risk and ambiguity is a 2x2 contingency table, where the columns are Risk Averse and Risk Seeking and the rows are Ambiguity Averse and Ambiguity Seeking. Entries in the cells are preferences for optimism derived from sufficient conditions for the composition of convex and concave functions as specified in the Composition theorem.

The Fourfold Pattern of (IR)rational Decision-Making under Risk and Ambiguity

	RISK	RISK
	AVERSE	SEEKING
AMBIGUITY	PESSIMISTIC	PESSIMISTIC
AVERSE	PREFERENCES	PREFERENCES
AMBIGUITY	OPTIMISTIC	OPTIMISTIC
SEEKING	PREFERENCES	PREFERENCES

(IR)rational Portfolio Analysis is an empirical, psychological theory of investing under risk and ambiguity, inspired by the Ellsberg paradox.

IR(rational) state probabilities differ from subjective state probabilities in that they may depend on the outcomes in different states of the world. In the Foundations of Statistics (1954) Savage, in postulate P2, explicitly excludes (IR)rational probabilities from his axiomatic derivation of subjective expected utility theory. In his seminal analysis of subjective probability theory, Risk, Ambiguity, and The Savage Axioms (1961), Daniel Ellsberg introduces the notion of ambiguity as an alternative to the notion of risk in decision making under uncertainty. That is, uncertainties that are not risks, where the state probability of future outcomes are unknown or may not exist. In this case, non-expected utility models by Huriwitz (1957) and Ellsberg (1962) provide an alternative characterization of the investor's attitudes regarding risk, ambiguity and optimism. Their models are the provenance of (IR)rational utility functions.

In a series of thought experiments using urns with known and unknown distributions of colored balls, he conjectured that some individuals may violate, Savage's Postulate the so-called SURE THING PRINCIPLE. These thought experiments have been conducted many times in many classrooms and Ellsberg's conjecture has been confirmed.

4 Diversification

This paper has 6 technical appendices comprised of 12 Cowles Foundation Discussion Papers (CFDP's). The appendices are listed as prior art in my pending non-provisional (utility) patent application: AFFECTIVE PORTFOLIO THEORY; Application/Control Number: 16/501,575; Filing Date:05/02/2019.

The appendices extend the benefits of diversification as a hedge against risk in portfolios of stocks and bonds, i.e., portfolios of risky assets, for investors endowed with objective or subjective state probabilities of asset-payoffs tomorrow. If these state probabilities are ill-defined or non-existent then investors may choose to invest in ambiguous assets where tomorrow's uncertain payoffs are characterized by (IR)rational state probabilities.

Nonsmooth affective portfolio theory, or nonsmooth APT, is a sequel to smooth affective portfolio theory, or smooth APT. This section prescribes a refutable generalization of smooth APT, for rationalizing a history of, elicited, optimal portfolios of risky and ambiguous assets of investors endowed with nonsmooth, affective utilities.

The approximation theorem for NP-hard rationalizations of elicited portfolio data in this section subsumes the linear approximation theorem for Np-hard rationalizations of investors endowed with smooth affective utilities.

The technical results are derived from two methodologies in convex analysis:

(a) Revealed Preference Analysis

(b) Legendre-Fenchel Duality Theory

The analysis in this section is an abridged summary of the specifications in my non-provisional (utility) patent application, Affective Portfolio Theory, patent pending May 23, 2019.

4 Smooth APT

The origin of smooth APT is the Keynesian notion of affective equilibrium in financial markets. Keynes viewed the equilibrium prices in asset markets as a balance of the sales of bears, the pessimists, and the purchases of bulls, the optimists.

That is, "equilibrium prices in asset markets will be fixed at the point at which the sales of the bears and the purchases of the bulls are balanced" (Keynes, 1930). Keynes believed that It is the optimism and pessimism of investors not the risk and return of assets that determine equilibrium in financial markets. This is a theory of affective investing, where the prices of assets today equilibrate the optimism and pessimism of bulls and bears regarding future asset-payoffs

In smooth APT, the equilibration of optimistic and pessimistic beliefs of investors is rationalized by investors maximizing affective utilities subject to budget constraints, defined by asset prices and the expenditures of investors.

Affective utilities represent the preferences of investors for optimism or pessimism, defined as the composition of the investor's preferences for risk and preferences for ambiguity.That is, an investor may be risk averse or risk seeking and ambiguity averse or ambiguity seeking and optimistic or pessimistic.

If $U(x)$ is a representation of the investor’s preferences for risk, and $J(y)$ is a representation of the investor’s preferences for ambiguity, where the state-utility vector $y = U(x)$ for some limited liability state-contingent claim x, then $V(x) = J(U(x))$, the composition of $U(x)$ and $J(y)$, is a representation of the investor’s preferences for optimism.

We follow the decision-theoretic literature, where averse preferences have strictly concave utility representations and seeking preferences have strictly convex utility representations. In addition, smooth APT assumes all representations of preferences are smooth. Following Keynes, smooth APT assumes that optimistic preferences have strictly convex utility representations and pessimistic preferences have strictly concave utility representations.

The fourfold pattern of affective decision making under risk and ambiguity is a 2x2 contingency table, where the columns are Risk Averse and Risk Seeking and the rows are Ambiguity Averse and Ambiguity Seeking. Entries in the cells are preferences for optimism derived from sufficient conditions for the composition of convex and concave functions, in the Composition Theorem for Convex/Concave function proved in Disciplined Convex Programming.The affective Afriat inequalities in smooth APT are defined as the first order conditions for maximizing the composite utility function, V(x), subject to a budget constraint, where the gradient of V is computed with the chain rule. Solving the affective Afriat inequalities for rationalizing asset demands of investors endowed with smooth affective utility functions is, in general, NP-hard. That is, in the worst case, the time it takes to solve a system of affective Afriat inequalities is exponential in the number of inequalities and unknowns.

If U is a diagonal NxN matrix, then DV(x) = DU(x) [ΔJ(y)] is the pointwise product of DU(x) and [ΔJ(y)]. That is, in general, DV(x) is bilinear, hence the ensuing NP-hard computational complexity.

The family of positive linear functions is a family of utility functions that are closed under composition, where L(x) is a positive linear function if L(x) = d·x , for some fixed d > 0 and all x > 0 in $(R)^N$.

If the utility functions for risk and ambiguity are positive linear functions, then their composition, the utility function for optimism, is also a positive linear function.

Suppose U(x) = b· x and J(k) = a· k, where a and b are positive, then V(x)=J(U(x)) is also a positive linear utility function, where V(x) = c· x and c is the pointwise product of a and b. Hence the marginal utility of expenditures in the affective Afriat inequalities for V can be normalized to 1 for all the investor's elicited optimal choices.

Arbitrary systems of linear inequalities can be solved in polynomial time as a function of the number of inequalities and unknowns, using interior-point algorithms. This observation suggests approximation theorems for NP-hard systems of affective Afriat inequalities, where linear systems of inequalities are used for the approximations.

5 The Affective Fourfold Pattern of Decision-Making under Risk and Ambiguity,

To derive the Affective Fourfold Pattern of Decision-Making under Risk and Ambiguity, we cite the Composition theorem on Convex/Concave Functions introduced in Disciplined Convex Programming.

Theorem (Boyd, et al)

If f*:* R→(R U + oo) is *convex and nondecreasing and*

g: R^N→(R U + oo) is convex, then h = fog is convex.

If f: R→(R U + oo) is convex and nonincreasing and

g*:* R^N→(R U + oo) is concave, then fog is convex.

If f: R→(R U + oo) is concave and nondecreasing and

g: R^N→(R^N U + oo) is concave, then f o g is concave.

If f: R→(R U + oo) is concave and nonincreasing and

g: R^N→(R^N U + oo) is convex, then f o g is concave.

For affective utilities their theorem implies:

If J is concave and nondecreasing and U is concave, then the investor is pessimistic.

If J is concave and nonincreasing and U is convex, then the investor is pessimistic.

If J is convex and nondecreasing and U is convex, then the investor is optimistic.

If J is convex and nonincreasing and U is concave, then the investor is optimistic

The Fourfold Pattern of Decision-Making under Risk and Ambiguity in smooth APT derives from the

Fourfold Pattern for Decision-Making under Risk in Prospect Theory

Fourfold Pattern of Decision-Making under Risk and Ambiguity

	RISK	RISK
	AVERSE	SEEKING
AMBIGUITY	PESSIMISTIC	PESSIMISTIC
AVERSE	PREFERENCE S	PREFERENCE S
AMBIGUITY	OPTIMISTIC	OPTIMISTIC
SEEKING	PREFERENCE S	PREFERENCE S

In smooth APT, equivalent representations of smooth affective utilities, are smooth affective expected utilities, derived using the Legendre duality theorem for smooth convex functions. Assuming that the gradient of V(x) is 1 to 1 on the interior of X, the positive orthant of R^N, the chain rule is used to compute the gradient of V(x)=J(U(x)), hence the NP- hard complexity of solving the affective Afriat inequalities.

6 Nonsmooth APT

Legendre-Fenchel Duality is an alternative theory of duality for nonsmooth affective utilities, $V(x)$, where the bi-conjugate of $V(x)$, denoted $V^{**}(x)$, is the sup of all the convex functions majorized by $V(x)$ and the bi-conjugate of -V (x) is the inf of all the concave functions minorized by - $V(x)$. That is,

$\sup\{f(x) < V(x), \text{ where } f(x) \text{ is convex}\} < V(x) < \inf\{g(x) > V(x), \text{ where } g(x) \text{ is a concave}\}$

Denote the LHS of the inequality as $V_{LB}(X)$ and the RHS of the inequality as $V_{UB}(x)$

Then $V_{LB}(x) < V(x) < V_{UB}(x)$ where $V_{LB}(x)$ is convex, hence a Bull and $V_{UB}(x)$ is concave, hence a Bear.These are affective utility bounds, in the sense of Keynes that “best” approximate the investor’s true tolerances for risk, ambiguity and optimism, denoted V(x), as a Bull or Bear. Unfortunately V(x) is unknown. A computable proxy for V(x) is W(x), a solution of a system of relaxed convex Afriat Inequalities, where the marginal utility of income for W(x) is 1 in every observation. W(x) minimizes the l_1 error of approximation subject to the investor’s elicited optimal choices over systems of relaxed convex Afriat inequalities, indexed by the nonnegative scalar variable t.

This model defines an infinite family of feasible linear Program P_t for the data set $D = \{(x_1,p_1),(x_2,p_2),...(x_N,p_N)\}$, where p_k are the asset prices in period k

and $\langle p_k,x_k \rangle$ is the investor's expenditure in period

$$t^* = \inf t$$

$$\text{S.T } 0 \leq t$$

$$W(x_i)-W(x_j) < p_j \cdot (x_i - x_j) + t_j$$

$t^* = 0$ iff the convex, relaxed affective Afriat inequalities are feasible

and $W(x_k)=V(x_k)$ for k=1,2,...N)

To test feasibility of concave, relaxed affective Afriat inequalities for Z(x), we solve for each s, the linear program Q_s

$$s^* = \sup s = -\inf -s$$

$$\text{S.T. } 0 \leq s_i$$

$$p_l \cdot (x_i - x_j) - s_i \leq Z(x_i)-Z(x_j)$$

where $s^*=0$ iff the concave, affective Afriat inequalities are feasible

(P_t) and (Q_t) are linear systems of inequalities solvable in polynomial time, with interior point algorithms. Using Afriat's construction we construct a convex function $W_{LB}(x)=\max\{1<k<N\} : W(x_k) + p\cdot(x - x_k)\}+ t^*$

Using Afriat's construction we construct a concave function

$Z_{UB}(x)=\min\{1<k<N\} : V(x_k) + p\cdot (x - x_k)\}+ s^*$

These are the Keynesian approximating linear affective utility functions, with explicit bounds on the approximation errors as solutions of the dual linear programs. 8

7 Affective Utility Functions

The set of affective utility functions is a new class of non-expected utility functions representing preferences of investors for optimism or pessimism, defined as the composition of the investor's preferences for risk and her preferences for ambiguity. Bulls and bears are defined respectively as optimistic and pessimistic investors. Simply put, bulls are investing optimists who believe that asset prices will go up tomorrow, and bears are investing pessimists who believe that asset prices will go down tomorrow.

The fourfold pattern of preferences discussed in chapter 29 of Thinking Fast and Slow (2011) by Daniel Kahneman is described as "one of the core achievements of prospect theory”. In a 2x2 contingency table, where the columns are high probability. (certainty effect) and low probability (possibility effect).and the rows are gains and losses from the status quo. The entries in the four cells are illustrative prospects. One cell is a surprise, where in the high probability/losses cell. Kahneman and Tversky observe risk seeking with negative prospects, commonly referred to as loss aversion. In his insightful monograph, Kahneman identifies “three cognitive features at the heart of prospect theory. They play an essential role in the evaluation of financial outcomes…. The third principle is loss aversion.” Prospect theory and its generalization cumulative prospect theory are descriptive, psychological theories of decision making under risk, inspired by the Allais paradox. In the social sciences they are the preferred alternatives to the normative, axiomatic expected utility model of decision making under risk in Theory of Games (1944) by Von Neumann and Morgenstern.

In this paper, Affective Portfolio Theory or APT is a, descriptive, psychological theory of investing under, risk and ambiguity, where investors maximize affective expected utility, using affective probabilities.These probabilities differ from objective or subjective probabilities, since they may depend on affective outcomes in different states of the world.

In the Foundations of Statistics (1954) Savage, in postulate P2, explicitly excludes affective probabilities from his axiomatic derivation of subjective expected utility theory. In his seminal analysis of subjective probability theory, Risk, Ambiguity, and The Savage Axioms (1961), Daniel Ellsberg introduces the notion of ambiguity as an alternative to the notion of risk in decision making under uncertainty. That is, uncertainties that are not risks, where the probability of outcomes tomorrow are unknown or may not exist. In this case, non-expected utility models by Huriwitz (1957) and Ellsberg (1962) provide an alternative characterization of the investor's attitudes regarding risk, ambiguity and optimism. Their models are the origins of affective utility functions.

8 Smooth APT

Smooth Affective Portfolio Theory,or Smooth APT, extends the mean-variance model for optimizing portfolios of risky assets to optimizing portfolios of risky and ambiguous assets, such as bitcoin, digital currencies,volatility indices or any IPO, where the uncertainties regarding the portfolio’s future payoffs are not risks. That is, ambiguous assets are characterized by affective states of the world, where objective or subjective probabilities of future returns are ill-defined and may not exist.

This generalization prescribes affective interactive web sites defined by the SEC as Robo-advisors, that are programmed with affective portfolio theory in a suite of three personalized apps allowing investors, based on their affective preferences for risk, ambiguity and optimism, to hold optimal portfolios of risky and ambiguous assets spanned by mutual funds of bonds, stocks, and bitcoin. Investors with loss aversion can hedge losses in their optimal portfolios with minimum - cost portfolio insurance, where the unrealistic assumption of complete asset markets in MPT is replaced by the weaker assumption of complete derivative markets In A

In this paper Affective Portfolio Theory or APT is an alternative, descriptive, psychological theory of investing under risk and ambiguity.

Savage in the Foundations of Statistics (1954), in postulate P2, explicitly excludes affective probabilities from his axiomatic derivation of subjective expected utility theory. In his seminal analysis of subjective probability theory, Risk, Ambiguity, and The Savage Axioms (1961) Daniel Ellsberg introduces the notion of ambiguity as an alternative to the notion of risk in decision making under uncertainty, that is, uncertainties that are not risks, where the probability of outcomes are unknown or may not exist. In a series of thought experiments using urns with known and unknown distributions of colored balls, he conjectured that some individuals may violate , Savage's Postulate 2, the so-called SURE THING PRINCIPLE. These thought experiments have now been conducted many times in many classrooms and Ellsberg's conjecture has been confirmed. To fully appreciate Ellsberg's paradigm changing contribution to decision making under uncertainty, read his recently published Ph.D. dissertation: Risk, Ambiguity, and Decision (1962), This paper prescribes a suite of three personalized digital investment apps, programmed with affective portfolio theory which advise investors who wish to hedge uncertainties of ambiguous assets, such as bitcoin or volatility indices, where the uncertainties regarding returns in future states of the world are not risks. The first app, for each of the four types of quasilinear approximations to the investor's true affective preferences, rationalizes a stated history of the investor's past optimal portfolio selections and selects the best "quasilinear" approximation of the investor's true preferences. Unfortunately, the composition of quasilinear utility functions for risk and ambiguity need not be quasilinear.

The example presented in this paper illustrate polynomial time approximations to NP-hard affective Afriat inequalities where utility functions for risk and ambiguity are linear functions, a special class of quasilinear utility functions, that are closed under composition. L(x) is said to be linear if $L(x) = b \cdot x$, where for fixed $a \geq 0$ and arbitrary $x \geq 0$ in $(R)^N$. Suppose $U(x)=r \cdot x$ and $J(k)= a \cdot x$, then $V(x)=J(U(x))$ is also a linear utility function, where $V(x)=c \cdot x$ and $c =a*r$, the pointwise product of a and r. Hence the marginal utility of income in the affective Afriat inequalities for V is one for all observed optimal choices. That is $\mu p =p =\Delta V(x)$. The second app selects the optimal portfolio from a stated menu of the investor's potential future investments, using the output of the first app, the best quasilinear approximation.

The third app, given the investor's loss aversion, a stated lower bound on the losses of chosen optimal portfolio, using the output of the second app, hedges the investor's losses by computing the premium for minimum-cost portfolio insurance, The three apps are Android apps, cited as "the world's most popular operating system", by Walter and Sherman in Learning MIT App Inventor, (2015). MIT App Inventor is a visual programming language. MIT App Inventor is the suggested programming language for the suite of apps. A Google account gives the inventor of an app the opportunity to use Google Services, Google Data Bases and upload Android apps to Google Play Store for distribution.

Affective utility functions are defined as the composition of an investor's preferences for risk, her preferences for ambiguity, and her preferences for optimism That is, an investor may be risk averse or risk seeking and ambiguity averse or ambiguity seeking and optimistic or pessimistic. U(x) is a representation of the investors preferences for risk, and J(y) is a representation of the investors preferences for ambiguity, where y = U(x) for some limited liability state-contingent claim x.

V(x) = J(U(x)), the composition of U(x) and J(y), is a representation of the investor's preferences for optimism. In the decision-theoretic literature, averse preferences have strictly concave utility representations; seeking preferences have strictly convex utility representations. Following Keynes's characterization of bulls and bears, optimistic preferences have strictly convex utility representations; pessimistic preferences have strictly concave utility representations. This specification defines 4 types of affective utility functions that are consistent with affective decision making. The fourfold pattern of affective decision -making under risk and ambiguity is a 2x2 contingency table, where the columns are Risk Averse and Risk Seeking and the rows are Ambiguity Averse and Ambiguity Seeking. Entries in the cells are preferences for optimism derived from sufficient conditions, as specified in Lemma 1.in Grant, et al (2006), for the compositions of convex/ concave functions to be convex or concave.

If f*:* R→(RU+ oo) is *convex and nondecreasing and*

g: R^N→(RU+ oo) is convex, then h = fog is convex.

If f: R→(R U+ oo) is convex and nonincreasing and

g*:* R^N→(RU+ oo) is concave, then fog is convex.

If f: R→(RU+ oo) is concave and nondecreasing and

g: R^N→(R^N U+ oo) is concave, then f o g is concave.

If f: R→(RU+ oo) is concave and nonincreasing and

g: R^N→(R^N U+ oo) is convex, then f o g is concave.

In addition, similar rules are described for functions with multiple arguments.

Let f=J and g=U.

If J is concave and nondecreasing and U is concave, then the investor is pessimistic.

If J is concave and nonincreasing and U is convex, then the investor is pessimistic.

If J is convex and nondecreasing and U is convex, then the investor is optimistic.

If J is convex and nonincreasing and U is concave, then the investor is optimistic.

The Fourfold Pattern of Affective Decision-Making under Risk and Ambiguity

	RISK AVERSE	RISK SEEKING
AMBIGUITY AVERSE	PESSIMISTIC PREFERENCE	PESSIMISTIC PREFERENCE
AMBIGUITY SEEKING	OPTIMISTIC PREFERENCE	OPTIMISTIC PREFERENCE

9 Linear Rationalizations of Affective Asset Demands

Solving the affective Afriat inequalities for rationalizing asset demands of investors endowed with an affective utility functions is, in general, NP-hard. That is, in the worst case, the time it takes to solve a system of affective Afriat inequalities is exponential in the number of inequalities and unknowns. Arbitrary systems of linear inequalities can be solved in polynomial time as a function of the number of inequalities and unknowns, using interior -point algorithms. This observation suggests approximation theorems where NP-hard systems of inequalities are approximated by linear systems of inequalities , with a prior computable degree of approximation.

The computational complexity of solving systems of affective Afriat inequalities is a consequence of the first order conditions for maximizing a composite utility function subject to a budget constraint and the chain rule. Assuming $V(x)=J(U(x))$, where $U:X\to Y$, $J: Y\to R$. X is the family of limited liability assets or state-contingent claims, and Y is a family of state-utility vectors. If U is a diagonal N x N matrix, then $DV(x) = DU(x)\,[\Delta J(y)]$ is the pointwise product of diag[DU(x)] and [ΔJ(y)]. That is, in general, DV(x) is bilinear, hence the ensuing computational complexity. To approximate the bilinear Afriat inequalities with a system of linear inequalities, assume the scalar Bernoulli state-utility functions $w_j(x_j)$, and J(y), the ambiguity utility function, are linear utility functions. If the space of limited liability state-contingent claims state space is $X= (R^{N+1})_+$ then

$U: X\to R$ is linear, if $U(x) = a \cdot x$ for $a \geq 0$, and $x = (x_1,\ldots,x_s,\ldots,x_{N+1})$ is in X.

Choose the N+1 state-contingent claim as numeraire, which is $a = (a_1, a_2,\ldots a_N,1)$.

If $J: Y \rightarrow R$ is linear, where $J(y) = b\cdot y$ for $b \geq 0$, and $y = (y_1,\ldots,y_s,\ldots,y_{N+1})$

A test of the feasibility of the affective Afriat inequalities, is the relaxed affective Afriat inequalities defining the convex optimization problem:

$$t^* = \text{Min } t$$

$$\text{S.T. } 0 \leq t$$

$$V(x_i)\text{-}V(x_j) \leq p\cdot (x_i - x_j) + t$$

$$w(x_{i,s})\text{-}w(x_{i,r}) \leq dw(x_{i,r})(x_{i,s} - x_{i,r})+t^*$$

$$J(U(x_i))\text{-}J(U(x_j)) \leq p_j \text{ diag } [dw(x_{j,r})]^{=1} (U(x_i)\text{-}U(x_J))+t$$

$$[p_j \text{ diag}[dw(x_{j,r})]^{=1} - \Delta J(U(x_j)]^2 \leq t \ :$$

This is a quadratic program, hence solvable in polynomial time in CVX

t^* is a measure of the degree of approximation. That is, $t^* = 0$ if and only if the affective Afriat inequalities are feasible.

10 Induced Value Theory

The principal references are Experimental Economics: Induced Value Theory by V.L.Smith (1976) and An Experimental Study of Competitive Market Behavior by V.L.Smith (1962). Smith shared the Nobel prize in Economics in 2002 with Daniel Kahneman for their seminal contributions to the methodology of experimental economics. Kahneman's well known contribution is his joint work with Amos Tversky on Prospect Theory, discussed in chapter 1. Smith's contribution is summarized in the following quotation:from Smith's (1976) paper, pg.275." The concept of induced valuation (Smith 1973) depends upon the postulate of *non-satiation:* Given a *costless* choice between two alternatives, identical except that the first yields more of the reward medium (usually currency) than the second, the first will always be chosen (preferred)over the second, by an *autonomous* individual, *i.e., utility is a monotone* increasing function of the monetary reward, U(M), U' > 0.[pg 22-23] "

Smith then induces demand functions for consumers, endowed with smooth, concave, monotone increasing, utility functions, and induces supply functions for producers endowed with smooth, convex,monotone decreasing cost functions.As is well known, under these assumptions, a producer ‘s behavior in competitive markets is characterized by the profit function, where the prices of inputs are fixed and prices of outputs the intersection of the market supply and market demand curves define the competitive equilibrium prices.Smith induces individual demand and supply schedules that are independent.In affect, a 1 good model for several different goods.

Less well known, is that the profit function is the Legendre transform of the cost function. This suggests that the biconjugate of V(x)= J(U(x)) can be induced, eliminating the need to approximate theoretical affective utility functions by solving the affective Afriat inequalities as first order conditions for maximizing V(x) subject to budget constraints. Conditions where the computational complexity is Np-Hard,as a consequence of applying the chain rule to compute the first order conditions.

for a composite function. Moreover, the polynomial-time approximation theorem derived using revealed preference analysis produces problematic bounds on the degree of approximation error even for the simplistic linear approximation model of V’’(x), the Legendre bi-conjugate of V(x). If V’’(x) is the intended efficiently computable proxy for the unknown and unobservable V(x), then the portfolios chosen using the linear approximation may be poor approximations to the counterfactual portfolios selected by the true V(x). Bottom Line:

Revealed Preference Analysis *approximates* V''(x);Induced Value Theory *induces* V''(x),

Now let's consider the non-smooth case

11 Non-Smooth Affective Portfolio Theory

Nonsmooth affective portfolio theory, or nonsmooth APT, is a sequel to smooth affective portfolio theory,or smooth APT. This paper prescribes a refutable generalization of smooth APT, for rationalizing the recent, elicited, optimal portfolios of risky and ambiguous assets of investors endowed with nonsmooth, affective utilities.

The approximation theorem for NP-hard rationalizations of elicited portfolio data in this paper subsumes the linear approximation theorem for Np-hard rationalizations of investors endowed with smooth affective utilities.

The technical results are derived from three methodologies in convex analysis:

(a) Revealed Preference Analysis

(b) Legendre-Fenchel Duality Theory

The analysis in this section is an abridged summary of the specifications in the non-provisional (utility) patent application, Affective Portfolio Theory, patent pending May 23, 2019.

The origin of smooth APT is the Keynesian notion of affective equilibrium in financial markets. Keynes viewed the equilibrium prices in asset markets as a balance of the sales of bears, the pessimists, and the purchases of bulls, the optimists.

That is, "equilibrium prices in asset markets will be fixed at the point at which the sales of the bears and the purchases of the bulls are balanced" (Keynes, 1930). Keynes believed that It is the optimism and pessimism of investors not the risk and return of assets that determine equilibrium in financial markets. This is a theory of affective investing, where the prices of assets today equilibrate the optimism and pessimism of bulls and bears regarding future asset-payoffs In smooth APT, the equilibration of optimistic and pessimistic beliefs of investors is rationalized by investors maximizing affective utilities subject to budget constraints, defined by asset prices and the expenditures of investors.

Affective utilities represent the preferences of investors for optimism or pessimism, defined as the composition of the investor's preferences for risk and preferences for ambiguity.That is, an investor may be risk averse or risk seeking and ambiguity averse or ambiguity seeking and optimistic or pessimistic.

If U(x) is a representation of the investor’s preferences for risk, and J(y) is a representation of the investor’s preferences for ambiguity, where the state-utility vector y = U(x) for some limited liability state-contingent claim x, then V(x) = J(U(x)), the composition of U(x) and J(y), is a representation of the investor’s preferences for optimism.

2

We follow the decision-theoretic literature, where averse preferences have strictly concave utility representations and seeking preferences have strictly convex utility representations. In addition, smooth APT assumes all representations of preferences are smooth. Following Keynes, smooth APT assumes that optimistic preferences have strictly convex utility representations and pessimistic preferences have strictly concave utility representations.The fourfold pattern of affective decision making under risk and ambiguity is a 2x2 contingency table, where the columns are Risk Averse and Risk Seeking and the rows are Ambiguity Averse and

Ambiguity Seeking. Entries in the cells are preferences for optimism derived from sufficient conditions for the composition of convex and concave functions, in the Composition Theorem for Convex/Concave function proved in Disciplined Convex Programming.The affective Afriat inequalities in smooth APT are defined as the first order conditions for maximizing the composite utility function, V(x), subject to a budget constraint, where the gradient of V is computed with the chain rule. Solving the affective Afriat inequalities for rationalizing asset demands of investors endowed with smooth affective utility functions is, in general, NP-hard. That is, in the worst case, the time it takes to solve a system of affective Afriat inequalities is exponential in the number of inequalities and unknowns.

If U is a diagonal NxN matrix, then $DV(x) = DU(x)\,[\Delta J(y)]$ is the pointwise product of DU(x) and [ΔJ(y)]. That is, in general, DV(x) is bilinear, hence the ensuing NP-hard computational complexity.

The family of positive linear functions is a family of utility functions that are closed under composition, where L(x) is a positive linear function if $L(x) = d \cdot x$, for some fixed $d > 0$ and all $x > 0$ in $(R)^N$.

If the utility functions for risk and ambiguity are positive linear functions, then their composition, the utility function for optimism, is also a positive linear function. Suppose U(x) = b· x and J(k) = a· k, where a and b are positive, then V(x)=J(U(x)) is also a positive linear utility function, where V(x) = c· x and c is the pointwise product of a and b. Hence the marginal utility of expenditures in the affective Afriat inequalities for V can be normalized to 1 for all the investor's elicited optimal choices.

Arbitrary systems of linear inequalities can be solved in polynomial time as a function of the number of inequalities and unknowns, using interior-point algorithms. This observation suggests approximation theorems for NP-hard systems of affective Afriat inequalities, where linear systems of inequalities are used for the approximations.

The Affective Fourfold Pattern of Decision-Making under Risk and Ambiguity,is derived from the Composition theorem on Convex/Concave Functions, introduced in Disciplined Convex Programming.

Theorem (Boyd, et al)

If f*:* R→(R U + oo) is *convex and nondecreasing and*

g: R^N→(R U + oo) is convex, then h = fog is convex.

If f: R→(R U + oo) is convex and nonincreasing and

g: R^N→(R U + oo) is concave, then fog is convex.

If f: R→(R U + oo) is concave and nondecreasing and

g: R^N→(R^N U + oo) is concave, then f o g is concave.

If f: R→(R U + oo) is concave and nonincreasing and

g: R^N→(R^N U + oo) is convex, then f o g is concave.

For affective utilities their theorem implies:

If J is concave and nondecreasing and U is concave, then the investor is pessimistic.

If J is concave and nonincreasing and U is convex, then the investor is pessimistic.

If J is convex and nondecreasing and U is convex, then the investor is optimistic.

If J is convex and nonincreasing and U is concave, then the investor is optimistic.

The Fourfold Pattern of Decision-Making under Risk and Ambiguity in smooth APT derives from the Fourfold Pattern for Decision-Making under Risk in Prospect Theory.

Fourfold Pattern of Decision-Making under Risk and Ambiguity

	RISK AVERSE	RISK SEEKING
AMBIGUITY AVERSE	PESSIMISTIC PREFERENCE	PESSIMISTIC PREFERENCE
AMBIGUITY SEEKING	OPTIMISTIC PREFERENCE	OPTIMISTIC PREFERENCE

In smooth APT, equivalent representations of smooth affective utilities, are smooth affective expected utilities, derived using the Legendre duality theorem for smooth convex functions. Assuming that the gradient of V(x) is 1 to 1 on the interior of X, the positive orthant of R^N, the chain rule is used to compute the gradient of V(x)=J(U(x)), hence the NP- hard complexity of solving the affective Afriat inequalities.

Legendre-Fenchel Duality is an alternative theory of duality for nonsmooth affective utilities, $V(x)$, where the bi-conjugate of $V(x)$, denoted $V^{**}(x)$, is the sup of all the convex functions majorized by $V(x)$ and the bi-conjugate of -V (x) is the inf of all the concave functions minorized by - $V(x)$. That is,

sup {f(x)<V(x), where f(x) is convex} < V(x) < inf {g(x)> V(x), where g(x) is a concave}

Denote the LHS of the inequality as $V_{LB}(X)$ and the RHS of the inequality as $V_{UB}(x)$

Then $V_{LB}(x) < V(x) < V_{UB}(x)$ where $V_{LB}(x)$ is convex, hence a Bull and $V_{UB}(x)$ is concave, hence a Bear.These are affective utility bounds, in the sense of Keynes that "best approximate" the investor's true tolerances for risk, ambiguity and optimism, denoted V(x), as a Bull or Bear. Unfortunately V(x) is unknown. A computable proxy for V(x) is W(x), a solution of a system of relaxed convex Afriat Inequalities, where the marginal utility of income for W(x) is 1 in every observation. W(x) minimizes the l_1 error of approximation subject to the investor's elicited optimal choices over systems of relaxed convex Afriat inequalities, indexed by the nonnegative scalar variable t. This model defines an infinite family of feasible linear Program P_t for the data set $D = \{(x_1,p_1),(x_2,p_2),...(x_N,p_N)\}$, where p_k are the asset prices in period k and $<p_k,x_k>$ is the investor's expenditure in period k.

$$t^* = \inf t$$

$$\text{S.T } 0 \leq t$$

$$W(x_i)\text{-}W(x_j) < p_j \cdot (x_i - x_j) + t_j$$

$t^*=0$ iff the convex, relaxed affective Afriat inequalities are feasible 7

and $W(x_k)=V(x_k)$ for k=1,2,...N)

To test feasibility of concave, relaxed affective Afriat inequalities for Z(x), we solve for each s, the linear program Q_s

$$s^* = \sup s = -\inf -s$$

$$\text{S.T. } 0 \leq s_i$$

$$p_l \cdot (x_i - x_j) - s_i \leq Z(x_i) - Z(x_j)$$

where $s^*=0$ iff the concave, affective Afriat inequalities are feasible

(P_t) and (Q_t) are linear systems of inequalities that can be solved in polynomial time, with interior point algorithms. Using Afriat's construction we construct a convex function $W_{LB}(x)=\max\{1<k<N\} : W(x_k) + p \cdot (x - x_k)\} + t^*$

Using Afriat's construction we construct a concave function

$Z_{UB}(x)=\min\{1<k<N\} : V(x_k) + p \cdot (x - x_k)\} + s^*$

These are the Keynesian approximating piecewise linear affective utility functions, with explicit bounds on the approximation errors as solutions of the dual linear programs.

Subjective expected utility theory, originally proposed by Savage as the foundation of Bayesian statistics, is a theory of decision-making under uncertainty that "... does not leave room for optimism or pessimism to play any role in the person's judgment" (Savage, 1954, p. 68). This viewpoint is not the perspective of Keynes who viewed the equilibrium prices in asset markets as a balance of the sales of bears, the pessimists, and the purchases of bulls, the optimists. That is, "equilibrium prices in asset markets will be fixed at the point at which the sales of the bears and the purchases of the bulls are balanced" (Keynes, 1930). In Keynes, equilibrium in asset markets is an affective notion. It is the optimism and pessimism of investors.

The set of affective utility functions is a new class of non-expected utility functions representing preferences of investors for optimism or pessimism, defined as the composition of the investor's preferences for risk and her preferences for ambiguity. Bulls and bears are defined respectively as optimistic and pessimistic investors. Simply put, bulls are investing optimists who believe that asset prices will go up tomorrow, and bears are investing pessimists who believe that asset prices will go down tomorrow.

The fourfold pattern of preferences discussed in chapter 29 of Thinking Fast and Slow (2011) by Daniel Kahneman is described as “one of the core achievements of prospect theory”. In a 2x2 contingency table, where the columns are high probability. (certainty effect) and low probability (possibility effect).and the rows are gains and losses from the status quo. The entries in the four cells are illustrative prospects. One cell is a surprise, where in the high probability/losses cell. Kahneman and Tversky observe risk seeking with negative prospects, commonly referred to as loss aversion. In his insightful monograph, Kahneman identifies “three cognitive features at the heart of prospect theory. They play an essential role in the evaluation of financial outcomes…. The third principle is loss aversion.” Prospect theory and its generalization cumulative prospect theory are descriptive, psychological theories of decision making under risk, inspired by the Allais paradox. In the social sciences they are the preferred alternatives to the normative, axiomatic expected utility model of decision making under risk in Theory of Games (1944) by Von Neumann and Morgenstern. Affective Portfolio Theory or APT is a, descriptive, psychological theory of investing under, risk and ambiguity.

state-contingent claims chosen by the rational self. Affective probabilities differ from subjective probabilities in that they may depend on the outcomes in different states of the world.

In the Foundations of Statistics (1954) Savage, in postulate P2, explicitly excludes affective probabilities from his axiomatic derivation of subjective expected utility theory. In his seminal analysis of subjective probability theory, Risk, Ambiguity, and The Savage Axioms (1961), Daniel Ellsberg introduces the notion of ambiguity as an alternative to the notion of risk in decision making under uncertainty. That is, uncertainties that are not risks, where the probability of outcomes tomorrow are unknown or may not exist.

In this case, non-expected utility models by Huriwitz (1957) and Ellsberg (1962) provide an alternative characterization of the investor’s attitudes regarding risk, ambiguity and optimism. Their models are the provenance of affective utility functions.

If the objective or subjective state probabilities that define objective and subjective distributions of returns.Knight, Keynes and Fisher recognized the importance and existence of uncertainties in the market prices of commodities and financial assets that are not risks.The intellectual provenance of this manuscript is the recently published Harvard PH.D dissertation of Ellsberg, Risk, Ambiguity and Decision**,** where the affective state of mind is optimism or pessimism**,** anticipated by Keynes.The analogous affective state of mind in Fisher is patience and impatience, also anticipated by Keynes. In the Theory of Games and Economic Behavior by Von Neumann and Morgenstern, an axiomatic theory of decision-making under objective risk is introduced, where players maximize objective expected utility. In The Foundations of Statistics by Savage, an axiomatic theory of decision-making under subjective risk is introduced, where Bayesian decision-makers maximize subjective expected utility. Savage's axioms explicitly preclude affective state probabilities.In Risk, Ambiguity and Decision, Ellsberg presents a theory of decision-making under risk and ambiguity, where decision-makers maximize affective expected utility. Both the Theory of Games and The Foundations of Statistics have an associated "paradox" due respectively to Allais and Ellsberg that violate the stated axioms. Recently, cognitive psychologists, using fMRI, found that the neural mechanisms which govern decision-making under risk and decision-making under ambiguity are independent and are therefore consistent with the model of affective decision-making presented in this manuscript. In general, experimental economics has confirmed the “Ellsberg paradox” that decision-makers are often ambiguity averse or ambiguity seeking in decision-making under uncertainty. Consequently they violate the Savage axioms in The Foundations of Statistics.

12 Post Script

Robo-Advisors: A Portfolio Management Perspective, 2016,Lam

(A Yale Senior Essay advised by David Swensen)

Risk, Ambiguity and Decision, 2016, Ellsberg

Thinking Fast and Slow, 2010, Kahnman

The Black Swan, 2010, Taleb

Prospect Theory for Risk and Ambiguity, 2010,Wakker

Refutable Theories of Value, 2008, Brown and Kubler

Nudge, 2008, Thaler

Irrational Exuberance, 2000, Shiller,

The Theory of Unemployment, 1936, Keynes

The Theory of Profit, 1921, Knight

The Theory of Interest, 1907, Fisher

Appendix 1

Affective Decision-Making (ADM)

[1.a] CFDP 1898R

[1.b] CFDP 1891

Appendix 2

Revealed Preference Analysis

[2.a] CFDP 1507

[2.b] CFDP 1399R

Appendix 3

Computational Complexity

[3.a] CFDP 1865

[3.b] CFDP 1395R2

Appendix 4

Approximation Theorems

[4.a] CFDP 1955R2

[4.b] CFDP 1955R

Appendix 5

Experimental Economics

[5.a] CFDP 1890R

[5.b] CFDP 1774

Appendix 6

Econometrics

[6.a] CFDP 1526

[6.b] CFDP 1518