DUAL-SELF REPRESENTATIONS OF AMBIGUITY PREFERENCES

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Dual-self representations of ambiguity preferences[*]

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Abstract

We propose a class of multiple-prior representations of preferences under ambiguity, where the belief the decision-maker (DM) uses to evaluate an uncertain prospect is the outcome of a game played by two conflicting forces, Pessimism and Optimism. The model does not restrict the sign of the DM's ambiguity attitude, and we show that it provides a unified framework through which to characterize different degrees of ambiguity aversion, and to represent the co-existence of negative and positive ambiguity attitudes within individuals as documented in experiments. We prove that our baseline representation, *dual-self expected utility (DSEU)*, yields a novel representation of the class of invariant biseparable preferences (Ghirardato, Maccheroni, and Marinacci, 2004), which drops uncertainty aversion from maxmin expected utility (Gilboa and Schmeidler, 1989). Extensions of DSEU allow for more general departures from independence.

1   Introduction

1.1   Motivation and Overview

A central approach to modeling preferences under ambiguity is based on the idea that the decision-maker (DM) quantifies uncertainty with a set of relevant beliefs and may use a different belief from this set to evaluate each uncertain prospect. A well-known limitation underlying many such multiple-prior models—notably Gilboa and Schmeidler's (1989)

maxmin expected utility model and several of its generalizations[1]—is a restrictive mechanism of belief selection, whereby the DM evaluates each prospect according to the *worst* possible relevant belief. Behaviorally, this restriction is reflected by Schmeidler's (1989) uncertainty aversion axiom, which captures a negative attitude towards ambiguity through a strong form of preference for hedging.

While consistent with Ellsberg's seminal two-color urn experiment, the uncertainty aversion axiom has been questioned both by subsequent theoretical work, which has proposed alternative formalizations and measures of ambiguity aversion,[2] as well as based on more recent experimental evidence. Indeed, this evidence points to more nuanced patterns of ambiguity attitudes: The same subjects may appear ambiguity-averse in some decision problems, but may also display *ambiguity-seeking* preferences in other notable settings, some of which we discuss below (for a survey, see Trautmann and van de Kuilen, 2015).

In this paper, we propose a decision-theoretic framework that provides a unified lens through which to formalize and contrast such mixed attitudes towards ambiguity and to explore their economic implications. To do so, we introduce a class of multiple-prior representations that allows for a flexible mechanism of belief selection: Instead of assuming that the DM uses the worst possible belief to evaluate any given prospect, our representations adopt a "dual-self" perspective on ambiguity, by modeling the DM's belief selection as the outcome of a game between two conflicting forces, *Pessimism* and *Optimism*.[3]

Our baseline representation generalizes maxmin expected utility by incorporating an ambiguity-seeking force via the addition of a maximization stage: Under *dual-self expected utility (DSEU)*, there is a compact collection $\mathbb{P}$ of closed and convex sets of beliefs and an affine utility u such that the DM evaluates each (Anscombe-Aumann) act f according to

$$W_{\text{DSEU}}(f) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mathbb{E}_\mu[u(f)]. \tag{1}$$

That is, the belief used to evaluate f is the outcome of a sequential zero-sum game: First, Optimism chooses a set of beliefs P from the collection $\mathbb{P}$ with the goal of maximizing the DM's expected utility to f; then Pessimism chooses a belief μ from P with the goal of minimizing expected utility. Maxmin expected utility corresponds to the extreme case in which Optimism has no choice, while the opposite extreme, maxmax expected utility, results

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[1]See, for example, Maccheroni, Marinacci, and Rustichini (2006); Chateauneuf and Faro (2009); Strzalecki (2011); Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011); Skiadas (2013).

[2]See, for instance, Epstein (1999); Ghirardato and Marinacci (2002); Baillon, L'Haridon, and Placido (2011); Dow and Werlang (1992); Baillon, Huang, Selim, and Wakker (2018).

[3]The idea that the DM consists of multiple strategic selves with conflicting motives is employed frequently in behavioral economics, for example to model risk preferences and intertemporal choices (e.g., Thaler and Shefrin, 1981; Fudenberg and Levine, 2006; Brocas and Carrillo, 2008).

when Pessimism has no choice.

Our main results are twofold. First, we provide foundations for the DSEU model. Theorem 1 shows that DSEU represents the class of preferences that satisfy all of Gilboa and Schmeidler's (1989) axioms except for uncertainty aversion; thus, the presence of ambiguity is captured solely by relaxing independence to certainty independence, without additionally restricting the DM's ambiguity attitude to be negative (or positive). Beyond maxmin and maxmax expected utility, this class of preferences—known as *invariant biseparable*—nests Choquet expected utility and α-MEU as notable special cases. Obtaining an easy-to-interpret representation of invariant biseparable preferences has been considered an important question in the ambiguity literature, and Section 1.2 spells out some advantages of DSEU relative to existing representations due to Ghirardato, Maccheroni, and Marinacci (2004) and Amarante (2009). Moreover, Section 4.3 shows that the dual-self approach extends beyond this class, as natural extensions of DSEU represent generalizations of invariant biseparable preferences that further relax certainty independence.

Proposition 1 notes that any DSEU preference $\succsim$ uniquely reveals a set of *relevant priors* $C = \bigcup_{P \in \mathbb{P}} P$, which represents all possible outcomes of the belief-selection game (up to convex closure and elimination of never-selected beliefs). Sections 4.1–4.2 further discuss the uniqueness properties and comparative statics of the DSEU model.

Our second contribution is to exploit the structure of the DSEU model to represent and organize a range of natural intermediate ambiguity attitudes: In line with the aforementioned experimental evidence, these successively relax uncertainty aversion, by accommodating some degree of ambiguity-seeking behavior. The main insight is that, under DSEU, there is a correspondence between the degree of ambiguity aversion of the DM, as captured by the strength of her preference for hedging, and the extent of overlap of sets in $\mathbb{P}$, which measures the relative "power" allocated to Pessimism vs. Optimism in the belief-selection game. Sections 3.1 and 3.2 formalize this as follows:

First, uncertainty aversion, i.e., a preference for *all* hedges, corresponds to the extreme case where the intersection of all sets in $\mathbb{P}$ coincides with C. That is, *all* relevant priors are available to Pessimism regardless of Optimism's action, thus rendering Optimism powerless.

Second, we show that allocating more power to Optimism by only requiring the intersection of all sets in $\mathbb{P}$ to be *nonempty* corresponds to Ghirardato and Marinacci's (2002) notion of *absolute ambiguity aversion*: This only imposes a preference for *complete* hedges, i.e., for hedges that fully eliminate uncertainty.

Third, we further relax absolute ambiguity aversion, motivated in part by the following preference pattern that was originally conjectured by Ellsberg (see Ellsberg, 2011) and subsequently confirmed in laboratory experiments (e.g., Dimmock, Kouwenberg, Mitchell, and

Peijnenburg, 2015; Kocher, Lahno, and Trautmann, 2018):

Example 1 (Ambiguity-seeking for small odds)**.** Consider an urn of unknown composition containing balls of up to 10 possible colors. A ball is drawn from the urn and its color observed. When given the choice between receiving \$10 if the observed color is one of *five* possible colors vs. receiving \$10 with probability 0.5, most subjects prefer the objective lottery, similar to the ambiguity-averse behavior predicted by Ellsberg's two-color urn experiment. By contrast, when the choice is between receiving \$10 if the observed color is a *single* possible color vs. receiving \$10 with probability 0.1, many subjects strictly prefer the former bet. That is, many individuals are simultaneously *ambiguity-averse for moderate-likelihood events* and *ambiguity-seeking for small-likelihood events*.   ▲

To capture this pattern, we introduce the notion of k-*ambiguity aversion* (for some $k = 2, 3, \ldots$), which weakens absolute ambiguity aversion by imposing a preference for complete hedges *only among any k acts*. As we discuss, the evidence in Example 1 is consistent with k-ambiguity aversion for small k (here, $k = 2$) but not for large k (here, $k = 10$). We show that under DSEU, k-ambiguity aversion is equivalent to the intersection of any k sets in $\mathbb{P}$ being nonempty and, as a result, the model can accommodate this preference pattern.

Finally, even 2-ambiguity aversion must be relaxed to accommodate another important behavioral pattern:

Example 2 (Source-dependent ambiguity attitudes)**.** One manifestation of home bias (French and Poterba, 1991; Coval and Moskowitz, 1999) is that many investors are more ambiguity-averse for bets pertaining to the foreign stock market than the domestic stock market; indeed, some investors are found to be ambiguity-averse for foreign investments but ambiguity-seeking for domestic investments (Anantanasuwong, Kouwenberg, Mitchell, and Peijnenberg, 2019, see Section 3.2 for more details). More generally, in many settings, individuals appear *ambiguity-averse with respect to unfamiliar sources* of uncertainty but *ambiguity-seeking with respect to familiar sources* (Heath and Tversky, 1991; Keppe and Weber, 1995).   ▲

To model such source-dependent ambiguity attitudes, we consider the sign of an event-based ambiguity aversion index that is commonly used in experimental work (Baillon and Bleichrodt, 2015; Baillon, Huang, Selim, and Wakker, 2018). We show that under DSEU the sign of this index is characterized by a "local" version of the binary intersection condition underlying 2-ambiguity aversion. As a result, the model can flexibly accommodate source-dependent negative and positive ambiguity attitudes. By contrast, we prove that this phenomenon is incompatible with α-MEU, a special case of DSEU that is often used to capture a mix of negative and positive ambiguity attitudes in applied work.

In Section 3.3, we build on the preceding results and present two applications that highlight some economic implications of intermediate ambiguity attitudes. First, we consider ambiguity sharing among multiple DSEU-maximizing agents. Using the above representation of absolute ambiguity aversion, we unify and generalize existing "minimal agreement" conditions for the inefficiency of parimutuel betting (Billot, Chateauneuf, Gilboa, and Tallon, 2000; Chateauneuf, Dana, and Tallon, 2000). Second, using a parametric specification of DSEU, we solve a single-agent insurance problem and show that further relaxing negative ambiguity attitudes to k-ambiguity aversion yields a potential explanation for the demand for partial insurance.

1.2   Related Literature

This paper contributes to the decision-theoretic literature on preferences under ambiguity (for a survey, see Gilboa and Marinacci, 2016). Our first main result—in particular, the finding that our baseline model, DSEU, represents the class of invariant biseparable preferences—complements Ghirardato, Maccheroni, and Marinacci (2004) (henceforth GMM) and Amarante (2009). Our second contribution of characterizing intermediate ambiguity attitudes relies heavily on the structure of DSEU and has no counterpart in these papers.

GMM introduce the class of invariant biseparable preferences and show that every such preference admits a representation

$$W(f) = \alpha(f) \min_{\mu \in C} \mathbb{E}_\mu[u(f)] + (1 - \alpha(f)) \max_{\mu \in C} \mathbb{E}_\mu[u(f)], \tag{2}$$

where α is a function from acts to $[0, 1]$ and C is the set of relevant priors of $\succsim$ (see Section 2.3). Importantly, as GMM point out, the converse of this result does not hold without further joint restrictions on the model parameters: Specifically, to ensure that the preference $\succsim$ induced by (2) satisfies certainty independence, the weight function α must be measurable with respect to a particular equivalence relation derived from u and C;[4] moreover, α, C, and u must be such that $\succsim$ is monotonic (see Remark 2 in GMM).[5] Similar to (2), DSEU provides a representation of invariant biseparable preferences that generalizes maxmin expected

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[4]That is, we must have $\alpha(f) = \alpha(g)$ whenever $f \asymp g$, where this equivalence relation is defined by

$$f \asymp g \iff \left[\mathbb{E}_\mu[u(f)] \geq \mathbb{E}_\nu[u(f)] \Leftrightarrow \mathbb{E}_\mu[u(g)] \geq \mathbb{E}_\nu[u(g)], \forall \mu, \nu \in C \right].$$

[5]Even when parameters (C, α, u) induce an invariant biseparable preference $\succsim$ via (2), C is not in general the set of relevant priors of $\succsim$, as this entails solving a fixed-point problem. For example, when α is constant and the state space is finite, Eichberger, Grant, Kelsey, and Koshevoy (2011) show that the only cases where C is the set of relevant priors of the induced preference are maxmin ($\alpha = 1$) and maxmax ($\alpha = 0$).

utility by incorporating a force for optimism, in the form of a max operator, into the DM's belief-selection process. However, the invariant biseparable axioms are not only sufficient, but also necessary for a DSEU representation: That is, in contrast with (2), *any* combination of model parameters $(\mathbb{P}, u)$ induces an invariant biseparable preference via (1). This is key in enabling our characterization of intermediate ambiguity attitudes in terms of the structure of $\mathbb{P}$. Our characterization of comparative ambiguity aversion also moves beyond GMM, in that it does not require holding fixed the set of relevant priors (see Section 4.2).

Amarante (2009) shows that the invariant biseparable axioms are both sufficient and necessary for a representation of the form

$$W(f) = \int_{\Delta(S)} \mathbb{E}_{\mu}[u(f)]\, d\nu(\mu), \tag{3}$$

where ν is a Choquet capacity over beliefs $\mu \in \Delta(S)$. This representation suggests the complementary interpretation of a robust Bayesian DM who uses a nonadditive prior over probabilistic models. However, in contrast with our results for DSEU, there are no known characterizations of absolute and comparative ambiguity attitudes in terms of the model parameters in (3): Notably, unlike for Choquet expected utility (Schmeidler, 1989), uncertainty aversion (resp., absolute ambiguity aversion) does not imply convexity (resp., non-emptiness of the core) of ν.

Our characterization of intermediate ambiguity attitudes is also an important difference from other papers that relax uncertainty aversion, including Schmeidler (1989), Klibanoff, Marinacci, and Mukerji's (2005) smooth model, and models of preferences over utility dispersion (e.g., Siniscalchi, 2009; Grant and Polak, 2013): While some of these papers provide representations of absolute ambiguity aversion, none use their models to characterize weaker degrees of ambiguity aversion.

Related to the structure of DSEU, several recent papers employ belief-set or utility-set collections in other contexts. While we maintain the weak order axiom and focus on relaxing independence, Lehrer and Teper (2011), Nascimento and Riella (2011), Nishimura and Ok (2016), Hara, Ok, and Riella (2019), and Aguiar, Hjertstrand, and Serrano (2020) study preferences that violate completeness and/or transitivity.[6] Whereas DSEU is a utility representation, these papers provide generalized unanimity representations à la Bewley (2002) and Dubra, Maccheroni, and Ok (2004), and the resulting proof methods are quite different. In the context of attitudes to randomization under ambiguity, Ke and Zhang (2019) consider

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[6]See also Kopylov (2019) for an extension of maxmin expected utility that relaxes transitivity by allowing the set of priors to depend upon the acts under consideration. Mononen (2020) generalizes the DSEU model (and some of its extensions) by relaxing monotonicity, and shows how to identify subjective probabilities and state-dependent utilities for the resulting representations.

preferences over *lotteries* over acts and propose a representation that adds *minimization* over belief-set collections to maxmin expected utility. When restricted to acts (i.e., degenerate lotteries), their representation is equivalent to Gilboa and Schmeidler (1989).

Beyond the ambiguity literature, representations based on a combination of max and min operators have been used to provide foundations for maxmin values in zero-sum games (Hart, Modica, and Schmeidler, 1994), utility aggregation (Chambers, 2007), and coarse reasoning (Saponara, 2020). These representations can be shown to be strict special cases of DSEU, suggesting that the DSEU model might serve as a unifying framework to capture additional phenomena, beyond the focus on ambiguity attitudes in the current paper.

Finally, Theorem 1 relates to recent results in mathematics on the linearization of positively homogeneous functions: These imply that a functional $I : \mathbb{R}^S \to \mathbb{R}$ admits a so-called "Boolean" representation, where $I(\phi) = \max_{U \in \mathbb{U}} \min_{\ell \in U} \ell \cdot \phi$ for some collection $\mathbb{U}$ of compact, convex subsets of $\mathbb{R}^S$, if and only if I is positively homogeneous, lower semicontinuous, and locally Lipschitz (see the survey by Rubinov and Dzalilov, 2002). We show that under the additional assumption that I is monotonic and constant-additive, $\mathbb{U}$ can be taken to be a belief-set collection. More importantly, our construction only makes use of beliefs μ in the Clarke differential $\partial I(\underline{0})$, which represents the DM's set of relevant priors (see Sections 2.2–2.3). As we discuss, this requires a different proof approach.

2 Dual-self expected utility

2.1 Setup

Let Z be a set of prizes and let $\Delta(Z)$ denote the space of probability measures with finite support over Z. We refer to typical elements $p, q \in \Delta(Z)$ as lotteries. Let S be a finite set of states. An *(Anscombe-Aumann) act* is a mapping $f : S \to \Delta(Z)$. Let $\mathcal{F}$ be the space of all acts, with typical elements f, g, h. For any $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$, define the mixture $\alpha f + (1 - \alpha)g \in \mathcal{F}$ to be the act that in each state $s \in S$ yields lottery $\alpha f(s) + (1 - \alpha)g(s) \in \Delta(Z)$. As usual, we identify each lottery $p \in \Delta(Z)$ with the constant act that yields lottery p in each state $s \in S$.

Let $\Delta(S)$ denote the set of all probability measures over S, which we embed in $\mathbb{R}^S$ and endow with the Euclidean topology. We refer to typical elements $\mu, \nu \in \Delta(S)$ as beliefs. Given any act $f \in \mathcal{F}$ and map $u : \Delta(Z) \to \mathbb{R}$, let $u(f)$ denote the element of $\mathbb{R}^S$ given by $u(f)(s) = u(f(s))$ for all $s \in S$, and let $\mathbb{E}_\mu[u(f)] := \mu \cdot u(f)$.

The DM's *preference* over $\mathcal{F}$ is given by a binary relation $\succsim$ on $\mathcal{F}$. As usual, $\succ$ and $\sim$ denote the asymmetric and symmetric parts of $\succsim$.

2.2 Representation

We now introduce our baseline representation, dual-self expected utility. Let $\mathcal{K}(\Delta(S))$ denote the space of all nonempty closed, convex sets of beliefs, endowed with the Hausdorff topology. A **belief-set collection** is a nonempty compact collection $\mathbb{P} \subseteq \mathcal{K}(\Delta(S))$; that is, each element $P \in \mathbb{P}$ is a nonempty closed, convex set of beliefs.

Definition 1. A **dual-self expected utility (DSEU)** representation of preference $\succsim$ consists of a belief-set collection $\mathbb{P}$ and a nonconstant affine utility $u : \Delta(Z) \to \mathbb{R}$ such that

$$W_{\text{DSEU}}(f) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mathbb{E}_{\mu}[u(f)] \tag{4}$$

represents $\succsim$.[7]

Just as Gilboa and Schmeidler's (1989) maxmin expected utility model, DSEU is a multiple-prior model of ambiguity preferences: The DM has in mind a set of relevant beliefs $\bigcup_{P \in \mathbb{P}} P$, and might use a different belief to evaluate each act. But unlike maxmin expected utility, the belief μ used to evaluate any given act f is not necessarily worst-case among all relevant beliefs. Instead, μ is the outcome of a sequential zero-sum game between two conflicting forces or "selves:" First, self 1 ("Optimism") chooses an action $P \in \mathbb{P}$ with the goal of maximizing expected utility to act f; then self 2 ("Pessimism") chooses an action $\mu \in P$ with the goal of minimizing expected utility to f.

Maxmin expected utility is given by the extreme case where Optimism's action set is trivial (i.e., $\mathbb{P} = \{P\}$ is a singleton), as in this case (4) reduces to $W(f) = \min_{\mu \in P} \mathbb{E}_{\mu}[u(f)]$. Likewise, maxmax expected utility, $W(f) = \max_{\mu \in P} \mathbb{E}_{\mu}[u(f)]$, corresponds to the opposite extreme where Pessimism's action set is always trivial (i.e., $\mathbb{P} = \{\{\mu\} : \mu \in P\}$ is a collection of singletons).

Our first main result is that DSEU represents the class of preferences (known as invariant biseparable) that satisfy all subjective expected utility axioms, except that independence is relaxed to certainty independence:

Axiom 1 (Weak Order). $\succsim$ is complete and transitive.

Axiom 2 (Monotonicity). If $f, g \in \mathcal{F}$ and $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.

Axiom 3 (Nondegeneracy). There exist $f, g \in \mathcal{F}$ such that $f \succ g$.

Axiom 4 (Archimedean). For all $f, g, h \in \mathcal{F}$ with $f \succ g \succ h$, there exist $\alpha, \beta \in (0, 1)$ such that

$$\alpha f + (1 - \alpha)h \succ g \succ \beta f + (1 - \beta)h.$$

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[7]The functional (4) is well-defined since $\mathbb{P}$ is nonempty and compact.

Axiom 5 (Certainty Independence). For all $f, g \in \mathcal{F}$, $p \in \Delta(Z)$, and $\alpha \in (0, 1]$,

$$f \succsim g \Longleftrightarrow \alpha f + (1 - \alpha)p \succsim \alpha g + (1 - \alpha)p.$$

Theorem 1. *Preference $\succsim$ satisfies Axioms 1–5 if and only if $\succsim$ admits a DSEU representation.*

Thus, like maxmin expected utility, DSEU captures the possible *presence* of ambiguity by imposing independence only for mixtures with constant acts (Axiom 5). However, unlike maxmin expected utility, DSEU does not additionally impose uncertainty aversion, which reflects a *negative attitude* toward ambiguity through a preference for hedging (see Axiom 6). Certainty independence is weak enough to allow the model to nest important special cases such as Choquet expected utility and α-MEU.[8] However, Section 4.3 will show that natural generalizations of DSEU represent classes of preferences that further relax certainty independence.

We prove Theorem 1 in Appendix B.1. We first invoke the well-known fact that $\succsim$ satisfies Axioms 1–5 if and only if $\succsim$ can be represented by $I \circ u$ for some nonconstant affine utility u and a functional $I : \mathbb{R}^S \to \mathbb{R}$ that is monotonic, positively homogeneous, and constant-additive (Appendix A.1 defines these terms). For the sufficiency direction of the proof, we then make use of the Clarke differential $\partial I(\underline{0}) \subseteq \Delta(S)$ of I at the constant vector $\underline{0}$ (Clarke, 1990, see Appendix A.2). The key step is to show that the belief-set collection $\mathbb{P}^*$ given by

$$\mathbb{P}^* := \mathrm{cl}\{P_\phi^* : \phi \in \mathbb{R}^S\} \quad \text{with} \quad P_\phi^* := \{\mu \in \partial I(\underline{0}) : \mathbb{E}_\mu[\phi] \geq I(\phi)\} \tag{5}$$

yields a DSEU representation of I, i.e., for all $\phi \in \mathbb{R}^S$,

$$I(\phi) = \max_{P \in \mathbb{P}^*} \min_{\mu \in P} \mathbb{E}_\mu[\phi]. \tag{6}$$

Our proof builds partly on a non-smooth generalization of results in Ovchinnikov (2001).

Remark 1. (i) **General action sets.** The specific form of action sets for Optimism and Pessimism in (4) is without loss of generality. Indeed, $\succsim$ admits a DSEU representation with utility u if and only if there exist *arbitrary* action sets A_1, A_2 and a mapping $\mu : A_1 \times A_2 \to \Delta(S)$ from action profiles to beliefs such that

$$W(f) = \max_{a_1 \in A_1} \min_{a_2 \in A_2} \mathbb{E}_{\mu(a_1, a_2)}[u(f)] \tag{7}$$

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[8]See also Ghirardato, Maccheroni, and Marinacci (2005), who argue why certainty independence is important for achieving a separation of tastes and beliefs.

is well-defined and represents $\succsim$.[9]

(ii) **Min-max form.** While DSEU takes the max-min form, where Optimism moves first, a natural alternative is to consider games where Pessimism is the first mover, as captured by the functional $W(f) = \min_{Q \in \mathbb{Q}} \max_{\mu \in Q} \mathbb{E}_\mu[u(f)]$ for some belief-set collection $\mathbb{Q}$. It can be shown that the latter class of representations is equivalent to DSEU, in the sense that preference $\succsim$ admits a DSEU representation $(\mathbb{P}, u)$ if and only if $\succsim$ admits a representation $(\mathbb{Q}, u)$ of the min-max form for some belief-set collection $\mathbb{Q}$. However, for a given preference $\succsim$, $\mathbb{Q}$ need not coincide with $\mathbb{P}$ in general. See Supplementary Appendix S.2 for details.

(iii) **Single-self interpretation.** In addition to the dual-self interpretation above, DSEU admits a single-self interpretation, whereby the DM optimally selects her own ambiguity preference from a feasible set.[10] Specifically, feasible ambiguity preferences take the maxmin expected utility form $\min_{\mu \in P} \mathbb{E}_\mu[u(f)]$ and depending on f, the DM optimally controls the parameter P, where $\mathbb{P}$ represents the constraints of the subjective optimization. $\blacktriangle$

2.3 Relevant Priors

A natural way to identify the DM's set of **relevant priors** under DSEU is to consider the union $\bigcup_{P \in \mathbb{P}} P$ of all sets in the belief-set collection. This captures all possible outcomes of the belief-selection game between Optimism and Pessimism. To eliminate redundant beliefs that are never selected, we focus on the *smallest* closed, convex set of beliefs that can arise under any DSEU representation. Proposition 1 shows that this set is uniquely identified:

Proposition 1. *Suppose $\succsim$ satisfies Axioms 1–5. There exists a unique closed, convex set $C \subseteq \Delta(S)$ such that*

$$C \subseteq \overline{\mathrm{co}} \bigcup_{P \in \mathbb{P}} P \tag{8}$$

for all DSEU representations $(\mathbb{P}, u)$ of $\succsim$ and such that (8) holds with equality for some $(\mathbb{P}, u)$.

We call a DSEU representation **tight** if (8) holds with equality. To prove Proposition 1 (Appendix B.2), we show that for any DSEU representation, $\overline{\mathrm{co}} \bigcup_{P \in \mathbb{P}} P$ includes the Clarke differential $\partial I(\underline{0})$ at $\underline{0}$ of the functional I from the proof of Theorem 1. Since the representation $\mathbb{P}^*$ in (5) satisfies $\overline{\mathrm{co}} \bigcup_{P \in \mathbb{P}^*} P = \partial I(\underline{0})$, this implies that the set of relevant priors C is precisely $\partial I(\underline{0})$ and that $\mathbb{P}^*$ is a tight representation.

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[9]To see this, suppose $(\mathbb{P}, u)$ is a DSEU representation of $\succsim$. Then (7) represents $\succsim$ with $A_1 := \mathbb{P}$, $A_2 := \prod_{P \in \mathbb{P}} P$, and $\mu(P, \sigma) := \sigma(P)$ for all $P \in A_1$, $\sigma \in A_2$. Conversely, suppose (7) represents $\succsim$ for some (A_1, A_2, μ, u). Then setting $\mathbb{P} := \mathrm{cl}\{\overline{\mathrm{co}}(\mu(a_1, A_2)) : a_1 \in A_1\}$ yields a DSEU representation of $\succsim$.

[10]See Sarver (2018) for an analogous model in the context of risk preferences.

An implication of this Clarke-differential characterization of C is that our definition of the DM's relevant priors as the possible outcomes of the belief-selection game is equivalent to the following behavioral definition due to GMM, which is based on quantifying departures from independence. For any preference $\succsim$ satisfying Axioms 1–5, GMM define the **unambiguous preference** $\succsim^*$ as the largest independent subrelation of $\succsim$; equivalently, $f \succsim^* g$ means that $\alpha f + (1 - \alpha)h \succsim \alpha g + (1 - \alpha)h$ holds for all $\alpha \in (0, 1]$ and $h \in \mathcal{F}$.

Note that $\succsim^*$ is incomplete whenever $\succsim$ violates independence. GMM show that $\succsim^*$ admits a unanimity representation à la Bewley (2002) and identify the unique closed, convex set of priors in the unanimity representation as the DM's relevant set of priors.[11] Since GMM show that the latter set again coincides with $\partial I(\underline{0})$, we obtain the following corollary:

Corollary 1. *If $\succsim$ admits a DSEU representation with utility u, then the set of relevant priors C is the unique closed, convex set such that*

$$f \succsim^* g \iff \mathbb{E}_\mu[u(f)] \geq \mathbb{E}_\mu[u(g)] \text{ for all } \mu \in C. \tag{9}$$

3 Intermediate ambiguity attitudes

In this section, we exploit the structure of the DSEU model to represent and organize a range of intermediate attitudes toward ambiguity. Our results can capture various ways in which a DM might display a mix of ambiguity-averse and ambiguity-seeking tendencies. We discuss the relevance of these results both for accommodating experimental evidence and for generating new economic predictions.

3.1 Shades of ambiguity aversion

We first show how DSEU can represent a range of different shades of ambiguity aversion that vary in the degree to which they impose a preference for hedging. First, Schmeidler's (1989) seminal uncertainty aversion axiom postulates that the DM *always* takes up an opportunity to hedge between two equally valued prospects:

Axiom 6 (Uncertainty Aversion)**.** If $f, g \in \mathcal{F}$ with $f \sim g$, then $\frac{1}{2}f + \frac{1}{2}g \succsim f$.

A second common definition of ambiguity aversion is due to Ghirardato and Marinacci (2002): Recall the standard comparative notion of ambiguity aversion, whereby $\succsim_1$ is **more**

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[11]Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) take an alternative approach by including $\succsim^*$ as part of the primitive. Ghirardato and Siniscalchi (2012) extend GMM's characterization of relevant priors beyond the invariant biseparable class. See Klibanoff, Mukerji, and Seo (2014) for a discussion of the interpretation of C.

ambiguity-averse than $\succsim_2$ if, whenever $f \succsim_1 p$ for some $f \in \mathcal{F}$ and $p \in \Delta(Z)$, then $f \succsim_2 p$. Analogous to the definition of absolutely risk-averse as more risk-averse than a risk-neutral preference, $\succsim$ is said to be ***absolutely ambiguity-averse*** if it is more ambiguity-averse than some nondegenerate subjective expected utility preference.[12]

We contrast these two formalizations of ambiguity aversion with the following third notion:

Axiom 7 (k-Ambiguity Aversion)**.** For all $f_1, ..., f_k \in \mathcal{F}$ with $f_1 \sim f_2 \sim \cdots \sim f_k$ and any $p \in \Delta(Z)$,

$$\frac{1}{k}f_1 + \cdots + \frac{1}{k}f_k = p \implies p \succsim f_1.$$

Axiom 7 only imposes a preference for *complete* hedging between k equally valued prospects, that is, for hedges that eliminate subjective uncertainty entirely. To see the relationship with absolute ambiguity aversion, say that $\succsim$ satisfies ∞-***ambiguity aversion*** if it satisfies k-ambiguity aversion for all k. This corresponds to the notion of preference for sure diversification used by Chateauneuf and Tallon (2002) to characterize absolute ambiguity aversion under Choquet expected utility. Arguments in Grant and Polak (2013) imply that this characterization extends to DSEU; moreover, we note that $|S|$-ambiguity aversion is sufficient for ∞-ambiguity aversion (where $|S|$ is the cardinality of the state space):

Lemma 1. *Suppose $\succsim$ admits a DSEU representation. The following are equivalent:* (i) *$\succsim$ is absolutely ambiguity-averse;* (ii) *$\succsim$ satisfies ∞-ambiguity aversion;* (iii) *$\succsim$ satisfies $|S|$-ambiguity aversion.*

The following result shows that under DSEU, the above notions of ambiguity aversion are characterized by the degree of overlap of sets in $\mathbb{P}$:

Theorem 2. *Suppose that $\succsim$ admits a DSEU representation $(\mathbb{P}, u)$. Then:*

1. *$\succsim$ satisfies uncertainty aversion if and only if $\bigcap\limits_{P \in \mathbb{P}} P = C$;*

2. *$\succsim$ is absolutely ambiguity-averse if and only if $\bigcap\limits_{P \in \mathbb{P}} P \neq \emptyset$;*

3. *$\succsim$ satisfies k-ambiguity aversion if and only if $\bigcap\limits_{i=1,...,k} P_i \neq \emptyset$ for all $P_1, ..., P_k \in \mathbb{P}$.*

Conditions 1–3 capture a natural sense in which these notions of ambiguity aversion allocate successively less relative "power" to Pessimism in the belief-selection game represented

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[12]See Epstein (1999) for another approach that takes as its benchmark probabilistic sophistication instead of subjective expected utility.

by $(\mathbb{P}, u)$, where "power" is measured by the extent of overlap of sets in $\mathbb{P}$. The first two conditions consider the intersection $\bigcap_{P \in \mathbb{P}} P$ of all sets in $\mathbb{P}$, which we note is uniquely identified from the preference $\succsim$.

Specifically, uncertainty aversion corresponds to the maximal allocation of power to Pessimism: Since $\bigcap_{P \in \mathbb{P}} P = C$, *all* relevant priors $\mu \in C$ are available to Pessimism, no matter which set $P \in \mathbb{P}$ Optimism chooses. The game thus boils down to Pessimism choosing a belief $\mu \in C$, yielding maxmin expected utility; indeed, note that if $(\mathbb{P}, u)$ is tight, then $\succsim$ satisfies uncertainty aversion if and only if $\mathbb{P} = \{C\}$.

Absolute ambiguity aversion allocates less power to Pessimism, requiring only that there is *some* prior $\mu \in \bigcap_{P \in \mathbb{P}} P$ that is always available to Pessimism regardless of Optimism's choice. As a result, the DM's valuation of any act f is bounded above by the expected utility $\mathbb{E}_\mu[u(f)]$ of f under prior μ. In the special case when $\succsim$ admits a Choquet expected utility representation with capacity ν, we note that $\bigcap_{P \in \mathbb{P}} P$ coincides with the core of ν; thus, our nonempty intersection condition generalizes the fact that under Choquet expected utility, absolute ambiguity aversion is characterized by the nonemptiness of the core of ν. In Section 3.3.1, we obtain a condition for ambiguity sharing in markets that extends this nonempty intersection condition to multiple agents.

Finally, while absolute ambiguity aversion requires the intersection of *all* sets in $\mathbb{P}$ to be nonempty, k-ambiguity aversion imposes this only for any k sets in $\mathbb{P}$. In other words, for any k actions $P_1, \cdots, P_k$ available to Optimism, there is at least one prior $\mu \in \bigcap_{i=1,\cdots,k} P_i$ that Pessimism can choose in response to all of these actions. Thus, k-ambiguity aversion further decreases the power allocated to Pessimism, and more so the smaller k. As highlighted in the introduction, one motivation for relaxing the DM's negative ambiguity attitude in this manner is experimental evidence on *ambiguity-seeking for small odds*:

Example 3 (Ambiguity-seeking for small odds—continued)**.** To formalize the relationship with Example 1, let the state space $S = \{1, \ldots, 10\}$ represent the observed color, let f_E denote the uncertain bet that pays \$10 if the observed color belongs to $E \subseteq S$ and \$0 otherwise, and let p_α denote the objective lottery that pays \$10 with probability α and \$0 otherwise. When the cardinality of E is 5, subjects' reported preference for the objective lottery $p_{0.5}$ over the uncertain bet f_E is consistent with 2-ambiguity aversion. However, assume that, by symmetry, subjects are indifferent between betting on any single color, i.e., $f_{\{1\}} \sim \ldots \sim f_{\{10\}}$. Then the commonly reported strict preference for any $f_{\{s\}}$ over $p_{0.1}$ is inconsistent with 10-ambiguity aversion, as $p_{0.1} = \frac{1}{10} f_{\{1\}} + \cdots \frac{1}{10} f_{\{10\}}$. ▲

Based on Theorem 2, we can show that DSEU allows for flexible degrees of k-ambiguity aversion, and hence can accommodate the evidence in Example 1. This contrasts, for in-

stance, with Siniscalchi's (2009) vector expected utility model, which also relaxes uncertainty aversion, but for which 2-ambiguity aversion and ∞-ambiguity aversion are equivalent.[13] As a simple example, consider the following parametrization of DSEU: Fix any $\pi \in \Delta(S)$ and $\varepsilon \in \mathbb{R}$ with $0 \leq \pi_s - \varepsilon \leq 1$ for all s. Then consider the representation $(\mathbb{P}, u)$ of the form $\mathbb{P} = \{P_s : s \in S\}$, where for each s,

$$P_s := \{\mu \in \Delta(S) : \mu(s) \geq \pi_s - \varepsilon\}. \tag{10}$$

For each $k \leq |S|$, Theorem 2 implies that k-ambiguity aversion is satisfied if and only if $\varepsilon \geq (\pi_{s_1} + \cdots + \pi_{s_k} - 1)/k$ for all distinct $s_1, \cdots, s_k \in S$.[14] In particular, the maximal degree of k-ambiguity aversion of the induced preference is non-decreasing in ε, and the preference is absolutely ambiguity-averse if and only if $\varepsilon \geq 0$. Moving beyond urn experiments, Section 3.3.2 will use this parametrization to highlight some implications of k-ambiguity aversion for optimal insurance coverage.

Remark 2. One can also exploit the structure of DSEU to represent other natural relaxations of uncertainty aversion. For example, the special case of DSEU where the belief-set collection $\mathbb{P}$ is finite is characterized by a weak form of uncertainty aversion, which imposes a preference for hedging only among acts f and g whose payoffs in all states are close enough. See Theorem 1 in the working paper version of Chandrasekher (2019). ▲

3.2 Source-dependent ambiguity attitudes

While the preceding notions of ambiguity aversion are "global," capturing the DM's attitude towards *any* uncertainty that can be generated in S, the experimental literature commonly takes a "local" approach, measuring the DM's ambiguity attitude relative to specific events or sources of uncertainty. As discussed in the introduction, an important finding is that a DM might display *source-dependent negative and positive ambiguity attitudes*, depending on whether she considers herself familiar or unfamiliar with a given source of uncertainty.

To formalize this idea, we use a local index of ambiguity attitude that was originally proposed by Schmeidler (1989) and subsequently employed in both theoretical work (Dow

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[13]Note that 2-ambiguity aversion is equivalent to Siniscalchi's (2009) Axiom 11, which he shows is equivalent to absolute ambiguity aversion (provided utilities are unbounded). Dillenberger and Segal (2017) show that a version of Segal's (1987) model is consistent with ambiguity-seeking for small odds.

[14]Indeed, suppose $\varepsilon \geq (\pi_{s_1} + \cdots + \pi_{s_k} - 1)/k$ for all distinct $s_1, \cdots, s_k$. Then for any $s_1, \cdots, s_k$, consider the belief μ such that $\mu(s_i) = \pi_{s_i} - \varepsilon$ for all $i = 1, \cdots, k$, and the remaining weight $1 - k\varepsilon - \pi_{s_1} - \cdots - \pi_{s_k}$ (which is nonnegative) is allocated uniformly across the other states. We have $\mu \in \cap_{i=1}^k P_{s_i}$. Thus, k-ambiguity aversion holds. Conversely, suppose $\varepsilon < (\pi_{s_1} + \cdots + \pi_{s_k} - 1)/k$ for some $s_1, \cdots, s_k$. Then any $\mu \in \bigcap_{i=1}^k P_{s_i}$ would satisfy $\mu(s_1) + \cdots + \mu(s_k) \geq \pi_{s_1} + \cdots + \pi_{s_k} - k\varepsilon > 1$, which is impossible. Thus, $\bigcap_{i=1}^k P_{s_i} = \emptyset$.

and Werlang, 1992) and experiments. One advantage of this index is that it can be defined for any event, without imposing symmetry on the state space as is common in urn experiments.

Definition 2. The ***matching probability*** $m(E) \in [0, 1]$ of an event E is defined by the indifference condition

$$x_E y \sim m(E)\delta_x + (1 - m(E))\delta_y,$$

where $x, y \in Z$ are two outcomes such that $\delta_x \succ \delta_y$ and $x_E y$ denotes the binary act that yields x for all $s \in E$ and y otherwise.[15] The ***ambiguity aversion index*** of E is

$$AA(E) := 1 - m(E) - m(E^c). \tag{11}$$

Whereas subjective expected utility implies $AA(E) = 0$ for all E, $AA(E) > 0$ (resp. $AA(E) < 0$) is interpreted as a negative (resp. positive) attitude to ambiguity associated with E. In particular, the aforementioned evidence suggests that a DM might display $AA(E) > 0$ when E is conditioned on an unfamiliar source of uncertainty, but might display $AA(E) < 0$ when she feels particularly competent about the relevant source:

Example 4 (Source-dependent ambiguity attitudes—continued). As a stylized formalization of Example 2, let $S_H = \{U, D\}$ be a state space specifying whether the domestic stock market goes up ("U") or down ("D"). Similarly, let $S_F = \{U, D\}$ describe the state of the stock market in a foreign country. Consider the product state space $S = S_H \times S_F$, and let $E_H = \{UU, UD\}$ be the event that the domestic stock market goes up, and $E_F = \{UU, DU\}$ be the corresponding event for the foreign stock market. The evidence in Anantanasuwong, Kouwenberg, Mitchell, and Peijnenberg (2019) suggests that some investors display $AA(E_F) > 0 > AA(E_H)$.[16] Similarly, in an experiment involving German subjects, Keppe and Weber (1995) find that the average ambiguity index is positive for bets concerning US geography, but negative for bets concerning German geography. ▲

To see how DSEU can capture this evidence, we note that the sign of $AA(E)$ is characterized by the following local analog of the binary intersection condition for 2-ambiguity aversion in Theorem 2. Given an event E and set of beliefs P, let $P_E := \{\mu(E) : \mu \in P\}$.

Proposition 2. *Suppose $\succsim$ admits a DSEU representation $(\mathbb{P}, u)$, and let E be any subset of S. Then:*

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[15]Under Axioms 1–5, $m(\cdot)$ is well-defined independent of the choice of x, y.

[16]Anantanasuwong, Kouwenberg, Mitchell, and Peijnenberg (2019) conduct an incentivized field survey among investors and find reversals in ambiguity attitudes as in Example 4, where H and F correspond to a domestic and foreign stock market index (see Figures 4 and 5). They also find a higher population average AA index for E_F than E_H, but the difference is relatively small, as some investors display the opposite reversal (which can also be accommodated by DSEU).

1. $AA(E) \geq 0 \iff P_E \cap P'_E \neq \emptyset$ *for all* $P, P' \in \mathbb{P}$;

2. $AA(E) > 0 \iff P_E \cap P'_E$ *is a non-degenerate interval for all* $P, P' \in \mathbb{P}$.

Thus, while 2-ambiguity aversion implies that $AA(E) \geq 0$ for all events E, further limiting the overlap of sets in $\mathbb{P}$ can accommodate the behavior in Example 4. Indeed, the following result shows that DSEU can capture source-dependent negative and positive ambiguity attitudes with respect to *any* families $\mathcal{E}$ and $\mathcal{F}$ of unfamiliar and familiar events:[17]

Corollary 2. *Fix any disjoint collections $\mathcal{E}$ and $\mathcal{F}$ of events, both of which are closed under complements and do not contain S. There exists a DSEU representation $(\mathbb{P}, u)$ whose induced preference satisfies $AA(E) > 0 > AA(F)$ for all $E \in \mathcal{E}, F \in \mathcal{F}$.*

Corollary 2 contrasts with a prominent special case of DSEU, the α-MEU model. This is given by the functional

$$W(f) = \alpha \min_{\mu \in P} \mathbb{E}_\mu[u(f)] + (1 - \alpha) \max_{\mu \in P} \mathbb{E}_\mu[u(f)] \tag{12}$$

for some $\alpha \in [0, 1]$, nonempty closed, convex set of beliefs P, and nonconstant affine u. While the α-MEU model is widely used in applied work to capture a mix of negative and positive ambiguity attitudes, the following result shows that it is incompatible with the source-dependent variation in ambiguity attitudes formalized above. Indeed, observe that (12) coincides with the DSEU representation $(\mathbb{P}, u)$ where $\mathbb{P} = \{\alpha P + (1 - \alpha)\{\mu\} : \mu \in P\}$. Then Proposition 2 implies that the sign of the ambiguity aversion index is the *same* for all events and is determined by the value of α:

Corollary 3. *Suppose $\succsim$ admits an α-MEU representation where P is not a singleton. Then $\alpha \geq 1/2$ (resp. $\alpha \leq 1/2$) if and only if $AA(E) \geq 0$ (resp. $AA(E) \leq 0$) for all E.*

Finally, these insights extend to another common formalization of source dependence (along the lines of experimental work by, e.g., Tversky and Fox, 1995; Heath and Tversky, 1991) that does not involve matching probabilities. As before, fix two outcomes $x, y \in Z$ such that $\delta_x \succ \delta_y$. Consider the preference pattern

$$x_E y \succ x_F y \succ x_G y \quad \text{and} \quad x_{E^c} y \succ x_{F^c} y \succ x_{G^c} y, \tag{13}$$

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[17]Several papers (e.g., Nau, 2006; Chew and Sagi, 2008; Ergin and Gul, 2009; Gul and Pesendorfer, 2015; Cappelli, Cerreia-Vioglio, Maccheroni, Marinacci, and Minardi, 2016) propose formalizations of source dependence based on the idea that the DM is probabilistically sophisticated over prospects that depend on a single common source, but exhibits varying attitudes toward uncertainty across sources. Corollary 2 considers a specific variation where the DM exhibits negative vs. positive attitudes depending on her familiarity with each source. See also Abdellaoui, Baillon, Placido, and Wakker (2011); Chew, Miao, and Zhong (2018) for experimental work using different notions of source dependence.

where event F is **unambiguous**, in the sense that $f \sim x_F y \Rightarrow \lambda f + (1 - \lambda) x_F y \sim x_F y$ for all $\lambda \in (0, 1)$.[18] In (13), the DM's preference to bet on both F vs. G and F^c vs. G^c captures a negative attitude towards the uncertainty underlying event G. At the same time, the preference for betting on E vs. F and E^c vs. F^c reflects a positive attitude towards the uncertainty underlying event E. It is easy to see that (13) implies $AA(E) < AA(F) = 0 < AA(G)$. Thus, it is immediate from Corollary 3 that this form of source dependence is also inconsistent with α-MEU, while it is again compatible with the general DSEU model:

Corollary 4. *Suppose $\succsim$ admits an α-MEU representation. Then there do not exist events E, F, G, where F is unambiguous, such that (13) is satisfied.*

Supplementary Appendix S.3 derives a similar incompatibility result to Corollary 4 for Klibanoff, Marinacci, and Mukerji's (2005) smooth model.

3.3   Applications

We conclude this section with two applications that highlight some economic implications of intermediate ambiguity attitudes, building on the preceding characterizations of such attitudes.

3.3.1   Ambiguity sharing

First, we consider ambiguity sharing among multiple DSEU-maximizing agents. There is a finite set $\mathcal{I}$ of agents and a single good (e.g., money). Each agent i's preference over state-dependent consumption bundles $x_i \in \mathbb{R}_+^S$ admits a DSEU representation $(\mathbb{P}_i, u_i)$, where all beliefs in $\mathbb{P}_i$ have full support and the consumption utility $u_i : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing and strictly concave. Each i's initial endowment is $\omega_i \in \mathbb{R}_+^S$, and the economy features no aggregate uncertainty, i.e., for some $w > 0$, $\sum_i \omega_i(s) = w$ for all s. A **feasible allocation** is a profile $x = (x_i)_{i \in \mathcal{I}} \in \mathbb{R}_+^{S \times \mathcal{I}}$ such that $\sum_i x_i(s) = w$ for each s. An allocation is **full insurance** if each x_i is constant across states s.

A classic literature asks when it is Pareto efficient for agents to bet against each other (i.e., to engage in trade that does not yield a full insurance allocation). Under subjective expected utility, betting is efficient if and only if there is some disagreement among the agents, meaning that they don't all share the same prior belief. This stark result raises the question why we don't see more parimutuel betting in everyday life, where opportunities for disagreement are widespread. The literature has proposed ambiguity aversion as a natural explanation

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[18]Under DSEU, this is equivalent to the condition that $x_F y$ is **crisp** as defined by GMM, which is in turn equivalent to requiring $\mu(F)$ to be constant across all beliefs $\mu \in C$ (Proposition 10 in GMM).

for the avoidance of such bets. Indeed, under maxmin expected utility, Billot, Chateauneuf, Gilboa, and Tallon (2000) show that all efficient allocations are full insurance as long as the intersection of all agents' sets of beliefs is nonempty; that is, if agents are uncertainty-averse, some *minimal agreement*, as opposed to full agreement, is sufficient to guarantee the optimality of full insurance. Chateauneuf, Dana, and Tallon (2000) prove a similar result for Choquet expected utility maximizers under the condition that the intersection of the cores of agents' capacities is nonempty.

Using the structure of DSEU, we can unify and generalize these two results by obtaining a minimal agreement condition for the case of *absolutely ambiguity-averse* agents.[19] By Theorem 2, agents are absolutely ambiguity-averse if and only if the sets $\bigcap_{P_i \in \mathbb{P}_i} P_i$ are nonempty for all i. The minimal agreement condition we propose requires the intersection of these sets across all agents to be nonempty. Under this condition, all efficient allocations are again full insurance; moreover, full insurance can be achieved through a market mechanism (despite the fact that preferences are not assumed convex):

Proposition 3. *Assume that $\bigcap_{i \in \mathcal{I}, P_i \in \mathbb{P}_i} P_i$ is nonempty. Then:*

1. *Any feasible allocation x is Pareto efficient if and only if x is full insurance.*

2. *There exists a competitive equilibrium that features full insurance.*

Under maxmin expected utility, Billot, Chateauneuf, Gilboa, and Tallon (2000) also prove a converse of Proposition 3; that is, if $\bigcap_{i \in \mathcal{I}} P_i$ is empty, then full insurance is strictly dominated by betting. However, the latter result does not hold in general for invariant biseparable preferences: For instance, Billot, Chateauneuf, Gilboa, and Tallon (2002) provide a counterexample under Choquet expected utility (see their Example 2).

3.3.2 Insurance choice

Second, we apply the DSEU model to a single-agent insurance problem and illustrate that further relaxing negative ambiguity attitudes to k-ambiguity aversion can provide a rationale for the demand for partial insurance. A DM with initial wealth w faces a potential loss. The size of the loss is uncertain and equals l_s in state s, with $l_1 = 0 < l_2 < \cdots < l_S < w$.

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[19]Proposition 3 generalizes Chateauneuf, Dana, and Tallon (2000), because, as noted following Theorem 2, if $\succsim_i$ admits a Choquet representation with capacity ν_i, then $\text{core}(\nu_i) = \bigcap_{P_i \in \mathbb{P}_i} P_i$. Rigotti, Shannon, and Strzalecki (2008) and Ghirardato and Siniscalchi (2018) generalize Billot, Chateauneuf, Gilboa, and Tallon (2000) to an abstract class of preferences, using notions of "subjective beliefs" to express minimal agreement. Proposition 3 does not follow from their results, as their assumptions—strictly convex preferences in Rigotti, Shannon, and Strzalecki (2008), differential quasiconcavity at certainty (DQC) in Ghirardato and Siniscalchi (2018)—imply that agents are maxmin expected utility maximizers in our setting. Note that DQC is required for Ghirardato and Siniscalchi's (2018) result to apply under any strictly increasing and strictly concave u.

An insurance policy is characterized by a premium p and a claim schedule $c = (c_1, \cdots, c_S)$ with $0 \leq c_s \leq l_s$ for all s. Let $\mathcal{C}$ denote the set of claim schedules. We assume that the premium associated with c is $p(c) := \mathbb{E}_\pi[c]$, where $\pi \in \Delta(S)$ has full support. A possible foundation for this assumption is that insurers are ambiguity-neutral and risk-neutral and attach probability π_s to each state s, in which case $p(c)$ is the actuarially fair premium. We also assume that $\pi_1 \geq \pi_s$ for all s, capturing that the likelihood of positive losses is small.

The DM is a DSEU maximizer who selects among insurance policies by solving

$$\max_{c \in \mathcal{C}, P \in \mathbb{P}} \min_{\mu \in P} \mathbb{E}_\mu \left[u(w - p(c) - l + c) \right], \tag{14}$$

for some increasing, continuously differentiable, and strictly concave utility u over money. As is well-known, under the subjective expected utility preference $\succsim_\pi$ with belief π, the DM chooses to fully insure, i.e., $c_s = l_s$ for all s. As a result, full insurance is also optimal for any absolutely ambiguity-averse DM who is more ambiguity-averse than $\succsim_\pi$.

The following proposition considers instead the parametrization of DSEU in (10), where for some $\varepsilon \in \mathbb{R}$ with $0 \leq \pi_s - \varepsilon \leq 1$ for all s, the belief-set collection $\mathbb{P}$ consists of the sets $P_s := \{\mu \in \Delta(S) : \mu(s) \geq \pi_s - \varepsilon\}$ for all s. We show that (14) continues to admit a tractable solution. Notably, full insurance is optimal if and *only if* the DM is absolutely ambiguity-averse (i.e., $\varepsilon \geq 0$): Any DM who is not absolutely ambiguity-averse (i.e., whose degree of k-ambiguity aversion is less than $|S|$) retains a positive share of the risk.

Proposition 4. *The optimal schedule is unique and has a straight deductible: There exists a unique $d \geq 0$ such that $c_s = \max\{l_s - d, 0\}$ for all s. The optimal deductible d is nonincreasing in ε (and hence in the degree of k-ambiguity aversion). Moreover, $d = 0$ (full insurance) is optimal if and only if $\varepsilon \geq 0$.*

The optimality of a positive straight deductible is a common finding in standard models of insurance. However, under expected utility, this finding requires additional forces, such as transaction costs (Arrow, 1971) or moral hazard (Hölmstrom, 1979), as otherwise full insurance is optimal.[20] Proposition 4 highlights that, under DSEU, intermediate ambiguity attitudes provide another explanation for the optimality of partial insurance: As long as the DM is not absolutely ambiguity-averse, she might prefer to take on some risk even if premia are actuarially fair. The optimal schedule balances the ambiguity-averse and ambiguity-seeking tendencies of the DM, by insuring her against large losses but leaving her exposed to small losses.

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[20]For instance, in the presence of transaction costs, a positive deductible achieves the best compromise between risk aversion and the willingness to reduce the deadweight loss from insurance. Alary, Gollier, and Treich (2013) extend this result to the smooth ambiguity model, under additional assumptions on the structure of ambiguity. In their model, the optimal deductible is again zero absent transaction costs.

4 Discussion and extensions

In this section, we briefly discuss the uniqueness properties and comparative statics of DSEU representations. We also show how relaxing certainty independence leads to natural generalizations of DSEU.

4.1 Uniqueness

While our results in the preceding sections apply to *all* DSEU representations $(\mathbb{P}, u)$ of a given preference $\succsim$, we briefly comment on the uniqueness properties of these representations. As shown in Proposition 1, $\succsim$ uniquely identifies the DM's set of relevant priors C. At the same time, the DM's belief-set collection $\mathbb{P}$ is not in general unique, analogous to other representations involving belief-set or utility-set collections.[21]

However, the following result shows how any two DSEU representations $(\mathbb{P}, u)$ and $(\mathbb{P}', u')$ of the same preference are related: The utilities must coincide up to some positive affine transformation (denoted $u \approx u'$), and the belief-set collections must coincide up to replacing all sets of beliefs in $\mathbb{P}$ and $\mathbb{P}'$ with the closed half-spaces that contain them. Formally, given any belief-set collection $\mathbb{P}$, define its ***half-space closure*** by

$$\overline{\mathbb{P}} := \mathrm{cl}\{H \subseteq \Delta(S) : H \text{ is a closed half-space in } \Delta(S) \text{ and } H \supseteq P \text{ for some } P \in \mathbb{P}\},$$

where we call H a closed half-space in $\Delta(S)$ if $H = H_{\phi, \lambda} := \{\mu \in \Delta(S) : \mu \cdot \phi \geq \lambda\}$ for some $\phi \in \mathbb{R}^S$ and $\lambda \in \mathbb{R}$.

Proposition 5. *Suppose $(\mathbb{P}, u)$ is a DSEU representation of $\succsim$. Then $(\overline{\mathbb{P}}, u)$ is also a DSEU representation of $\succsim$. Moreover, for any belief-set collection $\mathbb{P}'$ and utility u', $(\mathbb{P}', u')$ is a DSEU representation of $\succsim$ if and only if $\overline{\mathbb{P}} = \overline{\mathbb{P}'}$ and $u \approx u'$.*

The uniqueness of u up to positive affine transformation is standard. The uniqueness of $\mathbb{P}$ up to half-space closure parallels the identification result in Hara, Ok, and Riella (2019), who represent independent (but possibly incomplete and intransitive) preferences over lotteries using a collection of utility-sets. Analogous to Hara, Ok, and Riella (2019), the idea is that for any $P \in \mathbb{P}$, the closed half-spaces containing P capture all information about P that is relevant to the representation. Indeed, in determining how any given utility act $\phi \in \mathbb{R}^S$ is evaluated by the representation, the only relevant feature of P is the worst-case expectation

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[21]One might conjecture that $\succsim$ admits a unique representation $\tilde{\mathbb{P}}$ that is *minimal*, in the sense that it features no redundant actions for Optimism or Pessimism (formally, $\tilde{\mathbb{P}}$ is minimal if there is no alternative representation $\mathbb{P} \neq \tilde{\mathbb{P}}$ with either (i) $\mathbb{P} \subseteq \tilde{\mathbb{P}}$ or (ii) $\forall \tilde{P} \in \tilde{\mathbb{P}}, \exists P \in \mathbb{P}$ with $P \subseteq \tilde{P}$). However, this conjecture is not valid, as some preferences admit multiple minimal representations.

$\lambda_{P,\phi} := \min_{\mu \in P} \mathbb{E}_\mu[\phi]$, and this worst-case expectation is shared by the closed half-space $H_{\phi,\lambda_{P,\phi}} \supseteq P$. Thus, replacing each set P in $\mathbb{P}$ with the closed half-spaces $H_{\phi,\lambda_{P,\phi}}$ for all $\phi \in \mathbb{R}^S$ yields an alternative DSEU representation of $\succsim$. In Appendix D.1, we express the half-space closure $\overline{\mathbb{P}}$ as a function of the utility act functional I associated with $\succsim$ (see the proof of Theorem 1). Because I is uniquely determined by $\succsim$, this shows that the half-space closures of all representations of $\succsim$ must coincide.

4.2 Comparative ambiguity attitudes

Next, building on Proposition 5, we provide a representation under DSEU of the standard comparative notion of ambiguity aversion defined in Section 3.1:

Proposition 6. *Suppose $\succsim_1$ and $\succsim_2$ admit DSEU representations $(\mathbb{P}_1, u_1)$ and $(\mathbb{P}_2, u_2)$, respectively. The following are equivalent:*

1. *$\succsim_1$ is more ambiguity-averse than $\succsim_2$.*

2. *$u_1 \approx u_2$ and $\overline{\mathbb{P}}_1 \subseteq \overline{\mathbb{P}}_2$.*

To interpret, note that $\overline{\mathbb{P}}_1 \subseteq \overline{\mathbb{P}}_2$ means that Optimism's action set, and hence Optimism's ability to influence the DM's belief, is more limited under representation $\overline{\mathbb{P}}_1$ than under $\overline{\mathbb{P}}_2$. Thus, more ambiguity aversion corresponds (up to taking half-space closures) to DSEU representations that allocate less relative "power" to Optimism. This comparative notion of power is consistent with the absolute measures in Section 3.1, because $\overline{\mathbb{P}}_1 \subseteq \overline{\mathbb{P}}_2$ implies that $\bigcap_{P_1 \in \mathbb{P}_1} P_1 \supseteq \bigcap_{P_2 \in \mathbb{P}_2} P_2$ and that $\succsim_1$ displays a weakly higher degree of k-ambiguity aversion than $\succsim_2$.

In contrast with GMM's characterization of comparative ambiguity aversion, which only applies when the sets of relevant priors C_1 and C_2 associated with $\succsim_1$ and $\succsim_2$ are equal (Proposition 12 in GMM), Proposition 6 does not assume any relationship between C_1 and C_2. Indeed, there are natural cases in which one invariant biseparable preference is more ambiguity-averse than another, despite the fact that their sets of priors do not coincide (nor are nested). For example, as long as $C_1 \cap C_2 \neq \emptyset$, then for any u, the maxmin expected utility preference $\succsim_1$ induced by C_1 is more ambiguity-averse than the maxmax expected utility preference $\succsim_2$ induced by C_2, and the subjective expected utility preference corresponding to any $\mu \in C_1 \cap C_2$ is less ambiguity-averse than $\succsim_1$ but more ambiguity-averse than $\succsim_2$.

4.3 Generalizations

As we have seen, our baseline model, DSEU, corresponds to a relaxation of subjective expected utility where independence is weakened to certainty independence and, equivalently,

to dropping uncertainty aversion from Gilboa and Schmeidler's (1989) axioms. The representation adds a maximization stage into Gilboa and Schmeidler's (1989) model, suggesting an interpretation in terms of a game between Optimism and Pessimism.

We highlight that this dual-self approach extends beyond certainty independence, yielding intuitive representations that further relax independence but still allow for a flexible mix of negative and positive ambiguity attitudes. To illustrate, consider the following two common relaxations of certainty independence. First, Maccheroni, Marinacci, and Rustichini's (2006) (henceforth MMR's) "variational" preferences generalize Gilboa and Schmeidler (1989) by replacing certainty independence with weak certainty independence. This axiom retains the "location invariance" property implied by certainty independence but relaxes the "scale invariance" property; we refer the reader to MMR for a detailed discussion:

Axiom 8 (Weak Certainty Independence). For any $f, g \in \mathcal{F}$, $p, q \in \Delta(Z)$, and $\alpha \in (0, 1)$,

$$\alpha f + (1 - \alpha)p \succsim \alpha g + (1 - \alpha)p \implies \alpha f + (1 - \alpha)q \succsim \alpha g + (1 - \alpha)q.$$

Second, Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio's (2011) (henceforth CMMM's) model of "uncertainty-averse" preferences imposes an even weaker form of independence that only holds for objective lotteries:

Axiom 9 (Risk Independence). For any $p, q, r \in \Delta(Z)$ and $\alpha \in (0, 1)$,

$$p \succsim q \implies \alpha p + (1 - \alpha)r \succsim \alpha q + (1 - \alpha)r.$$

While MMR and CMMM maintain uncertainty aversion, the following two results show that dropping uncertainty aversion from their axioms yields dual-self representations that extend DSEU to more general games between Optimism and Pessimism:

Theorem 3. *Preference $\succsim$ satisfies Axioms 1–4 and Axiom 8 if and only if $\succsim$ admits a **dual-self variational** representation; that is, there exists a nonconstant affine utility $u : \Delta(Z) \to \mathbb{R}$ and a collection $\mathbb{C}$ of convex cost functions $c : \Delta(S) \to \mathbb{R} \cup \{\infty\}$ with $\max_{c \in \mathbb{C}} \min_{\mu \in \Delta(S)} c(\mu) = 0$ such that*

$$W(f) := \max_{c \in \mathbb{C}} \min_{\mu \in \Delta(S)} \mathbb{E}_\mu[u(f)] + c(\mu) \tag{15}$$

is well-defined and represents $\succsim$.

In (15), Optimism first chooses a cost function $c : \Delta(S) \to \mathbb{R} \cup \{\infty\}$ from some collection $\mathbb{C}$, and Pessimism then chooses a belief subject to this cost. This model adds a maximization

stage into MMR's variational representation, which corresponds to the special case in which $\mathbb{C}$ is a singleton.[22] Likewise, the following representation incorporates a maximization stage into CMMM's representation:[23]

Theorem 4. *Preference $\succsim$ satisfies Axioms 1–4 and Axiom 9 if and only if $\succsim$ admits a **rational dual-self** representation; that is, there exists a nonconstant affine utility $u : \Delta(Z) \to \mathbb{R}$ and a collection $\mathbb{G}$ of quasiconvex functions $G : \mathbb{R} \times \Delta(S) \to \mathbb{R} \cup \{\infty\}$ that are increasing in their first argument and satisfy $\max_{G \in \mathbb{G}} \inf_{\mu \in \Delta(S)} G(a, \mu) = a$ for all a such that*

$$W(f) := \max_{G \in \mathbb{G}} \inf_{\mu \in \Delta(S)} G(\mathbb{E}_\mu[u(f)], \mu) \tag{16}$$

is well-defined, continuous, and represents $\succsim$.

The generalizations of DSEU in Theorems 3 and 4 can accommodate additional experimental evidence. For instance, by relaxing the positive homogeneity of I implied by certainty independence but preserving constant-additivity, the dual-self variational model can accommodate Machina's (2009) paradoxes (see also Baillon, L'Haridon, and Placido, 2011).[24] Another important finding is that ambiguity attitudes can differ for gains and losses: For example, in urn experiments subjects who are ambiguity-averse for bets with positive payoffs are often ambiguity-seeking when the sign of the bet is reversed (Trautmann and Wakker, 2018). This is inconsistent with any model that displays constant-additivity, but can be accommodated by the rational dual-self representation in (16).

Appendix: Proofs

This appendix presents the proofs of all results in Sections 2–4.2. The supplementary appendix contains proofs for the generalizations in Section 4.3, as well as other omitted material.

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[22]A special case of (15), imposing the stronger requirement that $\min_{\mu \in \Delta(S)} c(\mu) = 0$ for all $c \in \mathbb{C}$, appears in the conclusion of Castagnoli, Cattelan, Maccheroni, and Tebaldi (in preparation), who note that this special case is characterized by the following axiom in addition to our axioms: for all $f \in \mathcal{F}$, $p \in \Delta(Z)$ and $\alpha \in (0,1)$, $f \succsim p \implies \alpha f + (1-\alpha)p \succsim p$ (F. Maccheroni, personal communication, June 2019).

[23]Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2011) provide an alternative representation of this class of preferences that generalizes (2). As for GMM, the necessity of the axioms requires joint restrictions on the weight function $\alpha(\cdot)$ and other model parameters in (2) to ensure that W is monotone.

[24]This follows from the fact that Siniscalchi's (2009) vector expected utility model can accommodate these paradoxes and is a special case of (15).

A Preliminaries

Throughout this section, we fix any interval $\Gamma \subseteq \mathbb{R}$ and let $U := \Gamma^S$. For any $a \in \mathbb{R}$, let $\underline{a}$ denote the vector in $\mathbb{R}^S$ with $\underline{a}(s) = a$ for all $s \in S$. For any $\phi, \psi \in \mathbb{R}^S$, write $\phi \geq \psi$ if $\phi(s) \geq \psi(s)$ for all s.

A.1 Properties of functionals

Fix any functional $I : U \to \mathbb{R}$. We call I: *monotonic* if $I(\phi) \geq I(\psi)$ for all $\phi, \psi \in U$ with $\phi \geq \psi$; *normalized* if $I(\underline{a}) = a$ for all $a \in \Gamma$; *constant-additive* if $I(\phi + \underline{a}) = I(\phi) + a$ for all $\phi \in U$ and $a \in \Gamma$ with $\phi + \underline{a} \in U$; *positively homogeneous* if $I(a\phi) = aI(\phi)$ for all $\phi \in U$ and $a \in \mathbb{R}_+$ with $a\phi \in U$; and *constant-linear* if I is constant-additive and positively homogeneous. It is easy to see that if $\underline{0} \in \Gamma$, then any constant-linear functional I is normalized.

A.2 Clarke derivative and differential

Consider a locally Lipschitz functional $I : U \to \mathbb{R}$. For every $\phi \in \text{int}U$ and $\xi \in \mathbb{R}^S$, the *Clarke (upper) derivative* of I in ϕ in the direction of ξ is

$$I^\circ(\phi; \xi) := \limsup_{\psi \to \phi, t \downarrow 0} \frac{I(\psi + t\xi) - I(\psi)}{t}.$$

The *Clarke (sub)differential* of I at ϕ is the set

$$\partial I(\phi) := \{\chi \in \mathbb{R}^S : \chi \cdot \xi \leq I^\circ(\phi; \xi), \forall \xi \in \mathbb{R}^S\}.$$

We will frequently invoke the following properties of the Clarke differential. First, if I is locally Lipschitz, then Rademacher's theorem yields a subset $\hat{U} \subseteq \text{int}U$ such that $U \setminus \hat{U}$ has Lebesgue measure zero and I is differentiable on $\hat{U}$. Combining this with Theorem 2.5.1 in Clarke (1990), we obtain the following approximation of the Clarke differential:

Lemma A.1 (Theorem 2.5.1 in Clarke (1990)). *Suppose $I : U \to \mathbb{R}$ is locally Lipschitz. Then there exists $\hat{U} \subseteq \text{int}U$ such that $U \setminus \hat{U}$ has Lebesgue measure zero, I is differentiable at each $\psi \in \hat{U}$, and for every $\phi \in \text{int}U$, we have*

$$\partial I(\phi) = \text{co}\{\lim_n \nabla I(\phi_n) : \phi_n \to \phi, \phi_n \in \hat{U}\}. \tag{17}$$

The next result is an "envelope theorem" for Clarke differentials:

Lemma A.2 (Theorem 2.8.6 in Clarke (1990)). *Suppose functional $I : U \to \mathbb{R}$ is given by*

$$I(\cdot) = \sup_{t \in T} I_t(\cdot)$$

for some indexed family of functionals $(I_t)_{t \in T}$ with domain U. Assume that there exists some $K > 0$ such that $|I_t(\psi) - I_t(\xi)| \leq K \|\psi - \xi\|$ for every $t \in T$ and $\psi, \xi \in \text{int} U$. Then for every $\phi \in \text{int} U$, we have $\partial I(\phi) \subseteq \text{co}\{\lim_{i \to \infty} \nabla I_{t_i}(\phi_i) : \phi_i \to \phi, t_i \in T, I_{t_i}(\phi) \to I(\phi)\}$.

Last, we note the following relationship between properties of I and its Clarke differential:

Lemma A.3 (Part 1 of Proposition A.3 in GMM). *If $I : U \to \mathbb{R}$ is locally Lipschitz, positively homogeneous, and $\underline{0} \in \text{int} U$, then $\partial I(\phi) \subseteq \partial I(\underline{0})$ for all $\phi \in \text{int} U$.*

Lemma A.4 (Parts 2–3 of Proposition A.3 in GMM). *If $I : U \to \mathbb{R}$ is locally Lipschitz, monotonic, and constant-additive, then $\partial I(\phi) \subseteq \Delta(S)$ for all $\phi \in \text{int} U$.*

A.3 Boolean representation of locally Lipschitz I

Throughout this subsection, we assume that $I : U \to \mathbb{R}$ is locally Lipschitz. Let $\hat{U}$ be the generic subset given by Lemma A.1.

Lemma A.6 below shows that, restricted to $\hat{U}$, I admits a so-called "Boolean" representation in terms of a family of affine functionals whose slopes correspond to gradients of I. This result extends Ovchinnikov (2001), who establishes Lemma A.6 under the assumption that I is continuously differentiable. Our non-smooth generalization is necessary for the proof of Theorem 1, where the utility-act functional I is non-differentiable (except in the case of subjective expected utility). We begin with a preliminary result:

Lemma A.5. *For every $\phi, \psi \in \hat{U}$ and $\varepsilon > 0$, there exists $\xi \in \hat{U}$ such that*

$$I(\xi) - I(\psi) + \nabla I(\xi) \cdot (\psi - \xi) \geq 0, \quad I(\xi) - I(\phi) + \nabla I(\xi) \cdot (\phi - \xi) \leq \varepsilon.$$

Proof. Take any $\phi, \psi \in \hat{U}$ and $\varepsilon > 0$. Let $m := I(\psi) - I(\phi)$. If $\nabla I(\phi) \cdot (\psi - \phi) \geq m$, we can set $\xi = \phi$. Likewise if $\nabla I(\psi) \cdot (\psi - \phi) \geq m$, we can set $\xi = \psi$. It remains to consider the case

$$\nabla I(\phi) \cdot (\psi - \phi), \nabla I(\psi) \cdot (\psi - \phi) < m. \tag{18}$$

Define

$$H(\lambda) := I(\phi + \lambda(\psi - \phi)) - \lambda m - I(\phi)$$

for each $\lambda \in \mathbb{R}$ with $\phi + \lambda(\psi - \phi) \in U$. Since $\phi, \psi \in \hat{U}$, H is differentiable at $\lambda \in \{0, 1\}$, with $H(0) = H(1) = 0$ and $H'(0), H'(1) < 0$ by assumption (18). Hence, H is negative for small

enough $\lambda > 0$ and positive for $\lambda < 1$ close enough to 1. Thus, the set $\{\lambda \in (0,1) : H(\lambda) = 0\}$ is nonempty and closed; let λ^* denote its supremum.

Since H is locally Lipschitz, we have $H(\lambda) = \int_{\lambda^*}^{\lambda} H'(\lambda')d\lambda'$ for all $\lambda > \lambda^*$. As $H(\lambda) > 0$ for all $\lambda \in (\lambda^*, 1)$, we can choose $\lambda^{**} \in (\lambda^*, 1)$ close enough to λ^* such that H is differentiable at λ^{**} with $H'(\lambda^{**}) > 0$ and $H(\lambda^{**}) \in (0, \varepsilon)$. But then

$$H'(\lambda^{**}) = \lim_{t \to 0} \frac{I(\phi + (\lambda^{**} + t)(\psi - \phi)) - I(\phi + \lambda^{**}(\psi - \phi))}{t} - m > 0,$$

which implies that

$$I^\circ(\phi + \lambda^{**}(\psi - \phi); \psi - \phi) - m \geq H'(\lambda^{**}) > 0.$$

Since $I^\circ(\xi; \zeta) = \max_{\mu \in \partial I(\xi)} \mu \cdot \zeta$ for any ζ, ξ (e.g., Proposition 2.1.2 in Clarke, 1990), this yields some $\mu \in \partial I(\phi + \lambda^{**}(\psi - \phi))$ such that

$$\mu \cdot (\psi - \phi) - m \geq H'(\lambda^{**}) > 0.$$

By (17), there exists a sequence $\xi_n \to \phi + \lambda^{**}(\psi - \phi)$ such that $\xi_n \in \hat{U}$ for each n and $\lim_n \nabla I(\xi_n) = \mu$. Then

$$
\begin{aligned}
\lim_n \left(I(\xi_n) - I(\psi) + \nabla I(\xi_n) \cdot (\psi - \xi_n) \right) &= I(\phi + \lambda^{**}(\psi - \phi)) - I(\psi) + (1 - \lambda^{**})\mu \cdot (\psi - \phi) \\
&= H(\lambda^{**}) - (1 - \lambda^{**})m + (1 - \lambda^{**})\mu \cdot (\psi - \phi) > 0
\end{aligned}
$$

where the inequality uses the fact that $H(\lambda^{**}) > 0$ and that $\mu \cdot (\psi - \phi) - m \geq H'(\lambda^{**}) > 0$. Similarly,

$$
\begin{aligned}
\lim_n \left(I(\xi_n) - I(\phi) + \nabla I(\xi_n) \cdot (\phi - \xi_n) \right) &= I(\phi + \lambda^{**}(\psi - \phi)) - I(\phi) - \lambda^{**}\mu \cdot (\psi - \phi) \\
&= H(\lambda^{**}) + \lambda^{**}m - \lambda^{**}\mu \cdot (\psi - \phi) < \varepsilon
\end{aligned}
$$

where the inequality uses $H(\lambda^{**}) < \varepsilon$ and $\mu \cdot (\psi - \phi) - m \geq H'(\lambda^{**}) > 0$. Thus, for any large enough n, $\xi_n \in \hat{U}$ is as desired. $\qquad \square$

We now establish the Boolean representation of I:

Lemma A.6. *For each $\phi \in \hat{U}$, we have*

$$I(\phi) = \max_{\psi \in \hat{U}} \inf_{\xi \in K_\psi} I(\xi) + \nabla I(\xi) \cdot (\phi - \xi),$$

where $K_\psi := \{\xi \in \hat{U} : I(\xi) + \nabla I(\xi) \cdot (\psi - \xi) \geq I(\psi)\}$ for all $\psi \in \hat{U}$.

Proof. For each $\phi, \psi \in \hat{U}$ and $\varepsilon > 0$, Lemma A.5 yields some $\xi \in K_\psi$ such that $I(\xi) + \nabla I(\xi) \cdot (\phi - \xi) \leq I(\phi) + \varepsilon$. Thus, $\inf_{\xi \in K_\psi} I(\xi) + \nabla I(\xi) \cdot (\phi - \xi) \leq I(\phi)$. Moreover, by definition of K_ϕ, $\inf_{\xi \in K_\phi} I(\xi) + \nabla I(\xi) \cdot (\phi - \xi) \geq I(\phi)$. Hence, $I(\phi) = \max_{\psi \in \hat{U}} \inf_{\xi \in K_\psi} I(\xi) + \nabla I(\xi) \cdot (\phi - \xi)$, as required. $\square$

B Proofs for Section 2

B.1 Proof of Theorem 1

We invoke the following standard result:

Lemma B.1 (Lemma 1 in GMM). *Preference $\succsim$ satisfies Axioms 1–5 if and only if there exists a monotonic, constant-linear functional $I : \mathbb{R}^S \to \mathbb{R}$ and a nonconstant affine function $u : \Delta(Z) \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}$,*

$$f \succsim g \iff I(u(f)) \geq I(u(g)). \tag{19}$$

Moreover, I is unique and u is unique up to positive affine transformation.

The necessity proof for Theorem 1 is standard and we omit it. To prove sufficiency, suppose $\succsim$ satisfies Axioms 1–5. Let I and u be as given by Lemma B.1. Consider the collection $\mathbb{P}^*$ given by (5), i.e.,

$$\mathbb{P}^* := \mathrm{cl}\{P_\phi^* : \phi \in \mathbb{R}^S\} \quad \text{with} \quad P_\phi^* := \{\mu \in \partial I(\underline{0}) : \mu \cdot \phi \geq I(\phi)\},$$

where cl denotes the topological closure in $\mathcal{K}(\Delta(S))$ under the Hausdorff topology.

Note that since I is monotonic and constant-linear, it is 1-Lipschitz. Thus, $\partial I(\underline{0}) \subseteq \Delta(S)$ by Lemma A.4, so that each P_ϕ^* is indeed a closed, convex set of beliefs. Moreover, $\mathbb{P}^*$ is compact, as it is a closed subset of the compact space $\mathcal{K}(\Delta(S))$. Thus, $\mathbb{P}^*$ is a belief-set collection. We will show that for all $\phi \in \mathbb{R}^S$,

$$I(\phi) = \max_{P \in \mathbb{P}^*} \min_{\mu \in P} \mu \cdot \phi, \tag{20}$$

which by (19) ensures that $(\mathbb{P}^*, u)$ is a DSEU representation of $\succsim$.

Lemma A.1 yields a set $\hat{U} \subseteq \mathbb{R}^S$ such that $\mathbb{R}^S \setminus \hat{U}$ has Lebesgue measure zero and I is differentiable on $\hat{U}$. Moreover, since I is positively homogeneous, Lemma A.3 implies that $\partial I(\phi) \subseteq \partial I(\underline{0})$ for all $\phi \in \mathbb{R}^S$, so that for all $\phi \in \hat{U}$, we have $\mu_\phi := \nabla I(\phi) \in \partial I(\underline{0})$. We will invoke the following lemma:

Lemma B.2. *For each $\phi \in \hat{U}$, $I(\phi) = \mu_\phi \cdot \phi$.*

Proof. Take any $\phi \in \hat{U}$. By positive homogeneity of I, $\alpha\phi \in \hat{U}$ and $\nabla I(\phi) = \nabla I(\alpha\phi)$ for any $\alpha \in (0,1)$. Thus, the function $h : [0,1] \to \mathbb{R}$ defined by $h(\alpha) = I(\alpha\phi)$ is differentiable at every $\alpha \in (0,1)$ and Lipschitz. Hence, $I(\phi) = h(1) - h(0) = \int_0^1 h'(\alpha')d\alpha' = \int_0^1 (\nabla I(\alpha\phi) \cdot \phi)d\alpha' = \phi \cdot \mu_\phi$. $\qquad\square$

To complete the proof of (20), first take any $\phi, \psi \in \hat{U}$ and let $K_\psi := \{\xi \in \hat{U} : I(\xi) + \mu_\xi \cdot (\psi - \xi) \geq I(\psi)\}$ be as in Lemma A.6. Then

$$I(\phi) = \max_{\psi \in \hat{U}} \inf_{\xi \in K_\psi} I(\xi) + \mu_\xi \cdot (\phi - \xi) = \max_{\psi \in \hat{U}} \inf_{\xi \in K_\psi} \mu_\xi \cdot \phi, \qquad (21)$$

where the first equality holds by Lemma A.6 and the second by Lemma B.2. Letting $P_\psi := \{\mu_\xi : \xi \in \hat{U}, \mu_\xi \cdot \psi \geq I(\psi)\}$, Lemma B.2 ensures that $\xi \in K_\psi$ if and only if $\mu_\xi \in P_\psi$. Moreover, (17) implies that $\overline{\text{co}}P_\psi = P_\psi^*$. Combining these two observations with (21) yields

$$I(\phi) = \max_{\psi \in \hat{U}} \inf_{\mu \in P_\psi} \mu \cdot \phi = \max_{\psi \in \hat{U}} \min_{\mu \in \overline{\text{co}}P_\psi} \mu \cdot \phi = \max_{\psi \in \hat{U}} \min_{\mu \in P_\psi^*} \mu \cdot \phi. \qquad (22)$$

Next, take any $\phi, \psi \in \mathbb{R}^S$. Then there exist sequences $\phi_n \to \phi, \psi_n \to \psi$ such that $\phi_n, \psi_n \in \hat{U}$. For each n, pick $\mu_n \in P_{\psi_n}^*$ such that $\min_{\mu \in P_{\psi_n}^*} \mu \cdot \phi_n = \mu_n \cdot \phi_n$ and consider a convergent subsequence (μ_{n_k}) with $\lim_{k \to \infty} \mu_{n_k} = \mu^*$. Note that $\mu^* \in P_\psi^*$: Indeed, for each k, we have $\mu_{n_k} \cdot \psi_{n_k} \geq I(\psi_{n_k})$, which by continuity of I implies $\mu^* \cdot \psi \geq I(\psi)$.

Moreover, for each k, we have $\mu_{n_k} \cdot \phi_{n_k} = \min_{\mu \in P_{\psi_{n_k}}^*} \mu \cdot \phi_{n_k} \leq I(\phi_{n_k})$, where the inequality holds by (22). Hence, continuity of I implies $\mu^* \cdot \phi \leq I(\phi)$, so that

$$\min_{\mu \in P_\psi^*} \mu \cdot \phi \leq \mu^* \cdot \phi \leq I(\phi). \qquad (23)$$

Since (23) holds for all $\psi \in \mathbb{R}^S$, it follows from the definition of $\mathbb{P}^*$ that

$$\min_{\mu \in P} \mu \cdot \phi \leq I(\phi)$$

holds for all $P \in \mathbb{P}^*$. Finally, applying (23) with $\psi = \phi$ yields $\min_{\mu \in P_\phi^*} \mu \cdot \phi \leq I(\phi) \leq \min_{\mu \in P_\phi^*} \mu \cdot \phi$, where the second inequality holds by definition of P_ϕ^*. Thus,

$$I(\phi) = \min_{\mu \in P_\phi^*} \phi \cdot \mu = \max_{P \in \mathbb{P}^*} \min_{\mu \in P} \mu \cdot \phi,$$

as required. $\qquad\square$

B.2 Proof of Proposition 1

We begin with the following lemma:

Lemma B.3. *Consider any functional* $I : \mathbb{R}^S \to \mathbb{R}$ *and belief-set collection* $\mathbb{P}$ *such that* $I(\phi) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu \cdot \phi$ *for all* $\phi \in \mathbb{R}^S$. *Then*

$$\partial I(\underline{0}) \subseteq \overline{\text{co}} \bigcup_{P \in \mathbb{P}} P.$$

Proof. For each $P \in \mathbb{P}$, let $I_P(\phi) := \min_{\mu \in P} \mu \cdot \phi$ for each ϕ. Thus, $I(\phi) = \max_{P \in \mathbb{P}} I_P(\phi)$ for each ϕ. Note that each I_P is 1-Lipschitz and $\partial I_P(0) = P$.

Take any convergent sequence $(\nabla I_{P_i}(\phi_i))$ where $\phi_i \to 0$, $P_i \in \mathbb{P}$, and $\nabla I_{P_i}(\phi_i)$ exists for each i. Then

$$\nabla I_{P_i}(\phi_i) \in \partial I_{P_i}(\phi_i) \subseteq \partial I_{P_i}(0) = P_i$$

where the set inclusion holds by Lemma A.3. Thus, $\lim_i \nabla I_{P_i}(\phi_i) \in \overline{\text{co}} \bigcup_{P \in \mathbb{P}} P$. Hence, the desired conclusion follows by applying Lemma A.2 to I. $\square$

Suppose $\succsim$ satisfies Axioms 1–5. Let I and u be as given by Lemma B.1. For $\mathbb{P}^*$ as in the sufficiency proof of Theorem 1, we have $\overline{\text{co}} \bigcup_{P \in \mathbb{P}^*} P \subseteq \partial I(\underline{0})$. Thus, Lemma B.3 immediately implies that $C = \partial I(\underline{0})$ is the unique closed, convex set satisfying (8) for all DSEU representations of $\succsim$, with equality for representation $\mathbb{P}^*$. $\square$

B.3 Proof of Corollary 1

Since the proof of Proposition 1 identifies the set of relevant priors as $C = \partial I(\underline{0})$, Corollary 1 is immediate from the following result in GMM:

Lemma B.4 (Theorem 14 in GMM). *Suppose* $\succsim$ *satisfies Axioms 1–5 and let* I *and* u *be as in Lemma B.1. Then the unique closed, convex set* D *satisfying*

$$f \succsim^* g \iff \mathbb{E}_\mu[u(f)] \geq \mathbb{E}_\mu[u(g)] \text{ for all } \mu \in D$$

is given by $D = \partial I(\underline{0})$.

C Proofs for Section 3

C.1 Proof of Lemma 1

We combine the proof of Lemma 1 with the proof of Theorem 2 (part 2) below.

C.2 Proof of Theorem 2

Throughout the proof, let I be the functional given by Lemma B.1.

C.2.1 Proof of part 1

To prove the "only if" direction, suppose that $\succsim$ satisfies uncertainty aversion. Since it admits the maxmin expected utility representation of Gilboa and Schmeidler (1989), $I(\phi) = \min_{\mu \in C} \mu \cdot \phi$ holds for all ϕ.

We first show that $\cap_{P \in \mathbb{P}} P \supseteq C$. If not, there exists $P \in \mathbb{P}$ such that $P \not\supseteq C$. By the standard property of support functions, this implies the existence of ϕ such that $\min_{\mu \in C} \phi \cdot \mu < \min_{\mu \in P} \phi \cdot \mu$. This leads to $I(\phi) > \min_{\mu \in C} \mu \cdot \phi$, a contradiction.

We now show that $\cap_{P \in \mathbb{P}} P \subseteq C$. If not, there exists $\mu^* \in \cap_{P \in \mathbb{P}} P \setminus C$. Then there exists ϕ such that $\min_{\mu \in C} \mu \cdot \phi > \mu^* \cdot \phi$. But this implies $I(\phi) \leq \mu^* \cdot \phi < \min_{\mu \in C} \mu \cdot \phi$, a contradiction.

To prove the " if" direction, suppose that $\cap_{P \in \mathbb{P}} P = C$. Take any ϕ. It suffices to show that $I(\phi) = \min_{\mu \in C} \mu \cdot \phi$. Note that by construction of the representation $\mathbb{P}^*$ defined by (5), we have $I(\phi) \geq \min_{\mu \in C} \mu \cdot \phi$. But the representation based on $\mathbb{P}$ yields the inequality $I(\phi) \leq \min_{\mu \in \cap_{P \in \mathbb{P}} P} \mu \cdot \phi = \min_{\mu \in C} \mu \cdot \phi$, which ensures the desired claim.

C.2.2 Proof of part 2 and Lemma 1

We prove the equivalence

$$\text{absolute ambiguity aversion} \Leftrightarrow \infty\text{-ambiguity aversion} \Leftrightarrow |S|\text{-ambiguity aversion}$$

$$\Leftrightarrow \bigcap_{P \in \mathbb{P}} P \neq \emptyset,$$

which implies both part 2 of Theorem 2 and Lemma 1.

The implication *absolute ambiguity aversion* $\Rightarrow \infty$-*ambiguity aversion* follows from the proofs of Theorem 2a and Corollary 3a in Grant and Polak (2013), which imply the equivalence of absolute ambiguity aversion and ∞-ambiguity aversion for any preference with a normalized, monotonic, continuous, constant-additive, and unbounded utility act functional I (as is the case for DSEU). The implication ∞- *ambiguity aversion* $\Rightarrow |S|$- *ambiguity aversion* is trivial.

We now turn to the implication $|S|$-*ambiguity aversion* $\Rightarrow \cap_{P \in \mathbb{P}} P \neq \emptyset$. If $\succsim$ satisfies $|S|$-ambiguity aversion, then by part 3 of the theorem (see the proof below) any DSEU representation $(\mathbb{P}, u)$ of $\succsim$ is such that every subcollection of $\mathbb{P}$ of cardinality at most $|S|$ has nonempty intersection. Since each P_i is convex and compact, Helly's theorem implies that

the whole collection $\mathbb{P}$ has nonempty intersection.[25] This proves the implication.

Finally, we prove the implication $\bigcap_{P \in \mathbb{P}} P \neq \emptyset \Rightarrow$ *absolute ambiguity aversion*. Suppose that there exists $\mu^* \in \bigcap_{P \in \mathbb{P}} P$ for some DSEU representation $(\mathbb{P}, u)$ of $\succsim$. For any $f \in \mathcal{F}$ and any $P \in \mathbb{P}$, this implies that $\min_{\mu \in P} \mu \cdot u(f) \leq \mu^* \cdot u(f)$, and hence $\max_{P \in \mathbb{P}} \min_{\mu \in P} \mu \cdot u(f) \leq \mu^* \cdot u(f)$. As a result,

$$f \succsim p \implies \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu \cdot u(f) \geq u(p) \implies \mu^* \cdot u(f) \geq u(p) \implies f \succsim_{\mu^*} p$$

where $\succsim_{\mu^*}$ is the subjective expected utility preference with belief μ^* and utility function u. Hence, $\succsim$ is more ambiguity-averse than $\succsim_{\mu^*}$, which proves the result.

C.2.3 Proof of part 3

The proof relies on the following lemma.

Lemma C.1. *Suppose that preference $\succsim$ admits a DSEU representation $(\mathbb{P}, u)$. Then $\succsim$ satisfies k-ambiguity aversion if and only if*

$$\sum_{i=1}^{k-1} \max_{P_i \in \mathbb{P}} \min_{\mu_i \in P_i} \mu_i \cdot \phi_i \leq \min_{P \in \mathbb{P}} \max_{\mu \in \mathbb{P}} \mu \cdot \sum_{i=1}^{k-1} \phi_i, \quad \text{for all } \phi_1, \cdots, \phi_{k-1} \in \mathbb{R}^S. \quad (24)$$

Proof. To prove the "if" part, suppose inequality (24) is satisfied. Consider any $f_1, \cdots, f_k \in \mathcal{F}$ such that $f_1 \sim f_i$ for all i and $\frac{1}{k} f_1 + \cdots + \frac{1}{k} f_k = p$ for some $p \in \Delta(Z)$. We have

$$I(\frac{1}{k} u(f_k)) = I(\underline{u(p)} - \sum_{i=1}^{k-1} \frac{1}{k} u(f_i)) = u(p) - \min_{P \in \mathbb{P}} \max_{\mu \in P} \sum_{i=1}^{k-1} \frac{1}{k} u(f_i) \cdot \mu$$

$$\leq u(p) - \sum_{i=1}^{k-1} \max_{P_i \in \mathbb{P}} \min_{\mu_i \in P_i} \frac{1}{k} u(f_i) \cdot \mu_i = u(p) - \sum_{i=1}^{k-1} I(\frac{1}{k} u(f_i)),$$

where the inequality holds by (24). Rearranging yields $\sum_{i=1}^{k} I(\frac{1}{k} u(f_i)) \leq u(p)$, which is simply $I(u(f_1)) \leq u(p)$ since $I(u(f_i)) = I(u(f_1))$ for all i. This is turn implies $p \succsim f_1$, and thus $\succsim$ satisfies k-ambiguity aversion.

To prove the "only if" part, suppose that there exist some vectors $\phi_1, \cdots, \phi_{k-1}$ such that the inequality (24) is violated. By the constant linearity of the max-min and min-max functionals, we can assume without loss of generality that $I(\phi_i) = I(\phi_1)$ for all i, and that each ϕ_i belongs to $[-1, 1]^S$.

---

[25]Recall that $\Delta(S)$ has dimension $|S| - 1$.

Let $c \in \mathbb{R}$ be given by $c = -I(-\phi_1 - \ldots - \phi_{k-1}) + I(\phi_1)$, so that $I(\underline{c} - \phi_1 - \cdots - \phi_{k-1}) = I(\phi_1)$. Note that $c \in [-k, k]$. Let $\phi_k \in \mathbb{R}^S$ be defined by $\phi_k = \underline{c} - \phi_1 - \cdots - \phi_{k-1}$, which implies $\phi_1 + \cdots + \phi_k = \underline{c}$. Up to rescaling all the ϕ_is and c by a common factor, this vector ϕ_k also belongs to $[-1, 1]^S$. By definition of c, we have $I(\phi_k) = I(\phi_1)$, and

$$I(\phi_k) = I(\underline{c} - \sum_{i=1}^{k-1} \phi_i) = c - \min_{P \in \mathbb{P}} \max_{\mu \in P} \mu \cdot \sum_{i=1}^{k-1} \phi_i > c - \sum_{i=1}^{k-1} \max_{P_i \in \mathbb{P}} \min_{\mu_i \in P_i} \mu_i \cdot \phi_i$$

$$= c - \sum_{i=1}^{k-1} I(\phi_i).$$

Rearranging yields $\sum_{i=1}^k I(\phi_i) > c$, which implies $I(\phi_1) > \frac{c}{k}$.

To conclude the proof, we assume that $u(\overline{z}) \geq 1, u(\underline{z}) \leq -1$ for some outcomes $\overline{z}, \underline{z} \in Z$. (This is without loss of generality by taking a positive affine transformation of u if necessary.) Since each ϕ_i belongs to $[-1, 1]^S$, it is possible to find weights (ε_i^s) such that the act f_i that maps each state s into the lottery $\varepsilon_i^s \delta_{\underline{z}} + (1 - \varepsilon_i^s)\delta_{\overline{z}}$ satisfies $u(f_i) = \phi_i$. In addition, the fact that $\sum_{i=1}^k u(f_i)$ is a constant vector equal to c shows that $\sum_{i=1}^k \frac{1}{k} f_i$ is a constant act that delivers a lottery p supported on $\{\overline{z}, \underline{z}\}$, where $u(p) = \frac{c}{k}$. The collection $(f_1, \cdots, f_k)$ thus satisfies $\frac{1}{k} f_1 + \cdots + \frac{1}{k} f_k = p$, $f_i \sim f_1$ for all i since $I(\phi_i) = I(\phi_1)$, and $f_1 \succ p$ since $I(\phi_1) > \frac{c}{k} = u(p)$. Hence, $\succsim$ does not satisfy k-ambiguity aversion. $\qquad\square$

Let us now prove part 3 of the theorem.

Sufficiency. Suppose that $P_1 \cap \cdots \cap P_k \neq \emptyset$ for all $P_1, \cdots, P_k \in \mathbb{P}$. Consider any $P_1, \cdots, P_k$ and some vectors $(\phi_1, \cdots, \phi_{k-1})$. Let $\mu \in P_1 \cap \cdots \cap P_k$. We have

$$\min_{\mu_1 \in P_1, \cdots, \mu_{k-1} \in P_{k-1}} \sum_{i=1}^{k-1} \mu_i \cdot \phi_i \leq \sum_{i=1}^{k-1} \mu \cdot \phi_i \leq \max_{\mu_k \in P_k} \sum_{i=1}^{k-1} \mu_k \cdot \phi_i$$

where the first inequality is due to the fact that $\mu \in P_i$ for all $i \leq k - 1$, and the second inequality is due to the fact that $\mu \in P_k$. Since this is true for any $P_1, \cdots, P_k$, this implies

$$\max_{(P_1, \cdots, P_{k-1}) \in \mathbb{P}^{k-1}} \min_{\mu_1 \in P_1, \cdots, \mu_{k-1} \in P_{k-1}} \sum_{i=1}^{k-1} \mu_i \cdot \phi_i \leq \min_{P_k \in \mathbb{P}} \max_{\mu_k \in P_k} \sum_{i=1}^{k-1} \mu_k \cdot \phi_i,$$

i.e.,

$$\sum_{i=1}^{k-1} \max_{P_i \in \mathbb{P}} \min_{\mu_i \in P_i} \mu_i \cdot \phi_i \leq \min_{P_k \in \mathbb{P}} \max_{\mu_k \in \mathbb{P}} \mu_k \cdot \sum_{i=1}^{k-1} \phi_i.$$

Thus, by Lemma C.1 $\succsim$ satisfies k-ambiguity aversion.

Necessity. Suppose that there exist $P_1, \cdots, P_k \in \mathbb{P}$ such that $P_1 \cap \cdots \cap P_k = \emptyset$. Consider

the sets $A, B \subseteq \mathbb{R}^{S(k-1)}$ defined by

$$A = \{(\mu_1, \cdots, \mu_{k-1}) : \mu_i \in P_i\} \text{ and } B = \{(\mu_k, \cdots, \mu_k) : \mu_k \in P_k\}.$$

The sets A and B are compact and convex. In addition, $A \cap B = \emptyset$ since any $(\mu_k, \cdots, \mu_k) \in A \cap B$ would satisfy $\mu_k \in P_1 \cap \cdots \cap P_k$, which is a contradiction. By the separating hyperplane theorem there exists a vector $\phi = (\phi_1, \cdots, \phi_{k-1}) \in \mathbb{R}^{S(k-1)}$, where each $\phi_i \in \mathbb{R}^S$, such that $\min_{a \in A} a \cdot \phi > \max_{b \in B} b \cdot \phi$, which is equivalent to

$$\min_{\mu_1 \in P_1, \cdots, \mu_{k-1} \in P_{k-1}} \sum_{i=1}^{k-1} \mu_i \cdot \phi_i > \max_{\mu \in P_k} \sum_{i=1}^{k-1} \mu \cdot \phi_i.$$

Hence,

$$\sum_{i=1}^{k-1} \max_{P_i \in \mathbb{P}} \min_{\mu_i \in P_i} \mu_i \cdot \phi_i \geq \min_{\mu_1 \in P_1, \cdots, \mu_{k-1} \in P_{k-1}} \sum_{i=1}^{k-1} \mu_i \cdot \phi_i > \max_{\mu \in P_k} \sum_{i=1}^{k-1} \mu \cdot \phi_i \geq \min_{P \in \mathbb{P}} \max_{\mu \in \mathbb{P}} \mu \cdot \sum_{i=1}^{k-1} \phi_i.$$

Thus, by Lemma C.1 $\succsim$ does not satisfy k-ambiguity aversion. $\qquad\square$

C.3   Proof of Proposition 2

Note that $m(E) = \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(E)$, while $m(E^c) = 1 - \min_{P \in \mathbb{P}} \max_{\mu \in P} \mu(E)$. Thus, $AA(E) = \min_{P \in \mathbb{P}} \max_{\mu \in P} \mu(E) - \max_{P \in \mathbb{P}} \min_{\mu \in P} \mu(E)$.

This implies that $AA(E) \geq 0$ if and only if all $P, P' \in \mathbb{P}$ satisfy $\max_{\mu \in P} \mu(E) \geq \min_{\mu' \in P'} \mu'(E)$, i.e., if and only if $P_E \cap P'_E \neq \emptyset$. Similarly, $AA(E) > 0$ if and only if all $P, P' \in \mathbb{P}$ satisfy $\max_{\mu \in P} \mu(E) > \min_{\mu' \in P'} \mu'(E)$, i.e., if and only if $P_E \cap P'_E$ is a non-degenerate interval. $\qquad\square$

C.4   Proof of Corollary 2

Pick any $\beta > 0$ and $\nu \in \Delta(S)$ with $\beta < \min_{s \in S} \nu(s)$. Define $\mathbb{P}$ by $\mathbb{P} = \{P^F : F \in \mathcal{F}\}$, where for each $F \in \mathcal{F}$,

$$P^F := \{\mu \in \Delta(S) : \mu(F) = \nu(F) + \frac{\beta}{2}, \mu(E) \in [\nu(E) - \beta, \nu(E) + \beta] \forall E \subseteq S\}.$$

Note that each P^F is nonempty: Indeed, pick any $s \in F$ and $s' \in F^c$ (which exist since $F \notin \{S, \emptyset\}$). Then setting $\mu(s) = \nu(s) + \frac{\beta}{2}$, $\mu(s') = \nu(s') - \frac{\beta}{2}$, and $\mu(s'') = \nu(s'')$ for all $s'' \neq s, s'$ yields $\mu \in P^F$. Since P^F is also closed and convex, $\mathbb{P}$ is a well-defined belief-set collection.

Take any $F \in \mathcal{F}$, and observe that $P_F^F = \{\nu(F) + \beta/2\}$, while $P_F^{F^c} = \{\nu(F) - \beta/2\}$. Therefore, $P_F^F \cap P_F^{F^c} = \emptyset$, which implies by Proposition 2 that $AA(F) < 0$.

Consider now any $E \in \mathcal{E}$, and any $F \in \mathcal{F}$. Since $E \neq F$ (as $\mathcal{E}$ and $\mathcal{F}$ are disjoint), we either have (a) $F \setminus E \neq \emptyset \neq E \setminus F$; (b) $E \subsetneq F$; or (c) $F \subsetneq E$. In each case, we show that there exist $\mu, \mu' \in P^F$ with $\mu(E) = \nu(E) - \frac{\beta}{2}$ and $\mu'(E) = \nu(E) + \frac{\beta}{2}$. Since this is true for any F, this implies that $P_E^F \cap P_E^{F'} \supseteq [\nu(E) - \frac{\beta}{2}, \nu(E) + \frac{\beta}{2}]$ is a nondegenerate interval for any $F, F' \in \mathcal{F}$, which in turn implies that $AA(E) > 0$ by Proposition 2.

In case (a), pick $s \in F \setminus E$ and $s' \in E \setminus F$. Since $E \neq F^c$ (as $F^c \in \mathcal{F}$), there also exists $s'' \in S \setminus (E \cup F)$. Then define μ by $\mu(s) = \nu(s) + \frac{\beta}{2}$, $\mu(s') = \nu(s') - \frac{\beta}{2}$, and $\mu(s''') = \nu(s''')$ for all $s''' \neq s, s'$; and μ' by $\mu'(s) = \nu(s) + \frac{\beta}{2}$, $\mu'(s') = \nu(s') + \frac{\beta}{2}$, $\mu'(s'') = \nu(s'') - \beta$, and $\mu'(s''') = \nu(s''')$ for all $s''' \neq s, s', s''$.

In case (b), pick $s \in F \setminus E$, $s' \in E$, and $s'' \in F^c \subseteq E^c$. Then define μ by $\mu(s) = \nu(s) + \beta$, $\mu(s') = \nu(s') - \frac{\beta}{2}$, $\mu(s'') = \nu(s'') - \frac{\beta}{2}$, and $\mu(s''') = \nu(s''')$ for all $s''' \neq s, s', s''$; and μ' by $\mu'(s) = \nu(s)$, $\mu(s') = \nu(s') + \frac{\beta}{2}$, $\mu(s'') = \nu(s'') - \frac{\beta}{2}$, and $\mu(s''') = \nu(s''')$ for all $s''' \neq s, s', s''$.

In case (c), pick $s \in F$, $s' \in E \setminus F$, and $s'' \in E^c \subseteq F^c$. Then define μ by $\mu(s) = \nu(s) + \frac{\beta}{2}$, $\mu(s') = \nu(s') - \beta$, $\mu(s'') = \nu(s'') + \frac{\beta}{2}$, and $\mu(s''') = \nu(s''')$ for all $s''' \neq s, s', s''$; and μ' by $\mu'(s) = \nu(s) + \frac{\beta}{2}$, $\mu'(s'') = \nu(s'') - \frac{\beta}{2}$, and $\mu'(s''') = \nu(s''')$ for all $s''' \neq s, s''$. $\qquad\square$

C.5   Proof of Corollary 3

Recall that the α-MEU functional (12) coincides with the DSEU representation $(\mathbb{P}, u)$ where $\mathbb{P} = \{\alpha P + (1 - \alpha)\{\mu\} : \mu \in P\}$. Let $P^\mu = \alpha P + (1 - \alpha)\{\mu\}$ for any $\mu \in P$. For any event E and $\mu \in P$, the interval $P_E^\mu = \{\nu(E) : \nu \in P^\mu\}$ is thus given by $[\alpha \min_{\nu \in P} \nu(E) + (1 - \alpha)\mu(E), \alpha \max_{\nu \in P} \nu(E) + (1 - \alpha)\mu(E)]$.

Suppose that $\alpha \geq 1/2$. Then, for any $\mu \in P$ and any event E, we have

$$
\alpha \min_{\nu \in P} \nu(E) + (1 - \alpha)\mu(E) \leq \alpha \min_{\nu \in P} \nu(E) + (1 - \alpha) \max_{\nu \in P} \nu(E)
$$
$$
\leq \frac{1}{2} \min_{\nu \in P} \nu(E) + \frac{1}{2} \max_{\nu \in P} \nu(E) \leq (1 - \alpha) \min_{\nu \in P} \nu(E) + \alpha \max_{\nu \in P} \nu(E)
$$
$$
\leq (1 - \alpha)\mu(E) + \alpha \max_{\nu \in P} \nu(E).
$$

Hence, $1/2 \min_{\nu \in P} \nu(E) + 1/2 \max_{\nu \in P} \nu(E) \in P_E^\mu$. Since this is true for every $\mu \in P$, this implies $P_E^\mu \cap P_E^{\mu'} \neq \emptyset$ for all $\mu, \mu' \in P$. Thus, $AA(E) \geq 0$ by the first part of Proposition 2. Moreover, consider the case $\alpha > 1/2$. Since P is not a singleton, there exists an event E such that $\min_{\nu \in P} \nu(E) < \max_{\nu \in P} \nu(E)$. Then the above inequality is strict for each μ, i.e.,

$$
\alpha \min_{\nu \in P} \nu(E) + (1 - \alpha)\mu(E) < \frac{1}{2} \min_{\nu \in P} \nu(E) + \frac{1}{2} \max_{\nu \in P} \nu(E) < (1 - \alpha)\mu(E) + \alpha \max_{\nu \in P} \nu(E).
$$

Thus, for each $\mu, \mu' \in P$, $P_E^\mu \cap P_E^{\mu'}$ is a non-degenerate interval, which implies $AA(E) > 0$ by the second part of Proposition 2.

Next, suppose that $\alpha \leq 1/2$. Take any E and let μ be a minimizer of $\mu(E)$ on P, and μ' be a maximizer. Since $\alpha \leq 1/2$, we have $\alpha\mu'(E) + (1-\alpha)\mu(E) \leq \alpha\mu(E) + (1-\alpha)\mu'(E)$. Since $P_E^\mu = [\mu(E), \alpha\mu'(E) + (1-\alpha)\mu(E)]$ and $P_E^{\mu'} = [\alpha\mu(E) + (1-\alpha)\mu'(E), \mu'(E)]$, this proves that $P_E^\mu \cap P_E^{\mu'}$ is not a non-degenerate interval. Thus, by the second part of Proposition 2, $AA(E) \leq 0$. Moreover, consider the case $\alpha < 1/2$. Since P is not a singleton, there exists an event E such that $\min_{\mu \in P} \mu(E) < \max_{\mu \in P} \mu(E)$. Then the above inequality is strict, i.e., $\alpha\mu'(E) + (1-\alpha)\mu(E) < \alpha\mu(E) + (1-\alpha)\mu'(E)$. Thus $P_E^\mu \cap P_E^{\mu'} = \emptyset$, which implies $AA(E) < 0$ by the first part of Proposition 2. $\qquad\square$

C.6 Proof of Proposition 3

Fix $\mu^* \in \bigcap_{i \in \mathcal{I}, P_i \in \mathbb{P}_i} P_i$. For any $i \in \mathcal{I}$ and $x_i \in \mathbb{R}_+^S$, define $U_i(x_i) = \max_{P_i \in \mathbb{P}_i} \min_{\mu_i \in P_i} \mathbb{E}_{\mu_i}[u_i(x_i)]$.

First part: Suppose x is a feasible allocation that is not full insurance, i.e., x_i is nonconstant for some $i \in \mathcal{I}$. Let $\bar{x}_i := \sum_s \mu^*(s) x_i(s)$ for each i. Observe that $\sum_i \bar{x}_i = \sum_s \mu^*(s) \sum_i x_i(s) = w$, so the profile $(\mathbf{1}\bar{x}_i)_{i \in \mathcal{I}}$ is a feasible allocation, where $\mathbf{1} \in \mathbb{R}_+^S$ is a vector of ones. For any i,

$$U_i(\mathbf{1}\bar{x}_i) = u_i\left(\sum_s \mu^*(s) x_i(s)\right) \geq \sum_s \mu^*(s) u_i(x_i(s)) \geq \max_{P_i \in \mathbb{P}_i} \min_{\mu_i \in P_i} \sum_s \mu_i(s) u_i(x_i(s)) = U_i(x_i),$$

where the first inequality holds by concavity of u_i and the second because $\mu^* \in \bigcap_{P_i \in \mathbb{P}_i} P_i$. Moreover, the first inequality is strict for any i with nonconstant x_i, as μ^* is full support and u_i is strictly concave. This shows that x is Pareto dominated by $(\mathbf{1}\bar{x}_i)_{i \in \mathcal{I}}$.

Conversely, suppose x is a feasible allocation that is full insurance. Consider any feasible allocation $y \neq x$. Then, for some i, we have $\sum_s \mu^*(s) x_i(s) \geq \sum_s \mu^*(s) y_i(s)$ and $y_i \neq x_i$, because $\sum_i \sum_s \mu^*(s) x_i(s) = w = \sum_i \sum_s \mu^*(s) y_i(s)$ by feasibility. Thus,

$$U_i(x_i) = u_i\left(\sum_s \mu^*(s) x_i(s)\right) \geq u_i\left(\sum_s \mu^*(s) y_i(s)\right) \geq \sum_s \mu^*(s) u_i(y_i(s))$$

$$\geq \max_{P_i \in \mathbb{P}_i} \min_{\mu_i \in P_i} \sum_s \mu_i(s) u_i(y_i(s)) = U_i(y_i),$$

where the inequalities hold because u_i is increasing and concave and $\mu^* \in \bigcap_{P_i \in \mathbb{P}_i} P_i$. If y_i is constant, the first inequality is strict as u_i is strictly increasing. If y_i is nonconstant, the second inequality is strict as u_i is strictly concave and μ^* is full support. Hence, x is not Pareto dominated by y.

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Second part: Consider a hypothetical economy in which each agent is a subjective expected utility maximizer with belief μ^* and consumption utility u_i. Since each agent's preference over $\mathbb{R}_+^S$ is continuous, strictly convex and strongly monotone, standard results imply that this economy admits a competitive equilibrium allocation x and corresponding price $p \in \mathbb{R}_{++}^S$. Thus, for any i and $y_i \in \mathbb{R}_+^S$ with $y_i \cdot p \leq \omega_i \cdot p$, we have $\sum_s \mu^*(s) u_i(x_i(s)) \geq \sum_s \mu^*(s) u_i(y_i(s))$. Since x is Pareto efficient (by the first welfare theorem), it is full insurance by the first part.

Now, consider the original economy. For any i and $y_i \in \mathbb{R}_+^S$ with $y_i \cdot p \leq \omega_i \cdot p$, we have

$$U_i(x_i) = \sum_s \mu^*(s) u_i(x_i(s)) \geq \sum_s \mu^*(s) u_i(y_i(s)) \geq \max_{P_i \in \mathbb{P}_i} \min_{\mu_i \in P_i} \sum_s \mu_i(s) u_i(y_i(s)) = U_i(y_i),$$

where the first inequality holds by the previous paragraph and the second because $\mu^* \in \bigcap_{P_i \in \mathbb{P}_i} P_i$. Thus, x is a competitive equilibrium allocation. $\qquad \square$

C.7 Proof of Proposition 4

The result follows from the following three lemmas.

Lemma C.2. *Any optimal schedule is such that $l_S - c_S \geq l_s - c_s$ for all s.*

Proof. Suppose otherwise, and let $s^* \in \{1, \cdots, S-1\}$ be a state that maximizes $l_s - c_s$. For any $s \geq 1$,

$$\min_{\mu \in P_s} \mathbb{E}_\mu u[w - p(c) - l + c]$$
$$= (\pi_s - \varepsilon) u[w - p(c) - l_s + c_s] + (1 - \pi_s + \varepsilon) u[w - p(c) - l_{s^*} + c_{s^*}]$$
$$\leq (\pi_1 - \varepsilon) u[w - p(c) - l_s + c_s] + (1 - \pi_1 + \varepsilon) u[w - p(c) - l_{s^*} + c_{s^*}]$$
$$\leq (\pi_1 - \varepsilon) u[w - p(c)] + (1 - \pi_1 + \varepsilon) u[w - p(c) - l_{s^*} + c_{s^*}]$$
$$= \min_{\mu \in P_1} \mathbb{E}_\mu u[w - p(c) - l + c],$$

where the first inequality uses $\pi_1 \geq \pi_s$ and $l_s - c_s \leq l_{s^*} - c_{s^*}$, and the second inequality uses $l_s - c_s \geq 0$. This shows that the agent's value of schedule c is

$$V(c) = (\pi_1 - \varepsilon) u[w - p(c)] + (1 - \pi_1 + \varepsilon) u[w - p(c) - l_{s^*} + c_{s^*}].$$

Consider the claim schedule c' where $c'_s = c_s$ for all $s < S$, and $c'_S = l_S - l_{s^*} + c_{s^*} < c_S$. Similar arguments prove that the value of schedule c' equals

$$V(c') = (\pi_1 - \varepsilon) u[w - p(c')] + (1 - \pi_1 + \varepsilon) u[w - p(c') - l_{s^*} + c_{s^*}],$$

which is strictly larger than $V(c)$ since $p(c') < p(c)$ due to $c'_S < c_S$. As a result, $V(c') > V(c)$, which contradicts the optimality of c. $\qquad\square$

Lemma C.3. *Any optimal schedule has a straight deductible.*

Proof. Consider a tentative optimal schedule c. By Lemma C.2, we have $l_S - c_S \geq l_s - c_s$ for all s, and arguments similar to those used in the proof of Lemma C.2 show that

$$V(c) = (\pi_1 - \varepsilon)u[w - p(c)] + (1 - \pi_1 + \varepsilon)u[w - p(c) - l_S + c_S]. \tag{25}$$

Let $d = l_S - c_S$, and consider $s < S$. By choice of d, $c_s \geq \max\{0, l_s - d\}$. If $c_s > \max\{0, l_s - d\}$, then replacing c_s with $\max\{0, l_s - d\}$ would strictly decrease $p(c)$ without affecting any of the other terms in (25), which would strictly increase $V(c)$ and contradict the optimality of c. As a result, we have $c_s = \max\{0, l_s - d\}$ for all $s < S$, and also for $s = S$. Thus, c is the claim schedule with straight deductible d. $\qquad\square$

Lemma C.4. *The optimal deductible is unique, nonincreasing in ε and equal to 0 if and only if $\varepsilon \geq 0$.*

Proof. The value of the claim schedule with deductible d equals

$$(\pi_1 - \varepsilon)u[w - \mathbb{E}_\pi \max\{0, l - d\}] + (1 - \pi_1 + \varepsilon)u[w - \mathbb{E}_\pi \max\{0, l - d\} - d].$$

The above expression is continuous in d and thus admits a maximizer on $[0, l_S]$. To simplify notation, let $g(d) = w - \mathbb{E}_\pi \max\{0, l - d\}$ and $h(d) = g(d) - d$. Both g and h are concave, and g is increasing. Since u is increasing and strictly concave, the function $(\pi_1 - \varepsilon)u \circ g + (1 - \pi_1 + \varepsilon)u \circ h$ itself is strictly concave, yielding a unique maximizer.

To prove monotonicity, take $\varepsilon < \varepsilon'$, and write d, d' for the optimal deductible in each case. Suppose toward a contradiction that $d < d'$, which implies $g(d) \leq g(d')$ and $h(d) \geq h(d')$. By optimality of d and d' at ε and ε', respectively, we have

$$(\pi_1 - \varepsilon)u[g(d)] + (1 - \pi_1 + \varepsilon)u[h(d)] > (\pi_1 - \varepsilon)u[g(d')] + (1 - \pi_1 + \varepsilon)u[h(d')],$$

and

$$(\pi_1 - \varepsilon')u[g(d')] + (1 - \pi_1 + \varepsilon')u[h(d')] > (\pi_1 - \varepsilon')u[g(d)] + (1 - \pi_1 + \varepsilon')u[h(d)].$$

Note that the first inequality implies that $u[h(d)] > u[h(d')]$. Combining these inequalities

and rearranging yields

$$\frac{1 - \pi_1 + \varepsilon}{\pi_1 - \varepsilon} > \frac{u(g(d')) - u(g(d))}{u(h(d)) - u(h(d'))} > \frac{1 - \pi_1 + \varepsilon'}{\pi_1 - \varepsilon'},$$

which is a contradiction since $(1 - \pi_1 + \varepsilon)/(\pi_1 - \varepsilon) < (1 - \pi_1 + \varepsilon')/(\pi_1 - \varepsilon')$.

Suppose that $\varepsilon \geq 0$. Then the agent is absolutely ambiguity averse and is more ambiguity-averse than the subjective expected utility maximizer with utility u and belief π. Thus, as noted in the text, since insurance is actuarially fair at belief π, full insurance is optimal, i.e., $d = 0$.

Now, suppose that $\varepsilon < 0$. For any $d < l_2$, the value of the schedule with deductible d equals $(\pi_1 - \varepsilon)u[w - \mathbb{E}_\pi l + (1 - \pi_1)d] + (1 - \pi_1 + \varepsilon)u[w - \mathbb{E}_\pi l - \pi_1 d]$. This expression is differentiable with respect to d and the derivative at $d = 0$ equals $[(\pi_1 - \varepsilon)(1 - \pi_1) - \pi_1(1 - \pi_1 + \varepsilon)]u'[w - \mathbb{E}_\pi l]$, which is strictly positive since $\varepsilon < 0$ and u is strictly increasing. Thus, the optimal deductible is strictly positive. $\qquad\square$

D Proofs for Section 4

D.1 Proof of Proposition 5

Below we fix the unique functional $I : \mathbb{R}^S \to \mathbb{R}$ associated with $\succsim$, as given by Lemma B.1. We begin with the following lemma:

Lemma D.1. *Suppose* $(\mathbb{P}, u)$ *is a DSEU representation of* $\succsim$. *Then* $\overline{\mathbb{P}} = \mathrm{cl}\{H_{\phi,\lambda} : \phi \in \mathbb{R}^S, \lambda \leq I(\phi)\}$.

Proof. First, take any $\phi \in \mathbb{R}^S$, $\lambda \in \mathbb{R}$ such that $\lambda \leq I(\phi)$. Since $(\mathbb{P}, u)$ represents $\succsim$, there exists $P \in \mathbb{P}$ such that $\min_{\mu \in P} \mu \cdot \phi = I(\phi)$. Thus, $P \subseteq H_{\phi,I(\phi)} \subseteq H_{\phi,\lambda}$, which implies $H_{\phi,\lambda} \in \overline{\mathbb{P}}$. This proves that $\overline{\mathbb{P}} \supseteq \mathrm{cl}\{H_{\phi,\lambda} : \phi \in \mathbb{R}^S, \lambda \leq I(\phi)\}$.

Conversely, consider any $\phi \in \mathbb{R}^S$, $\lambda \in \mathbb{R}$ such that there exists $P' \in \mathbb{P}$ with $P' \subseteq H_{\phi,\lambda}$. Since $(\mathbb{P}, u)$ represents $\succsim$, $I(\phi) \geq \min_{\mu \in P'} \mu \cdot \phi \geq \min_{\mu \in H_{\phi,\lambda}} \phi \cdot \mu$. Hence, $\lambda \leq I(\phi)$. This proves that $\overline{\mathbb{P}} \subseteq \mathrm{cl}\{H_{\phi,\lambda} : \phi \in \mathbb{R}^S, \lambda \leq I(\phi)\}$. $\qquad\square$

Proof of Proposition 5. Suppose first that $(\mathbb{P}', u')$ is another DSEU representation of $\succsim$. Then the fact that $\overline{\mathbb{P}} = \overline{\mathbb{P}'}$ is immediate from Lemma D.1 and the uniqueness of I. The proof that $u \approx u'$ is standard.

Conversely, suppose that $u \approx u'$ and $\overline{\mathbb{P}} = \overline{\mathbb{P}'}$. To show that $(\mathbb{P}', u')$ represents $\succsim$, it suffices to show that $\max_{P' \in \mathbb{P}'} \min_{\mu \in P'} \mu \cdot \phi = I(\phi)$ for all $\phi \in \mathbb{R}^S$. To prove this, observe first that since (by Lemma D.1) $H_{\phi,I(\phi)} \in \overline{\mathbb{P}} = \overline{\mathbb{P}'}$, there exist sequences of $P'_n \in \mathbb{P}'$ and half-spaces $H_n \supseteq$

P'_n with $H_n \to H_{\phi, I(\phi)}$. Then, for all ϕ, we have $\min_{\mu \in H_{\phi, I(\phi)}} \mu \cdot \phi = I(\phi) = \lim_n \min_{\mu \in H_n} \mu \cdot \phi$ and $\min_{\mu \in H_n} \mu \cdot \phi \leq \min_{\mu \in P'_n} \mu \cdot \phi$ for all n. This implies that $\max_{P' \in \mathbb{P}} \min_{\mu \in P'} \mu \cdot \phi \geq I(\phi)$. Suppose next that $\min_{\mu \in P''} \mu \cdot \phi - I(\phi) =: \varepsilon > 0$ for some $P'' \in \mathbb{P}'$. Then $H_{\phi, I(\phi)+\varepsilon} \supseteq P''$, which implies $H_{\phi, I(\phi)+\varepsilon} \in \overline{\mathbb{P}'}$. Since $\overline{\mathbb{P}'} = \overline{\mathbb{P}}$, this contradicts Lemma D.1.

Finally, note that the half-space closure of $\overline{\mathbb{P}}$ is $\overline{\mathbb{P}}$ itself. Thus, by the previous paragraph, $(\overline{\mathbb{P}}, u)$ is itself a DSEU representation of $\succsim$. $\qquad\square$

D.2 Proof of Proposition 6

For each preference $\succsim_i$, let utility u_i and functional I_i be as given by Lemma B.1. Note that $\succsim_1$ is more ambiguity-averse than $\succsim_2$ if and only if $u_1 \approx u_2$ and $I_1(\phi) \leq I_2(\phi)$ for all $\phi \in \mathbb{R}^S$. Thus, it suffices to show that $I_1(\phi) \leq I_2(\phi)$ for all ϕ if and only if $\overline{\mathbb{P}}_1 \subseteq \overline{\mathbb{P}}_2$.

Suppose first that $I_1(\phi) \leq I_2(\phi)$ for all ϕ. Then $\{H_{\phi, \lambda} : \phi \in \mathbb{R}^S, \lambda \leq I_1(\phi)\} \subseteq \{H_{\phi, \lambda} : \phi \in \mathbb{R}^S, \lambda \leq I_2(\phi)\}$. By Lemma D.1, this implies that $\overline{\mathbb{P}}_1 \subseteq \overline{\mathbb{P}}_2$.

Conversely, if $\overline{\mathbb{P}}_1 \subseteq \overline{\mathbb{P}}_2$, then $\max_{P \in \overline{\mathbb{P}}_1} \min_{\mu \in P} \mu \cdot \phi \leq \max_{P \in \overline{\mathbb{P}}_2} \min_{\mu \in P} \mu \cdot \phi$ for all ϕ. Since $(\overline{\mathbb{P}}_i, u_i)$ is a DSEU representation of $\succsim_i$ for $i = 1, 2$, this inequality means that $I_1(\phi) \leq I_2(\phi)$ for all ϕ. $\qquad\square$

References

ABDELLAOUI, M., A. BAILLON, L. PLACIDO, AND P. P. WAKKER (2011): "The rich domain of uncertainty: Source functions and their experimental implementation," *American Economic Review*, 101(2), 695–723.

AGUIAR, V., P. HJERTSTRAND, AND R. SERRANO (2020): "A Rationalization of the Weak Axiom of Revealed Preference," .

ALARY, D., C. GOLLIER, AND N. TREICH (2013): "The effect of ambiguity aversion on insurance and self-protection," *The Economic Journal*, 123(573), 1188–1202.

AMARANTE, M. (2009): "Foundations of neo-Bayesian statistics," *Journal of Economic Theory*, 144(5), 2146–2173.

ANANTANASUWONG, K., R. KOUWENBERG, O. S. MITCHELL, AND K. PEIJNENBERG (2019): "Ambiguity attitudes about investments: Evidence from the field," Working Paper.

ARROW, K. (1971): *Essays in the theory of risk-bearing.* Markam Chicago.

BAILLON, A., AND H. BLEICHRODT (2015): "Testing ambiguity models through the measurement of probabilities for gains and losses," *American Economic Journal: Microeconomics*, 7(2), 77–100.

BAILLON, A., Z. HUANG, A. SELIM, AND P. P. WAKKER (2018): "Measuring ambiguity attitudes for all (natural) events," *Econometrica*, 86(5), 1839–1858.

BAILLON, A., O. L'HARIDON, AND L. PLACIDO (2011): "Ambiguity models and the Machina paradoxes," *American Economic Review*, 101(4), 1547–60.

BEWLEY, T. F. (2002): "Knightian decision theory. Part I," *Decisions in Economics and Finance*, 25(2), 79–110.

BILLOT, A., A. CHATEAUNEUF, I. GILBOA, AND J.-M. TALLON (2000): "Sharing beliefs: between agreeing and disagreeing," *Econometrica*, 68(3), 685–694.

——— (2002): "Sharing beliefs and the absence of betting in the Choquet expected utility model," *Statistical Papers*, 43(1), 127–136.

BROCAS, I., AND J. D. CARRILLO (2008): "The brain as a hierarchical organization," *American Economic Review*, 98(4), 1312–46.

CAPPELLI, V., S. CERREIA-VIOGLIO, F. MACCHERONI, M. MARINACCI, AND S. MINARDI (2016): "Sources of uncertainty and subjective prices," Working paper.

CASTAGNOLI, E., G. CATTELAN, F. MACCHERONI, AND C. TEBALDI (in preparation): "Competitive risk measures," mimeo.

CERREIA-VIOGLIO, S., P. GHIRARDATO, F. MACCHERONI, M. MARINACCI, AND M. SINISCALCHI (2011): "Rational preferences under ambiguity," *Economic Theory*, 48(2-3), 341–375.

CERREIA-VIOGLIO, S., F. MACCHERONI, M. MARINACCI, AND L. MONTRUCCHIO (2011): "Uncertainty averse preferences," *Journal of Economic Theory*, 146(4), 1275–1330.

CHAMBERS, C. P. (2007): "Ordinal aggregation and quantiles," *Journal of Economic Theory*, 137(1), 416–431.

CHANDRASEKHER, M. (2019): "Source-dependent uncertainty aversion," Working paper.

CHATEAUNEUF, A., R.-A. DANA, AND J.-M. TALLON (2000): "Optimal risk-sharing rules and equilibria with Choquet-expected-utility," *Journal of Mathematical Economics*, 34(2), 191–214.

CHATEAUNEUF, A., AND J. H. FARO (2009): "Ambiguity through confidence functions," *Journal of Mathematical Economics*, 45(9-10), 535–558.

CHATEAUNEUF, A., AND J.-M. TALLON (2002): "Diversification, convex preferences and non-empty core in the Choquet expected utility model," *Economic Theory*, 19(3), 509–523.

CHEW, S. H., B. MIAO, AND S. ZHONG (2018): "Ellsberg meets Keynes at an urn," Working paper.

CHEW, S. H., AND J. S. SAGI (2008): "Small worlds: Modeling attitudes toward sources of uncertainty," *Journal of Economic Theory*, 139(1), 1–24.

CLARKE, F. H. (1990): *Optimization and nonsmooth analysis*, vol. 5. Siam.

COVAL, J. D., AND T. J. MOSKOWITZ (1999): "Home bias at home: Local equity preference in domestic portfolios," *The Journal of Finance*, 54(6), 2045–2073.

DILLENBERGER, D., AND U. SEGAL (2017): "Skewed noise," *Journal of Economic Theory*, 169, 344–364.

DIMMOCK, S. G., R. KOUWENBERG, O. S. MITCHELL, AND K. PEIJNENBURG (2015): "Estimating ambiguity preferences and perceptions in multiple prior models: Evidence from the field," *Journal of Risk and Uncertainty*, 51(3), 219–244.

DOW, J., AND S. R. WERLANG (1992): "Uncertainty aversion, risk aversion, and the optimal choice of portfolio," *Econometrica*, pp. 197–204.

DUBRA, J., F. MACCHERONI, AND E. A. OK (2004): "Expected utility theory without the completeness axiom," *Journal of Economic Theory*, 115(1), 118–133.

EICHBERGER, J., S. GRANT, D. KELSEY, AND G. A. KOSHEVOY (2011): "The α-MEU model: A comment," *Journal of Economic Theory*, 146(4), 1684–1698.

ELLSBERG, D. (2011): "Notes on the origin of the Ellsberg Urns (introduction to the symposium issue)," *Economic Theory*, 48, 221–227.

EPSTEIN, L. G. (1999): "A definition of uncertainty aversion," *Review of Economic Studies*, 66(3), 579–608.

ERGIN, H., AND F. GUL (2009): "A subjective theory of compound lotteries," *Journal of Economic Theory*, 144(3), 899–929.

FRENCH, K. R., AND J. M. POTERBA (1991): "Investor diversification and international equity markets," *American Economic Review*, 81(2), 222–226.

FRICK, M., R. IIJIMA, AND Y. LE YAOUANQ (2019): "Boolean representations of preferences under ambiguity," Cowles Foundation Discussion Paper.

FUDENBERG, D., AND D. K. LEVINE (2006): "A dual-self model of impulse control," *American Economic Review*, 96(5), 1449–1476.

GHIRARDATO, P., F. MACCHERONI, AND M. MARINACCI (2004): "Differentiating ambiguity and ambiguity attitude," *Journal of Economic Theory*, 118(2), 133–173.

——— (2005): "Certainty independence and the separation of utility and beliefs," *Journal of Economic Theory*, 120(1), 129–136.

GHIRARDATO, P., AND M. MARINACCI (2002): "Ambiguity made precise: A comparative foundation," *Journal of Economic Theory*, 102(2), 251–289.

GHIRARDATO, P., AND M. SINISCALCHI (2012): "Ambiguity in the small and in the large," *Econometrica*, 80(6), 2827–2847.

——— (2018): "Risk sharing in the small and in the large," *Journal of Economic Theory*, 175, 730–765.

GILBOA, I., F. MACCHERONI, M. MARINACCI, AND D. SCHMEIDLER (2010): "Objective and subjective rationality in a multiple prior model," *Econometrica*, 78(2), 755–770.

GILBOA, I., AND M. MARINACCI (2016): "Ambiguity and the Bayesian paradigm," in *Readings in Formal Epistemology*, pp. 385–439. Springer.

GILBOA, I., AND D. SCHMEIDLER (1989): "Maxmin expected utility with non-unique prior," *Journal of Mathematical Economics*, 18(2), 141–153.

GRANT, S., AND B. POLAK (2013): "Mean-dispersion preferences and constant absolute uncertainty aversion," *Journal of Economic Theory*, 148(4), 1361–1398.

GUL, F., AND W. PESENDORFER (2015): "Hurwicz expected utility and subjective sources," *Journal of Economic Theory*, 159, 465–488.

HARA, K., E. A. OK, AND G. RIELLA (2019): "Coalitional expected multi-utility theory," *Econometrica*, 87(3), 933–980.

HART, S., S. MODICA, AND D. SCHMEIDLER (1994): "A neo^2 Bayesian foundation of the maxmin value for two-person zero-sum games," *International Journal of Game Theory*, 23(4), 347–358.

HEATH, C., AND A. TVERSKY (1991): "Preference and belief: Ambiguity and competence in choice under uncertainty," *Journal of Risk and Uncertainty*, 4(1), 5–28.

HÖLMSTROM, B. (1979): "Moral hazard and observability," *The Bell journal of economics*, pp. 74–91.

KE, S., AND Q. ZHANG (2019): "Randomization and ambiguity aversion," Working paper.

KEPPE, H.-J., AND M. WEBER (1995): "Judged knowledge and ambiguity aversion," *Theory and Decision*, 39(1), 51–77.

KLIBANOFF, P., M. MARINACCI, AND S. MUKERJI (2005): "A smooth model of decision making under ambiguity," *Econometrica*, 73(6), 1849–1892.

KLIBANOFF, P., S. MUKERJI, AND K. SEO (2014): "Perceived ambiguity and relevant measures," *Econometrica*, 82(5), 1945–1978.

KOCHER, M. G., A. M. LAHNO, AND S. T. TRAUTMANN (2018): "Ambiguity aversion is not universal," *European Economic Review*, 101, 268–283.

KOPYLOV, I. (2019): "Multiple priors and comparative ignorance," Working paper.

LEHRER, E., AND R. TEPER (2011): "Justifiable preferences," *Journal of Economic Theory*, 146(2), 762–774.

MACCHERONI, F., M. MARINACCI, AND A. RUSTICHINI (2006): "Ambiguity aversion, robustness, and the variational representation of preferences," *Econometrica*, 74(6), 1447–1498.

MACHINA, M. J. (2009): "Risk, ambiguity, and the rank-dependence axioms," *American Economic Review*, 99(1), 385–92.

MONONEN, L. (2020): "State-dependent utility and ambiguity," Working paper.

NASCIMENTO, L., AND G. RIELLA (2011): "A class of incomplete and ambiguity averse preferences," *Journal of Economic Theory*, 146(2), 728–750.

NAU, R. F. (2006): "Uncertainty aversion with second-order utilities and probabilities," *Management Science*, 52(1), 136–145.

NISHIMURA, H., AND E. A. OK (2016): "Utility representation of an incomplete and nontransitive preference relation," *Journal of Economic Theory*, 166, 164–185.

OVCHINNIKOV, S. (2001): "Boolean representation of manifolds and functions," *Journal of Mathematical Analysis and Applications*, 263, 294–300.

RIGOTTI, L., C. SHANNON, AND T. STRZALECKI (2008): "Subjective beliefs and ex ante trade," *Econometrica*, 76(5), 1167–1190.

RUBINOV, A., AND Z. DZALILOV (2002): "Abstract convexity of positively homogeneous functions," *Journal of Statistics and Management Systems*, 5(1-3), 1–20.

SAPONARA, N. (2020): "Revealed reasoning," *Journal of Economic Theory*, forthcoming.

SARVER, T. (2018): "Dynamic mixture-averse preferences," *Econometrica*, 86(4), 1347–1382.

SCHMEIDLER, D. (1989): "Subjective probability and expected utility without additivity," *Econometrica*, pp. 571–587.

SEGAL, U. (1987): "The Ellsberg paradox and risk aversion: An anticipated utility approach," *International Economic Review*, pp. 175–202.

Siniscalchi, M. (2009): "Vector expected utility and attitudes toward variation," *Econometrica*, 77(3), 801–855.

Skiadas, C. (2013): "Scale-invariant uncertainty-averse preferences and source-dependent constant relative risk aversion," *Theoretical Economics*, 8(1), 59–93.

Strzalecki, T. (2011): "Axiomatic foundations of multiplier preferences," *Econometrica*, 79(1), 47–73.

Thaler, R. H., and H. M. Shefrin (1981): "An economic theory of self-control," *Journal of Political Economy*, 89(2), 392–406.

Trautmann, S., and P. P. Wakker (2018): "Making the Anscombe-Aumann approach to ambiguity suitable for descriptive applications," *Journal of Risk and Uncertainty*, 56, 83–116.

Trautmann, S. T., and G. van de Kuilen (2015): "Ambiguity attitudes," *The Wiley Blackwell Handbook of Judgment and Decision Making*, 2, 89–116.

Tversky, A., and C. R. Fox (1995): "Weighing risk and uncertainty.," *Psychological review*, 102(2), 269.

Supplementary Appendix to "Dual-self representations of ambiguity preferences"

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This supplementary appendix is organized as follows. Appendix S.1 presents the proofs for the generalizations of DSEU considered in Section 4.3. Appendix S.2 considers the representation obtained by inverting the order of moves of Optimism and Pessimism. Appendix S.3 presents an incompatibility result for source dependence under Klibanoff, Marinacci, and Mukerji's (2005) smooth model.

S.1 Proofs for Section 4.3

S.1.1 Proof of Theorem 3

We will invoke the following result from MMR:

Lemma S.1.1 (Lemma 28 in MMR). *Preference $\succsim$ satisfies Axioms 1–4 and Axiom 8 if and only if there exists a nonconstant affine function $u : \Delta(Z) \to \mathbb{R}$ with $U := (u(\Delta(Z)))^S$ and a normalized niveloid $I : U \to \mathbb{R}$ such that $I \circ u$ represents $\succsim$.*

Recall that functional $I : U \to \mathbb{R}$ is a *niveloid* if $I(\phi) - I(\psi) \leq \max_s(\phi_s - \psi_s)$ for all $\phi, \psi \in U$. Lemma 25 in MMR shows that I is a niveloid if and only if it is monotonic and constant-additive.

Based on this result, the necessity direction of Theorem 3 is standard. We now prove the sufficiency direction. Suppose $\succsim$ satisfies Axioms 1–4 and Axiom 8. Let I, u, and U be as given by Lemma S.1.1. Since I is a niveloid, it is 1-Lipschitz. Hence, Lemma A.1 yields a subset $\hat{U} \subseteq \text{int}U$ with $U \setminus \hat{U}$ of Lebesgue measure 0 such that I is differentiable on $\hat{U}$. Define $\mu_\psi := \nabla I(\psi)$ and $w_\psi := I(\psi) - \nabla I(\psi) \cdot \psi$ for each $\psi \in \hat{U}$. By Lemma A.4 and the fact that niveloids are monotonic and constant-additive, $\mu_\psi \in \Delta(S)$ for all $\psi \in \hat{U}$. For each $\psi \in U$, define

$$D_\psi := \{(\mu, w) \in \Delta(S) \times \mathbb{R} : \mu \cdot \psi + w \geq I(\psi)\} \cap \overline{\text{co}}\{(\mu_\xi, w_\xi) : \xi \in \hat{U}\},$$

and let $\mathbb{D} := \{D_\psi : \psi \in U\}$. The following lemma implies that each D_ψ is nonempty; note also that it is closed, convex, and bounded below.

Lemma S.1.2. *For every $\phi, \psi \in U$, $\min_{(\mu,w) \in D_\psi} \mu \cdot \phi + w \leq I(\phi)$ with equality if $\phi = \psi$.*

Proof. First, consider any $\phi, \psi \in \hat{U}$. Let $K_\psi := \{\xi \in \hat{U} : \mu_\xi \cdot \psi + w_\xi \geq I(\psi)\}$ be as in Lemma A.6. Note that $D_\psi = \overline{\text{co}}\{(\mu_\xi, w_\xi) : \xi \in K_\psi\}$, so that

$$\inf_{\xi \in K_\psi} \mu_\xi \cdot \phi + w_\xi = \min_{(\mu,w) \in D_\psi} \mu \cdot \phi + w,$$

where the minimum is attained as D_ψ is closed and bounded below. Thus, Lemma A.6 implies that

$$\min_{(\mu,w) \in D_\psi} \mu \cdot \phi + w \leq I(\phi), \tag{26}$$

where, by definition of D_ψ, (26) holds with equality if $\psi = \phi$.

Next, consider any $\phi, \psi \in U$. Take sequences $\phi_n \to \phi$, $\psi_n \to \psi$ such that $\phi_n, \psi_n \in \hat{U}$ for each n, where we choose $\phi_n = \psi_n$ if $\phi = \psi$. For each n, the previous paragraph yields some $(\mu_n, w_n) \in D_{\psi_n}$ such that $\mu_n \cdot \phi_n + w_n = \min_{(\mu,w) \in D_{\psi_n}} \mu \cdot \phi_n + w \leq I(\phi_n)$, with equality if $\phi = \psi$. Thus, for each n, we have $I(\psi_n) - \mu_n \cdot \psi_n \leq w_n \leq I(\phi_n) - \mu_n \cdot \phi_n$. Since $\phi_n \to \phi$, $\psi_n \to \psi$, and I is continuous, this implies that sequence (w_n) is bounded. Thus, up to restricting to a suitable subsequence, we can assume that $(\mu_n, w_n) \to (\mu_\infty, w_\infty)$ for some $(\mu_\infty, w_\infty) \in \Delta(S) \times \mathbb{R}$. Then $(\mu_\infty, w_\infty) \in D_\psi$ and $\mu_\infty \cdot \phi + w_\infty \leq I(\phi)$ by continuity of I, with equality if $\phi = \psi$. Thus, $\min_{(\mu,w) \in D_\psi} \mu \cdot \phi + w = \inf_{(\mu,w) \in D_\psi} \mu \cdot \phi + w \leq I(\phi)$, with equality if $\phi = \psi$, where the minimum is attained since D_ψ is closed and bounded below. $\qquad\square$

Finally, we obtain a dual-self variational representation of $\succsim$ as follows. For each $D \in \mathbb{D}$, define $c_D : \Delta(S) \to \mathbb{R} \cup \{\infty\}$ by $c_D(\mu) := \inf\{w \in \mathbb{R} : (\mu, w) \in D\}$ for each $\mu \in \Delta(S)$, where by convention the infimum of the empty set is ∞. Note that c_D is convex for all D by convexity of D. Moreover, for all $\phi \in U$, $\min_{(\mu,w) \in D} \mu \cdot \phi + w = \min_{\mu \in \Delta(S)} \mu \cdot \phi + c_D(\mu)$. Thus, Lemma S.1.2 implies

$$I(\phi) = \max_{D \in \mathbb{D}} \min_{\mu \in \Delta(S)} \mu \cdot \phi + c_D(\mu) \tag{27}$$

for all $\phi \in U$. Since I is normalized, applying (27) to any constant vector $\underline{a} \in U$, yields $I(\underline{a}) = a + \max_{D \in \mathbb{D}} \min_{\mu \in \Delta(S)} c_D(\mu) = a$. Hence, $\mathbb{C}^* := \{c_D : D \in \mathbb{D}\}$ satisfies $\max_{c \in \mathbb{C}^*} \min_{\mu \in \Delta(S)} c(\mu) = 0$ and $(\mathbb{C}^*, u)$ is a dual-self variational representation of $\succsim$ by Lemma S.1.1. $\qquad\square$

Remark 3. We note that our characterization of the set of relevant priors under DSEU generalizes to the dual-self variational model. Specifically, let $\text{dom}(c) := \{\mu : c(\mu) \in \mathbb{R}\}$ denote the effective domain of any cost function. Then there exists a unique closed, convex set C such that $C \subseteq \overline{\text{co}}\left(\bigcup_{c \in \mathbb{C}} \text{dom}(c)\right)$ for all dualf-self variational representations of $\succsim$, with equality for the representation $\mathbb{C}^*$ we constructed in the proof of Theorem 3. Moreover, it can again be shown that C is the Bewley set of the unambigous preference $\succsim^*$. The argument

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relies on the observation that $C = \overline{\text{co}} \left(\bigcup_{\phi \in \text{int} U} \partial I(\phi) \right)$, where I is the utility act functional obtained in the proof of Theorem 3 and U its domain. Details are available on request. ▲

S.1.2 Proof of Theorem 4

The following result follows from a minor modification of the proof of Lemma 57 in CMMM:

Lemma S.1.3. *Preference $\succsim$ satisfies Axioms 1–4 and 9 if and only if there exists a non-constant affine function $u : \Delta(Z) \to \mathbb{R}$ with $U := (u(\Delta(Z)))^S$ and a monotonic, normalized and continuous functional $I : U \to \mathbb{R}$ such that $I \circ u$ represents $\succsim$.*

Based on this result, the necessity direction of Theorem 4 is standard. We now prove the sufficiency direction. Suppose $\succsim$ satisfies Axioms 1–4 and 9. Let I, u, and U be as given by Lemma S.1.3. Define $D_\psi := \{ (\mu, I(\psi) - \mu \cdot \psi) \in \mathbb{R}_+^S \times \mathbb{R} : \mu \in \mathbb{R}_+^S \}$ for each $\psi \in U$. Note that D_ψ is nonempty and convex. Let $I_\psi(\phi) := \inf_{(\mu,w) \in D_\psi} \mu \cdot \phi + w$ for each $\phi, \psi \in U$.

Take any $\phi, \psi \in U$. Observe that

$$I_\psi(\phi) = \inf_{\alpha > 0, s \in S} I(\psi) + \alpha(\phi_s - \psi_s) = \begin{cases} I(\psi) \text{ if } \phi \geq \psi \\ -\infty \text{ if } \phi \not\geq \psi \end{cases}$$

Thus, $I(\phi) \geq I_\psi(\phi)$ by monotonicity of I, with equality if $\phi = \psi$. That is, for each $\phi \in U$,

$$I(\phi) = \max_{\psi \in U} I_\psi(\phi). \tag{28}$$

For each $\psi \in U$, define a function $G_\psi : \mathbb{R} \times \Delta(S) \to \mathbb{R} \cup \{\infty\}$ by

$$G_\psi(t, \mu) = \sup\{ I_\psi(\xi) : \xi \in U, \xi \cdot \mu \leq t \}$$

for each (t, μ). The map is quasi-convex (Lemma 31 in CMMM) and increasing in t.

Lemma S.1.4. *We have $I_\psi(\phi) = \inf_{\mu \in \Delta(S)} G_\psi(\mu \cdot \phi, \mu)$ for each $\phi, \psi \in U$.*

Proof. Observe that RHS $= \inf_{\mu \in \Delta(S)} \sup\{ I_\psi(\xi) : \xi \cdot \mu \leq \phi \cdot \mu \}$. To see that LHS $\leq$ RHS, observe that $I_\psi(\phi) \leq \sup\{ I_\psi(\xi) : \xi \cdot \mu \leq \phi \cdot \mu \}$ holds for any $\mu \in \Delta(S)$. To see that LHS $\geq$ RHS, note first that if $\phi \geq \psi$ then LHS $= I(\psi)$ and RHS $\in \{ I(\psi), -\infty \}$, so the inequality clearly holds. If $\phi \not\geq \psi$ then $\phi_s < \psi_s$ for some $s \in S$. Thus, by taking $\mu = \delta_s$, any ξ with $\xi \cdot \mu \leq \phi \cdot \mu$ satisfies $\xi_s \leq \phi_s$, which implies $\xi \not\geq \psi$, whence $I_\psi(\xi) = -\infty$. □

Setting $\mathbb{G} = \{ G_\phi : \phi \in U \}$, Lemma S.1.4 and (28) ensure that the functional W given by (16) represents $\succsim$ and is continuous. Finally, note that since I is normalized, we have $a = I(\underline{a}) = \max_{G \in \mathbb{G}} \inf_{\mu \in \Delta(S)} G(a, \mu)$ for any $a \in \mathbb{R}$, as required. □

3

S.2 Minmax DSEU representation

While DSEU assumes that Optimism plays first and Pessimism plays second, this is equivalent to a model with the opposite order of moves. We omit all proofs for this section, as they can be obtained as minor modifications of the original proofs for DSEU.

Theorem S.2.1. *Preference $\succsim$ satisfies Axioms 1–5 if and only if $\succsim$ admits a minmax DSEU representation, i.e., there exists a belief-set collection $\mathbb{Q}$ and a nonconstant affine utility $u : \Delta(Z) \to \mathbb{R}$ such that*

$$W(f) = \min_{Q \in \mathbb{Q}} \max_{\mu \in Q} \mathbb{E}_\mu[u(f)]$$

represents $\succsim$.

Our construction of the maxmin DSEU representation considered in the text uses the belief-set collection $\mathbb{P}^* = \mathrm{cl}\{P_\phi^* : \phi \in \mathbb{R}^S\}$ with $P_\phi^* := \{\mu \in \partial I(\underline{0}) : \mu \cdot \phi \geq I(\phi)\}$. Analogously, it can be shown that the belief-set collection $\mathbb{Q}^* := \mathrm{cl}\{Q_\phi^* : \phi \in \mathbb{R}^S\}$ with $Q_\phi^* := \{\mu \in \partial I(\underline{0}) : \mu \cdot \phi \leq I(\phi)\}$ yields a minmax DSEU representation. Paralleling Section 2.3, it is straightforward to show that $C := \partial I(\underline{0})$ again corresponds to the smallest set of priors that is contained in $\overline{\mathrm{co}} \bigcup_{Q \in \mathbb{Q}} Q$ for all minmax DSEU representations $\mathbb{Q}$ of $\succsim$, with equality for representation $\mathbb{Q}^*$.

While the different notions of ambiguity aversion are most conveniently characterized using the maxmin DSEU representation (cf. Theorem 2), the minmax DSEU representation is useful for characterizing their ambiguity-seeking counterparts. Axioms 10 and 11 and Theorem S.2.2 below provide the analogs of Axioms 6 and 7 and Theorem 2, respectively.

Axiom 10 (Uncertainty Seeking). *If $f, g \in \mathcal{F}$ with $f \sim g$, then $\dfrac{1}{2}f + \dfrac{1}{2}g \precsim f$.*

Axiom 11 (k-Ambiguity Seeking). *For all $f_1, ..., f_k \in \mathcal{F}$ with $f_1 \sim f_2 \sim \cdots \sim f_k$ and any $p \in \Delta(Z)$,*

$$\frac{1}{k}f_1 + \cdots + \frac{1}{k}f_k = p \Rightarrow p \precsim f_1.$$

We say that $\succsim$ is *absolutely ambiguity-seeking* if there exists a nondegenerate subjective expected utility preference that is more ambiguity-averse than $\succsim$. Analogous to Lemma 1, this is characterized by ∞-*ambiguity seeking*, i.e., k-ambiguity seeking for all k.

Theorem S.2.2. *Suppose that $\succsim$ admits a minmax DSEU representation $(\mathbb{Q}, u)$. Then:*

1. $\succsim$ satisfies uncertainty seeking if and only if $\displaystyle\bigcap_{Q \in \mathbb{Q}} Q = C$;

2. $\succsim$ *is absolutely ambiguity-seeking if and only if* $\bigcap\limits_{Q\in\mathbb{Q}} Q \neq \emptyset$;

3. $\succsim$ *satisfies k-ambiguity seeking if and only if* $\bigcap_{i=1,\cdots,k} Q_i \neq \emptyset$ *for all* $Q_1,\cdots,Q_k \in \mathbb{Q}$.

S.3   Source dependence and the smooth model

Recall that under Klibanoff, Marinacci, and Mukerji's (2005) (henceforth, KMM's) smooth model, $\succsim$ is represented by the functional

$$W(f) = \int \phi(u(f)\cdot\mu)\,d\nu(\mu), \tag{29}$$

for some Borel probability measure $\nu \in \Delta(\Delta(S))$ over beliefs, nonconstant affine $u : \Delta(Z) \to \mathbb{R}$, and strictly increasing $\phi : u(Z) \to \mathbb{R}$. For expositional simplicity, we consider $Z = [0,1]$. Assume that u is strictly increasing and continuous on Z with $u(0) = 0$, and that ϕ is twice continuously differentiable with $\phi'(0), \phi''(0) \neq 0$.

Analogous to Corollary 4 for the α-MEU model, the following claim establishes a sense in which the smooth model is incompatible with source-dependent negative and positive ambiguity attitudes:

Claim 1. *Suppose that* $\succsim$ *admits a representation (29). Then there do not exist events* $E, F, G \subseteq S$ *such that for all* $x > 0$,

$$x_E 0 \succ x_F 0 \succ x_G 0 \quad and \quad x_{E^c} 0 \succ x_{F^c} 0 \succ x_{G^c} 0 \tag{30}$$

and such that $\mu(F)$ *is constant across all* μ *in the support of* ν.[26]

Proof. Suppose toward a contradiction that such events E, F, G exist. For each event $A \subseteq S$ and $\Delta \in [0, u(1)]$, let

$$W_A(\Delta) := \int \phi(\mu(A)\Delta)\,d\nu(\mu).$$

Then $W(x_A 0) = W_A(u(x))$ for all $x > 0$. Thus, (30) implies that, for all $\Delta \in [0, u(1)]$,

$$W_E(\Delta) > W_F(\Delta) > W_G(\Delta) \quad and \quad W_{E^c}(\Delta) > W_{F^c}(\Delta) > W_{G^c}(\Delta). \tag{31}$$

---

[26] See Theorem 3 in KMM for a behavioral characterization of such unambiguous events F.

Observe that, for each A, we have $W_A(0) = \phi(0)$, and

$$
\begin{aligned}
\frac{\partial}{\partial \Delta} W_A(\Delta) &= \int \phi'(\mu(A)\Delta)\mu(A)\,d\nu(\mu) \\
&= \phi'(0) \int \mu(A)\,d\nu(\mu) \text{ at } \Delta = 0, \\
\frac{\partial^2}{\partial \Delta^2} W_A(\Delta) &= \int \phi''(\mu(A)\Delta)\mu(A)^2\,d\nu(\mu) \\
&= \phi''(0) \int \mu(A)^2\,d\nu(\mu) \text{ at } \Delta = 0.
\end{aligned}
$$

Let α be the constant such that $\alpha = \mu(F)$ for all μ in the support of ν. Then, performing a first-order Taylor approximation, the first inequalities in (31) imply $\int \mu(E)\,d\nu(\mu) \geq \alpha \geq \int \mu(G)\,d\nu(\mu)$. Likewise, the second inequalities in (31) imply $\int \mu(E^c)\,d\nu(\mu) \geq 1 - \alpha \geq \int \mu(G^c)\,d\nu(\mu)$. Thus,

$$
\int \mu(E)\,d\nu(\mu) = \alpha = \int \mu(G)\,d\nu(\mu). \tag{32}
$$

Note that it is not the case that $\mu(E) = \alpha$ for ν-almost every μ, as this would imply $W_E(\Delta) = W_F(\Delta)$, contradicting $W_E(\Delta) > W_F(\Delta)$. Likewise, it is not the case that $\mu(G) = \alpha$ for ν-almost every μ, as this would contradict $W_F(\Delta) > W_G(\Delta)$. Thus, by Jensen's inequality

$$
\int \mu(E)^2\,d\nu(\mu), \int \mu(G)^2\,d\nu(\mu) > \alpha^2.
$$

Hence, performing a second-order Taylor approximation, $W_E(\Delta) > W_F(\Delta)$ and (32) implies that $\phi''(0) > 0$. Likewise, $W_F(\Delta) > W_G(\Delta)$ and (32) implies that $\phi''(0) < 0$. This is a contradiction. $\qquad\square$