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Abstract

Consider the following "informational robustness" question: what can we say about the set of outcomes that may arise in equilibrium of a Bayesian game if players may observe some additional information? This set of outcomes will correspond to a solution concept that is weaker than equilibrium, with the solution concept depending on what restrictions are imposed on the additional information.

We describe a unified approach encompassing prior informational robustness results, as well as identifying the solution concept that corresponds to no restrictions on the additional information; this version of rationalizability depends only on the support of players' beliefs and implies novel predictions in classic economic environments of coordination and trading games.

Our results generalize from complete to incomplete information the classical results in Aumann (1974, 1987) and Brandenburger and Dekel (1987) which can be (and were) given informational robustness interpretations. We discuss the relation between informational robustness and "epistemic" foundations of solution concepts.

Jel Classification: C72, C79, D82, D83.

Keywords: Incomplete Information, Informational Robustness, Bayes Correlated Equilibrium, Interim Corrrelated Rationalizability, Belief Free Rationalizability

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1 Introduction

Classical analysis of Bayesian games treats the information structure of the players as given, and examines the consequences of equilibrium behavior given that information structure. But the exact information structure is often not known to the analyst, and thus it is interesting to examine the implications of equilibrium in all information structures that the analyst thinks possible, and thus identify predictions that are robust to informational assumptions.[1] There is a close connection between relaxing informational assumptions and relaxing solution concepts. Consider the solution concept of Nash equilibrium in a complete information game. Suppose that we allow players to observe arbitrary (payoff-irrelevant) signals. If the common prior assumption is maintained, then Aumann (1987) showed that distribution of equilibrium behavior would correspond to an (objective) correlated equilibrium. Without the common prior assumption, Brandenburger and Dekel (1987) and Tan and Werlang (1988) showed that all one can say about the resulting equilibrium behavior is that each player will choose a (correlated) rationalizable action.[2]

What are the Bayesian analogues of these results? Suppose now that payoffs depend on a "payoff state" and that players may also observe *payoff-irrelevant* signals that do not change their beliefs and higher-order beliefs about the payoff state. Some Bayesian analogues of the complete information results are known in this setting. If the common prior assumption is maintained, and we study (Bayes Nash) equilibria on the expanded type space with payoff-irrelevant signals, then the distribution of equilibrium behavior corresponds to a *belief invariant Bayes correlated equilibrium* (Liu (2015)). Without the common prior assumption, all one can say about the resulting equilibrium behavior is that players will choose *interim correlated rationalizable* actions (Dekel, Fudenberg, and Morris (2007)). Alternative extensions of the complete information results to Bayesian games arise if we allow players to observe *payoff-relevant* signals, i.e., signals

[1]As discussed in more detail below, we have examined - in prior work - such informational robustness questions both in the context of mechanism design (Bergemann and Morris (2012)) and in the context of general games (Bergemann and Morris (2015)).

[2]Aumann (1974) introduced correlated equilibrium without the common prior assumption. Brandenburger and Dekel (1987) showed that correlated rationalizability characterizes the set of actions that could be played in subjective correlated equilibrium. We mostly avoid discussing subjective correlated equilibrium, although our non-common prior results could also be expressed as incomplete information generalizations of subjective correlated equilibrium.

that refine their initial beliefs and higher-order beliefs about the payoff state. If the common prior assumption is maintained, and we study equilibria on the expanded type space with payoff-relevant signals, then the distribution of equilibrium behavior corresponds to a *Bayes correlated equilibrium* (Bergemann and Morris (2015)). In recent work (Bergemann and Morris (2013)), we have argued that this tool is useful not only for characterizing robust predictions in games, but also for addressing other questions, such as the information design problem of finding the optimal information structure in a given strategic setting.[3]

The first contribution of this paper is to present a unified, "informational robustness", interpretation of these results. The second contribution is to identify and analyze the novel solution concept of belief-free rationalizability that corresponds to a case not described above (and has been previously studied only in special cases and without the relation to informational robustness).[4] An action is said to be belief-free rationalizable if it survives the following iterative deletion process using only the support of a type's beliefs, i.e., the set of profiles of other players' types and states that he thinks possible. At each round, we delete an action for a particular type if there is no conjecture about the other players' actions and states, such that the action is a best response to the conjecture; and with the restriction that the conjecture assigns zero probability to (1) profiles of the other players' actions and types that have already been deleted; and (2) profiles of other players' types and states that are not in the support of his original beliefs. We then establish in Proposition 8 that an action can be played in equilibrium by a given type who may observe extra payoff-relevant signals (not necessarily consistent with the common prior assumption) if and only if it is belief-free rationalizable for that type.

The following table now summarizes the consequences of equilibrium under incomplete information if we allow players to observe additional signals that may or may not be consistent with the common prior and may or may not be payoff-relevant:

	payoff-relevant signals	payoff-irrelevant signals only
common prior	Bayes correlated equilibrium	belief invariant Bayes correlated equilibrium
non common prior	belief-free rationalizability	interim correlated rationalizability

[3]Thus it provides a many player analogue of the Bayesian persuasion problem studied by Kamenica and Gentzkow (2011).

[4]We discuss the use of this name in Battigalli, Di Tillio, Grillo, and Penta (2011) in a more restricted environment below.

In the special case of complete information (i.e., a unique payoff state), both rationalizability results reduce to the result of Brandenburger and Dekel (1987) and both correlated equilibrium results reduce to the result of Aumann (1987).

We study the implications of belief-free rationalizability in two player two action games where each player is choosing between a risky action and a safe action. The safe action always gives a payoff of zero. The payoff to the risky action depends on the other player's action and the payoff state. The payoff to the risky action if the other player takes the safe action is always negative, and can be interpreted as the cost of taking the risky action. Two important applications, within this class of games, are studied. In coordination games - where if both players take the risky action, the payoffs of the players always have the same sign - we interpret the risky action as "invest". In trading games - where if both players take the risky action, the payoffs of the players have different signs - we interpret the risky action as accepting a trade. The safe action ("don't invest" or "reject trade") is always belief-free rationalizable in these games. We characterize when the risky action ("invest" or "accept trade") is belief-free rationalizable.

An event is said to be *commonly possible* for a player if he thinks that the event is possible (i.e., assigns it strictly positive probability), thinks that it is possible that both the event is true and that the other player thinks it is possible; and so on. Invest (the risky action in the coordination game) is belief-free rationalizable for a player if and only if it is a common possibility for that player that the payoff from both players investing is positive. Accepting trade (the risky action in the trading game) is belief-free rationalizable for a player if and only if an analogous iterated statement about possibility is true: (i) each player thinks that it is possible that he gains from trade; (ii) each player thinks it is possible that both he gains from trade and that (i) holds for the other player; and so on. This property can be given an alternative characterization in terms of the existence of cycles of types and generalizes an analysis of rationalizable trade in Morris and Skiadas (2000). We also compare these characterizations with the more refined predictions under the more refined solution concepts discussed above.

All these solution concepts have simpler statements and interpretations in the special case of "payoff type" environments, where it is assumed that the payoff state can be represented as a vector of player specific "payoff types", and each player is certain of his own payoff type. This "payoff type" assumption corresponds to the assumption that there is "distributed certainty," i.e., the join of players' information reveals the true state (while noting that this assumption is not without loss of generality). But under this assumption, the "correlation" in interim *correlated* rationalizability is no longer relevant, and it is equivalent to interim *independent* rationalizability; the belief invariant Bayes correlated equilibrium reduces to the belief invariant Bayesian solution of Forges (2006) and Lehrer, Rosenberg, and Shmaya (2010); and Bayes correlated equilibrium reduces to the Bayesian solution of Forges (1993).

Much of the literature - for one reason or another - focusses on the special case of "payoff type" environments. This assumption is implicit in much of the literature on incomplete information correlated equilibrium, e.g., in the solution concepts and papers cited in the previous paragraph. There are two important special cases where belief-free rationalizability has already been applied in "payoff type" environments. A leading example of a payoff type environment is a private values environment (where a player's payoff depends only on his own payoff type), and Chen, Micali, and Pass (2015) have proposed what we are calling belief-free rationalizability in this context and used it for novel results on robust revenue maximization. Payoff type environments without private values were the focus of earlier work of ours on robust mechanism design collected in Bergemann and Morris (2012). In that work, we (implicitly) considered the special case where all players believed that all payoff type profiles of other players were possible. In this special case, belief-free rationalizability has a particularly simple characterization. For each payoff type, iteratively delete actions for that payoff type which are not a best response against any conjecture over others players' actions and types that have survived the iterated deletion procedure so far. This solution concept (with appropriate informational robustness foundations) was used in a number of our papers on robust mechanism design (see Bergemann and Morris (2009a), (2009b), (2011)). Battigalli, Di Tillio, Grillo, and Penta (2011) labelled this solution concept "belief-free rationalizability". We used Bayes correlated equilibrium (in the special case of payoff type environments) in Bergemann and Morris (2008). The unified treatment of informational robustness thus also embeds both our earlier work on robust mechanism design and our more recent work on robust predictions in games (Bergemann and Morris (2013), (2015)).

Battigalli and Siniscalchi (2003b) introduced the notion of "Δ-rationalizability" for both complete and incomplete information environments, building in arbitrary restrictions on the beliefs of any type about other players' types and actions, and states. Battigalli, Di Tillio, Grillo, and Penta (2011) describes how interim correlated rationalizability (in general) and belief-free rationalizability (in the case of payoff type spaces) are special cases of "Δ-rationalizability", where particular restrictions are placed on beliefs about other players' types and states. Belief-free rationalizability could also be given a Δ-rationalizability formulation, outside of payoff type environments, where the corresponding type dependent restriction on beliefs would be on the support of beliefs only.

For purposes of the informational robustness approach in this paper, we take as given the standard solution concept of (Bayes Nash) equilibrium and examine how outcomes under this solution concept vary as we change the information structure. We do not provide a separate justification for using equilibrium as a solution concept. We use equilibrium as a solution concept even when the information structure is not consistent with the common prior assumption. There is a tension between assuming equilibrium - a solution concept that has correct common beliefs built into it - in environments where the common prior assumption is not satisfied. Thus Dekel, Fudenberg, and Levine (2004) argue that natural learning justifications that would explain equilibrium in an incomplete information setting would also give rise to a learning justification of common prior beliefs. We agree with these concerns. We follow Brandenburger and Dekel (1987) in showing *even if* one makes the strong (and perhaps unjustified) assumption of equilibrium, one cannot remove the possibility of (appropriately defined) rationalizable actions being played if non-common prior payoff irrelevant signals may be observed. In this sense, our results - like those of Brandenburger and Dekel (1987) - provide an alternative argument against using equilibrium as a solution concept on non common-prior type spaces.

We framed the complete information results of Brandenburger and Dekel (1987) and Aumann (1987) as "informational robustness" results, i.e., what happens to equilibrium predictions if we allow players to observe additional (payoff-irrelevant) signals, and this corresponds to the formal statements of their results.[5] However, both papers interpret their results informally as establishing foundations for solution concepts by establishing that they correspond to the implications of common certainty of rationality,[6] with or without extra epistemic assumptions, and the later

[5] Thus Proposition 2.1 of Brandenburger and Dekel (1987), while stated in the language of interim payoffs, established that the set of actions played in an appropriate version of subjective correlated equilibrium were equal to the correlated rationalizable actions. The main theorem of Aumann (1987) showed that under assumptions equivalent to Bayes Nash equilibrium on a common prior type space with payoff-irrelevant signals, the ex ante distribution of play corresponds to an (objective) correlated equilibrium. Aumann (1974) has an explicit informational robustness motivation.

[6] Aumann (1987) notes in the introduction that he assumes "common knowledge that each player chooses a strategy that maximizes his expected utility given his information". Brandenburger and Dekel (1987) write in

"epistemic foundations" has developed more formal statements of these results as consequence of common certainty of rationality.[7] In this paper, we deliberately focus on a narrower informational robustness interpretation of the results both because this is the interpretation that is relevant for our applications and because the modern epistemic foundations literature addresses a wide set of important but subtle issues that are relevant for the epistemic interpretation but moot for our informational robustness interpretation. Desiderata that are important in the modern epistemic foundations literature are therefore not addressed, including (i) the removal of reference to players' beliefs about their own types or counterfactual belief of types (Aumann and Brandenburger (1995)); (ii) restricting attention to state spaces that reflect "expressible" statements about the model (Brandenburger and Friedenberg (2008) and Battigalli, Di Tillio, Grillo, and Penta (2011)); (iii) giving an interim interpretation of the common prior assumption (Dekel and Siniscalchi (2014)).

The informational robustness results in this paper involve asking what happens if players observe extra signals about payoffs, but without allowing payoff perturbations. A related but different strand of the literature (Fudenberg, Kreps, and Levine (1988), Kajii and Morris (1997) and Weinstein and Yildiz (2007)) examines the robustness of equilibrium predictions to payoff perturbations about which players face uncertainty.

We discuss the four solution concepts in Section 2 and applications in Section 3. In Section 4, we describe how the solution concepts specialize to complete information rationalizability and correlated equilibrium in the case of complete information games, and widely used and simpler solution concepts in the case of payoff-type environments. Informational robustness foundations of the solution concepts are reported in Section 5, as well as the relation to the epistemic foundations literature.

the introduction that their approach "starts from the assumption that the rationality of the players is common knowledge." We follow the recent literature in replacing the term "knowledge" in the expression common knowledge because it corresponds to "belief with probability 1," rather than "true belief" (the meaning of knowledge in philosophy and general discourse). We use "certainty" to mean "belief with probability 1".

[7] Thus Dekel and Siniscalchi (2014) state a modern version of the main result of Brandenburger and Dekel (1987) as Theorem 1 and a (somewhat) more modern statement of Aumann (1987) in Section 4.6.2.

2 Four Solution Concepts

We will fix a finite set of players $1, ..., I$ and a finite set of payoff-relevant states Θ.

We divide a standard description of an incomplete information game into a "basic game" and a "type space". A basic game $\mathcal{G} = (A_i, u_i)_{i=1}^I$ consists of, for each player, a finite set of possible actions A_i and a payoff function $u_i : A \times \Theta \to \mathbb{R}$ where $A = A_1 \times \cdots \times A_I$.[8] A type space $\mathcal{T} = (T_i, \pi_i)_{i=1}^I$ consists of, for each player, a finite set of types T_i and, for each player, a belief over others' types and the state, $\pi_i : T_i \to \Delta(T_{-i} \times \Theta)$.[9] An incomplete information game consists of a basic game $\mathcal{G} = (A_i, u_i)_{i=1}^I$ and a type space $\mathcal{T} = (T_i, \pi_i)_{i=1}^I$. We begin by discussing "classical solution concepts" for the fixed incomplete information game $(\mathcal{G}, \mathcal{T})$, meaning that we define solution concepts without referring to informational robustness (or epistemic) foundations.

We consider two alternative definitions of rationalizability in game $(\mathcal{G}, \mathcal{T})$. First consider *interim correlated rationalizability* (Dekel, Fudenberg, and Morris (2007)). An action is interim correlated rationalizable for a type t_i if we iteratively delete actions which are not a best response to any supporting conjecture over other players' actions and types, as well as states, which (1) puts probability 1 on action type profiles which have survived the iterated deletion procedure so far, and (2) has a marginal belief over others' types and states which is consistent with that type's beliefs on the type space. Crucially, this definition allows arbitrary correlation in the supporting conjecture as long as (1) and (2) are satisfied. Formally, let $ICR_i^0(t_i) = A_i$ and let $ICR_i^{n+1}(t_i)$ equal the set of actions for which there exists $\nu_i \in \Delta(A_{-i} \times T_{-i} \times \Theta)$ such that

$$\begin{aligned}
&(1)\ \nu_i(a_{-i}, t_{-i}, \theta) > 0 \Rightarrow a_j \in ICR_j^n(t_j) \text{ for each } j \neq i; \\
&(2)\ \sum_{a_{-i}} \nu_i(a_{-i}, t_{-i}, \theta) = \pi_i(t_{-i}, \theta | t_i) \text{ for each } t_{-i}, \theta; \\
&(3)\ a_i \in \arg\max_{a_i'} \sum_{a_{-i}, t_{-i}, \theta} \nu_i(a_{-i}, t_{-i}, \theta)\, u_i((a_i', a_{-i}), \theta);
\end{aligned} \tag{1}$$

and let

$$ICR_i(t_i) = \bigcap_{n \geq 1} ICR_i^n(t_i).$$

[8] In Bergemann and Morris (2015) we included a common prior on states in the description of the basic game. Because we are relaxing the common prior assumption, it is convenient to use a slightly different definition in this paper.

[9] Even if one assumes that there is a true ex ante stage, as we sometimes will, a player's prior belief over his own type will not be relevant for our analysis.

Definition 1 (Interim Correlated Rationalizable)
Action a_i is interim correlated rationalizable for type t_i (in game $(\mathcal{G},\mathcal{T})$) if $a_i \in ICR_i(t_i)$.

Now consider a more permissive rationalizability notion, *belief-free rationalizability*. The definition is the same as iterated correlated rationalizability except that we relax assumption (2) in (1) to the requirement that the rationalizing conjecture be consistent with the player's belief on the type space to the weaker requirement that its support is a subset of the player's belief on the type space. Thus we have $BFR_i^0(t_i) = A_i$ and let $BFR_i^{n+1}(t_i)$ equal the set of actions for which there exists $\nu_i \in \Delta(T_{-i} \times A_{-i} \times \Theta)$ such that

$$\begin{aligned} &(1)\ \nu_i(a_{-i}, t_{-i}, \theta) > 0 \Rightarrow a_j \in BFR_j^n(t_j) \text{ for each } j \neq i; \\ &(2)\ \sum_{a_{-i}} \nu_i(a_{-i}, t_{-i}, \theta) > 0 \Rightarrow \pi_i(t_{-i}, \theta | t_i) > 0 \text{ for each } t_{-i}, \theta; \\ &(3)\ a_i \in \arg\max_{a_i'} \sum_{a_{-i}, t_{-i}, \theta} \nu_i(a_{-i}, t_{-i}, \theta)\, u_i((a_i', a_{-i}), \theta); \end{aligned} \tag{2}$$

and let

$$BFR_i(t_i) = \bigcap_{n \geq 1} BFR_i^n(t_i).$$

Definition 2 (Belief-Free Rationalizable)
Action a_i is belief-free rationalizable for type t_i (in game $(\mathcal{G},\mathcal{T})$) if $a_i \in BFR_i(t_i)$.

Note that this definition is independent of a type's quantized beliefs and depends only on which profiles of other players' types and states he considers possible, i.e., the support of his beliefs.

We now consider two parallel definitions of (objective) incomplete information correlated equilibrium for the same incomplete information game. Type space $\mathcal{T} = (T_i, \pi_i)_{i=1}^I$ satisfies the *common prior assumption* if there exists $\pi^* \in \Delta(T \times \Theta)$ such that

$$\sum_{t_{-i}', \theta'} \pi^*\left(\left(t_i, t_{-i}'\right), \theta'\right) > 0$$

for all i and t_i, and

$$\pi_i(t_{-i}, \theta | t_i) = \frac{\pi^*((t_i, t_{-i}), \theta)}{\sum\limits_{t_{-i}', \theta'} \pi^*\left(\left(t_i, t_{-i}'\right), \theta'\right)}$$

for all i, (t_i, t_{-i}) and θ.[10]

Now we have a common prior incomplete information game $(\mathcal{G}, \mathcal{T})$. Behavior in this incomplete information game can be described by a *decision rule* mapping players' types and states to a probability distribution over players' actions, $\sigma : T \times \Theta \to \Delta(A)$. A decision rule σ satisfies *belief invariance* if, for each player i,

$$\sigma_i\left(a_i | \left(t_i, t_{-i}\right), \theta\right) \triangleq \sum_{a_{-i}} \sigma\left(\left(a_i, a_{-i}\right) | \left(t_i, t_{-i}\right), \theta\right) \tag{3}$$

is independent of (t_{-i}, θ). Thus a decision rule satisfies belief invariance if a player's action recommendation does not reveal any additional information to him about others' types and the state. This property has played an important role in the literature on incomplete information correlated equilibrium, see, Forges (1993), Forges (2006) and Lehrer, Rosenberg, and Shmaya (2010). Notice that property (2) in the iterative definition of interim correlated rationalizability in (1) was a belief invariance assumption.

Decision rule σ satisfies *obedience* if

$$\begin{aligned} & \sum_{a_{-i}, t_{-i}, \theta} \pi^*\left(t_i, t_{-i}\right) \sigma\left(\left(a_i, a_{-i}\right) | \left(t_i, t_{-i}\right), \theta\right) u_i\left(\left(a_i, a_{-i}\right), \theta\right) \\ \geq & \sum_{a_{-i}, t_{-i}, \theta} \pi^*\left(t_i, t_{-i}\right) \sigma\left(\left(a_i, a_{-i}\right) | \left(t_i, t_{-i}\right), \theta\right) u_i\left(\left(a_i', a_{-i}\right), \theta\right). \end{aligned}$$

for all i, $t_i \in T_i$ and $a_i, a_i' \in A_i$. Obedience has the following mediator interpretation. Suppose that an omniscient mediator knew players' types and the true state, randomly selected an action profile according to σ and privately informed each player of his recommended action. Would a player who knew his own type and heard the mediator's recommendation have an incentive to follow the recommendation? Obedience says that he would want to follow the recommendation.

Definition 3 (Belief Invariant Bayes Correlated Equilibrium (BIBCE)) *Decision rule σ is a belief invariant Bayes correlated equilibrium (BIBCE) if it satisfies obedience and belief invariance.*

[10] When the common prior assumption is maintained, we understand the common prior π^* to be implicitly defined by the type space. In the (special) case where multiple common priors satisfy the above properties, our results will hold true for any choice of common prior. By requiring that all types are assigned positive probability, we are making a slightly stronger assumption than some formulations of results in the literature. This version simplifies the statement of results and will also tie in with the support assumption that we impose in the informational robustness foundations in Section 5.

Liu (2015) described the subjective correlated equilibrium analogue of interim correlated rationalizability. If one then imposes the common prior assumption (as he discusses in Section 5.2), then the version of incomplete information correlated equilibrium that one obtains is given by Definition 3.[11] Its relation to the incomplete information correlated equilibrium literature is further discussed in Bergemann and Morris (2015): it is in general a weaker requirement than the belief invariant Bayesian solution of Forges (2006) and Lehrer, Rosenberg, and Shmaya (2010), because - like interim correlated rationalizability - it allows unexplained correlation between types and payoff states. It is immediate from the definitions that any action played with positive probability by a type in a belief invariant Bayes correlated equilibrium is interim correlated rationalizable.

Definition 4 (Bayes Correlated Equilibrium (BCE))
Decision rule σ is a Bayes correlated equilibrium (BCE) if it satisfies obedience.

This solution concept is studied in Bergemann and Morris (2015). It is immediate from the definitions that any action played with positive probability by a type in a Bayes correlated equilibrium is belief-free rationalizable.

3 Applications of Belief-Free Rationalizability

In Section 4, we will discuss the specialization of belief-free rationalizability to the payoff-type environment. Belief-free rationalizability in this environment has been studied in many applications. Our prior work on robust mechanism design (Bergemann and Morris (2012)) did so under a full support assumption on payoff types. Chen, Micali, and Pass (2015) report elegant robust revenue maximization results using belief-free rationalizability under private values (they also work with finite level version of the solution concept).

In this Section, we look at some classic economic problems - coordination and trade - outside of the payoff-type environment, so that there is distributed uncertainty in the sense that the join of players' information does not reveal the state. In particular, we consider a class of two player

[11] Liu (2015) actually refers to the belief invariant Bayes correlated equilibrium as the common-prior correlated equilibrium, see Definition 4 in Liu (2015). We use the current language to emphasize the belief invariance property relative to the Bayes correlated equilibrium itself.

two action games where the payoffs in state $\theta \in \Theta$ are given by

θ	Risky	Safe
Risky	$x_1(\theta) - c, x_2(\theta) - c$	$-c, 0$
Safe	$0, -c$	$0, 0$

. (4)

We will characterize belief-free rationalizable actions (as well as other solution concepts) in this class of games with restrictions giving coordination and trading interpretations. While the characterizations of belief-free rationalizability are analogous in the two classes of games, the extensions are very different under other solution concepts. In particular, in the coordination game case, interim correlated rationalizability will strictly refine belief-free rationalizability and essentially characterize belief invariant Bayes correlated equilibrium, while in the trading game case, belief-free and interim correlated rationalizability will be equivalent but no trade is the unique belief invariant Bayes correlated equilibrium.

Before presenting these characterizations, we report a general language for discussing possibility and common possibility that is useful in the characterization of rationalizable behavior in both classes of games.

3.1 Possibility and Common Possibility

For a fixed type space $\mathcal{T}$, an event E is a subset of $T \times \Theta$. "Possibility operators" are defined as follows. We write $B_i(E)$ for the set of types of player i that think that E is possible:

$$B_i(E) = \left\{ t_i \in T_i \;\middle|\; \begin{array}{l} \exists t_j \in T_j \text{ and } \theta \in \Theta \text{ such that} \\ ((t_i, t_j), \theta) \in E \text{ and } \pi_i(t_j, \theta | t_i) > 0 \end{array} \right\}.$$

For a pair of events E_1 and E_2, (E_1, E_2) are a *common possibility* for player i if:

1. player i thinks it is possible that E_i is true,

2. player i thinks it is possible that both (i) E_i is true; and (ii) player j thinks that E_j is possible,

3. and so on... .

Thus if we write $C_i(E_1, E_2)$ for the set of types of player i for whom (E_1, E_2) are a common possibility, we have

$$C_i(E_1, E_2) = B_i(E_i) \cap B_i(E_i \cap B_j(E_j)) \cap B_i(E_i \cap B_j(E_j \cap B_i(E_i))) \cap .....$$

We inductively define operators B_1^k and B_2^k on pairs of events by $B_i^0(E_1, E_2) = T_i$ and $B_i^{k+1}(E_1, E_2) = B_i\left(E_i \cap B_j^k(E_j)\right)$ for each $k = 1, 2...,$ and we define

$$C_i(E_1, E_2) = \bigcap_{k \geq 1} B_i^k(E_1, E_2). \tag{5}$$

The sequence $B_i^k(E_1, E_2)$ is decreasing under set inclusion for each i. Thus this definition of common possibility also has a well defined fixed point characterization:

Lemma 1 (Common Possibility as Fixed Point)
Let $F_1 \subseteq T_1$ and $F_2 \subseteq T_2$ be the largest sets of types satisfying $F_1 \subseteq B_1(E_1 \cap F_2)$ and $F_2 \subseteq B_2(E_2 \cap F_2)$. Then $C_i(E_1, E_2) = F_i$.

The definition given by (5) describes a concept of common possibility for a *pair* of events (E_1, E_2) for two players. If we are only interested in a single event, and we can specialize the above definitions to a single event $E_1 = E_2 = E$, then event E is a common possibility for player i if:

1. player i thinks it is possible that E is true,
2. player i thinks it is possible that both (i) E is true; and (ii) player j thinks that E is possible,
3. and so on... .

We will write $C_i(E)$ as shorthand for $C_i(E, E)$, and so

$$C_i(E) = B_i(E) \cap B_i(E \cap B_j(E)) \cap B_i(E \cap B_j(E \cap B_i(E))) \cap .....;$$

and this is equivalent to inductively defining

$$B_i^0(E) = T_i \text{ and } B_i^{k+1}(E) = B_i\left(E \cap B_j^k(E)\right)$$

and setting

$$C_i(E) = \bigcap_{k \geq 1} B_i^k(E).$$

At this point it is informative to compare the (single event) possibility operators to belief operators and common possibility to common p-belief as defined by Monderer and Samet (1989). We will use both possibility and p-belief operators to analyze the coordination and trading game subsequently. Following Monderer and Samet (1989), we write $B_i^p(E)$ for the set of types of player i who assign probability at least p to event E,

$$B_i^p(E) = \left\{ t_i \in T_i \left| \sum_{\{(t_j,\theta)|((t_i,t_j),\theta)\in E\}} \pi_i(t_j, \theta | t_i) \geq p \right. \right\}.$$

Now

$$B_i(E) = \bigcup_{p>0} B_i^p(E).$$

Event E is *repeated common p-belief* for player i if:

1. player i assigns probability at least p to event E,
2. player i assigns probability at least p to the event that both (i) E is true; and (ii) player j assigns probability at least p to event E,
3. and so on... .

Now writing $C_i^p(E)$ for the set of types of player i for whom event E is *repeated common p-belief*, we have that

$$C_i^p(E) = B_i^p(E) \cap B_i^p\left(E \cap B_j^p(E)\right) \cap B_i^p\left(E \cap B_j^p(E \cap B_i^p(E))\right) \cap .....;$$

and this is equivalent to inductively defining

$$B_i^{p,0}(E) = T_i \text{ and } B_i^{p,k+1}(E) = B_i^p\left(E \cap B_j^{p,k}(E)\right)$$

and setting

$$C_i^p(E) = \cap_{k \geq 1} B_i^{p,k}(E).$$

This definition of belief operators follows Monderer and Samet (1989) while the definition of repeated common p-belief comes from Monderer and Samet (1996).[12]

3.2 Coordination Games

We now return to the two person two action game described by (4) above For the class of coordination games, define Θ_G to be the set of "good" payoff states where both players strictly benefit if both take the risky action ("invest"); thus

$$\Theta_G = \{\theta \in \Theta \,| x_1(\theta) > c \text{ and } x_2(\theta) > c\}.$$

Define Θ_B to be the set of "bad" payoff states where both players are strictly made worse off even if both take the risky action; thus

$$\Theta_B = \{\theta \in \Theta \,| x_1(\theta) < c \text{ and } x_2(\theta) < c\}.$$

We will define a coordination game to be a situation where all states are either good or bad, so that

$$\Theta = \Theta_G \cup \Theta_B.$$

To remove uninteresting cases based on indifference, this definition excludes the possibility that $x_i(\theta) = c$. Thus we have a pure coordination game if we also require that $x_1(\theta) = x_2(\theta)$ for all θ, but this restriction does not matter in our characterization of belief-free rationalizability. We write E_G and E_B for the set of states where the payoff state is good and bad, respectively, so

$$E_G = \{(t,\theta) \,| \theta \in \Theta_G\} \text{ and } E_B = \{(t,\theta) \,| \theta \in \Theta_B\}.$$

In coordination games, at all good states, the corresponding complete information game has two strict Nash equilibria (both invest and both don't invest), while at all bad states, both players have a strictly dominant strategy to not invest. Now we have:

[12] The definition of repeated common p-belief is closely related to the more widely used concept of common p-belief introduced in Monderer and Samet (1989) given by

$$\widetilde{C}_i^p(E) = B_i^p(E) \cap B_i^p(B_1^p(E) \cap B_2^p(E)) \cap B_i^p(B_1^p(B_1^p(E) \cap B_2^p(E)) \cap B_2^p(B_1^p(E) \cap B_2^p(E))) \cap .....;$$

Monderer and Samet (1996) describe the close relationship between common p-belief and repeated common p-belief, which we omit here.

Proposition 1 (Belief-Free Rationalizability in Coordination Game)
In a coordination game, the safe action (not invest) is always belief-free rationalizable; the risky action (invest) is belief-free rationalizable for player i if and only if the event E_G is a common possibility for player i.

The first claim follows immediately because both not invest is always a strict Nash equilibrium of the underlying complete information game. For the second claim, observe that $B_i^k(E_G)$ is the set of types of agent i for whom invest is kth level belief-free rationalizable. This follows by induction since $B_i^0(E) = T_i$ corresponds to the set of types from whom invest is 0th level belief-free rationalizable; and, if $B_j^k(E)$ is the set of types of player j for whom invest is kth level belief-free rationalizable, then invest is $(k+1)$th level rationalizable for player i only if he attaches positive probability to $E_G \cap B_j^k(E_G)$. But - by definition - the set of types of player i for which this is true is exactly $B_i^{k+1}\left(E_G \cap B_i^k(E_G)\right)$ so we have our induction.

We briefly compare this characterization of belief-free rationalizable actions to the alternative solution concepts we have discussed. To do so, we will use a more specialized class of coordination games. Suppose that

$$\begin{aligned}\theta &\in \Theta_G \Rightarrow x_1(\theta) = x_2(\theta) = x^* > c,\\ \theta &\in \Theta_B \Rightarrow x_1(\theta) = x_2(\theta) = 0.\end{aligned}$$

Call this an x^*-coordination game.

Proposition 2 (Belief-Free Rationalizability in x^* Coordination Game)
In an x^*-coordination game, the safe action (not invest) is always interim correlated rationalizable; the risky action (invest) is interim correlated rationalizable if and only if the event E_G is repeated common c/x^*-belief for player i.

Again, the first claim follows immediately because since both not invest is always a strict Nash equilibrium of the underlying complete information game. For the second claim, note that invest is a best response for a player only if he attaches probability at least c/x^* to both the state being good and his opponent choosing to invest. Now, analogously to the previous proposition, we can show by induction that $B_i^{\frac{c}{x^*},k}(E_G)$ is the set of types of agent i for whom invest is kth level belief-free rationalizable: $B_i^{\frac{c}{x^*},0}(E) = T_i$ is the set of types from whom invest is 0th level belief-free rationalizable, and, if $B_j^{\frac{c}{x^*},k}(E)$ is the set of types of player j for whom invest is kth level belief-free rationalizable, then invest is $(k+1)$th level rationalizable for player i only if he attaches probability at least c/x^* to $E \cap B_j^{\frac{c}{x^*},k}(E)$ and so, again, we have our induction.

Because this is a game of strategic complementarities, and the largest and smallest rationalizable strategies (in the natural order) constitute equilibria, we have:

Proposition 3 (Belief Invariant BCE in x^* Coordination Game)
In an x^* coordination game, there is a belief invariant Bayes correlated equilibrium where the safe action (not invest) is always played. There is another belief invariant Bayes correlated equilibrium where the risky action (not invest) is played by player i if and only if the event E_G is repeated common c/x^*-belief. All other belief invariant Bayes correlated equilibria are "in between" these two, in the sense that invest is never played if the event E_G is not repeated common c/x^*-belief.

The structure of Bayes correlated equilibria is more subtle in this example; see Bergemann and Morris (2015) for a discussion of the structure of Bayes correlated equilibria in two player two action games of incomplete information.

3.3 Trading Games

We now want to consider a class of trading games where the safe action is interpreted as no trade and the risky action is interpreted as (agreeing to) trade. For this exercise, we think of c as being very small and corresponding to a small transaction cost associated with agreeing to trade. But trade will only take place if both players agree to trade. Let $\Theta_i \subseteq \Theta$ to be the set of "i gain (payoff) states" where trade is beneficial for player i, but not for player j, so

$$\Theta_i = \{\theta \in \Theta \,|x_i(\theta) > c \text{ and } x_j(\theta) < c\}.$$

Define a trading game to be a situation where all states are gain states for exactly one player, so that

$$\Theta = \Theta_1 \cup \Theta_2.$$

True zero sum trade would require that $x_1(\theta) + x_2(\theta) = 0$ for all θ, while a weaker non-positive sum trade requirement would be that $x_1(\theta) + x_2(\theta) \leq 0$ for all θ. We do not use either of these restrictions for our results and we would not get sharper results if we imposed either of them.

Now write E_i for the set of states and types corresponding to i-gain payoff state for player i,

$$E_i = \{(t, \theta) \, | \theta \in \Theta_i \} .$$

Now we have:

Proposition 4 (Belief-Free Rationalizability in Trading Game)
In a trading game, the safe action (reject trade) is always belief-free rationalizable; the risky action (accept trade) is belief-free rationalizable for player i if and only if the events (E_1, E_2) are a common possibility for player i.

The first claim is immediate because, in a trading game, the strictly positive cost c implies that there is always a strict equilibrium where each player never trades, which in turn implies that rejecting trade must be belief-free rationalizable. For the second claim, we observe that $B_i^{+,k}(E_i, E_j)$ is the set of types of player i for whom trade is kth level belief-free rationalizable. This follows by induction: $B_i^{+,0}(E_i, E_j) = T_i$ corresponds to the set of types from whom accepting trade is 0th level belief-free rationalizable; and if $B_j^{+,k}(E_j, E_i)$ is the set of types of player j for whom accepting trade is k-th level belief-free rationalizable, then trade is $(k+1)$-th level rationalizable for player i only if he attaches positive probability to $E_i \cap B_j^{+,k}(E_j, E_i)$. But - by definition - the set of types of player i for which this is true is exactly $B_i^{+,k+1}\left(E_i \cap B_j^{+,k}(E_j, E_i)\right)$ so we have our induction.

This characterization extends almost immediately to interim correlated rationalizability:

Proposition 5 (Interim Correlated Rationalizability in Trading Game)
In a trading game, the safe action (reject trade) is always interim correlated rationalizable; the risky action (accept trade) is interim correlated rationalizable for player i if and only if the events (E_1, E_2) are a common possibility of player i.

To see why, it is enough to show that the inductive step that worked for belief-free rationalizability continues to work for interim correlated rationalizability. In particular, suppose that E_j^k is the set of types of player j for whom accept trade is k-th level rationalizable (recall that rejecting trade is always kth level rationalizable). Now consider a type t_i of player i. He will have an interim belief $\pi_i(\cdot|t_i)$ over (t_j, θ), the type of the other player and the payoff state. Suppose $\left(t_j^*, \theta^*\right) \in E_j^k \times \Theta_i$, i.e., a type payoff state pair where accept trade is k-th level interim correlated rationalizable for player j and the payoff state is an i-gain state. Now we can endow type t_i of agent with a belief $\nu_i \in \Delta(A_j \times T_j \times \Theta)$ given by

$$\nu_i(a_j, t_j, \theta) = \begin{cases} \pi_i(t_j, \theta | t_i), & \text{if} \quad a_j = \text{reject trade and } (t_j, \theta) \neq (t_j^*, \theta^*), \\ \pi_i(t_j, \theta | t_i), & \text{if} \quad a_j = \text{accept trade and } (t_j, \theta) = (t_j^*, \theta^*), \\ 0, & \text{if} \quad \text{otherwise.} \end{cases}$$

Clearly, accept trade is best response to this conjecture and thus $(k+1)$-th level rationalizable for type t_i. Thus the induction argument for belief-free rationalizability goes through unchanged for interim correlated rationalizability.

Essentially this game was analyzed earlier by Morris and Skiadas (2000).[13] While an interim version of rationalizability was used as a solution concept in Morris and Skiadas (2000), it turned out that only supports mattered for the results and, in that sense, the equivalence of belief-free rationalizability and interim rationalizability was implicit. The characterization of the possibility of rationalizable trade in Morris and Skiadas (2000) was expressed somewhat differently, using the construction of cycles, but the following lemma shows the connection.

Lemma 2 (Common Possibility)
There is common possibility of (E_1, E_2) for type t_1 of player 1 if and only if there exists a sequence $(t_1^k, \theta_1^k, t_2^k, \theta_2^k) \in T_1 \times \Theta_1 \times T_2 \times \Theta_2$ for $k = 1, ..., K$ forming a cycle so that (i) $t_1^1 = t_1$; (ii) $\pi_1(t_2^k, \theta_2^k | t_1^k) > 0$ for each $k = 1, ..., K$; and (iii) $\pi_2(t_1^{k+1}, \theta_1^{k+1} | t_2^k) > 0$ for each $k = 1, ..., K$ (with the convention that $K + 1 = 1$)

For each type of player i for whom there is common possibility of (E_1, E_2), i must think it is possible that both there is common possibility of (E_1, E_2) for the other player j and the payoff type is an i-gain state. We can use this observation to construct the cycle of types in the Lemma above. Morris and Skiadas (2000) used various versions of this cyclic condition to characterize the possibility of rationalizable trade.

In the common prior case, we have

Proposition 6 (BCE in Trading Game)
In a trading game, there is a unique Bayes correlated equilibrium (and thus a unique belief

[13] Morris and Skiadas (2000) maintained the payoff type assumption, so that trades were conditional on only the type profile.

invariant Bayes correlated equilibrium) where both players always choose the safe action (reject trade).

It is well known that trade is not possible under the common prior assumption: see Sebenius and Geanakoplos (1983) for a statement in the bilateral risk neutral setting discussed here and Milgrom and Stokey (1982) in a more general environment. Arguments from this literature immediately apply.

4 Two Important Special Cases

4.1 Complete Information

If Θ is a singleton, then interim correlated rationalizability and belief-free rationalizability will both reduce to (complete information) correlated rationalizability (Brandenburger and Dekel (1987)); and belief invariant Bayes correlated equilibrium and Bayes correlated equilibrium reduce to complete information (objective) correlated equilibrium (Aumann (1987)). In this sense, we are looking at natural generalizations of the classical complete information results, when they are given an informational robustness interpretation.

4.2 Payoff Type Spaces

Consider the special case where the payoff-relevant states have a product structure, i.e.,

$$\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_I,$$

and each player knows his own "payoff type" $\theta_i \in \Theta_i$, and nothing more. Thus this corresponds to an assumption of no distributed uncertainty in the sense that the join of players' information reveals everything. Thus we have $T_i = \Theta_i$. This "naive" or "payoff" type space is a particular example of a type space as used in the preceding analysis. On this space, beliefs will reduce to $\pi_i : \Theta_i \rightarrow \Delta(\Theta_{-i})$.

The definition of interim correlated rationalizability reduces as follows. Let $ICR_i^0(\theta_i) = A_i$

and let $ICR_i^{n+1}(\theta_i)$ equal the set of actions for which there exists $\nu_i \in \Delta(A_{-i} \times \Theta_{-i})$ such that

$$\begin{aligned}
&(1)\ \nu_i(a_{-i}, \theta_{-i}) > 0 \Rightarrow \theta_j \in ICR_j^k(\theta_j) \text{ for each } j \neq i, \\
&(2)\ \sum_{a_{-i}} \nu_i(a_{-i}, \theta_{-i}) = \pi_i(\theta_{-i}|\theta_i) \text{ for each } \theta, \\
&(3)\ a_i \in \arg\max_{a_i'} \sum_{a_{-i}, \theta_{-i}} \nu_i(a_{-i}, \theta_{-i},) \, u_i((a_i', a_{-i}), (\theta_i, \theta_{-i}));
\end{aligned}$$

and let

$$ICR_i(\theta_i) = \bigcap_{n \geq 1} ICR_i^n(\theta_i).$$

In a payoff type environment, allowing correlation between others' types and payoff states makes no difference here, and this version of interim rationalizability has been widely used in payoff-type environments (for example, Battigalli and Siniscalchi (2003a) and Dekel and Wolinsky (2003)).

Belief-free rationalizability reduces as follows. Let $BFR_i^0(\theta_i) = A_i$ and let $BFR_i^{n+1}(\theta_i)$ equal the set of actions for which there exists $\nu \in \Delta(A_{-i} \times \Theta_{-i})$ such that

$$\begin{aligned}
&(1)\ \nu(a_{-i}, \theta_{-i},) > 0 \Rightarrow a_j \in BFR_j^n(\theta_j) \text{ for each } j \neq i, \\
&(2)\ \sum_{a_{-i}} \nu(a_{-i}, \theta_{-i},) > 0 \Rightarrow \pi_i(\theta_{-i}|\theta_i) > 0, \\
&(3)\ a_i \in \arg\max_{a_i'} \sum_{a_{-i}, \theta_{-i}} \nu(a_{-i}, \theta_{-i}) \, u_i((a_i', a_{-i}), (\theta_i, \theta_{-i}));
\end{aligned}$$

and let

$$BFR_i(\theta_i) = \bigcap_{n \geq 1} BFR_i^n(\theta_i).$$

A decision rule will now be a mapping $\sigma : \Theta \to \Delta(A)$ and will be obedient if

$$\begin{aligned}
&\sum_{a_{-i}, \theta_{-i}} \pi^*(\theta_i, \theta_{-i}) \, \sigma((a_i, a_{-i}) \,|\, (\theta_i, \theta_{-i})) \, u_i((a_i, a_{-i}), (\theta_i, \theta_{-i})) \\
\geq\ &\sum_{a_{-i}, \theta_{-i}} \pi^*(\theta_i, \theta_{-i}) \, \sigma((a_i, a_{-i}) \,|\, (\theta_i, \theta_{-i})) \, u_i((a_i', a_{-i}), (\theta_i, \theta_{-i})).
\end{aligned}$$

for all i, $\theta_i \in \Theta_i$, and $a_i, a_i' \in A_i$; and belief invariant if

$$\sigma_i(a_i| (\theta_i, \theta_{-i})) \triangleq \sum_{a_{-i}} \sigma((a_i, a_{-i}) \,|\, (\theta_i, \theta_{-i}))$$

is independent of θ_{-i}. In this case, Bayes correlated equilibrium reduces to the Bayesian solution of Forges (1993) and the belief invariant Bayes correlated equilibrium reduces to belief invariant Bayesian solution of Forges (2006) and Lehrer, Rosenberg, and Shmaya (2010).

Within payoff type spaces, we can consider two further restrictions in order to relate belief-free rationalizability to existing approaches:

1. There are *private values* if $u_i((a_i, a_{-i}), (\theta_i, \theta_{-i}))$ is independent of θ_{-i}. Under the private values assumption, the solution concept of belief-free rationalizability is studied by Chen, Micali, and Pass (2015) and used to develop novel results about robust revenue maximization.

2. The *full (payoff type) support assumption* is satisfied if $\pi_i(\theta_{-i}|\theta_i) > 0$ for all i, θ_i and θ_{-i}. Under the full support assumption, restriction (2) in the definition of belief-free rationalizability becomes redundant. We referred to the resulting notion as "incomplete information rationalizability" in Bergemann and Morris (2008). This is the solution concept analyzed in much of our mechanism design work (Bergemann and Morris (2012)). Note that we did not report beliefs over payoff types in our robust mechanism design work, but if we had, they would be irrelevant to our analysis and we were implicitly assuming full support by always allowing any payoff type profile of others to be associated with a given payoff type of a player. We studied Bayes correlated equilibrium in this context in Bergemann and Morris (2008) where we called it "incomplete information correlated equilibrium".

5 Informational Robustness Foundations of Four Solution Concepts

5.1 Expansions and Informational Robustness Foundations

Now suppose that we start out with type space $\mathcal{T}$ and we allow each player i to observe an additional signal $s_i \in S_i$. Each player i has a subjective belief ϕ_i about the distribution of signals conditional on the type profiles and the payoff state:

$$\phi_i : T \times \Theta \to \Delta(S).$$

We make the *support assumption* that, for all players i and $t_i \in T_i$, there exists $\overline{S}_i(t_i) \subseteq S_i$ such that

$$\sum_{s_{-i}, t_{-i}, \theta} \phi_i((s_i, s_{-i}) \mid (t_i, t_{-i}), \theta) \pi_i(t_{-i}, \theta | t_i) > 0 \tag{6}$$

for each $s_i \in \overline{S}_i(t_i)$ and

$$\phi_j((s_i, s_{-i})|t, \theta) = 0 \tag{7}$$

for all $j \neq i$, $s_i \notin \overline{S}_i(t_i)$, s_{-i}, t and θ. The interpretation is that if player i does not think it is possible that he will observe an additional signal $s_i \notin \overline{S}_i(t_i)$ if he is type t_i, then no player j ever thinks it is possible that player i observes signal s_i when his type is t_i. This support assumption ensures that whenever a player other than i thinks that (t_i, s_i) is possible, the beliefs of player i conditional on (t_i, s_i) are well-defined by Bayes rule. If this assumption is not made, then players can attach positive probability to other players being types with undefined beliefs. Aumann (1974) discussed why an assumption like this was necessary in a sensible definition of subjective correlated equilibrium with an informational robustness interpretation. This assumption was implicit in the formulation of a correlating device in Liu (2015). We briefly discuss in Section 5.2 alternative ways of addressing this issue and the relation to "a posteriori equilibrium" in the complete information case.

We refer to any conditional distribution of signals, $(S_i, \phi_i)_{i=1}^I$, satisfying the support restriction as an *expansion* of type space $\mathcal{T}$. An expansion is *belief-invariant* if, for each player i,

$$\sum_{s_{-i} \in S_{-i}} \phi_i((s_i, s_{-i}) \,|\, (t_i, t_{-i}), \theta) \tag{8}$$

is independent of (t_{-i}, θ). Note that this is the same definition as (3) applied to expansions rather than decision rules, and it will immediately translate into belief invariance of decision rules in our information robustness results. Liu (2015) has shown that this definition characterizes payoff irrelevance in the sense that players can observe signals without altering their beliefs and higher-order beliefs about the state (see also Bergemann and Morris (2015)).

Now a basic game G, a type space $\mathcal{T}$ and an expansion $(S_i, \phi_i)_{i=1}^I$ jointly define a game of incomplete information. A (pure) strategy for player i in this game of incomplete information is a mapping $\beta_i : T_i \times S_i \to A_i$. Now strategy profile β is a (Bayes Nash) equilibrium if, for each player i, t_i and $s_i \in \overline{S}_i(t_i)$, we have

$$\begin{aligned} & \sum_{t_{-i}, s_{-i}, \theta} \pi_i(t_{-i}, \theta | t_i)\, \phi_i(s_i, s_{-i}| ((t_i, t_{-i}), \theta))\, u_i\left(\left(\beta_i(t_i, s_i), \beta_{-i}(t_{-i}, s_{-i})\right), \theta\right) \\ \geq & \sum_{t_{-i}, s_{-i}, \theta} \pi_i(t_{-i}, \theta | t_i)\, \phi_i(s_i, s_{-i}| ((t_i, t_{-i}), \theta))\, u_i\left(\left(a_i, \beta_{-i}(t_{-i}, s_{-i})\right), \theta\right) \end{aligned} \tag{9}$$

for all $a_i \in A_i$.

Now we can formally state the informational robustness foundations for the two rationalizability solution concepts we discussed:

Proposition 7 (Informational Robustness to Payoff-Irrelevant Signals)
Action a_i is interim correlated rationalizable for type t_i of player i in $(G,\mathcal{T})$ if and only if there exists a payoff-irrelevant expansion $\left(S_j,\phi_j\right)_{j=1}^I$ of $\mathcal{T}$, an equilibrium β of $\left(G,\mathcal{T},\left(S_j,\phi_j\right)_{j=1}^I\right)$ and a signal $s_i\in\overline{S}_i\left(t_i\right)$ such that $\beta_i\left(t_i,s_i\right)=a_i$.

Versions of this observation appear as Proposition 2 in Dekel, Fudenberg, and Morris (2007) and as Lemma 2 in Liu (2015). For completeness, and for comparison with the next Proposition, we report a proof in the Appendix for the Proposition under the current notation and interpretation.

Proposition 8 (Informational Robustness to Payoff-Relevant Signals)
Action a_i is belief-free rationalizable for type t_i of player i in $(G,\mathcal{T})$ if and only if there exists an expansion $\left(S_j,\phi_j\right)_{j=1}^I$ of $\mathcal{T}$, an equilibrium β of $\left(G,\mathcal{T},\left(S_j,\phi_j\right)_{j=1}^I\right)$ and signal $s_i\in\overline{S}_i\left(t_i\right)$ such that $\beta_i\left(t_i,s_i\right)=a_i$.

Proof. Suppose that action a_i is belief-free rationalizable for type t_i in $(G,\mathcal{T})$. By the definition of belief-free rationalizability, there exists, for each $a_j\in BFR_j\left(t_j\right)$, a conjecture $\nu_j^{a_j,t_j}\in\Delta\left(T_{-j}\times A_{-j}\times\Theta\right)$ such that

$$\begin{array}{l}(1)\ \nu_j^{a_j,t_j}\left(t_{-j},a_{-j},\theta\right)>0\Rightarrow a_k\in BFR_k\left(t_k\right)\ \text{ for each } k\neq j;\\ (2)\ \sum\limits_{a_{-j}}\nu_j^{a_j,t_j}\left(t_{-j},a_{-j},\theta\right)>0\Rightarrow\pi_j\left(t_{-j},\theta|t_j\right)>0 \text{ for each } t_{-j},\theta;\text{ and}\\ (3)\ a_j\in\arg\max\limits_{a_j'}\sum\limits_{t_{-j},a_{-j},\theta}\nu_j^{a_j,t_j}\left(t_{-j},a_{-j},\theta\right)u_j\left(\left(a_j',a_{-j}\right),\theta\right).\end{array}\tag{10}$$

Now consider the expansion $\left(S_j,\phi_j\right)_{j=1}^I$ of $\mathcal{T}$, where $S_j=A_j\cup\left\{s_j^*\right\}$ and $\phi_j:T\times\Theta\to\Delta\left(S\right)$ is given by

$$\phi_j\left(\left(s_j,s_{-j}\right)|\left(t_j,t_{-j}\right),\theta\right)=\begin{cases}\frac{\varepsilon}{\#BFR_j(t_j)}\nu_j^{s_j,t_j}\left(t_{-j},s_{-j},\theta\right), & \text{if}\quad s\in BFR\left(t\right),\\ \pi_j\left(t_{-j},\theta|t_j\right)-\varepsilon\sum\limits_{s_{-j}\in A_{-j}}\nu_j^{s_j,t_j}\left(t_{-j},s_{-j},\theta\right), & \text{if}\quad s=s^*,\\ 0, & \text{if}\quad \text{otherwise,}\end{cases}$$

for some $\varepsilon>0$. It is always possible to construct such an expansion for sufficiently small $\varepsilon>0$ because of property (2) in (10) above. Now, by construction, there is an equilibrium of the game $\left(G, \mathcal{T}, \left(S_j, \phi_j\right)_{j=1}^I\right)$ where if $s_j \in \overline{S}_j\left(t_j\right)$, $\beta_j\left(t_j, s_j\right) = s_j$, and $\beta_j\left(t_j, s_j^*\right)$ can be arbitrarily set equal to any element of

$$\underset{a_j'}{\arg\max} \sum_{t_{-j}, a_{-j}} \pi_j\left(t_{-j}, \theta | t_j\right) \phi_j\left(s_j^*, a_{-j} | \left(\left(t_j, t_{-j}\right), \theta\right)\right) u_j\left(\left(a_j', a_{-j}\right), \theta\right).$$

For the converse, suppose that there exists an expansion $\left(S_j, \phi_j\right)_{j=1}^I$ of $\mathcal{T}$ and an equilibrium β of $\left(G, \mathcal{T}, \left(S_j, \phi_j\right)_{j=1}^I\right)$. We will show inductively in n that, for all players j, $a_j \in BFR_j^n\left(t_j\right)$ whenever $s_j \in \overline{S}_j\left(t_j\right)$ and $\beta_j\left(t_j, s_j\right) = a_j$. It is true by construction for $n = 0$. Suppose that it is true for n. Since $s_j \in \overline{S}_j\left(t_j\right)$, equilibrium condition (9) implies that a_j is a best response to a conjecture over others' types and actions and the state. By the inductive hypothesis, this conjecture assigns zero probability to type action profiles $\left(t_j, a_j\right)$ of player j where $a_j \notin BFR_j^n\left(t_j\right)$. By construction, the marginal of this conjecture on $T_{-j} \times \Theta$ has support contained in the support of $\pi_j\left(\cdot | t_j\right)$. Thus $a_j \in BFR_j^{n+1}\left(t_j\right)$. ■

An expansion $\left(S_i, \phi_i\right)_{i=1}^I$ satisfies the common prior assumption if ϕ_i is independent of i. An expanded game $\left(G, \mathcal{T}, \left(S_i, \phi_i\right)_{i=1}^I\right)$ and a strategy profile β for that game will *induce* a decision rule $\sigma : T \times \Theta \rightarrow \Delta\left(A\right)$:

$$\sigma\left(a | t, \theta\right) = \sum_{\{(t,s):\beta(t,s)=a\}} \phi\left(s | \left(t, \theta\right)\right).$$

We record for completeness the corresponding results for expansions that satisfy the common prior assumption.

Proposition 9 (Informational Robustness to Common Prior Payoff-Irrelevant Signals) *If $\mathcal{T}$ is a common prior type space, then σ is a belief invariant Bayes correlated equilibrium of $\left(G, \mathcal{T}\right)$ if and only if there exists a payoff-irrelevant common prior expansion $\left(S_i, \phi_i\right)_{i=1}^I$ of $\mathcal{T}$ and equilibrium β of $\left(G, \mathcal{T}, \left(S_i, \phi_i\right)_{i=1}^I\right)$ such that β induces σ.*

A subjective version of Proposition 9 appears in Liu (2015) (and the common prior case is discussed in Section 5.2).

Proposition 10 (Informational Robustness to Common Prior Payoff-Relevant Signals) *If $\mathcal{T}$ is a common prior type space, then σ is a Bayes correlated equilibrium of $\left(G, \mathcal{T}\right)$ if and only if there exists a common prior expansion $\left(S_i, \phi_i\right)_{i=1}^I$ of $\mathcal{T}$ and equilibrium β of $\left(G, \mathcal{T}, \left(S_i, \phi_i\right)_{i=1}^I\right)$ such that β induces σ.*

Proposition 10 appears as Theorem 2 in Bergemann and Morris (2015).

5.2 The Support Assumption, Interim Statements and a Posteriori Equilibrium

In our informational robustness foundations, an expansion was characterized by each player's subjective belief about how players' signals were being (stochastically) chosen as a function of players' types and the payoff state. Thus expansions were being explicitly identified with new signals that players observed. In this Section, we will discuss an alternative way of describing an expansion of the type space, one that works directly with a player i's interim beliefs conditional on t_i and s_i. There are a number of reasons for doing so. First, this will highlight the significance and interpretation of the support assumption in the previous Section. Second, it will clarify the connection to the prior literature. Finally, it will provide a step towards explaining the relation between "informational robustness" and "epistemic" foundations of solution concepts.

Suppose that we started with a type space $\mathcal{T} = (T_i, \pi_i)_{i=1}^I$ but now consider a different definition of an expanded type space (which will reduce to the previous one under additional assumptions). An expanded type space will take the form $\widetilde{\mathcal{T}} = \left(\widetilde{T}_i, \widetilde{\pi}_i\right)_{i=1}^I$ where $\widetilde{T}_i \subseteq T_i \times S_i$ and, for each i and $t_i \in T_i$, there exists $s_i \in S_i$ such that $\widetilde{t}_i = (t_i, s_i)$. What can we say about possible equilibrium behavior on such an expanded type space? We have built into this formulation the assumption that all possible types are rational with respect to some beliefs, and, in this sense, this formulation captures the idea of *a posteriori equilibrium*, the version of subjective correlated equilibrium introduced by Aumann (1974) and applied in Brandenburger and Dekel (1987). If we impose no restrictions on how the beliefs of (t_i, s_i) on the expanded type space relate to the beliefs of t_i on the original type space, then the original type space becomes irrelevant. In particular, say that an action is *ex post rationalizable* in basic game $\mathcal{G}$ if it survives an iterative deletion procedure where, at each round, we delete actions that are not a best response given any conjecture over surviving actions and payoff states. Formally, let $EPR_i^0 = A_i$, let EPR_i^{n+1} be the set of actions for which there exists $\nu_i \in \Delta(A_{-i} \times \Theta)$ s.t.

$$\begin{aligned}&(1)\ \nu_i(a_{-i}, \theta) > 0 \Rightarrow a_j \in EPR_j^n \text{ for each } j \neq i,\\ &(2)\ a_i \in \arg\max_{a_i'} \sum_{a_{-i}, \theta} \nu_i(a_{-i}, \theta)\, u_i((a_i', a_{-i}), \theta);\end{aligned}$$

and let

$$EPR_i = \bigcap_{n \geq 1} EPR_i^n.$$

This solution concept characterizes actions that can be played in equilibrium on any expanded type space and corresponds to dropping the support assumption in the analysis of the previous Subsection.

This motivates putting additional restrictions on the expanded type space. We start by imposing a weak restriction that will correspond to the support assumption in the previous Subsection: a player's beliefs on the original type space are not contradicted by his beliefs on the expanded type space. Thus

$$\sum_{s_{-i}} \widetilde{\pi}_i \left((t_{-i}, s_{-i}, \theta) \left| t_i, s_i \right. \right) > 0 \Rightarrow \pi_i \left((t_{-i}, \theta) \left| t_i \right. \right) > 0. \tag{11}$$

Restriction (11) reduces to the support restriction as defined in the previous Subsection. Define

$$S_i(t_i) = \left\{ s_i \in S_i \left| (t_i, s_i) \in \widetilde{T}_i \right. \right\},$$

and

$$\phi_i \left((s_i, s_{-i}) \,|\, (t_i, t_{-i}), \theta \right) = \frac{1}{\# S_i(t_i)} \frac{\widetilde{\pi}_i \left((t_{-i}, s_{-i}, \theta) \left| t_i, s_i \right. \right)}{\pi_i \left((t_{-i}, \theta) \left| t_i \right. \right)} \tag{12}$$

whenever $\pi_i \left((t_{-i}, \theta) \left| t_i \right. \right) > 0$ and $\phi_i \left(\cdot | (t_i, t_{-i}), \theta \right)$ is an arbitrary distribution otherwise. Now (11) implies

$$\sum_{s_{-i}} \phi_i \left((s_i, s_{-i}) \,|\, (t_i, t_{-i}), \theta \right) \pi_i \left((t_{-i}, \theta) \left| t_i \right. \right) = \frac{1}{\# S_i(t_i)} \sum_{s_{-i}} \widetilde{\pi}_i \left((t_{-i}, s_{-i}, \theta) \left| t_i, s_i \right. \right)$$

for each t_i and $s_i \in S_i(t_i)$, and so

$$\begin{aligned} \sum_{s_{-i}, t_{-i}, \theta} \phi_i \left((s_i, s_{-i}) \,|\, (t_i, t_{-i}), \theta \right) \pi_i \left((t_{-i}, \theta) \left| t_i \right. \right) &= \frac{1}{\# S_i(t_i)} \sum_{s_{-i}, t_{-i}, \theta} \widetilde{\pi}_i \left((t_{-i}, s_{-i}, \theta) \left| t_i, s_i \right. \right) \\ &= \frac{1}{\# S_i(t_i)} \\ &> 0, \end{aligned}$$

which is the support assumption. Belief invariance in the formulation of the previous Subsection adds the requirement on the current expanded type space that

$$\sum_{s_{-i}} \widetilde{\pi}_i \left(t_{-i}, s_{-i}, \theta | t_i, s_i \right) = \widetilde{\pi}_i \left(t_{-i}, \theta | t_i \right)$$

for each i, t_{-i}, θ, t_i and s_i.

We noted earlier that a posteriori equilibrium from Aumann (1974) and Brandenburger and Dekel (1987) was equivalent to asking what can happen on all expanded type spaces in the case of complete information. But in the case of incomplete information – in the sense of there being many payoff states – we saw that the original type no longer mattered. Imposing either the weaker support assumption or belief invariance assumption are the natural generalizations of a posteriori equilibrium. Ex post rationalizability, like belief-free rationalizability and interim correlated rationalizability, reduces to correlated rationalizability in complete information games.

5.3 Epistemic Foundations

Instead of examining the consequences of equilibrium on expanded type spaces, we could instead have looked at the implications of common certainty of rationality on epistemic type spaces where players' epistemic types include a description of their beliefs, their action choices and other pertinent characteristics. As is well-known for the complete information case of Aumann (1987) and Brandenburger and Dekel (1987), there is a formal equivalence between these two questions. And what we have described as information robustness foundations for the incomplete information case reduce to the formal results in those classic references.

But the modern epistemic foundations literature raises a number of novel issues with the epistemic interpretation of such results and we will briefly discuss how those issues can or cannot and have or have not been addressed in the results surveyed here. We will discuss three issues in turn. [14]

A first concern is that epistemic results should refer only to interim beliefs and should not refer to either ex ante beliefs or counterfactual beliefs about what a type's beliefs would have been in another circumstance. Aumann and Brandenburger (1995) address this concern by removing reference to a player's beliefs about his own action or his own type, and the epistemic literature has followed this approach since then. Our informational robustness foundation for belief-free rationalizability (Proposition 8) cannot easily be adapted to deal with this concern. In Section

[14] In a working paper version of the paper, Bergemann and Morris (2014), we translated the present results into the language of the epistemic foundations literature, but did not deal with the three conceptual issues from the modern epistemic foundations literature to be discussed now.

5.1, it was explicitly assumed that there was a distribution of signals conditional on players' types and the state, and thus there was a stage prior to players observing signals in the expanded type space, and we made the "support assumption" which relies on this interpretation.

This concern can be dealt with in the case of interim correlated rationality (Proposition 7). It is also possible to deal with another concern, namely that for an epistemic foundations result, types in the type space should have an interpretation independent of the type space in which they live, and thus reflect "expressible" statements about the model (Brandenburger and Friedenberg (2008) and Battigalli, Di Tillio, Grillo, and Penta (2011)). Battigalli, Di Tillio, Grillo, and Penta (2011) introduce a richer language (including "signals") for discussing incomplete information games and give an epistemic foundation for interim correlated rationalizability in that language. Dekel and Siniscalchi (2014) give an alternative epistemic foundation for interim correlated rationalizability by explicitly identifying the beliefs and higher order beliefs about payoff types of a type in the epistemic space.

The common prior assumption raises further issues for the epistemic foundations of solution concepts. If we re-interpret our information robustness results for BIBCE and BCE as epistemic foundations results, then they take the same form as Aumann (1987) and Forges (1993): what can we say about the ex ante distribution of play under the assumption that there is common certainty of rationality in the play of the game? There are now a number of ways of giving an interim interpretation of the common prior assumption based on no trade (Morris (1994), Samet (1998a) and Feinberg (2000)) and iterated expectations (Samet (1998b)), but epistemic foundations based on these interim interpretations of the common prior assumption have not been fully developed.[15] We would need incomplete information extensions to give purely interim interpretations of Propositions 9 and 10.

[15] See Dekel and Siniscalchi (2014) for a discussion and a conjecture concerning (interim) epistemic foundations of correlated equilibrium. Nau and McCardle (1990) and Nau (1992) showed that a no trade condition and common certainty of rationality implied correlated equilibrium (under complete and incomplete information, respectively) and, in this sense, gave an epistemic foundation without reference to the common prior assumption.

6 Appendix

Proof of Proposition 7. Suppose that action a_i is interim correlated rationalizable for type t_i in $(G, \mathcal{T})$. By the definition of interim correlated rationalizability, there exists, for each player j and $a_j \in ICR_j(t_j)$, a conjecture $\nu_j^{a_j,t_j} \in \Delta(T_{-j} \times A_{-j} \times \Theta)$ s.t.

$$\begin{aligned}
&(1)\ \nu_j^{a_j,t_j}(t_{-j}, a_{-j}, \theta) > 0 \Rightarrow a_k \in ICR_k(t_k) \text{ for each } k \neq j; \\
&(2)\ \sum_{a_{-j}} \nu_j^{a_j,t_j}(t_{-j}, a_{-j}, \theta) = \pi_j(t_{-j}, \theta | t_j) \text{ for each } t_{-j}, \theta; \text{ and} \\
&(3)\ a_j \in \arg\max_{a'_j} \sum_{t_{-j}, a_{-j}, \theta} \nu_j^{a_j,t_j}(t_{-j}, a_{-j}, \theta)\, u_j\left(\left(a'_j, a_{-j}\right), \theta\right).
\end{aligned}$$

Now consider the expansion $\left(S_j, \phi_j\right)_{j=1}^{I}$ of $\mathcal{T}$, where $S_j = A_j$ and $\phi_j : T \times \Theta \to \Delta(S)$ satisfies

$$\phi_j\left(\left(a_j, a_{-j}\right) | \left(t_j, t_{-j}\right), \theta\right) = \begin{cases} \frac{\nu_j^{a_j,t_j}(t_{-j}, a_{-j}, \theta)}{\pi_j(t_{-j}, \theta | t_j) \cdot \# ICR_j(t_j)}, & \text{if } a \in ICR(t), \\ 0, & \text{if otherwise;} \end{cases}$$

whenever $\pi_j(t_{-j}, \theta | t_j) > 0$. Now, by construction, there is an equilibrium of the game $(G, \mathcal{T}, \left(S_j, \phi_j\right)_{j=1}^{I})$ where

$$\beta_j(t_j, a_j) = a_j$$

for all j, t_j and $a_j \in ICR_j(t_j)$.

For the converse, suppose that there exists an expansion $\left(S_j, \phi_j\right)_{j=1}^{I}$ of $\mathcal{T}$, an equilibrium β of $(G, \mathcal{T}, \left(S_j, \phi_j\right)_{j=1}^{I})$. We will show inductively in n that, for all players j, $a_j \in ICR_j^n(t_j)$ whenever $\beta_j(t_j, s_j) = a_j$ for some $s_j \in \overline{S}_j(t_j)$. It is true by construction for $n = 0$. Suppose that it is true for n. Equilibrium condition (9) implies that a_j is a best response to a conjecture over others' types and actions and the state. By the inductive hypothesis, this conjecture assigns zero probability to type action profile (t_k, a_k) of player $k \neq j$ with $a_k \notin ICR_k^n(t_k)$. By construction, the marginal on $T_{-j} \times \Theta$ is equal to $\pi_j(\cdot | t_j)$. Thus $a_j \in ICR_j^{n+1}(t_j)$. ■

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