APPROXIMATE SOLUTIONS OF THE WALRASIAN AND GORMAN POLAR FORM EQUILIBRIUM INEQUALITIES

By

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Approximate Solutions of Walrasian and Gorman Polar Form Equilibrium Inequalities

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Abstract

Recently Cherchye et al. (2011) reformulated the Walrasian equilibrium inequalities, introduced by Brown and Matzkin (1996), as an integer programming problem and proved that solving the Walrasian equilibrium inequalities is NP-hard. Following Brown and Shannon (2000), we reformulate the Walrasian equilibrium inequalities as the dual Walrasian equilibrium inequalities.

Brown and Shannon proved that the Walrasian equilibrium inequalities are solvable iff the dual Walrasian equilibrium inequalities are solvable. We show that solving the dual Walrasian equilibrium inequalities is equivalent to solving a NP-hard minimization problem. Approximation theorems are polynomial time algorithms for computing approximate solutions of NP-hard minimization problems.

The primary contribution of this paper is an approximation theorem for the equivalent NP-hard minimization problem. In this theorem, we propose a polynomial time algorithm for computing an approximate solution to the dual Walrasian equilibrium inequalities, where the marginal utilities of income are uniformly bounded. We derive explicit bounds on the degree of approximation from observable market data.

The second contribution is the derivation of the Gorman polar form equilibrium inequalities for an exchange economy, where each consumer is endowed with an indirect utility function in Gorman polar form. If the marginal utilities of income are uniformly bounded then we prove a similar approximation theorem for the Gorman polar form equilibrium inequalities.
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1 Introduction

The Brown-Matzkin (1996) theory of rationalizing market data with Walrasian markets, where consumers are price-taking, utility maximizers subject to budget constraints, consists of market data sets and the Walrasian equilibrium inequalities. A market data set is a finite number of observations on market prices, income distributions and social endowments. The Walrasian equilibrium inequalities are the Afriat inequalities for each consumer, the budget constraints for each consumer and the market clearing conditions in each observation. The unknowns in the Walrasian equilibrium inequalities are the utility levels, the marginal utilities of income and the Marshallian demands of individual consumers in each observation. The parameters are the observable market data: market prices, income distributions and social endowments in each observation. The Walrasian equilibrium inequalities are said to rationalize the observable market data if the Walrasian equilibrium inequalities are solvable for some family of utility levels, marginal utilities of income and Marshallian demands of individual consumers, where aggregate Marshallian demands are equal to the social endowments in every observation. Brown and Matzkin show that the observed market data is consistent with the Walrasian paradigm, as articulated by Arrow and Debreu (1954), iff the Walrasian equilibrium inequalities rationalize the observed market data. As such, the Brown–Matzkin theory of rationalizing market data with Walrasian markets requires an efficient algorithm for solving the Walrasian equilibrium inequalities.

The Walrasian equilibrium inequalities are multivariate polynomial inequalities. The Tarski–Seidenberg theorem, Tarski (1951), provides an algorithm, "quantifier elimination," that can be used to derive a finite family of multivariate polynomial inequalities, i.e., the "revealed Walrasian equilibrium inequalities" from the Walrasian equilibrium inequalities, where the unknowns are the observable market data: market prices, income distributions and the social endowments in each observation. It follows from the Tarski–Seidenberg theorem that the revealed Walrasian equilibrium inequalities are solvable for the observed market data iff the Walrasian equilibrium inequalities are solvable for some family of utility levels, marginal utilities of income and Marshallian demands of consumers.

An important example is the special case of the Walrasian equilibrium inequalities, recently introduced by Brown and Calsamiglia (2014). They propose necessary and sufficient conditions on observable market data to rationalize the market data with consumers endowed with utility functions, where the marginal utilities of income are constant: the so-called "strong law of demand". The strong law of demand is a finite family of linear inequalities on the observed market data, hence solvable in polynomial time. See their paper for details.

Unfortunately, in general, the computational complexity of the Tarski–Seidenberg algorithm, is known to be doubly exponential in the worse case. See Basu (2011) for a discussion of the Tarski–Seidenberg theorem and the computational complexity of quantifier elimination. Hence we are forced to consider approximate solutions of the Walrasian equilibrium inequalities.

A decision problem in computer science is a problem where the answer is "yes" or "no." In this paper, the decision problem is: Can the observed market data set be rationalized with Walrasian equilibrium inequalities? That is, are the Walrasian equilibrium inequalities solvable if the values of the parameters are derived from the observed market data? A decision problem is said to have polynomial complexity, i.e., the problem is in class P, if there exists an algorithm that solves each instance of the problem in time that is polynomial in some measure of the size of the problem instance. In the literature on computational complexity, polynomial time algorithms are referred to as "efficient" algorithms. A decision problem is said to be in NP, if there exists an algorithm that verifies, in polynomial time, if a proposal is a solution of the problem instance. Clearly,

$$P \subset NP$$

but it is widely conjectured by computer scientists that

$$P \neq NP$$

The decision problem A is said to be NP-*hard*, if every problem in NP can be reduced in polynomial time to A. That is, if we can decide the NP-*hard* problem A in polynomial time then we can decide every NP problem in polynomial time. In this case, contrary to the current beliefs of computer scientists,

$$P = NP.$$

What is the computational complexity of solving the Walrasian equilibrium inequalities? This important question was first addressed by Cherchye et al. (2011). They proved that solving the Walrasian equilibrium inequalities, reformulated as an integer programming problem, is NP-*hard*. We show that approximate solutions of the Walrasian equilibrium inequalities, reformulated as the dual Walrasian equilibrium inequalities introduced by Shannon and Brown (2000), can be computed in polynomial time. In the Brown-Shannon theory of rationalizing market data with Walrasian markets, the Afriat inequalities are replaced by the dual Afriat inequalities for minimizing the consumer's smooth, monotone, strictly convex, indirect utility function over prices subject to her budget constraint, defined by her Marshallian demand at the equilibrium market prices. The dual Walrasian equilibrium inequalities are said to rationalize the observed market data if the inequalities are solvable for some family of indirect utility levels, marginal indirect utilities and Marshallian demands of individual consumers, derived from Roy's identity, where the aggregate Marshallian demands are equal to the social endowments in every observation.

Brown and Shannon proved that the Walrasian equilibrium inequalities are solvable iff the dual Walrasian equilibrium inequalities are solvable. We show that solving the dual Walrasian equilibrium inequalities is equivalent to solving a $NP-hard$ minimization problem. Approximation theorems are polynomial time algorithms for computing approximate solutions of a $NP-hard$ minimization problem, where there are explicit a priori bounds on the degree of approximation. The primary contribution of this paper is an approximation theorem for a NP-*hard* minimization problem equivalent to solving Walrasian equilibrium inequalities with uniformly bounded marginal utilities of income.

Gorman introduced his polar form indirect utility functions in (1961). Recall that quasilinear, homothetic and CES indirect utility functions are all special cases of Gorman polar form indirect utility functions, and that endowing consumers with Gorman polar form indirect utility functions is a necessary and sufficient condition for the existence of a representative consumer. The second contribution of this paper is the derivation of the Gorman polar form equilibrium inequalities, where we prove an approximation theorem for a *NP-hard* minimization problem equivalent to solving the Gorman polar form equilibrium inequalities with uniformly bounded marginal utilities of income.

Using the two approximation theorems, we test two simple hypotheses: (1) The null hypothesis $H_{0,W}$: The observed market data is rationalized by the Walrasian equilibrium inequalities with uniformly bounded marginal utilities of income, where the alternative hypothesis is $H_{A,W}$: The Walrasian equilibrium inequalities with uniformly bounded marginal utilities of income are refuted by the observed market data. (2) The null hypothesis $H_{0,G}$: The observed market data is rationalized by the Gorman polar form equilibrium inequalities with uniformly bounded marginal utilities of income, where the alternative hypothesis is $H_{A,G}$: The Gorman polar form equilibrium inequalities with uniformly bounded marginal utilities of income are refuted by the observed market data.

There are four logical outcomes of testing two simple hypotheses, but only the following three outcomes are possible in our model: (a) We reject both null hypotheses and accept both alternative hypotheses. (b) We fail to reject the null hypothesis $H_{0,W}$, but reject the null hypothesis $H_{0,G}$and accept the alternative hypothesis $H_{A,G}$ (c) We fail to reject either null hypothesis. Hence (b) is the most interesting and important outcome. That is, failing to reject the null hypothesis $H_{0,W}$ and accepting the alternative hypothesis $H_{A,G}$ means the observed market data cannot be rationalized by a representative consumer economy, but may be rationalized by an exchange economy with heterogeneous consumers.

2 The Dual Walrasian Equilibrium Inequalities

In this section, we review and summarize the dual Walrasian equilibrium inequalities proposed by Brown and Shannon. We consider an exchange economy, with $i \in \{1, 2, ..., M\}$ consumers. For each observation $j \in \{1, 2, ..., N\}$, p_j is a vector of prices in R^L_{++} , η_j is a vector of social endowments of commodities in R^L_{++} and $\{I_{1,j}, I_{2,j}, ..., I_{M,j}\}$ is the distribution of positive incomes of consumers in observation j, where$\sum_{i=1}^{i=M} I_{i,j} = p_j \cdot \eta_j$ for $j = 1, 2, .., N$. Brown and Shannon show that there exist smooth, monotone, strictly convex indirect utility functions $V_i(\frac{p}{I})$ for the i^{th} consumer and Marshallian demand vectors $x_{ij} \in R^L_{++}$ for the i^{th} consumer in the j^{th} observation that constitute a competitive equilibrium in the j^{th} observation with respect to the observed data iff there exists numbers $V_{i,j} > 0$ and $\lambda_{i,j} > 0$ and vectors $q_{i,j} << 0$ such that Eqs. (1), (2) and (3) hold

$$(1)\ V_{i,k} > V_{i,j} + q_{i,j} \cdot (\frac{1}{I_{i,k}} p_k - \frac{1}{I_{i,j}} p_j) \ \text{Dual Afriat Inequalities}$$

$$(2)\ \sum_{i=1}^{i=M} \frac{-1}{\lambda_{i,j} I_{i,j}} q_{i,j} \ \leq \eta_j \quad \text{Market Clearing}$$

$$(3)\ \frac{p_j \cdot -q_{i,j}}{I_{i,j}^2} = \lambda_{i,j} \quad \text{FOC}$$

for all $i \in \{1, 2, ..., M\}$ and for all $j, k \in \{1, 2, ..., N\}$, $j \neq k$, where the expression for the Marshallian demand vector of consumer i in observation j: $x_{ij} = \frac{-1}{\lambda_{i,j} I_{i,j}} q_{i,j}$ follows from Roy's identity.

The intuition of this specification is immediate: $V_{i,j}$ is the i^{th} consumer's utility of $x_{i,j}$ in observation j; $\lambda_{i,j}$ is her marginal utility of income in observation j; $q_{i,j}$ is the gradient of her indirect utility function with respect to $(\frac{p_j}{I_{i,j}})$ in observation j; Eq. (1) is the dual Afriat inequalities for minimizing her smooth, monotone, strictly convex, indirect utility function subject to her budget constraint in each observation; Eq. (2) are the market clearing conditions in observation j; Eq.(3) is the first order conditions for the minimization problem of consumer i in observation j, where she minimizes her smooth, monotone, strictly convex indirect utility function subject to her budget constraint, defined by market prices, her income and her Marshallian demand at the given market prices and her income.

The system of inequalities defined by Eq. (1) and (3) are linear in the unknown utility levels $V_{i,j}$, marginal utilities of income $\lambda_{i,j}$ and marginal indirect utilities $q_{i,j}$ .Unfortunately, Eq. (2) is nonconvex in $\lambda_{i,j}$ and $q_{i,j}$ .In fact, this nonconvexity is the cause of the $NP - hard$ computational complexity first observed by Cherchye et al.

3 Uniform Bounds on the Marginal Utilities of Income

There is a special case of the dual Walrasian equilibrium inequalities where the computational complexity is polynomial. If we restrict attention to quasilinear exchange economies where $\lambda_{i,j} = 1$ for all i and j, as in Brown and Calsamiglia, then Equation (2) can be rewritten as $\sum_{i=1}^{i=M} \frac{-1}{I_{i,j}} q_{i,j} \ \leq \eta_j$. In this case, the dual Walrasian equilibrium inequalities : Eqs. (1),(2) and (3), are linear inequalities in the unknowns: $\omega \in \Omega$, where

$$\Omega \equiv \{(V_{i,j}, q_{i,j}) |\ V_i(\frac{p}{I}) \text{ is a smooth, monotone, strictly convex, indirect utility function}\}$$

and Eq. (1) holds for $i = 1, 2, ..., M$; $j = 1, 2, ..., N$. Hence the dual Walrasian equilibrium inequalities for quasilinear exchange economies are solvable in polynomial time. We next normalize the indirect utility functions that are NOT quasilinear, i.e., indirect utility functions where the marginal utilities of income vary over the observed incomes, $I_{i,j}$, and market prices, p_j. For each such indirect utility function, $V(\frac{p}{I})$, we compute the 2-norm of the gradient, with respect to $(\frac{p}{I})$ at $(\frac{p_s}{I_{r,s}})$, for $r = 1, 2, ..., M$; $s = 1, 2, ..., N$. If $[\max_{r,s} \left\| \frac{-\nabla_{(\frac{p}{I})} V(\frac{p_s}{I_{r,s}})}{I_{r,s}} \right\|_2]^{-1}$ is the normalizing constant for $V(\frac{p}{I})$, then the corresponding normalized indirect utility function

$$\overline{V(\frac{p}{I})} \equiv [\max_{r,s} \left\| \frac{-\nabla_{(\frac{p}{I})} V(\frac{p_s}{I_{r,s}})}{I_{r,s}} \right\|_2]^{-1} [V(\frac{p}{I})$$

If

$$\frac{q_{i,j}}{I_{i,j}} = \nabla_p V_i(\frac{p_j}{I_{i,j}})$$

$$\lambda_{i,j} = \partial_I V_i(\frac{p_j}{I_{i,j}}) = \frac{p_j \quad \cdot \quad -q_{i,j}}{I_{i,j}^2} = \frac{p_j}{I_{i,j}} \cdot \frac{-q_{i,j}}{I_{i,j}}$$

then the gradient of $\overline{V_i(\frac{p}{I})}$ at $(\frac{p_j}{I_{i,j}})$ is $< \nabla_p \overline{V_i(\frac{p_j}{I_{i,j}})}, \overline{\partial_l V_i(\frac{p_j}{I_{i,j}})} >$, where

$$\nabla_p \overline{V_i(\frac{p_j}{I_{i,j}})} = [\max_{r,s} \left\| \frac{-\ q_{r,s}}{I_{r,s}} \right\|_2]^{-1} \nabla_p V_i(\frac{p_j}{I_{i,j}}) = [\max_{r,s} \left\| \frac{-\ q_{r,s}}{I_{r,s}} \right\|_2]^{-1} \frac{q_{i,j}}{I_{i,j}}$$

$$\overline{\lambda_{i,j}} = \overline{\partial_l V_i(\frac{p_j}{I_{i,j}})} = [\max_{r,s} \left\| \frac{-\ q_{r,s}}{I_{r,s}} \right\|_2]^{-1} \partial_I \mathbf{V}_i(\frac{p_j}{I_{i,j}}) = [\max_{r,s} \left\| \frac{-\ q_{r,s}}{I_{r,s}} \right\|_2]^{-1} \boldsymbol{\lambda}_{i,j} =$$

$$[\max_{r,s} \left\| \frac{-\ q_{r,s}}{I_{r,s}} \right\|_2]^{-1} \frac{p_j \quad \cdot \quad -q_{i,j}}{I_{i,j}^2} = [\max_{r,s} \left\| \frac{-\ q_{r,s}}{I_{r,s}} \right\|_2]^{-1} \frac{p_j}{I_{i,j}} \cdot \frac{-q_{i,j}}{I_{i,j}}$$

Theorem 1 *If the observed data on market prices, income distribution and social endowments are described by* β *,where*

$$\Theta_W(\beta) \equiv \max\{1, \max_{r,s} \left\| \frac{p_s}{I_{r,s}} \right\|_1 \}$$

and

$$\overline{\lambda_{i,j}} \equiv [\max_{r,s} \left\| \frac{-\ q_{r,s}}{I_{r,s}} \right\|_2]^{-1} \lambda_{i,j} \quad = [\max_{r,s} \|-q_{r,s}\|_2]^{-1} [-q_{i,j} \cdot \frac{p_j}{I_{i,j}}]$$

then

$$\Theta_W(\beta) \geq \max\{1, \overline{\lambda_{i,j}} : i = 1, 2, ..., M; j = 1, 2, ..., N\}$$

and

$$(4) \ \overline{\lambda_j} \leq \max\{1, \max_{r,s} \left\| \frac{p_s}{I_{r,s}} \right\|_1 \} \equiv \Theta_W(\beta) \quad \textit{Upper Bound} \ ,$$

That is, $\Theta_W(\beta)$ *is a uniform upper bound on the normalized marginal utilities of income,* $\overline{\lambda_{i,j}}$ *, for the normalized indirect utility functions ,where the* $\lambda_{i,j}$ *vary over the observed market prices and income distributions. Moreover,* $\Theta_W(\beta)$ *is an upper bound on the constant marginal utility of income for quasilinear utility functions, where* $\lambda_{i,j} = 1$ *for all* i *and* j*. .*

4 Approximation Theorem

Definition 2 *An approximation theorem for a $NP-$ hard minimization problem, with optimal value $OPT(\beta)$ for each input β, is a polynomial time algorithm for computing $\widehat{OPT(\beta)}$, the optimal value of the approximating minimization problem for the input β, and the approximation ratio $\alpha(\beta) \geq 1$*

$$OPT(\beta) \leq \widehat{OPT(\beta)} \leq \alpha(\beta) OPT(\beta),$$

This definition was taken from the survey paper by Arora (1998) on the theory and application of approximation theorems in combinatorial optimization now prove an approximation theorem for the dual Walrasian equilibrium inequalities. In the Walrasian model $\alpha(\beta) \equiv \Theta_W(\beta)$,and

$$\Omega \equiv \{(V_{i,j}, q_{i,j}) | V_i \ (\frac{p}{I}) \text{is a smooth, monotone, strictly convex, indirect utility function}\}$$

,where we approximate the nonconvex family of dual Walrasian equilibrium inequalities with a family of linear equilibrium inequalities derived from an exchange economy where consumers are endowed with quasilinear utility functions..

Theorem 3 *If $\Theta \geq \Theta_W \geq 1$ and Δ_W is the optimal value of the nonconvex program S_W, where*

$$\textit{(5)}\ \Delta_W \equiv \min_{\omega \in \Omega, s_j \geq 1} \frac{1}{N} \{ \sum_{j=1}^{j=N} s_j : Eqs.\ \textit{(1), (3), (4)}\ hold\ and \sum_{i=1}^{i=M} \frac{-1}{I_{i,j i} \overline{\lambda_{i,j}}(\omega)} \overline{q_{i,j}} \leq s_j \eta_j$$
$$for\ 1 \leq j \leq N\} : S_W$$

Γ_W is the optimal value of the approximating linear program R_W,where

$$\textit{(6)}\ \Gamma_W \equiv \min_{\omega \in \Omega, r_j \geq 1} \frac{1}{N} \{ \sum_{j=1}^{j=N} r_j : Eqs.\textit{(1), (3), (4)}\ hold\ and \sum_{i=1}^{i=M} \frac{-1}{I_{i,j}} \overline{q_{i,j}} \leq r_j \eta_j$$
$$for\ 1 \leq j \leq N\} : R_W$$

Ψ_W is the optimal value of the nonconvex program T_W, where

$$\textit{(7)}\ \Psi_W \equiv \min_{\omega \in \Omega, t_j \geq 1} \frac{1}{N} \{ \sum_{j=1}^{j=N} t_j : Eqs.\ \textit{(1), (3), (4)}\ hold\ and\ \Theta \sum_{i=1}^{i=M} \frac{-1}{I_{i,j} \overline{\lambda_{i,j}}(\omega)} \overline{q_{i,j}} \leq t_j \eta_j$$
$$for\ 1 \leq j \leq N\} : T_W$$

then

$$\textit{(8)}\ \Psi_W \geq \Gamma_W \geq \Delta_W$$

and

$$\textit{(9)}\ \Psi_W = \Theta \Delta_W$$

Hence

$$\textit{(10)}\ \Theta \Delta_W \geq \Gamma_W \geq \Delta_W \Leftrightarrow \Gamma_W \geq \Delta_W \geq \frac{\Gamma_W}{\Theta}$$

Proof. Eq.6 is a special case of Eq.5, with the additional constraint that $\overline{\lambda_{i,j}}(\omega) = 1$. Hence if r_j is feasible in R_W, then r_j is feasible in S_W. Since $\frac{\Theta}{\overline{\lambda_{i,j}(\omega)}} \geq 1$,it follows that if t_j is feasible in T_W then t_j is feasible in R_W.This proves (8).If

$$\frac{\Psi_W}{\Theta} \equiv \min_{\omega\in\Omega, t_j\geq 1} \frac{1}{N}\{\sum_{j=1}^{j=N} t_j : \text{Eqs. } \mathbf{(1)}, \mathbf{(3)}, \mathbf{(4)} \text{ hold and } \sum_{i=1}^{i=M} \frac{-1}{I_{ij}\overline{\lambda_{i,j}}(\omega)}\overline{q_{i,j}} \leq \frac{t_j}{\Theta}\eta_j$$

for $1 \leq j \leq N\}$

and

$$\Delta_W \equiv \min_{\omega\in\Omega, s_j\geq 1} \frac{1}{N}\{\sum_{j=1}^{j=N} s_j : \text{Eqs. } \mathbf{(1)}, \mathbf{(3)}, \mathbf{(4)} \text{ hold and } \sum_{i=1}^{i=M} \frac{-1}{I_{i,j}\overline{\lambda_{i,j}}(\omega)}\overline{q_{i,j}} \leq s_j\eta_j$$

for $1 \leq j \leq N\}$

then

$$\Psi_W = \Theta\Gamma_W.$$

This proves (9). ∎

Corollary 4 *(a)* $\Gamma_W = 1$, *iff the Walrasian equilibrium inequalities with constant marginal utilities of income rationalize the observed market data.*β *(b) It follows from the Brown and Calsamiglia paper that* $\Gamma_W = 1$ *iff the observed market data* β *satisfies the strong law of demand.*

5 The Gorman Polar Form Equilibrium Inequalities

In Gorman's seminal (1961) paper on the existence of a representative consumer in an exchange economy with a finite number of consumers, he derived necessary and sufficient conditions that a representative consumer exists iff all consumers in the exchange economy are endowed with indirect utility functions, $V_i(\frac{p}{I})$ in polar form: $V_i(\frac{p}{I}) = \frac{I - a_i(p)}{b(p)}$where $a_i(p)$ and $b(p)$ are concave and homogeneous of degree one functions of the market prices. Moreover, he assumed that the marginal utilities of income are the same for all consumers and only depend on the market prices. That is, $\lambda_{i,j} = \frac{1}{b(p_j)}$ for all i. As suggested by Varian (1992), we represent Gorman polar form indirect utility functions as

$$\mathbf{(G)}\ V_i(\frac{p}{I}) = Ie(p) + f_i(p),$$

We define $e(p) \equiv \frac{1}{b(p)}$ and $f_i(p) \equiv \frac{a_i(p)}{b(p)}$as convex functions of p, where $e(p)$ *is* homogeneous of degree minus one and $f(p)$ is homogeneous of degree zero.

Theorem 5 *The Gorman polar form equilibrium inequalities: Eqs. (G1) to (G6) and (11), (12) and (13) are necessary and sufficient conditions for rationalizing the observed market data with an exchange economy where consumers are endowed with*

Gorman polar form indirect utility functions, $V_i(\frac{p}{I}) \equiv Ie(p) + f_i(p)$*, where* $e(p)$ *and* $f_i(p)$ *are smooth, monotone and strictly convex. For* $k \neq j$ *and for* $1 \leq i \leq M$ *and* $1 \leq j, k \leq N$ *:*

(G1) $e(p_k) > \ e(p_j) + \nabla_p e(p_j)] \cdot (p_k - p_j)$

(G2) $f_i(p_k) > \ f_i(p_j) + \nabla_p f_i(p_j)] \cdot (p_k - p_j)$

(G3) $[Ie(p_k) + f_i(p_k)] >$
$[Ie(p_j) + f_i(p_j)] + \nabla_p[Ie(p_j) + f_i(p_j)] \cdot (p_k - p_j) + \ e(p_j)(I_{i,k} - I_{i,j})]$

(G4) $V_{i,j} = I_{i,j}e(p_j) + f_i(p_j)$

(G5) $\frac{q_{i,j}}{I_{i,j}} = \nabla_p[I_{i,j}e(p_j) + f_i(p_j)]$

(G6) $\lambda_{i,j} = e(p_j) \equiv \lambda_j$

(11) $V_{i,k} > V_{i,j} - \frac{q_{i,j}}{I_{i,j}} \cdot (p_k - p_j) + \frac{p_j \cdot -q_{i,j}}{I_{i,j}^2}(I_{i,k} - I_{i,j})$ *Dual Afriat Inequalities*

(12) $\sum_{i=1}^{i=M} \frac{-1}{\lambda_{i,j}} q_{i,j} \leq \eta_j$ *Market Clearing*

(13) $\frac{p_j \cdot -q_{i,j}}{I_{i,j}^2} = \lambda_{i,j}$ *FOC*

Proof. Necessity is obvious. For sufficiency, if

$$W_i(\frac{p}{I}) \equiv \max_{1 \leq \ k \ \leq N} [[Ie(p_k) + f_i(p_k)] + \nabla_p[Ie(p_k) + f_i(p_k)] \cdot (p - p_k)] + e(p_k)(I - I_{i,k})]$$

then

$$W_i(\frac{p_j}{I_{i,j}}) = [I_{i,j}e(p_j) + f_i(p_j)]$$

If (p, I) satisfies the budget constraint defined by $(p_j, I_{i,j})$: $p_j \cdot x_{i,j} = I_{i,j}$, where

$$x_{i,j} = \frac{-q_{i,j}}{I_{i,j}} \frac{1}{\lambda_{i,j}} = -\nabla_p[I_{i,j}e(p_j) + f_i(p_j)]/e(p_j).$$

then

$$-p_{i,j} \cdot \nabla_p[I_{i,j}e(p_j) + f_i(p_j)] = I_{i,j}e(p_j)$$

and

$$-p \cdot \nabla_p[I_{i,j}e(p_j) + f_i(p_j)] \leq Ie(p_j)$$

Hence

$$\nabla_p[Ie(p_j) + f_i(p_j)] \cdot (p - p_j)] + e(p_k)(I - I_{i,j})] \geq 0$$

and

$$W_i(\frac{p}{I}) \geq [I_{i,j}e(p_j) + f_i(p_j)] = W_i(\frac{p_j}{I_{i,j}})$$

Recall that each consumer minimizes her indirect utility function subject to her budget constraint. That is, $W_i(\frac{P_j}{I_{i,j}})$ rationalizes the observed market data: $(p_j, I_{i,j})$. $W_i(\frac{p}{I})$ is a lower bound for the Gorman polar form indirect utility function:

$$V_i(\frac{p}{I}) \equiv \max_{1 \ \leq k \leq \ N} [I[e(p_k) + \nabla_p e(p_k) \cdot (p - p_k)] + e(p_k)(I - I_{i,k})] + \max_{1 \leq k \leq N} [f_i(p_k) + \nabla_p f_i(p_k)] \cdot (p - p_k)]$$

and $W_i(\frac{p_j}{I_{i,j}}) = V_i(\frac{p_j}{I_{i,j}})$. Hence the Gorman polar form indirect utility function $V_i(\frac{p}{I})$ rationalizes the observed market data: $(p_j, I_{i,j})$. That is,

$$V_i(\frac{p}{I}) \geq W_i(\frac{p}{I}) \geq W_i(\frac{p_j}{I_{i,j}}) = V_i(\frac{p_j}{I_{i,j}})$$

■

Theorem 6 *If* $\overline{\lambda_j} \equiv [\max_{r,s} \left\| \frac{-q_{r,s}}{I_{r,s}} \right\|_2]^{-1} \lambda_j \ = [\max_{r,s} \|-q_{r,s}\|_2]^{-1}[-q_{i,j} \cdot \frac{p_j}{I_{i,j}}]$ *then*

$$\textbf{\textit{(14)}}\ \overline{\lambda_j} \leq \max\{1, \max_{r,s} \left\| \frac{p_s}{I_{r,s}} \right\|_1\} \equiv \Theta_G(\beta) \quad \textbf{\textit{Upper Bound}}\ ,$$

That is, $\Theta_G(\beta)$ *is a uniform upper bound on the normalized marginal utilities of income,* $\overline{\lambda_{i,j}}$*, for normalized indirect utility functions ,where the* $\lambda_{i,j}$ *vary over the observed market prices and income distributions. Moreover,* $\Theta_G(\beta)$ *is an upper bound on the constant marginal utility of income for quasilinear utility functions, where* $\lambda_{i,j} = 1$ *for all* i *and* j*.*

Proof. See the argument preceding Theorem 1. ■

We now prove an approximation theorem for the Gorman polar form equilibrium inequalities. In the Gorman model $\alpha(\beta) \equiv \Theta_G(\beta)$,where $\Omega \equiv \{V_{i,j}, q_{i,j} | \ V_i\ (\frac{p}{I})$ is a smooth, monotone, strictly convex, indirect utility function$\}$

Theorem 7 *If* $\Theta \geq \Theta_G \geq 1$ *and* Δ_G *is the optimal value of the nonconvex program* S_G*, where*

$$\textit{(15)}\ \Delta_G \ \equiv \ \min_{\omega \in \Omega, s_j \geq 1} \frac{1}{N}\{\sum_{j=1}^{j=N} s_j : \textit{Eqs.}\ \textbf{\textit{(G1)}}\ \textit{to}\ \textbf{\textit{G(6)}}\ \textbf{\textit{and (11), (13) and (14)}}$$

$$\textit{hold and}\ \frac{1}{\overline{\lambda_j}(\omega)} \sum_{i=1}^{i=M} \frac{-1}{I_{i,j}}\overline{q_{i,j}} \ \leq \ s_j\eta_j \ \textit{for}\ 1 \leq j \leq N\} : S_G$$

Γ_G *is the optimal value of the approximating linear program* R_G*,where*

$$\textit{(16)}\ \Gamma_G \ \equiv \ \min_{\omega \in \Omega, r_j \geq 1} \frac{1}{N}\{\sum_{j=1}^{j=N} r_j : \textit{Eqs.}\ \textbf{\textit{(G1)}}\ \textit{to}\ \textbf{\textit{G(6)}}\ \textbf{\textit{and (11), (13) and (14)}}$$

$$\textit{hold and} \sum_{i=1}^{i=M} \frac{-1}{I_{i,j}}\overline{q_{i,j}} \ \leq \ r_j\eta_j \ \textit{for}\ 1 \leq j \leq N\} : R_G$$

Ψ_G *is the optimal value of the nonconvex program* T_G*, where*

$$\textit{(17)}\ \Psi_G \ \equiv \ \min_{\omega \in \Omega, t_j \geq 1} \frac{1}{N}\{\sum_{j=1}^{j=N} t_j\textit{: Eqs.}\textbf{\textit{(G1)}}\ \textit{to}\ \textbf{\textit{G(6)}}\ \textbf{\textit{and (11), (13) and (14)}}$$

$$\textit{hold and}\ \frac{\Theta}{\overline{\lambda_j}(\omega)} \sum_{i=1}^{i=M} \frac{-1}{I_{i,j}}\overline{q_{i,j}} \ \leq \ t_j\eta_j \ \textit{for}\ 1 \leq j \leq N\} : T_G$$

then

$$\textit{(18)}\ \Psi_G \geq \Gamma_G \geq \Delta_G$$

and

$$\textit{(19)}\ \Psi_G = \Theta\Delta_G$$

Hence

$$\textit{(20)}\ \Theta\Delta_G \geq \Gamma_G \geq \Delta_G \Leftrightarrow \Gamma_G \geq \Delta_G \geq \frac{\Gamma_G}{\Theta}$$

Proof. See the proof of Theorem 3 ■

Corollary 8 *(a) $\Gamma_G = 1$, iff the Gorman Polar Form equilibrium inequalities with constant marginal utilities of income rationalize the observed market data β. (b) It follows from the Brown and Calsamiglia paper that $\Gamma_G = 1$ iff the observed market data β satisfies the strong law of demand.*

The Gorman polar form equilibrium inequalities are a superset of the Walrasian equilibrium inequalities. As such, $\Gamma_W \leq \Gamma_G$. Since $\Theta_W = \Theta_G$, it follows that $\frac{\Gamma_W}{\Theta_W} \leq \frac{\Gamma_G}{\Theta_G}$. To test the two hypotheses, described in the introduction, we define the "confidence intervals" C_W and C_G, where

$$C_W \equiv [\frac{\Gamma_W}{\Theta_W}, \Gamma_W] \text{ and } C_G \equiv [\frac{\Gamma_G}{\Theta_G}, \Gamma_G]$$

It follows from the approximation theorems that:

$$\Delta_W \in C_W \text{ and } \Delta_G \in C_G$$

(a) If $1 \notin C_W \cup C_G$, then we reject both null hypotheses and accept both alternative hypotheses.
(b) If $1 \in C_W \cap C_G^c$, then we fail to reject the null hypothesis $H_{0,W}$, but reject the null hypothesis $H_{0,G}$ and accept the alternative hypothesis $H_{A,G}$
(c) If $1 \in C_W \cap C_G$,then we fail to reject either null hypothesis.

6 Discussion

In this final section of the paper, we describe our contribution to the literature on Algorithmic Game Theory (AGT) or more precisely our contribution to the literature on Algorithmic General Equilibrium (AGE), that predates AGT. AGE begins with Scarf's seminal (1967) article on computing approximate fixed points, followed by his classic (1973) monograph: The Computation of Economic Equilibria. Codenotti and Varadarajan (2007) review the literature on polynomial time algorithms for computing competitive equilibria of restricted classes of exchange economies, where the set of competitive equilibria is a convex set. It is the convexity of the equilibrium set that allows the use of polynomial time algorithms devised for solving convex optimization problems. The authors conclude that the computational complexity of general equilibrium models, where the set of equilibria is nonconvex, is unlikely to be polynomial.

Scarf does not explicitly address the issue of computational complexity of what is now called the Scarf algorithm, for computing competitive equilibria. His primary research agenda is the computation of economic equilibria in real world economies. This project is best illustrated by the (1992) monograph of Shoven and Whalley, two of Scarf's graduate students, on Computable General Equilibrium (CGE) models. CGE models are now the primary models for counterfactual economic policy analysis, used by policy makers for estimating the economic impact of proposed taxes, quotas, tariffs, price controls, global warming, agricultural subsidies,... See the (2012) Handbook of Computable General Equilibrium Modeling edited by Dixon and Jorgenson.

CGE models use parametric specifications of utility functions and production functions. The parameters are often estimated using a method called"calibration". That is, choosing parameter values such that the CGE model replicates the observed equilibrium prices and observed market demands in a single benchmark data set. As you might expect there is some debate about the efficacy of this methodology among academic economists. In response to the obvious limitations of calibration and parametric specification of tastes and technology, Brown and Matzkin proposed the Walrasian equilibrium inequalities as a methodology for nonparametric estimation of CGE models, using several benchmark data sets. That is, Refutable General Equilibrium (RGE) models – see Brown and Kubler (2008). Brown and Kannan (2008) initiated the complexity analysis of searching for solutions of the Walrasian equilibrium inequalities. Subsequently, Cherchye et al (2011) showed that solving the Walrasian equilibrium inequalities, formulated as an integer programming problem is NP- hard.

Hence AGE consists of two computable classes of general equilibrium models. The parametric CGE models of Scarf-Shoven-Whalley and the nonparametric RGE models of Brown-Matzkin- Shannon. Both classes of models admit counterfactual policy analysis. Both classes of models contain special cases solvable in polynomial time. In general, both classes of models lack the polynomial time algorithms necessary for the efficient computation of solutions, hence they require approximation theorems to carry out effective counterfactual policy analysis. The contribution of this paper to the literature on refutable general equilibrium models are the proposed approximation theorems for the Walrasian and Gorman Polar Form equilibrium inequalities.

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