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Nonlinear Pricing with Finite Information*

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Abstract

We analyze nonlinear pricing with finite information. A seller offers a menu to a continuum of buyers with a continuum of possible valuations. The menu is limited to offering a finite number of choices representing a finite communication capacity between buyer and seller.

We identify necessary conditions that the optimal finite menu must satisfy, either for the socially efficient or for the revenue-maximizing mechanism. These conditions require that information be bundled, or "quantized" optimally. We show that the loss resulting from using the n-item menu converges to zero at a rate proportional to $1/n^2$.

We extend our model to a multi-product environment where each buyer has preferences over a d dimensional variety of goods. The seller is limited to offering a finite number n of d-dimensional choices. By using repeated scalar quantization, we show that the losses resulting from using the d-dimensional n-class menu converge to zero at a rate proportional to $d/n^{2/d}$. We introduce vector quantization and establish that the losses due to finite menus are significantly reduced by offering optimally chosen bundles.

Keywords: Mechanism Design, Nonlinear Pricing, Multi-Dimension, Multi-Product, Private Information, Limited Information, Quantization, Information Theory.

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1 Introduction

The theory of mechanism design addresses a wide set of questions, ranging from the design of markets and exchanges to the design of constitutions and political institutions. A central result in the theory of mechanism design is the "revelation principle" which establishes that if an allocation can be implemented incentive compatible in any mechanism, then it can be truthfully implemented in the direct revelation mechanism, where every agent reports his private information, his type, truthfully. Yet, when the amount of private information (the type space) of the agents is large, the direct revelation mechanism requires the agents to have abundant capacity to communicate with the principal, and the principal to have abundant capacity to process information. By contrast, the objective of this paper is to study the performance of optimal mechanisms, when the agents can communicate only limited information and/or when the principal can process only limited information. We pursue the analysis in the context of a representative, but suitably tractable, mechanism design environment, namely the canonical problem of nonlinear pricing. Here the principal, the seller, is offering a variety of choices to the agent, the buyer, who has private information about his willingness-to-pay (preference or type) for the product.

The distinct point of view, relative to the seminal analysis by Mussa and Rosen (1978) and Maskin and Riley (1984), resides in the fact that the information conveyed by the agents, and subsequently the menu of possible choices offered by the seller, is finite, rather than uncountable as in the earlier analysis. The limits to information may arise for various, direct or indirect, reasons. On the demand side, it may be too difficult or too complex for the buyer to communicate his exact preferences and resulting willingness to pay to the seller. On the supply side, it may be too time-consuming for the seller to process the fine detail of the consumer's preferences, or to identify the consumer's preferences across many goods with close attributes and only subtle differences.

Our analysis adopts a linear-quadratic specification (analogous to that of Mussa and Rosen (1978) and Maskin and Riley (1984)) in which the consumer's gross utility is the product of his willingness-to-pay (his type θ) and the consumed quantity (or quality) q of the product, whereas the cost of production is quadratic in the quantity (quality). For this important case, we reveal an interesting connection between the problem of optimal nonlinear pricing with limited information to the problem of optimally quantizing a source signal by using a finite number of representation levels in information theory. In our setting, the socially efficient quantity (quality) q for a customer should be equated to his valuation θ if a continuum of choices were available. In the case where a finite number of choices are accessible, q can take on only some values. If we see θ as the source signal and q as the representation level, then the total social welfare can be written in terms of the mean square error between the source signal and the representation signal. Given this, the social welfare maximization problem can be characterized by the Lloyd-Max optimality conditions, a well-established result in the theory of quantization. Furthermore, we can extend the analysis to the revenue maximization problem, after replacing the customer's true valuation by the corresponding virtual valuation. We estimate the welfare and revenue loss resulting from the use of a finite n-item menu (relative to the continuum menu). In particular, we characterize the rate of convergence for the welfare and revenue loss as a function of n. We examine this problem first for a given distribution on the customer's type, and then over all possible type distributions with finite support. We establish that the maximum welfare loss and the maximum revenue loss shrink towards zero at the rate proportional to $1/n^2$. To our best knowledge, this is the first time that quantization theory has been applied to solve the problem of mechanism design with limited information in economics.

Our approach extends naturally, via vector quantization, to the multi-dimensional nonlinear pricing problem where the seller is offering a variety of heterogeneous products to the buyer who has private information about his preferences (types) for these products characterized by a vector. Under the linear-quadratic specification in multiple dimensions, if we view the private information (the preference or type vector) as the signal vector and his choice (quantity or quality vector) as the representation vector, the social welfare maximization problem and the revenue maximization problem can be characterized by the Lloyd-Max optimality conditions for vector quantization. We estimate the welfare and revenue loss resulting from the use of d-dimensional finite menu with n choices. We establish an upper bound on the welfare loss and the revenue loss by using repeated scalar quantization where we simply partition the type space with orthotopes (Cartesian product of intervals in multiple dimensions), and treat each dimension independently. However, this method is not consistent with the Lloyd-Max conditions in general, and thus is not optimal. Lookabaugh and Gray (1989) provided a significant decomposition theorem, which indicates that we can considerably improve the upper bound by using more subtle vector quantization methods to design the multi-product finite menus over the entire type space. This improvement results from the vector quantization gain which consists of three components: space-filling advantage, shape advantage, and dependence advantage. Most notably, even in the extreme case when the types are distributed independently and uniformly across all dimensions, the vector quantization method can still reduce the welfare loss and the revenue loss, due to the space-filling advantage. This is the main reason why we bundle the consumer's preferences over multiple goods as a vector, instead of viewing them separately as independent types. We then establish the vector-quantization-based upper bound and the lower bounds on the welfare loss and the revenue loss.

The role of limited information in mechanism design has recently attracted increased attention. McAfee (2002) phrases the priority rationing problem as a two-sided matching problem (between consumer and services) and shows that a binary priority contract ("coarse matching") can already achieve at least half of the social welfare that could be generated by a continuum of priorities. Hoppe, Moldovanu, and Ozdenoren (2010) extend the matching analysis and explicitly consider monetary transfers between the agents. In particular, they present lower bounds on the revenue which can be achieved with specific, not necessarily optimal, binary contracts. By contrast, Madarasz and Prat (2010) suggest a specific allocation, the "profit-participation" mechanism to establish approximation results, rather than finite optimality results, in the nonlinear pricing environment. While the above contributions are concerned with single agent environments, there have been a number of contributions to multi-agent mechanisms, specifically single-item auctions among many bidders. Blumrosen, Nisan, and Segal (2007) consider the effect of restricted communication in auctions with either two agents or binary messages for every agent. Kos (2012) generalizes the analysis by allowing for a finite number of messages and agents. In turn, their equilibrium characterization in terms of partitions shares features with the optimal information structures in auctions as derived by Bergemann and Pesendorfer (2007).

Closer to our approach is Wilson (1989) which considers the impact of a finite number of priority classes on the efficient rationing of services. His analysis is less concerned with the optimal priority ranking for a given finite class, and more with the approximation properties of the finite priority classes. Wilson (1989) showed that the social welfare loss due to the use of a finite number of priority classes converges to zero at a rate no faster than $1/n^2$, where n is the number of classes. The analysis in Wilson (1989), however, is limited to one-dimensional social welfare maximization, and is not easily generalizable to the multi-dimensional social welfare maximization problem or the revenue maximization problem. The latter problems have remained open in general. In earlier and preliminary work, Bergemann, Shen, Xu, and Yeh (2012a), we suggested the use of the quantization technique to obtain upper and lower bounds on worst-case welfare and revenue loss. In an extension, Bergemann, Shen, Xu, and Yeh (2012b), we suggested that the welfare analysis may extend to the multi-dimensional welfare maximization problem. The present contribution represents the first systematic and comprehensive solution to these problems in many dimensions.

Even in the absence of communication constraints, the multi-dimensional mechanism design does not represent a trivial generalization of its one-dimensional counterpart. In many environments of interest, the preference of an individual agent cannot be summarized by a mere scalar but is more suitably represented as a vector. A real-life example would be a customer who has to make his choices in a supermarket where a large variety of commodities are available. Hence, designing a smart pricing strategy (e.g., product bundling by offering a combination of several distinct products for joint sale rather than selling each item separately) is of first-order concern in practice. In this respect, Wilson (1993) and Armstrong (1996) are two notable early contributions with explicit solutions to specific multi-dimensional screening problems. Rochet and Chone (1998) developed a systematic approach, coined the dual approach, to a general class of environments and pointed to the prevalence of bunching (agents with different type profile making the same choices). We refer readers to Rochet and Stole (2003) for a detailed survey of multi-dimensional screening problems.

The rest of the paper is organized as follows. We introduce the basic nonlinear pricing model in the next section. In Section 3 and 4, we establish the link to the quantization problem in information theory. With this perspective, we introduce the Lloyd-Max conditions that the optimal finite menu have to satisfy. By designing a sequence of specific menus, we estimate the convergence rate of the losses due to information constraints in finite menus. In Section 5, we generalize our approach to the multi-product environment. Using the repeated scalar quantization method, we provide an upper bound which is helpful for estimating the convergence rate of the losses. We then introduce the vector quantization gain and the decomposition theorem, and derive the vector-quantization-based upper bound and the lower bounds on the losses. We present some examples to illustrate the benefit of the multi-product pricing. In Section 6, we conclude with a brief summary and some open issues for our future research.

2 Model

We consider a monopolist facing a continuum of heterogeneous consumers. Each consumer is characterized by a quasi-linear utility function:

$$U(\theta, q) = u(\theta, q) - t(q),$$

where q is the quantity (or quality) of his consumption purchased from the monopolist, θ measures his particular taste for this good (called his "type"), and t is the monetary transfer. The monopoly seller offers q units of the product at a cost $c(q)$, resulting in the revenue

$$\Pi(q, t) = t(q) - c(q).$$

We assume that $\partial u/\partial q > 0, \partial u/\partial \theta > 0$ and $\partial^2 u/\partial q \partial \theta > 0$. The distribution function of θ is given by F with the support $\left[0, \bar{\theta}\right]$, where $0 < \bar{\theta} \leq \infty$. It is assumed that F is commonly known and that θ has finite first and second moments: $\mathbb{E}[\theta] < \infty, \mathbb{E}\left[\theta^2\right] < \infty$.

Throughout the paper, we assume that:

$$u(\theta, q) = \theta q, \quad c(q) = \frac{1}{2} q^2,$$

i.e., a multiplicative, separable utility and a quadratic cost function. This setting, usually called the linear-quadratic model, has been extensively used in the literature (see Mussa and Rosen (1978)). In this case, the type θ represents the willingness to pay for an additional unit of the object. With the quadratic cost function, the socially efficient quantity (quality) for a consumer is equal to willingness to pay, his type θ.

3 Welfare Maximization

We first consider the social welfare maximization problem. In the presence of private information, it is well known that the socially efficient allocation can be implemented by a direct mechanism $(q(\theta), t(\theta))$, where the transfers $t(\theta)$ represent the (social) cost of providing the object. Thus, in the efficient direct mechanism, the consumers are offered a menu $\{q(\theta)\}_{\theta \in [0, \bar{\theta}]}$ in which the consumer with type θ is allocated the quantity (quality) $q(\theta)$. This efficient mechanism, as a special case of the Vickrey-Clark-Groves mechanism, satisfies two sets of constraints, namely the individual rationality (or participation) constraint: $\theta q(\theta) - t(\theta) \geq 0$, for all $\theta \in \left[0, \bar{\theta}\right]$, and the incentive constraint: $\theta q(\theta) - t(\theta) \geq \theta q(\theta') - t(\theta')$

for all $\theta, \theta' \in \left[0, \bar{\theta}\right]$. The participation constraint guarantees that the buyer receives a non-negative net utility from his choice, and the incentive constraints account for the fact that θ is private information to the buyer, and hence the revelation of the information is required to be incentive compatible.

The expected social welfare is defined by the sum of the customer's net utility and the seller's revenue:

$$\mathbb{E}_{\theta}\left[\theta q\left(\theta\right) - \frac{1}{2} q\left(\theta\right)^2\right].$$

It follows that for a consumer with type θ, it is socially optimal to provide a production level equal to his type:

$$q^*\left(\theta\right) = \theta.$$

The resulting maximized social welfare is given by:

$$SW^*\left(\infty\right) \triangleq \mathbb{E}_{\theta}\left[\theta q^*\left(\theta\right) - \frac{1}{2} q^*\left(\theta\right)^2\right] = \frac{1}{2}\mathbb{E}\left[\theta^2\right]. \tag{1}$$

The optimal menu thus requires an infinity, a continuum, of reports $\theta \in \left[0, \bar{\theta}\right]$ and a corresponding continuum of allocations, hence our definition for $SW^*\left(\infty\right)$. By contrast, we are interested in finding the optimal menu when each buyer can communicate his type only in a finite language, or equivalently, when the seller can process only finitely many different messages, or equivalently, when the seller can produce only finitely many alternative versions of his product.

3.1 n-Item Menu

Thus we consider the optimal design of a finite menu. The menu is composed of $n < \infty$ different allocations $\{q_k\}$, $k = 1, \ldots, n$, where q_k is the quantity (quality) of the $k-$th item of the menu. Let $\{A_k = [\theta_{k-1}, \theta_k)\}_{k=1}^n$ represent a corresponding partition of the set of consumer types, $\left[0, \bar{\theta}\right]$, where[1]

$$0 = \theta_0 \leq \theta_1 \leq \ldots \leq \theta_n = \bar{\theta}.$$

A consumer with type $\theta \in A_k$ selects $q\left(\theta\right) = q_k$. $\{A_k, q_k\}_{k=1}^n$ describes a finite menu (and its associated assignment), called an n-item menu henceforth. Let $\mathcal{M}_F$ be the set of all n-item

[1] In general, a partition of $\left[0, \bar{\theta}\right]$ need not consist of intervals. Due to the nature of the optimization problem in (3), however, it can be shown that the optimal partition consists of intervals.

menus for a given distribution F:

$$\mathcal{M}_F \triangleq \left\{ \{A_k, q_k\}_{k=1}^n : A_k = [\theta_{k-1}, \theta_k), 0 = \theta_0 \leq \theta_1 \leq \ldots \leq \theta_n = \bar{\theta} \right\}.$$

We wish to choose a finite menu from $\mathcal{M}_F$ to maximize the expected social welfare:

$$SW^*(n) \triangleq \max_{\{A_k, q_k\}_{k=1}^n \in \mathcal{M}_F} \left\{ \mathbb{E}_\theta \left[\theta q - \frac{1}{2} q^2 \right] \right\}. \tag{2}$$

Since the distribution function F of θ is known, maximizing the expected social welfare in (2) is equivalent to minimizing

$$\mathbb{E}_\theta \left[\theta^2 - 2\theta q + q^2 \right] = \mathbb{E}_\theta \left[(\theta - q)^2 \right].$$

The key to our approach is the realization that if we view θ as a continuous signal which must be represented by a representation point q_k in the interval A_k, then this is an instance of the quantization problem in information theory, where the intervals $\{A_k\}_{k=1}^n$ and the corresponding representation points $\{q_k\}_{k=1}^n$ are jointly chosen to minimize the mean square error (MSE):

$$MSE(n) \triangleq \min_{\{A_k, q_k\}_{k=1}^n \in \mathcal{M}_F} \left\{ \mathbb{E}_\theta \left[(\theta - q)^2 \right] \right\}. \tag{3}$$

Thus we may view any finite menu $\{A_k, q_k\}_{k=1}^n$ as the solution to a scalar quantization problem. Henceforth, we use the terms quantizer and finite menu interchangeably.

3.2 Scalar Quantizer

It is well known that the optimal scalar quantizer or finite menu $\{A_k^*, q_k^*\}_{k=1}^n$ must satisfy the following Lloyd-Max conditions (see Chapter 6.2, Gersho and Gray (2007)).

Theorem 1 (Lloyd-Max Optimality Conditions)
The optimal n-item menu $\{A_k^*, q_k^*\}_{k=1}^n$, $A_k^* = \left[\theta_{k-1}^*, \theta_k^*\right)$, which maximizes the social welfare defined in (2) must satisfy:

$$\theta_k^* = \frac{1}{2}\left(q_k^* + q_{k+1}^*\right), \qquad k = 1, \ldots, n-1; \tag{4}$$

$$q_k^* = \mathbb{E}\left[\theta | \theta \in A_k^*\right], \qquad k = 1, \ldots, n; \tag{5}$$

where $\theta_0^* = 0$ and $\theta_n^* = \bar{\theta}$.

Thus, θ_k^*, which separates two neighboring intervals A_k^* and A_{k+1}^*, must be the arithmetic average of q_k^* and q_{k+1}^*. At the same time, q_k^*, the representation level for the interval $A_k^* = \left[\theta_{k-1}^*, \theta_k^*\right)$, must be the conditional mean for θ given that θ falls in this interval.

We now consider the optimal scalar quantization for the uniform distribution as an example to illustrate how the Lloyd-Max conditions are used to obtain the optimal finite menu. The resulting boundary points $\{\theta_k^*\}_{k=0}^n$ as well as the representation points $\{q_k^*\}_{k=0}^n$ share the uniformity with the underlying distribution of the values.

Corollary 1 (Optimal Finite Menu for Uniform Distribution)
With a uniform distribution of types, $\theta \sim \mathcal{U}[0,1]$, the optimal n-item menu $\{A_k^*, q_k^*\}_{k=1}^n, A_k^* = \left[\theta_{k-1}^*, \theta_k^*\right)$ is given by:

$$
\begin{aligned}
\theta_k^* &= \frac{k}{n}, & k &= 0,\ldots,n; \\
q_k^* &= \frac{k-\frac{1}{2}}{n}, & k &= 1,\ldots,n.
\end{aligned}
$$

To obtain Corollary 1, note that the conditional mean in any interval A_k is clearly given by $\mathbb{E}[\theta|\theta \in A_k] = (\theta_k + \theta_{k-1})/2$. From Theorem 1, the optimal menu $\{A_k^*, q_k^*\}_{k=1}^n$ must satisfy:

$$
\begin{aligned}
\theta_k^* &= \frac{q_k^* + q_{k+1}^*}{2}, & k &= 1,\ldots,n-1; \\
q_k^* &= \frac{\theta_{k-1}^* + \theta_k^*}{2}, & k &= 1,\ldots,n.
\end{aligned}
$$

Hence, $\theta_{k+1}^* - 2\theta_k^* + \theta_{k-1}^* = 0$. Note that $\theta_0^* = 0, \theta_n^* = 1$, and thus we have a unique solution to the Lloyd-Max conditions, which must be optimal:

$$
\theta_k^* = \frac{k}{n}, \qquad k = 1,\ldots,n-1,
$$

and thus

$$
q_k^* = \frac{\theta_{k-1}^* + \theta_k^*}{2} = \frac{k-\frac{1}{2}}{n}, \qquad k = 1,\ldots,n.
$$

For certain simple distributions (e.g., the uniform distribution and simple discrete distributions), it is possible to obtain the closed form of the optimal finite menus from the Lloyd-Max conditions. For general distributions, closed form solutions are often difficult to obtain. On the other hand, Lloyd-Max conditions imply a useful iterative algorithm for searching for the optimal menu $\{A_k^*, q_k^*\}_{k=1}^n$ (see Chapter 6.2, Gersho and Gray (2007)).

3.3 Welfare Loss of n-Item Menu

We are interested in how well the optimal n-item menu $\{A_k^*, q_k^*\}_{k=1}^n$ can approximate the performance of the optimal continuous menu $\{q^*(\theta)\}_{\theta\in[0,\bar{\theta}]}$ as measured by the welfare loss $SW^*(\infty) - SW^*(n)$.

Definition 1 (Welfare Loss)
For a given distribution function F, the welfare loss induced by the optimal n-item menu, relative the optimal continuous menu, is defined by:

$$L(F;n) \triangleq SW^*(\infty) - SW^*(n).$$

It is easy to see that a tight lower bound on the welfare loss over all distributions is zero, i.e. $\inf_F L(F;n) = 0$. This can be achieved by the discrete uniform distribution, $\Pr\left[\theta = \frac{k}{n}\right] = \frac{1}{n}$ for $k = 1, ..., n$. In what follows, we will focus on the upper bound on the welfare loss over all distributions with finite support, which we normalize without loss of generality to be $[0, 1]$. Thus, let $\mathcal{F}$ be the set of all distribution functions on $[0, 1]$. Our main task is to estimate the worst-case behavior of the welfare loss over all distributions $F \in \mathcal{F}$.

Definition 2 (Maximum Welfare Loss)
The maximum welfare loss induced by the optimal n-item menu over all distribution function $F \in \mathcal{F}$ is defined by

$$L(n) \triangleq \sup_{F\in\mathcal{F}} L(F;n).$$

Before we discuss the general case, we first consider the uniform distribution, for which the welfare loss can be exactly established, and a fortiori the resulting convergence rate as n increases.

Proposition 1 (Welfare Loss for Uniform Distribution)
With a uniform distribution of types, $\theta \sim \mathcal{U}[0,1]$, the welfare loss is $L(F_{\mathcal{U}};n) = 1/(24n^2)$.

Proof. By Corollary 1, the optimal n-item menu is characterized by $\theta_k^* = k/n$, $q_k^* = \left(k - \frac{1}{2}\right)/n$, and thus the expected social welfare is

$$SW^*(n) = \mathbb{E}_\theta\left[\theta q^*(\theta) - \frac{1}{2}q^*(\theta)^2\right] = \sum_{k=1}^n \int_{\theta_{k-1}^*}^{\theta_k^*} \left[\theta q_k^* - \frac{1}{2}q_k^{*2}\right] d\theta = \frac{1}{6} - \frac{1}{24n^2}.$$

By contrast, the social welfare realized by the optimal continuous menu is $SW^*(\infty) = \frac{1}{2}\mathbb{E}\left[\theta^2\right] = \frac{1}{6}$, which gives us the welfare loss: $L(F_{\mathcal{U}}; n) = 1/24n^2$. ■

A direct calculation of the welfare loss requires the explicit form of the optimal finite menu, as determined by the Lloyd-Max conditions. In general, however, this is a difficult problem. We therefore design a sequence of finite menus to obtain an upper bound on the welfare loss. In these menus, the boundary points are uniformly spaced, and the quantities $\{q_k\}$ are consistent with the Lloyd-Max condition (5). This construction allows us to estimate how fast the maximum welfare loss converges to zero as the number of classes n tends to infinity.

For any $F \in \mathcal{F}$, the social welfare corresponding to an n-class menu $\{A_k, q_k\}_{k=1}^n \in \mathcal{M}_F$ is:

$$SW(n) = \mathbb{E}_\theta\left[\theta q - \frac{1}{2}q^2\right] = \frac{1}{2}\mathbb{E}\left[\theta^2\right] - \frac{1}{2}\mathbb{E}_\theta\left[(\theta - q)^2\right].$$

By (1),

$$SW^*(\infty) - SW(n) = \frac{1}{2}\mathbb{E}_\theta\left[(\theta - q)^2\right].$$

Therefore, the welfare loss induced by the optimal n-class menu is

$$\begin{aligned} L(F; n) &= \inf_{\{A_k, q_k\}_{k=1}^n \in \mathcal{M}_F} \frac{1}{2}\mathbb{E}_\theta\left[(\theta - q)^2\right] \\ &= \inf_{\{A_k, q_k\}_{k=1}^n \in \mathcal{M}_F} \frac{1}{2}\sum_{k=1}^n \left[F(\theta_k) - F(\theta_{k-1})\right]\int_{A_k}(\theta - q_k)^2\, dF_k(\theta), \end{aligned}$$

where $F_k(\theta) = \frac{F(\theta)}{\Pr[\theta \in A_k]} = \frac{F(\theta)}{F(\theta_k) - F(\theta_{k-1})}$ is the conditional distribution function on the interval $A_k = [\theta_{k-1}, \theta_k)$.

To proceed, we consider a more constrained set of n-class menus:

$$\mathcal{M}'_F \equiv \left\{\{A_k, q_k\}_{k=1}^n \in \mathcal{M}_F : q_k = \mathbb{E}\left[\theta | \theta \in A_k\right], 1 \le k \le n\right\}.$$

That is, $\mathcal{M}'_F$ includes all n-class menus $\{A_k, q_k\}_{k=1}^n$ consistent with the optimality condition (5). Note that the optimal menu $\{A_k^*, q_k^*\}_{k=1}^n$ lies in $\mathcal{M}'_F$. Then we can write

$$\begin{aligned} L(F; n) &= \inf_{\{A_k, q_k\}_{k=1}^n \in \mathcal{M}'_F} \frac{1}{2}\sum_{k=1}^n \left[F(\theta_k) - F(\theta_{k-1})\right]\int_{A_k}(\theta - q_k)^2\, dF_k(\theta) \\ &= \inf_{\{A_k, q_k\}_{k=1}^n \in \mathcal{M}'_F} \frac{1}{2}\sum_{k=1}^n \left[F(\theta_k) - F(\theta_{k-1})\right] var\left[\theta | \theta \in A_k\right]. \end{aligned} \tag{6}$$

Equation (6) holds since $q_k = \mathbb{E}\left[\theta | \theta \in A_k\right]$ for any $\{A_k, q_k\}_{k=1}^n \in \mathcal{M}'_F$.

To obtain an upper bound on $L(F;n)$, we develop a key upper bound on the conditional variance of θ on any given interval.

Lemma 1 *If θ is a random variable on $[a,b]$, $-\infty < a < b < \infty$, then* $\mathrm{var}[\theta] \le \frac{1}{4}(b-a)^2$.

Proof. Consider the random variable $\hat{\theta} = \frac{1}{b-a}(\theta - a)$ on $[0,1]$. Note that $\mathrm{var}[\theta] = (b-a)^2\, var[\hat{\theta}]$, so we just need to show $\mathrm{var}[\hat{\theta}] \le \frac{1}{4}$. Since $\hat{\theta}^2 \le \hat{\theta}$ for any $\hat{\theta} \in [0,1]$, $\mathbb{E}[\hat{\theta}^2] \le \mathbb{E}[\hat{\theta}]$. Let $u = \mathbb{E}[\hat{\theta}]$. Then $\mathrm{var}[\hat{\theta}] = \mathbb{E}[\hat{\theta}^2] - (\mathbb{E}[\hat{\theta}])^2 \le u - u^2 \le \frac{1}{4}$. ■

Proposition 2 (Upper Bound on Welfare Loss)
For any $F \in \mathcal{F}$, and $n \ge 1$, the welfare loss of the optimal n-item menu $L(F;n) \le 1/8n^2$.

Proof. For any $F \in \mathcal{F}$, consider a particular sequence of n-item menus $\{A'_k, q'_k\}_{k=1}^n \in \mathcal{M}'_F$, where $A'_k = \left[\theta'_{k-1}, \theta'_k\right)$, and

$$\begin{aligned} \theta'_k &= \frac{k}{n}, && k = 0, \ldots, n; \\ q'_k &= \mathbb{E}\left[\theta | \theta \in \left[\theta'_{k-1}, \theta'_k\right)\right], && k = 1, \ldots, n. \end{aligned}$$

That is, the boundary points $\{\theta'_k\}_{k=0}^n$ are uniformly spaced over $[0,1]$, and the quantities $\{q'_k\}_{k=1}^n$ are consistent with the Lloyd-Max condition (5). Then by (6), we have

$$L(F;n) \le \frac{1}{2} \sum_{k=1}^n \left[F(\theta'_k) - F\left(\theta'_{k-1}\right)\right] var\left[\theta | \theta \in \left[\theta'_{k-1}, \theta'_k\right)\right].$$

By Lemma 1, we have

$$var\left[\theta | \theta \in \left[\theta'_{k-1}, \theta'_k\right)\right] \le \frac{1}{4}\left(\theta'_k - \theta'_{k-1}\right)^2 = \frac{1}{4n^2}.$$

Hence, the welfare loss

$$L(F;n) \le \frac{1}{8n^2} \sum_{k=1}^n \left[F(\theta'_k) - F\left(\theta'_{k-1}\right)\right] = \frac{1}{8n^2}$$

which establishes the result. ■

Combining Proposition 2 with our result in Proposition 1 for the uniform distribution, we see that the maximum welfare loss is of order $1/n^2$.

Proposition 3 (Bounds on Maximum Welfare Loss)
For any $n \geq 1$, the maximum welfare loss $L(n) = \Theta\left(\frac{1}{n^2}\right)$.[2] *Specifically,* $\frac{1}{24n^2} \leq L(n) \leq \frac{1}{8n^2}$.

Proof. Let $F_{\mathcal{U}}$ be the uniform distribution function on $[0,1]$. From Proposition 1, we have $L(F_{\mathcal{U}}; n) = \frac{1}{24n^2}$. Therefore, $L(n) \geq L(F_{\mathcal{U}}; n) = \frac{1}{24n^2}$. The upper bound results from Proposition 2. ∎

A version of the one-dimensional social welfare maximization problem in (2) was considered earlier in Wilson (1989). He obtained an approximate solution to (2) by uniformly quantizing the distribution function of θ, and by expanding the social welfare on each quantization interval by the Taylor series around zero up to the order of $1/n^3$, where n is the total number of intervals. He also established that the efficiency loss resulting from an n-item menu is of order no more than $1/n^2$, i.e., $SW(n) \geq SW^*(\infty) - O(1/n^2)$. This approach, however, is difficult to generalize to higher dimensions, and cannot be easily used to solve the revenue maximization problem.

In contrast to the approach in Wilson (1989), we establish an underlying connection between the problem of nonlinear pricing with limited information and the quantization problem in information theory. We quantize the type space directly, and introduce the Lloyd-Max conditions that the optimal finite menu must satisfy. We then choose a specific sequence of finite menus $\left\{\left[\theta'_{k-1}, \theta'_k\right), q'_k\right\}_{k=1}^n$, where the q'_k's are consistent with the Lloyd-Max conditions and the θ'_k's are uniformly spaced, to provide the upper bound on the welfare loss. To our knowledge, we are the first to explore the connection between pricing and quantization.

Compared with Wilson (1989)'s technique, our quantization approach is more direct, and has several significant advantages. First, quantization theory provides not only the Lloyd-Max conditions that the optimal finite menu must satisfy, but also an iterative algorithm to construct an optimal finite menu. Second, by using quantization theory in the type space directly, we establish a connection between the finite menu for revenue maximization and that for welfare maximization. This allows us to use similar techniques to prove the convergence rate of the revenue loss. Finally, our approach extends naturally, via the general technique of vector quantization, to the multi-dimensional environment.

[2] Given two functions $f(n)$ and $g(n)$, we write $f(n) = \Theta(g(n))$ if $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for some constants c_1 and c_2, as n becomes large.

4 Revenue Maximization

We now analyze the problem of revenue maximization. In contrast to the social welfare problem, here, the seller wishes to design a menu $\{q(\theta), t(\theta)\}_{\theta\in[0,\bar{\theta}]}$ to maximize his expected net revenue, i.e. the difference between the gross revenue that he receives from the buyer minus the cost of providing the demanded quantity (quality):

$$\Pi^*(\infty) = \max_{\{q(\theta),t(\theta)\}} \left\{ \mathbb{E}_\theta \left[t(\theta) - \frac{1}{2} q(\theta)^2 \right] \right\}.$$

As before, the contract offered has to satisfy two sets of constraints, namely the incentive constraints and the individual rationality (or participation) constraints. As usual, we can use the incentive constraints to eliminate the transfers and rewrite the problem in terms of the allocation alone. The revenue maximization problem can therefore be transformed into a welfare maximization problem (without incentive constraints) as long as we replace the true valuation θ of the buyer with the corresponding virtual valuation:

$$\psi(\theta) \triangleq \theta - \frac{1 - F(\theta)}{f(\theta)}. \tag{7}$$

The virtual valuation is below the true valuation, and the inverse of the hazard rate $[1 - F(\theta)]/f(\theta)$ accounts for the information rent, the cost of the private information, as perceived by the principal in the optimal mechanism. With this standard transformation of the problem, the expected revenue of the seller (without communication constraints) is:

$$\Pi^*(\infty) = \mathbb{E}_\theta \left[q^*(\theta)\psi(\theta) - \frac{1}{2}(q^*(\theta))^2 \right],$$

and the resulting optimal contract exhibits:

$$q^*(\theta) = \max\{\psi(\theta), 0\} \tag{8}$$

We identify the lowest value θ at which the virtual valuation attains a nonnegative value as

$$\underline{\theta} \triangleq \min\{\theta \,|\psi(\theta) \geq 0\},$$

and hence the corresponding revenue is

$$\Pi^*(\infty) = \mathbb{E}_\theta \left[q^*(\theta)\psi(\theta) - \frac{1}{2} q^*(\theta)^2 \right] = \frac{1}{2}\int_{\underline{\theta}}^{1} \psi^2(\theta)\, dF(\theta). \tag{9}$$

We hasten to add the caveat that the above solution is subject to the requirement that the virtual valuation (7) is monotone. By contrast, if the virtual valuation fails to be monotone, then the optimal solution $q^*(\theta)$ has to display flat parts due to the familiar ironing argument of Myerson (1979). For our analysis, it turns out the critical bounds are established by distributions that generate monotone virtual valuations, see the discussion following Proposition 6, and thus the restriction to monotone or "regular environments" in the words of Myerson (1979) is without loss of generality.

4.1 n-Item Menu

The current problem is then identical to the seminal analysis by Mussa and Rosen (1978) and Maskin and Riley (1984) with the following important exception: the buyer can access only a finite number of choices due to the limited communication with the seller. Now, a menu of quantity (quality)-price bundles is designed by the monopolistic seller to extract as much profit as possible. In a finite menu, the seller can offer only a finite number of pairs $\{q_k, t_k\}_{k=1}^n$ to the consumer. Let $A_k = [\theta_{k-1}, \theta_k), k = 0, \ldots, n$, represent the partition of $[0, \bar{\theta}]$ with boundary values $\theta_{-1} = 0, \theta_n = \bar{\theta} \leq \infty$. If $\theta \in A_k, 1 \leq k \leq n$, the consumer chooses $q(\theta) = q_k$ and pays $t(\theta) = t_k$. In the revenue maximization setting, the seller will exclude consumers with low type from the market so that θ_0 is bounded away from zero and is endogenously determined. All consumers whose types are lower than θ_0 are meant to choose $q_0 = 0$ and pay $t_0 = 0$.

Due to the monotonicity of $\psi(\theta)$, we can relabel the type θ directly in terms of the corresponding virtual valuation $\widehat{\theta}$:

$$\hat{\theta} \triangleq \psi(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)},$$

and define the associated intervals $\{A_k\}_{k=1}^n$ directly in terms of the new variable $\hat{\theta}$:

$$\theta \in A_k = [\theta_{k-1}, \theta_k) \Leftrightarrow \hat{\theta} \in \hat{A}_k = \left[\hat{\theta}_{k-1}, \hat{\theta}_k\right), 1 \leq k \leq n,$$

where $\hat{\theta}_k = \psi(\theta_k), 1 \leq k \leq n$. After this change of variable, we define a distribution function $G\left(\hat{\theta}\right)$ in terms of the original distribution function $F(\theta)$:

$$G\left(\hat{\theta}\right) = G(\psi(\theta)) \triangleq F(\theta).$$

Then the revenue of an n-item menu can be written in terms of the virtual type $\hat{\theta}$:

$$\Pi(n) = \mathbb{E}_{\theta}\left[q\psi(\theta) - \frac{1}{2}q^2\right] = \mathbb{E}_{\hat{\theta}}\left[q\hat{\theta} - \frac{1}{2}q^2\right], \tag{10}$$

and the revenue of the optimal n-item menu is given by

$$\Pi^*(n) \triangleq \max_{\left\{\widehat{A}_k, q_k\right\}_{k=0}^{n} \in \hat{\mathcal{M}}_F} \Pi(n),$$

where the set of all n-class menus is rewritten as:

$$\hat{\mathcal{M}}_F = \left\{\left\{\hat{A}_k, q_k\right\}_{k=0}^{n} : q_0 = 0, \hat{A}_k = \left[\hat{\theta}_{k-1}, \hat{\theta}_k\right), 0 = \hat{\theta}_{-1} < \hat{\theta}_0 \leq \hat{\theta}_1 \leq \ldots \leq \hat{\theta}_n = \bar{\theta}\right\}.$$

The problem is now formally equivalent to the earlier welfare maximization problem (2). As before, we consider the revenue loss induced by the optimal n-item menu in terms of the distribution function F and the number n of allowed items.

Definition 3 (Revenue Loss)
For a given distribution function F, the revenue loss induced by the optimal n-item menu compared with the optimal continuous menu is:

$$\widehat{L}(F; n) \triangleq \Pi^*(\infty) - \Pi^*(n).$$

We denote the revenue loss by $\widehat{L}(F; n)$ to emphasize that the relevant random variable is now the virtual valuation $\widehat{\theta}$ rather than the valuation θ itself.

Definition 4 (Maximum Revenue Loss)
The maximum revenue loss induced by the optimal n-item menu over all $F \in \mathcal{F}$ is:

$$\widehat{L}(n) \triangleq \sup_{F \in \mathcal{F}} \widehat{L}(F; n).$$

We briefly describe the optimal finite menu for the uniform distribution before investigating general distributions.

Proposition 4 (Uniform Distribution)
With a uniform distribution, $\theta \sim \mathcal{U}[0,1]$, the optimal n-item menu $\{A_k^*, q_k^*\}_{k=0}^{n}, A_k^* = \left[\theta_{k-1}^*, \theta_k^*\right)$ is:

$$\begin{aligned} \theta_k^* &= \frac{n+k+1}{2n+1}, \qquad k = 0, \ldots, n; \\ q_k^* &= \frac{2k}{2n+1}, \qquad k = 0, \ldots, n; \end{aligned}$$

and the corresponding revenue loss is:

$$\widehat{L}\left(F_{\mathcal{U}};n\right)=\frac{1}{12\left(2n+1\right)^{2}}.$$

Proof. In this case, $F\left(\theta_k^*\right)=\theta_k^*$, $f\left(\theta_k^*\right)=1$. From Theorem 1, we know that the optimal menu must satisfy:

$$\begin{aligned}2\theta_k^*-1 &= \frac{1}{2}\left(q_k^*+q_{k+1}^*\right), \qquad && k=0,\ldots,n-1;\\ q_k^* &= \theta_{k-1}^*+\theta_k^*-1, && k=1,\ldots,n.\end{aligned}$$

We therefore have the following recursive equation:

$$\theta_{k+1}^*-2\theta_k^*+\theta_{k-1}^*=0, \qquad k=1,\ldots,n-1.$$

This implies that $\theta_{k+1}^*-\theta_k^*=\theta_k^*-\theta_{k-1}^*=\Delta\theta$, and thus

$$\theta_k^*=\theta_0^*+k\Delta\theta \qquad k=0,\ldots,n.$$

Note that $\theta_0^*=\frac{1}{4}q_1^*+\frac{1}{2}$, and $q_1^*=\theta_1^*+\theta_0^*-1=2\theta_0^*+\Delta\theta-1$. Therefore, $\Delta\theta=2\theta_0^*-1$. Since $\theta_n=1$, we have $\theta_0^*+n\Delta\theta=1$. From the above two equations, we have

$$\theta_0^*=\frac{n+1}{2n+1}, \qquad \Delta\theta=\frac{1}{2n+1}.$$

Therefore,

$$\theta_k^*=\frac{n+k+1}{2n+1}, \qquad k=0,\ldots,n; \qquad q_k^*=\frac{2k}{2n+1}, \qquad k=0,\ldots,n.$$

The expected revenue is

$$\Pi^*\left(n\right)=\mathbb{E}_\theta\left[q^*\left(\theta\right)\psi\left(\theta\right)-\frac{1}{2}q^*\left(\theta\right)^2\right]=\sum_{k=1}^{n}\int_{\theta_{k-1}^*}^{\theta_k^*}\left[q_k^*\left(2\theta-1\right)-\frac{1}{2}q_k^{*2}\right]d\theta=\frac{1}{3}\frac{n\left(n+1\right)}{\left(2n+1\right)^2}.$$

The optimal continuous menu is: $q^*\left(\theta\right)=\max\left\{2\theta-1,0\right\}$, and the maximum revenue is:

$$\Pi^*\left(\infty\right)=\mathbb{E}_\theta\left[q^*\left(\theta\right)\left(2\theta-1\right)-\frac{1}{2}q^*\left(\theta\right)^2\right]=\frac{1}{12},$$

and it follows that the revenue loss induced by the optimal n-item menu is given as stated by $\widehat{L}\left(F_{\mathcal{U}};n\right)$. ■

Thus, the convergence rate of the revenue loss induced by the optimal n-item menu for the uniform distribution is of order $\frac{1}{n^2}$, and thus, as expected, identical to the finding of Proposition 1 for the social welfare maximization environment.

In addition, we find that the seller tends to serve fewer consumers as compared to the case when a continuous menu is used: $\theta_0^*(n) = \frac{n+1}{2n+1} > \frac{1}{2} = \theta_0^*(\infty)$. The difference $\theta_0^*(n) - \theta_0^*(\infty)$ shrinks to 0 as n goes to infinity. This is a consequence of the fact that the seller's ability of extracting revenue is more limited in the case of finite menus. To compensate, the seller would like to reduce the service coverage in order to pursue higher profits.

4.2 Revenue Loss for n-Item Menu

We can now obtain the upper bound on the revenue loss induced by the optimal finite menus, and then estimate the convergence rate of the maximum revenue loss as the number n of items tends to infinity. For any n-item menu $\left\{\hat{A}_k, q_k\right\}_{k=0}^{n} \in \hat{\mathcal{M}}_F$, since $q_0 = 0$ for all $\hat{\theta} \leq \hat{\theta}_0$, the revenue of the seller can be written as:

$$\begin{aligned}\Pi(n) &= \mathbb{E}_{\hat{\theta}}\left[\hat{\theta}q - \frac{1}{2}q^2\right] = \int_{\hat{\theta}_0}^{1}\left[\hat{\theta}q - \frac{1}{2}q^2\right] dG\left(\hat{\theta}\right) \\ &= \frac{1}{2}\int_{\hat{\theta}_0}^{1}\hat{\theta}^2 dG\left(\hat{\theta}\right) - \frac{1}{2}\int_{\hat{\theta}_0}^{1}\left(\hat{\theta} - q\right)^2 dG\left(\hat{\theta}\right).\end{aligned}$$

Recall that the revenue resulting from the optimal continuous menu is:

$$\Pi^*(\infty) = \frac{1}{2}\int_{\underline{\theta}}^{1}\psi^2(\theta)\, dF(\theta) = \frac{1}{2}\int_{0}^{1}\hat{\theta}^2 dG\left(\hat{\theta}\right).$$

Therefore, the revenue loss can be written as

$$\begin{aligned}\widehat{L}(F;n) &= \inf_{\left\{\hat{A}_k, q_k\right\}_{k=0}^{n} \in \hat{\mathcal{M}}_F} \left[\Pi^*(\infty) - \Pi(n)\right] \\ &= \inf_{\left\{\hat{A}_k, q_k\right\}_{k=0}^{n} \in \hat{\mathcal{M}}_F} \left[\frac{1}{2}\int_{0}^{\hat{\theta}_0}\hat{\theta}^2 dG\left(\hat{\theta}\right) + \frac{1}{2}\int_{\hat{\theta}_0}^{1}\left(\hat{\theta} - q\right)^2 dG\left(\hat{\theta}\right)\right].\end{aligned}$$

As before, when estimating the upper bound on the revenue losses, we restrict our attention to a subset of n-item menus where the second of the Lloyd-Max optimality conditions is satisfied:

$$\hat{\mathcal{M}}_F' \triangleq \left\{\left\{\hat{A}_k, q_k\right\}_{k=0}^{n} \in \hat{\mathcal{M}}_F : q_k = \mathbb{E}\left[\hat{\theta} | \hat{\theta} \in \hat{A}_k\right], 1 \leq k \leq n\right\}.$$

Note that the optimal n-item menu $\left\{\hat{A}_k^*, q_k^*\right\}_{k=0}^n$ is also included in $\hat{\mathcal{M}}'_F$. Since $q_k = \mathbb{E}\left[\hat{\theta}|\hat{\theta} \in \hat{A}_k\right]$ for all $\left\{\hat{A}_k, q_k\right\}_{k=0}^n \in \hat{\mathcal{M}}'_F$, we can verify that

$$\widehat{L}\left(F;n\right) = \inf_{\left\{\hat{A}_k, q_k\right\}_{k=0}^n \in \hat{\mathcal{M}}'_F} \left\{\frac{1}{2}\int_0^{\hat{\theta}_0} \hat{\theta}^2 dG\left(\hat{\theta}\right) + \frac{1}{2}\sum_{k=1}^n \left[G\left(\hat{\theta}_k\right) - G\left(\hat{\theta}_{k-1}\right)\right] \operatorname{var}\left[\hat{\theta}|\hat{\theta} \in \hat{A}_k\right]\right\}. \tag{11}$$

The first term in the square bracket in (11) captures the revenue loss by reducing the service coverage. The second term is similar to the welfare loss $L\left(F;n\right)$ in (6). The only difference is the boundary condition $\hat{\theta}_0 > 0$. Thus, a technique similar to that used in proving Proposition 2 can be adapted to this new formulation, leading to an identical upper bound.

Proposition 5 (Upper Bound on Revenue Loss)
For a given $F \in \mathcal{F}$, and $n \geq 1$, the revenue loss induced by the optimal n-item menu $\widehat{L}\left(F;n\right) \leq \frac{1}{8n^2}$.

Proof. For the given $F \in \mathcal{F}$, and $n \geq 1$, choose a sequence $\{\delta_k\}_{k=1}^\infty$ s.t. $0 < \delta_k < \frac{1}{n}$, and $\lim_{k\to\infty} \delta_k = 0$. For any fixed $k \geq 1$, consider a particular sequence of n-item menu $\left\{A'_{k,k}, q'_{k,m}\right\}_{k=1}^n \in \hat{\mathcal{M}}'_F$, $\hat{A}'_{k,m} = \left[\hat{\theta}'_{k-1,m}, \hat{\theta}'_{k,m}\right)$, where

$$\begin{aligned}
\hat{\theta}'_{0,m} &= \delta_m, \\
\hat{\theta}'_{k,m} &= \frac{k}{n}, \qquad k = 1, \ldots, n; \\
q'_{k,m} &= \mathbb{E}\left[\hat{\theta}|\hat{\theta} \in \left[\hat{\theta}'_{k-1,m}, \hat{\theta}'_{k,m}\right)\right], \qquad k = 1, \ldots, n;
\end{aligned}$$

where $\hat{\theta}'_{-1,m} = 0, q'_{0,m} = 0$.

Thus by (11), the welfare loss

$$\widehat{L}\left(F;n\right) \leq \frac{1}{2}\int_0^{\delta_m} \hat{\theta}^2 dG\left(\hat{\theta}\right) + \frac{1}{2}\sum_{k=1}^n \left[G\left(\hat{\theta}'_{k,m}\right) - G\left(\hat{\theta}'_{k-1,m}\right)\right] \operatorname{var}\left[\hat{\theta}|\hat{\theta} \in \left[\hat{\theta}'_{k-1,m}, \hat{\theta}'_{k,m}\right)\right].$$

By Proposition 1, we have

$$\begin{aligned}
\operatorname{var}\left[\hat{\theta}|\hat{\theta} \in \left[\hat{\theta}'_{k-1,m}, \hat{\theta}'_{k,m}\right)\right] &\leq \frac{1}{4}\left(\hat{\theta}'_{k,m} - \hat{\theta}'_{k-1,m}\right)^2 = \frac{1}{4n^2}, \qquad k = 2, \ldots, n; \\
\operatorname{var}\left[\hat{\theta}|\hat{\theta} \in \left[\hat{\theta}'_{0,m}, \hat{\theta}'_{1,m}\right)\right] &\leq \frac{1}{4}\left(\hat{\theta}'_{1,m} - \hat{\theta}'_{0,m}\right)^2 = \frac{1}{4}\left(\frac{1}{n} - \delta_m\right)^2 \leq \frac{1}{4n^2}.
\end{aligned}$$

Hence, for any fixed $m \geq 1$, the revenue loss

$$\widehat{L}(F;n) \leq \frac{1}{2}\int_0^{\delta_m} \hat{\theta}^2 dG\left(\hat{\theta}\right) + \frac{1}{8n^2}\left[1 - G(\delta_m)\right] \leq \frac{1}{2}\delta_m^2 + \frac{1}{8n^2}.$$

Let $m \to \infty$. Since $\lim_{m\to\infty} \delta_m = 0$, we have $\widehat{L}(F;n) \leq \frac{1}{8n^2}$. ■

By Proposition 4 and Proposition 5, the maximum revenue loss converges to zero at a rate proportional to $\frac{1}{n^2}$.

Proposition 6 (Bounds on Maximum Revenue Loss)
For any $n \geq 1$, the maximum revenue loss $\widehat{L}(n) = \Theta\left(\frac{1}{n^2}\right)$, specifically,

$$\frac{1}{12(2n+1)^2} \leq \widehat{L}(n) \leq \frac{1}{8n^2}.$$

We mentioned earlier that the exact solution of the continuous menu problem, see (8), is only valid if the virtual valuation is monotone. If it fails to be monotone, then the solution $q^*(\theta)$ has to display flat parts due to the familiar ironing argument of Myerson (1979), as any incentive compatible allocation has to be monotone in the type θ. For our analysis below, this has two implications: first, in the area where the continuous menu is constant, there is no loss from using a finite menu; second, in the absence of a monotone virtual valuation, the corresponding revenue is below the solution indicated by (9). This means that the bounds on the revenue losses that we obtain for all the distributions with monotone virtual valuation hold a fortiori for any problem with non-monotone virtual valuation. Thus, the critical distributions for the bounds on the revenue loss are those with monotone virtual valuations, for which the above solution (8) of the continuous problem is exact.

5 Multi-Dimensional Type Space

In this section, we consider a multi-dimensional version of the nonlinear pricing problem which leads to the design of finite menus over multiple products. We demonstrate that our quantization view generalizes to the multi-dimensional environment, where the optimal design of finite menus requires the technique of vector quantization. We present bounds on the welfare and revenue loss arising from the communication constraints. In particular, we show that in many cases, it is beneficial to bundle the consumer's preferences over multiple goods as a vector, instead of treating them separately as independent quantities, thereby enabling the true joint design of finite menus over multiple goods.

5.1 Multi-Product Model

We consider a monopolistic firm facing a continuum of consumers and providing d heterogeneous goods. Each consumer's preferences (types) over these goods is characterized by a d-dimensional vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d)^T \in \mathbb{R}_+^d$, called the consumer's type vector, where for $1 \leq l \leq d$, θ_l represents his preference (type) for good l. Let $\Theta \subseteq \mathbb{R}_+^d$ denote the compact d-dimensional type space. The joint probability distribution of $\boldsymbol{\theta}$, denoted by $F(\boldsymbol{\theta})$, is assumed to be commonly known. We denote by F_l the marginal distribution function of type θ_l. We assume that $\mathbb{E}[\boldsymbol{\theta}] < \infty$,[3] $\mathbb{E}\left[\boldsymbol{\theta}^T\boldsymbol{\theta}\right] < \infty$. We assume that the joint density function f is continuous almost everywhere (a.e.) in the support (i.e., the type space): $\Theta = \left\{\boldsymbol{\theta} \in \mathbb{R}_+^d : f(\boldsymbol{\theta}) > 0\right\}$.

A customer with type $\boldsymbol{\theta}$ consumes the bundle of d goods with the quantity (or quality) vector $\boldsymbol{q} = (q_1, \ldots, q_d)^T$ by transferring a payment $t(\boldsymbol{q})$. We assume that the consumer is characterized by the following quasi-linear utility function:

$$u(\boldsymbol{\theta}, \boldsymbol{q}) - t(\boldsymbol{q}) = \boldsymbol{\theta}^T \Phi \boldsymbol{q} - t(\boldsymbol{q}),$$

where $\Phi = \left(\phi_{ij}\right)_{d\times d}$ is a $d \times d$ matrix which captures the interactions among different goods. We assume that $\phi_{ii} > 0$ for all i so that $\frac{\partial^2 u}{\partial\theta_i \partial q_i} > 0$. It turns out that no further assumptions, such as invertibility, symmetry or positive-definiteness of Φ, are needed for the analysis which follows. We further assume that the firm incurs a quadratic cost:

$$c(\boldsymbol{q}) = \frac{1}{2}\boldsymbol{q}^T \Sigma \boldsymbol{q}$$

by providing the vector $\boldsymbol{q}$. Here, $\Sigma = (\sigma_{ij})_{d\times d}$ is a $d \times d$ symmetric positive-definite matrix which characterizes the interactions in the production of multiple products. All of its diagonal elements must be positive: $\sigma_{ii} > 0$ for all i. If producing good i raises (reduces) the marginal cost of producing good j, then we set $\sigma_{ij} = \sigma_{ji} > (<)\, 0$. If $\sigma_{ij} = \sigma_{ji} = 0$, the technologies for producing good i and j are independent. The seller's revenue is given by:

$$\Pi(\boldsymbol{q}, t) = t(\boldsymbol{q}) - c(\boldsymbol{q}) = t(\boldsymbol{q}) - \frac{1}{2}\boldsymbol{q}^T \Sigma \boldsymbol{q}.$$

Standard Form We say that the utility and the cost function have the standard form if $\Phi = \Sigma = I_d$ (the $d \times d$ identity matrix). In fact, we can always transform the utility and

[3] By $\mathbb{E}[\boldsymbol{\theta}] < \infty$, we mean $\mathbb{E}[\theta_l] < \infty$ for all $1 \leq l \leq d$.

the cost function into the standard form as follows. We diagonalize the symmetric positive-definite matrix Σ: $\Sigma = P^T \Lambda P$, where $\Lambda = diag(\lambda_1, ..., \lambda_d)$, $\lambda_i > 0$ is the i-th eigenvalue of Σ, and P is a unitary matrix (i.e., $P^T P = I_d$). Let $B = \Lambda^{1/2} P$ and $A = \Lambda^{-1/2} P \Phi^T$, where $\Lambda^i = diag\left(\lambda_1^i, ..., \lambda_d^i\right)$, $i = \pm\frac{1}{2}$. Then it is easy to show that $A^T B = \Phi$ and $B^T B = \Sigma$. If we introduce the new type vector $\hat{\boldsymbol{\theta}} = A\boldsymbol{\theta}$ and the new quantity (quality) vector $\hat{\boldsymbol{q}} = B\boldsymbol{q}$, then the consumer's net utility and the cost function can be written in the standard form in terms of $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{q}}$:

$$\begin{aligned} u(\boldsymbol{\theta}, \boldsymbol{q}) &= \boldsymbol{\theta}^T \Phi \boldsymbol{q} = \boldsymbol{\theta}^T A^T B \boldsymbol{q} = \hat{\boldsymbol{\theta}}^T \hat{\boldsymbol{q}}, \\ c(\boldsymbol{q}) &= \frac{1}{2} \boldsymbol{q}^T \Sigma \boldsymbol{q} = \frac{1}{2} \boldsymbol{q}^T B^T B \boldsymbol{q} = \frac{1}{2} \hat{\boldsymbol{q}}^T \hat{\boldsymbol{q}}. \end{aligned}$$

Thus, without loss of generality, we focus on the standard form, assuming that $\Phi = \Sigma = I_d$.

5.2 Welfare Maximization

With a continuous menu, the social welfare is maximized by solving the $d-$dimensional linear quadratic program:

$$SW^*(\infty) = \max_{\mathbf{q}(\boldsymbol{\theta})} \mathbb{E}_{\boldsymbol{\theta}} \left[\boldsymbol{\theta}^T \boldsymbol{q} - \frac{1}{2} \boldsymbol{q}^T \boldsymbol{q} \right].$$

and it is socially optimal to provide a quantity (quality) vector equal to the type vector:

$$\boldsymbol{q}^*(\boldsymbol{\theta}) = \boldsymbol{\theta}.$$

The maximal social welfare therefore equals:

$$SW^*(\infty) = \mathbb{E}_{\boldsymbol{\theta}} \left[\boldsymbol{\theta}^T \boldsymbol{q}^*(\boldsymbol{\theta}) - \frac{1}{2} \boldsymbol{q}^*(\boldsymbol{\theta})^T \boldsymbol{q}^*(\boldsymbol{\theta}) \right] = \frac{1}{2} \mathbb{E}_{\boldsymbol{\theta}} \left[\boldsymbol{\theta}^T \boldsymbol{\theta} \right]. \tag{12}$$

By contrast, in the presence of information (or communication) constraints, the customers face a finite menu composed of $n < \infty$ different items $\{\boldsymbol{q}_k\}, k = 1, \ldots, n$. Let $\{B_k\}_{k=1}^n$ represent a partition of the consumer's type space Θ, i.e., $B_i \cap B_j = \emptyset$ if $i \neq j$, and $\cup_{k=1}^n B_k = \Theta$. All consumers with type vector $\boldsymbol{\theta} \in B_k$ will be allocated the kth quantity (quality) vector $\boldsymbol{q}_k$. Now, $\{B_k, \boldsymbol{q}_k\}_{k=1}^n$ describes a finite multi-product menu, called the n-item menu. As before, let $\mathcal{M}_F$ be the set of all n-item menus for a given distribution F:

$$\mathcal{M}_F = \left\{ \{B_k, \boldsymbol{q}_k\}_{k=1}^n : B_i \cap B_j = \emptyset \text{ if } i \neq j, \text{ and } \cup_{k=1}^n B_k = \Theta \right\}.$$

We choose $\{B_k, \boldsymbol{q}_k\}_{k=1}^n$ to maximize the expected social welfare:

$$SW^*(n) = \max_{\{B_k, \boldsymbol{q}_k\}_{k=1}^n \in \mathcal{M}_F} \left\{ \mathbb{E}_{\boldsymbol{\theta}} \left[\boldsymbol{\theta}^T \boldsymbol{q} - \frac{1}{2} \boldsymbol{q}^T \boldsymbol{q} \right] \right\}. \tag{13}$$

5.2.1 Vector Quantizer

When the joint probability distribution of $\boldsymbol{\theta}$ is known, maximizing the social welfare in (13) is equivalent to minimizing

$$\mathbb{E}_{\boldsymbol{\theta}}\left[\boldsymbol{\theta}^T\boldsymbol{\theta} - 2\boldsymbol{\theta}^T\boldsymbol{q} + \boldsymbol{q}^T\boldsymbol{q}\right] = \mathbb{E}_{\boldsymbol{\theta}}\left[(\boldsymbol{\theta} - \boldsymbol{q})^T(\boldsymbol{\theta} - \boldsymbol{q})\right] = \mathbb{E}_{\boldsymbol{\theta}}\left[\|\boldsymbol{\theta} - \boldsymbol{q}\|^2\right],$$

where $\|\cdot\|$ is the Euclidean norm. If we view $\boldsymbol{\theta}$ as the signal vector and $\boldsymbol{q}_k$ as the representation vector of $\boldsymbol{\theta}$ in the region B_k, then this becomes the d-dimensional n-region vector quantization problem, where the partition $\{B_k\}_{k=1}^n$ and the set of representation points $\{\boldsymbol{q}_k\}_{k=1}^n$ are jointly chosen to minimize the mean square error (MSE):

$$MSE^*(n) = \min_{\{B_k,\boldsymbol{q}_k\}_{k=1}^n \in \mathcal{M}_F} \left\{\mathbb{E}_{\boldsymbol{\theta}}\left[\|\boldsymbol{\theta} - \boldsymbol{q}\|^2\right]\right\}. \tag{14}$$

In this manner, any multi-product finite menu $\{B_k, \boldsymbol{q}_k\}_{k=1}^n$ can be viewed as a vector quantizer. We can therefore use the two terms "vector quantizer" and "finite multi-product menu" interchangeably.

As in the one-dimensional case, we investigate how well the optimal n-item menu $\{B_k^*, \boldsymbol{q}_k^*\}_{k=1}^n$ can approximate the performance of the optimal continuous menu $\{\boldsymbol{q}^*(\boldsymbol{\theta})\}_{\boldsymbol{\theta}\in\Theta}$. We quantify the welfare loss induced by the optimal finite menu in terms of the joint distribution function F, the number of items n, and the dimension d.

Definition 5 (Welfare Loss)
For a given joint distribution function F, the welfare loss induced by the optimal n-item menu compared with the optimal continuous menu is:

$$L(F;n;d) \triangleq SW^*(\infty) - SW^*(n).$$

The welfare loss induced by the optimal n-item menu can be written more explicitly as:

$$\begin{aligned} L(F;n;d) &= \inf_{\{B_k,\boldsymbol{q}_k\}_{k=1}^n \in \mathcal{M}_F} \frac{1}{2}\mathbb{E}_{\boldsymbol{\theta}}\left[\|\boldsymbol{\theta} - \boldsymbol{q}\|^2\right] \\ &= \inf_{\{B_k,\boldsymbol{q}_k\}_{k=1}^n \in \mathcal{M}_F} \frac{1}{2}\sum_{k=1}^n \int_{B_k} \|\boldsymbol{\theta} - \boldsymbol{q}_k\|^2 \, dF(\boldsymbol{\theta}). \end{aligned} \tag{15}$$

We are interested in the worst-case behavior of the welfare loss over all joint distributions over a d-dimensional support (i.e., the type space) $\Theta \subseteq R_+^d$ with positive and finite volume.

Let $b = \sup_{\boldsymbol{\theta}\in\Theta} \|\boldsymbol{\theta}\|$. Then $0 < b < \infty$. We can normalize all type vectors by b so that the support is $\Theta \subseteq [0,1]^d$. Note that this normalization has no effect on the order of the convergence rate of the welfare loss. Let $\mathcal{F}$ be the set of all joint distribution functions with the support $\Theta \subseteq [0,1]^d$. Our main task is to quantify the worst-case behavior of $L(F;n;d)$ over all distributions $F \in \mathcal{F}$.

Definition 6 (Maximal Welfare Loss)
The maximal welfare loss under the optimal n-item menu over all distributions $F \in \mathcal{F}$ is:

$$L(n;d) \triangleq \sup_{F\in\mathcal{F}} L(F;n;d).$$

In order to prove an upper bound on the welfare loss, we construct a simple d-dimensional K^d-region vector quantizer by combining d independent K-level scalar quantizers, where $K = \lfloor n^{1/d} \rfloor$. Such a vector quantizer is called a repeated scalar quantizer.

5.2.2 Repeated Scalar Quantization

Given the joint distribution $F \in \mathcal{F}$, for each type θ_l, $1 \le l \le d$, consider a K-level scalar quantizer $\{A_{l,t}, r_{l,t}\}_{t=1}^K \in \mathcal{M}_{F_l}$ on $[0,1]$ where $\mathcal{M}_{F_l}$ is the set of all scalar quantizers for the marginal distribution F_l:

$$\mathcal{M}_{F_l} = \left\{ \{A_{l,k}, r_{l,k}\}_{k=1}^K : A_{l,k} = [\theta_{l,k-1}, \theta_{l,k}), 0 = \theta_{l,0} \le \theta_{l,1} \le \ldots \le \theta_{l,n} = 1 \right\}.$$

We construct the corresponding d-dimensional K^d-region repeated scalar quantizer $\{B'_k, \boldsymbol{q}'_k\}_{k=1}^{K^d}$ in the type space $[0,1]^d$ as:

$$\begin{aligned} \{B'_k\}_{k=1}^{K^d} &= \{A_{1,k_1} \times \ldots \times A_{d,k_d} : k_l \in \{1,\ldots,K\}, 1 \le l \le d\}, \\ \{\boldsymbol{q}'_k\}_{k=1}^{K^d} &= \left\{ (r_{1,k_1}, \ldots, r_{d,k_d})^T : k_l \in \{1,\ldots,K\}, 1 \le l \le d \right\}. \end{aligned}$$

One can see that the set of regions $\{B'_k\}_{k=1}^{K^d}$ are orthotopes, i.e., the Cartesian product of intervals in d dimensions. In the following lemma, we use the repeated scalar quantizer to obtain a simple upper bound on the welfare loss in multiple dimensions.

Lemma 2
For any $F \in \mathcal{F}$, and $K \ge 1$, $L(F;K^d;d) \le \sum_{l=1}^d L(F_l;K)$, where F_l is the marginal distribution function of the type θ_l.

Proof. See the Appendix. ■

Proposition 7 (Upper Bound on Welfare Loss)
For any $F \in \mathcal{F}$, $n \geq 1$, $d \geq 1$, the welfare loss induced by the optimal n-item menu

$$L(F;n;d) \leq \frac{d}{8\left(n^{1/d}-1\right)^2} \triangleq L.$$

Proof. Let $K = \left\lfloor n^{1/d} \right\rfloor$. Since $n \geq K^d$, and $L(\cdot;n;\cdot)$ is a decreasing function of n according to Definition 5, the welfare loss

$$L(F;n;d) \leq L\left(F;K^d;d\right) \leq \sum_{l=1}^{d} L(F_l;K).$$

By Proposition 2, $L(F_l;K) \leq \frac{1}{8K^2}$, for all $1 \leq l \leq d$. Therefore,

$$L(F;n;d) \leq \frac{d}{8K^2} \overset{(a)}{\leq} \frac{d}{8\left(n^{1/d}-1\right)^2},$$

where (a) holds because $K = \left\lfloor n^{1/d} \right\rfloor \geq n^{1/d} - 1$. ■

Repeated scalar quantization does not result in the optimal n-item menu in general. Indeed, in higher dimensions, we can use more subtle vector quantization methods to design better finite menus. To achieve this, we bundle the consumer's preferences over multiple goods as a vector, instead of viewing them separately as independent choices. In the following, we first introduce the Lloyd-Max conditions that the optimal multi-dimensional finite menu must satisfy. We then discuss the vector quantization gain, and a significant decomposition theorem. We use these results to derive a vector-quantization-based upper bound and lower bound on the welfare loss.

5.2.3 Lloyd-Max Optimality Conditions

As mentioned above, the social welfare maximization problem can be viewed as a d-dimensional n-region vector quantization problem. Therefore, the optimal menu $\{B_k^*, \boldsymbol{q}_k^*\}_{k=1}^n$ must satisfy the following Lloyd-Max optimality conditions for vector quantization.

Theorem 2 (Lloyd-Max Conditions (Gersho and Gray (2007)))
The optimal n-item menu $\{B_k^*, \boldsymbol{q}_k^*\}_{k=1}^n$ which maximizes the expected social welfare (13) *must satisfy:*

$$B_k^* = \{\boldsymbol{\theta} \in \Theta : \|\boldsymbol{\theta} - \boldsymbol{q}_k^*\| \leq \|\boldsymbol{\theta} - \boldsymbol{q}_l^*\| \text{ for all } l = 1, \ldots, n\}, \tag{16}$$

$$\boldsymbol{q}_k^* = \mathbb{E}[\boldsymbol{\theta} | \boldsymbol{\theta} \in B_k^*]. \tag{17}$$

In other words, $\{B_k^*\}_{k=1}^n$ is chosen as a Voronoi partition (a set of the nearest-neighbor regions) with respect to $\{\boldsymbol{q}_k^*\}_{k=1}^n$, and $\boldsymbol{q}_k^*$ is chosen as the conditional mean of $\boldsymbol{\theta}$ given that $\boldsymbol{\theta}$ lies in the region B_k^*.

As in the one-dimensional case, it is difficult to get the closed form of the optimal finite menus from the Lloyd-Max conditions for general distributions. Nevertheless, Lloyd-Max conditions imply a useful algorithm for designing the optimal finite menu in many dimensions as well (see. Chapter 11.2, Gersho and Gray (2007)).

5.2.4 Vector Quantization Gain

Recall that in Lemma 2, we showed that $L\left(F; K^d; d\right) \leq \sum_{l=1}^d L\left(F_l; K\right)$. Note that $L\left(F; K^d; d\right)$ is the welfare loss induced by the optimal d-dimensional K^d-item menu, or equivalently the optimal vector quantizer which satisfies the Lloyd-Max conditions in Theorem 2. In contrast, $\sum_{l=1}^d L\left(F_l; K\right)$ is the welfare loss induced by the repeated scalar quantizer, discussed in Section 5.2.2. Lemma 2 implies that we can typically reduce the welfare loss by using true vector quantization rather than repeated scalar quantization to design the finite menus. The ratio of $\sum_{l=1}^d L\left(F_l; K\right)$ to $L\left(F; K^d; d\right)$ captures this gain, defined as the vector quantization gain.

Definition 7 (Vector Quantization Gain for Social Welfare)
The vector quantization gain for social welfare is defined by the ratio of the welfare loss induced by the repeated scalar quantizer to the welfare loss induced by the optimal vector quantizer:

$$G_{SW} = \frac{\sum_{l=1}^d L\left(F_l; K\right)}{L\left(F; K^d; d\right)}.$$

To simplify our analysis, we assume from now on that the consumer's preferences over d goods, $\theta_1, \ldots, \theta_d$, are identically, but not necessarily independently distributed. Denote by f and $\hat{f}$ the joint density function and the marginal density function, respectively. Lookabaugh and Gray (1989) showed that, when the number of items per dimension is sufficiently large, the vector quantization gain can be decomposed into three terms.

Theorem 3 (Decomposition)
Suppose that the consumer's preferences over d goods are identically distributed. When the

number K of items in each dimension becomes sufficiently large, the vector quantization gain can be decomposed as follows:

$$G_{SW} \approx SF(d) \times S(\hat{f}, d) \times DP(\hat{f}, f, d),$$

where $SF(d)$, $S(\hat{f}, d)$, and $DP(\hat{f}, f, d)$ are called the space-filling advantage, shape advantage and dependence advantage, respectively.

Space-Filling Advantage Recall that in repeated scalar quantization, we simply choose orthotopes (Cartesian product of intervals) to partition the type space. When $d \geq 2$, however, we have the freedom to select more complex quantization regions. This leads to the space-filling advantage. For example, Gersho (1979) showed that the optimal admissible polygons $\{B_k^*\}_{k=1}^n$ yielding the minimum welfare loss for the i.i.d. uniform distribution in $\mathbb{R}^2$ are regular hexagons, rather than equilateral triangles or squares.

More generally, when $d \geq 3$, it is optimal to choose the admissible polytopes as close as possible to the d-dimensional sphere. Indeed as the dimension increases, the optimal admissible polytope becomes geometrically closer and closer to the sphere, leading to an asymptotic space-filling advantage, $\lim_{d\to\infty} SF(d) = \frac{\pi e}{6}$, as established by Conway and Sloane (1985).

Shape Advantage The shape advantage can be written as

$$S\left(\hat{f}, d\right) = \frac{\left[\int \left(\hat{f}(\theta_l)\right)^{1/3} d\theta_l\right]^3}{\left[\int \left(\hat{f}(\theta_l)\right)^{d/d+2} d\theta_l\right]^{d+2}}. \tag{18}$$

Given the dimension d, the shape advantage depends solely on the marginal density function $\hat{f}$. The uniform distribution does not provide any shape advantage, and we can easily verify that $S(\hat{f}, d) = 1$, if $\hat{f}$ is the uniform density.

Dependence Advantage The dependence advantage can be written as

$$DP\left(\hat{f}, f, d\right) = \frac{\left[\int \left(\hat{f}(\theta_l)\right)^{d/d+2} d\theta_l\right]^{d+2}}{\left[\int \ldots \int \left(f(\theta_1, \ldots, \theta_d)\right)^{d/d+2} d\theta_1 \ldots d\theta_d\right]^{(d+2)/d}}. \tag{19}$$

Given the dimension d, the dependence advantage is determined by both the joint density f and the marginal density $\hat{f}$, and thus implicitly by the correlation among the types $\theta_1, \ldots, \theta_d$. As expected, we can easily verify that there is no dependence advantage for i.i.d. random variables, i.e., $DP(\hat{f}, f, d) = 1$.

5.2.5 Vector-Quantization-Based Upper Bound on Welfare Loss

By taking into account the vector quantization gain, we obtain an upper bound on the welfare loss, called the vector-quantization-based upper bound, which is (asymptotically) tighter than the bound given in Proposition 7.

Proposition 8 (Vector-Quantization-Based Upper Bound)
Suppose that the consumer's types $\theta_1, \ldots, \theta_d$ are identically distributed with the joint distribution function $F \in \mathcal{F}$. When $n^{1/d}$ is sufficiently large,

$$L(F; n; d) \leq \frac{1}{SF \times S \times DP} \times \frac{d}{8\left(n^{1/d} - 1\right)^2} = \frac{1}{SF \times S \times DP} L \triangleq L_{VQ}.$$

Proof. Let $K = \left\lfloor n^{1/d} \right\rfloor$ approximate the number of items per dimension, and $\hat{F}$ be the marginal distribution function for each type. When K is sufficiently large, the vector quantization gain $G_{SW} = \frac{d \times L\left(\hat{F}; K\right)}{L\left(F; K^d; d\right)} = SF \times S \times DP$. By Proposition 2, $L\left(\hat{F}; K\right) \leq \frac{1}{8K^2}$. Therefore,

$$L(F; n; d) \overset{(a)}{\leq} L\left(F; K^d; d\right) \leq \frac{1}{G_{SW}} \times \frac{d}{8K^2} \overset{(b)}{\leq} \frac{1}{SF \times S \times DP} \times \frac{d}{8\left(n^{1/d} - 1\right)^2},$$

where (a) holds because $n \geq K^d$ and (b) holds because $K \geq n^{1/d} - 1$. ■

Compared with the upper bound L derived in Proposition 7, the upper bound L_{VQ} differs in several respects. First, the consumer's types over multiple products are assumed to be identically distributed, which is not necessary for L. In addition, unlike the case for L, L_{VQ} depends on the distributions of the types. Finally, L_{VQ} is an asymptotic upper bound on $L(F; n; d)$ when $n^{1/d}$ is sufficiently large, whereas L is an upper bound for any $n \geq 1, d \geq 1$. The vector quantization based upper bound L_{VQ} captures the vector quantization gain $G_{SW} = SF \times S \times DP$, and is useful for explaining the benefit of jointly designing finite menus in multiple dimensions.

Example 1 (Multi-Dimensional Uniform Distribution)
Suppose the types $\theta_1, \ldots, \theta_d$ are i.i.d. uniformly distributed on $[0,1]$. If $n^{1/d}$ is sufficiently large, then $L(F_{\mathcal{U}}; n; d) \leq L_{VQ,1} = \frac{1}{SF(d)} L$. When d is also sufficiently large, $L_{VQ,1} \approx \frac{6}{\pi e} L \approx 0.703L$. Since $\theta_1, \ldots, \theta_d$ are i.i.d. uniformly distributed, there are no shape and dependence advantages, i.e., $S = 1, DP = 1$. Therefore, $L_{VQ,1} = \frac{1}{SF(d)} L$.

Thus, even for the i.i.d. uniform distribution, by choosing better (more spherical) quantization regions than orthotopes, we obtain an asymptotic upper bound L_{VQ} which is tighter than L by a factor of $\frac{1}{SF}$, due to the space-filling advantage. This example shows that it is beneficial, especially in high dimensions, to bundle the consumer's preferences over multiple goods as a vector, instead of treating them separately as independent scalar quantities.

5.2.6 Lower Bound on Welfare Loss

From the above discussion, one can see that for the i.i.d. uniform distribution, vector quantization can provide only the space-filling advantage, which is upper bounded by $\frac{\pi e}{6}$. We combine this result with the welfare loss for the uniform distribution in one dimension to obtain a lower bound on the welfare loss in higher dimensions.

Lemma 3 (Welfare Loss for I.I.D. Uniform Distribution)
Suppose the types $\theta_1, \ldots, \theta_d$ are i.i.d. uniformly distributed on $[0,1]$. If $n^{1/d}$ is sufficiently large, then $L(F_{\mathcal{U}}; n; d) \geq \frac{1}{4\pi e} \frac{d}{\left(n^{1/d}+1\right)^2}$.

Proof. Let $\hat{K} = \lceil n^{1/d} \rceil$ approximate the number of items per dimension, and $\hat{F}_{\mathcal{U}}$ be the (marginal) uniform distribution on $[0,1]$. When $\hat{K}$ is sufficiently large, the vector quantization gain for the social welfare $G_{SW} = \frac{d \times L\left(\hat{F}_{\mathcal{U}}; \hat{K}\right)}{L\left(F_{\mathcal{U}}; \hat{K}^d; d\right)} = SF(d) \times S\left(\hat{f}_{\mathcal{U}}, d\right) \times DP\left(\hat{f}_{\mathcal{U}}, f_{\mathcal{U}}, d\right)$. By Conway and Sloane (1985), the space-filling advantage is $SF(d) \leq \frac{\pi e}{6}$. Clearly, $S\left(\hat{f}_{\mathcal{U}}, d\right) = 1$, and $DP\left(\hat{f}_{\mathcal{U}}, f_{\mathcal{U}}, d\right) = 1$, i.e., there are no shape and dependence advantages for the i.i.d. uniform distribution. Therefore,

$$\frac{d \times L\left(\hat{F}_{\mathcal{U}}; \hat{K}\right)}{L\left(F_{\mathcal{U}}; \hat{K}^d; d\right)} \leq \frac{\pi e}{6}.$$

Note that $L\left(\hat{F}_{\mathcal{U}}; \hat{K}\right) = \frac{1}{24\hat{K}^2}$ by Lemma 1. Thus we have

$$L(F_{\mathcal{U}}; n; d) \overset{(a)}{\geq} L\left(F_{\mathcal{U}}; \hat{K}^d; d\right) \geq \frac{1}{4\pi e} \frac{d}{\hat{K}^2} \overset{(b)}{\geq} \frac{1}{4\pi e} \frac{d}{\left(n^{1/d}+1\right)^2},$$

where (a) holds since $n \leq \hat{K}^d$ and $L(\cdot; n; \cdot)$ is a decreasing function of n, and (b) holds since $\hat{K} \leq n^{1/d} + 1$. ■

Proposition 7 provides an upper bound on the welfare loss for any joint distribution $F \in \mathcal{F}$, and Lemma 3 provides a lower bound on the welfare loss for the i.i.d. uniform distribution, which is a lower bound on the maximum welfare loss $L(n; d)$. Hence, we have the following result, which states that the maximal welfare loss induced by the n-item menu converges to zero at a rate proportional to $\frac{d}{n^{2/d}}$ as the number of items n tends to infinity.

Proposition 9 (Bounds on Maximum Welfare Loss)
If $n^{1/d}$ is sufficiently large, then the maximum welfare loss satisfies:

$$\frac{1}{4\pi e} \frac{d}{\left(n^{1/d} + 1\right)^2} \leq L(n; d) \leq \frac{1}{8} \frac{d}{\left(n^{1/d} - 1\right)^2}.$$

5.3 Multi-Dimensional Revenue Maximization

We complete our analysis by finally considering the revenue maximization problem in many dimensions. The problem for the seller in the direct mechanism without communication constraints is given by maximizing

$$\Pi^* (\infty) = \max_{\{\boldsymbol{q}(\boldsymbol{\theta}), \boldsymbol{t}(\boldsymbol{\theta})\}} \mathbb{E}_{\boldsymbol{\theta}} \left[t(\boldsymbol{\theta}) - \frac{1}{2} \boldsymbol{q}(\boldsymbol{\theta})^T \boldsymbol{q}(\boldsymbol{\theta}) \right],$$

subject to the individual rationality (participation) constraints: $\boldsymbol{\theta}^T \boldsymbol{q}(\boldsymbol{\theta}) - t(\boldsymbol{\theta}) \geq 0$ and the incentive constraints: $\boldsymbol{\theta}^T \boldsymbol{q}(\boldsymbol{\theta}) - t(\boldsymbol{\theta}) \geq \boldsymbol{\theta}^T \boldsymbol{q}(\boldsymbol{\theta}') - t(\boldsymbol{\theta}')$ for all $\boldsymbol{\theta}, \boldsymbol{\theta}' \in \Theta$, where $\Theta \subseteq \mathbb{R}^d_+$ is the type space. In a seminal contribution, Armstrong (1996) showed that the firm's revenue can be written as:

$$\Pi^* (\infty) = \mathbb{E}_{\boldsymbol{\theta}} \left[\boldsymbol{\psi}(\boldsymbol{\theta})^T \boldsymbol{q}(\boldsymbol{\theta}) - \frac{1}{2} \boldsymbol{q}(\boldsymbol{\theta})^T \boldsymbol{q}(\boldsymbol{\theta}) \right],$$

where

$$\boldsymbol{\psi}(\boldsymbol{\theta}) = h(\boldsymbol{\theta}) \boldsymbol{\theta}, \quad h(\boldsymbol{\theta}) = 1 - \frac{\beta(\boldsymbol{\theta})}{f(\boldsymbol{\theta})}, \quad \beta(\boldsymbol{\theta}) = \int_1^{+\infty} f(r\boldsymbol{\theta}) r^{d-1} dr. \tag{20}$$

The optimal continuous menu satisfies:

$$\boldsymbol{q}^* (\boldsymbol{\theta}) = \begin{cases} \boldsymbol{\psi}(\boldsymbol{\theta}) & \text{if } \boldsymbol{\theta} \in \tilde{\Theta} \\ \boldsymbol{0} & \text{if } \boldsymbol{\theta} \in \Theta \backslash \tilde{\Theta} \end{cases},$$

where $\tilde{\Theta} = \{\boldsymbol{\theta} \in \Theta : h(\boldsymbol{\theta}) \geq 0\}$ is the active type space. The maximum revenue can therefore be expressed as:

$$\Pi^*(\infty) = \mathbb{E}_{\boldsymbol{\theta}}\left[\boldsymbol{\psi}(\boldsymbol{\theta})^T \boldsymbol{q}^*(\boldsymbol{\theta}) - \frac{1}{2}\boldsymbol{q}^*(\boldsymbol{\theta})^T \boldsymbol{q}^*(\boldsymbol{\theta})\right] = \frac{1}{2}\int_{\tilde{\Theta}} \boldsymbol{\psi}(\boldsymbol{\theta})^T \boldsymbol{\psi}(\boldsymbol{\theta})\, dF(\boldsymbol{\theta}). \tag{21}$$

The finite version of the revenue maximization problem specifies a menu which contains $n < \infty$ different items. Armstrong (1996) already observed that some consumers with low type vectors in the active type space $\tilde{\Theta}$ will leave the market when a finite menu is offered. Thus, there exists a region $B_0 \subseteq \tilde{\Theta}$, determined endogenously, such that all consumers with $\boldsymbol{\theta} \in B_0$ will choose $\boldsymbol{q}_0 = \mathbf{0}, t_0 = 0$. In this case, $\{B_k, \boldsymbol{q}_k\}_{k=1}^n$ characterizes a multi-product finite menu, called the n-item menu, and let $\mathcal{M}_F$ be the set of all n-item menus for a given distribution F:

$$\mathcal{M}_F = \left\{\{B_k, \boldsymbol{q}_k\}_{k=0}^n : \boldsymbol{q}_0 = \mathbf{0}, B_i \cap B_j = \emptyset \text{ if } i \neq j, \text{ and } \cup_{k=0}^n B_k = \tilde{\Theta}\right\}.$$

The seller chooses $\{B_k, \boldsymbol{q}_k\}_{k=0}^n$ to maximize the expected revenue:

$$\Pi^*(n) = \max_{\{B_k, \boldsymbol{q}_k\}_{k=0}^n \in \mathcal{M}_F} \left\{\mathbb{E}_{\boldsymbol{\theta}}\left[\boldsymbol{\psi}(\boldsymbol{\theta})^T \boldsymbol{q} - \frac{1}{2}\boldsymbol{q}^T \boldsymbol{q}\right]\right\}. \tag{22}$$

Define for $\boldsymbol{\theta} \in \tilde{\Theta} = \{\boldsymbol{\theta} \in \Theta : h(\boldsymbol{\theta}) \geq 0\}$, the virtual type vector

$$\hat{\boldsymbol{\theta}} = \boldsymbol{\psi}(\boldsymbol{\theta}) = h(\boldsymbol{\theta})\,\boldsymbol{\theta}. \tag{23}$$

Let the virtual type space be denoted by $\hat{\Theta} = \boldsymbol{\psi}[\Theta] = \left\{\boldsymbol{\psi}(\boldsymbol{\theta}) : \boldsymbol{\theta} \in \tilde{\Theta}\right\}$. As in the one-dimensional analysis, we can transform the partition $\{B_k\}_{k=0}^n$ of the active type space into a partition $\left\{\hat{B}_k\right\}_{k=0}^n$ of the virtual type space $\hat{\Theta}$ as follows:

$$\boldsymbol{\theta} \in B_k \Leftrightarrow \hat{\boldsymbol{\theta}} \in \hat{B}_k = \{\boldsymbol{\psi}(\boldsymbol{\theta}) : \boldsymbol{\theta} \in B_k\}.$$

In the virtual space, the expected revenue for an n-item menu can be written as:

$$\Pi(n) = \mathbb{E}_{\boldsymbol{\theta}}\left[\boldsymbol{\psi}(\boldsymbol{\theta})^T \boldsymbol{q} - \frac{1}{2}\boldsymbol{q}^T \boldsymbol{q}\right] = \mathbb{E}_{\hat{\boldsymbol{\theta}}}\left[\hat{\boldsymbol{\theta}}^T \boldsymbol{q} - \frac{1}{2}\boldsymbol{q}^T \boldsymbol{q}\right],$$

and the expected revenue of the optimal n-item menu is given as:

$$\Pi^*(n) = \max_{\{B_k, \boldsymbol{q}_k\}_{k=0}^n \in \hat{\mathcal{M}}_F} \Pi(n),$$

where the set of all n-item menus in the virtual type space is given by:

$$\hat{\mathcal{M}}_F = \left\{ \left\{ \hat{B}_k, \boldsymbol{q}_k \right\}_{k=0}^{n} : \boldsymbol{q}_0 = \mathbf{0}, \hat{B}_i \cap \hat{B}_j = \emptyset \text{ if } i \neq j, \text{ and } \cup_{k=0}^{n} \hat{B}_k = \hat{\Theta} \right\}.$$

The problem is now formally equivalent to earlier welfare maximization problem (14). We now consider how well the optimal n-item menu can approximate the performance of the optimal continuous menu. That is, we consider the revenue loss in terms of the distribution function F, the number of items n, and the dimension d.

Definition 8 (Revenue Loss)
For any joint distribution function F, the revenue loss induced by the optimal n-item menu compared with the optimal continuous menu is:

$$\widehat{L}(F; n; d) \triangleq \Pi^*(\infty) - \Pi^*(n).$$

Similar to the case for welfare maximization, we are interested in the worst-case behavior of the revenue loss over all joint distributions with a d-dimensional support set (i.e., the type space) with positive and finite volume. Without loss of generality, we may assume the type space $\Theta \subseteq [0,1]^d$. Our main task is to quantify the worst-case behavior of $\widehat{L}(F; n; d)$ over all distributions $F \in \mathcal{F}$, where $\mathcal{F}$ is the set of all joint distribution functions in type space $\Theta \subseteq [0,1]^d$.

Definition 9 (Maximum Revenue Loss)
The maximum revenue loss induced by the optimal n-item menu over all $F \in \mathcal{F}$ is:

$$\widehat{L}(n; d) \triangleq \sup_{F \in \mathcal{F}} \widehat{L}(F; n; d).$$

5.3.1 Bounds on Revenue Loss

Using the connection between the revenue maximization problem and the vector quantization problem, we can obtain upper and lower bounds on the revenue loss, as in the social welfare case. For instance, by using repeated scalar quantization, we obtain the following upper bound on the revenue loss:

Proposition 10 (Upper Bound on Revenue Loss)
For any $F \in \mathcal{F}$, $n \geq 1$, $d \geq 1$, the revenue loss induced by the optimal n-item menu satisfies:

$$\widehat{L}(F; n; d) \leq \frac{d}{8\left(n^{1/d} - 1\right)^2} \triangleq \bar{\Lambda}.$$

As in the welfare case, we can introduce the vector quantization gain provided by the optimal finite menu for the seller's revenue. The decomposition result, Theorem 3, naturally also holds for the revenue problem, and thus we obtain the following vector-quantization-based upper bound on the revenue loss.

Proposition 11 (Vector-Quantization-Based Upper Bound)
Suppose that the consumer's types $\theta_1, \ldots, \theta_d$ are identically distributed with the joint distribution function $F \in \mathcal{F}$. When $n^{1/d}$ is sufficiently large,

$$\widehat{L}\left(F; n; d\right) \leq \frac{1}{SF \times S \times DP} \times \frac{d}{8\left(n^{1/d} - 1\right)^2} = \frac{1}{SF \times S \times DP} \bar{\Lambda} \triangleq \bar{\Lambda}_{VQ}.$$

By considering the vector quantization gain specifically for the i.i.d. uniform distribution, we can obtain a lower bound on the revenue loss. We can then show that the maximum revenue loss induced by the n-item menu converges to zero at a rate proportional to $\frac{d}{n^{2/d}}$ as the number of items n tends to infinity.

Proposition 12 (Revenue Loss for I.I.D. Uniform Distribution)
Suppose the types $\theta_1, \ldots, \theta_d$ are i.i.d. uniformly distributed on $[0, 1]$ with the joint distribution function $F_{\mathcal{U}}$. If $n^{1/d}$ is sufficiently large, then

$$\widehat{L}\left(F_{\mathcal{U}}; n; d\right) \geq \frac{1}{8\pi e} \frac{d}{\left(n^{1/d} + \frac{3}{2}\right)^2}.$$

Proposition 13 (Bounds on Maximum Revenue Loss)
If $n^{1/d}$ is sufficiently large, then the maximum revenue loss satisfies

$$\frac{1}{8\pi e} \frac{d}{\left(n^{1/d} + \frac{3}{2}\right)^2} \leq \widehat{L}\left(n; d\right) \leq \frac{1}{8} \frac{d}{\left(n^{1/d} - 1\right)^2}.$$

6 Conclusion

We explored the consequences of economic transactions with limited information within the concrete setting of the nonlinear pricing model. Using the linear-quadratic specification, we relate both social welfare maximization and revenue maximization to the quantization problem in information theory. Using this link, we introduce the Lloyd-Max conditions that the optimal finite menu for the socially efficient and the revenue-maximizing mechanism must satisfy. In addition, we study the performance of the finite menus relative to the optimal continuous menu. Our analysis shows that for both social welfare and the seller's revenue, the losses due to the usage of the n-item finite menu converge to zero at the rate proportional to $1/n^2$.

Based on the information-theoretic approach in the one-dimensional environment, we generalize our results to the multi-product environment. We provide a general upper bound on the losses for both the social welfare and the seller's revenue by using the repeated scalar quantization method. This bound is used to prove that the losses, due to the usage of the d-dimensional n-item finite menu, converge to zero at the rate proportional to $d/n^{2/d}$. Although such treatment is simple and helpful for estimating the order of the convergence rate of the losses, it ignores some significant features in multi-dimensional mechanism design. Therefore, we introduce the vector quantization gain and the decomposition theorem, and obtain a vector-quantization-based upper bound and a lower bound on the welfare loss and the revenue loss. The vector-quantization-based upper bound is tighter than the repeated scalar upper bound, and the improvement becomes significant in high dimensions, and/or when the correlation among the consumer's preferences over multiple products exists. This shows that it is beneficial to bundle the consumer's preferences over multiple goods, and then design the finite menus jointly in multiple dimensions.

While the nonlinear pricing environment is of interest by itself, it also represents an elementary instance of the general mechanism design environment. The simplicity of the nonlinear pricing problem arises from the fact that it can viewed as a relationship between the principal, here the seller, and a single agent, here the buyer, even in the presence of many buyers. The reason for the simplicity is that the principal does not have to solve allocative externalities. By contrast, in auctions, and other multi-agent allocation problems, the allocation (and hence the relevant information) with respect to a given agent constrains and is constrained by the allocation to the other agents.

Finally, the current analysis focused on limited information, and the ensuing problem of efficient source coding. But clearly, from an information-theoretic as well as economic viewpoint, it is natural to augment the analysis to reliable communication between agent and principal over noisy channels, the problem of channel coding, which we plan to address in future work.

7 Appendix

Proof of Lemma 2. Since a repeated scalar quantizer is a feasible K^d-region vector quantizer, we have

$$L\left(F;K^d;d\right) \leq \frac{1}{2}\sum_{k=1}^{K^d}\int_{B'_k}\left\|\boldsymbol{\theta}-\boldsymbol{q}'_k\right\|^2 dF\left(\boldsymbol{\theta}\right) = \frac{1}{2}\sum_{l=1}^{d}\left\{\sum_{k=1}^{K^d}\int_{B'_k}\left(\theta_l-q'_{k,l}\right)^2 dF\left(\boldsymbol{\theta}\right)\right\}.$$

Based on the construction of $\{B'_k\}_{k=1}^{K^d}$ and $\{\boldsymbol{q}'_k\}_{k=1}^{K^d}$, we have

$$\begin{aligned}
&\sum_{k=1}^{K^d}\int_{B'_k}\left(\theta_l-q'_{k,l}\right)^2 dF\left(\boldsymbol{\theta}\right)\\
=\ &\sum_{k_1=1}^{K}\cdots\sum_{k_d=1}^{K}\int_{A_{1,k_1}\times\ldots\times A_{d,k_d}}\left(\theta_l-r_{l,k_l}\right)^2 dF\left(\boldsymbol{\theta}\right)\\
=\ &\sum_{k_l=1}^{K}\int_{A_{l,k_l}}\left(\theta_l-r_{l,k_l}\right)^2\left\{\sum_{k_1=1}^{K}\cdots\sum_{k_{l-1}=1}^{K}\sum_{k_{l+1}=1}^{K}\cdots\sum_{k_d=1}^{K}\int_{A_{l,-k_l}} dF_{-l}\left(\boldsymbol{\theta}_{-l}|\theta_l\right)\right\} dF_l\left(\theta_l\right),
\end{aligned}$$

where $A_{l,-k_l} = A_{1,k_1}\times\ldots\times A_{l-1,k_{l-1}}\times A_{l+1,k_{l+1}}\ldots\times A_{d,k_d}$, $\boldsymbol{\theta}_{-l} = (\theta_1,\ldots,\theta_{l-1},\theta_{l+1},\ldots,\theta_d)$, and $F_{-l}\left(\cdot|\cdot\right)$ is the conditional distribution function of $\boldsymbol{\theta}_{-l}$ given θ_l. Note that

$$\sum_{k_1=1}^{K}\cdots\sum_{k_{l-1}=1}^{K}\sum_{k_{l+1}=1}^{K}\cdots\sum_{k_d=1}^{K}\int_{A_{l,-k_l}} dF_{-l}\left(\boldsymbol{\theta}_{-l}|\theta_l\right) = \int_0^1\cdots\int_0^1 dF_{-l}\left(\boldsymbol{\theta}_{-l}|\theta_l\right) = 1,$$

for any θ_l. Thus,

$$\sum_{k=1}^{K^d}\int_{B'_k}\left(\theta_l-q'_{k,l}\right)^2 dF\left(\boldsymbol{\theta}\right) = \sum_{k_l=1}^{K}\int_{A_{l,k_l}}\left(\theta_l-r_{l,k_l}\right)^2 dF_l\left(\theta_l\right).$$

Therefore,

$$L\left(F;n;d\right) \leq \frac{1}{2}\sum_{l=1}^{d}\left\{\sum_{k_l=1}^{K}\int_{A_{l,k_l}}\left(\theta_l-r_{l,k_l}\right)^2 dF_l\left(\theta_l\right)\right\}.$$

This is true for any set of d independent scalar quantizers $\{A_{l,k_l}, r_{l,k_l}\}_{k_l=1}^{K} \in \mathcal{M}_{F_l}, 1\leq l\leq d$. If we independently choose the d optimal quantizers for the types $\theta_1,\ldots,\theta_d$, we obtain the upper bound:

$$L\left(F;n;d\right) \leq \sum_{l=1}^{d}\left\{\inf_{\left\{A_{l,k_l},r_{l,k_l}\right\}_{k_l=1}^{K}\in\mathcal{M}_{F_l}}\frac{1}{2}\sum_{k_l=1}^{K}\int_{A_{l,k_l}}\left(\theta_l-r_{l,k_l}\right)^2 dF_l\left(\theta_l\right)\right\} = \sum_{l=1}^{d}L\left(F_l;K\right),$$

which completes the proof. ■

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