

# DUAL-SELF REPRESENTATIONS OF AMBIGUITY PREFERENCES

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# Dual-self representations of ambiguity preferences\*

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## Abstract

We propose a class of multiple-prior representations of preferences under ambiguity, where the belief the decision-maker (DM) uses to evaluate an uncertain prospect is the outcome of a game played by two conflicting forces, Pessimism and Optimism. The model does not restrict the sign of the DM’s ambiguity attitude, and we show that it provides a unified framework through which to characterize different degrees of ambiguity aversion, and to represent the co-existence of negative and positive ambiguity attitudes within individuals as documented in experiments. We prove that our baseline representation, *dual-self expected utility (DSEU)*, yields a novel representation of the class of invariant biseparable preferences (Ghirardato, Maccheroni, and Marinacci, 2004), which drops uncertainty aversion from maxmin expected utility (Gilboa and Schmeidler, 1989), while extensions of DSEU allow for more general departures from independence. We also provide foundations for a generalization of prior-by-prior belief updating to our model.

## 1 Introduction

### 1.1 Motivation and overview

A central approach to modeling preferences under ambiguity is based on the idea that the decision-maker (DM) quantifies uncertainty with a set of beliefs and may use a different belief from this set to evaluate each uncertain prospect. A well-known limitation underlying many such multiple-prior models—notably Gilboa and Schmeidler’s (1989) maxmin expected

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utility model and several of its generalizations<sup>1</sup>—is a restrictive mechanism of belief selection, whereby the DM evaluates each prospect according to the *worst* possible belief in her set. Behaviorally, this restriction is reflected by Schmeidler’s (1989) uncertainty aversion axiom, which captures a negative attitude towards ambiguity through a strong form of preference for hedging.

While consistent with Ellsberg’s seminal two-color urn experiment, the uncertainty aversion axiom has been questioned both by subsequent theoretical work, which has proposed alternative formalizations and measures of ambiguity aversion,<sup>2</sup> and by more recent experimental evidence. Indeed, this evidence points to more nuanced patterns of ambiguity attitudes: The same subjects may appear ambiguity-averse in some decision problems, but may also display *ambiguity-seeking* preferences in other notable settings, some of which we discuss below (for a survey, see Trautmann and van de Kuilen, 2015).

In this paper, we propose a decision-theoretic framework that provides a unified lens through which to represent and organize such mixed attitudes towards ambiguity. To do so, we introduce a class of multiple-prior representations that allows for a flexible mechanism of belief selection: Instead of assuming that the DM uses the worst possible belief to evaluate any given prospect, our representations adopt a “dual-self” perspective on ambiguity, by modeling the DM’s belief selection as the outcome of a game between two conflicting forces, *Pessimism* and *Optimism*.<sup>3</sup>

Our baseline representation generalizes maxmin expected utility by incorporating an ambiguity-seeking force via the addition of a maximization stage: Under *dual-self expected utility (DSEU)*, there is a compact collection  $\mathbb{P}$  of closed and convex sets of beliefs and an affine utility  $u$  such that the DM evaluates each (Anscombe-Aumann) act  $f$  according to

$$W_{\text{DSEU}}(f) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mathbb{E}_{\mu}[u(f)].$$

That is, the belief used to evaluate  $f$  is the outcome of a sequential zero-sum game: First, Optimism chooses a set of beliefs  $P$  from the collection  $\mathbb{P}$  with the goal of maximizing the DM’s expected utility to  $f$ ; then Pessimism chooses a belief  $\mu$  from  $P$  with the goal of minimizing expected utility. Maxmin expected utility corresponds to the extreme case in which Optimism has no choice, while the opposite extreme, maxmax expected utility, results

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<sup>1</sup>See, for example, Maccheroni, Marinacci, and Rustichini (2006); Chateauneuf and Faro (2009); Strzalecki (2011); Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011); Skiadas (2013).

<sup>2</sup>See, for instance, Epstein (1999); Ghirardato and Marinacci (2002); Baillon, L’Haridon, and Placido (2011); Dow and Werlang (1992); Baillon, Huang, Selim, and Wakker (2018).

<sup>3</sup>The idea that the DM consists of multiple strategic selves with conflicting motives is employed frequently in behavioral economics, for example to model risk preferences and intertemporal choices (e.g., Thaler and Shefrin, 1981; Bénabou and Pycia, 2002; Fudenberg and Levine, 2006; Brocas and Carrillo, 2008).

when Pessimism has no choice.

Our main results are threefold. First, we provide foundations for the DSEU model. Theorem 1 shows that DSEU represents the class of preferences that satisfy all of Gilboa and Schmeidler’s (1989) axioms except for uncertainty aversion; thus, the presence of ambiguity is captured solely by relaxing independence to certainty independence, without additionally restricting the DM’s ambiguity attitude to be negative (or positive). Beyond maxmin and maxmax expected utility, this important class of preferences—known as *invariant biseparable*—nests Choquet expected utility and  $\alpha$ -MEU as notable special cases. Section 1.2 contrasts DSEU with existing representations of invariant biseparable preferences due to Ghirardato, Maccheroni, and Marinacci (2004) and Amarante (2009); moreover, Section 4.3 shows that the dual-self approach extends beyond this class, as extensions of DSEU represent generalizations of invariant biseparable preferences that further relax certainty independence. Proposition 1 notes that any DSEU preference  $\succsim$  uniquely reveals a set of *relevant priors*  $C = \bigcup_{P \in \mathbb{P}} P$ , which represents all possible outcomes of the belief-selection game (up to convex closure and elimination of redundant beliefs). Sections 4.1–4.2 further discuss the uniqueness properties and comparative statics of the DSEU model.

Our second contribution is to exploit the structure of the DSEU model to represent and organize a range of natural intermediate ambiguity attitudes: In line with the aforementioned experimental evidence, these successively relax uncertainty aversion, by accommodating some degree of ambiguity-seeking behavior. The main insight is that, under DSEU, there is a correspondence between the degree of ambiguity aversion of the DM, as captured by the strength of her preference for hedging, and the extent of overlap of sets in  $\mathbb{P}$ , which measures the relative “power” allocated to Pessimism vs. Optimism in the belief-selection game. Section 3.1 formalizes this as follows:

- First, uncertainty aversion, i.e., a preference for *all* hedges, corresponds to the extreme case where the intersection of all sets in  $\mathbb{P}$  coincides with  $C$ . That is, *all* relevant priors are available to Pessimism regardless of Optimism’s action, thus rendering Optimism powerless.
- Second, we show that allocating more power to Optimism by only requiring the intersection of all sets in  $\mathbb{P}$  to be *nonempty* corresponds to Ghirardato and Marinacci’s (2002) notion of *absolute ambiguity aversion*: This only imposes a preference for *complete* hedges, i.e., for hedges that fully eliminate subjective uncertainty.
- Third, we further relax absolute ambiguity aversion, motivated in part by evidence that many individuals are simultaneously *ambiguity-averse for large/moderate-likelihood events* but *ambiguity-seeking for small-likelihood events*. For instance, in Example 1,

we illustrate this pattern in the context of a *many-color* urn experiment considered in the literature: Here, subjects are found to display ambiguity-seeking preferences when betting on the (small-likelihood) event of drawing a ball of any one color, in contrast with the ambiguity-averse behavior in Ellsberg’s two-color urn experiment.

We introduce the notion of *k-ambiguity aversion* (for some  $k = 2, 3, \dots$ ), which weakens absolute ambiguity aversion by imposing a preference for complete hedges *only among any k acts*. As we discuss, this makes it possible to formalize the above odds-dependent ambiguity attitudes, by imposing *k-ambiguity aversion* for small  $k$ , but not for large  $k$ . We show that under DSEU, *k-ambiguity aversion* is equivalent to the intersection of any  $k$  sets in  $\mathbb{P}$  being nonempty and, as a result, the model can accommodate flexible degrees of *k-ambiguity aversion*.

- Last, even 2-ambiguity aversion must be relaxed to accommodate another important behavioral pattern: In many settings, individuals appear *ambiguity-averse with respect to unfamiliar sources* of uncertainty (e.g., for investments in foreign stocks) but *ambiguity-seeking with respect to familiar sources* (e.g., for investments in domestic stocks). To model this pattern, we consider the sign of an event-based ambiguity aversion index that is commonly used in experimental work. We show that DSEU can flexibly accommodate source-dependent ambiguity attitudes, as the sign of the ambiguity aversion index is characterized by a “local” version of the binary intersection condition underlying 2-ambiguity aversion. By contrast, we prove that this phenomenon is incompatible with  $\alpha$ -MEU, a special case of DSEU that is often used to capture a mix of negative and positive ambiguity attitudes in applied work.

Finally, our third contribution (Section 3.2) is to propose and characterize a belief-updating rule for DSEU, paving the way for dynamic applications of the model. While updating rules have been defined and used in applied work for some special subclasses of invariant biseparable preferences, how to update this general class of preferences has remained an important open question in the literature. For maxmin expected utility, one of the most widely used updating rules is *prior-by-prior updating*, where the DM’s updated preference conditional on any event is obtained by Bayesian-updating each prior in her ex-ante belief-set  $P$ . Theorem 2 shows that this updating rule extends naturally to DSEU: In particular, while Pires (2002) characterizes prior-by-prior updating for maxmin expected utility based on a weak form of dynamic consistency (Axiom 9), we prove that imposing this same axiom under DSEU amounts to updating each belief-set  $P$  in the ex-ante collection  $\mathbb{P}$  prior-by-prior.

## 1.2 Related literature

This paper contributes to the decision-theoretic literature on preferences under ambiguity (for a survey, see [Gilboa and Marinacci, 2016](#)). Our first main result—in particular, the finding that our baseline model, DSEU, represents the class of invariant biseparable preferences—complements [Ghirardato, Maccheroni, and Marinacci \(2004\)](#) (henceforth GMM) and [Amarante \(2009\)](#). In contrast, our second and third contributions of characterizing intermediate ambiguity attitudes and defining an updating rule have no counterpart in these papers and rely heavily on the structure of DSEU: We briefly spell out this point for intermediate ambiguity attitudes below, while we illustrate some complications with defining potential analogs of prior-by-prior updating under GMM and Amarante’s models at the end of Section 3.2.

GMM introduce the class of invariant biseparable preferences to allow for nuanced ambiguity attitudes and to give a common framework to several important subcases. One of their key contributions is to provide a behavioral interpretation and analytical characterization (using Clarke differentials) of the set of relevant priors of an invariant biseparable preference, on which we build to construct a DSEU representation without redundant beliefs (see Section 2.3). GMM also show that every invariant biseparable preference  $\succsim$  admits a representation

$$W(f) = \alpha(f) \min_{\mu \in C} \mathbb{E}_\mu[u(f)] + (1 - \alpha(f)) \max_{\mu \in C} \mathbb{E}_\mu[u(f)], \quad (1)$$

where  $\alpha(\cdot)$  is a function from acts to  $[0, 1]$  and  $C$  is the set of relevant priors of  $\succsim$ . However, as GMM point out, the converse of this result does not hold without further joint restrictions on the model parameters  $(\alpha(\cdot), C, u)$ .<sup>4</sup> Similar to (1), DSEU provides a representation of invariant biseparable preferences that generalizes maxmin expected utility by incorporating a force for optimism, in the form of a max operator, into the DM’s belief-selection process. In contrast with (1), the DSEU representation is exact, in that *any* combination of the model parameters  $(\mathbb{P}, u)$  induces an invariant biseparable preference. This is key in enabling our characterization of intermediate ambiguity attitudes in terms of the structure of  $\mathbb{P}$ .

[Amarante \(2009\)](#) shows that the invariant biseparable axioms are both sufficient and necessary for a representation of the form

$$W(f) = \int_P \mathbb{E}_\mu[u(f)] d\nu(\mu), \quad (2)$$

where  $\nu$  is a Choquet capacity defined on some set of beliefs  $P \subseteq \Delta(S)$ . This representation suggests an alternative interpretation in terms of a robust Bayesian DM who uses a non-

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<sup>4</sup>Specifically,  $\alpha(\cdot)$  must be measurable with respect to a particular equivalence relation derived from  $u$  and  $C$  (to guarantee certainty independence), and  $\alpha(\cdot)$  and  $C$  must be such that  $\succsim$  is monotonic (see Remark 2 in GMM). Moreover, ensuring that  $C$  is the set of relevant priors of  $\succsim$  entails solving a fixed-point problem.

additive prior over probabilistic models. In contrast with our results for DSEU, there are no known characterizations of absolute and comparative ambiguity attitudes in terms of the model parameters in (2): Notably, unlike for Choquet expected utility (Schmeidler, 1989), uncertainty aversion (resp., absolute ambiguity aversion) does not imply convexity (resp., non-emptiness of the core) of  $\nu$ .

Our characterization of intermediate ambiguity attitudes is also an important difference from other papers that relax uncertainty aversion, including Schmeidler (1989), Klibanoff, Marinacci, and Mukerji’s (2005) smooth model, and models of preferences over utility dispersion (e.g., Siniscalchi, 2009; Grant and Polak, 2013): While some of these papers provide representations of absolute ambiguity aversion, none use their models to characterize weaker degrees of ambiguity aversion.

Related to the structure of DSEU, several recent papers employ belief-set or utility-set collections in other contexts. While we maintain the weak order axiom and focus on relaxing independence, Lehrer and Teper (2011), Nascimento and Riella (2011), Nishimura and Ok (2016), Hara, Ok, and Riella (2019), and Aguiar, Hjertstrand, and Serrano (2020) study preferences that violate completeness and/or transitivity.<sup>5</sup> Whereas DSEU is a utility representation, these papers provide generalized unanimity representations à la Bewley (2002) and Dubra, Maccheroni, and Ok (2004), and the resulting proof methods are quite different. In the context of attitudes to randomization under ambiguity, Ke and Zhang (2019) consider preferences over *lotteries* over acts and propose a representation that adds *minimization* over belief-set collections to maxmin expected utility. When restricted to acts (i.e., degenerate lotteries), their representation is equivalent to Gilboa and Schmeidler (1989).

Finally, Theorem 1 relates to results in mathematics on the linearization of positively homogeneous functions: These imply that a functional  $I : \mathbb{R}^S \rightarrow \mathbb{R}$  admits a so-called “Boolean” representation, where  $I(\phi) = \max_{U \in \mathbb{U}} \min_{\ell \in U} \ell \cdot \phi$  for some collection  $\mathbb{U}$  of compact, convex subsets of  $\mathbb{R}^S$ , if and only if  $I$  is positively homogeneous, lower semicontinuous, and locally Lipschitz (see the survey by Rubinov and Dzalilov, 2002). We show that under the additional assumption that  $I$  is monotonic and constant-additive,  $\mathbb{U}$  can be taken to be a belief-set collection. More importantly, our construction only makes use of beliefs  $\mu$  in the Clarke differential  $\partial I(\underline{0})$ , which represents the DM’s set of relevant priors. As we discuss (see Section 2.3), this requires a different proof approach.

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<sup>5</sup>See also Kopylov (2019) for an extension of maxmin expected utility that relaxes transitivity by allowing the set of priors to depend upon the acts under consideration. Mononen (2020) generalizes the DSEU model (and some of its extensions) by relaxing monotonicity, and shows how to identify subjective probabilities and state-dependent utilities for the resulting representations.

## 2 Dual-self expected utility

### 2.1 Setup

Let  $Z$  be a set of prizes and let  $\Delta(Z)$  denote the space of probability measures with finite support over  $Z$ .<sup>6</sup> We refer to typical elements  $p, q \in \Delta(Z)$  as lotteries. Let  $S$  be a finite set of states.<sup>7</sup> An (*Anscombe-Aumann*) act is a mapping  $f : S \rightarrow \Delta(Z)$ . Let  $\mathcal{F}$  be the space of all acts, with typical elements  $f, g, h$ . For any  $f, g \in \mathcal{F}$  and  $\alpha \in [0, 1]$ , define the mixture  $\alpha f + (1 - \alpha)g \in \mathcal{F}$  to be the act that in each state  $s \in S$  yields lottery  $\alpha f(s) + (1 - \alpha)g(s) \in \Delta(Z)$ . As usual, we identify each lottery  $p \in \Delta(Z)$  with the constant act that yields lottery  $p$  in each state  $s \in S$ .

Let  $\Delta(S)$  denote the set of all probability measures over  $S$ , which we embed in  $\mathbb{R}^S$  and endow with the Euclidean topology. We refer to typical elements  $\mu, \nu \in \Delta(S)$  as beliefs. Given any act  $f \in \mathcal{F}$  and map  $u : \Delta(Z) \rightarrow \mathbb{R}$ , let  $u(f)$  denote the element of  $\mathbb{R}^S$  given by  $u(f)(s) = u(f(s))$  for all  $s \in S$ , and let  $\mathbb{E}_\mu[u(f)] := \mu \cdot u(f)$ .

The DM's preference over  $\mathcal{F}$  is given by a binary relation  $\succsim$  on  $\mathcal{F}$ . As usual,  $\succ$  and  $\sim$  denote the asymmetric and symmetric parts of  $\succsim$ .

### 2.2 Representation

We now introduce our baseline representation, dual-self expected utility. Let  $\mathcal{K}(\Delta(S))$  denote the space of all nonempty closed, convex sets of beliefs, endowed with the Hausdorff topology. A *belief-set collection* is a nonempty compact collection  $\mathbb{P} \subseteq \mathcal{K}(\Delta(S))$ ; that is, each element  $P \in \mathbb{P}$  is a nonempty closed, convex set of beliefs.

**Definition 1.** A *dual-self expected utility (DSEU)* representation of preference  $\succsim$  consists of a belief-set collection  $\mathbb{P}$  and a nonconstant affine utility  $u : \Delta(Z) \rightarrow \mathbb{R}$  such that

$$W_{\text{DSEU}}(f) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mathbb{E}_\mu[u(f)] \quad (3)$$

represents  $\succsim$ .<sup>8</sup>

Just as Gilboa and Schmeidler's (1989) maxmin expected utility model, DSEU is a multiple-prior model of ambiguity preferences: The DM has in mind a set of beliefs  $\bigcup_{P \in \mathbb{P}} P$ , and might use a different belief from this set to evaluate each act. However, unlike maxmin

<sup>6</sup>All results also hold more generally if  $\Delta(Z)$  is replaced with any convex subset  $X$  of a vector space.

<sup>7</sup>The DSEU representation was also subsequently explored by Xia (2020), who shows that our main representation results (Theorems 1 and 3) extend to the case of an infinite state space.

<sup>8</sup>The functional (3) is well-defined since  $\mathbb{P}$  is nonempty and compact.

expected utility, the belief  $\mu$  used to evaluate any given act  $f$  is not necessarily worst-case among all the DM’s beliefs. Instead,  $\mu$  is the outcome of a sequential zero-sum game between two conflicting forces or “selves:” First, self 1 (“Optimism”) chooses an action  $P \in \mathbb{P}$  with the goal of maximizing expected utility to act  $f$ ; then self 2 (“Pessimism”) chooses an action  $\mu \in P$  with the goal of minimizing expected utility to  $f$ .

Maxmin expected utility is given by the extreme case where Optimism’s action set is trivial (i.e.,  $\mathbb{P} = \{P\}$  is a singleton), as in this case (3) reduces to  $W(f) = \min_{\mu \in P} \mathbb{E}_{\mu}[u(f)]$ . Likewise, maxmax expected utility,  $W(f) = \max_{\mu \in P} \mathbb{E}_{\mu}[u(f)]$ , corresponds to the opposite extreme where Pessimism’s action set is always trivial (i.e.,  $\mathbb{P} = \{\{\mu\} : \mu \in P\}$  is a collection of singletons).

Our first main result is that DSEU represents the class of preferences—known as *invariant biseparable*—that satisfy all subjective expected utility axioms, except that independence is relaxed to certainty independence:

**Axiom 1** (Weak Order).  $\succsim$  is complete and transitive.

**Axiom 2** (Monotonicity). If  $f, g \in \mathcal{F}$  and  $f(s) \succsim g(s)$  for all  $s \in S$ , then  $f \succsim g$ .

**Axiom 3** (Nondegeneracy). There exist  $f, g \in \mathcal{F}$  such that  $f \succ g$ .

**Axiom 4** (Archimedean). For all  $f, g, h \in \mathcal{F}$  with  $f \succ g \succ h$ , there exist  $\alpha, \beta \in (0, 1)$  such that

$$\alpha f + (1 - \alpha)h \succ g \succ \beta f + (1 - \beta)h.$$

**Axiom 5** (Certainty Independence). For all  $f, g \in \mathcal{F}$ ,  $p \in \Delta(Z)$ , and  $\alpha \in (0, 1]$ ,

$$f \succsim g \iff \alpha f + (1 - \alpha)p \succsim \alpha g + (1 - \alpha)p.$$

**Theorem 1.** *Preference  $\succsim$  satisfies Axioms 1–5 if and only if  $\succsim$  admits a DSEU representation.*

Thus, like maxmin expected utility, DSEU captures the possible *presence* of ambiguity by imposing independence only for mixtures with constant acts (Axiom 5). However, unlike maxmin expected utility, DSEU does not additionally impose uncertainty aversion, which reflects a *negative attitude* toward ambiguity through a preference for hedging (see Section 3.1.1). Certainty independence is weak enough to allow the model to nest important special cases such as Choquet expected utility and  $\alpha$ -MEU.<sup>9</sup> However, Section 4.3 will

<sup>9</sup>See also Ghirardato, Maccheroni, and Marinacci (2005), who argue why certainty independence is important for achieving a separation of tastes and beliefs.

show that natural generalizations of DSEU represent classes of preferences that further relax certainty independence.

We prove Theorem 1 in Appendix B.1. To understand the basic idea behind our construction of a DSEU representation, consider, for any act  $f$  and belief  $\mu$ , the constant act  $p_{\mu,f} := \sum_{s \in S} \mu(s) f(s) \in \Delta(Z)$ . That is, the distribution over outcomes in  $Z$  induced by  $p_{\mu,f}$  is equal to the distribution over outcomes that the DM expects under act  $f$  if her belief is  $\mu$ . Let

$$P_f := \{\mu \in \Delta(S) : p_{\mu,f} \succsim f\}.$$

Under Axioms 1–5, one can show that (the closure of) the collection

$$\mathbb{P} := \{P_f : f \in \mathcal{F}\}, \tag{4}$$

together with the utility  $u$  obtained from the DM’s preference over constant acts, yields a DSEU representation of  $\succsim$ . However, this representation potentially features some *redundant priors*, i.e., beliefs  $\mu$  that are never selected as an outcome of the game between Optimism and Pessimism. Thus, our proof of Theorem 1 adapts this construction, by replacing each belief-set  $P_f$  with its restriction  $P_f^*$  to the set of *relevant priors*, which we define in Section 2.3.

**Remark 1.** (i) **General action sets.** The specific form of action sets for Optimism and Pessimism in (3) is without loss of generality. Indeed,  $\succsim$  admits a DSEU representation with utility  $u$  if and only if there exist *arbitrary* action sets  $A_1, A_2$  and a mapping  $\mu : A_1 \times A_2 \rightarrow \Delta(S)$  from action profiles to beliefs such that

$$W(f) = \max_{a_1 \in A_1} \min_{a_2 \in A_2} \mathbb{E}_{\mu(a_1, a_2)}[u(f)] \tag{5}$$

is well-defined and represents  $\succsim$ .<sup>10</sup>

(ii) **Min-max form.** While DSEU takes the max-min form, where Optimism moves first, a natural alternative is to consider games where Pessimism is the first mover, as captured by the functional  $W(f) = \min_{Q \in \mathbb{Q}} \max_{\mu \in Q} \mathbb{E}_{\mu}[u(f)]$  for some belief-set collection  $\mathbb{Q}$ . It can be shown that the latter class of representations is equivalent to DSEU, in the sense that preference  $\succsim$  admits a DSEU representation  $(\mathbb{P}, u)$  if and only if  $\succsim$  admits a representation  $(\mathbb{Q}, u)$  of the min-max form for some belief-set collection  $\mathbb{Q}$ . However, for a given preference  $\succsim$ ,  $\mathbb{Q}$  need not coincide with  $\mathbb{P}$  in general. See Supplementary Appendix S.3 for details.

(iii) **Single-self interpretation.** In addition to the dual-self interpretation above, DSEU

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<sup>10</sup>To see this, suppose  $(\mathbb{P}, u)$  is a DSEU representation of  $\succsim$ . Then (5) represents  $\succsim$  with  $A_1 := \mathbb{P}$ ,  $A_2 := \prod_{P \in \mathbb{P}} P$ , and  $\mu(P, \sigma) := \sigma(P)$  for all  $P \in A_1$ ,  $\sigma \in A_2$ . Conversely, suppose (5) represents  $\succsim$  for some  $(A_1, A_2, \mu, u)$ . Then setting  $\mathbb{P} := \text{cl}\{\overline{\text{co}}(\mu(a_1, A_2)) : a_1 \in A_1\}$  yields a DSEU representation of  $\succsim$ .

admits a single-self interpretation, whereby the DM optimally selects her own ambiguity preference from a feasible set.<sup>11</sup> Specifically, feasible ambiguity preferences take the maxmin expected utility form  $\min_{\mu \in P} \mathbb{E}_\mu[u(f)]$  and depending on  $f$ , the DM optimally controls the parameter  $P$ , where  $\mathbb{P}$  represents the constraints of the subjective optimization.

(iv) **Finite representation.** The special case of DSEU where the belief-set collection  $\mathbb{P}$  is finite is characterized by a weak form of uncertainty aversion that imposes a preference for hedging only among acts  $f$  and  $g$  whose payoffs in all states are close enough. See Theorem 1 in the working paper version of [Chandrasekher \(2019\)](#). ▲

## 2.3 Relevant priors

For any DSEU representation  $(\mathbb{P}, u)$  of  $\succsim$ , the union  $\bigcup_{P \in \mathbb{P}} P$  captures the set of beliefs that might be selected as an outcome of the corresponding game between Optimism and Pessimism. To eliminate redundant beliefs that are never selected, we consider the *smallest* closed, convex set of beliefs that can arise under any representation. The next result shows that this set is uniquely identified from  $\succsim$ :

**Proposition 1.** *Suppose  $\succsim$  satisfies Axioms 1–5. There exists a unique closed, convex set  $C \subseteq \Delta(S)$  such that, for all DSEU representations  $(\mathbb{P}, u)$  of  $\succsim$ ,*

$$C \subseteq \overline{\text{co}} \bigcup_{P \in \mathbb{P}} P, \tag{6}$$

*and such that (6) holds with equality for some representation  $(\mathbb{P}, u)$ .*

We refer to the set  $C$  as the DM’s set of **relevant priors**, and call a DSEU representation **tight** if (6) holds with equality.

GMM provide an alternative behavioral definition of the set of relevant priors, which is based on quantifying departures from independence. For any invariant biseparable preference  $\succsim$ , GMM define the associated **unambiguous preference**  $\succsim^*$  as the largest independent subrelation of  $\succsim$ ; equivalently,  $f \succsim^* g$  means that  $\alpha f + (1 - \alpha)h \succsim \alpha g + (1 - \alpha)h$  holds for all  $\alpha \in (0, 1]$  and  $h \in \mathcal{F}$ . Note that  $\succsim^*$  is incomplete whenever  $\succsim$  violates independence. GMM prove that  $\succsim^*$  admits a unanimity representation à la [Bewley \(2002\)](#) and identify the DM’s set of relevant priors with the unique closed, convex set of beliefs in this unanimity representation (i.e., the Bewley set of  $\succsim^*$ ).<sup>12</sup> The following result shows that our approach

<sup>11</sup>See [Sarver \(2018\)](#) for an analogous model in the context of risk preferences.

<sup>12</sup>[Ghirardato and Siniscalchi \(2012\)](#) extend GMM’s characterization of relevant priors beyond the invariant biseparable class.

of identifying the set of relevant priors with the (smallest) set of outcomes of the DSEU belief-selection game is equivalent to GMM’s behavioral definition:

**Corollary 1.** *If  $\succsim$  admits a DSEU representation with utility  $u$ , then the set of relevant priors  $C$  identified in Proposition 1 is the Bewley set of  $\succsim^*$ , i.e., for any  $f, g \in \mathcal{F}$ ,*

$$f \succsim^* g \iff \mathbb{E}_\mu[u(f)] \geq \mathbb{E}_\mu[u(g)] \text{ for all } \mu \in C. \quad (7)$$

Both Proposition 1 and Corollary 1 rely on the following important observations due to GMM. GMM show that any invariant biseparable preference  $\succsim$  can be represented by  $I \circ u$  for some nonconstant affine utility  $u$  and a functional  $I : \mathbb{R}^S \rightarrow \mathbb{R}$  that is monotonic, positively homogeneous, and constant-additive (Appendix A.1 defines these terms). They then prove that the Clarke differential  $\partial I(\underline{0}) \subseteq \Delta(S)$  of  $I$  evaluated at the constant vector  $\underline{0}$  (Clarke, 1990, see Appendix A.2) coincides with the Bewley set of  $\succsim^*$ .

To prove Proposition 1 (Appendix B.2), we show that for any DSEU representation of  $\succsim$ ,  $\overline{\text{co}} \bigcup_{P \in \mathbb{P}} P$  includes the Clarke differential  $\partial I(\underline{0})$ . Moreover, building partly on a non-smooth generalization of results in Ovchinnikov (2001), our proof of the sufficiency direction of Theorem 1 obtains a DSEU representation  $(\mathbb{P}^*, u)$  of  $\succsim$  that replaces each belief-set  $P_f$  in (4) with its subset  $P_f^* := P_f \cap \partial I(\underline{0})$ . Thus,  $\mathbb{P}^*$  is a tight representation and the set of relevant priors  $C$  identified by Proposition 1 also coincides with the Clarke differential  $\partial I(\underline{0})$ . Combined with GMM’s observations above, this implies Corollary 1.

### 3 Properties of the DSEU representation

In this section, we highlight two important properties of the DSEU model: Section 3.1 illustrates how varying the degree of overlap of sets in  $\mathbb{P}$  allows one to represent and organize a range of natural intermediate ambiguity attitudes, motivated in part by evidence that individuals display a mix of ambiguity-averse and ambiguity-seeking tendencies. Section 3.2 shows that the DSEU model yields a natural way to perform belief updating under invariant biseparable preferences.

#### 3.1 Intermediate ambiguity attitudes

##### 3.1.1 Shades of ambiguity aversion

We first show how DSEU can represent a range of different shades of ambiguity aversion that vary in the degree to which they impose a preference for hedging. First, Schmeidler’s (1989)

seminal uncertainty aversion axiom postulates that the DM *always* takes up an opportunity to hedge between any equally valued prospects:

**Axiom 6** (Uncertainty Aversion). If  $f, g \in \mathcal{F}$  with  $f \sim g$ , then  $\alpha f + (1 - \alpha)g \succsim f$  for any  $\alpha \in [0, 1]$ .

A second common definition of ambiguity aversion is due to [Ghirardato and Marinacci \(2002\)](#): Recall the standard comparative notion of ambiguity aversion, whereby  $\succsim_1$  is *more ambiguity-averse* than  $\succsim_2$  if, whenever  $f \succsim_1 p$  for some  $f \in \mathcal{F}$  and  $p \in \Delta(Z)$ , then  $f \succsim_2 p$ . Analogous to the definition of absolutely risk-averse as more risk-averse than a risk-neutral preference,  $\succsim$  is said to be *absolutely ambiguity-averse* if it is more ambiguity-averse than some nondegenerate subjective expected utility (SEU) preference.<sup>13</sup> Arguments in [Grant and Polak \(2013\)](#) imply that, under DSEU, absolute ambiguity aversion is equivalent to relaxing uncertainty aversion to preference for sure diversification, generalizing an observation due to [Chateauneuf and Tallon \(2002\)](#) for Choquet expected utility. The latter condition only postulates a preference for complete hedges, i.e., mixtures of acts that eliminate subjective uncertainty entirely. Formally, a *complete hedge* for acts  $f_1, \dots, f_k \in \mathcal{F}$  is a constant act  $p \in \Delta(Z)$  such that  $\sum_{i=1}^k \alpha_i f_i = p$  for some  $\alpha_i \in [0, 1]$  with  $\sum_{i=1}^k \alpha_i = 1$ .

**Axiom 7** (Preference for Sure Diversification). For all  $k$  and  $f_1, \dots, f_k \in \mathcal{F}$  with  $f_1 \sim \dots \sim f_k$ , if  $p \in \Delta(Z)$  is a complete hedge for  $f_1, \dots, f_k$ , then  $p \succsim f_1$ .

The following result shows that, under DSEU, these two notions of ambiguity aversion are characterized by different amounts of overlap between the sets in  $\mathbb{P}$ :

**Proposition 2.** *Suppose that  $\succsim$  admits a DSEU representation  $(\mathbb{P}, u)$ . Then:*

1.  $\bigcap_{P \in \mathbb{P}} P = C$  if and only if  $\succsim$  satisfies uncertainty aversion;
2.  $\bigcap_{P \in \mathbb{P}} P \neq \emptyset$  if and only if  $\succsim$  is absolutely ambiguity-averse, which in turn holds if and only if  $\succsim$  satisfies preference for sure diversification.

We note that the intersection  $\bigcap_{P \in \mathbb{P}} P$  of all sets in  $\mathbb{P}$  is uniquely identified from the preference  $\succsim$ , and that a greater amount of overlap captures a sense in which Pessimism is allocated more “power” in the underlying belief-selection game.

In particular, by the first part of Proposition 2, uncertainty aversion corresponds to the maximal allocation of power to Pessimism: Since  $\bigcap_{P \in \mathbb{P}} P = C$ , all relevant priors  $\mu \in C$

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<sup>13</sup>See [Epstein \(1999\)](#) for another approach that takes as its benchmark probabilistic sophistication instead of subjective expected utility.

are available to Pessimism, no matter which set  $P \in \mathbb{P}$  Optimism chooses. The game thus boils down to Pessimism choosing a belief  $\mu \in C$ , yielding maxmin expected utility; indeed, if  $(\mathbb{P}, u)$  is tight, then  $\succsim$  satisfies uncertainty aversion if and only if  $\mathbb{P} = \{C\}$ .

In contrast, by the second part, absolute ambiguity aversion allocates less power to Pessimism, requiring only that there is *some* prior  $\mu \in \bigcap_{P \in \mathbb{P}} P$  that is always available to Pessimism regardless of Optimism's choice. The DM's evaluation of any act  $f$  is then bounded above by the expected utility  $\mathbb{E}_\mu[u(f)]$  of  $f$  under belief  $\mu$ . In the special case when  $\succsim$  admits a Choquet expected utility representation with capacity  $\nu$ , we note that  $\bigcap_{P \in \mathbb{P}} P$  coincides with the core of  $\nu$ ; thus, our nonempty intersection condition generalizes the fact that, in this case, absolute ambiguity aversion is characterized by the nonemptiness of the core of  $\nu$ .

However, absolute ambiguity aversion is still too strong to capture the following behavior that was originally conjectured by Ellsberg (see Ellsberg, 2011) and subsequently confirmed in laboratory experiments (e.g., Dimmock, Kouwenberg, Mitchell, and Peijnenburg, 2015; Kocher, Lahno, and Trautmann, 2018):

**Example 1** (Many-color Ellsberg urn). An urn of unknown composition contains balls of up to 10 possible colors. A ball is drawn from the urn and its color observed. When given the choice between receiving \$10 if the observed color is one of *five* possible colors vs. receiving \$10 with probability 0.5, most subjects prefer the objective lottery, similar to the ambiguity-averse behavior predicted by Ellsberg's two-color urn experiment. However, when the choice is between receiving \$10 if the observed color is a *single* possible color vs. receiving \$10 with probability 0.1, many subjects strictly prefer the former uncertain bet.

Formally, let the state space  $S = \{1, \dots, 10\}$  represent the observed color, let  $f_E$  denote the uncertain bet that pays \$10 if the observed color belongs to  $E \subseteq S$  and \$0 otherwise, and let  $p_\alpha$  denote the objective lottery that pays \$10 with probability  $\alpha$  and \$0 otherwise. Assume that, by symmetry, subjects are indifferent between betting on any two sets of colors with the same cardinality, i.e.,  $f_E \sim f_F$  whenever  $|E| = |F|$ .

Then, when  $|E| = 5$ , the above evidence can be written as

$$p_{0.5} = \frac{1}{2}f_E + \frac{1}{2}f_{E^c} \succ f_E \sim f_{E^c}.$$

However, when  $|E| = 1$ , the above evidence implies the opposite preference pattern,

$$p_{0.1} = \frac{1}{10}f_{\{1\}} + \dots + \frac{1}{10}f_{\{10\}} \prec f_{\{1\}} \sim \dots \sim f_{\{10\}},$$

which violates preference for sure diversification. ▲

To capture this behavior, we introduce the following axiom, which postulates a preference for complete hedges only between a *given number*  $k$  of equally valued acts:

**Axiom 8** ( $k$ -Ambiguity Aversion). For all  $f_1, \dots, f_k \in \mathcal{F}$  with  $f_1 \sim \dots \sim f_k$ , if  $p \in \Delta(Z)$  is a complete hedge for  $f_1, \dots, f_k$ , then  $p \succsim f_1$ .

Preference for sure diversification requires  $k$ -ambiguity aversion for *all*  $k$ .<sup>14</sup> In contrast, the behavior in Example 1 is consistent with 2-ambiguity aversion, but not with 10-ambiguity aversion. More generally, if a DM displays  $k$ -ambiguity aversion for small  $k$  but not for large  $k$ , this formalizes a sense in which the DM is *ambiguity-averse for large/moderate-likelihood events* but *ambiguity-seeking for small-likelihood events*.

This pattern can be interpreted as an analog for choice under ambiguity of the inverse S-shaped probability distortion that underlies prospect theory for choice under risk.<sup>15</sup> Indeed, generalizing Example 1, consider a partition of the state space into (possibly asymmetric) events  $E_1, \dots, E_k$ , where  $k$  parametrizes how fine the partition is. Consider acts  $f_{E_i}$  that yield a positive prize  $x$  with probability  $q_i$  in event  $E_i$  and yield 0 otherwise. If  $f_{E_1} \sim \dots \sim f_{E_k}$ , the DM's subjective probability  $\mu^k$  of winning the prize is the same under each bet  $f_{E_i}$ . At the same time, a complete hedge for acts  $f_{E_1}, \dots, f_{E_k}$  must yield the prize with some certain probability  $\pi^k$ , which is independent of the DM's preference and thus can be thought of as a benchmark objective winning probability associated with bets  $f_{E_i}$ . While  $\mu^k = \pi^k$  under SEU,  $k$ -ambiguity aversion requires that  $\mu^k \leq \pi^k$ . Moreover, the larger  $k$  (i.e., the finer the partition), the smaller are both  $\mu^k$  and  $\pi^k$ . Thus, a DM who is  $k$ -ambiguity averse for small  $k$  but not for large  $k$  subjectively underweights moderate or large objective winning odds but might overweight small odds.<sup>16</sup> Beyond the above urn example, such patterns of probability distortion have been found to be relevant in many economic applications, from financial investments to betting markets (see, e.g., the survey by Barberis, 2013).

Under DSEU,  $k$ -ambiguity aversion is characterized by further limiting the power of Pessimism. Indeed, relative to absolute ambiguity aversion, which requires all belief-sets in  $\mathbb{P}$  to overlap, the following result requires this only for any  $k$  sets in  $\mathbb{P}$ :

**Proposition 3.** *Suppose that  $\succsim$  admits a DSEU representation  $(\mathbb{P}, u)$ . Then the following are equivalent:*

<sup>14</sup>This can in turn be shown to be equivalent to  $|S|$ -ambiguity aversion, where  $|S|$  is the cardinality of the state space. See Lemma 1 in the previous version, Chandrasekher, Frick, Iijima, and Le Yaouanq (2020).

<sup>15</sup>See Chapter 12 in Wakker (2010) for another formalization of probability weighting in the domain of ambiguity.

<sup>16</sup>Capturing overweighting (resp. underweighting) of small (resp. large) *losing* odds (i.e., if  $x < 0$ ) would instead require  $k$ -ambiguity *seeking* for small  $k$  but not large  $k$ . Consistent with this, there is evidence that subjects display opposite ambiguity attitudes for gains vs. losses, which can be accommodated by generalizations of DSEU that relax certainty independence; see Section 4.3.

1.  $\bigcap_{i=1,\dots,k} P_i \neq \emptyset$  for all  $P_1, \dots, P_k \in \mathbb{P}$ ;
2.  $\succsim$  satisfies  $k$ -ambiguity aversion;
3. for all  $f_1, \dots, f_k \in \mathcal{F}$ , there exists a nondegenerate SEU preference  $\hat{\succsim}$  such that, whenever  $f \succsim p$  for some  $f \in \{\alpha f_i + (1 - \alpha)q : i = 1, \dots, k, q \in \Delta(Z), \alpha \in [0, 1]\}$  and  $p \in \Delta(Z)$ , then  $f \hat{\succsim} p$ .

To interpret the first part, suppose the DM faces  $k$  uncertain acts  $f_1, \dots, f_k$ . While Optimism might potentially choose different belief-sets  $P_1, \dots, P_k \in \mathbb{P}$  depending on each act, the condition requires there to be at least one prior  $\mu \in \bigcap_{i=1,\dots,k} P_i$  that Pessimism can choose in response to all  $P_i$ . Thus, the DM's evaluation of each  $f_i$  is bounded above by its expected utility  $\mathbb{E}_\mu[u(f_i)]$  under  $\mu$ . Equivalently, while absolute ambiguity aversion requires the DM to be more ambiguity-averse than some benchmark SEU preference in *all* decision problems, the third part requires this comparison only for restricted decision problems whose uncertainty can be summarized by  $k$  acts. Further generalizing the above example on odds-dependent ambiguity attitudes, this allows the DM's ambiguity attitude to vary with the “richness” of the uncertainty underlying each decision problem.

Based on Proposition 3, it is easy to see that DSEU allows for flexible degrees of  $k$ -ambiguity aversion, and hence can accommodate behavior such as Example 1.<sup>17</sup> To illustrate, consider the following important special case of DSEU: For any  $\alpha \in [0, 1]$  and nonempty closed, convex set of beliefs  $P$ , the representation  $(\mathbb{P}, u)$  with  $\mathbb{P} = \{\alpha P + (1 - \alpha)\{\mu\} : \mu \in P\}$  yields the widely used  $\alpha$ -**MEU model**, where  $\succsim$  is represented by the functional

$$W(f) = \alpha \min_{\mu \in P} \mathbb{E}_\mu[u(f)] + (1 - \alpha) \max_{\mu \in P} \mathbb{E}_\mu[u(f)]. \quad (8)$$

Then, Proposition 3 implies that (8) satisfies  $k$ -ambiguity aversion for all  $P$  if and only if  $\alpha \geq 1 - 1/k$ .<sup>18</sup>

<sup>17</sup>This contrasts, for instance, with Siniscalchi's (2009) vector expected utility model, which also relaxes uncertainty aversion, but for which 2-ambiguity aversion and preference for sure diversification are equivalent. Dillenberger and Segal (2017) show that a version of Segal's (1987) model is consistent with ambiguity-seeking for small odds.

<sup>18</sup>If  $\alpha \geq 1 - \frac{1}{k}$ , take any  $P$  and  $\nu_1, \dots, \nu_k \in P$ . Then  $\mu_\ell := \frac{1}{\alpha}(\frac{1}{k} - 1 + \alpha)\nu_\ell + \sum_{j \neq \ell} \frac{1}{\alpha k} \nu_j \in P$  for all  $\ell = 1, \dots, k$ , as  $P$  is convex and  $\frac{1}{k} - 1 + \alpha \geq 0$ . Also,  $\alpha \mu_\ell + (1 - \alpha)\nu_\ell = \sum_{j=1}^k \frac{1}{k} \nu_j$  for all  $\ell$ . Thus,  $\sum_{j=1}^k \frac{1}{k} \nu_j \in \bigcap_{\ell=1}^k \alpha P + (1 - \alpha)\{\nu_\ell\}$ , whence Proposition 3 implies  $k$ -ambiguity aversion. Conversely, if  $\alpha < 1 - \frac{1}{k}$ , consider  $P = \Delta(S)$  and  $P_\ell := \alpha P + (1 - \alpha)\{\delta_{s_\ell}\}$  for distinct  $s_1, \dots, s_k \in S$ . If  $\mu \in \bigcap_{\ell=1}^k P_\ell$ , then  $\mu(s_\ell) \geq 1 - \alpha > \frac{1}{k}$  for all  $\ell$ , which is impossible. Thus, by Proposition 3,  $k$ -ambiguity aversion is violated.

### 3.1.2 Source-dependent ambiguity attitudes

While the preceding notions of ambiguity aversion are “global,” capturing the DM’s attitude towards *any* uncertainty that can be generated in  $S$ , the experimental literature commonly takes a “local” approach, measuring the DM’s ambiguity attitude relative to specific events or sources of uncertainty. As noted in the introduction, an important finding is that a DM might display *source-dependent negative and positive ambiguity attitudes*, depending on whether she considers herself familiar or unfamiliar with a given source of uncertainty.

To formalize this idea, we use a local index of ambiguity attitude that was originally proposed by [Schmeidler \(1989\)](#) and subsequently employed in both theoretical work ([Dow and Werlang, 1992](#)) and experiments ([Baillon and Bleichrodt, 2015](#); [Baillon, Huang, Selim, and Wakker, 2018](#)):

**Definition 2.** The *matching probability*  $m(E) \in [0, 1]$  of an event  $E$  is defined by the indifference condition

$$xEy \sim m(E)\delta_x + (1 - m(E))\delta_y,$$

where  $x, y \in Z$  are two outcomes such that  $\delta_x \succ \delta_y$  and  $xEy$  denotes the binary act that yields  $x$  for all  $s \in E$  and  $y$  otherwise.<sup>19</sup> The *ambiguity aversion index* of  $E$  is

$$AA(E) := 1 - m(E) - m(E^c). \tag{9}$$

Whereas SEU implies  $AA(E) = 0$  for all  $E$ ,  $AA(E) > 0$  (resp.  $AA(E) < 0$ ) is interpreted as a negative (resp. positive) attitude to ambiguity associated with  $E$ . In particular, the aforementioned evidence suggests that a DM might display  $AA(E) > 0$  when  $E$  is conditioned on an unfamiliar source of uncertainty, but might display  $AA(E) < 0$  when she feels particularly competent about the relevant source:

**Example 2** (Source-dependent ambiguity attitudes). As a stylized example related to the “home bias” phenomenon ([French and Poterba, 1991](#); [Coval and Moskowitz, 1999](#)), let  $S_H = \{U, D\}$  be a state space specifying whether the domestic stock market goes up (“U”) or down (“D”). Similarly, let  $S_F = \{U, D\}$  describe the state of the stock market in a foreign country. Consider the product state space  $S = S_H \times S_F$ , and let  $E_H = \{UU, UD\}$  be the event that the domestic stock market goes up, and  $E_F = \{UU, DU\}$  be the corresponding event for the foreign stock market. Evidence in [Anantanasuwong, Kouwenberg, Mitchell, and Peijnenberg \(2019\)](#) suggests that some investors display  $AA(E_F) > 0 > AA(E_H)$ , capturing negative ambiguity attitudes towards foreign investments but positive attitudes towards domestic

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<sup>19</sup>Under Axioms 1–5,  $m(\cdot)$  is well-defined independent of the choice of  $x, y$ .

investments.<sup>20</sup> ▲

To see how DSEU can capture this pattern, we note that the sign of  $AA(E)$  is characterized by the following local analog of the binary intersection condition for 2-ambiguity aversion in Proposition 3. Given an event  $E$  and set of beliefs  $P$ , let  $P(E) := \{\mu(E) : \mu \in P\}$ .

**Proposition 4.** *Suppose  $\succsim$  admits a DSEU representation  $(\mathbb{P}, u)$ , and let  $E \subseteq S$ . Then:*

1.  $AA(E) \geq 0 \iff P(E) \cap P'(E) \neq \emptyset$  for all  $P, P' \in \mathbb{P}$ ;
2.  $AA(E) > 0 \iff P(E) \cap P'(E)$  is a non-degenerate interval for all  $P, P' \in \mathbb{P}$ .

Thus, while 2-ambiguity aversion implies that  $AA(E) \geq 0$  for all events  $E$ , further limiting the overlap of sets in  $\mathbb{P}$  can accommodate the behavior in Example 2. Indeed, the following result shows that DSEU can capture source-dependent negative and positive ambiguity attitudes with respect to *any* families  $\mathcal{E}$  and  $\mathcal{F}$  of unfamiliar and familiar events:<sup>21</sup>

**Corollary 2.** *Fix any disjoint collections  $\mathcal{E}$  and  $\mathcal{F}$  of events, both of which are closed under complements and do not contain  $S$ . There exists a DSEU representation  $(\mathbb{P}, u)$  whose induced preference satisfies  $AA(E) > 0 > AA(F)$  for all  $E \in \mathcal{E}, F \in \mathcal{F}$ .*

Corollary 2 highlights an important distinction between the general DSEU model and its special case given by the  $\alpha$ -MEU representation (8). Indeed, while the  $\alpha$ -MEU model is widely used in applied work to capture a mix of negative and positive ambiguity attitudes, the following result shows that it is incompatible with the source-dependent variation in ambiguity attitudes formalized above. This is because Proposition 4 applied to  $\mathbb{P} = \{\alpha P + (1 - \alpha)\{\mu\} : \mu \in P\}$  implies that under  $\alpha$ -MEU the sign of the ambiguity aversion index is the *same* for all events and is determined by the value of  $\alpha$ :

**Corollary 3.** *Suppose  $\succsim$  admits an  $\alpha$ -MEU representation where  $P$  is not a singleton. Then  $\alpha \geq 1/2$  (resp.  $\alpha \leq 1/2$ ) if and only if  $AA(E) \geq 0$  (resp.  $AA(E) \leq 0$ ) for all  $E$ .*

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<sup>20</sup>Anantanasuwong, Kouwenberg, Mitchell, and Peijnenberg (2019) conduct an incentivized field survey among investors and find reversals in ambiguity attitudes as in Example 2, where  $H$  and  $F$  correspond to a domestic and foreign stock market index (see Figures 4 and 5). They also find a higher population average  $AA$  index for  $E_F$  than  $E_H$ , but the difference is relatively small, as some investors display the opposite reversal. Similarly, in an experiment involving German subjects, Keppe and Weber (1995) find that the average ambiguity index is negative (resp. positive) for bets concerning German (resp. US) geography.

<sup>21</sup>Several papers (e.g., Nau, 2006; Chew and Sagi, 2008; Ergin and Gul, 2009; Gul and Pesendorfer, 2015; Cappelli, Cerreia-Vioglio, Maccheroni, Marinacci, and Minardi, 2016) propose formalizations of source dependence based on the idea that the DM is probabilistically sophisticated over prospects that depend on a single common source, but exhibits varying attitudes toward uncertainty across sources. Corollary 2 considers a specific variation where the DM exhibits negative vs. positive attitudes depending on her familiarity with each source. See also Abdellaoui, Baillon, Placido, and Wakker (2011); Chew, Miao, and Zhong (2018) for experimental work using different notions of source dependence.

At the same time, we point to another parametric special case of DSEU that retains much of the tractability of  $\alpha$ -MEU, but is flexible enough to accommodate source-dependent negative and positive ambiguity attitudes (as well as all shades of ambiguity aversion discussed in Section 3.1.1): Consider  $(\mathbb{P}, u)$  where  $\mathbb{P} = \{\alpha P_1 + (1 - \alpha)\{\mu\} : \mu \in P_2\}$  for some closed, convex sets of beliefs  $P_1, P_2$  and  $\alpha \in [0, 1]$ . This yields an *asymmetric  $\alpha$ -MEU* representation, where the belief-sets  $P_1$  and  $P_2$  for Pessimism and Optimism might differ:

$$W(f) = \alpha \min_{\mu \in P_1} \mathbb{E}_\mu[u(f)] + (1 - \alpha) \max_{\mu \in P_2} \mathbb{E}_\mu[u(f)]. \quad (10)$$

Using (10), the behavior in Example 2 can be captured in an intuitive manner, by allowing Pessimism (resp. Optimism) to control beliefs about the foreign (resp. domestic) stock market. For instance, if  $P_1 = \{\mu : \mu(E_H) = \frac{1}{2}\}$  and  $P_2 = \{\mu : \mu(E_F) = \frac{1}{2}\}$ , then for any  $\alpha \in (0, 1)$ ,  $AA(E_H) = \alpha - 1 < 0$  and  $AA(E_F) = \alpha > 0$ .

Finally, the above insights extend to another common formalization of source dependence (along the lines of experimental work by, e.g., Tversky and Fox, 1995; Heath and Tversky, 1991) that does not involve matching probabilities. As before, fix two outcomes  $x, y \in Z$  such that  $\delta_x \succ \delta_y$ . Consider the preference pattern

$$xEy \succ xFy \succ xGy \quad \text{and} \quad xE^c y \succ xF^c y \succ xG^c y, \quad (11)$$

where event  $F$  is *unambiguous*, in the sense that  $f \sim xFy \Rightarrow \lambda f + (1 - \lambda)xFy \sim xFy$  for all  $\lambda \in (0, 1)$ .<sup>22</sup> In (11), the DM's preference to bet on both  $F$  vs.  $G$  and  $F^c$  vs.  $G^c$  captures a negative attitude towards the uncertainty underlying event  $G$ . At the same time, the preference for betting on  $E$  vs.  $F$  and  $E^c$  vs.  $F^c$  reflects a positive attitude towards the uncertainty underlying event  $E$ . It is easy to see that this implies  $AA(E) < AA(F) = 0 < AA(G)$ . Thus, it is immediate from Corollary 3 that this form of source dependence is also inconsistent with  $\alpha$ -MEU, while it is again compatible with the general DSEU model:

**Corollary 4.** *Suppose  $\succsim$  admits an  $\alpha$ -MEU representation. Then there do not exist events  $E, F, G$ , where  $F$  is unambiguous, such that (11) is satisfied.*

Supplementary Appendix S.4 derives a similar incompatibility result to Corollary 4 for Klibanoff, Marinacci, and Mukerji's (2005) smooth model.

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<sup>22</sup>Under DSEU, this is equivalent to the condition that  $xFy$  is *crisp* as defined by GMM, which is in turn equivalent to requiring  $\mu(F)$  to be constant across all beliefs  $\mu \in C$  (Proposition 10 in GMM).

## 3.2 Belief updating

Next, we use the DSEU representation to provide an answer to the question how to update invariant biseparable preferences. In particular, we show that one of the most widely used updating rules for maxmin expected utility, prior-by-prior updating, extends naturally to DSEU.

Formally, given any event  $E \subseteq S$ , we enrich the primitives of the static model with a conditional preference  $\succsim_E$ . We interpret  $\succsim$  as the DM's ex-ante preference and  $\succsim_E$  as the DM's updated preference conditional on learning that the state  $s$  is in  $E$ . For any acts  $f$  and  $g$ , we write  $fEg$  for the act that yields  $f(s)$  in any  $s \in E$  and  $g(s)$  in any  $s \notin E$ .

Suppose first that the ex-ante preference  $\succsim$  admits a maxmin expected utility representation with belief-set  $P$  and utility  $u$ . Under the **prior-by-prior updating** (or *full Bayesian updating*) rule, each conditional preference  $\succsim_E$  is induced by the maxmin representation  $(P_E, u)$ , whose belief-set  $P_E := \{\mu_E : \mu \in P, \mu(E) > 0\}$  consists of the Bayesian updates  $\mu_E$  of all priors  $\mu \in P$ . Pires (2002) shows that prior-by-prior updating for maxmin preferences is characterized by the following axiom:

**Axiom 9** (C-Dynamic Consistency). For all  $f \in \mathcal{F}$  and  $p \in \Delta(Z)$ ,  $f \succsim_E p \Leftrightarrow fEp \succsim p$ .

While full dynamic consistency requires that  $f \succsim_E g \Leftrightarrow fEg \succsim g$  for all acts  $f$  and  $g$ , Axiom 9 imposes this equivalence only when  $g$  is a constant act. This axiom guarantees that, for any act  $f$ , if  $p$  is a constant equivalent for  $f$  conditional on event  $E$  evaluated from the ex-ante perspective (i.e.,  $fEp \sim p$ ), then  $p$  remains a constant equivalent for  $f$  after event  $E$  has realized (i.e.,  $f \sim_E p$ ), and vice versa.

We now show that, for general invariant biseparable preferences  $\succsim$ , imposing Axiom 9 characterizes the following natural extension of prior-by-prior updating to DSEU. We say that event  $E$  is **non-null** if  $p \succ qEp$  for some  $p, q \in \Delta(Z)$ .<sup>23</sup>

**Theorem 2.** Suppose that  $\succsim$  admits a DSEU representation  $(\mathbb{P}, u)$ , that  $E$  is non-null, and that  $\succsim_E$  is an Archimedean weak order (i.e., satisfies Axioms 1 and 4). Then, the following are equivalent:

1.  $(\succsim, \succsim_E)$  satisfies Axiom 9.
2.  $\succsim_E$  is represented by the DSEU representation  $(\mathbb{P}_E, u)$ , where  $\mathbb{P}_E := \{P_E : P \in \mathbb{P}\}$ .

Thus, under DSEU, Axiom 9 amounts to requiring that each belief-set  $P$  in the ex-ante representation  $\mathbb{P}$  is updated prior-by-prior to  $P_E$ . In other words, the game played by

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<sup>23</sup>Under DSEU, this is equivalent to the following property: For any representation  $(\mathbb{P}, u)$  of  $\succsim$ , each  $P \in \mathbb{P}$  contains some  $\mu \in P$  with  $\mu(E) > 0$ .

Optimism and Pessimism after updating is obtained from the original game by replacing each prior with its associated posterior.

Several features of prior-by-prior updating under DSEU are worth noting. First, even though the ex-ante preference  $\succsim$  can admit multiple DSEU representations (see Section 4.1), Theorem 2 implies that the conditional preference  $\succsim_E$  is uniquely pinned down from the ex-ante preference  $\succsim$ . That is, if  $(\mathbb{P}, u)$  and  $(\mathbb{P}', u')$  both represent the ex-ante preference  $\succsim$ , then  $(\mathbb{P}_E, u)$  and  $(\mathbb{P}'_E, u')$  induce the same conditional preference  $\succsim_E$ . Moreover, just as prior-by-prior updating for maxmin, prior-by-prior updating for DSEU implies the normatively appealing property of *consequentialism*, i.e.,  $fEg \sim_E fEh$  for all acts  $f, g, h$ .

Second, for maxmin, Epstein and Schneider (2003) show that Axiom 9 can be strengthened to full dynamic consistency for a given partition of events if and only if the supporting set of priors  $P$  satisfies a property called *rectangularity*. In Supplementary Appendix S.2.1, we show that this result also extends to prior-by-prior updating for DSEU, under an appropriate generalization of the notion of rectangularity to belief-set collections  $\mathbb{P}$ .

Finally, prior-by-prior updating for DSEU yields a new and well-behaved way to update the important special case when  $\succsim$  admits an  $\alpha$ -MEU representation (8). Treating the  $\alpha$ -MEU representation  $(\alpha, P, u)$  as the DSEU representation  $(\mathbb{P}, u)$  with  $\mathbb{P} = \{\alpha P + (1 - \alpha)\{\mu\} : \mu \in P\}$  and updating  $\mathbb{P}$  prior-by-prior yields the conditional preference  $\succsim_E$  represented by

$$\max_{\mu \in P} \min_{\nu \in P} \mathbb{E}_{(\alpha\nu + (1-\alpha)\mu)_E}[u(f)]. \quad (12)$$

The resulting  $\succsim_E$  is different from the conditional preference induced by the  $\alpha$ -MEU representation  $(\alpha, P_E, u)$ , where the belief-set  $P$  is updated prior-by-prior while holding  $\alpha$  and  $u$  fixed: The latter representation can be written as  $\max_{\mu \in P} \min_{\nu \in P} \mathbb{E}_{(\alpha\nu_E + (1-\alpha)\mu_E)}[u(f)]$ ; relative to (12), this reverses the order of the  $\alpha$ -mixture and Bayesian updating operations. While the latter updating rule is sometimes used in applied work, Frick, Iijima, and Le Yaouanq (2020) show that it is ill-defined at the level of preferences: It is possible to find two distinct  $\alpha$ -MEU representations  $(\alpha, P, u)$  and  $(\alpha', P', u)$  of the same ex-ante preference, but such that the updated models  $(\alpha, P_E, u)$  and  $(\alpha', P'_E, u)$  represent different conditional preferences. By contrast, (12) does not suffer from this issue, because, as noted above, prior-by-prior updating for DSEU uniquely pins down the conditional preference from the ex-ante preference.

**Remark 2.** No updating rules have thus far been proposed for the alternative representations of invariant biseparable preferences due to GMM and Amarante (see Section 1.2). In Supplementary Appendix S.2.2, we illustrate some complications that arise in defining potential extensions of prior-by-prior updating for these representations.

For example, a seemingly natural extension of prior-by-prior updating for GMM's rep-

resentation (1) might be to update the set of relevant priors  $C$  prior-by-prior to  $C_E$ , while holding the weight function  $\alpha(\cdot)$  and utility  $u$  fixed.<sup>24</sup> However, as we show, this updating rule does not satisfy Axiom 9, and thus less naturally extends prior-by-prior updating for maxmin. Indeed, the resulting conditional preference is not necessarily invariant biseparable, as it can violate monotonicity (Axiom 2), and consequentialism can fail.<sup>25</sup>

For Amarante’s representation (2), natural candidates for extending prior-by-prior updating are less clear. We show that one extension, which holds fixed  $u$  but updates the capacity  $\nu$  by shifting all weight from any prior belief to its posterior, suffers from the same issue discussed for  $\alpha$ -MEU above, i.e., the conditional preference is not pinned down by the ex-ante preference and instead depends on the choice of the (non-unique) ex-ante representation.  $\blacktriangle$

## 4 Discussion and extensions

In this section, we briefly discuss the uniqueness properties and comparative statics of DSEU representations. We also show how relaxing certainty independence leads to natural generalizations of DSEU.

### 4.1 Uniqueness

While our results in the preceding sections apply to *all* DSEU representations  $(\mathbb{P}, u)$  of a given preference  $\succsim$ , we briefly comment on the uniqueness properties of these representations. As observed previously,  $\succsim$  uniquely identifies the DM’s set of relevant priors  $C$  (i.e., the smallest union  $\overline{\text{co}} \bigcup_{P \in \mathbb{P}} P$ ) and the intersection  $\bigcap_{P \in \mathbb{P}} P$ . At the same time, the DM’s belief-set collection  $\mathbb{P}$  itself is not in general unique, analogous to other representations involving belief-set or utility-set collections.<sup>26</sup>

However, the following result shows how any two DSEU representations  $(\mathbb{P}, u)$  and  $(\mathbb{P}', u')$  of the same preference are related: The utilities must coincide up to some positive affine transformation (denoted  $u \approx u'$ ), and the belief-set collections must coincide up to replacing all sets of beliefs in  $\mathbb{P}$  and  $\mathbb{P}'$  with the closed half-spaces that contain them. Formally, given

<sup>24</sup>Note that, in contrast with the aforementioned updating rule for  $\alpha$ -MEU, this is well-defined at the level of preferences, as the set of relevant priors  $C$  and weight function  $\alpha(\cdot)$  are uniquely pinned down by  $\succsim$ .

<sup>25</sup>As we also discuss, if a GMM representation is instead updated by imposing Axiom 9, the parameters  $\alpha^E(\cdot)$  and  $C^E$  of the conditional GMM representation must each depend jointly on both the ex-ante parameters  $\alpha(\cdot)$  and  $C$ —in particular,  $C^E$  is *not* in general the prior-by-prior update of  $C$ .

<sup>26</sup>One might conjecture that  $\succsim$  admits a unique representation  $\tilde{\mathbb{P}}$  that is *minimal*, in the sense that it features no redundant actions for Optimism or Pessimism (formally,  $\tilde{\mathbb{P}}$  is minimal if there is no alternative representation  $\mathbb{P} \neq \tilde{\mathbb{P}}$  with either (i)  $\mathbb{P} \subseteq \tilde{\mathbb{P}}$  or (ii)  $\forall \tilde{P} \in \tilde{\mathbb{P}}, \exists P \in \mathbb{P}$  with  $P \subseteq \tilde{P}$ ). However, this conjecture is not valid, as some preferences admit multiple minimal representations.

any belief-set collection  $\mathbb{P}$ , define its *half-space closure* by

$$\bar{\mathbb{P}} := \text{cl}\{H \subseteq \Delta(S) : H \text{ is a closed half-space in } \Delta(S) \text{ and } H \supseteq P \text{ for some } P \in \mathbb{P}\},$$

where we call  $H$  a closed half-space in  $\Delta(S)$  if  $H = H_{\phi, \lambda} := \{\mu \in \Delta(S) : \mu \cdot \phi \geq \lambda\}$  for some  $\phi \in \mathbb{R}^S$  and  $\lambda \in \mathbb{R}$ .

**Proposition 5.** *Suppose  $(\mathbb{P}, u)$  is a DSEU representation of  $\succsim$ . Then  $(\bar{\mathbb{P}}, u)$  is also a DSEU representation of  $\succsim$ . Moreover, for any belief-set collection  $\mathbb{P}'$  and utility  $u'$ ,  $(\mathbb{P}', u')$  is a DSEU representation of  $\succsim$  if and only if  $\bar{\mathbb{P}} = \bar{\mathbb{P}'}$  and  $u \approx u'$ .*

The uniqueness of  $u$  up to positive affine transformation is standard. The uniqueness of  $\bar{\mathbb{P}}$  up to half-space closure parallels the identification result in Hara, Ok, and Riella (2019), who represent independent (but possibly incomplete and intransitive) preferences over lotteries using a collection of utility-sets. Analogous to Hara, Ok, and Riella (2019), the idea is that for any  $P \in \mathbb{P}$ , the closed half-spaces containing  $P$  capture all information about  $P$  that is relevant to the representation. Indeed, in determining how any given utility act  $\phi \in \mathbb{R}^S$  is evaluated by the representation, the only relevant feature of  $P$  is the worst-case expectation  $\lambda_{P, \phi} := \min_{\mu \in P} \mathbb{E}_{\mu}[\phi]$ , and this worst-case expectation is shared by the closed half-space  $H_{\phi, \lambda_{P, \phi}} \supseteq P$ . Thus, replacing each set  $P$  in  $\mathbb{P}$  with the closed half-spaces  $H_{\phi, \lambda_{P, \phi}}$  for all  $\phi \in \mathbb{R}^S$  yields an alternative DSEU representation of  $\succsim$ . Finally, we show in Appendix D.1 that the half-space closure  $\bar{\mathbb{P}}$  of any  $\mathbb{P}$  is uniquely determined by (the utility act functional  $I$  associated with) the preference  $\succsim$ .

## 4.2 Comparative ambiguity attitudes

Next, building on Proposition 5, we provide a representation under DSEU of the standard comparative notion of ambiguity aversion defined in Section 3.1.1:

**Proposition 6.** *Suppose  $\succsim_1$  and  $\succsim_2$  admit DSEU representations  $(\mathbb{P}_1, u_1)$  and  $(\mathbb{P}_2, u_2)$ , respectively. The following are equivalent:*

1.  $\succsim_1$  is more ambiguity-averse than  $\succsim_2$ .
2.  $u_1 \approx u_2$  and  $\bar{\mathbb{P}}_1 \subseteq \bar{\mathbb{P}}_2$ .

To interpret, note that  $\bar{\mathbb{P}}_1 \subseteq \bar{\mathbb{P}}_2$  means that Optimism's action set, and hence Optimism's ability to influence the DM's belief, is more limited under representation  $\bar{\mathbb{P}}_1$  than under  $\bar{\mathbb{P}}_2$ . Thus, more ambiguity aversion corresponds (up to taking half-space closures) to DSEU representations that allocate less relative "power" to Optimism. This comparative notion of

power is consistent with the absolute measures in Section 3.1.1, because  $\overline{\mathbb{P}}_1 \subseteq \overline{\mathbb{P}}_2$  implies that  $\bigcap_{P_1 \in \mathbb{P}_1} P_1 \supseteq \bigcap_{P_2 \in \mathbb{P}_2} P_2$  and that  $\succsim_1$  displays a weakly higher degree of  $k$ -ambiguity aversion than  $\succsim_2$ .

In contrast with GMM’s characterization of comparative ambiguity aversion, which only applies when the sets of relevant priors  $C_1$  and  $C_2$  associated with  $\succsim_1$  and  $\succsim_2$  are equal (Proposition 12 in GMM), Proposition 6 does not assume any relationship between  $C_1$  and  $C_2$ . Indeed, there are natural cases in which one invariant biseparable preference is more ambiguity-averse than another, despite the fact that their sets of priors do not coincide (nor are nested). For example, as long as  $C_1 \cap C_2 \neq \emptyset$ , then for any  $u$ , the maxmin expected utility preference  $\succsim_1$  induced by  $C_1$  is more ambiguity-averse than the maxmax expected utility preference  $\succsim_2$  induced by  $C_2$ .

### 4.3 Generalizations

As we have seen, our baseline model, DSEU, corresponds to a relaxation of subjective expected utility where independence is weakened to certainty independence and, equivalently, to dropping uncertainty aversion from Gilboa and Schmeidler’s (1989) axioms. The representation adds a maximization stage into Gilboa and Schmeidler’s (1989) model, suggesting an interpretation in terms of a game between Optimism and Pessimism.

We highlight that this dual-self approach extends beyond certainty independence, yielding intuitive representations that further relax independence but still allow for a flexible mix of negative and positive ambiguity attitudes. To illustrate, consider the following two common relaxations of certainty independence. First, Maccheroni, Marinacci, and Rustichini’s (2006) (henceforth MMR’s) *variational* preferences generalize Gilboa and Schmeidler (1989) by replacing certainty independence with weak certainty independence. This axiom retains the “location invariance” property implied by certainty independence but relaxes the “scale invariance” property; we refer the reader to MMR for a detailed discussion:

**Axiom 10** (Weak Certainty Independence). For any  $f, g \in \mathcal{F}$ ,  $p, q \in \Delta(Z)$ , and  $\alpha \in (0, 1)$ ,

$$\alpha f + (1 - \alpha)p \succsim \alpha g + (1 - \alpha)p \implies \alpha f + (1 - \alpha)q \succsim \alpha g + (1 - \alpha)q.$$

Second, Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio’s (2011) (henceforth CMMM’s) model of *uncertainty-averse* preferences imposes an even weaker form of independence that only holds for objective lotteries:

**Axiom 11** (Risk Independence). For any  $p, q, r \in \Delta(Z)$  and  $\alpha \in (0, 1)$ ,

$$p \succsim q \implies \alpha p + (1 - \alpha)r \succsim \alpha q + (1 - \alpha)r.$$

While MMR and CMMM maintain uncertainty aversion, the following two results show that dropping uncertainty aversion from their axioms yields dual-self representations that extend DSEU to more general games between Optimism and Pessimism:

**Theorem 3.** *Preference  $\succsim$  satisfies Axioms 1–4 and Axiom 10 if and only if  $\succsim$  admits a **dual-self variational** representation; that is, there exists a nonconstant affine utility  $u : \Delta(Z) \rightarrow \mathbb{R}$  and a collection  $\mathbb{C}$  of convex cost functions  $c : \Delta(S) \rightarrow \mathbb{R} \cup \{\infty\}$  with  $\max_{c \in \mathbb{C}} \min_{\mu \in \Delta(S)} c(\mu) = 0$  such that*

$$W(f) := \max_{c \in \mathbb{C}} \min_{\mu \in \Delta(S)} \mathbb{E}_\mu[u(f)] + c(\mu) \quad (13)$$

*is well-defined and represents  $\succsim$ .*

In (13), Optimism first chooses a cost function  $c : \Delta(S) \rightarrow \mathbb{R} \cup \{\infty\}$  from some collection  $\mathbb{C}$ , and Pessimism then chooses a belief subject to this cost. This model adds a maximization stage into MMR’s variational representation, which corresponds to the special case in which  $\mathbb{C}$  is a singleton.<sup>27</sup> Likewise, the following representation incorporates a maximization stage into CMMM’s representation:<sup>28</sup>

**Theorem 4.** *Preference  $\succsim$  satisfies Axioms 1–4 and Axiom 11 if and only if  $\succsim$  admits a **rational dual-self** representation; that is, there exists a nonconstant affine utility  $u : \Delta(Z) \rightarrow \mathbb{R}$  and a collection  $\mathbb{G}$  of quasiconvex functions  $G : \mathbb{R} \times \Delta(S) \rightarrow \mathbb{R} \cup \{\infty\}$  that are increasing in their first argument and satisfy  $\max_{G \in \mathbb{G}} \inf_{\mu \in \Delta(S)} G(a, \mu) = a$  for all  $a$  such that*

$$W(f) := \max_{G \in \mathbb{G}} \inf_{\mu \in \Delta(S)} G(\mathbb{E}_\mu[u(f)], \mu) \quad (14)$$

*is well-defined, continuous, and represents  $\succsim$ .*

The generalizations of DSEU in Theorems 3 and 4 can accommodate additional experimental evidence. For instance, by relaxing the positive homogeneity of  $I$  implied by certainty independence but preserving constant-additivity, the dual-self variational model can accommodate Machina’s (2009) paradoxes (see also Baillon, L’Haridon, and Placido,

<sup>27</sup>Castagnoli, Cattelan, Maccheroni, Tebaldi, and Wang (2021) consider a special case of (13) that imposes the stronger normalization that  $\min_{\mu \in \Delta(S)} c(\mu) = 0$  for all  $c \in \mathbb{C}$ , ensuring that each choice of Optimism induces a variational preference. They show that this special case is characterized by additionally requiring a weak form of preference for hedging, where for all  $f \in \mathcal{F}$ ,  $p \in \Delta(Z)$ , and  $\alpha \in (0, 1)$ ,  $f \succsim p \implies \alpha f + (1 - \alpha)p \succsim p$ . In contrast, Theorem 3 shows that our weaker normalization,  $\max_{c \in \mathbb{C}} \min_{\mu \in \Delta(S)} c(\mu) = 0$ , corresponds to fully dropping any preference for hedging from the variational model, which is necessary in order to nest the case of uncertainty-seeking preferences.

<sup>28</sup>Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011) provide an alternative representation of this class of preferences that generalizes (1). As for GMM, the necessity of the axioms requires joint restrictions on the weight function  $\alpha(\cdot)$  and other model parameters in (1) to ensure that  $W$  is monotone.

2011).<sup>29</sup> Another important finding is that ambiguity attitudes can differ for gains and losses: For example, in urn experiments subjects who are ambiguity-averse for bets with positive payoffs are often ambiguity-seeking when the sign of the bet is reversed (Trautmann and Wakker, 2018). This is inconsistent with any model that displays constant-additivity, but can be accommodated by the rational dual-self representation in (14).

## 5 Conclusion

We adopt a dual-self perspective on ambiguity preferences, by proposing a class of multiple-prior representations where the belief the DM uses to evaluate each act is the outcome of a game between Optimism and Pessimism. Our baseline model, DSEU, provides a novel representation of the class of invariant biseparable preferences, which drops uncertainty aversion from maxmin expected utility; further relaxing certainty independence yields dual-self generalizations of variational and uncertainty-averse preferences. Relative to existing work, we highlight two key properties of the DSEU model: In the static context, DSEU provides a unified framework to represent a range of intermediate ambiguity attitudes, motivated for instance by evidence that individuals' attitudes towards ambiguity can be odds-dependent or source-dependent. In the dynamic context, DSEU yields a natural way to perform belief updating under invariant biseparable preferences.

More broadly, representations based on a combination of max and min operators have been used to provide foundations for maxmin values in zero-sum games (Hart, Modica, and Schmeidler, 1994), utility aggregation (Chambers, 2007), and coarse reasoning (Saponara, 2020). These representations can be shown to be strict special cases of DSEU, suggesting that the DSEU model and its extensions might serve as a unifying framework to capture additional phenomena, beyond the focus on ambiguity attitudes in the current paper.

## Appendix: Proofs

This appendix presents the proofs of all results in Sections 2–4.2. The supplementary appendix contains proofs for the generalizations in Section 4.3, as well as other omitted material.

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<sup>29</sup>This follows from the fact that Siniscalchi's (2009) vector expected utility model can accommodate these paradoxes and is a special case of (13).

# A Preliminaries

Throughout this section, we fix any interval  $\Gamma \subseteq \mathbb{R}$  and let  $U := \Gamma^S$ . For any  $a \in \mathbb{R}$ , let  $\underline{a}$  denote the vector in  $\mathbb{R}^S$  with  $\underline{a}(s) = a$  for all  $s \in S$ . For any  $\phi, \psi \in \mathbb{R}^S$ , write  $\phi \geq \psi$  if  $\phi(s) \geq \psi(s)$  for all  $s$ .

## A.1 Properties of functionals

Fix any functional  $I : U \rightarrow \mathbb{R}$ . We call  $I$  **monotonic** if  $I(\phi) \geq I(\psi)$  for all  $\phi, \psi \in U$  with  $\phi \geq \psi$ ; **normalized** if  $I(\underline{a}) = a$  for all  $a \in \Gamma$ ; **constant-additive** if  $I(\phi + \underline{a}) = I(\phi) + a$  for all  $\phi \in U$  and  $a \in \Gamma$  with  $\phi + \underline{a} \in U$ ; **positively homogeneous** if  $I(a\phi) = aI(\phi)$  for all  $\phi \in U$  and  $a \in \mathbb{R}_+$  with  $a\phi \in U$ ; and **constant-linear** if  $I$  is constant-additive and positively homogeneous. It is easy to see that if  $\underline{0} \in \Gamma$ , then any constant-linear functional  $I$  is normalized.

## A.2 Clarke derivative and differential

Consider a locally Lipschitz functional  $I : U \rightarrow \mathbb{R}$ . For every  $\phi \in \text{int}U$  and  $\xi \in \mathbb{R}^S$ , the **Clarke (upper) derivative** of  $I$  in  $\phi$  in the direction of  $\xi$  is

$$I^\circ(\phi; \xi) := \limsup_{\psi \rightarrow \phi, t \downarrow 0} \frac{I(\psi + t\xi) - I(\psi)}{t}.$$

The **Clarke (sub)differential** of  $I$  at  $\phi$  is the set

$$\partial I(\phi) := \{\chi \in \mathbb{R}^S : \chi \cdot \xi \leq I^\circ(\phi; \xi), \forall \xi \in \mathbb{R}^S\}.$$

We will frequently invoke the following properties of the Clarke differential. First, if  $I$  is locally Lipschitz, then Rademacher's theorem yields a subset  $\hat{U} \subseteq \text{int}U$  such that  $U \setminus \hat{U}$  has Lebesgue measure zero and  $I$  is differentiable on  $\hat{U}$ . Combining this with Theorem 2.5.1 in [Clarke \(1990\)](#), we obtain the following approximation of the Clarke differential:

**Lemma A.1** (Theorem 2.5.1 in [Clarke \(1990\)](#)). *Suppose  $I : U \rightarrow \mathbb{R}$  is locally Lipschitz. Then there exists  $\hat{U} \subseteq \text{int}U$  such that  $U \setminus \hat{U}$  has Lebesgue measure zero,  $I$  is differentiable at each  $\psi \in \hat{U}$ , and for every  $\phi \in \text{int}U$ , we have*

$$\partial I(\phi) = \text{co}\{\lim_n \nabla I(\phi_n) : \phi_n \rightarrow \phi, \phi_n \in \hat{U}\}. \quad (15)$$

The next result is an “envelope theorem” for Clarke differentials:

**Lemma A.2** (Theorem 2.8.6 in Clarke (1990)). *Suppose functional  $I : U \rightarrow \mathbb{R}$  is given by*

$$I(\cdot) = \sup_{t \in T} I_t(\cdot)$$

*for some indexed family of functionals  $(I_t)_{t \in T}$  with domain  $U$ . Assume that there exists some  $K > 0$  such that  $|I_t(\psi) - I_t(\xi)| \leq K\|\psi - \xi\|$  for every  $t \in T$  and  $\psi, \xi \in \text{int}U$ . Then for every  $\phi \in \text{int}U$ , we have  $\partial I(\phi) \subseteq \text{co}\{\lim_{i \rightarrow \infty} \nabla I_{t_i}(\phi_i) : \phi_i \rightarrow \phi, t_i \in T, I_{t_i}(\phi) \rightarrow I(\phi)\}$ .*

Last, we note the following relationship between properties of  $I$  and its Clarke differential:

**Lemma A.3** (Part 1 of Proposition A.3 in GMM). *If  $I : U \rightarrow \mathbb{R}$  is locally Lipschitz, positively homogeneous, and  $\underline{0} \in \text{int}U$ , then  $\partial I(\phi) \subseteq \partial I(\underline{0})$  for all  $\phi \in \text{int}U$ .*

**Lemma A.4** (Parts 2–3 of Proposition A.3 in GMM). *If  $I : U \rightarrow \mathbb{R}$  is locally Lipschitz, monotonic, and constant-additive, then  $\partial I(\phi) \subseteq \Delta(S)$  for all  $\phi \in \text{int}U$ .*

### A.3 Boolean representation of locally Lipschitz $I$

Throughout this subsection, we assume that  $I : U \rightarrow \mathbb{R}$  is locally Lipschitz. Let  $\hat{U}$  be the generic subset given by Lemma A.1.

Lemma A.6 below shows that, restricted to  $\hat{U}$ ,  $I$  admits a so-called ‘‘Boolean’’ representation in terms of a family of affine functionals whose slopes correspond to gradients of  $I$ . This result extends Ovchinnikov (2001), who establishes Lemma A.6 under the assumption that  $I$  is continuously differentiable. Our non-smooth generalization is necessary for the proof of Theorem 1, where the utility-act functional  $I$  is non-differentiable (except in the case of subjective expected utility). We begin with a preliminary result:

**Lemma A.5.** *For every  $\phi, \psi \in \hat{U}$  and  $\varepsilon > 0$ , there exists  $\xi \in \hat{U}$  such that*

$$I(\xi) - I(\psi) + \nabla I(\xi) \cdot (\psi - \xi) \geq 0, \quad I(\xi) - I(\phi) + \nabla I(\xi) \cdot (\phi - \xi) \leq \varepsilon.$$

*Proof.* Take any  $\phi, \psi \in \hat{U}$  and  $\varepsilon > 0$ . Let  $m := I(\psi) - I(\phi)$ . If  $\nabla I(\phi) \cdot (\psi - \phi) \geq m$ , we can set  $\xi = \phi$ . Likewise if  $\nabla I(\psi) \cdot (\psi - \phi) \geq m$ , we can set  $\xi = \psi$ . It remains to consider the case

$$\nabla I(\phi) \cdot (\psi - \phi), \nabla I(\psi) \cdot (\psi - \phi) < m. \tag{16}$$

Define

$$H(\lambda) := I(\phi + \lambda(\psi - \phi)) - \lambda m - I(\phi)$$

for each  $\lambda \in \mathbb{R}$  with  $\phi + \lambda(\psi - \phi) \in U$ . Since  $\phi, \psi \in \hat{U}$ ,  $H$  is differentiable at  $\lambda \in \{0, 1\}$ , with  $H(0) = H(1) = 0$  and  $H'(0), H'(1) < 0$  by assumption (16). Hence,  $H$  is negative for small

enough  $\lambda > 0$  and positive for  $\lambda < 1$  close enough to 1. Thus, the set  $\{\lambda \in (0, 1) : H(\lambda) = 0\}$  is nonempty and closed; let  $\lambda^*$  denote its supremum.

Since  $H$  is locally Lipschitz, we have  $H(\lambda) = \int_{\lambda^*}^{\lambda} H'(\lambda') d\lambda'$  for all  $\lambda > \lambda^*$ . As  $H(\lambda) > 0$  for all  $\lambda \in (\lambda^*, 1)$ , we can choose  $\lambda^{**} \in (\lambda^*, 1)$  close enough to  $\lambda^*$  such that  $H$  is differentiable at  $\lambda^{**}$  with  $H'(\lambda^{**}) > 0$  and  $H(\lambda^{**}) \in (0, \varepsilon)$ . But then

$$H'(\lambda^{**}) = \lim_{t \rightarrow 0} \frac{I(\phi + (\lambda^{**} + t)(\psi - \phi)) - I(\phi + \lambda^{**}(\psi - \phi))}{t} - m > 0,$$

which implies that

$$I^\circ(\phi + \lambda^{**}(\psi - \phi); \psi - \phi) - m \geq H'(\lambda^{**}) > 0.$$

Since  $I^\circ(\xi; \zeta) = \max_{\mu \in \partial I(\xi)} \mu \cdot \zeta$  for any  $\zeta, \xi$  (e.g., Proposition 2.1.2 in Clarke, 1990), this yields some  $\mu \in \partial I(\phi + \lambda^{**}(\psi - \phi))$  such that

$$\mu \cdot (\psi - \phi) - m \geq H'(\lambda^{**}) > 0.$$

By (15), there exists a sequence  $\xi_n \rightarrow \phi + \lambda^{**}(\psi - \phi)$  such that  $\xi_n \in \hat{U}$  for each  $n$  and  $\lim_n \nabla I(\xi_n) = \mu$ . Then

$$\begin{aligned} \lim_n (I(\xi_n) - I(\psi) + \nabla I(\xi_n) \cdot (\psi - \xi_n)) &= I(\phi + \lambda^{**}(\psi - \phi)) - I(\psi) + (1 - \lambda^{**})\mu \cdot (\psi - \phi) \\ &= H(\lambda^{**}) - (1 - \lambda^{**})m + (1 - \lambda^{**})\mu \cdot (\psi - \phi) > 0 \end{aligned}$$

where the inequality uses the fact that  $H(\lambda^{**}) > 0$  and that  $\mu \cdot (\psi - \phi) - m \geq H'(\lambda^{**}) > 0$ . Similarly,

$$\begin{aligned} \lim_n (I(\xi_n) - I(\phi) + \nabla I(\xi_n) \cdot (\phi - \xi_n)) &= I(\phi + \lambda^{**}(\psi - \phi)) - I(\phi) - \lambda^{**}\mu \cdot (\psi - \phi) \\ &= H(\lambda^{**}) + \lambda^{**}m - \lambda^{**}\mu \cdot (\psi - \phi) < \varepsilon \end{aligned}$$

where the inequality uses  $H(\lambda^{**}) < \varepsilon$  and  $\mu \cdot (\psi - \phi) - m \geq H'(\lambda^{**}) > 0$ . Thus, for any large enough  $n$ ,  $\xi_n \in \hat{U}$  is as desired.  $\square$

We now establish the Boolean representation of  $I$ :

**Lemma A.6.** *For each  $\phi \in \hat{U}$ , we have*

$$I(\phi) = \max_{\psi \in \hat{U}} \inf_{\xi \in K_\psi} I(\xi) + \nabla I(\xi) \cdot (\phi - \xi),$$

where  $K_\psi := \{\xi \in \hat{U} : I(\xi) + \nabla I(\xi) \cdot (\psi - \xi) \geq I(\psi)\}$  for all  $\psi \in \hat{U}$ .

*Proof.* For each  $\phi, \psi \in \hat{U}$  and  $\varepsilon > 0$ , Lemma A.5 yields some  $\xi \in K_\psi$  such that  $I(\xi) + \nabla I(\xi) \cdot (\phi - \xi) \leq I(\phi) + \varepsilon$ . Thus,  $\inf_{\xi \in K_\psi} I(\xi) + \nabla I(\xi) \cdot (\phi - \xi) \leq I(\phi)$ . Moreover, by definition of  $K_\phi$ ,  $\inf_{\xi \in K_\phi} I(\xi) + \nabla I(\xi) \cdot (\phi - \xi) \geq I(\phi)$ . Hence,  $I(\phi) = \max_{\psi \in \hat{U}} \inf_{\xi \in K_\psi} I(\xi) + \nabla I(\xi) \cdot (\phi - \xi)$ , as required.  $\square$

## B Proofs for Section 2

### B.1 Proof of Theorem 1

We invoke the following standard result:

**Lemma B.1** (Lemma 1 in GMM). *Preference  $\succsim$  satisfies Axioms 1–5 if and only if there exists a monotonic, constant-linear functional  $I : \mathbb{R}^S \rightarrow \mathbb{R}$  and a nonconstant affine function  $u : \Delta(Z) \rightarrow \mathbb{R}$  such that for all  $f, g \in \mathcal{F}$ ,*

$$f \succsim g \iff I(u(f)) \geq I(u(g)). \quad (17)$$

Moreover,  $I$  is unique and  $u$  is unique up to positive affine transformation.

The necessity proof for Theorem 1 is standard and we omit it. To prove sufficiency, suppose  $\succsim$  satisfies Axioms 1–5. Let  $I$  and  $u$  be as given by Lemma B.1. Consider the following collection  $\mathbb{P}^*$ :

$$\mathbb{P}^* := \text{cl}\{P_\phi^* : \phi \in \mathbb{R}^S\} \quad \text{with} \quad P_\phi^* := \{\mu \in \partial I(\underline{0}) : \mu \cdot \phi \geq I(\phi)\}, \quad (18)$$

where  $\text{cl}$  denotes the topological closure in  $\mathcal{K}(\Delta(S))$  under the Hausdorff topology.

Note that since  $I$  is monotonic and constant-linear, it is 1-Lipschitz. Thus,  $\partial I(\underline{0}) \subseteq \Delta(S)$  by Lemma A.4, so that each  $P_\phi^*$  is indeed a closed, convex set of beliefs. Moreover,  $\mathbb{P}^*$  is compact, as it is a closed subset of the compact space  $\mathcal{K}(\Delta(S))$ . Thus,  $\mathbb{P}^*$  is a belief-set collection. We will show that for all  $\phi \in \mathbb{R}^S$ ,

$$I(\phi) = \max_{P \in \mathbb{P}^*} \min_{\mu \in P} \mu \cdot \phi, \quad (19)$$

which by (17) ensures that  $(\mathbb{P}^*, u)$  is a DSEU representation of  $\succsim$ .

Lemma A.1 yields a set  $\hat{U} \subseteq \mathbb{R}^S$  such that  $\mathbb{R}^S \setminus \hat{U}$  has Lebesgue measure zero and  $I$  is differentiable on  $\hat{U}$ . Moreover, since  $I$  is positively homogeneous, Lemma A.3 implies that  $\partial I(\phi) \subseteq \partial I(\underline{0})$  for all  $\phi \in \mathbb{R}^S$ , so that for all  $\phi \in \hat{U}$ , we have  $\mu_\phi := \nabla I(\phi) \in \partial I(\underline{0})$ . We will invoke the following lemma:

**Lemma B.2.** For each  $\phi \in \hat{U}$ ,  $I(\phi) = \mu_\phi \cdot \phi$ .

*Proof.* Take any  $\phi \in \hat{U}$ . By positive homogeneity of  $I$ ,  $\alpha\phi \in \hat{U}$  and  $\nabla I(\phi) = \nabla I(\alpha\phi)$  for any  $\alpha \in (0, 1)$ . Thus, the function  $h : [0, 1] \rightarrow \mathbb{R}$  defined by  $h(\alpha) = I(\alpha\phi)$  is differentiable at every  $\alpha \in (0, 1)$  and Lipschitz. Hence,  $I(\phi) = h(1) - h(0) = \int_0^1 h'(\alpha') d\alpha' = \int_0^1 (\nabla I(\alpha\phi) \cdot \phi) d\alpha' = \phi \cdot \mu_\phi$ .  $\square$

To complete the proof of (19), first take any  $\phi, \psi \in \hat{U}$  and let  $K_\psi := \{\xi \in \hat{U} : I(\xi) + \mu_\xi \cdot (\psi - \xi) \geq I(\psi)\}$  be as in Lemma A.6. Then

$$I(\phi) = \max_{\psi \in \hat{U}} \inf_{\xi \in K_\psi} I(\xi) + \mu_\xi \cdot (\phi - \xi) = \max_{\psi \in \hat{U}} \inf_{\xi \in K_\psi} \mu_\xi \cdot \phi, \quad (20)$$

where the first equality holds by Lemma A.6 and the second by Lemma B.2. Letting  $P_\psi := \{\mu_\xi : \xi \in \hat{U}, \mu_\xi \cdot \psi \geq I(\psi)\}$ , Lemma B.2 ensures that  $\xi \in K_\psi$  if and only if  $\mu_\xi \in P_\psi$ . Moreover, (15) implies that  $\overline{\text{co}}P_\psi = P_\psi^*$ . Combining these two observations with (20) yields

$$I(\phi) = \max_{\psi \in \hat{U}} \inf_{\mu \in P_\psi} \mu \cdot \phi = \max_{\psi \in \hat{U}} \min_{\mu \in \overline{\text{co}}P_\psi} \mu \cdot \phi = \max_{\psi \in \hat{U}} \min_{\mu \in P_\psi^*} \mu \cdot \phi. \quad (21)$$

Next, take any  $\phi, \psi \in \mathbb{R}^S$ . Then there exist sequences  $\phi_n \rightarrow \phi, \psi_n \rightarrow \psi$  such that  $\phi_n, \psi_n \in \hat{U}$ . For each  $n$ , pick  $\mu_n \in P_{\psi_n}^*$  such that  $\min_{\mu \in P_{\psi_n}^*} \mu \cdot \phi_n = \mu_n \cdot \phi_n$  and consider a convergent subsequence  $(\mu_{n_k})$  with  $\lim_{k \rightarrow \infty} \mu_{n_k} = \mu^*$ . Note that  $\mu^* \in P_\psi^*$ : Indeed, for each  $k$ , we have  $\mu_{n_k} \cdot \psi_{n_k} \geq I(\psi_{n_k})$ , which by continuity of  $I$  implies  $\mu^* \cdot \psi \geq I(\psi)$ .

Moreover, for each  $k$ , we have  $\mu_{n_k} \cdot \phi_{n_k} = \min_{\mu \in P_{\psi_{n_k}}^*} \mu \cdot \phi_{n_k} \leq I(\phi_{n_k})$ , where the inequality holds by (21). Hence, continuity of  $I$  implies  $\mu^* \cdot \phi \leq I(\phi)$ , so that

$$\min_{\mu \in P_\psi^*} \mu \cdot \phi \leq \mu^* \cdot \phi \leq I(\phi). \quad (22)$$

Since (22) holds for all  $\psi \in \mathbb{R}^S$ , it follows from the definition of  $\mathbb{P}^*$  that

$$\min_{\mu \in P} \mu \cdot \phi \leq I(\phi)$$

holds for all  $P \in \mathbb{P}^*$ . Finally, applying (22) with  $\psi = \phi$  yields  $\min_{\mu \in P_\phi^*} \mu \cdot \phi \leq I(\phi) \leq \min_{\mu \in P_\phi^*} \mu \cdot \phi$ , where the second inequality holds by definition of  $P_\phi^*$ . Thus,

$$I(\phi) = \min_{\mu \in P_\phi^*} \phi \cdot \mu = \max_{P \in \mathbb{P}^*} \min_{\mu \in P} \mu \cdot \phi,$$

as required.  $\square$

## B.2 Proof of Proposition 1

We begin with the following lemma:

**Lemma B.3.** *Consider any functional  $I : \mathbb{R}^S \rightarrow \mathbb{R}$  and belief-set collection  $\mathbb{P}$  such that  $I(\phi) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu \cdot \phi$  for all  $\phi \in \mathbb{R}^S$ . Then*

$$\partial I(\underline{0}) \subseteq \overline{\text{co}} \bigcup_{P \in \mathbb{P}} P.$$

*Proof.* For each  $P \in \mathbb{P}$ , let  $I_P(\phi) := \min_{\mu \in P} \mu \cdot \phi$  for each  $\phi$ . Thus,  $I(\phi) = \max_{P \in \mathbb{P}} I_P(\phi)$  for each  $\phi$ . Note that each  $I_P$  is 1-Lipschitz and  $\partial I_P(0) = P$ .

Take any convergent sequence  $(\nabla I_{P_i}(\phi_i))$  where  $\phi_i \rightarrow 0$ ,  $P_i \in \mathbb{P}$ , and  $\nabla I_{P_i}(\phi_i)$  exists for each  $i$ . Then

$$\nabla I_{P_i}(\phi_i) \in \partial I_{P_i}(\phi_i) \subseteq \partial I_{P_i}(0) = P_i$$

where the set inclusion holds by Lemma A.3. Thus,  $\lim_i \nabla I_{P_i}(\phi_i) \in \overline{\text{co}} \bigcup_{P \in \mathbb{P}} P$ . Hence, the desired conclusion follows by applying Lemma A.2 to  $I$ .  $\square$

Suppose  $\succsim$  satisfies Axioms 1–5. Let  $I$  and  $u$  be as given by Lemma B.1. For  $\mathbb{P}^*$  as in the sufficiency proof of Theorem 1, we have  $\overline{\text{co}} \bigcup_{P \in \mathbb{P}^*} P \subseteq \partial I(\underline{0})$ . Thus, Lemma B.3 immediately implies that  $C = \partial I(\underline{0})$  is the unique closed, convex set satisfying (6) for all DSEU representations of  $\succsim$ , with equality for representation  $\mathbb{P}^*$ .  $\square$

## B.3 Proof of Corollary 1

Since the proof of Proposition 1 identifies the set of relevant priors as  $C = \partial I(\underline{0})$ , Corollary 1 is immediate from the following result in GMM:

**Lemma B.4** (Theorem 14 in GMM). *Suppose  $\succsim$  satisfies Axioms 1–5 and let  $I$  and  $u$  be as in Lemma B.1. Then the unique closed, convex set  $D$  satisfying*

$$f \succsim^* g \iff \mathbb{E}_\mu[u(f)] \geq \mathbb{E}_\mu[u(g)] \text{ for all } \mu \in D$$

*is given by  $D = \partial I(\underline{0})$ .*

## C Proofs for Section 3

### C.1 Proof of Proposition 2

Throughout the proof, let  $I$  be the functional given by Lemma B.1.

### C.1.1 Proof of part 1

To prove the “only if” direction, suppose that  $\succsim$  satisfies uncertainty aversion. Since it admits the maxmin expected utility representation of Gilboa and Schmeidler (1989),  $I(\phi) = \min_{\mu \in C} \mu \cdot \phi$  holds for all  $\phi$ .

We first show that  $\bigcap_{P \in \mathbb{P}} P \supseteq C$ . If not, there exists  $P \in \mathbb{P}$  such that  $P \not\supseteq C$ . By the standard property of support functions, this implies the existence of  $\phi$  such that  $\min_{\mu \in C} \phi \cdot \mu < \min_{\mu \in P} \phi \cdot \mu$ . This leads to  $I(\phi) > \min_{\mu \in C} \mu \cdot \phi$ , a contradiction.

We now show that  $\bigcap_{P \in \mathbb{P}} P \subseteq C$ . If not, there exists  $\mu^* \in \bigcap_{P \in \mathbb{P}} P \setminus C$ . Then there exists  $\phi$  such that  $\min_{\mu \in C} \mu \cdot \phi > \mu^* \cdot \phi$ . But this implies  $I(\phi) \leq \mu^* \cdot \phi < \min_{\mu \in C} \mu \cdot \phi$ , a contradiction.

To prove the “if” direction, suppose that  $\bigcap_{P \in \mathbb{P}} P = C$ . Take any  $\phi$ . It suffices to show that  $I(\phi) = \min_{\mu \in C} \mu \cdot \phi$ . Note that by construction of the representation  $\mathbb{P}^*$  defined by (18), we have  $I(\phi) \geq \min_{\mu \in C} \mu \cdot \phi$ . But the representation based on  $\mathbb{P}$  yields the inequality  $I(\phi) \leq \min_{\mu \in \bigcap_{P \in \mathbb{P}} P} \mu \cdot \phi = \min_{\mu \in C} \mu \cdot \phi$ , which ensures the desired claim.

### C.1.2 Proof of part 2

**Absolute ambiguity aversion  $\Rightarrow$  preference for sure diversification:** This implication follows from the proofs of Theorem 2a and Corollary 3a in Grant and Polak (2013), which imply the equivalence of absolute ambiguity aversion and preference for sure diversification for any preference with a normalized, monotonic, continuous, constant-additive, and unbounded utility act functional  $I$  (as is the case for DSEU).

**Preference for sure diversification  $\Rightarrow \bigcap_{P \in \mathbb{P}} P \neq \emptyset$ :** If  $\succsim$  satisfies preference for sure diversification, then by Proposition 3 (see the proof below) any DSEU representation  $(\mathbb{P}, u)$  of  $\succsim$  is such that every finite subcollection of  $\mathbb{P}$  has nonempty intersection. Since each  $P \in \mathbb{P}$  is convex and compact, Helly’s theorem implies that the whole collection  $\mathbb{P}$  has nonempty intersection.

$\bigcap_{P \in \mathbb{P}} P \neq \emptyset \Rightarrow$  **absolute ambiguity aversion:** Suppose that there exists  $\mu^* \in \bigcap_{P \in \mathbb{P}} P$  for some DSEU representation  $(\mathbb{P}, u)$  of  $\succsim$ . For any  $f \in \mathcal{F}$  and any  $P \in \mathbb{P}$ , this implies that  $\min_{\mu \in P} \mu \cdot u(f) \leq \mu^* \cdot u(f)$ , and hence  $\max_{P \in \mathbb{P}} \min_{\mu \in P} \mu \cdot u(f) \leq \mu^* \cdot u(f)$ . As a result,

$$f \succsim p \implies \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu \cdot u(f) \geq u(p) \implies \mu^* \cdot u(f) \geq u(p) \implies f \succsim_{\mu^*} p$$

where  $\succsim_{\mu^*}$  is the subjective expected utility preference with belief  $\mu^*$  and utility function  $u$ . Hence,  $\succsim$  is more ambiguity-averse than  $\succsim_{\mu^*}$ , which proves the result.  $\square$

## C.2 Proof of Proposition 3

We will invoke the following result, due to Samet (1998):

**Lemma C.1.** *Let  $P_1, \dots, P_k$  be nonempty closed, convex subsets of  $\Delta(S)$ . Then  $\bigcap_{i=1, \dots, k} P_i = \emptyset$  if and only if there exist  $\phi_1, \dots, \phi_k \in \mathbb{R}^S$  such that  $\sum_{i=1, \dots, k} \phi_i = \underline{0}$  and  $\min_{\mu \in P_i} \mu \cdot \phi_i > 0$  for each  $i = 1, \dots, k$ .*

To prove Proposition 3, let  $I$  be the utility act functional of  $\succsim$  given by Lemma B.1.

(1.)  $\implies$  (3.): Take any acts  $f_1, \dots, f_k$ . For each  $i = 1, \dots, k$ , let  $P_i \in \mathbb{P}$  be such that  $I(u(f_i)) = \min_{\mu \in P_i} \mu \cdot u(f)$ . Take  $\hat{\mu} \in \bigcap_{i=1, \dots, k} P_i$ . Then  $I(u(f_i)) \leq \hat{\mu} \cdot u(f_i)$  for each  $i = 1, \dots, k$ . This ensures that part 3 holds for the SEU preference  $\hat{\succsim}$  represented by  $(u, \hat{\mu})$ .

(3.)  $\implies$  (2.): Take any acts  $f_1, \dots, f_k$  such that  $f_1 \sim f_i$  for each  $i$  and such that  $p = \sum_i \alpha_i f_i$  is a complete hedge for  $f_1, \dots, f_k$ . Let  $\hat{\succsim}$  be the corresponding SEU preference from part 3. By assumption,  $\hat{\succsim}$  is represented by  $(u, \hat{\mu})$  for some belief  $\hat{\mu}$  such that  $I(u(f_i)) \leq \hat{\mu} \cdot u(f_i)$  for each  $i = 1, \dots, k$ . Thus,  $u(p) = \hat{\mu} \cdot \sum_{i=1, \dots, k} \alpha_i u(f_i) = \sum_i \alpha_i \hat{\mu} \cdot u(f_i) \geq \sum_i \alpha_i I(u(f_i))$ , which implies  $u(p) \geq I(u(f_1))$ , as  $f_i \sim f_1$  for all  $i$ . Hence,  $p \succsim f_1$ .

(2.)  $\implies$  (1.): We prove the contrapositive. Suppose there exist  $P_1, \dots, P_k \in \mathbb{P}$  with  $\bigcap_{i=1, \dots, k} P_i = \emptyset$ . By Lemma C.1, there exist  $\phi_1, \dots, \phi_k \in \mathbb{R}^S$  such that  $\sum_{i=1, \dots, k} \phi_i = \underline{0}$  and  $\min_{\mu \in P_i} \mu \cdot \phi_i > 0$  for each  $i = 1, \dots, k$ . Let  $\beta_i := I(\phi_i) \geq \min_{\mu \in P_i} \mu \cdot \phi_i > 0$  for each  $i$ . By constant-linearity of  $I$ , we have  $I(\phi_i - \underline{\beta}_i) = 0$  for each  $i$  but  $I(\frac{1}{k} \sum_i (\phi_i - \underline{\beta}_i)) = I(-\frac{1}{k} \sum_i \underline{\beta}_i) < 0$ . Then, for any acts  $f_1, \dots, f_k$  with  $u(f_i) = \phi_i - \underline{\beta}_i$  for each  $i$  (such acts exist up a positive affine transformation of  $u$ ), we have that  $f_1 \sim \dots \sim f_k$  and that  $p := \sum_{i=1}^k \frac{1}{k} f_i$  is a complete hedge but  $p \prec f_1$ , violating  $k$ -ambiguity aversion.  $\square$

## C.3 Proof of Proposition 4

Note that  $m(E) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(E)$ , while  $m(E^c) = 1 - \min_{P \in \mathbb{P}} \max_{\mu \in P} \mu(E)$ . Thus,  $AA(E) = \min_{P \in \mathbb{P}} \max_{\mu \in P} \mu(E) - \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(E)$ .

This implies that  $AA(E) \geq 0$  if and only if all  $P, P' \in \mathbb{P}$  satisfy  $\max_{\mu \in P} \mu(E) \geq \min_{\mu' \in P'} \mu'(E)$ , i.e., if and only if  $P(E) \cap P'(E) \neq \emptyset$ . Similarly,  $AA(E) > 0$  if and only if all  $P, P' \in \mathbb{P}$  satisfy  $\max_{\mu \in P} \mu(E) > \min_{\mu' \in P'} \mu'(E)$ , i.e., if and only if  $P(E) \cap P'(E)$  is a non-degenerate interval.  $\square$

## C.4 Proof of Corollary 2

Pick any  $\beta > 0$  and  $\nu \in \Delta(S)$  with  $\beta < \min_{s \in S} \nu(s)$ . Define  $\mathbb{P}$  by  $\mathbb{P} = \{P^F : F \in \mathcal{F}\}$ , where for each  $F \in \mathcal{F}$ ,

$$P^F := \{\mu \in \Delta(S) : \mu(F) = \nu(F) + \frac{\beta}{2}, \mu(E) \in [\nu(E) - \beta, \nu(E) + \beta] \forall E \subseteq S\}.$$

Note that each  $P^F$  is nonempty: Indeed, pick any  $s \in F$  and  $s' \in F^c$  (which exist since  $F \notin \{S, \emptyset\}$ ). Then setting  $\mu(s) = \nu(s) + \frac{\beta}{2}$ ,  $\mu(s') = \nu(s') - \frac{\beta}{2}$ , and  $\mu(s'') = \nu(s'')$  for all  $s'' \neq s, s'$  yields  $\mu \in P^F$ . Since  $P^F$  is also closed and convex,  $\mathbb{P}$  is a well-defined belief-set collection.

Take any  $F \in \mathcal{F}$ , and observe that  $P^F(F) = \{\nu(F) + \beta/2\}$ , while  $P^{F^c}(F) = \{\nu(F) - \beta/2\}$ . Therefore,  $P^F(F) \cap P^{F^c}(F) = \emptyset$ , which implies by Proposition 2 that  $AA(F) < 0$ .

Consider now any  $E \in \mathcal{E}$  and any  $F \in \mathcal{F}$ . Since  $E \neq F$  (as  $\mathcal{E}$  and  $\mathcal{F}$  are disjoint), we either have (a)  $F \setminus E \neq \emptyset \neq E \setminus F$ ; (b)  $E \subsetneq F$ ; or (c)  $F \subsetneq E$ . In each case, we show that there exist  $\mu, \mu' \in P^F$  with  $\mu(E) = \nu(E) - \frac{\beta}{2}$  and  $\mu'(E) = \nu(E) + \frac{\beta}{2}$ . Since this is true for any  $F$ , this implies that  $P^F(E) \cap P^{F'}(E) \supseteq [\nu(E) - \frac{\beta}{2}, \nu(E) + \frac{\beta}{2}]$  is a nondegenerate interval for any  $F, F' \in \mathcal{F}$ , which in turn implies that  $AA(E) > 0$  by Proposition 4.

In case (a), pick  $s \in F \setminus E$  and  $s' \in E \setminus F$ . Since  $E \neq F^c$  (as  $F^c \in \mathcal{F}$ ), there also exists  $s'' \in S \setminus (E \cup F)$ . Then define  $\mu$  by  $\mu(s) = \nu(s) + \frac{\beta}{2}$ ,  $\mu(s') = \nu(s') - \frac{\beta}{2}$ , and  $\mu(s''') = \nu(s''')$  for all  $s''' \neq s, s'$ ; and  $\mu'$  by  $\mu'(s) = \nu(s) + \frac{\beta}{2}$ ,  $\mu'(s') = \nu(s') + \frac{\beta}{2}$ ,  $\mu'(s'') = \nu(s'') - \beta$ , and  $\mu'(s''') = \nu(s''')$  for all  $s''' \neq s, s', s''$ .

In case (b), pick  $s \in F \setminus E$ ,  $s' \in E$ , and  $s'' \in F^c \subseteq E^c$ . Then define  $\mu$  by  $\mu(s) = \nu(s) + \beta$ ,  $\mu(s') = \nu(s') - \frac{\beta}{2}$ ,  $\mu(s'') = \nu(s'') - \frac{\beta}{2}$ , and  $\mu(s''') = \nu(s''')$  for all  $s''' \neq s, s', s''$ ; and  $\mu'$  by  $\mu'(s) = \nu(s)$ ,  $\mu'(s') = \nu(s') + \frac{\beta}{2}$ ,  $\mu'(s'') = \nu(s'') - \frac{\beta}{2}$ , and  $\mu'(s''') = \nu(s''')$  for all  $s''' \neq s, s', s''$ .

In case (c), pick  $s \in F$ ,  $s' \in E \setminus F$ , and  $s'' \in E^c \subseteq F^c$ . Then define  $\mu$  by  $\mu(s) = \nu(s) + \frac{\beta}{2}$ ,  $\mu(s') = \nu(s') - \beta$ ,  $\mu(s'') = \nu(s'') + \frac{\beta}{2}$ , and  $\mu(s''') = \nu(s''')$  for all  $s''' \neq s, s', s''$ ; and  $\mu'$  by  $\mu'(s) = \nu(s) + \frac{\beta}{2}$ ,  $\mu'(s'') = \nu(s'') - \frac{\beta}{2}$ , and  $\mu'(s''') = \nu(s''')$  for all  $s''' \neq s, s''$ .  $\square$

## C.5 Proof of Corollary 3

Recall that the  $\alpha$ -MEU functional (8) coincides with the DSEU representation  $(\mathbb{P}, u)$  where  $\mathbb{P} = \{\alpha P + (1 - \alpha)\{\mu\} : \mu \in P\}$ . Let  $P^\mu = \alpha P + (1 - \alpha)\{\mu\}$  for any  $\mu \in P$ . For any event  $E$  and  $\mu \in P$ , the interval  $P^\mu(E) = \{\nu(E) : \nu \in P^\mu\}$  is thus given by  $[\alpha \min_{\nu \in P} \nu(E) + (1 - \alpha)\mu(E), \alpha \max_{\nu \in P} \nu(E) + (1 - \alpha)\mu(E)]$ .

Suppose that  $\alpha \geq 1/2$ . Then, for any  $\mu \in P$  and any event  $E$ , we have

$$\begin{aligned} & \alpha \min_{\nu \in P} \nu(E) + (1 - \alpha)\mu(E) \leq \alpha \min_{\nu \in P} \nu(E) + (1 - \alpha) \max_{\nu \in P} \nu(E) \\ & \leq \frac{1}{2} \min_{\nu \in P} \nu(E) + \frac{1}{2} \max_{\nu \in P} \nu(E) \leq (1 - \alpha) \min_{\nu \in P} \nu(E) + \alpha \max_{\nu \in P} \nu(E) \\ & \leq (1 - \alpha)\mu(E) + \alpha \max_{\nu \in P} \nu(E). \end{aligned}$$

Hence,  $1/2 \min_{\nu \in P} \nu(E) + 1/2 \max_{\nu \in P} \nu(E) \in P_E^\mu$ . Since this is true for every  $\mu \in P$ , this implies  $P^\mu(E) \cap P^{\mu'}(E) \neq \emptyset$  for all  $\mu, \mu' \in P$ . Thus,  $AA(E) \geq 0$  by the first part of Proposition 4. Moreover, consider the case  $\alpha > 1/2$ . Since  $P$  is not a singleton, there exists an event  $E$  such that  $\min_{\nu \in P} \nu(E) < \max_{\nu \in P} \nu(E)$ . Then the above inequality is strict for each  $\mu$ , i.e.,

$$\alpha \min_{\nu \in P} \nu(E) + (1 - \alpha)\mu(E) < \frac{1}{2} \min_{\nu \in P} \nu(E) + \frac{1}{2} \max_{\nu \in P} \nu(E) < (1 - \alpha)\mu(E) + \alpha \max_{\nu \in P} \nu(E).$$

Thus, for each  $\mu, \mu' \in P$ ,  $P^\mu(E) \cap P^{\mu'}(E)$  is a non-degenerate interval, which implies  $AA(E) > 0$  by the second part of Proposition 4.

Next, suppose that  $\alpha \leq 1/2$ . Take any  $E$  and let  $\mu$  be a minimizer of  $\mu(E)$  on  $P$ , and  $\mu'$  be a maximizer. Since  $\alpha \leq 1/2$ , we have  $\alpha\mu'(E) + (1 - \alpha)\mu(E) \leq \alpha\mu(E) + (1 - \alpha)\mu'(E)$ . Since  $P^\mu(E) = [\mu(E), \alpha\mu'(E) + (1 - \alpha)\mu(E)]$  and  $P^{\mu'}(E) = [\alpha\mu(E) + (1 - \alpha)\mu'(E), \mu'(E)]$ , this proves that  $P^\mu(E) \cap P^{\mu'}(E)$  is not a non-degenerate interval. Thus, by the second part of Proposition 4,  $AA(E) \leq 0$ . Moreover, consider the case  $\alpha < 1/2$ . Since  $P$  is not a singleton, there exists an event  $E$  such that  $\min_{\mu \in P} \mu(E) < \max_{\mu \in P} \mu(E)$ . Then the above inequality is strict, i.e.,  $\alpha\mu'(E) + (1 - \alpha)\mu(E) < \alpha\mu(E) + (1 - \alpha)\mu'(E)$ . Thus  $P^\mu(E) \cap P^{\mu'}(E) = \emptyset$ , which implies  $AA(E) < 0$  by the first part of Proposition 4.  $\square$

## C.6 Proof of Theorem 2

Suppose that  $\succsim$  admits a DSEU representation  $(\mathbb{P}, u)$  and  $E$  is non-null. For each act  $f$ , let

$$W_E(f) = \max_{P \in \mathbb{P}} \min_{\mu_E \in P_E} \sum_{s \in E} \mu_E(s) u(f(s)),$$

where for any  $\mu$  with  $\mu(E) > 0$ ,  $\mu_E$  is the Bayesian update of  $\mu$  defined by  $\mu_E(F) = \mu(F \cap E) / \mu(E)$  for all  $F \subseteq S$ . Note that  $W_E$  is well-defined because  $E$  is non-null.

**Lemma C.2.** *For any  $f \in \mathcal{F}$  and  $p \in \Delta(Z)$ , we have  $fEp \succsim p \Leftrightarrow W_E(f) \geq u(p)$ .*

*Proof.* Observe that

$$\begin{aligned}
fEp \succsim p &\Leftrightarrow \max_{P \in \mathbb{P}} \min_{\mu \in P} \left[ \sum_{s \in E} \mu(s)u(f(s)) + (1 - \mu(E))u(p) \right] \geq u(p) \\
&\Leftrightarrow \exists P \in \mathbb{P}, \forall \mu \in P, \sum_{s \in E} \mu(s)u(f(s)) \geq \mu(E)u(p) \\
&\Leftrightarrow \exists P \in \mathbb{P}, \forall \mu \in P \text{ with } \mu(E) > 0, \sum_{s \in E} \frac{\mu(s)}{\mu(E)}u(f(s)) \geq u(p) \\
&\Leftrightarrow \max_{P \in \mathbb{P}} \min_{\mu_E \in P_E} \sum_{s \in E} \mu_E(s)u(f(s)) \geq u(p) \\
&\Leftrightarrow W_E(f) \geq u(p),
\end{aligned}$$

where the fourth equivalence uses the fact that for any  $P \in \mathbb{P}$ , there is  $\mu \in P$  such that  $\mu(E) > 0$ .  $\square$

We now prove Theorem 2. For the implication (2.)  $\Rightarrow$  (1.), note that  $W_E$  is the functional associated with the DSEU representation  $(\mathbb{P}_E, u)$ . Thus, if  $\succsim_E$  is represented by  $(\mathbb{P}_E, u)$ , the equivalence in Lemma C.2 can be rewritten as  $fEp \succsim p \Leftrightarrow f \succsim_E p$ . Thus, Axiom 9 holds.

To prove that (1.)  $\Rightarrow$  (2.), note that for each act  $f$ , there exists  $p_f \in \Delta(Z)$  such that  $W_E(f) = u(p_f)$ . We claim that  $f \sim_E p_f$ : Indeed, Lemma C.2 and Axiom 9 imply that  $f \succsim_E p_f$ . Suppose for a contradiction that  $f \succ_E p_f$ . By Lemma C.2 and Axiom 9,  $\succsim_E$  restricted to constant acts is represented by  $u$ . Thus, since  $u$  is nonconstant, either (i) there exists  $q \in \Delta(Z)$  with  $p_f \succ_E q$ , or (ii) there exists  $q \in \Delta(Z)$  with  $q \succ_E p_f$ . In case (i), consider  $f' := (1 - \varepsilon)f + \varepsilon q$  for some small enough  $\varepsilon \in (0, 1)$ . Then,  $W_E(f') < u(p_f)$ , but  $f' \succ_E p_f$  (by Archimedean continuity of  $\succsim_E$ ), contradicting Lemma C.2 and Axiom 9. In case (ii), consider  $p' := (1 - \varepsilon)p + \varepsilon q$  for some small enough  $\varepsilon \in (0, 1)$ . Then,  $W_E(f) < u(p')$ , but  $f \succ_E p'$  (by Archimedean continuity), again contradicting Lemma C.2 and Axiom 9.

Hence, for any  $f, g \in \mathcal{F}$ ,  $f \succsim_E g$  iff  $p_f \succsim_E p_g$  (as  $\succsim_E$  is a weak order) iff  $u(p_f) \geq u(p_g)$  iff  $W_E(f) \geq W_E(g)$ . Thus,  $\succsim_E$  is represented by  $W_E$ , i.e., by the DSEU representation  $(\mathbb{P}_E, u)$ .  $\square$

## D Proofs for Sections 4.1–4.2

### D.1 Proof of Proposition 5

Below we fix the unique functional  $I : \mathbb{R}^S \rightarrow \mathbb{R}$  associated with  $\succsim$ , as given by Lemma B.1. We begin with the following lemma:

**Lemma D.1.** *Suppose  $(\mathbb{P}, u)$  is a DSEU representation of  $\succsim$ . Then  $\overline{\mathbb{P}} = \text{cl}\{H_{\phi, \lambda} : \phi \in \mathbb{R}^S, \lambda \leq I(\phi)\}$ .*

*Proof.* First, take any  $\phi \in \mathbb{R}^S$ ,  $\lambda \in \mathbb{R}$  such that  $\lambda \leq I(\phi)$ . Since  $(\mathbb{P}, u)$  represents  $\succsim$ , there exists  $P \in \mathbb{P}$  such that  $\min_{\mu \in P} \mu \cdot \phi = I(\phi)$ . Thus,  $P \subseteq H_{\phi, I(\phi)} \subseteq H_{\phi, \lambda}$ , which implies  $H_{\phi, \lambda} \in \overline{\mathbb{P}}$ . This proves that  $\overline{\mathbb{P}} \supseteq \text{cl}\{H_{\phi, \lambda} : \phi \in \mathbb{R}^S, \lambda \leq I(\phi)\}$ .

Conversely, consider any  $\phi \in \mathbb{R}^S$ ,  $\lambda \in \mathbb{R}$  such that there exists  $P' \in \mathbb{P}$  with  $P' \subseteq H_{\phi, \lambda}$ . Since  $(\mathbb{P}, u)$  represents  $\succsim$ ,  $I(\phi) \geq \min_{\mu \in P'} \mu \cdot \phi \geq \min_{\mu \in H_{\phi, \lambda}} \mu \cdot \phi$ . Hence,  $\lambda \leq I(\phi)$ . This proves that  $\overline{\mathbb{P}} \subseteq \text{cl}\{H_{\phi, \lambda} : \phi \in \mathbb{R}^S, \lambda \leq I(\phi)\}$ .  $\square$

We now prove Proposition 5. Suppose first that  $(\mathbb{P}', u')$  is another DSEU representation of  $\succsim$ . Then the fact that  $\overline{\mathbb{P}} = \overline{\mathbb{P}'}$  is immediate from Lemma D.1 and the uniqueness of  $I$ . The proof that  $u \approx u'$  is standard.

Conversely, suppose that  $u \approx u'$  and  $\overline{\mathbb{P}} = \overline{\mathbb{P}'}$ . To show that  $(\mathbb{P}', u')$  represents  $\succsim$ , it suffices to show that  $\max_{P' \in \mathbb{P}'} \min_{\mu \in P'} \mu \cdot \phi = I(\phi)$  for all  $\phi \in \mathbb{R}^S$ . To prove this, observe first that since (by Lemma D.1)  $H_{\phi, I(\phi)} \in \overline{\mathbb{P}} = \overline{\mathbb{P}'}$ , there exist sequences of  $P'_n \in \mathbb{P}'$  and half-spaces  $H_n \supseteq P'_n$  with  $H_n \rightarrow H_{\phi, I(\phi)}$ . Then, for all  $\phi$ , we have  $\min_{\mu \in H_{\phi, I(\phi)}} \mu \cdot \phi = I(\phi) = \lim_n \min_{\mu \in H_n} \mu \cdot \phi$  and  $\min_{\mu \in H_n} \mu \cdot \phi \leq \min_{\mu \in P'_n} \mu \cdot \phi$  for all  $n$ . This implies that  $\max_{P' \in \mathbb{P}'} \min_{\mu \in P'} \mu \cdot \phi \geq I(\phi)$ . Suppose next that  $\min_{\mu \in P''} \mu \cdot \phi - I(\phi) =: \varepsilon > 0$  for some  $P'' \in \mathbb{P}'$ . Then  $H_{\phi, I(\phi) + \varepsilon} \supseteq P''$ , which implies  $H_{\phi, I(\phi) + \varepsilon} \in \overline{\mathbb{P}'}$ . Since  $\overline{\mathbb{P}'} = \overline{\mathbb{P}}$ , this contradicts Lemma D.1.

Finally, note that the half-space closure of  $\overline{\mathbb{P}}$  is  $\overline{\mathbb{P}}$  itself. Thus, by the previous paragraph,  $(\overline{\mathbb{P}}, u)$  is itself a DSEU representation of  $\succsim$ .  $\square$

## D.2 Proof of Proposition 6

For each preference  $\succsim_i$ , let utility  $u_i$  and functional  $I_i$  be as given by Lemma B.1. Note that  $\succsim_1$  is more ambiguity-averse than  $\succsim_2$  if and only if  $u_1 \approx u_2$  and  $I_1(\phi) \leq I_2(\phi)$  for all  $\phi \in \mathbb{R}^S$ . Thus, it suffices to show that  $I_1(\phi) \leq I_2(\phi)$  for all  $\phi$  if and only if  $\overline{\mathbb{P}}_1 \subseteq \overline{\mathbb{P}}_2$ .

Suppose first that  $I_1(\phi) \leq I_2(\phi)$  for all  $\phi$ . Then  $\{H_{\phi, \lambda} : \phi \in \mathbb{R}^S, \lambda \leq I_1(\phi)\} \subseteq \{H_{\phi, \lambda} : \phi \in \mathbb{R}^S, \lambda \leq I_2(\phi)\}$ . By Lemma D.1, this implies that  $\overline{\mathbb{P}}_1 \subseteq \overline{\mathbb{P}}_2$ .

Conversely, if  $\overline{\mathbb{P}}_1 \subseteq \overline{\mathbb{P}}_2$ , then  $\max_{P \in \overline{\mathbb{P}}_1} \min_{\mu \in P} \mu \cdot \phi \leq \max_{P \in \overline{\mathbb{P}}_2} \min_{\mu \in P} \mu \cdot \phi$  for all  $\phi$ . Since  $(\overline{\mathbb{P}}_i, u_i)$  is a DSEU representation of  $\succsim_i$  for  $i = 1, 2$ , this inequality means that  $I_1(\phi) \leq I_2(\phi)$  for all  $\phi$ .  $\square$

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# Supplementary Appendix to “Dual-self representations of ambiguity preferences”

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This supplementary appendix is organized as follows. Appendix S.1 provides the proofs for the generalizations of DSEU considered in Section 4.3. Appendix S.2 presents additional content for Section 3.2: a characterization of full dynamic consistency under DSEU, and some supporting examples for Remark 2 on updating under the Amarante and GMM representations. Appendix S.3 considers the representation obtained by inverting the order of moves of Optimism and Pessimism. Appendix S.4 presents an incompatibility result for source dependence under Klibanoff, Marinacci, and Mukerji’s (2005) smooth model.

## S.1 Proofs for Section 4.3

### S.1.1 Proof of Theorem 3

We will invoke the following result from MMR:

**Lemma S.1.1** (Lemma 28 in MMR). *Preference  $\succsim$  satisfies Axioms 1–4 and Axiom 10 if and only if there exists a nonconstant affine function  $u : \Delta(Z) \rightarrow \mathbb{R}$  with  $U := (u(\Delta(Z)))^S$  and a normalized niveloid  $I : U \rightarrow \mathbb{R}$  such that  $I \circ u$  represents  $\succsim$ .*

Recall that functional  $I : U \rightarrow \mathbb{R}$  is a **niveloid** if  $I(\phi) - I(\psi) \leq \max_s(\phi_s - \psi_s)$  for all  $\phi, \psi \in U$ . Lemma 25 in MMR shows that  $I$  is a niveloid if and only if it is monotonic and constant-additive.

Based on this result, the necessity direction of Theorem 3 is standard. We now prove the sufficiency direction. Suppose  $\succsim$  satisfies Axioms 1–4 and Axiom 10. Let  $I$ ,  $u$ , and  $U$  be as given by Lemma S.1.1. Since  $I$  is a niveloid, it is 1-Lipschitz. Hence, Lemma A.1 yields a subset  $\hat{U} \subseteq \text{int}U$  with  $U \setminus \hat{U}$  of Lebesgue measure 0 such that  $I$  is differentiable on  $\hat{U}$ . Define  $\mu_\psi := \nabla I(\psi)$  and  $w_\psi := I(\psi) - \nabla I(\psi) \cdot \psi$  for each  $\psi \in \hat{U}$ . By Lemma A.4 and the fact that niveloids are monotonic and constant-additive,  $\mu_\psi \in \Delta(S)$  for all  $\psi \in \hat{U}$ . For each  $\psi \in U$ , define

$$D_\psi := \{(\mu, w) \in \Delta(S) \times \mathbb{R} : \mu \cdot \psi + w \geq I(\psi)\} \cap \overline{\text{co}}\{(\mu_\xi, w_\xi) : \xi \in \hat{U}\},$$

and let  $\mathbb{D} := \{D_\psi : \psi \in U\}$ . The following lemma implies that each  $D_\psi$  is nonempty; note also that it is closed, convex, and bounded below.

**Lemma S.1.2.** For every  $\phi, \psi \in U$ ,  $\min_{(\mu, w) \in D_\psi} \mu \cdot \phi + w \leq I(\phi)$  with equality if  $\phi = \psi$ .

*Proof.* First, consider any  $\phi, \psi \in \hat{U}$ . Let  $K_\psi := \{\xi \in \hat{U} : \mu_\xi \cdot \psi + w_\xi \geq I(\psi)\}$  be as in Lemma A.6. Note that  $D_\psi = \overline{\text{co}}\{(\mu_\xi, w_\xi) : \xi \in K_\psi\}$ , so that

$$\inf_{\xi \in K_\psi} \mu_\xi \cdot \phi + w_\xi = \min_{(\mu, w) \in D_\psi} \mu \cdot \phi + w,$$

where the minimum is attained as  $D_\psi$  is closed and bounded below. Thus, Lemma A.6 implies that

$$\min_{(\mu, w) \in D_\psi} \mu \cdot \phi + w \leq I(\phi), \quad (23)$$

where, by definition of  $D_\psi$ , (23) holds with equality if  $\psi = \phi$ .

Next, consider any  $\phi, \psi \in U$ . Take sequences  $\phi_n \rightarrow \phi$ ,  $\psi_n \rightarrow \psi$  such that  $\phi_n, \psi_n \in \hat{U}$  for each  $n$ , where we choose  $\phi_n = \psi_n$  if  $\phi = \psi$ . For each  $n$ , the previous paragraph yields some  $(\mu_n, w_n) \in D_{\psi_n}$  such that  $\mu_n \cdot \phi_n + w_n = \min_{(\mu, w) \in D_{\psi_n}} \mu \cdot \phi_n + w \leq I(\phi_n)$ , with equality if  $\phi = \psi$ . Thus, for each  $n$ , we have  $I(\psi_n) - \mu_n \cdot \psi_n \leq w_n \leq I(\phi_n) - \mu_n \cdot \phi_n$ . Since  $\phi_n \rightarrow \phi$ ,  $\psi_n \rightarrow \psi$ , and  $I$  is continuous, this implies that sequence  $(w_n)$  is bounded. Thus, up to restricting to a suitable subsequence, we can assume that  $(\mu_n, w_n) \rightarrow (\mu_\infty, w_\infty)$  for some  $(\mu_\infty, w_\infty) \in \Delta(S) \times \mathbb{R}$ . Then  $(\mu_\infty, w_\infty) \in D_\psi$  and  $\mu_\infty \cdot \phi + w_\infty \leq I(\phi)$  by continuity of  $I$ , with equality if  $\phi = \psi$ . Thus,  $\min_{(\mu, w) \in D_\psi} \mu \cdot \phi + w = \inf_{(\mu, w) \in D_\psi} \mu \cdot \phi + w \leq I(\phi)$ , with equality if  $\phi = \psi$ , where the minimum is attained since  $D_\psi$  is closed and bounded below.  $\square$

Finally, we obtain a dual-self variational representation of  $\succsim$  as follows. For each  $D \in \mathbb{D}$ , define  $c_D : \Delta(S) \rightarrow \mathbb{R} \cup \{\infty\}$  by  $c_D(\mu) := \inf\{w \in \mathbb{R} : (\mu, w) \in D\}$  for each  $\mu \in \Delta(S)$ , where by convention the infimum of the empty set is  $\infty$ . Note that  $c_D$  is convex for all  $D$  by convexity of  $D$ . Moreover, for all  $\phi \in U$ ,  $\min_{(\mu, w) \in D} \mu \cdot \phi + w = \min_{\mu \in \Delta(S)} \mu \cdot \phi + c_D(\mu)$ . Thus, Lemma S.1.2 implies

$$I(\phi) = \max_{D \in \mathbb{D}} \min_{\mu \in \Delta(S)} \mu \cdot \phi + c_D(\mu) \quad (24)$$

for all  $\phi \in U$ . Since  $I$  is normalized, applying (24) to any constant vector  $\underline{a} \in U$ , yields  $I(\underline{a}) = \underline{a} + \max_{D \in \mathbb{D}} \min_{\mu \in \Delta(S)} c_D(\mu) = \underline{a}$ . Hence,  $\mathbb{C}^* := \{c_D : D \in \mathbb{D}\}$  satisfies  $\max_{c \in \mathbb{C}^*} \min_{\mu \in \Delta(S)} c(\mu) = 0$  and  $(\mathbb{C}^*, u)$  is a dual-self variational representation of  $\succsim$  by Lemma S.1.1.  $\square$

**Remark 3.** We note that our characterization of the set of relevant priors under DSEU generalizes to the dual-self variational model. Specifically, let  $\text{dom}(c) := \{\mu : c(\mu) \in \mathbb{R}\}$  denote the effective domain of any cost function. Then there exists a unique closed, convex set  $C$  such that  $C \subseteq \overline{\text{co}}\left(\bigcup_{c \in \mathbb{C}} \text{dom}(c)\right)$  for all dual-self variational representations of  $\succsim$ , with

equality for the representation  $\mathbb{C}^*$  we constructed in the proof of Theorem 3. Moreover, it can again be shown that  $C$  is the Bewley set of the unambiguous preference  $\succsim^*$ . The argument relies on the observation that  $C = \overline{\text{co}} \left( \bigcup_{\phi \in \text{int}U} \partial I(\phi) \right)$ , where  $I$  is the utility act functional obtained in the proof of Theorem 3 and  $U$  its domain. Details are available on request.  $\blacktriangle$

### S.1.2 Proof of Theorem 4

The following result follows from a minor modification of the proof of Lemma 57 in CMMM:

**Lemma S.1.3.** *Preference  $\succsim$  satisfies Axioms 1–4 and 11 if and only if there exists a non-constant affine function  $u : \Delta(Z) \rightarrow \mathbb{R}$  with  $U := (u(\Delta(Z)))^S$  and a monotonic, normalized and continuous functional  $I : U \rightarrow \mathbb{R}$  such that  $I \circ u$  represents  $\succsim$ .*

Based on this result, the necessity direction of Theorem 4 is standard. We now prove the sufficiency direction. Suppose  $\succsim$  satisfies Axioms 1–4 and 11. Let  $I$ ,  $u$ , and  $U$  be as given by Lemma S.1.3. Define  $D_\psi := \{(\mu, I(\psi) - \mu \cdot \psi) \in \mathbb{R}_+^S \times \mathbb{R} : \mu \in \mathbb{R}_+^S\}$  for each  $\psi \in U$ . Note that  $D_\psi$  is nonempty and convex. Let  $I_\psi(\phi) := \inf_{(\mu, w) \in D_\psi} \mu \cdot \phi + w$  for each  $\phi, \psi \in U$ .

Take any  $\phi, \psi \in U$ . Observe that

$$I_\psi(\phi) = \inf_{\alpha > 0, s \in S} I(\psi) + \alpha(\phi_s - \psi_s) = \begin{cases} I(\psi) & \text{if } \phi \geq \psi \\ -\infty & \text{if } \phi \not\geq \psi \end{cases}$$

Thus,  $I(\phi) \geq I_\psi(\phi)$  by monotonicity of  $I$ , with equality if  $\phi = \psi$ . That is, for each  $\phi \in U$ ,

$$I(\phi) = \max_{\psi \in U} I_\psi(\phi). \tag{25}$$

For each  $\psi \in U$ , define a function  $G_\psi : \mathbb{R} \times \Delta(S) \rightarrow \mathbb{R} \cup \{\infty\}$  by

$$G_\psi(t, \mu) = \sup\{I_\psi(\xi) : \xi \in U, \xi \cdot \mu \leq t\}$$

for each  $(t, \mu)$ . The map is quasi-convex (Lemma 31 in CMMM) and increasing in  $t$ .

**Lemma S.1.4.** *We have  $I_\psi(\phi) = \inf_{\mu \in \Delta(S)} G_\psi(\mu \cdot \phi, \mu)$  for each  $\phi, \psi \in U$ .*

*Proof.* Observe that  $\text{RHS} = \inf_{\mu \in \Delta(S)} \sup\{I_\psi(\xi) : \xi \cdot \mu \leq \phi \cdot \mu\}$ . To see that  $\text{LHS} \leq \text{RHS}$ , observe that  $I_\psi(\phi) \leq \sup\{I_\psi(\xi) : \xi \cdot \mu \leq \phi \cdot \mu\}$  holds for any  $\mu \in \Delta(S)$ . To see that  $\text{LHS} \geq \text{RHS}$ , note first that if  $\phi \geq \psi$  then  $\text{LHS} = I(\psi)$  and  $\text{RHS} \in \{I(\psi), -\infty\}$ , so the inequality clearly holds. If  $\phi \not\geq \psi$  then  $\phi_s < \psi_s$  for some  $s \in S$ . Thus, by taking  $\mu = \delta_s$ , any  $\xi$  with  $\xi \cdot \mu \leq \phi \cdot \mu$  satisfies  $\xi_s \leq \phi_s$ , which implies  $\xi \not\geq \psi$ , whence  $I_\psi(\xi) = -\infty$ .  $\square$

Setting  $\mathbb{G} = \{G_\phi : \phi \in U\}$ , Lemma S.1.4 and (25) ensure that the functional  $W$  given by (14) represents  $\succsim$  and is continuous. Finally, note that since  $I$  is normalized, we have  $a = I(\underline{a}) = \max_{G \in \mathbb{G}} \inf_{\mu \in \Delta(S)} G(a, \mu)$  for any  $a \in \mathbb{R}$ , as required.  $\square$

## S.2 Additional material for Section 3.2

### S.2.1 Characterization of dynamic consistency

Fix any partition  $\Pi$  of  $S$  and a family of conditional preferences  $\{\succsim_E\}_{E \in \Pi}$ . Consider the following strengthening of C-dynamic consistency (Axiom 9):

**Axiom 12** (Dynamic Consistency). For all  $f, g \in \mathcal{F}$ ,  $f \succsim_E g \Leftrightarrow fEg \succsim g$ .

Epstein and Schneider (2003) show that prior-by-prior updating under the maxmin model satisfies Axiom 12<sup>30</sup> for each  $E \in \Pi$  if and only if the ex-ante set of priors  $P$  is *rectangular* with respect to partition  $\Pi$ , meaning that there exist belief-sets  $Q^0 \subseteq \Delta(\Pi)$  and  $Q^E \subseteq \Delta(E)$  for each  $E \in \Pi$  such that<sup>31</sup>

$$P = Q^0 \times (Q^E)_{E \in \Pi} := \{\mu \in \Delta(S) : \mu(\cdot) = \sum_{E \in \Pi} \nu^0(E) \nu^E(\cdot) \text{ for some } \nu^0 \in Q^0, \nu^E \in Q^E\}.$$

We show that for prior-by-prior updating under DSEU, Axiom 12 in turn characterizes the following extension of the notion of rectangularity to belief-set collections. Say that  $\mathbb{P}$  is a **rectangular belief-set collection** (with respect to  $\Pi$ ) if there exist belief-set collections  $\mathbb{Q}^0 \subseteq \mathcal{K}(\Delta(\Pi))$  and  $\mathbb{Q}^E \subseteq \mathcal{K}(\Delta(E))$  for each  $E \in \Pi$  such that

$$\mathbb{P} = \mathbb{Q}^0 \times (\mathbb{Q}^E)_{E \in \Pi} := \{Q^0 \times (Q^E)_{E \in \Pi} : Q^0 \in \mathbb{Q}^0, Q^E \in \mathbb{Q}^E \forall E \in \Pi\}.$$

Note that this is stronger than requiring each  $P \in \mathbb{P}$  to be rectangular. Say that  $E \in \Pi$  is **strongly non-null** if for all  $f \in \mathcal{F}$  and  $p, q \in \Delta(Z)$  with  $p \succ q$ , we have  $pEf \succ qEf$ .

**Theorem S.2.1.** *Suppose that  $\succsim$  satisfies Axioms 1–5, that each  $E \in \Pi$  is strongly non-null, and that each  $(\succsim_E)_{E \in \Pi}$  is an Archimedean weak order. Then, the following are equivalent:*

1. Each pair  $(\succsim, \succsim_E)_{E \in \Pi}$  satisfies Axiom 12.
2. There exists a rectangular belief-set collection  $\mathbb{P}$  and a nonconstant affine utility  $u$  such that  $(\mathbb{P}, u)$  is a DSEU representation of  $\succsim$  and  $(\mathbb{P}_E, u)$  is a DSEU representation of  $\succsim_E$  for each  $E \in \Pi$ .

<sup>30</sup>Epstein and Schneider (2003) use an alternative formulation of dynamic consistency, which is equivalent to Axiom 12 in our setting (cf. Lemma S.2.1).

<sup>31</sup>In the following, we identify  $\Delta(E)$  with the subset  $\{\mu \in \Delta(S) : \mu(E) = 1\} \subseteq \Delta(S)$ .

### S.2.1.1 Proof of Theorem S.2.1

We will invoke the following lemma:<sup>32</sup>

**Axiom 13** (Consequentialism). If  $f(s) = g(s)$  for all  $s \in E$ , then  $f \sim_E g$ .

**Lemma S.2.1.** Suppose  $\succsim$  and each  $(\succsim_E)_{E \in \Pi}$  are weak orders. The following are equivalent:

1. Each pair  $(\succsim, \succsim_E)_{E \in \Pi}$  satisfies Axiom 12.
2. Each  $(\succsim_E)_{E \in \Pi}$  satisfies Axiom 13 and, for all  $f, g \in \mathcal{F}$ ,

$$[f \succsim_E g \forall E \in \Pi] \implies f \succsim g; \quad (26)$$

$$[f \succsim_E g \forall E \in \Pi \text{ and } f \succ_E g \text{ for some } E \in \Pi] \implies f \succ g. \quad (27)$$

*Proof. (1.)  $\implies$  (2.):* Suppose each  $(\succsim, \succsim_E)_{E \in \Pi}$  satisfies Axiom 12. To show Axiom 13, consider any  $f, g \in \mathcal{F}$  and  $E \in \Pi$  with  $f(s) = g(s)$  for all  $s \in E$ . Then  $fEg \sim gEg$  since  $\succsim$  is reflexive, which implies  $f \sim_E g$  by Axiom 12.

Then, for any  $f, g, h \in \mathcal{F}$  and  $E \in \Pi$ , Axioms 12 and 13 imply

$$f \succsim_E g \underbrace{\iff}_{\text{Ax. 13}} fEh \succsim_E gEh \underbrace{\iff}_{\text{Ax. 12}} fEh \succsim gEh. \quad (28)$$

To show (26) suppose  $f \succsim_E g \forall E \in \Pi$ . Then enumerating  $\Pi = \{E_1, \dots, E_n\}$  and applying (28) iteratively, we have

$$f = fE_1f \succsim gE_1f \succsim gE_1(gE_2f) \succsim gE_1(gE_2(gE_3f)) \succsim \dots \succsim g,$$

as required. Moreover, if  $f \succ_{E_i} g$  for some  $i$ , then the above ensures  $f \succ g$ , so (27) holds.

**(2.)  $\implies$  (1.):** For each  $f, g \in \mathcal{F}$  and  $E \in \Pi$ , since  $\succsim_E$  is a weak order and satisfies Axiom 13, we have

$$f \succsim_E g \iff fEg \succsim_E g;$$

moreover, for each  $F \in \Pi \setminus \{E\}$ ,

$$fEg \sim_F g$$

Thus, if  $f \succsim_E g$  then  $fEg \succsim g$  by (26). If not  $f \succsim_E g$ , then  $g \succ_E f$  since  $\succsim_E$  is a weak order, which implies  $g \succ fEg$  by (27).  $\square$

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<sup>32</sup>For the direction (1.)  $\implies$  (2.), Hubmer and Ostrizek (2015) observe that dynamic consistency implies consequentialism.

*Proof of Theorem S.2.1.*

**(2.)  $\implies$  (1.):** Since each  $\succsim_E$  admits the updated DSEU representation  $(\mathbb{P}_E, u)$ , it satisfies Axiom 13. Thus, to prove that  $(\succsim, \succsim_E)_{E \in \Pi}$  satisfies Axiom 12, it suffices by Lemma S.2.1 to verify (26)-(27).

Observe that since  $\mathbb{P} = \mathbb{Q}^0 \times (\mathbb{Q}^E)_{E \in \Pi}$  is rectangular, the prior-by-prior updates  $\mathbb{P}_E$  satisfy  $\mathbb{P}_E = \mathbb{Q}^E$  for each  $E \in \Pi$ . Thus, each  $\succsim_E$  is represented by the functional  $W_E(f) = \max_{Q^E \in \mathbb{Q}^E} \min_{\nu^E \in Q^E} \nu^E \cdot u(f)$ . Moreover,  $\succsim$  is represented by the functional

$$\begin{aligned} W(f) &= \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu \cdot u(f) = \max_{Q^0 \in \mathbb{Q}^0} \min_{\nu^0 \in Q^0} \sum_E \nu^0(E) \max_{Q^E \in \mathbb{Q}^E} \min_{\nu^E \in Q^E} \nu^E \cdot u(f) \\ &= \max_{Q^0 \in \mathbb{Q}^0} \min_{\nu^0 \in Q^0} \sum_E \nu^0(E) W_E(f). \end{aligned}$$

Thus, for any  $f, g \in \mathcal{F}$ , if  $W_E(f) \geq W_E(g)$  for all  $E \in \Pi$ , then  $W(f) \geq W(g)$ , verifying (26). To verify (27), suppose  $W_E(f) > W_E(g)$  for some  $E \in \Pi$  and  $W_F(f) \geq W_F(g)$  for all  $F \in \Pi \setminus \{E\}$ . Pick  $p, q \in \Delta(Z)$  such that  $u(p) = W_E(f)$  and  $u(q) = W_E(g)$ . Then

$$W(f) = W(pEf) > W(qEf) \geq W(qEg) = W(g),$$

where the strict inequality holds since each  $E$  is strongly non-null.

**(1.)  $\implies$  (2.):** Since  $\succsim$  satisfies Axioms 1–5, Lemma B.1 yields a nonconstant, affine  $u$  and monotonic, constant-linear functional  $I : \mathbb{R}^S \rightarrow \mathbb{R}$  such that  $f \succsim g$  iff  $I(u(f)) \geq I(u(g))$ . Up to applying a positive affine transformation, we can assume that  $u(\Delta(Z)) \supseteq [-1, 1]$ . Since Axiom 12 implies Axiom 9, each  $\succsim_E$  admits some DSEU representation  $(\mathbb{Q}^E, u)$  by Theorem 2. Let  $I_E : \mathbb{R}^S \rightarrow \mathbb{R}$  denote the corresponding monotonic, constant-linear functional given by  $I_E(\phi) = \max_{Q^E \in \mathbb{Q}^E} \min_{\nu^E \in Q^E} \nu^E \cdot \phi$ .

For each  $\phi^0, \psi^0 \in \mathbb{R}^\Pi$ , write  $\phi^0 \succsim^* \psi^0$  if there exist  $\phi, \psi \in \mathbb{R}^S$  such that  $I(\phi) \geq I(\psi)$  and

$$\phi^0(E) = I_E(\phi), \quad \psi^0(E) = I_E(\psi), \quad \forall E \in \Pi. \quad (29)$$

Note that  $\succsim^*$  is a weak order. Indeed, for any  $\phi^0 \in \mathbb{R}^\Pi$  define  $G(\phi^0) = \phi \in \mathbb{R}^S$  by  $\phi(s) = \phi^0(E)$  for each  $E \in \Pi$  and  $s \in E$ . Then, by construction of  $I_E$ , we have  $\phi^0(E) = I_E(\phi)$  for all  $E$ . Moreover, note that for any other  $\phi' \in \mathbb{R}^S$  with  $\phi^0(E) = I_E(\phi')$ , we have  $I(\phi) = I(\phi')$ : To see this, take  $\alpha > 0$  small enough that  $\alpha\phi, \alpha\phi' \in (u(\Delta(Z)))^S$ . Since  $I_E(\alpha\phi) = I_E(\alpha\phi')$  for each  $E$  (as  $I_E$  is constant-linear), the implication (26) of Axiom 12 in Lemma S.2.1 yields  $I(\alpha\phi) = I(\alpha\phi')$ . Thus,  $I(\phi) = I(\phi')$  (as  $I$  is constant-linear). Taken together, this shows that for any  $\phi^0, \psi^0 \in \mathbb{R}^\Pi$ ,  $\phi^0 \succsim^* \psi^0$  if and only if  $I(G(\phi^0)) \geq I(G(\psi^0))$ , i.e.,  $\succsim^*$  is represented by the functional  $I_0 := I \circ G : \mathbb{R}^\Pi \rightarrow \mathbb{R}$ .

Note that  $I_0$  is monotonic, as  $I$  is monotonic and  $\phi^0 \geq \psi^0$  implies  $G(\phi^0) \geq G(\psi^0)$ . Moreover,  $I_0$  is constant-linear, as  $I$  is constant-linear and for any  $\phi^0 \in \mathbb{R}^\Pi$ ,  $\alpha > 0$ , and  $\beta \in \mathbb{R}$ , we have  $G(\alpha\phi^0 + \beta) = \alpha G(\phi^0) + \beta$ . Thus, by the proof of Theorem 1, there is a belief-set collection  $\mathbb{Q}^0 \subseteq 2^{\Delta(\Pi)}$  such that  $I_0(\phi^0) = \max_{Q^0 \in \mathbb{Q}^0} \min_{\nu^0 \in Q^0} \nu^0 \cdot \phi^0$  for each  $\phi^0 \in \mathbb{R}^\Pi$ .

Set  $\mathbb{P} := \{Q^0 \times (Q^E)_{E \in \Pi} : Q^0 \in \mathbb{Q}^0, Q^E \in \mathbb{Q}^E \forall E \in \Pi\}$ , which is rectangular. Then for each  $\phi \in \mathbb{R}^S$ ,

$$\begin{aligned} \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu \cdot \phi &= \max_{Q^0 \in \mathbb{Q}^0} \max_{Q^E \in \mathbb{Q}^E, \forall E} \min_{\nu^0 \in Q^0} \sum_E \nu^0(E) \min_{\nu^E \in Q^E} \nu^E \cdot \phi \\ &= \max_{Q^0 \in \mathbb{Q}^0} \min_{\nu^0 \in Q^0} \sum_E \nu^0(E) \max_{Q^E \in \mathbb{Q}^E} \min_{\nu^E \in Q^E} \nu^E \cdot \phi. \end{aligned}$$

We claim that  $(\mathbb{P}, u)$  is a DSEU representation of  $\succsim$ . Indeed, for any  $f, g$  with  $\phi = u(f), \psi = u(g)$ , define  $\phi^0, \psi^0 \in \mathbb{R}^\Pi$  by  $\phi^0(E) = I_E(\phi), \psi^0(E) = I_E(\psi)$  for each  $E \in \Pi$ . Then

$$\begin{aligned} f \succsim g &\iff \phi^0 \succsim^* \psi^0 \\ &\iff \max_{Q^0 \in \mathbb{Q}^0} \min_{\nu^0 \in Q^0} \nu^0 \cdot \phi^0 \geq \max_{Q^0 \in \mathbb{Q}^0} \min_{\nu^0 \in Q^0} \nu^0 \cdot \psi^0 \\ &\iff \max_{Q^0 \in \mathbb{Q}^0} \min_{\nu^0 \in Q^0} \sum_E \nu^0(E) \max_{Q^E \in \mathbb{Q}^E} \min_{\nu^E \in Q^E} \nu^E \cdot \phi \\ &\quad \geq \max_{Q^0 \in \mathbb{Q}^0} \min_{\nu^0 \in Q^0} \sum_E \nu^0(E) \max_{Q^E \in \mathbb{Q}^E} \min_{\nu^E \in Q^E} \nu^E \cdot \psi \\ &\iff \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu \cdot \phi \geq \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu \cdot \psi. \end{aligned}$$

Finally, by construction, we have  $\mathbb{Q}^E = \mathbb{P}_E$  for each  $E \in \Pi$ , and thus  $(\mathbb{P}_E, u)$  is a DSEU representation of  $\succsim_E$ .  $\square$

## S.2.2 Details for Remark 2

We elaborate on some difficulties, outlined in Remark 2, with extending prior-by-prior updating to GMM and Amarante's representations of invariant biseparable preferences.

### S.2.2.1 GMM

Suppose the ex-ante preference  $\succsim$  admits a GMM representation (1) with parameters  $(\alpha(\cdot), C, u)$ . As in Remark 2, consider the following potential extension of prior-by-prior updating: Define the conditional preference  $\succsim_E$  by updating the set of relevant priors  $C$  prior-by-prior to  $C_E$ ,

while holding the weight function  $\alpha(\cdot)$  and utility  $u$  fixed; that is,  $\succsim_E$  is represented by

$$W_E(f) = \alpha(f) \min_{\mu \in C_E} \mathbb{E}_\mu[u(f)] + (1 - \alpha(f)) \max_{\mu \in C_E} \mathbb{E}_\mu[u(f)].$$

The following example highlights several difficulties that arise for this updating rule: (i) the induced  $\succsim_E$  need not be invariant biseparable, as it can violate monotonicity; and (ii)  $\succsim_E$  may violate consequentialism. In particular, this implies (by Theorem 2) that this updating rule does not in general satisfy C-dynamic consistency (Axiom 9).

**Example 3.** Take  $S = \{1, 2, 3\}$ , and a nonconstant affine utility  $u$  with range  $[0, 1]$ . Write  $f = (f_1, f_2, f_3)$  for the act  $f$  that yields the lottery  $f_s$  in state  $s$ .

Suppose  $\succsim$  is induced by an  $\alpha$ -MEU representation (8) with  $\alpha = 1/2$ , utility  $u$ , and belief-set  $P = \Delta(S)$ . Then  $\succsim$  equivalently admits a GMM representation  $(\alpha(\cdot), C, u)$ , where:<sup>33</sup>

- The set of relevant priors is  $C = \text{co} \left\{ \left(\frac{1}{2}, \frac{1}{2}, 0\right), \left(0, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, 0, \frac{1}{2}\right) \right\}$ .
- The function  $\alpha(\cdot)$  is defined, for all  $f$  with nonconstant utility profile  $(u(f_1), u(f_2), u(f_3))$ , by

$$\alpha(f) = \frac{\text{med}(u(f)) - \min(u(f))}{\max(u(f)) - \min(u(f))},$$

where  $\max(u(f)) = \max\{u(f_1), u(f_2), u(f_3)\}$ ,  $\min(u(f)) = \min\{u(f_1), u(f_2), u(f_3)\}$ , and  $\text{med}(u(f))$  is the median value in  $\{u(f_1), u(f_2), u(f_3)\}$ . For instance, if  $f$  satisfies  $u(f_1) > u(f_2) > u(f_3)$ , then  $\alpha(f) = (u(f_2) - u(f_3))/(u(f_1) - u(f_3))$ .

Consider the event  $E = \{1, 2\}$ . The prior-by-prior update of  $C$  is  $C_E = \text{co} \{(1, 0, 0), (0, 1, 0)\}$ . Thus, the conditional preference  $\succsim_E$  induced by the above prior-by-prior updating rule for GMM is represented by the functional

$$W_E(f) = \alpha(f) \min\{u(f_1), u(f_2)\} + (1 - \alpha(f)) \max\{u(f_1), u(f_2)\}.$$

Consider two acts  $f$  and  $g$  such that  $u(f_1) = u(g_1) = 1$ ,  $u(f_2) = 1/2$ , and  $u(f_3) = u(g_2) = u(g_3) = 0$ . Then  $\alpha(f) = 1/2$  and  $\alpha(g) = 0$ . Hence,  $W_E(f) = 3/4$  and  $W_E(g) = 1$ . This shows that  $g \succ_E f$  despite the fact that  $f(s) \succsim_E g(s)$  for all  $s \in S$ . Thus,  $\succsim_E$  violates monotonicity (Axiom 2) and hence is not an invariant biseparable preference.

Next, consider the same act  $f$  as above and some  $\tilde{g}$  with  $\tilde{g}_1 = f_1$ ,  $\tilde{g}_2 = f_2$ , and  $u(\tilde{g}_3) = 1/2$ . We have  $\alpha(\tilde{g}) = 0$ , and hence  $W_E(\tilde{g}) = 1 > W_E(f)$ , which implies  $\tilde{g} \succ_E f$ . This shows that  $\succsim_E$  violates consequentialism (Axiom 13), as  $f(s) = \tilde{g}(s)$  for all  $s \in E = \{1, 2\}$ .  $\blacktriangle$

<sup>33</sup>Indeed, note that the corresponding utility act functional  $I(v) = \frac{1}{2} \min_{i=1,2,3} v_i + \frac{1}{2} \max_{i=1,2,3} v_i$  is piecewise linear with three slopes given by  $\mu \in \left\{ \left(\frac{1}{2}, \frac{1}{2}, 0\right), \left(0, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, 0, \frac{1}{2}\right) \right\}$ , so  $C$  is the convex hull of these three beliefs. Given this,  $\alpha(\cdot)$  is determined by setting  $\alpha(f) \min_C \mu \cdot u(f) + (1 - \alpha(f)) \max_C \mu \cdot u(f) = I(u(f))$ .

An alternative approach to extend prior-by-prior updating to GMM's representation is to impose C-dynamic consistency on  $(\succsim, \succsim_E)$ . This uniquely pins down a conditional preference  $\succsim_E$ , which is invariant biseparable (as can be seen from Theorem 2). Thus, the conditional preference  $\succsim_E$  induced in this manner must admit some GMM representation  $(\alpha^E(\cdot), C^E, u)$ . However, we note that obtaining the conditional parameters  $\alpha^E(\cdot)$  and  $C^E$  directly from the parameters  $\alpha(\cdot)$  and  $C$  of the ex-ante representation can be difficult, as  $\alpha^E(\cdot)$  and  $C^E$  can each depend jointly on both  $\alpha(\cdot)$  and  $C$  (in a way that involves solving a fixed-point problem).<sup>34</sup> Notably, the following example illustrates that when  $\alpha(\cdot) \not\equiv 0, 1$ , the set  $C^E$ , i.e., the set of relevant priors of the conditional preference  $\succsim_E$ , need *not* be equal to the prior-by-prior update  $C_E$  of the ex-ante set of relevant priors  $C$ :

**Example 4.** As in Example 3, let  $S = \{1, 2, 3\}$  and suppose the ex-ante preference  $\succsim$  is an  $\alpha$ -MEU preference with  $\alpha = 1/2$ , nonconstant utility  $u$ , and belief-set  $P = \Delta(S)$ . As noted, the set of relevant priors of  $\succsim$  is  $C = \text{co} \left\{ \left( \frac{1}{2}, \frac{1}{2}, 0 \right), \left( 0, \frac{1}{2}, \frac{1}{2} \right), \left( \frac{1}{2}, 0, \frac{1}{2} \right) \right\}$ .

Again, consider event  $E = \{1, 2\}$ , but now suppose the conditional preference  $\succsim_E$  is pinned down from  $\succsim$  by C-dynamic consistency. Note that, for any act  $f$  with utility profile  $(u(f_1), u(f_2), u(f_3))$ , the condition  $fEp \sim p$  is equivalent to

$$\frac{1}{2} \min\{u(f_1), u(f_2), u(p)\} + \frac{1}{2} \max\{u(f_1), u(f_2), u(p)\} = u(p),$$

i.e., to

$$\frac{1}{2}u(f_1) + \frac{1}{2}u(f_2) = u(p).$$

Thus, by C-dynamic consistency, the conditional preference  $\succsim_E$  is the SEU preference with belief  $(1/2, 1/2, 0)$ . Hence, the set of relevant priors of  $\succsim_E$  is  $C^E = \{(1/2, 1/2, 0)\}$ , which is a strict subset of the prior-by-prior update  $C_E = \text{co} \{(1, 0, 0), (0, 1, 0)\}$  of  $C$ . ▲

### S.2.2.2 Amarante

We first restate an example from [Frick, Iijima, and Le Yaouanq \(2020\)](#), which illustrates that, under the  $\alpha$ -MEU model, if belief-sets are updated prior-by-prior, then conditional preferences are not uniquely pinned down from the ex-ante preference and instead depend on the choice of ex-ante representation:

**Example 5.** Suppose  $S = \{1, 2, 3\}$ . Fix any nonconstant affine utility  $u$ , and consider the

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<sup>34</sup>Specifically, to obtain  $(\alpha^E(\cdot), C^E)$  directly from  $(\alpha(\cdot), C)$ , one must first obtain  $\succsim_E$  from  $\succsim$  via C-dynamic consistency. For each act  $f$ , this involves finding a constant act  $p_f$  that solves the fixed point problem  $fEp_f \sim p_f$ , and then defining  $f \succsim_E g \Leftrightarrow p_f \succsim_E p_g$ .

two  $\alpha$ -MEU representations  $(\alpha, P, u)$  and  $(\alpha', P', u)$ , where

$$\begin{aligned}\alpha &= \frac{3}{4}, & P &= \text{co} \left\{ \left( \frac{5}{6}, \frac{1}{12}, \frac{1}{12} \right), \left( \frac{1}{6}, \frac{5}{12}, \frac{5}{12} \right) \right\}, \\ \alpha' &= 1, & P' &= \text{co} \left\{ \left( \frac{2}{3}, \frac{1}{6}, \frac{1}{6} \right), \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \right\}.\end{aligned}$$

The two representations represent the same ex-ante preference  $\succsim$ , since for all  $f$ ,

$$\begin{aligned}& \frac{3}{4} \min_{\mu \in \text{co} \left\{ \left( \frac{5}{6}, \frac{1}{12}, \frac{1}{12} \right), \left( \frac{1}{6}, \frac{5}{12}, \frac{5}{12} \right) \right\}} \mathbb{E}_\mu[u(f)] + \frac{1}{4} \max_{\mu \in \text{co} \left\{ \left( \frac{5}{6}, \frac{1}{12}, \frac{1}{12} \right), \left( \frac{1}{6}, \frac{5}{12}, \frac{5}{12} \right) \right\}} \mathbb{E}_\mu[u(f)] \\ &= \min_{\mu \in \text{co} \left\{ \left( \frac{2}{3}, \frac{1}{6}, \frac{1}{6} \right), \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \right\}} \mathbb{E}_\mu[u(f)].\end{aligned}$$

Now, consider the event  $E = \{1, 2\}$ . The prior-by-prior Bayesian updates of  $P$  and  $P'$  are

$$P_E = \text{co} \left\{ \left( \frac{10}{11}, \frac{1}{11}, 0 \right), \left( \frac{2}{7}, \frac{5}{7}, 0 \right) \right\}, \quad P'_E = \text{co} \left\{ \left( \frac{4}{5}, \frac{1}{5}, 0 \right), \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \right\}.$$

Consider an act  $f$  with utility profile  $u(f) = (1, 0, 0)$ . The value of this act under the updated model  $(\alpha, P_E, u)$  equals

$$\frac{3}{4} \min \left\{ \frac{10}{11}, \frac{2}{7} \right\} + \frac{1}{4} \max \left\{ \frac{10}{11}, \frac{2}{7} \right\} = \frac{34}{77},$$

and therefore the DM is ex-post indifferent between  $f$  and the constant act  $p$  with utility  $34/77$ . However, under the updated model  $(\alpha', P'_E, u)$ , the value of  $f$  equals  $1/2$ , and thus the DM strictly prefers  $p$  to  $f$  ex post under this model. This shows that  $(\alpha, P_E, u)$  and  $(\alpha', P'_E, u)$  do not represent the same conditional preference.  $\blacktriangle$

Now, consider an Amarante representation (2) with utility  $u$  and capacity  $\nu$  defined on some  $P \subseteq \Delta(S)$ . Natural updating rules for this representation seem less apparent: The literature has considered several updating rules for the special case of Choquet expected utility (see the survey by [Gilboa and Marinacci, 2016](#)), but directly applying these rules to Amarante's model would require one to observe ex-post preferences  $\succsim_Q$  conditional on subsets  $Q \subseteq P$  of beliefs, rather than conditional on subsets  $E$  of states.

One potential extension of prior-by-prior updating might be to hold fixed the utility  $u$  and consider the updated capacity  $\nu_E$ , which is defined on the set  $P_E$  by  $\nu_E(Q) := \nu(\{\mu \in P : \mu_E \in Q\})$  for each  $Q \subseteq P_E$ ; that is,  $\nu_E$  transfers all weight that  $\nu$  assigns to any prior belief to its posterior. However, this rule gives rise to the same issue as in Example 5, i.e., conditional preferences are not uniquely pinned down from the ex-ante preference. To see

this, we use the observation from [Amarante \(2009\)](#) that any  $\alpha$ -MEU representation  $(\alpha, P, u)$  is equal to the Amarante representation with utility  $u$  and capacity  $\nu$  defined on  $P$  by  $\nu(Q) = \alpha$  for all  $\emptyset \neq Q \subsetneq P$ ,  $\nu(\emptyset) = 0$ , and  $\nu(P) = 1$ . This induces an updated capacity  $\nu_E$  that is defined on  $P_E$  and satisfies  $\nu_E(Q) = \alpha$  for all  $\emptyset \neq Q \subsetneq P_E$ ,  $\nu_E(\emptyset) = 0$ , and  $\nu_E(P_E) = 1$ . Thus, the induced conditional Amarante representation is equal to the  $\alpha$ -MEU representation  $(\alpha, P_E, u)$ . Given this, the multiplicity of conditional preferences in [Example 5](#) also applies to this updating rule for the Amarante model.

### S.3 Minmax DSEU representation

While DSEU assumes that Optimism plays first and Pessimism plays second, this is equivalent to a model with the opposite order of moves. We omit all proofs for this section, as they can be obtained as minor modifications of the original proofs for DSEU.

**Theorem S.3.1.** *Preference  $\succsim$  satisfies Axioms 1–5 if and only if  $\succsim$  admits a minmax DSEU representation, i.e., there exists a belief-set collection  $\mathbb{Q}$  and a nonconstant affine utility  $u : \Delta(Z) \rightarrow \mathbb{R}$  such that*

$$W(f) = \min_{Q \in \mathbb{Q}} \max_{\mu \in Q} \mathbb{E}_\mu[u(f)]$$

represents  $\succsim$ .

Our construction of the maxmin DSEU representation in the proof of [Theorem 1](#) uses the belief-set collection  $\mathbb{P}^* = \text{cl}\{P_\phi^* : \phi \in \mathbb{R}^S\}$  with  $P_\phi^* := \{\mu \in \partial I(\underline{0}) : \mu \cdot \phi \geq I(\phi)\}$ . Analogously, it can be shown that the belief-set collection  $\mathbb{Q}^* := \text{cl}\{Q_\phi^* : \phi \in \mathbb{R}^S\}$  with  $Q_\phi^* := \{\mu \in \partial I(\underline{0}) : \mu \cdot \phi \leq I(\phi)\}$  yields a minmax DSEU representation. Paralleling [Section 2.3](#), it is straightforward to show that  $C := \partial I(\underline{0})$  again corresponds to the smallest set of priors that is contained in  $\overline{\text{co}} \bigcup_{Q \in \mathbb{Q}} Q$  for all minmax DSEU representations  $\mathbb{Q}$  of  $\succsim$ , with equality for representation  $\mathbb{Q}^*$ .

While the different shades of ambiguity aversion in [Section 3.1.1](#) are most conveniently characterized using the maxmin DSEU representation, the minmax DSEU representation is useful for characterizing ambiguity-seeking attitudes. Indeed, one can derive analogs of [Propositions 2 and 3](#) that characterize the ambiguity-seeking counterparts of [Axioms 6, 7, and 8](#) in terms of the intersection of belief-sets in  $\mathbb{Q}$ .

#### S.4 Source dependence and the smooth model

Recall that under [Klibanoff, Marinacci, and Mukerji's \(2005\)](#) (henceforth, KMM's) smooth model,  $\succsim$  is represented by the functional

$$W(f) = \int \phi(u(f) \cdot \mu) d\nu(\mu), \quad (30)$$

for some Borel probability measure  $\nu \in \Delta(\Delta(S))$  over beliefs, nonconstant affine  $u : \Delta(Z) \rightarrow \mathbb{R}$ , and strictly increasing  $\phi : u(Z) \rightarrow \mathbb{R}$ . For expositional simplicity, we consider  $Z = [0, 1]$ . Assume that  $u$  is strictly increasing and continuous on  $Z$  with  $u(0) = 0$ , and that  $\phi$  is twice continuously differentiable with  $\phi'(0), \phi''(0) \neq 0$ .

Analogous to [Corollary 4](#) for the  $\alpha$ -MEU model, the following claim establishes a sense in which the smooth model is incompatible with source-dependent negative and positive ambiguity attitudes:

**Claim 1.** *Suppose that  $\succsim$  admits a representation (30). Then there do not exist events  $E, F, G \subseteq S$  such that for all  $x > 0$ ,*

$$xE0 \succ xF0 \succ xG0 \quad \text{and} \quad xE^c0 \succ xF^c0 \succ xG^c0 \quad (31)$$

and such that  $\mu(F)$  is constant across all  $\mu$  in the support of  $\nu$ .<sup>35</sup>

*Proof.* Suppose toward a contradiction that such events  $E, F, G$  exist. For each event  $A \subseteq S$  and  $\Delta \in [0, u(1)]$ , let

$$W_A(\Delta) := \int \phi(\mu(A)\Delta) d\nu(\mu).$$

Then  $W(xA0) = W_A(u(x))$  for all  $x > 0$ . Thus, (31) implies that, for all  $\Delta \in [0, u(1)]$ ,

$$W_E(\Delta) > W_F(\Delta) > W_G(\Delta) \quad \text{and} \quad W_{E^c}(\Delta) > W_{F^c}(\Delta) > W_{G^c}(\Delta). \quad (32)$$

Observe that, for each  $A$ , we have  $W_A(0) = \phi(0)$ , and

$$\begin{aligned} \frac{\partial}{\partial \Delta} W_A(\Delta) &= \int \phi'(\mu(A)\Delta) \mu(A) d\nu(\mu) \\ &= \phi'(0) \int \mu(A) d\nu(\mu) \text{ at } \Delta = 0, \end{aligned}$$

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<sup>35</sup>See [Theorem 3](#) in KMM for a behavioral characterization of such unambiguous events  $F$ .

$$\begin{aligned}
\frac{\partial^2}{\partial \Delta^2} W_A(\Delta) &= \int \phi''(\mu(A)\Delta) \mu(A)^2 d\nu(\mu) \\
&= \phi''(0) \int \mu(A)^2 d\nu(\mu) \text{ at } \Delta = 0.
\end{aligned}$$

Let  $\alpha$  be the constant such that  $\alpha = \mu(F)$  for all  $\mu$  in the support of  $\nu$ . Then, performing a first-order Taylor approximation, the first inequalities in (32) imply  $\int \mu(E) d\nu(\mu) \geq \alpha \geq \int \mu(G) d\nu(\mu)$ . Likewise, the second inequalities in (32) imply  $\int \mu(E^c) d\nu(\mu) \geq 1 - \alpha \geq \int \mu(G^c) d\nu(\mu)$ . Thus,

$$\int \mu(E) d\nu(\mu) = \alpha = \int \mu(G) d\nu(\mu). \tag{33}$$

Note that it is not the case that  $\mu(E) = \alpha$  for  $\nu$ -almost every  $\mu$ , as this would imply  $W_E(\Delta) = W_F(\Delta)$ , contradicting  $W_E(\Delta) > W_F(\Delta)$ . Likewise, it is not the case that  $\mu(G) = \alpha$  for  $\nu$ -almost every  $\mu$ , as this would contradict  $W_F(\Delta) > W_G(\Delta)$ . Thus, by Jensen's inequality

$$\int \mu(E)^2 d\nu(\mu), \int \mu(G)^2 d\nu(\mu) > \alpha^2.$$

Hence, performing a second-order Taylor approximation,  $W_E(\Delta) > W_F(\Delta)$  and (33) implies that  $\phi''(0) > 0$ . Likewise,  $W_F(\Delta) > W_G(\Delta)$  and (33) implies that  $\phi''(0) < 0$ . This is a contradiction.  $\square$