INFORMATION AND VOLATILITY

By

Dirk Bergemann, Tibor Heumann, and Stephen Morris

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Dirk Bergemann[a], Tibor Heumann[b], Stephen Morris[c]

[a]*Department of Economics, Yale University, New Haven, CT 06520, U.S.A. dirk.bergemann@yale.edu.*
[b]*Department of Economics, Yale University, New Haven, CT 06520, U.S.A. tibor.heumann@yale.edu.*
[c]*Department of Economics, Princeton University, Princeton, NJ 08544, U.S.A. smorris@princeton.edu.*

Abstract

In an economy of interacting agents with both idiosyncratic and aggregate shocks, we examine how the structure of private information influences aggregate volatility. The maximal aggregate volatility is attained in a noise free information structure in which the agents confound idiosyncratic and aggregate shocks, and display excess response to the aggregate shocks, as in Lucas [14]. For any given variance of aggregate shocks, the upper bound on aggregate volatility is linearly increasing in the variance of the idiosyncratic shocks. Our results hold in a setting of symmetric agents with linear best responses and normal uncertainty. We establish our results by providing a characterization of the set of all joint distributions over actions and states that can arise in equilibrium under any information structure. This tractable characterization, extending results in Bergemann and Morris [8], can be used to address a wide variety of questions linking information with the statistical moments of the economy.

Key words: Incomplete Information, Idiosyncratic Shocks, Aggregate Shocks, Volatility, Confounding Information, Moment Restrictions, Linear Best Responses, Quadratic Payoffs, Bayes Correlated Equilibrium.

1. Introduction

Consider an economy of interacting agents, each of whom picks an action. Agents are subject to idiosyncratic and aggregate shocks. A classical economic question in this environment is to ask how aggregate and idiosyncratic shocks map into "aggregate volatility" - the variance of the average action. Versions of this question arise in many different economic contexts. In particular, a central question in macroeconomics is how aggregate and individual productivity shocks translate into variation in GDP. Another classical question is when and how asymmetric information can influence this mapping, and in particular exacerbate aggregate volatility. A difficulty addressing this question is that the answer depends on the nature of the asymmetric information, something that is not easily observed. Thus results may be sensitive to the exact information structure assumed.

This paper considers a very simple stylized economy where we can completely characterize what can happen for all information structures. In particular, we consider a setting with a continuum of agents with linear best responses that depend on the average action of others and idiosyncratic and aggregate shocks. We assume that shocks, actions and signals are symmetrically normally distributed across agents, maintaining symmetry and normality of the information structure. Our sharp characterization of what can happen across all information structures in this symmetric normal class can be used to address many economic questions of interest. In particular, we can study the two classical questions described above, providing an upper bound on aggregate volatility as a function of fundamentals and identifying the critical information structures that give rise to maximal volatility.

The information structure that maximizes aggregate volatility turns out to be "noise free": each agent observes a one-dimensional signal which is a deterministic function of his idiosyncratic and the aggregate shock. While there is no noise in such signals, they are imperfect because they leave the agent uncertain about the size of the idiosyncratic and aggregate shocks. The shocks are confounded in the agent's signal. Aggregate volatility is highest when signals overweigh the aggregate shock relative to the idiosyncratic shock. In this case, agents who want to tailor their actions to their idiosyncratic shocks have no choice but to overweigh the aggregate shock, generating aggregate volatility. We show how maximum aggregate volatility increases linearly in the variance of the idiosyncratic shocks even if the variance of aggregate shocks is held constant. The critical noise free signal generating the maximal aggregate volatility puts proportionately more weight on the aggregate shock of constant variance as the variance of the idiosyncratic shock becomes larger.

These noise free information structures are also critical for many questions of interest, including for dispersion (the variance of individual actions around the mean action) and individual volatility (the variance of individual actions, which is equal to the sum of aggregate volatility and dispersion). But different noise free information structures maximize these variables. Thus dispersion is highest when signals overweigh the idiosyncratic shock relative to the aggregate shock. In this case, agents who want to tailor their actions to their aggregate shocks have no choice but to overweigh the idiosyncratic shock, generating dispersion.

The fact that confounding shocks can lead to overreaction has been long recognized, notably by Lucas [14] and more recently by Hellwig and Venkateswaran [11] and Venkateswaran [21]. Our contribution is to highlight that, in this setting with idiosyncratic and aggregate shocks, noise free confounding information structures are extremal and provide global bounds on how much volatility can arise via the information structure. The intuition for the bounding result is simple and comes in two parts. First, suppose that agents observed a one-dimensional signal that was a linear function of the idiosyncratic shock, the aggregate shock and a noise term which may be correlated across agents. Equilibrium actions must be linear in the signal. The impact of the noise in the signal must be to dampen the response of agents to the signal and thus to both the idiosyncratic and aggregate shocks. Thus among one-dimensional symmetric information structures, noise free information structures generate the most volatility. Second, imagine any other, perhaps multidimensional and perhaps noisy, symmetric information structure. By symmetry, each agent's equilibrium action choice (assumed to be one-dimensional) can be expressed as a linear function of the idiosyncratic shock, the aggregate shock and a noise term which may be correlated across agents. Now we can replace the original information structure by the one-dimensional one where each agent observes a signal which is linear in the *equilibrium action* he would have chosen under the old information structure. Equilibrium in this new information structure will now generate the same outcomes as the equilibrium with the richer information structure. Thus it is enough to study one-dimensional information structures where signals are a linear function of the idiosyncratic shock, the aggregate shock and a noise term which may be correlated across agents.

We provide a complete characterization of the symmetric equilibrium distributions of individual actions, aggregate actions, idiosyncratic shocks and aggregate shocks that can arise for a given distribution of payoff shocks across all possible information structures. The direct characterization of all feasible equilibrium distributions - independent of the information structure - makes use of a solution concept, Bayes correlated equilibria, introduced in earlier work by two of us, Bergemann

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and Morris [8], [8]. Namely, the set of Bayes correlated equilibrium distributions is equal to the set of Bayes Nash equilibrium distributions under all possible information structures. And in particular, the boundary of the set of Bayes correlated, and hence Bayes Nash equilibria, can be written as the (Bayes Nash) equilibrium of some noise free and one-dimensional information structure. The resulting bounds of the aggregate volatility are independent of the assumption of normality of the shocks, and only depend on the variance of the idiosyncratic and the aggregate shocks.

While noise free information structures generate maximal aggregate volatility with *both* idiosyncratic and aggregate shocks, this is not longer true if there are only idiosyncratic shocks. In this case, if each agent responded to his idiosyncratic shock only, there would be no aggregate volatility by the law of large numbers. On the other hand, if each agent had no information about his idiosyncratic shock, his action would be constant and there would again be no aggregate volatility. The information structure which maximizes aggregate volatility would be one where each agent observes his idiosyncratic shock with an intermediate level of noise, and where the noise in agents' signals is perfectly correlated. Angeletos and La'O [4] have analyzed the role of such aggregate shocks to beliefs about purely idiosyncratic uncertainty in a macroeconomic model, describing them as "sentiment" shocks. How can we relate this finding that - with only idiosyncratic shocks - adding noise maximizes aggregate volatility to our finding that - adding aggregate shocks - noise can only decrease aggregate volatility? We can reconcile the results by considering what happens if we let the variance of aggregate shocks decline towards zero in our model. In this case, our results show that the information structure that maximizes aggregate volatility is a noise free information structure where the signal puts a larger and larger weight on the aggregate shock and a smaller and smaller weight on the idiosyncratic shock. The agent - in order to respond to the idiosyncratic shock at all - must put a larger and larger weight on the signal. The total sensitivity to the aggregate shock (multiplying the weight on the aggregate shock in the signal with the weight on the signal in the equilibrium strategy) converges to a constant as the aggregate shock disappears. In the limit the dependence on the aggregate payoff shock becomes dependence on a common, payoff irrelevant, noise term, i.e., the sentiment shock. Thus this paper highlights a tight connection between noise free, but confounding information structures and sentiment shocks.

The results described thus far hold independently of whether the weight on the average action, r, is negative (the strategic substitutes case), zero (the purely decision theoretic case) or positive (the strategic complementarities case). A striking property of our characterization of Bayes correlated equilibria - i.e., what can happen in all symmetric information structures - is that the set of feasible

correlations between individual and average actions and individual and aggregate shocks is independent of r and determined only by statistical constraints. There are three degrees of freedom in describing the correlation structure: the correlation between each agent's action and his individual shock (the sum of the aggregate shock and his idiosyncratic shock), the correlation between any two agents' actions, and the correlation between any agent's action and any other agent's individual shock. On the other hand, once one pins down the correlation structure, the mean and variance of individual actions are pinned down. The best response parameter r describes how to translate correlation structures into first and second moments. Thus there is a three-dimensional class of Bayes correlated equilibria which is extremely tractable.

While we can restrict attention to one-dimensional information structures in deriving bounds on volatility, we may want to assume that agents have access to particular class of (possibly multidimensional) signals. In general, this will impose restrictions on the set of Bayes Nash equilibria and the upper bounds on volatility cannot necessarily be obtained. For example, in Angeletos and La'O [4], each agent knows his individual payoff shock but remains uncertain about the payoff shock of the other agents. In Angeletos and Pavan [6] the agents have access to a rich three-dimensional class of normal signals, and even though the set of Bayes correlated equilibria is also three-dimensional, the information structures of Angeletos and Pavan [6] do not give rise to maximal volatility.

This paper is an application of a general approach to analyzing equilibrium behavior of agents for a given description of the fundamentals for *all* possible information structures. In Bergemann and Morris [9], we consider this problem in a canonical game theoretic setting and show that a general version of Bayes correlated equilibrium characterizes the set of outcomes that could arise in any Bayes Nash equilibrium of an incomplete information game where agents may have access to more information beyond the given common prior over the fundamentals. In Bergemann and Morris [8] we pursue this argument in detail and characterize the set of Bayes correlated equilibria in the class of games with quadratic payoffs and normally distributed uncertainty, but there we restrict our attention to the special environment with aggregate shocks only. By providing a complete analysis of the impact of information in a simple linear quadratic model with both idiosyncratic and aggregate payoff shocks, we can connect with richer macroeconomic and other applied models that work with parameterized classes of information structures.

The remainder of the paper is organized as follows. Section 2 introduces the model. Section 3 introduces the noise free information structures and analyzes the Bayes Nash equilibrium behavior for this class of information structures. Section 4 derives the maximal volatility and dispersion in

a benchmark model without strategic interaction. We identify the specific noise free information structure that attains the maximal volatility, and thus establish the link between information and volatility. Section 5 examines how the strategic interaction affects how the information structure impacts the statistical moments of the economy. In Section 6, we introduce symmetric Bayes correlated equilibria and establish an equivalence between the set of Bayes correlated equilibria and the set of Bayes Nash equilibria under any symmetric normal information structure. Section 7 then considers some particular information structures that have been studied, and highlights the subtle restrictions that such modelling choices imposes on the equilibrium behavior. Section 8 indicates how the present analysis extends beyond aggregate interaction to allow for richer, network like, interaction structures. Finally, we apply the currents insights to the related literature on information sharing and obtain optimal information sharing rules that differ and improve upon those appearing in the literature. Section 9 constitutes the appendix and contains most of the proofs.

2. Model

We consider a continuum of agents, with mass normalized to 1. Agent $i \in [0,1]$ chooses an action $a_i \in \mathbb{R}$ and is assumed to have a quadratic payoff function,

$$u_i : \mathbb{R}^3 \to \mathbb{R},$$

which is function of his action a_i, the mean action taken by all agents, A,

$$A \triangleq \int_j a_j dj,$$

and the individual payoff shock, $\theta_i \in \mathbb{R}$, thus $u_i (a_i, A, \theta_i)$.

We assume that the individual payoff shock θ_i is given by the sum of an *aggregate shock* $\bar{\theta}$ and an *idiosyncratic shock* $\Delta \theta_i$:

$$\theta_i = \bar{\theta} + \Delta \theta_i.$$

The aggregate shock $\bar{\theta}$ is common to all agents and the idiosyncratic shock $\Delta \theta_i$ is identically and independently distributed across agents, as well as independent of the aggregate shock. Each component of the payoff shock θ_i is normally distributed. The payoff environment is thus completely described by the pair $(\bar{\theta}, \Delta \theta_i)$ of random variables,

$$\begin{pmatrix} \bar{\theta} \\ \Delta \theta_i \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} \mu_{\bar{\theta}} \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{\bar{\theta}}^2 & 0 \\ 0 & \sigma_{\Delta \theta_i}^2 \end{pmatrix} \right), \tag{1}$$

which form the commonly known *common prior*. The sample average of the idiosyncratic shocks across all agents always equals zero. We denote the sample average across the entire population, that is across all i, as $\mathbb{E}_i[\cdot]$, and so $\mathbb{E}_i[\Delta\theta_i] = 0$. The aggregate shock can be interpreted as the sample mean or average payoff shock, and so $\bar{\theta} = \mathbb{E}_i[\theta_i]$.

Given the independence and the symmetry of the idiosyncratic shock $\Delta\theta_i$ across agents, the variance of the individual payoff shock θ_i can be expressed in terms of the variance of the sum of the idiosyncratic and the aggregate shock:

$$\sigma_\theta^2 \triangleq \sigma_{\bar{\theta}}^2 + \sigma_{\Delta\theta_i}^2. \tag{2}$$

The correlation (coefficient) $\rho_{\theta\theta}$ between the payoff shocks of any two agents i and j, θ_i and θ_j is given by,

$$\rho_{\theta\theta} = \frac{\sigma_{\bar{\theta}}^2}{\sigma_{\bar{\theta}}^2 + \sigma_{\Delta\theta_i}^2}, \tag{3}$$

and the covariance between individual payoff shocks θ_i and θ_j equals the covariance between the individual shock θ_i and the aggregate shock $\bar{\theta}$. Thus, the correlation coefficient $\rho_{\theta\theta}$ represents a measure of the "commonality" of the shocks across agents. Separately, from the perspective of a given agent i, the correlation coefficient $\rho_{\theta\theta}$ measures the relative contribution of the aggregate shock $\bar{\theta}$ to the individual shock of agent i, $\theta_i = \bar{\theta} + \Delta\theta_i$.

Given the quadratic property of the payoff function, each agent i has a linear best response function:

$$a_i = r\mathbb{E}[A|\mathcal{I}_i] + \mathbb{E}[\theta_i|\mathcal{I}_i], \tag{4}$$

where $\mathbb{E}[\cdot|\mathcal{I}_i]$ is the expectation conditional on the information $\mathcal{I}_i$ that agent i has prior to taking an action a_i. The parameter $r \in \mathbb{R}$ of the best response function represents the strategic interaction among the agents. If $r < 0$, then we have a game of *strategic substitutes*, if $r > 0$, then we have a game of *strategic complements*.

For future reference, we briefly describe the Nash equilibrium under complete information. If each agent observes the idiosyncratic and the aggregate shock separately, and hence receives the *two-dimensional* signal $(\Delta\theta_i, \bar{\theta})$, then the linear best response function gives rise the symmetric Nash equilibrium strategy:

$$a_i(\Delta\theta_i, \bar{\theta}) \triangleq \Delta\theta_i + \frac{\bar{\theta}}{1-r}. \tag{5}$$

In the complete information equilibrium, each agent assigns weight 1 to the idiosyncratic shock $\Delta\theta_i$, and weight $1/(1-r)$ to the aggregate shock $\bar{\theta}$. We assume that the interaction parameter r is

bounded above, or $r \in (-\infty, 1)$, which guarantees that there is a unique interior Nash equilibrium.
[1]

The present model of a continuum of players with quadratic payoffs and normally distributed idiosyncratic and aggregate shocks was first proposed by Vives [23] to analyze information sharing among agents with private, but noisy, information about the fundamentals. While the focus of the present paper is rather different, we shall briefly indicate in the conclusion how our approach also yields new insights to the large literature on information sharing. In the information sharing literature, the *aggregate* and the *idiosyncratic shock,* are often referred to as the common value and the private value *component* of the payoff shock θ_i.

3. Noise Free Bayes Nash Equilibrium

We begin the analysis by considering a class of noise free information structures and then derive the Bayes Nash equilibrium behavior under these one-dimensional information structures. We consider the following *one-dimensional* class of signals:

$$s_i \triangleq \lambda \Delta \theta_i + (1 - \lambda) \bar{\theta}, \tag{6}$$

where the linear composition of the signal s_i is determined by the parameter $\lambda \in [0, 1]$. We restrict attention to symmetric information structures (across agents), all agents have the same parameter λ, and hence simply refer to the noise free information structure λ. In the present section, we consider the case of idiosyncratic *and* aggregate shocks, thus $\rho_{\theta\theta} \in (0, 1)$, and we discuss the limit cases of pure idiosyncratic and pure aggregate shocks, thus $\rho_{\theta\theta} \in \{0, 1\}$, in the next section.

The information structure λ is *noise free* in the sense that every signal s_i is a linear combination of the idiosyncratic and the aggregate shock, $\Delta \theta_i$ and $\bar{\theta}$, and no extraneous noise or error term enters the signal of each agent. Nonetheless, since the signal s_i combines the idiosyncratic and the

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[1]The best response (4) normalizes the weight on the payoff shock θ_i to 1 and assigns the weight r to the average action A. A subset of us adopted this normalization, Bergemann and Morris [8], and we can directly contrast the results of the pure aggregate shock model there with the model of idiosyncratic *and* aggregate shocks here. An alternative normalization would assign weights $1 - r$ to the payoff shock and r to the average action, as in Morris and Shin [16] or Angeletos and Pavan [5]. With *pure aggregate shocks*, the later normalization has the advantage that under complete information, the equilibrium volatility of the individual or the aggregate action is independent of r, whereas in the present model r scales the volatility. Yet, in a general model with idiosyncratic and aggregate shocks even the complete information volatility is affected by r (since then $a_i \left(\Delta \theta_i, \bar{\theta} \right) = (1 - r) \Delta \theta_i + \bar{\theta}$) and hence this normalization would also fail to preserve the invariance property for the individual volatility.

aggregate shock, each signal s_i leaves agent i with residual uncertainty about the contribution of the idiosyncratic and aggregate shock respectively. Moreover, unless the weight λ in the information structure exactly mirrors the composition of the payoff shock, $\theta_i = \bar{\theta} + \Delta\theta_i$, and hence exactly equals $1/2$, agent i still faces residual uncertainty about his payoff shock θ_i. Thus, the signal *confounds* the two sources of payoff uncertainty.

Given the information structure λ, we can compute the conditional expectation of agent i given the signal s_i about the idiosyncratic shock $\Delta\theta_i$:[2]

$$\mathbb{E}\left[\Delta\theta_i \,|s_i\right] = \frac{\text{cov}\left(\Delta\theta_i, s_i\right)}{\text{var}\left(s_i\right)} = \frac{\left(1 - \rho_{\theta\theta}\right)\lambda}{\rho_{\theta\theta}\left(1 - \lambda\right)^2 + \left(1 - \rho_{\theta\theta}\right)\lambda^2} s_i, \tag{7}$$

the aggregate shock $\bar{\theta}$:

$$\mathbb{E}\left[\bar{\theta} \,|s_i\right] = \frac{\text{cov}\left(\bar{\theta}, s_i\right)}{\text{var}\left(s_i\right)} = \frac{\rho_{\theta\theta}\left(1 - \lambda\right)}{\rho_{\theta\theta}\left(1 - \lambda\right)^2 + \left(1 - \rho_{\theta\theta}\right)\lambda^2} s_i, \tag{8}$$

and the payoff shock θ_i of agent i :

$$\mathbb{E}\left[\theta_i \,|s_i\right] = \frac{\text{cov}\left(\theta_i, s_i\right)}{\text{var}\left(s_i\right)} = \frac{\rho_{\theta\theta}\left(1 - \lambda\right) + \left(1 - \rho_{\theta\theta}\right)\lambda}{\rho_{\theta\theta}\left(1 - \lambda\right)^2 + \left(1 - \rho_{\theta\theta}\right)\lambda^2} s_i. \tag{9}$$

For the above conditional expectations, it is only the relative size of idiosyncratic and aggregate shocks, thus $\rho_{\theta\theta}$, that matters rather than the level of the shocks, the variance $\sigma^2_{\Delta\theta_i}$ and $\sigma^2_{\bar{\theta}}$.

A few noise free information structures are of particular interest. If $\lambda = 1/2$, then each agent knows his own payoff shock θ_i, as (9) reduces to $\mathbb{E}\left[\theta_i \,|s_i\right] = 2s_i = \Delta\theta_i + \bar{\theta}$, but remains uncertain about the exact value of the idiosyncratic and aggregate shock. Similarly, if $\lambda = 0$, then the aggregate shock is known with certainty by each agent, as $\mathbb{E}\left[\bar{\theta} \,|s_i\right] = s_i = \bar{\theta}$, but there remains residual uncertainty about the idiosyncratic shock $\Delta\theta_i$ and a fortiori about the payoff shock θ_i. Likewise, if $\lambda = 1$, then the idiosyncratic shock is known with certainty, as $\mathbb{E}\left[\Delta\theta_i \,|s_i\right] = s_i = \Delta\theta_i$, but there remains residual uncertainty about the aggregate shock $\bar{\theta}$ and a fortiori about the payoff shock θ_i.

We record the standard solution concept for games of incomplete information.

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[2]For the remainder of the paper, we report all of the conditional expectations under the normalization of $\mu_{\bar{\theta}} = 0$. With $\mu_{\bar{\theta}} \neq 0$, the conditional expectations, such as (7)-(9) below, are given by a convex combination of the signal s_i and prior mean $\mu_{\bar{\theta}}$. By normalizing $\mu_{\bar{\theta}} = 0$, the statistical expressions become easier to read with minor loss of generality. By contrast, the description of the equilibrium in terms of mean and variance, as in Proposition 2, are always stated for $\mu_{\bar{\theta}} \in \mathbb{R}$.

Definition 1 (Bayes Nash Equilibrium).
The strategy profile $a^* : \mathbb{R} \to \mathbb{R}$ forms a pure strategy symmetric Bayes Nash equilibrium if and only if:

$$a^*\left(s_i\right) = \mathbb{E}[\theta_i + r \int a_j^*\left(s_j\right) dj \mid s_i], \ \forall s_i \in \mathbb{R}.$$

The construction of a linear and symmetric equilibrium strategy in the multivariate normal environment is by now standard, see Vives [24] and Veldkamp [20]. Given an information structure λ, we denote the responsiveness, and in the linear strategy, the slope of the strategy in the signal s_i, by $w(\lambda)$.

Proposition 1 (Noise Free BNE).
For every noise free information structure λ, there is a unique Bayes Nash equilibrium and the strategy of each agent i is linear in the signal s_i :

$$a_i^*\left(s_i\right) = w\left(\lambda\right) s_i, \tag{10}$$

with weight $w\left(\lambda\right)$:

$$w(\lambda) = \frac{\rho_{\theta\theta}(1 - \lambda) + (1 - \rho_{\theta\theta})\lambda}{(1 - r)\rho_{\theta\theta}(1 - \lambda)^2 + \lambda^2(1 - \rho_{\theta\theta})}. \tag{11}$$

The responsiveness of the individual strategy is in general affected by the interaction parameter r, but in the special case of $r = 0$, each agent solves an pure statistical prediction problem and the optimal weight corresponds to the Bayesian updating rule given by (9). If $r > 0$, then the agents are in a game with strategic complements and respond stronger to the signal than Bayesian updating would suggest because of the inherent coordination motive with respect to the aggregate shock represented by the weight $\rho_{\theta\theta}(1 - \lambda)^2$.

In every noise free information structure, the best response of each agent i is always summarized by a scalar, $w\left(\lambda\right)$. To understand the impact the information structure has on the outcome of the equilibrium we provide a statistical description in terms of first and second moments of the variables (actions and payoff shocks) induced by an equilibrium. Given the information structure λ and the linearity of the unique Bayes Nash equilibrium, we can immediately derive the properties of the joint distribution of the equilibrium variables.

Proposition 2 (Moments of the Noise Free BNE).
For every noise free information structure λ:

1. *the mean of the individual and the aggregate action is:*

$$\mathbb{E}[a_i] \triangleq \mu_a = \mu_\theta / (1 - r) = \mu_A \triangleq \mathbb{E}[A];$$

2. *the variance of the individual action is:*

$$\text{var}\,(a_i) \triangleq \sigma_a^2 = w(\lambda)^2 (\rho_{\theta\theta}(1-\lambda)^2 + (1-\rho_{\theta\theta})\lambda^2)\sigma_\theta^2; \tag{12}$$

3. *and the variance of the aggregate action is:*

$$\text{var}\,(A) = \text{cov}(a_i, a_j) \triangleq \rho_{aa}\sigma_a^2 = w(\lambda)^2 \rho_{\theta\theta}(1-\lambda)^2 \sigma_\theta^2. \tag{13}$$

The mean of the individual and the aggregate action is only a function of the mean μ_θ of the payoff shock θ_i and the interaction parameter r, and thus is invariant with respect to the composition of the payoff shocks in terms of idiosyncratic and aggregate shocks and invariant with respect to the information structure λ. By contrast, the second moments, respond to the composition of the shocks as represented by the correlation coefficient $\rho_{\theta\theta}$ and to the information structure λ. Given the normal distribution of the payoff shocks, and the linearity of the strategy, the variance and covariance terms are naturally the products of the weights, λ and $w(\lambda)$, and the variance σ_θ^2 of the fundamental uncertainty.

In the symmetric environment, the variance of the aggregate action, $\text{var}\,(A)$, coincides with covariance between any pair of actions by agents i and j, a_i and a_j, $\text{cov}(a_i, a_j)$, and we denote the correlation coefficients by ρ_{aa}. The covariance between the action a_i and the payoff shock θ_i of agent i, is given by:

$$\text{cov}(a_i, \theta_i) \triangleq \rho_{a\theta}\sigma_a\sigma_\theta = w(\lambda)\left(\rho_{\theta\theta}(1-\lambda) + (1-\rho_{\theta\theta})\lambda\right)\sigma_\theta^2 = \frac{\rho_{\theta\theta}(1-\lambda) + (1-\rho_{\theta\theta})\lambda}{\sqrt{\rho_{\theta\theta}(1-\lambda)^2 + (1-\rho_{\theta\theta})\lambda^2}}\sigma_\theta^2, \tag{14}$$

whereas the covariance between the action a_i of agent i and the payoff shock θ_j of another agent j is given by:

$$\text{cov}(a_i, \theta_j) \triangleq \rho_{a\phi}\sigma_a\sigma_\theta = w(\lambda)(1-\lambda)\rho_{\theta\theta}\sigma_\theta^2 = \text{cov}(a_i, \bar{\theta}) = \text{cov}(A, \bar{\theta}). \tag{15}$$

We denote the corresponding correlation coefficients by $\rho_{a\theta}$ and $\rho_{a\phi}$, respectively.

As the above equalities indicate, in the symmetric environment, the covariance between the individual action a_i of agent i and the payoff shock θ_j of another agent j, $\text{cov}(a_i, \theta_j)$, is equal to the covariance between the aggregate action A and aggregate shock $\bar{\theta}$. After all, the idiosyncratic element vanishes in the covariance between a_i and θ_j, and what remains is the covariance between the common terms, A and $\bar{\theta}$. Importantly though, the correlation coefficients $\rho_{a\phi}$ and $\rho_{A\bar{\theta}}$ differ as we can compute that

$$\rho_{a\phi} = \frac{(1-\lambda)\rho_{\theta\theta}}{\sqrt{(\rho_{\theta\theta}(1-\lambda)^2 + (1-\rho_{\theta\theta})\lambda^2)}},$$

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but

$$\rho_{A\bar\theta} = 1.$$

Thus under any noise free information structure parametrized by λ, the aggregate action A is perfectly correlated with the aggregate shock $\bar\theta$.

In their statistical content, the correlation *coefficients* reflect the direction in the relationship of any two random variables. Hence, the interaction parameter r which only affects the slope of the equilibrium strategy, but not the composition of the signals, does not appear in the expression of the equilibrium correlation coefficients. Thus, for a given noise free information structure λ, we could, as an alternative to Proposition 2 completely characterize the Bayes Nash equilibrium in terms of the correlation coefficients $\left(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi}\right)$. This triple only depends on the composition of the payoff shocks, $\rho_{\theta\theta}$ and the information structure λ. We could then recover all the moments of the equilibrium distribution, such as mean and variance, from the triple of correlation coefficients and the interaction parameter r.

4. Individual Decisions and Aggregate Volatility

We first consider aggregate volatility in the absence of any strategic interaction, and thus we are setting the strategic parameter r equal to zero. While we focus on aggregate volatility, we will also report results for individual volatility and dispersion.

With $r = 0$, the best response of each agent simply reflects a statistical prediction problem, namely to predict the payoff shock θ_i given the signal s_i:

$$a_i = \mathbb{E}\left[\theta_i \,|\, s_i\right] = \frac{(1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta})}{(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta})} s_i. \tag{16}$$

The individual prediction problem is more responsive to the signal s_i if and only if the signal contains more information about the payoff shock θ_i. As we observed, the noise free information structure $\lambda = 1/2$ allows each agent to perfectly infer the individual payoff shock θ_i. It follows that the responsiveness, and hence the variance of the *individual action* σ_a^2 is maximized at $\lambda = 1/2$:

$$\sigma_a^2 = \sigma_{\bar\theta}^2 + \sigma_{\Delta\theta_i}^2.$$

Now, to the extent that the individual payoff shocks θ_i and θ_j are correlated, we find that even though each agent i only solves an individual prediction problem, their actions are correlated by

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means of the underlying correlation of the individual payoff shocks. Under the information structure $\lambda = 1/2$ the aggregate volatility is

$$\sigma_A^2 = \rho_{aa}\sigma_a^2 = \sigma_{\bar{\theta}}^2.$$

We can now ask whether the aggregate volatility can reach higher levels under information structures different from $\lambda = 1/2$. As the information structure departs from $\lambda = 1/2$, we necessarily introduce a bias in the signal s_i towards one of the two components of the payoff shock θ_i. Clearly, the signal s_i is losing informational quality with respect to the payoff shock θ_i as λ moves away from $1/2$. Thus the individual prediction problem (16) is becoming noisier, and in consequence the response of the individual agent to the signal s_i is *attenuated*. But a larger weight, $1 - \lambda$, on the aggregate shock $\bar{\theta}$, may support correlation in the actions across agents, and thus support larger aggregate volatility. At the same time, the response of the agent is likely to be attenuated, and thus a trade-off appears between bias and loss of information. We then ask what is the maximal aggregate volatility that can be sustained across all noise free information structures.

Proposition 3 (Maximal Aggregate Volatility).
The maximal aggregate volatility:

$$\max_{\lambda} \{\text{var}(A)\} = \frac{\left(\sigma_{\bar{\theta}} + \sqrt{\sigma_{\bar{\theta}}^2 + \sigma_{\Delta\theta_i}^2}\right)^2}{4}, \tag{17}$$

is achieved by the information structure λ^* :

$$\lambda^* \triangleq \arg\max_{\lambda} \{\text{var}(A)\} = \frac{\sigma_{\bar{\theta}}}{2\sigma_{\bar{\theta}} + \sqrt{\sigma_{\bar{\theta}}^2 + \sigma_{\Delta\theta_i}^2}} < \frac{1}{2}. \tag{18}$$

The aggregate volatility is maximized by an information structure which biases the signal towards the aggregate shock.

We recall from (2) that the total variance of the payoff shock θ_i is given by $\sigma_{\theta}^2 = \sigma_{\bar{\theta}}^2 + \sigma_{\Delta\theta_i}^2$. By a change of variable, we can express the variance of the idiosyncratic and aggregate components of the shocks in terms of the total variance of the payoff shock θ_i and the correlation coefficient $\rho_{\theta\theta}$, thus

$$\sigma_{\bar{\theta}}^2 = \rho_{\theta\theta}\sigma_{\theta}^2, \qquad \sigma_{\Delta\theta_i}^2 = (1 - \rho_{\theta\theta})\sigma_{\theta}^2. \tag{19}$$

We can then express the information structure that maximizes the aggregate volatility in terms of the correlation coefficient $\rho_{\theta\theta}$:

$$\arg\max_{\lambda} \{\text{var}(A)\} = \frac{\sqrt{\rho_{\theta\theta}}}{1 + 2\sqrt{\rho_{\theta\theta}}},$$

and the maximal volatility given by (17) can be expressed as:

$$\max_{\lambda} \{\operatorname{var}(A)\} = \frac{1}{4}\left(1 + \sqrt{\rho_{\theta\theta}}\right)^2 \sigma_\theta^2.$$

Thus, as we approach the environment with aggregate shocks only, $\rho_{\theta\theta} \to 1$ (and equivalently as the contribution from the idiosyncratic shock vanishes with $\sigma_{\Delta\theta_i}^2 \to 0$), the maximal aggregate volatility of the actions coincides with the variance of the aggregate shock. This is achieved in a complete information equilibrium in which the action of each agent matches the realization of the payoff shock θ_i. As all of the variance in the payoff shock θ_i stems from the aggregate shock, the variance of the individual action is attained by the variance of the average action.

More surprisingly, as we approach an environment with purely idiosyncratic shocks, the maximal *aggregate* volatility does not converge to zero, rather it is bounded away from 0, and given by $\sigma_{\Delta\theta_i}^2/4$, as stated in the following Corollary of Proposition 3.

Corollary 1 (Maximal Volatility with Aggregate or Idiosyncratic Shocks Only).
The maximal volatility with aggregate shocks only is $\lim_{\sigma_{\Delta\theta_i}^2 \to 0} \max_\lambda \{\operatorname{var}(A)\} = \sigma_\theta^2$, *with idiosyncratic shocks only is* $\lim_{\sigma_\theta^2 \to 0} \max_\lambda \{\operatorname{var}(A)\} = \sigma_{\Delta\theta_i}^2/4.$

If the individual payoff shock is increasingly dominated by the idiosyncratic shock, then the information structure puts more and more weight on the aggregate shock which itself has diminishing variance. The volatility maximizing information structure amplifies the response to the small aggregate shock and hence maintains a substantial correlation in the signals (and actions) across the agents, even though the payoff shocks are almost purely idiosyncratic and thus almost independent. If we consider the ratio of the weighted standard deviations of the aggregate and idiosyncratic shocks, where the weights are determined by the information structure λ^*, then we find that

$$\frac{(1 - \lambda^*)\sigma_{\bar{\theta}}}{\lambda^* \sigma_{\Delta\theta_i}} = \frac{1 + \sqrt{\rho_{\theta\theta}}}{\sqrt{1 - \rho_{\theta\theta}}},$$

and in the limit as $\rho_{\theta\theta} \to 0$ (or equivalently as $\sigma_{\bar{\theta}} \to 0$),

$$\lim_{\sigma_{\bar{\theta}} \to 0} \left\{ \frac{(1 - \lambda^*)\sigma_{\bar{\theta}}}{\lambda^* \sigma_{\Delta\theta_i}} \right\} = 1. \tag{20}$$

Thus, the economy can maintain a large aggregate volatility even in the presence of vanishing aggregate payoff shocks by confounding the payoff relevant information about the idiosyncratic shock with the (in the limit) payoff irrelevant information about the aggregate shock.

In Section 3 we analyzed the equilibrium behavior in the noise free information structures assuming idiosyncratic *and* aggregate shocks, and hence confining the analysis to $\rho_{\theta\theta} \in (0, 1)$. The above

limit argument towards idiosyncratic shocks only suggests that as long as there is some *arbitrarily small* variation in the payoff shock, the signal can always amplify the informational importance of the shock much beyond its payoff importance. But, if there is no variance in either the idiosyncratic or the aggregate shock, then of course no amplification of those shocks is possible. Nonetheless, the limits in Corollary 1 can still be attained with zero variance in either one of the shocks, but now require noise in the signal that is *payoff irrelevant*. We illustrate this for the case of idiosyncratic payoff shocks only. Thus, consider an information structure in which each agent i observes a signal that contains an error ε, common to all agents, with mean 0 and variance σ_ε^2,

$$s_i = \Delta\theta_i + \varepsilon. \tag{21}$$

Given the signal s_i, the best response of agent i is

$$a_i = \mathbb{E}\left[\Delta\theta_i | s_i = \Delta\theta_i + \varepsilon\right] = \frac{\sigma_{\Delta\theta_i}^2}{\sigma_{\Delta\theta_i}^2 + \sigma_\varepsilon^2} s_i, \tag{22}$$

and it follows that the realized average action A is, as the idiosyncratic shock $\Delta\theta_i$ integrates out,

$$A = \frac{\sigma_{\Delta\theta_i}^2}{\sigma_{\Delta\theta_i}^2 + \sigma_\varepsilon^2} \varepsilon.$$

The resulting variance of the average action, the aggregate volatility is:

$$\sigma_A^2 = \left(\frac{\sigma_{\Delta\theta_i}^2}{\sigma_{\Delta\theta_i}^2 + \sigma_\varepsilon^2}\right)^2 \sigma_\varepsilon^2,$$

and the aggregate volatility is maximized by setting the variance of the error term equal to the variance of the idiosyncratic shock:

$$\sigma_\varepsilon^2 = \sigma_{\Delta\theta_i}^2. \tag{23}$$

This results in a positive level of aggregate volatility σ_A^2 driven by purely idiosyncratic payoff shocks $\sigma_{\Delta\theta_i}^2$:

$$\sigma_A^2 = \frac{1}{4}\sigma_{\Delta\theta_i}^2.$$

The noisy information structure (21) thus achieves the limit of Corollary 1 with the noise to signal ratio of 1 implied by (23), which we derived earlier in (20) as the limiting ratio.

The maximal aggregate volatility is therefore achieved by an information structure that finds an optimal trade-off between biasing the information towards the aggregate shock, and here simply the common error, and maintaining responsiveness of agent i towards the signal s_i as given by the best

response condition (22). Specifically, an increase in the variance σ_ε^2 of the error term leads to larger aggregate volatility only if the response of each agent to the signal does not become too attenuated. As the slope $\sigma_{\Delta\theta_i}^2 / \left(\sigma_{\Delta\theta_i}^2 + \sigma_\varepsilon^2\right)$ of the best response is decreasing in the variance σ_ε^2 of the error term, the idiosyncratic payoff shock can only absorb a finite variance of the error term, namely $\sigma_\varepsilon^2 = \sigma_{\Delta\theta_i}^2$, before the response to the signal becomes too weak to generate additional aggregate volatility.

In the special cases of pure idiosyncratic or pure aggregate shocks, the payoff uncertainty is described completely by either $\Delta\theta_i$ or $\bar\theta$, and reduces from a two-dimensional to a one-dimensional space of uncertainty. In either case, across all $\lambda \in [0,1]$, there are only *two* possible noise free equilibrium outcomes. Namely, either players respond perfectly to the shock of the world (complete information) or players do not respond at all (zero information). For example, with idiosyncratic shocks only, that is $\rho_{\theta\theta} = 0$, we have $\sigma_{\Delta\theta_i}^2 = \sigma_\theta^2$, and $\sigma_{\bar\theta}^2 = 0$. Then, the signal s_i is perfectly informative for all $\lambda \neq 0$ about the idiosyncratic shock, and we are effectively in a complete information setting. By contrast, if $\lambda = 0$, then the signal s_i is completely uninformative, and each agent makes a deterministic choice given the expected value $\mathbb{E}[\Delta\theta_i] = 0$ of the shock. Correspondingly, for purely aggregate shocks, the critical value for which the information structure is completely uninformative is $\lambda = 1$. Therefore there is a discontinuity at $\rho_{\theta\theta} \in \{0,1\}$ in the set of *noise free* Bayes Nash equilibria, but as the construction of the noisy information structure (21) suggests there is no discontinuity in the set of outcomes. The reason is simple and stems from the fact that as $\rho_{\theta\theta}$ approaches 0 or 1, one of the dimensions of payoff uncertainty vanishes. Yet we should emphasize, that even as the payoff shocks approach the case of pure aggregate or pure idiosyncratic shocks, the part of the fundamental that becomes small can be arbitrarily amplified by the informational weight λ. For example, as $\rho_{\theta\theta} \to 1$, the idiosyncratic shock $\Delta\theta_i$ can still be amplified by letting $\lambda \to 1$ in the construction of the signal (6) above. Thus, the idiosyncratic shock $\Delta\theta_i$ acts like a purely idiosyncratic noise in an environment with aggregate shocks. After all, the idiosyncratic shock $\Delta\theta_i$ only affects the payoffs in a negligible way, but with a large enough weight, it has a non-negligible effect on the actions that the players take. This suggests that for the case in which the correlation of shocks approaches the case of pure aggregate or pure idiosyncratic shocks, there is no longer a sharp distinction between what is noise and what is fundamentals.

The dispersion of the individual action, $\Delta a_i \triangleq a_i - A$, is the volatility of the individual action beyond the aggregate volatility:

$$\text{var}(\Delta a_i) = (1 - \rho_{aa})\,\sigma_a^2.$$

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The analysis of the maximal dispersion is entirely symmetric to the one for aggregate volatility after we redefine the relevant variables in the obvious way:

$$\tilde{\lambda} \triangleq (1 - \lambda), \ \tilde{\rho}_{\theta\theta} \triangleq (1 - \rho_{\theta\theta}).$$

The result below then follows directly from Proposition 3.

Corollary 2 (Maximal Dispersion).
The maximal dispersion:

$$\max_{\lambda} \{\text{var}(\Delta a_i)\} = \frac{1}{4} \left(1 + \sqrt{1 - \rho_{\theta\theta}} \right)^2 \sigma_\theta^2;$$

is achieved by the information structure λ :

$$\arg\max_{\lambda} \{\text{var}(\Delta a_i)\} = \frac{1 + \sqrt{1 - \rho_{\theta\theta}}}{1 + 2\sqrt{1 - \rho_{\theta\theta}}} > \frac{1}{2}.$$

5. Strategic Decisions and Aggregate Volatility

We proceed to analyze the aggregate volatility in the presence of strategic interaction. We thus return to the general model with $r \in (-\infty, 1)$ rather than $r = 0$ as in the previous section. For each individual agent the responsiveness of the action to the signal will now depend on the informational content of the signal as well as the structure of the interaction.

An instructive benchmark for the responsiveness of the action to the signal is the behavior in the complete information equilibrium. Under complete information, each agent separately observes the idiosyncratic and aggregate shock, thus each player observes a noise free, but *two-dimensional* signal $(\bar{\theta}, \Delta\theta_i)$. The resulting equilibrium strategy is linear in the shocks and assigns weight 1 to the idiosyncratic shock $\Delta\theta_i$, and weight $1/(1 - r)$ to the aggregate shock $\bar{\theta}$, see (5). Given the linearity of the complete information strategy, a specific *one-dimensional* noise free information structure can replicate the outcome under the two-dimensional complete information structure. In particular, the one-dimensional information structure $\widehat{\lambda}$ with

$$\widehat{\lambda} \triangleq \frac{1 - r}{2 - r}, \tag{24}$$

is the unique information structure $\lambda \in [0, 1]$ such that for every realized pair of shocks, $(\Delta\theta_i, \bar{\theta})$ and every signal realization s_i,

$$s_i = \widehat{\lambda}\Delta\theta_i + (1 - \lambda)\bar{\theta},$$

the equilibrium action

$$a_i^*(s_i) = \Delta\theta_i + \frac{\bar{\theta}}{1-r}$$

coincides with the complete information equilibrium action. Thus for every level of strategic interaction, r, there is a noise free information structure $\widehat{\lambda}$ under which the agents' equilibrium behavior exactly mimics the complete information Nash equilibrium. Moreover, by inspecting the responsiveness to the signal, $w(\lambda)$, derived earlier in Proposition 1, we find that $\widehat{\lambda}$ is the unique information structure among all $\lambda \in [0,1]$ such that the responsiveness $w(\lambda)$ is invariant to the relative contribution of the aggregate and idiosyncratic shock, and hence invariant with respect to the value of $\rho_{\theta\theta}$, namely

$$w\left(\widehat{\lambda}\right) = \frac{1}{\widehat{\lambda}} = \frac{2-r}{1-r}.$$

In particular, the Bayes Nash equilibrium under the information structure $\widehat{\lambda}$ always reproduces the complete information outcome, independent of the composition of the shocks.

We now decompose the responsiveness of the individual action a_i into the components of the payoff shock θ_i, namely the idiosyncratic shock $\Delta\theta_i$ and the aggregate shock $\bar{\theta}$. Given the multivariate normal distribution, the responsiveness of the action can be expressed in terms of the covariance:

$$\frac{\partial \mathbb{E}[a_i|\Delta\theta_i]}{\partial \Delta\theta_i} = \frac{\mathrm{cov}(a_i, \Delta\theta_i)}{\sigma_{\Delta\theta}^2} = \lambda w(\lambda), \quad \frac{\partial \mathbb{E}[a_i|\bar{\theta}]}{\partial\bar{\theta}} = \frac{\mathrm{cov}(a_i, \bar{\theta})}{\sigma_{\bar{\theta}}^2} = (1-\lambda)w(\lambda). \tag{25}$$

Proposition 4 (Responsiveness to Fundamentals).
In the noise free Bayes Nash equilibrium with information structure λ:

1. $\lambda \in (\widehat{\lambda}, 1) \Leftrightarrow \frac{\mathrm{cov}(a_i, \Delta\theta_i)}{\sigma_{\Delta\theta}^2} > 1$;
2. $\lambda \in (0, \widehat{\lambda}) \Leftrightarrow \frac{\mathrm{cov}(a_i, \bar{\theta})}{\sigma_{\bar{\theta}}^2} > \frac{1}{1-r}$.

Thus, the responsiveness of the action to each component of the payoff shock is determined by the weight λ that the signal assigns relative to complete information benchmark $\hat{\lambda}$. For any given information structure λ, the responsiveness is stronger than in the complete information environment for exactly one of the components, and weaker for the other.

We recall from the analysis in the previous section that with purely aggregate shocks, any residual uncertainty about the payoff shock inevitably reduced the responsiveness of the individual agent to the aggregate shock, and ultimately reduced the aggregate responsiveness. Similarly, with idiosyncratic shocks only, the residual uncertainty attenuated the responsiveness to his payoff

shock θ_i. By contrast, in the joint presence of idiosyncratic and aggregate shocks, the interaction between the idiosyncratic and the aggregate shock can correlate the responsiveness of the agents without attenuating the individual response to one of the components, thus leading to a greater responsiveness to *one* component of the shock than could be achieved under complete information. Intuitively, this stems from the fact that part of the responsiveness to one of the components is loaded onto the other component.

In Figure 1 we plot the responsiveness for the case of $\rho_{\theta\theta} = 0.5$ and different interaction parameters r. The threshold values $\widehat{\lambda}_r$ simply corresponds to the critical value $\widehat{\lambda}$ for each of the considered interaction parameters $r \in \left\{ -\frac{3}{4}, 0, +\frac{3}{4} \right\}$. The horizontal black lines represent the responsiveness to the aggregate shock $\bar{\theta}$ in the complete information equilibrium which is equal to $1/(1-r)$, and the responsiveness to the idiosyncratic part, which is always equal to 1. By contrast, the red curves represent the responsiveness to the aggregate shock along the noise free equilibrium, and the blue curves represent the responsiveness to the idiosyncratic shock. Thus if $\lambda < \widehat{\lambda}$, then the responsive-
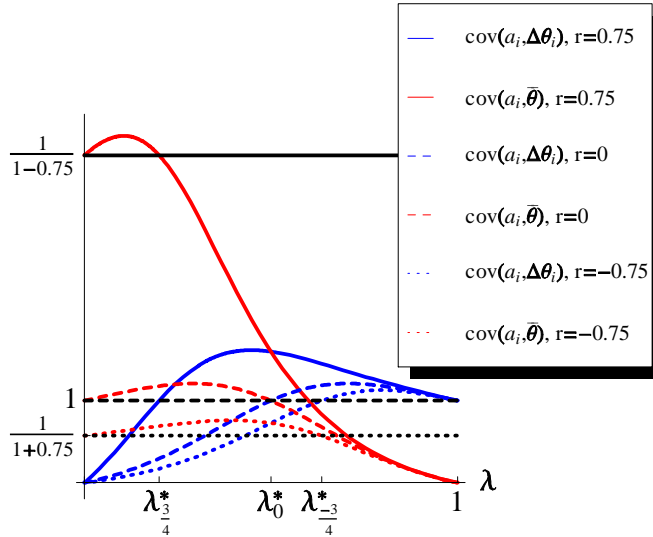

Figure 1: Responsiveness to Fundamentals for $\rho_{\theta\theta} = 1/2$

ness to the aggregate shock $\bar{\theta}$ is larger than in the complete information equilibrium, and conversely for $\Delta\theta_i$. Moreover, we observe that the maximum responsiveness to the aggregate shock is never attained under either the complete information equilibrium or at the boundary values of λ, at 0 or 1. This immediately implies that the responsiveness is not monotonic in the informational content. We now provide some comparative static results with respect to the strategic environment represented by r.

Proposition 5 (Informational Weight and Maximal Volatility).
For all $\rho_{\theta\theta} \in (0,1)$:

1. *the informational weights λ that maximize the second moments satisfy:*

$$\text{argmax}_\lambda \ \left\{(1 - \rho_{aa})\sigma_a^2\right\} > \text{argmax}_\lambda \ \left\{\sigma_a^2\right\} > \text{argmax}_\lambda \ \left\{\rho_{aa}\sigma_a^2\right\};$$

2. *the informational weights λ that maximize the second moments: $\text{argmax}_\lambda \sigma_a^2$, $\text{argmax}_\lambda \rho_{aa}\sigma_a^2$, $\text{argmax}_\lambda(1 - \rho_{aa})\sigma_a^2$ are strictly decreasing in r;*
3. *the maximal second moments: $\max_\lambda \sigma_a^2$, $\max_\lambda \rho_{aa}\sigma_a^2$, $\max_\lambda(1 - \rho_{aa})\sigma_a^2$ are strictly increasing in r.*

Thus, the maximal volatility, both individual and aggregate, is increasing in the level of complementarity r. Even the maximal dispersion is increasing in r. In the equilibrium with maximum dispersion, the agents confound the idiosyncratic and aggregate component of the payoff shock and overreact to the idiosyncratic part, this effect increases with r. This implies that the responsiveness to the aggregate shock $\bar{\theta}$ increases, and hence the overreaction to the idiosyncratic shock $\Delta\theta_i$ increases as well. Moreover, the optimal weight on the aggregate shock increases in r for all of the second moments.

Proposition 6 (Aggregate Volatility).
The maximal aggregate volatility is given by:

$$\max_\lambda \left\{\rho_{aa}\sigma_a^2\right\} = \frac{\sigma_{\Delta\theta_i}^4}{4\left(\sqrt{\sigma_{\bar{\theta}}^2 + (1 - r)\sigma_{\Delta\theta_i}^2} - \sigma_{\bar{\theta}}\right)^2} \tag{26}$$

and is strictly increasing and without bound in the idiosyncratic uncertainty $\sigma_{\Delta\theta_i}^2$.

Importantly, the maximum aggregate volatility is not bounded anymore by the aggregate volatility of the complete information equilibrium. In fact, the aggregate volatility is increasing without bounds in the variance of the idiosyncratic shock even in the absence of variance of the aggregate shock $\bar{\theta}$. These results are in stark contrast to the complete information equilibrium in which the aggregate volatility is unaffected by the variance of the idiosyncratic shock. It illustrates in a simple way that the aggregate volatility may stem from uncertainty about *either* the aggregate *or* the idiosyncratic fundamental. As in the case of individual decision-making, we can consider the limits of the aggregate volatility as we approach a model of aggregate or idiosyncratic shocks only.

Corollary 3 (Maximal Volatility with Aggregate or Idiosyncratic Shocks Only).
The maximal volatility with only aggregate shocks is $\lim_{\sigma_{\Delta\theta_i}^2 \to 0} \max_\lambda \left\{\text{var}(A)\right\} = \sigma_{\bar{\theta}}^2 / (1 - r)^2$ and with only idiosyncratic shocks is $\lim_{\sigma_{\bar{\theta}}^2 \to 0} \max_\lambda \left\{\text{var}(A)\right\} = \sigma_{\Delta\theta_i}^2 / (4(1 - r))$.

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Confounding Information. The idea that confounding shocks can lead to overreaction goes back at least to Lucas [14]. In a seminal contribution, he shows how monetary shocks can have a real effect in the economy, even when under complete information monetary shocks would have no real effect. As agents observe just a one-dimensional signal (a price) that confounds two shocks, namely the labor market shock and the monetary supply shock, they necessarily respond to the two shocks in the same way. By contrast, under complete information they would condition their hiring decisions only on the labor market conditions. Yet the one-dimensional signal does not allow them to disentangle both shocks and in equilibrium they respond to both shocks. Thus, this can be seen as an overreaction to monetary shocks due to "informational frictions". The idea has been present also in more recent papers. For example, Venkateswaran [21] uses a similar idea to show how firms can have an excess reaction to aggregate shocks when these are confounded with idiosyncratic shocks or Mackowiak and Wiederholt [15] who derive informational frictions in a model of rational inattention.

In a recent contribution, Angeletos and La'O [4] consider an economy with purely idiosyncratic payoff shocks that still displays aggregate fluctuations. Each agent is assumed to *know* the realization of his individual payoff shock but only interacts with a specific trading partner rather than the aggregate market. Now, even though the payoff uncertainty is purely idiosyncratic, and each agent knows his own payoff shock, the pairwise interaction leaves each agent uncertain about the action of his trading partner. It is this uncertainty that can be affected by a common noise term, and hence generate aggregate volatility across the agents. They interpret this common noise term as *sentiments* which generate aggregate fluctuations.

The current analysis allow us to extract some very simple intuitions on when informational frictions can have a big effect on aggregate outcomes. We saw that as we approach a world of idiosyncratic uncertainty ($\rho_{\theta\theta} \to 0$), the maximum aggregate volatility is bounded away from 0 and it is achieved by a noise free equilibria. The information structure amplifies the aggregate shock that has small variance and leads to a big response by the agents provided that the informational weight on the aggregate shock is sufficiently large. Thus, we find that in a model with idiosyncratic uncertainty such as Angeletos and La'O [4], any arbitrarily small aggregate shock can have a huge effect, that it can be amplified if it receives a sufficiently large weight in the signal of the agents.

6. Equilibrium Behavior for All Information Structures

Until now we have analyzed the outcomes under the Bayes Nash equilibrium for a very special class of information structures. By contrast, we now describe the equilibrium behavior for all (symmetric and normally distributed) information structures. Thus we now allow for *noisy* rather than noise free information structures, and we allow for *multidimensional* rather than one-dimensional signals. Moreover, the signals, both across dimensions and across agents can have arbitrary correlation structures.

6.1. Definition of Bayes Correlated Equilibrium

In order to describe the equilibrium behavior across all possible information structures, we introduce a solution concept, Bayes correlated equilibrium, that describes the behavior (and outcomes) independent of the specific information structure that the agents may have access to. The set of Bayes correlated equilibria has the advantage that it can be completely described by a small set of inequalities on the first and second moments of the equilibrium distribution.

Definition 2 (Bayes Correlated Equilibrium).
The variables $(\theta_i, \bar{\theta}, a_i, A)$ form a symmetric and normally distributed Bayes correlated equilibrium if their joint distribution is given by a multivariate normal distribution and for all i and a_i :

$$a_i = r\mathbb{E}[A|a_i] + \mathbb{E}[\theta_i|a_i]. \tag{27}$$

The Bayes correlated equilibrium requires that the joint distribution of shocks and actions is such that for every action a_i in the support of the joint distribution the best response satisfies condition (27). We emphasize that the equilibrium notion does not refer to any information structure or signals. It is intentionally defined without reference to any specific information structure in contrast to the notion of Bayes Nash equilibrium that is always defined with reference to a specific information structure. The only informational restrictions that are imposed by the above equilibrium notion are that: (i) the marginal distribution over the payoff shocks $(\theta_i, \bar{\theta})$ coincides with common prior, and (ii) each agent conditions on the information contained in joint distribution, and hence when choosing action a_i forms the conditional expectation. The notion of Bayes correlated equilibrium was defined earlier in Bergemann and Morris [8], [9]. In the present context, we use it to obtain bounds on the volatility of equilibrium without reference to a specific information structure to begin with.

6.2. Characterization of Bayes Correlated Equilibria

We begin the analysis of the Bayes correlated equilibrium by reducing the dimensionality of the variance-covariance matrix. We appeal to the symmetry condition to express the aggregate variance in terms of the individual variance and the correlation between individual terms. Just as we described the variance $\sigma_{\bar{\theta}}^2$ of the aggregate shock $\bar{\theta}$ in terms of the covariance between any two individual payoff shocks, or $\sigma_{\bar{\theta}}^2 = \rho_{\theta\theta}\sigma_\theta^2$, we can describe the variance of aggregate action σ_A^2 in terms of the covariance of any two individual actions, or $\sigma_A^2 = \rho_{aa}\sigma_a^2$. In the discussion following Proposition 2, we introduced the correlation coefficient between action a_i and payoff shock θ_i of player i by $\rho_{a\theta}$:

$$\mathrm{cov}\,(a_i, \theta_i) \triangleq \rho_{a\theta}\sigma_a\sigma_\theta,$$

and the correlation coefficient between the action a_i of agent i and the payoff shock θ_j of a different agent j by $\rho_{a\phi}$:

$$\mathrm{cov}\,(a_i, \theta_j) \triangleq \rho_{a\phi}\sigma_a\sigma_\theta.$$

These three correlation coefficients, $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$, parameterize the entire variance-covariance matrix, denoted by $\mathbb{V}$, of the joint distribution of $(\theta_i, \bar{\theta}, a_i, A)$. Now, the covariance between a purely idiosyncratic random variable and an aggregate random variable is always 0. This implies that both the covariance between the aggregate action A and the payoff shock θ_j of player j and the covariance between the agent i's action, a_i, and the aggregate shock to the payoff shock, $\bar{\theta}$, are the same as the covariance between the action of player i and the payoff shock θ_j of player j, or $\rho_{a\phi}\sigma_a\sigma_\theta$. Thus we can reduce the number of variance terms, and in particular the number of correlation coefficients needed to describe the variance-covariance matrix $\mathbb{V}$ without any loss of generality.

Lemma 1 (Symmetric Bayes Correlated Equilibrium).
The variables $(\theta_i, \bar{\theta}, a_i, A)$ form a symmetric and normally distributed Bayes correlated equilibrium if and only if there exist parameters of the first and second moments, $(\mu_a, \sigma_a, \rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$, such that the joint distribution is given by:

$$\begin{pmatrix}\theta_i \\ \bar{\theta} \\ a_i \\ A\end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix}\mu_\theta \\ \mu_\theta \\ \mu_a \\ \mu_a\end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 & \rho_{\theta\theta}\sigma_\theta^2 & \rho_{a\theta}\sigma_a\sigma_\theta & \rho_{a\phi}\sigma_a\sigma_\theta \\ \rho_{\theta\theta}\sigma_\theta^2 & \rho_{\theta\theta}\sigma_\theta^2 & \rho_{a\phi}\sigma_a\sigma_\theta & \rho_{a\phi}\sigma_a\sigma_\theta \\ \rho_{a\theta}\sigma_a\sigma_\theta & \rho_{a\phi}\sigma_a\sigma_\theta & \sigma_a^2 & \rho_{aa}\sigma_a^2 \\ \rho_{a\phi}\sigma_a\sigma_\theta & \rho_{a\phi}\sigma_a\sigma_\theta & \rho_{aa}\sigma_a^2 & \rho_{aa}\sigma_a^2\end{pmatrix}\right), \tag{28}$$

and for all i and a_i:

$$a_i = r\mathbb{E}[A|a_i] + \mathbb{E}[\theta_i|a_i]. \tag{29}$$

We can now find restrictions on the first and second moments of a_i using the best response condition (29). By taking expectations of (29) and using the law of iterated expectations, we get:

$$\mu_a = r\mu_a + \mu_\theta \Leftrightarrow \mu_a = \frac{\mu_\theta}{1-r}. \tag{30}$$

If we multiply (29) by a_i and take expectations we get:

$$\mathbb{E}[a_i^2] = r\mathbb{E}[a_i A] + \mathbb{E}[a_i \theta_i].$$

Using (30) we obtain a similar condition for the variance of a_i:

$$\sigma_a^2 = r\,cov(a_i, A) + cov(a_i, \theta_i) \Leftrightarrow \sigma_a = \frac{\rho_{a\theta}\sigma_\theta}{1 - r\rho_{aa}}. \tag{31}$$

We thus have a complete determination of the individual mean and variance. Note that these conditions do not depend at all on the joint normality of $(a_i, A, \theta_i, \bar{\theta})$ as they only used the law of iterated expectations. Moreover, it is easy to see how one could get similar additional restrictions for the higher moments of a_i which would be relevant for the non-normal case. Now, for $\mathbb{V}$ to be a valid variance-covariance matrix, it has to be positive semidefinite, and this imposes restrictions on the remaining covariance terms.

Proposition 7 (Characterization).
A multivariate normal distribution of $\left(\theta_i, \bar{\theta}, a_i, A\right)$ is a symmetric Bayes correlated equilibrium if and only if :

1. *the mean of the individual action is:*

$$\mu_a = \mu_\theta / \left(1 - r\right); \tag{32}$$

2. *the standard deviation of the individual action is:*

$$\sigma_a = \frac{\rho_{a\theta}\sigma_\theta}{1 - r\rho_{aa}} \geq 0; \tag{33}$$

3. *the correlation coefficients $\rho_{aa}, \rho_{a\theta}, \rho_{a\phi}$ satisfy $\rho_{aa}, \rho_{a\theta} \geq 0$ and the inequalities:*

$$(i) \quad (\rho_{a\phi})^2 \leq \rho_{\theta\theta}\rho_{aa}, \quad (ii) \quad (1 - \rho_{aa})(1 - \rho_{\theta\theta}) \geq (\rho_{a\theta} - \rho_{a\phi})^2. \tag{34}$$

We thus identified necessary and sufficient conditions for the random variables $(\theta_i, \bar{\theta}, a_i, A)$ to form a Bayes correlated equilibrium. These conditions can be separated into two distinct sets of requirements: the first set of conditions follow directly from the best response condition (27) and merely rely on the linearity of the best response; the second set of conditions are purely statistical

conditions that require that the variance-covariance matrix $\mathbb{V}$ of the joint multivariate distribution constitutes a valid variance-covariance matrix, namely that it is positive semidefinite.

Importantly, both sets of conditions are necessary *independent* of the assumption of normally distributed payoff uncertainty. The normality assumption simply ensures that the equilibrium distributions are completely determined by the first and second moments. Thus, the normality assumptions allows us to describe the set of Bayes correlated equilibria in terms of restrictions that are necessary *and* sufficient. Nevertheless, for any arbitrary distribution of payoff shocks and equilibrium actions, the aggregate volatility provided in Proposition 6 will remain a valid bound on aggregate volatility.

There are two aspects of Proposition 7 that we should highlight. First, the mean μ_a of the individual action (and a fortiori the mean of the aggregate action A) is completely pinned down by the aggregate payoff shock. This implies that any differences across Bayes correlated equilibria must manifest themselves in the second moments only. Second, the restrictions on the equilibrium correlation coefficients do not at all depend on the interaction parameter r. The restrictions on the set of equilibrium correlations are purely statistical and stem from the condition that the variance-covariance matrix $\mathbb{V}$ forms a positive semidefinite matrix. By contrast, the mean μ_a and the variance σ_a^2 of the individual actions do depend on the interaction parameter r, as they are determined by the best response condition (29).

We will show in Section 7 that the disentanglement of the set of feasible correlations and the interaction parameter is possible only if we allow for all possible information structures, i.e. when we do not impose any restrictions on the private information that agents may have.

In the special case of pure idiosyncratic or pure aggregate shocks the set of outcomes in terms of the correlation coefficients $\left(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi}\right)$ reduces to a two-dimensional set. The reduction in dimensionality arises as the correlation coefficient $\rho_{a\phi}$ of action a_i and the payoff shock θ_j is either zero (in the absence of aggregate shocks) or equal $\rho_{a\theta}$ (in the absence of idiosyncratic shocks), and thus redundant in either case. In the case of pure aggregate shocks, $\rho_{\theta\theta} = 1$, the conditions in (34) reduce to $\rho_{a\phi} = \rho_{a\theta}$, and $\rho_{a\theta}^2 \leq \rho_{aa}$ as established earlier in Bergemann and Morris [8].

6.3. Equivalence between Bayes Nash and Bayes Correlated Equilibrium

Next we describe the relationship between the joint distributions of $\left(\theta_i, \bar{\theta}, a_i, A\right)$ that can arise as Bayes correlated equilibria and the distributions that can arise as Bayes Nash equilibria for some information structure $\mathcal{I} = \{S_i\}_{i \in [0,1]}$. In contrast to the restriction to one-dimensional noise free information structure made in Section 3, we now allow for a much larger class of information

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structures, including noisy and multidimensional information structures. In fact, for the present purpose, it is sufficient to merely require that the associated symmetric equilibrium strategy $\{a_i\}_{i \in [0,1]}$:
$a_i : S_i \to \mathbb{R}$ forms a multivariate normal distribution.

Proposition 8 (Equivalence).
The variables $(\theta_i, \bar{\theta}, a_i, A)$ form a (normal) Bayes correlated equilibrium if and only if there exists some information structure $\mathcal{I}$ under which the variables $(\theta_i, \bar{\theta}, a_i, A)$ form a Bayes Nash equilibrium.

The important insight of the equivalence is that the set of outcomes that can be achieved as a Bayes Nash equilibrium for *some* information structure can equivalently be described as a Bayes correlated equilibrium. Thus, the solution concept of Bayes correlated equilibrium allows us to study the set of outcomes that can be achieved as a Bayes Nash equilibrium, importantly without the need to specify a specific information structure. In Bergemann and Morris [9], we establish the equivalence between Bayes correlated equilibrium and Bayes Nash equilibrium for canonical finite games and arbitrary information structures (see Theorem 1 there). The above proposition specializes the proof to an environment with linear best responses and symmetrically normally distributed payoff shocks and actions.

We will discuss specific information structures and their associated equilibrium behavior in Section 7. Here, we describe a one-dimensional class of signals that is already sufficiently rich to "decentralize" the entire set of Bayes correlated equilibria as Bayes Nash equilibria. For this, we enlarge the set of noise free structures studied in Section 3 by allowing the weights on $\Delta\theta_i$ and $\bar{\theta}$ to have different signs, and adding noise:

$$s_i = \lambda \Delta\theta_i + (1 - |\lambda|)\bar{\theta} + \varepsilon_i, \tag{35}$$

where $\lambda \in [-1, 1]$ and ε_i is normally distributed with mean zero and variance σ_ε^2.[3] Similarly to the definition of the payoff shocks, the individual error term ε_i can have a common and an idiosyncratic component, respectively:

$$\bar{\varepsilon} \triangleq \mathbb{E}_i[\varepsilon_i], \quad \Delta\varepsilon_i \triangleq \varepsilon_i - \bar{\varepsilon}.$$

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[3]The weight λ on the idiosyncratic shock is now allowed to take on negative values, and thus the weights on the idiosyncratic and the aggregate shock can have different signs. By expanding the class of information structures relative to those considered in Section 3, we expand the set of feasible information structures. This permits for equilibrium correlation structures in which either $\rho_{a\theta} < \rho_{a\phi}$ or $\rho_{a\phi} < 0$.

Thus, the joint distribution of the shocks and signals is given by:

$$
\begin{pmatrix} \Delta\theta_i \\ \bar{\theta} \\ \Delta\varepsilon_i \\ \bar{\varepsilon} \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} 0 \\ \mu_\theta \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} (1-\rho_{\theta\theta})\sigma_\theta^2 & 0 & 0 & 0 \\ 0 & \rho_{\theta\theta}\sigma_\theta^2 & 0 & 0 \\ 0 & 0 & (1-\rho_{\varepsilon\varepsilon})\sigma_\varepsilon^2 & 0 \\ 0 & 0 & 0 & \rho_{\varepsilon\varepsilon}\sigma_\varepsilon^2 \end{pmatrix} \right),
\tag{36}
$$

and the standard deviation $\sigma_\varepsilon > 0$ and the correlation coefficient $\rho_{\varepsilon\varepsilon} \in [0,1]$ are the parameters of the fully specified information structure $\mathcal{I} = \{\mathcal{I}_i\}_{i\in[0,1]}$, together with the confounding parameter λ. We observe that the dimensionality of information structure $\mathcal{I}$, given by (35) and (36), and thus parametrized by $(\lambda, \sigma_\varepsilon, \rho_{\varepsilon\varepsilon})$, matches the dimensionality of the Bayes correlated equilibrium expressed in terms of the correlation coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$.

Proposition 9 (Informational Decentralization).
The variables $(\theta_i, \bar{\theta}, a_i, A)$ form a (normal) Bayes correlated equilibrium if and only if there exist some information structure $(\lambda, \sigma_\varepsilon^2, \rho_{\varepsilon\varepsilon})$ under which the variables $(\theta_i, \bar{\theta}, a_i, A)$ form a Bayes Nash equilibrium.

The class of one-dimensional signals described by (35) is thus rich enough to "informationally decentralize" in the sense that every Bayes correlated equilibrium distribution can be reproduced as a Bayes Nash equilibrium distribution for some choice of the parameters $(\lambda, \sigma_\varepsilon, \rho_{\varepsilon\varepsilon})$ that describe the family of information structures given by (35). This illustrates how the equilibrium conditions of the Bayes correlated equilibrium, in particular the conditioning in the best response (27) encode in the joint distribution information that is made explicit by the information structure of the corresponding Bayes Nash equilibrium.

6.4. The Boundary of Bayes Correlated Equilibria

We characterized the entire set of Bayes correlated equilibria in Proposition 7. The mean and the variance of the individual action were determined by the equalities (32) and (33) and the correlation coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ were restricted by the two inequalities given by (34). We now provide a characterization of the boundary of the set of Bayes correlated equilibria in terms of the two inequalities which only involve the correlation coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$.

Definition 3 (Boundary of Bayes Correlated Equilibrium).
The correlation coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ are on the (upper) boundary of the Bayes correlated equilibrium set if they satisfy (34) and there is no other triple of coefficients $(\rho_{aa}, \rho'_{a\theta}, \rho'_{a\phi})$, also satisfying (34), such that:

$$
(\rho'_{a\theta}, \rho'_{a\phi}) \geq (\rho_{a\theta}, \rho_{a\phi}), \ \ and \ (\rho'_{a\theta}, \rho'_{a\phi}) \neq (\rho_{a\theta}, \rho_{a\phi}).
$$

We refer to any triple $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ that satisfies the conditions of Definition 3 as an element of the upper boundary. For any attainable correlation between actions, ρ_{aa}, there is no other Bayes correlated equilibrium $(\rho_{aa}, \rho'_{a\theta}, \rho'_{a\phi})$ that achieves a weakly higher correlation with the idiosyncratic shock $\Delta\theta_i$ and the aggregate shock $\overline{\theta}$, and strictly higher for at least one of the two shocks.

Now, necessary conditions for a triple $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ to be on the boundary of the Bayes correlated equilibrium set is that both inequalities (34) are satisfied as equalities, and the restriction to the positive root of the quadratic equations yields the sufficient condition.

Proposition 10 (Boundary of Bayes Correlated Equilibria).
The correlation coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ form the boundary of the Bayes correlated equilibrium if and only if

$$\{(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi}) \in [0,1]^3 : \rho_{a\theta} = \sqrt{\rho_{aa}\rho_{\theta\theta}} + \sqrt{(1-\rho_{\theta\theta})(1-\rho_{aa})}, \ \rho_{a\phi} = \sqrt{\rho_{aa}\rho_{\theta\theta}} \}. \qquad (37)$$

The boundary of the Bayes correlated equilibria is described by equation (37) which identifies $\rho_{a\theta}$ and $\rho_{a\phi}$ as a function of ρ_{aa} for a given composition of idiosyncratic and aggregate shocks, $\rho_{\theta\theta}$. Accordingly, the correlation coefficient $\rho_{a\theta}$ between action a_i and shock θ_i of agent i is the sum of two roots. The roots arise from the aggregate and the idiosyncratic shock in the agents' actions and the agents' shocks, $\rho_{aa}\rho_{\theta\theta}$ and $(1-\rho_{aa})(1-\rho_{\theta\theta})$, respectively. The set of attainable correlations is restricted by the composition of the payoff shocks as (37) depends explicitly on $\rho_{\theta\theta}$. Figure 2 visualizes the boundary of the equilibrium set in the two-dimensional space of correlation coefficients $(\rho_{aa}, \rho_{a\theta})$ for different compositions of the shocks, $\rho_{\theta\theta}$. The left panel represents the case of $\rho_{\theta\theta} = 1/2$, whereas the right panel represents $\rho_{\theta\theta} = 1/4$ and $\rho_{\theta\theta} = 3/4$. The right panel illustrates how the composition of the payoff shocks, $\rho_{\theta\theta}$, impacts the relationship between ρ_{aa} and $\rho_{a\theta}$ relative to the symmetric condition with $\rho_{\theta\theta} = 1/2$.
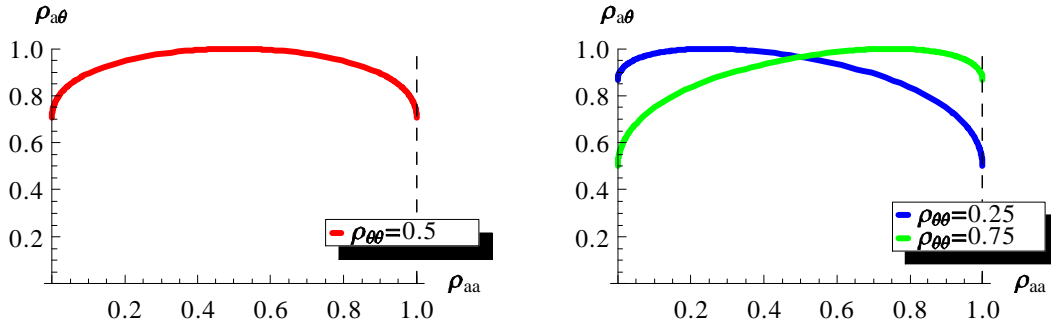

Figure 2: The Boundary of the Bayes Correlated Equilibrium Set

We can decompose the action of each agent in terms of his responsiveness to the idiosyncratic shock $\Delta\theta_i$ and the aggregate shock $\bar{\theta}$, and any residual responsiveness has to be attributed to noise, see (25). The action a_i itself also has an idiosyncratic and an aggregate component as $a_i = A + \Delta a_i$. The conditional variance of these components of a_i can be expressed in terms of the correlation coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi}, \rho_{\theta\theta})$, which are subject to the restrictions of Proposition 7. By using the familiar property of the multivariate normal distribution for the *conditional variance*, we obtain a diagonal matrix:

$$\text{var}\left[\begin{array}{c|c} \Delta a_i & \Delta\theta_i \\ A & \bar{\theta} \end{array}\right] = \sigma_a^2 \left(\begin{array}{cc} (1 - \rho_{aa}) - \frac{(\rho_{a\theta} - \rho_{a\phi})^2}{1 - \rho_{\theta\theta}} & 0 \\ 0 & \rho_{aa} - \frac{\rho_{a\phi}^2}{\rho_{\theta\theta}} \end{array}\right). \tag{38}$$

If the components A and Δa_i of the agent's individual action are completely explained by the components of the individual payoff shock, $\bar{\theta}$ and $\Delta\theta_i$, then the conditional variance of the action components, and a fortiori of the action itself, is equal to zero.[4] In fact, the above conditional variances are equal to zero if the inequalities of Proposition 7 are satisfied as equalities.[5]

This restatement of the conditions for the boundary of the Bayes correlated equilibrium in terms of zero conditional variance provides a hint as to which information structures might attain the above boundary of the Bayes correlated equilibrium set as a Bayes Nash equilibrium. After all, the condition of zero conditional variance says that the action of each agent is completely explained by variations in the idiosyncratic and aggregate shock. But precisely this property was guaranteed by the noise free information structures that we considered in Section 3. Indeed, we can now provide an equivalence result akin to Proposition 8, but this time specialized to the boundary of the Bayes correlated equilibrium set.

Proposition 11 (Equivalence).
The coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ form a boundary Bayes correlated equilibrium if and only if they form a noise free Bayes Nash equilibrium for some information structure λ.

The early restriction to noise free information structure in Section 3 may have appeared ad hoc. Proposition 11 establishes that the noise free information structures are economically significant in

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[4]The matrix of conditional variances, (38), is well-defined only in the joint presence of idiosyncratic and aggregate shocks, that is for $\rho_{\theta\theta} \in (0, 1)$. For the case of pure idiosyncratic or pure aggregate shocks, $\rho_{\theta\theta} = 0$ or $\rho_{\theta\theta} = 1$, one of the conditioning terms, $\bar{\theta}$ or $\Delta\theta_i$, has zero variance by definition.

[5]The negative root of the equality version of (34) also leads to the zero conditional variance. The resulting correlation coefficients could be generated by noise free information structures with $\lambda < 0$. But the correlation coefficients are never part of the upper boundary of the Bayes correlated equilibrium set as Footnote 3 indicates.

that they attain the maximally achievable equilibrium correlations among all possible information structures. Thus, the boundary of the Bayes correlated equilibria can be informationally decentralized by signals of the form (35) after imposing the requirement that $\sigma_\varepsilon = 0$ *and* $\lambda \in [0, 1]$, and hence precisely the one-dimensional noise free information structures investigated in Section 3.

A crucial implication of the characterization of the boundary in terms of the correlation coefficients is that if we seek to identify the equilibrium distribution that maximizes volatility or dispersion in the economy, then it is without loss of generality to focus on the boundary of the Bayes correlated equilibrium set. Moreover, by Proposition 11, this means that we can restrict attention to the class of one-dimensional noise free information structure.

Proposition 12 (Maximal Volatility and Dispersion).
Among all Bayes correlated equilibria, individual volatility, aggregate volatility, and dispersion are all maximized by a boundary Bayes correlated equilibrium.

More generally, the construction of the boundary suggests that any monotone function of the correlation coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ should achieve its maximum across all possible Bayes correlated equilibria somewhere on the boundary. This indicates that the noise free information structures remain the critical ones if we were to conduct a more comprehensive welfare analysis beyond the analysis of the second moments here. Notably, an auxiliary result for Proposition 12, Lemma 2 in the Appendix, establishes that any continuous function, ψ, with $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi}) \mapsto \psi(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$, that is strictly increasing in $\rho_{a\theta}$ and weakly increasing in $\rho_{a\phi}$, achieves its maximum in the set of all feasible Bayes correlated equilibria on the boundary of the set. As the conditions of Lemma 2 are silent about the correlation coefficient ρ_{aa}, we can accommodate strategic environments (and payoffs and associated objective functions) with either strategic substitutes or complements. Thus, we conclude that the special class of noise free information structures are indeed the relevant information structures even if we were to analyze a larger class of welfare functions.

In the discussion following Proposition 7, we argued that the moment restrictions remain necessary conditions even in the absence of any distributional assumptions of normality. Therefore, we can actually state a stronger version of Proposition 12. Suppose we maintain the assumption of normality in the payoff shocks, but neither do we require the normality in actions nor the joint normality in actions and shocks. Then, we would still have the result that the volatility is maximized by the noise free and normally distributed equilibria of Proposition 12, as the necessary boundary conditions of the Bayes correlated equilibria given by Proposition 7 are indeed attained by linear

combinations of the idiosyncratic and the aggregate shocks.[6]

7. Information Structures and Equilibrium Behavior

We began our analysis with a class of specific information structures (noise free) and then established that these noise free information structures indeed form the boundary of equilibrium behavior with respect to all (symmetric normal) information structures. It is usual to assume some particular, but not noise free, information structure. We now ask how restrictive these commonly used classes of information structures are with respect to the entire set of feasible equilibrium behavior. Our purpose here is only to illustrate how assumptions about the information structure can be restrictive, rather than providing a comprehensive account of all information structure that appeared in the literature. Our representative examples illustrate subtle implications of restrictions on the information structure. The specific information structures that we study are a subset of the following three-dimensional information structures:

$$S_i \triangleq \{s_i^1 = \theta_i + \varepsilon_i^1 \ , \ s_i^2 = \bar{\theta} + \varepsilon_i^2 \ , \ s_i^3 = \bar{\theta} + \bar{\varepsilon}^3\}, \tag{39}$$

where $\varepsilon_i^1, \varepsilon_i^2$ are idiosyncratic noise terms and $\bar{\varepsilon}^3$ is a common noise term, all normally distributed, independent and with zero mean. This class of information structures appears in the analysis of Angeletos and Pavan [6] and is parameterized by three variables, namely the variances $(\sigma_{\varepsilon^1}^2, \sigma_{\varepsilon^2}^2, \sigma_{\varepsilon^3}^2)$ of the noise terms. We begin by characterizing the set of feasible correlations when agents only observe a noisy idiosyncratic signal of their payoff shock θ_i:[7]

$$s_i^1 = \theta_i + \varepsilon_i^1,$$

and thus we set $\sigma_{\varepsilon^2}^2 = \sigma_{\varepsilon^3}^2 = \infty$. This class of signals is frequently used in the literature on information sharing, see Vives [23] and Raith [18].

Proposition 13 (Noisy Signal of Payoff State).
A set of correlations $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ can be achieved as a Bayes Nash equilibrium with an information

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[6]We conjecture that Proposition 12 holds even more generally in environments without normally distributed payoff shocks. But in the absence of normally distributed payoff shocks, the associated noise free information structure is likely to be a nonlinear, rather than linear, function of the shocks.

[7]We emphasize the fact that conditions (32) and (33) must hold for any signal structure. Thus, by characterizing the set of feasible correlations, we are characterizing the set of feasible outcomes.
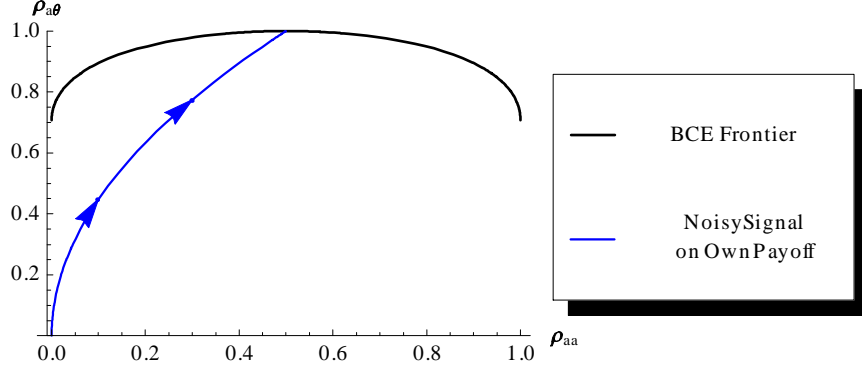
Figure 3: Feasible BNE correlation coefficients with noisy signals about payoff state θ_i

structure $\{s_i^1\}_{i \in [0,1]}$, *if and only if:*

$$\rho_{a\theta} = \sqrt{\frac{\rho_{aa}}{\rho_{\theta\theta}}}; \quad \rho_{a\phi} = \rho_{a\theta}\rho_{\theta\theta}; \quad \rho_{aa} \in [0, \rho_{\theta\theta}]. \tag{40}$$

We observe that the set of feasible correlations when the agents receive only a one-dimensional signal of the form s_i^1 does not depend on the interaction parameter r. In Figure 3 we illustrate the locus of attainable correlations with information structures $\{s_i^1\}_{i \in [0,1]}$ for $\rho_{\theta\theta} = 1/2$. The arrows point in the direction of greater precision (i.e., lower variance) of the error term. Notably, all the attainable equilibrium coefficients are below the frontier given by the Bayes correlated equilibria, except for a single point that is identified by zero noise, or $\sigma_{\varepsilon^1}^2 = 0$.

Next, we consider information structures in which each agent knows his own payoff shock, and thus we set $\sigma_{\varepsilon^1}^2 = 0$, but allow $\sigma_{\varepsilon^2}^2, \sigma_{\varepsilon^3}^2 \in [0, \infty)$. That is, all possible outcomes that are consistent with each agent knowing *at least* θ_i. Since each agent knows his own payoff shock, the residual uncertainty is with respect to the actions taken by the other players. The informational assumption that each agent knows his own payoff shock θ_i commonly appears in the macroeconomics literature. For example, Angeletos and La'O [2], [3] and Angeletos, Iovino, and La'O [1] consider models with idiosyncratic and aggregate shocks and imperfect information, but assume that each agent knows his own payoff shock θ_i. In a model with idiosyncratic rather than aggregate interaction, Angeletos and La'O [4] analyze the impact of informational friction on aggregate fluctuations. Again, they assume that each agent knows his own payoff shock θ_i, but is uncertain about the payoff shock θ_j of the trading partner j. Similarly, Lorenzoni [13] investigates the optimal monetary policy with dispersed information. He also considers a form of individual matching rather than aggregate interaction. The informational assumption common to all of these models is that every agent i

knows his own payoff shock θ_i, and thus all uncertainty is purely strategic.

The characterization of the set of attainable equilibrium correlations is achieved in two steps. First, we describe the set of feasible action correlations ρ_{aa}. If each agent only knows his own payoff shock θ_i, then the correlation ρ_{aa} is equal to $\rho_{\theta\theta}$ as the actions of any two agents can only be correlated to the extent that their payoff shocks are correlated. By contrast, if the agents have complete information, then the correlation is given by $\rho_{aa} = \widehat{\rho}_{aa}$,

$$\widehat{\rho}_{aa} \triangleq \frac{\rho_{\theta\theta}}{(1 - \rho_{\theta\theta})(1 - r)^2 + \rho_{\theta\theta}}, \tag{41}$$

where $\widehat{\rho}_{aa}$ is the correlation that is achieved by the information structure $\widehat{\lambda}$ that recovers the equilibrium moments of the complete information Nash equilibrium, see (24).

We find that the set of feasible action correlations is always between these two quantities, $\rho_{\theta\theta}$ and $\widehat{\rho}_{aa}$, providing the lower and upper bound. If $r > 0$, then the complete information bound is the upper bound, if $r < 0$, it is the lower bound. For $r = 0$ they coincide as θ_i is a sufficient statistic of the action taken by each agent under complete information.

Second, we describe the set of feasible correlations between action and individual payoff shock, $\rho_{a\theta}$, for any feasible ρ_{aa}. The set of feasible $\rho_{a\theta}$ is determined by two functions of ρ_{aa}, which provide the lower and upper bound for the feasible $\rho_{a\theta}$. We denote these functions by $\rho_{a\theta}^i(\rho_{aa})$ and $\rho_{a\theta}^c(\rho_{aa})$ as these bounds are achieved by information structures in which each agent knows his own payoff shock and receives a second signal, either an idiosyncratic signal of the aggregate shock $\bar{\theta} : s_i^2 \triangleq \bar{\theta} + \varepsilon_i^2$ or a common signal of $\bar{\theta} : s_i^3 \triangleq \bar{\theta} + \bar{\varepsilon}^3$.

Proposition 14 (Known Payoff State θ_i).
A set of correlations $(\rho_{aa}, \rho_{a\theta})$ can be induced by a linear Bayes Nash equilibrium in which each agent knows his payoff shock θ_i if and only if

$$\rho_{aa} \in [\min\{\widehat{\rho}_{aa}, \rho_{\theta\theta}\}, \max\{\widehat{\rho}_{aa}, \rho_{\theta\theta}\}]; \tag{42}$$

and for any ρ_{aa} satisfying (42):

$$\rho_{a\theta} \in [\min\{\rho_{a\theta}^c(\rho_{aa}), \ \rho_{a\theta}^i(\rho_{aa})\}, \max\{\rho_{a\theta}^c(\rho_{aa}), \ \rho_{a\theta}^i(\rho_{aa})\}].$$

In Figure 4, we illustrate the Bayes Nash equilibrium set for different values of r for a given correlation $\rho_{\theta\theta} = 1/2$. Each interaction value r is represented by a differently colored pair of lower and upper bounds. For each value of r, the entire set of Bayes Nash equilibria is given by the area enclosed by the lower and upper bounds. Notably, the bounds $\rho_{a\theta}^c(\rho_{aa})$ and $\rho_{a\theta}^i(\rho_{aa})$ intersect in two points, corresponding to each agent knowing his payoff shock θ_i only (at $\rho_{aa} = \rho_{\theta\theta} = 1/2$) and

to complete information, at the low or high end of ρ_{aa} depending on the nature of the interaction, respectively. In fact these, and only these, two points, are also boundary points of the unrestricted set of Bayes correlated equilibria. When $r \geq 0$, the upper bound is given by a signal with an idiosyncratic error term, s_i^2, while the lower bound is given by a signal with a common error term, s_i^3, and conversely for $r \leq 0$.

Thus, if each agent knows at least his own payoff shock, then we observe a dramatic reduction in the set of feasible Bayes Nash equilibria. Notably, every element, with the exception of the information structures mentioned in the above paragraph, are in the interior of the unrestricted set of Bayes correlated equilibria. Moreover, the nature of the interaction has a profound impact on the set of correlations $(\rho_{aa}, \rho_{a\theta})$ that can arise in equilibrium, both in terms of its size as well as its location in the unit square.
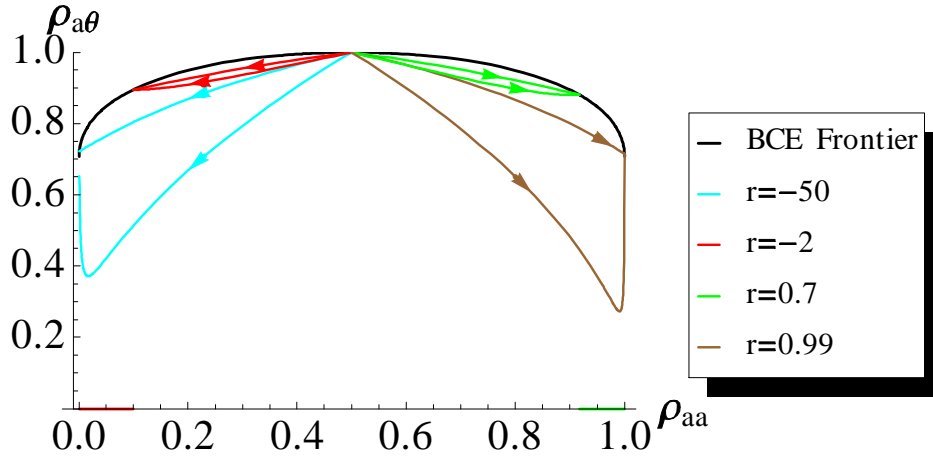

Figure 4: Boundary of the set of feasible correlations when agents know own payoff ($\rho_{\theta\theta} = 1/2$)

If each agent i is assumed to know his payoff shock θ_i, then we can restate the best response condition (4) with respect to a_i after the following change of variables in terms of deviations from the payoff shock:

$$\tilde{a}_i \triangleq a_i - \theta_i, \quad \widetilde{A} \triangleq A - \overline{\theta}. \tag{43}$$

The best response condition (4) in terms of a_i reduces to the following best response condition in terms of $\tilde{a}_i$:

$$\tilde{a}_i = r\mathbb{E}\left[\overline{\theta} \,|\theta_i, \mathcal{I}_i\right] + r\mathbb{E}\left[\widetilde{A} \,|\theta_i, \mathcal{I}_i\right], \tag{44}$$

where $\mathcal{I}_i$ is any information agent i gets beyond knowing θ_i. The resulting best response condition is now isomorphic to one where there are only aggregate shocks, *and* where the payoff shock and

average action receive the same weight in the best response condition of the individual agent. This provides a distinct intuition on the strong restrictions on behavior that arise from imposing that agents know their own payoff shock as stated in Proposition 14.

In the supplemental online material, we analyze a third subset of information structure, in which, besides the noisy signal $\{s_i^1\}_{i \in [0,1]}$, each player also knows the aggregate shock $\bar{\theta}$, thus $\sigma_{\varepsilon^2}^2 = \sigma_{\varepsilon^3}^2 = 0$. Although a priori this may not seem like an information structure that would arise exogenously, it is the information structure that arises when agents receive endogenous information on the average action taken by other players, such as in a rational expectations equilibrium with a continuum of sellers as studied by Vives [25] or Bergemann, Heumann, and Morris [7]. In the supplemental online material, we characterize the entire set of equilibrium correlations that can be achieved with the three-dimensional structures defined by (39). Surprisingly then the above class of three-dimensional information structures S_i fails to decentralize the entire set of Bayes correlated equilibria. By contrast, in the case of pure aggregate shocks Bergemann and Morris [8] show that any Bayes correlated equilibrium can be decentralized by considering a private and a public signal of the payoff shock, namely $s_i^1 = \theta_i + \varepsilon_i^1$ and $s_i^3 = \bar{\theta} + \bar{\varepsilon}^3$.

8. Discussion

We conclude by discussing the relevance of the current analysis to environments with heterogeneous rather than aggregate interaction. We end by relating our analysis to the large literature on information sharing among firms and suggest how the current tools might yield new results there as well.

Beyond Aggregate Interaction. We deliberately restricted our analysis to an environment with aggregate interaction. Every agent formed a best response against the average of the population. Yet, within the linear quadratic framework, it appears feasible to extend the analysis to much richer interaction structures, such as pairwise interaction or even general network interaction structures. In the macroeconomics literature, models of heterogenous interactions have appeared prominently, for example, in Lorenzoni [13] and Angeletos and La'O [4]. Notably, these models of pairwise interaction assume that each agent knows his own payoff shock θ_i but is still uncertain about the payoff shocks of other agents. As each agent i knows his own payoff shock θ_i, there is no *payoff uncertainty* anymore and so the residual uncertainty is all about the *strategic uncertainty*, namely the action of

the other agent. [8]

Interestingly, even if we were interested in strategic uncertainty in the absence of payoff uncertainty, the noise free information structures remain of central importance for the aggregate behavior. To see this, consider a simple model of pairwise interaction as in Angeletos and La'O [4]. We assume there is pairwise matching between i and j and that agents interact with their partner, r_a, as well as with the aggregate population, r_A. Thus, the first order condition of agent i's problem when he is matched with j is given by:

$$a_i = \mathbb{E}\left[\theta_i \,|\theta_i, \mathcal{I}_i\right] + r_a \mathbb{E}\left[a_j \,|\theta_i, \mathcal{I}_i\right] + r_A \mathbb{E}\left[A \,|\theta_i, \mathcal{I}_i\right].$$

If we make the same change of variables as earlier in (43), so that we express the choice variables in terms of their deviation from the payoff shock: $\tilde{a}_i \triangleq a_i - \theta_i, \quad \widetilde{A} \triangleq A - \bar{\theta}$, then the associated first order conditions are given by:

$$\tilde{a}_i = r_a \mathbb{E}\left[\Delta\theta_j \,|\theta_i, \mathcal{I}_i\right] + r_A \mathbb{E}\left[\bar{\theta} \,|\theta_i, \mathcal{I}_i\right] + r_a \mathbb{E}\left[\tilde{a}_j \,|\theta_i, \mathcal{I}_i\right] + r_A \mathbb{E}\left[\widetilde{A} \,|\theta_i, \mathcal{I}_i\right]. \tag{45}$$

Thus, we have a similar model as the one we have been studying so far, but with some differences. First, agents have some prior information on $\bar{\theta}$ which comes from knowing θ_i. Second, the size of the shocks $\Delta\theta_j$ and $\bar{\theta}$ are scaled by r_a and r_A, respectively, in the first order conditions. Besides these differences, a model with heterogeneous interaction in which each agent knows his own payoff shock is almost identical to our original model. Namely, each agent's uncertainty is still two-dimensional, with an aggregate and an idiosyncratic shock (equal to $\bar{\theta}$ and $\Delta\theta_j$ respectively). Thus, we see that even if we were interested in strategic uncertainty in the absence of fundamental uncertainty, the same basic intuitions and ideas would still apply. A key factor to consider would be to see how the signal leads each agent to be confuses about $\Delta\theta_j$ and $\bar{\theta}$. And as before, the confounding of the information would lead to overreaction and underreaction to some of these fundamentals, respectively. Thus, the pairwise interaction, or any other richer interaction structure, enriches the set of feasible outcomes and partially reverses the restrictions that come with the informational assumptions of knowing the individual payoff states that we earlier established in Proposition 14.

Information Sharing. We described the impact that the private information structure has on the second moments of the economy, in particular the volatility of the aggregate outcome. Naturally,

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[8]We thank our discussant, Marios Angeletos, for emphasizing the importance of the distinct contribution of each source of uncertainty to the aggregate volatility.
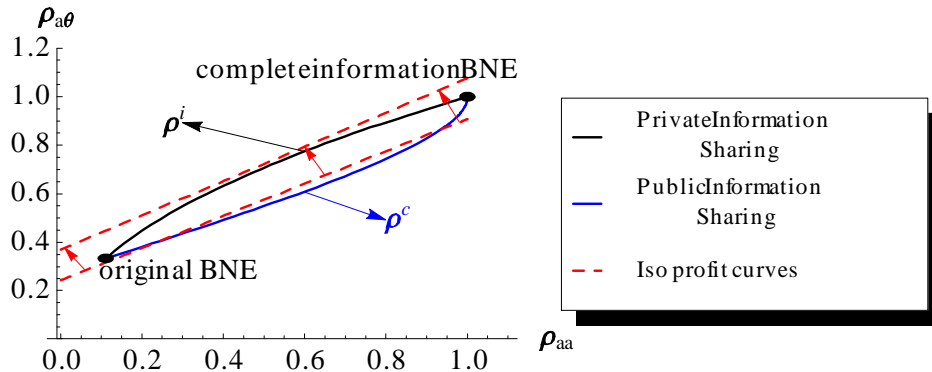
Figure 5: Information Sharing under Public and Private Disclosure Rules

we could expand the analysis to functions of the (second) moments of the economy. In the large literature on information sharing among firms, pioneered in work by Novshek and Sonnenschein [17], Clarke [10] and Vives [22], the expected profit function of the individual firm is a function of the volatility both of the individual and the aggregate outcome. In this class of models which is presented in a very general framework by Raith [18] and surveyed by Vives [24], each firm receives a private signal about a source of uncertainty, say a demand or cost shock. The central question is under which conditions the firms have an incentive to commit ex-ante to an agreement to share information in some form. The present analysis of the impact of information structures on the set of feasible correlations suggests novel insights into the nature of optimal information sharing policies.

We briefly illustrate this within a competitive equilibrium with a continuum of producers, each one of them with a quadratic cost of production $c(a_i) = a_i^2/2$, and facing a linear inverse demand function dependent on the aggregate demand shock $\overline{\theta}$ and the aggregate supply A: $p\left(\overline{\theta}, A\right) = \overline{\theta} + rA$, so that the resulting best response function is again given by (4).[9] We can depict the iso-profit curve $\overline{\pi}$ defined implicitly by a constant expected profit $\overline{\pi}$ of the representative firm, $\overline{\pi} \triangleq \mathbb{E}\left[a_i p - \frac{1}{2}a_i^2\right]$, in the space of the correlation coefficients $(\rho_{aa}, \rho_{a\theta})$. The iso-profit curve $\overline{\pi}$ can be shown to be linear in ρ_{aa}, as indicated by the red dashed line in Figure 5, and the slope is determined by the responsiveness r of the price to supply.

The maximal correlation $\rho_{a\theta}$ that is achievable with disclosure of a common signal, denoted

---

[9]The restriction to the pure aggregate shock environment, $\theta_i = \overline{\theta}$, allows us to directly use arguments in Bergemann and Morris [8], in particular Proposition 8, but the insights naturally extend to the environment with idiosyncratic *and* aggregate shocks.

earlier in Proposition 14 by $\rho^c_{a\theta}(\rho_{aa})$, is convex in ρ_{aa}, whereas the maximal correlation achievable with disclosure of an idiosyncratic signal is given by $\rho^i_{a\theta}(\rho_{aa}) \triangleq \sqrt{\rho_{aa}}$, and is concave in ρ_{aa}. In fact with aggregate shocks only, the idiosyncratic signals $s^1_i = \bar{\theta} + \varepsilon_i$ trace out the entire boundary of the Bayes correlated equilibrium coefficients, as illustrated in Figure 5. We therefore can conclude that the optimal disclosure policy with a *public signal* is either zero or complete disclosure, which was a central finding in Kirby [12], Vives [23] and Raith [18].[10] By contrast, the optimal disclosure policy of a *private signal* depends on r and can be noisy. The iso-profit curve generates a linear trade-off in the correlations of the individual supply decision a_i and the demand shock $\bar{\theta}$. A better match with the level of aggregate demand increases profit, but a better match with the supply of the other firms decreases the profit. With *public* disclosure of a noisy signal s_i, the convexity in the trade-off leads to *either* zero disclosure *or* complete disclosure of the aggregate information. With *private* disclosure, the trade-off is resolved in favor of a better match with the demand shock without an undue increase in the correlation of the supply decisions. Thus, we find that the industry-wide preferred disclosure policy frequently involves partial disclosure of information, by which disclosure is noisy and idiosyncratic, as opposed to the bang-bang solution that was previously obtained in the literature under the (implicit) restriction to public disclosure policies. Thus we find that a common and hence perfectly correlated disclosure policy is (always) weakly and (sometimes) strictly dominated by a private and hence imperfectly correlated disclosure policy.

We can rephrase this insight in terms of the macroeconomic language. An idiosyncratic "sentiment" shock may be needed to generate the largest individual volatility in the aggregate shock environment, just as a common "sentiment" shocks is necessary to generate the largest volatility in the idiosyncratic shock environment of Angeletos and La'O [4]. The analysis in Section 7 suggests that the above results for the pure aggregate shock environment (and similarly for the pure idiosyncratic shock) extend to the general environment with idiosyncratic as well as aggregate shocks. We leave a more comprehensive analysis for future research.

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[10]In Section 8.4 of Vives [24], the design of the optimal information sharing policy in a large market with a continuum of agents is posed as the problem of a mediator who elicits and then transmits the collected information to the agents. The analysis is thus close to the present perspective of the Bayes correlated equilibrium, but also restricts the transmission policy to public signals, and hence leads to the same conclusion as the above mentioned literature.

9. Appendix A

The Appendix A collects the omitted proofs from the main body of the text. The supplementary online material contains the Appendices B and C, which have additional results regarding the restrictions that information structures impose on the equilibrium behavior, complementing the results of Section 7 of the main paper.

Proof of Proposition 1. Since the actions of players must be measurable with respect to s_i, in any linear strategy the actions of players must be given by $a_i = w(\lambda)s_i + \nu$, where ν and $w(\lambda)$ are constants. Thus $A = w(\lambda)((1-\lambda)\bar{\theta}) + \nu$. Thus, we must have that:

$$a_i = w(\lambda)s_i + \nu = \mathbb{E}[r(w(\lambda)((1-\lambda)\bar{\theta}) + \nu) + \theta_i|s_i].$$

By taking expectations and using the law of iterated expectations, we get:

$$w(\lambda)(1-\lambda)\mu_\theta + \nu = rw(\lambda)((1-\lambda)\mu_\theta + \nu) + \mu_\theta.$$

Using that $\mu_\theta = 0$, we get that $\nu = 0$. Thus, we know that $a_i = w(\lambda)((1-\lambda)\bar{\theta} + \lambda\Delta\theta_i)$ and $A = w(\lambda)(1-\lambda)\bar{\theta}$. Multiplying by a_i we get: $a_i^2 = \mathbb{E}[rAa_i + \theta_i a_i|s_i]$, and appealing to the law of iterated expectations we get:

$$w(\lambda)((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta})) = ((1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta})),$$

and solving for $w(\lambda)$ yields the expression in (11). The uniqueness of the Bayes Nash equilibrium is established in Ui and Yoshizawa [19]. ∎

Proof of Proposition 2. By using the law of iterated expectations we obtain $\mu_a = \mu_A = \mu_\theta/(1-r)$. We can compute the variance and covariances by using (10) and (11). It is easy to see that:

$$\text{var}(a_i) = \sigma_a^2 = w(\lambda)^2 \text{var}(s_i) = w(\lambda)^2((1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))\sigma_\theta^2,$$

thus we get (12). Similarly, we obtain:

$$\text{var}(A) = \text{cov}(a_i, a_j) = \rho_{aa}\sigma_a^2 = w(\lambda)^2\mathbb{E}[((1-\lambda)\bar{\theta} + \lambda\Delta\theta_i)((1-\lambda)\bar{\theta} + \lambda\Delta\theta_j)] = w(\lambda)^2(1-\lambda)^2\rho_{\theta\theta}\sigma_\theta^2,$$

and

$$\text{cov}(a_i, \theta_i) = \rho_{a\theta}\sigma_a\sigma_\theta = w(\lambda)\mathbb{E}[((1-\lambda)\bar{\theta} + \lambda\Delta\theta_i)\theta_i] = w(\lambda)((1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta}))\sigma_\theta^2,$$

which establishes the result. ∎

Proof of Proposition 3.   The result follows directly from solving the following maximization problem:

$$\max_{\lambda} \left(\frac{(1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta})}{(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta})} \right)^2 (1-\lambda)^2\rho_{\theta\theta}.$$

The solution (17) follows from Proposition 6. ∎

Proof of Proposition 4. Given a noise free equilibrium parametrized by λ we have that:

$$\text{cov}(a_i, \bar{\theta}) = w(\lambda)(1-\lambda)\bar{\theta} = \frac{((1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}(1-\lambda)\bar{\theta},$$

$$\text{cov}(a_i, \Delta\theta_i) = w(\lambda)\lambda\Delta\theta_i = \frac{((1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}\lambda\Delta\theta_i.$$

But, note that if $\lambda < \widehat{\lambda}$, then $\frac{\lambda}{(1-r)} < (1-\lambda)$, but then

$$
\begin{aligned}
\text{cov}(a_i, \Delta\theta_i) &= \frac{((1-\lambda)\rho_{\theta\theta}\lambda + \lambda^2(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}\Delta\theta_i \\
&\geq \frac{((1-\lambda)^2(1-r)\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))} = 1,
\end{aligned}
$$

with strict inequality if $\lambda > \widehat{\lambda}$. Thus, the response to the idiosyncratic shock is greater than in the complete information equilibrium if $\lambda \in (\widehat{\lambda}, 1)$. For the second part we repeat the same argument. Note that if $\lambda < \widehat{\lambda}$, then $\lambda < (1-\lambda)(1-r)$, but then:

$$
\begin{aligned}
\text{cov}(a_i, \bar{\theta}) &= \frac{((1-\lambda)^2\rho_{\theta\theta} + (1-\lambda)\lambda(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}\bar{\theta} \\
&\geq \frac{1}{1-r}\frac{((1-\lambda)^2(1-r)\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))} = \frac{1}{1-r},
\end{aligned}
$$

with strict inequality if $\lambda < \widehat{\lambda}$. ∎

Proof of Proposition 5.   The comparative statics with respect to the argmax are shown by proving that the quantities have a unique maximum, which is interior, and then use the sign of the cross derivatives (the derivative with respect to λ and r). The ordering of the information structures that maximizes the different second moments is proved by comparing the derivatives.

(**2.**) We begin by rewriting the individual variance, and using (12) we can write it in terms of λ:

$$
\begin{aligned}
\sigma_a^2 &= \left(\frac{((1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))} \right)^2 ((1-\lambda)^2\rho_{\theta\theta} + \lambda^2(1-\rho_{\theta\theta}))\sigma_\theta^2 \\
&= \rho_{\theta\theta}\frac{(1+yx)^2}{((1-r)+x^2)^2}(1+x^2)\sigma_\theta^2,
\end{aligned}
$$

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where

$$x \triangleq \frac{\sqrt{(1 - \rho_{\theta\theta})}\lambda}{\sqrt{\rho_{\theta\theta}}(1 - \lambda)}, \quad y \triangleq \frac{\sqrt{1 - \rho_{\theta\theta}}}{\sqrt{\rho_{\theta\theta}}}. \tag{46}$$

Note that x is strictly increasing in λ, and if $\lambda \in [0, 1]$ then $x \in [0, \infty]$, and thus maximizing with respect to $x \in [0, \infty]$ is equivalent to maximizing with respect to $\lambda \in [0, 1]$. Finding the derivative we get:

$$\frac{\partial \sigma_a^2}{\partial x} = -\frac{2(xy + 1)\left(x^3 + (2r - 1)yx^2 + (r + 1)x - (1 - r)y\right)}{(x^2 + 1 - r)^3} \sigma_\theta^2.$$

It is easy to see that $\frac{d\sigma_a^2}{dx}$ is positive at $x = 0$ and negative if we take a x large enough, and thus the maximum must be in $x \in (0, \infty)$. We would like to show that the polynomial:

$$\left(x^3 + (2r - 1)yx^2 + (r + 1)x - (1 - r)y\right),$$

has a unique root in $x \in (0, \infty)$. If $r < -1$, then the function is increasing in x and has a negative value at $x = 0$, thus it has a unique root. If $x > 1/2$, then the function is negative and decreasing at x. Since it is a cubic polynomial and the term next to x^3 is positive, it must have a unique positive root. For $r \in [-1, 1/2]$ we define the determinant of the cubic equation:

$$\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2.$$

We know that if $\Delta < 0$ then the polynomial has a unique root. Replacing by the respective values of the cubic polynomial we get:

$$\Delta = 4y^4(2r - 1)^3(1 - r) + y^2((2r - 1)^2(1 + r)^2 - 18(1 - r^2)(2r - 1) - 27(1 - r)^2)) - 4(1 + r)^3,$$

using the fact that for $r \in [-1, 1/2]$ we have that $(2r - 1) \leq 0$ and $1 + r \geq 0$, we know that the term with y^4 and without y are negative. We just need to check the term with y^2, but this is also negative for $r \in [-1, 1/2]$. Thus, $\Delta < 0$, and thus for $r \in [-1, 1/2]$ the polynomial has a unique root.

Thus, we have that there exists a unique λ that maximizes σ_a^2. Finally, we have that:

$$\frac{\partial \sigma_a^2}{\partial r} = 2\frac{(1 - \lambda)^2\rho_{\theta\theta}}{((1 - r)(1 - \lambda)^2\rho_{\theta\theta} + \lambda^2(1 - \rho_{\theta\theta}))}\sigma_a^2.$$

Note that

$$\frac{\partial}{\partial \lambda}\frac{(1 - \lambda)^2\rho_{\theta\theta}}{((1 - r)(1 - \lambda)^2\rho_{\theta\theta} + \lambda^2(1 - \rho_{\theta\theta}))} < 0,$$

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and thus at the maximum:

$$\frac{\partial^2 \sigma_a^2}{\partial r \partial \lambda} = 2\sigma_a^2 \frac{\partial}{\partial \lambda} \frac{(1-\lambda)^2 \rho_{\theta\theta}}{((1-r)(1-\lambda)^2 \rho_{\theta\theta} + \lambda^2 (1-\rho_{\theta\theta}))} < 0,$$

and thus $\text{argmax}_\lambda \, \sigma_a^2$ is decreasing in r.

Next, we consider the aggregate variance $\rho_{aa}\sigma_a^2$, and write it in terms of λ:

$$\rho_{aa}\sigma_a^2 = \left(\frac{((1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2 \rho_{\theta\theta} + \lambda^2 (1-\rho_{\theta\theta}))}\right)^2 (1-\lambda)^2 \rho_{\theta\theta}\sigma_\theta^2 = \rho_{\theta\theta}\frac{(1+yx)^2}{((1-r)+x^2)^2}\sigma_\theta^2, \tag{47}$$

where x and y are defined as in (46). Maximizing with respect to $x \in [0, \infty]$ is equivalent to maximizing with respect to $\lambda \in [0, 1]$. Finding the derivative we get:

$$\frac{\partial \rho_{aa}\sigma_a^2}{\partial x} = -\frac{2(xy+1)(2x + (x^2 + r - 1)y)}{(x^2 + 1 - r)^3}\sigma_\theta^2. \tag{48}$$

Again, we have that $(2x + (x^2 + r - 1)y)$ has a unique root in $(0, \infty)$ Thus, we have that there exists a unique λ that maximizes $\rho_{aa}\sigma_a^2$. Finally, we have that:

$$\frac{\partial \rho_{aa}\sigma_a^2}{\partial r} = 2\frac{(1-\lambda)^2 \rho_{\theta\theta}}{((1-r)(1-\lambda)^2 \rho_{\theta\theta} + \lambda^2 (1-\rho_{\theta\theta}))}\rho_{aa}\sigma_a^2.$$

Note that,

$$\frac{\partial}{\partial \lambda} \frac{(1-\lambda)^2 \rho_{\theta\theta}}{((1-r)(1-\lambda)^2 \rho_{\theta\theta} + \lambda^2 (1-\rho_{\theta\theta}))} < 0,$$

and thus at the maximum $\frac{\partial^2 \sigma_a^2}{\partial r \partial \lambda} < 0$, and thus $\text{argmax}_\lambda \, \rho_{aa}\sigma_a^2$ is decreasing in r.

Finally, we consider the dispersion, $(1 - \rho_{aa})\sigma_a^2$, expressed in terms of λ:

$$\begin{aligned}
(1 - \rho_{aa})\sigma_a^2 &= \left(\frac{((1-\lambda)\rho_{\theta\theta} + \lambda(1-\rho_{\theta\theta}))}{((1-r)(1-\lambda)^2 \rho_{\theta\theta} + \lambda^2 (1-\rho_{\theta\theta}))}\right)^2 \lambda^2 (1-\rho_{\theta\theta})\sigma_\theta^2 \\
&= \rho_{\theta\theta}\frac{(1+yx)^2}{((1-r)+x^2)^2}x^2\sigma_\theta^2,
\end{aligned}$$

where x and y are defined in (46). As before, maximizing with respect to $x \in [0, \infty]$ is equivalent to maximizing with respect to $\lambda \in [0, 1]$. Finding the derivative we get:

$$\frac{\partial(1 - \rho_{aa})\sigma_a^2}{\partial x} = -\frac{2x(xy+1)(x^2 + 2(r-1)yx + r - 1)}{(x^2 + 1 - r)^3}\sigma_\theta^2.$$

Again, we have that $(x^2 + 2(r-1)yx + r - 1)$ has a unique root in $(0, \infty)$ Thus, there exists a unique λ that maximizes $(1 - \rho_{aa})\sigma_a^2$. Finally, we have that:

$$\frac{\partial(1 - \rho_{aa})\sigma_a^2}{\partial r} = 2\frac{(1-\lambda)^2 \rho_{\theta\theta}}{((1-r)(1-\lambda)^2 \rho_{\theta\theta} + \lambda^2 (1-\rho_{\theta\theta}))}(1 - \rho_{aa})\sigma_a^2.$$

Note that

$$\frac{\partial}{\partial\lambda}\frac{(1-\lambda)^2\rho_{\theta\theta}}{((1-r)(1-\lambda)^2\rho_{\theta\theta}+\lambda^2(1-\rho_{\theta\theta}))}<0,$$

and thus at the maximum $\frac{\partial^2\sigma_a^2}{\partial r\partial\lambda}<0$, and thus $\mathrm{argmax}_\lambda\,(1-\rho_{aa})\sigma_a^2$ is decreasing in r.

(1.) Finally, we want to show that $\mathrm{argmax}_\lambda\,(1-\rho_{aa})\sigma_a^2>\mathrm{argmax}_\lambda\,\sigma_a^2>\mathrm{argmax}_\lambda\,\rho_{aa}\sigma_a^2$. These inequalities follows from comparing the derivatives of $(1-\rho_{aa})\sigma_a^2$, σ_a^2 and $\rho_{aa}\sigma_a^2$ with respect to λ (or equivalently x). It is easy to see that:

$$\frac{\partial\log(1-\rho_{aa})\sigma_a^2}{\partial x}<\frac{\partial\log\sigma_a^2}{\partial x}<\frac{\partial\log\rho_{aa}\sigma_a^2}{\partial x}.$$

Since the derivatives satisfy the previous inequalities, and the quantities have a unique maximum, the argument of the maximum must also satisfy the same inequalities.

(3.) The comparative static results with respect to the maximum follow directly from the envelope theorem. ∎

Proof of Proposition 6. We first solve for $\max_\lambda\{\rho_{aa}\sigma_a^2\}$. By setting (48) equal to 0, we have that the aggregate volatility is maximized at,

$$x=\frac{\sqrt{1+y^2(1-r)}-1}{y}.$$

In terms of the original variables this can be written as follows:

$$\lambda=\frac{\rho_{\theta\theta}\left(\sqrt{\frac{1-r}{\rho_{\theta\theta}}+r}+r-2\right)}{(r-4)\rho_{\theta\theta}+1}.$$

Substituting the solution in (47) and using the definitions of x and y we get that the maximum volatility is equal to:

$$\frac{\sigma_\theta^2(1-\rho_{\theta\theta})^2}{4(\sqrt{\rho_{\theta\theta}}-\sqrt{\rho_{\theta\theta}+(1-r)(1-\rho_{\theta\theta})})^2}.$$

Using the definition of $\sigma_{\bar\theta}$ and σ_θ^2 we get (26). Note that by imposing $r=0$ we also get (17) and (18). ∎

Proof of Corollary 3. It follows directly from (26) that:

$$\lim_{\sigma_\theta^2\to 0}\max_\lambda\{\rho_{aa}\sigma_a^2\}=\frac{\sigma_\theta^2}{4(1-r)}.$$

Thus, we are only left with proving that,

$$\lim_{\sigma_{\Delta\theta_i}^2\to 0}\max_\lambda\{\rho_{aa}\sigma_a^2\}=\sigma_\theta^2/(1-r)^2.$$

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The limit can be easily calculated using L'Hopital's rule. That is, just note that as $\sigma^2_{\Delta\theta_i} \to 0$ we have that:

$$4(\sigma_{\bar\theta} - \sqrt{\sigma^2_{\bar\theta} + (1-r)\sigma^2_{\Delta\theta_i}})^2 \approx \sigma^4_{\Delta\theta_i}(1-r)^2/\sigma^2_{\bar\theta} + o(\sigma^6_{\Delta\theta_i}),$$

and hence we get the result. ∎

Proof of Lemma 1. We need to prove that given the assumption of symmetry, the parameters $(\mu_a, \rho_{aa}, \rho_{a\theta}, \rho_{a\phi}, \sigma_a)$ are sufficient to characterize the distribution of the random variables $(\theta_i, \bar\theta, a_i, A)$. Clearly, we have that $\mu_a = \mu_A$, as it follows from the law of iterated expectations. By the previous definition (and decomposition) of the idiosyncratic shock θ_i, we observe that the expectations of the following products all agree: $\mathbb{E}_i[a_i\bar\theta] = \mathbb{E}_i[A\theta\theta_i] = \mathbb{E}_i[A\bar\theta]$. This can be easily seen as follows:

$$\mathbb{E}[\theta_i A] = \mathbb{E}[\bar\theta A] + \mathbb{E}[\Delta\theta_i A] = \mathbb{E}[\bar\theta A] + \mathbb{E}[A \cdot \underbrace{\mathbb{E}[\Delta\theta_i | A]}_{=0}] = \mathbb{E}[\bar\theta A],$$

where we just use the law of iterated expectations and the fact that the expected value of a idiosyncratic variable conditioned on an aggregate variable must be 0. Thus:

$$\text{cov}(a_i, \bar\theta) = \text{cov}(A, \theta_i) = \text{cov}(A, \bar\theta) = \text{cov}(a_i, \theta_j) = \mathbb{E}[a_i\theta_j] - \mu_a\mu_\theta = \rho_{a\phi}\sigma_\theta\sigma_a.$$

Similarly, since we consider a symmetric Bayes correlated equilibrium, the covariance of the actions of any two individuals, a_i and a_j, which is denoted by $\rho_{aa}\sigma^2_a$, is equal to the aggregate variance. Once again, this can be easily seen as follows,

$$\mathbb{E}[a_ia_j] = \mathbb{E}[A^2] + \mathbb{E}[A\Delta a_j] + \mathbb{E}[\Delta a_i A] + \mathbb{E}[\Delta a_i \Delta a_j] = \mathbb{E}[A^2],$$

where in this case we need to use that the equilibrium is symmetric and thus $\mathbb{E}[\Delta a_i \Delta a_j] = 0$. Thus, we have $\sigma^2_A = \text{cov}(a_i, a_j) = \text{cov}(A, a_i) = \rho_{aa}\sigma^2_a$. ∎

Proof of Proposition 7. The moment equalities (1) and (2) were established in (30) and (31). Thus we proceed to verify that the inequality constraints (3) are necessary and sufficient to guarantee that the matrix $\mathbb{V}$ is positive semidefinite.

Here we express the equilibrium conditions, by a change of variables, in terms of different variables, which facilitates the calculation. Let:

$$M \triangleq \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, we have that:

$$\begin{pmatrix} \Delta\theta_i \\ \bar{\theta} \\ \Delta a_i \\ A \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ \mu_\theta \\ 0 \\ \mu_a \end{pmatrix}, M\mathbb{V}M' \right),$$

where

$$\mathbb{V}_\perp \triangleq M\mathbb{V}M' = \begin{pmatrix} (1-\rho_{\theta\theta})\sigma_\theta^2 & 0 & (\rho_{a\theta}-\rho_{a\phi})\sigma_a\sigma_\theta & 0 \\ 0 & \rho_{\theta\theta}\sigma_\theta^2 & 0 & \rho_{a\phi}\sigma_a\sigma_\theta \\ (\rho_{a\theta}-\rho_{a\phi})\sigma_a\sigma_\theta & 0 & (1-\rho_{aa})\sigma_a^2 & 0 \\ 0 & \rho_{a\phi}\sigma_a\sigma_\theta & 0 & \rho_{aa}\sigma_a^2 \end{pmatrix}. \tag{49}$$

We use $\mathbb{V}_\perp$ to denote the variance/covariance matrix expressed in terms of $(\Delta\theta_i, \bar{\theta}, \Delta a_i, A)$. It is easy to verify that $\mathbb{V}_\perp$ is positive semidefinite if and only if the inequality conditions (3) are satisfied. To check this it is sufficient to note that the leading principal minors are positive if and only if these conditions are satisfied, and thus $\mathbb{V}_\perp$ is positive semidefinite if and only if these conditions are satisfied. ∎

Proof of Proposition 8. ($\Leftarrow$) We first prove that if the variables $(\theta_i, \bar{\theta}, a_i, A)$ form a Bayes Nash equilibrium for some information structure $\mathcal{I}_i$ (and associated signals), then the variables $(\theta_i, \bar{\theta}, a_i, A)$ also form a Bayes correlated equilibrium. Consider the case in which agents receive normally distributed signals through the information structure $\mathcal{I}_i$, which by minor abuse of notation also serves as conditioning event. Then in any Bayes Nash equilibrium of the game, we have that the actions of the agents are given by:

$$a_i = r\mathbb{E}[A|\mathcal{I}_i] + \mathbb{E}[\theta_i|\mathcal{I}_i], \quad \forall i, \forall \mathcal{I}_i, \tag{50}$$

and since the information is normally distributed, the variables $(\theta_i, \bar{\theta}, a_i, A)$ are jointly normal as well. By taking the expectation of (50) conditional on the information set $\mathcal{I}_i' = \{\mathcal{I}_i, a_i\}$ we get:

$$\begin{aligned} \mathbb{E}[a_i|\mathcal{I}_i, a_i] = a_i &= r\mathbb{E}[\mathbb{E}[A|\mathcal{I}_i]|\mathcal{I}_i, a_i] + \mathbb{E}[\mathbb{E}[\theta_i|\mathcal{I}_i]|\mathcal{I}_i, a_i] \\ &= r\mathbb{E}[A|\mathcal{I}_i, a_i] + \mathbb{E}[\theta_i|\mathcal{I}_i, a_i]. \end{aligned} \tag{51}$$

In other words, agents know the recommended action they are supposed to take, and thus, we can assume that the agents condition on their own actions. By taking expectations of (51) conditional on $\{a_i\}$ we get:

$$\begin{aligned} \mathbb{E}[a_i|a_i] = a_i &= r\mathbb{E}[\mathbb{E}[A|\mathcal{I}_i, a_i]|a_i] + \mathbb{E}[\mathbb{E}[\theta_i|\mathcal{I}_i, a_i]|a_i] \\ &= r\mathbb{E}[A|a_i] + \mathbb{E}[\theta_i|a_i], \end{aligned} \tag{52}$$

where we used the law of iterated expectations. In other words, the information contained in $\{a_i\}$ is a sufficient statistic for agents to compute their best response, and thus the agents compute the same best response if they know $\{\mathcal{I}_i, a_i\}$ or if they just know $\{a_i\}$. Yet, looking at (52), by definition $(\theta_i, \bar{\theta}, a_i, A)$ form a Bayes correlated equilibrium.

($\Rightarrow$) We now prove that if $(\theta_i, \bar{\theta}, a_i, A)$ form a Bayes correlated equilibrium, then there exists an information structure $\mathcal{I}_i$ such that the variables $(\theta_i, \bar{\theta}, a_i, A)$ form a Bayes Nash equilibrium when agents receive this information structure. We consider the case in which the variables $(\theta_i, \bar{\theta}, a_i, A)$ form a Bayes correlated equilibrium, and thus the variables are jointly normal and

$$a_i = r\mathbb{E}[A|a_i] + \mathbb{E}[\theta_i|a_i]. \tag{53}$$

Since the variables are jointly normal we can always find $w \in \mathbb{R}$ and $\lambda \in [-1, 1]$, such that:

$$a_i = w(\lambda \Delta \theta_i + (1 - |\lambda|)\bar{\theta} + \varepsilon_i).$$

The variables (λ, w) and the random variable ε are defined by the following equations of the Bayes correlated equilibrium distribution:

$$w\lambda = \frac{\text{cov}(a_i, \Delta \theta_i)}{\sigma^2_{\Delta \theta_i}}, \quad w(1 - |\lambda|) = \frac{\text{cov}(a_i, \bar{\theta})}{\sigma^2_{\bar{\theta}}},$$

and

$$\varepsilon = \frac{a_i - \frac{\text{cov}(a_i, \Delta \theta_i)\Delta \theta_i}{\sigma^2_{\Delta \theta_i}} - \frac{\text{cov}(a_i, \bar{\theta})}{\sigma^2_{\bar{\theta}}})}{w}.$$

Now consider the case in which agents receive a one-dimensional signal

$$s_i \triangleq \frac{a_i}{w} = (\lambda \Delta \theta_i + (1 - \lambda)\bar{\theta} + \varepsilon_i). \tag{54}$$

Then, by definition, we have that:

$$a_i = ws_i = r\mathbb{E}[A|a_i] + \mathbb{E}[\theta_i|a_i] = r\mathbb{E}[A|s_i] + \mathbb{E}[\theta_i|s_i],$$

where we use the fact that conditioning on a_i is equivalent to conditioning on s_i. Thus, when agent i receives information structure (and associated signal s_i): $\mathcal{I}_i = \{s_i\}$, then agent i taking action $a_i = ws_i$ constitutes a Bayes Nash equilibrium, as it complies with the best response condition. Thus, the distribution $(\theta_i, \bar{\theta}, a_i, A)$ forms a Bayes Nash equilibrium when agents receive signals $\mathcal{I}_i = \{s_i\}$. ∎

Proof of Proposition 9. Note that (54) has the form stated in the Proposition, and thus this was implicitly established by the proof of Proposition 8. ∎

To establish Proposition 12, we use the following lemma that is of independent interest. Consider an arbitrary continuous function:

$$\psi : [0,1] \times [0,1] \times [-1,1] \to \mathbb{R},$$

whose domain is given by the triple of correlation coefficients: $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$.

Lemma 2.
If $\psi(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ is a continuous function, strictly increasing in $\rho_{a\theta}$ and weakly increasing in $\rho_{a\phi}$, then the Bayes correlated equilibrium that maximizes ψ is an noise free Bayes correlated equilibrium.

Proof. By rewriting the constraints (34) of Proposition 7 we obtain:

1. $\rho_{\theta\theta}\rho_{aa} - (\rho_{a\phi})^2 \geq 0$;
2. $(1 - \rho_{aa})(1 - \rho_{\theta\theta}) - (\rho_{a\theta} - \rho_{a\phi})^2 \geq 0$.

If $\psi(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ is strictly increasing, then in the optimum the above inequality (2) must bind. Moreover, if the constraint (1) does not bind, then we can just increase $\rho_{a\theta}$ and $\rho_{a\phi}$ in equal amounts, without violating (2) and increasing the value of ψ. Thus, in the maximum of ψ we must have that both bind. ∎

Proof of Proposition 10. First, it is easy to see that for any coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ in the boundary, we must have that that both inequalities in (34) must be satisfied with equality. Otherwise, we could always increase $\rho_{a\theta}$ and $\rho_{a\phi}$ in the right amounts without breaking either inequality, and thus achieving a higher value of $\rho_{a\theta}$. Second, it is easy to solve for $\rho_{a\theta}$ in terms of ρ_{aa} and $\rho_{\theta\theta}$. Using that $\rho_{a\theta} \geq 0$, we must have that:

$$\rho_{a\theta} = |\sqrt{\rho_{aa}\rho_{\theta\theta}} \pm \sqrt{(1 - \rho_{aa})(1 - \rho_{\theta\theta})}| \text{ and } \rho_{a\phi} = \pm\sqrt{\rho_{aa}\rho_{\theta\theta}}.$$

Finally, it is clear that the boundary will be given by:

$$\rho_{a\theta} = \sqrt{\rho_{aa}\rho_{\theta\theta}} + \sqrt{(1 - \rho_{aa})(1 - \rho_{\theta\theta})},$$

as this gives the maximum value for $\rho_{a\theta}$. ∎

Proof of Proposition 11. From Proposition 10, we must have that both inequalities in (34) must be satisfied with equality. From (38) it is clear that if both inequalities in (34) are satisfied

with equality, then a_i must be deterministic conditional on $\Delta\theta_i$ and $\bar{\theta}$. Moreover, give the properties of multivariate normal distributions, we must have that a_i is a linear combination of $\Delta\theta_i$ and $\bar{\theta}$. Thus, without loss of generality, for any coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ in the boundary, we must have that:

$$a_i = w(\lambda)((1 - |\lambda|)\bar{\theta} + \lambda\Delta\theta_i) \text{ with } \lambda \in [-1, 1]. \tag{55}$$

This comes from the fact that we can calculate the outcome of the Bayes correlated equilibrium as the outcome of a Bayes Nash equilibrium when agents get signals of the form (35), with $\sigma_\varepsilon^2 = 0$. For any λ such that players action is given by (55) we have that the correlation in actions is given by:

$$\rho_{aa} = \frac{(1 - |\lambda|)^2\rho_{\theta\theta}}{(1 - |\lambda|)^2\rho_{\theta\theta} + \lambda^2(1 - \rho_{\theta\theta})}.$$

Finally, for any $\lambda \in [0, 1]$ we have that the correlation of players action for information structure $-\lambda$ yields the same correlation of action as information structure λ, but strictly lower correlation between action and payoff state, $\rho_{a\theta}$. Thus, all noise free information structures with $\lambda \in (-1, 0)$ cannot be in the boundary as they yield strictly lower $\rho_{a\theta}$ than noise free information structures with $\lambda \in [0, 1]$. Thus, all Bayes correlated equilibrium on the boundary form a Bayes Nash equilibrium with a noise free information structure with $\lambda \in [0, 1]$.

On the other hand, all Bayes Nash equilibrium with a noise free information structure with $\lambda \in [0, 1]$ form a Bayes correlated equilibrium on the boundary. This just come from the fact that actions in Bayes Nash equilibrium with a noise free information structure are linear combination of $\bar{\theta}$ and $\Delta\theta$. Moreover, from (13) and (14) it is easy to see that $\lambda \in [0, 1]$ span all correlations satisfying,

$$\rho_{a\theta} = \sqrt{\rho_{aa}\rho_{\theta\theta}} + \sqrt{(1 - \rho_{aa})(1 - \rho_{\theta\theta})},$$

with $\rho_{aa} \in [0, 1]$. ∎

Proof of Proposition 12. From (33) the individual volatility, aggregate volatility and dispersion can be written as follows:

$$\frac{\rho_{a\theta}\sigma_\theta}{1 - r\rho_{aa}}, \ \rho_{aa}\frac{\rho_{a\theta}\sigma_\theta}{1 - r\rho_{aa}}, \ (1 - \rho_{aa})\frac{\rho_{a\theta}\sigma_\theta}{1 - r\rho_{aa}},$$

and the result follows directly.∎

Proof of Proposition 13. See supplemental online appendix.∎

Proof of Proposition 14. See supplemental online appendix.∎

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Supplementary Results for
Information and Volatility

Dirk Bergemann[a], Tibor Heumann[b], Stephen Morris[c]

[a]*Department of Economics, Yale University, New Haven, CT 06520, U.S.A. dirk.bergemann@yale.edu.*
[b]*Department of Economics, Yale University, New Haven, CT 06520, U.S.A. tibor.heumann@yale.edu.*
[c]*Department of Economics, Princeton University, Princeton, NJ 08544, U.S.A. smorris@princeton.edu.*

Abstract

This supplement to "Information and Volatility" contains auxiliary material and proofs to the results found in the main paper.

This supplement of Bergemann, Heumann, and Morris [4] contains Appendices B and C of our paper "Information and Volatility". Appendix A is found in the main paper.

Supplementary Appendices

B. Additional Results regarding Information Structures and Equilibrium Behavior, complementing the results in Section 7 of the main paper.

C. Proofs for the Additional Results of Appendix B.

Appendix B:
Additional Results for Section 7: Information Structures and Equilibrium Behavior

A third interesting information structure, next to those investigated in Section 7, is the case in which, besides signal $\{s_i^1\}_{i \in [0,1]}$, the players also know the average payoff state $\bar{\theta}$, and thus we set $\sigma_{\varepsilon 2}^2 = \sigma_{\varepsilon 3}^2 = 0$, and each agent observes $\bar{\theta}$ and s_i^1. Although a priori this may not seem like an information structure that would arise exogenously, it is the information structure that arises when agents receive endogenous information on the average action taken by other players. For example, in a rational expectations equilibrium with a continuum of sellers as studied by Vives [6], each seller chooses a supply function, given a signal about his private cost θ_i. The resulting equilibrium price p can be shown to be a linear function of the average supply, and in turn a linear function of the average cost $\bar{\theta}$. Similarly, in recent work by Benhabib, Wang, and Wen [1], (2013) and Bergemann, Heumann, and Morris [3] the equilibrium condition of the rational expectations equilibrium has a linear structure in which each agent conditions his decision on the common component $\bar{\theta}$.

Proposition 15 (Noisy Signal of Payoff State / Noiseless Signal of Common Component).
A set of correlations $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ can be achieved as a Bayes Nash equilibrium with an information structure $\{s_i^1, \bar{\theta}\}_{i \in [0,1]}$ if and only if:

$$\rho_{a\theta} = \sqrt{\frac{\rho_{\theta\theta}}{\rho_{aa}}} \frac{1 - r\rho_{aa}}{1 - r}; \quad \rho_{a\phi} = \sqrt{\rho_{\theta\theta}\rho_{aa}}; \quad \rho_{aa} \in [\widehat{\rho}_{aa}, 1]. \tag{56}$$

Now, the set of feasible correlations indeed depends on r, and the information structure $\left(\{s_i^1\}_{i \in [0,1]}, \bar{\theta}\right)$ generates a two-dimensional signal for every agent i. We also find that the correlation coefficient ρ_{aa} has a lower bound, which is the correlation coefficient $\widehat{\rho}_{aa}$ achieved in the complete information equilibrium. In Figure 6 we illustrate the set of correlations that can be achieved when each agent receives a noisy signal s_i^1 of his payoff state θ_i, and *in addition* knows the average payoff state $\bar{\theta}$. We observe that the class of two-dimensional information structures $\{s_i^1, \bar{\theta}\}_{i \in [0,1]}$ induces a one-dimensional subspace of $(\rho_{a\theta}, \rho_{aa})$ that does depend on the nature of the interaction. Indeed Proposition 15 establishes that these signals maintain a one-dimensional subspace even with respect to the full three dimensional space of correlation coefficients $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$. Importantly, the feasible correlation coefficients remained bounded away from the frontier, except for the two points of $\sigma_{\varepsilon 1}^2 = 0$ and $\sigma_{\varepsilon 1}^2 = \infty$, where the later leads to perfect correlation in the actions across agents: $\rho_{aa} = 1$, at the expense of low correlation with the payoff state: $\rho_{a\theta} = \sqrt{\rho_{\theta\theta}}$.
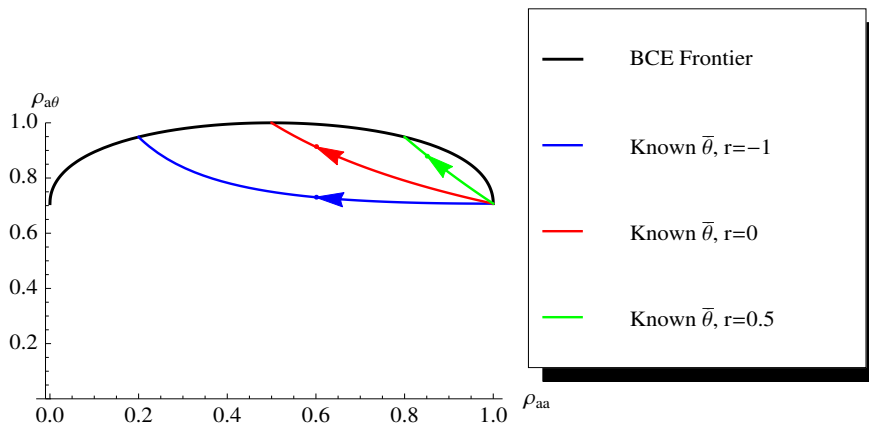
Figure 6: Set of feasible correlations for subset of information structures ($\rho_{\theta\theta} = 1/2$)

So far, we confined attention to strict subsets of the three-dimensional information structures summarized by Table 1:

	Noisy Idiosyncratic Information	Known Common Component	Known Own Payoff State
σ_{ε^1}	$\in [0, \infty)$	$\in [0, \infty)$	0
σ_{ε^2}	∞	0	$\in [0, \infty)$
σ_{ε^3}	∞	0	$\in [0, \infty)$
Proposition	13	15	14

Table 1: Information as Restriction on Equilibrium Behavior

Now that we understand how the observability of the individual state or components of the individual state affect the equilibrium correlation, we can ask what is the entire set of equilibrium correlations that can be achieved with the three dimensional signal structure. As the dimensionality of the information structure defined by (39) coincides with the dimensionality of the set of equilibrium correlations, one might expect that the set of information structures is sufficiently rich to span the set of equilibria defined by the BCE. In fact, in the case of pure common values Bergemann and Morris [5] show that any BCE can be decentralized by considering a private and a public signal of the payoff state, namely $s_i^1 = \theta_i + \varepsilon_i^1$ and $s_i^3 = \bar{\theta} + \bar{\varepsilon}^3$. By contrast, in the present general environment neither the class of binary nor the extended class of tertiary information structures can decentralize the entire sets of BCE. We characterize the set of feasible correlations in the $(\rho_{aa}, \rho_{a\theta})$ coefficient space. In order to describe the set of feasible correlations in the $(\rho_{aa}, \rho_{a\theta})$ space, we define

$\overline{\rho}_{a\theta}(\rho_{aa})$ to be the upper bound that can be attained by the correlation coefficient ρ_{aa} in a Bayes Nash equilibrium with an information structure in the set defined by (39).

Proposition 16 (Boundary of Correlations under $\mathbf{S}_i$).
The boundary of feasible correlations $\overline{\rho}_{a\theta}(\rho_{aa})$ for Bayes Nash equilibria when the information structure is in $\mathbf{s}_i$ is given by:

1. *If $\rho_{aa} \in [0, \min\{\widehat{\rho}_{aa}, \rho_{\theta\theta}\}]$, then $\overline{\rho}_{a\theta}(\rho_{aa})$ is attained by an information structure in which each agent gets a noisy signal on his own payoff state: $\sigma_{\varepsilon^1} \in [0, \infty)$, $\sigma_{\varepsilon^2} = \sigma_{\varepsilon^3} = \infty$ with $\overline{\rho}_{a\theta}(\rho_{aa}) = \sqrt{\rho_{aa}/\rho_{\theta\theta}}$.*
2. *If $\rho_{aa} \in [\min\{\widehat{\rho}_{aa}, \rho_{\theta\theta}\}, \max\{\widehat{\rho}_{aa}, \rho_{\theta\theta}\}]$, then $\overline{\rho}_{a\theta}(\rho_{aa})$ is attained by an information structure in which each agent knows his own payoff state: $\sigma_{\varepsilon^1} = 0, \sigma_{\varepsilon^2}, \sigma_{\varepsilon^3} \in [0, \infty)$ with $\overline{\rho}_{a\theta}(\rho_{aa}) = \max\{\rho_{a\theta}^c, \rho_{a\theta}^i\}$.*
3. *If $\rho_{aa} \in [\max\{\widehat{\rho}_{aa}, \rho_{\theta\theta}\}, 1]$, then $\overline{\rho}_{a\theta}(\rho_{aa})$ is attained by an information structure in which each agent knows the common component: $\sigma_{\varepsilon^1} \in [0, \infty)$, $\sigma_{\varepsilon^2} = \sigma_{\varepsilon^3} = 0$ with:*

$$\overline{\rho}_{a\theta}(\rho_{aa}) = \sqrt{\frac{\rho_{\theta\theta}}{\rho_{aa}}} \frac{1 - r\rho_{aa}}{1 - r}.$$

The upper bound $\overline{\rho}_{a\theta}(\rho_{aa})$ can therefore be constructed from the union of the information structures that we considered in Propositions 13, 14 and 15. In Figure 7, the left panel illustrates the behavior that can be achieved for $r = 0$. In this special case, the correlation coefficient of the complete information equilibrium $\widehat{\rho}_{aa}$ coincides with the correlation coefficient of the payoff states $\rho_{\theta\theta}$, and hence $\min\{\widehat{\rho}_{aa}, \rho_{\theta\theta}\} = \max\{\widehat{\rho}_{aa}, \rho_{\theta\theta}\}$. Thus the entire boundary can be achieved by noisy and idiosyncratic signals of the payoff state θ_i with either zero or complete information about the common component $\overline{\theta}$, appealing to Propositions 16.1 and 16.3 respectively. The right panel illustrates the Proposition for $r \neq 0$. As stated earlier in Proposition 14, the intermediate segment of the boundary is obtained with a noisy signal of the common components that comes with a common error in the case of strategic substitutes: $\sigma_{\varepsilon^1} = \sigma_{\varepsilon^2} = \infty, \sigma_{\varepsilon^3} \in [0, \infty)$; or an idiosyncratic error in the case of strategic complements: $\sigma_{\varepsilon^1} = \sigma_{\varepsilon^3} = \infty, \sigma_{\varepsilon^2} \in [0, \infty)$. Using positive variances of the error terms, it is easy to show that for all $(\rho_{aa}, \rho_{a\theta})$ such that $\rho_{a\theta} \in [0, \overline{\rho}_{a\theta}(\rho_{aa})]$, there exists an information structure in the set $\mathbf{s}_i$ that achieves theses correlations as a Bayes Nash equilibrium.

A common feature across all strategic environments is the property that the boundary $\overline{\rho}_{a\theta}$ is strictly below the frontier of all Bayes correlated equilibria, which implies that the set of three-dimensional information structures given by (39) is indeed restrictive. We find that the boundary $\overline{\rho}_{a\theta}$ attains the frontier of the BCE at exactly three points: (i) the complete information equilibrium, (ii) the equilibrium in which each agent knows θ_i and (iii) the equilibrium in which each agent only knows $\overline{\theta}$. It is perhaps worth highlighting that the boundary $\overline{\rho}_{a\theta}$ is discontinuous for strategic

substitutes, that is $r < 0$. This result emphasizes the subtle role that the information structure has on equilibrium outcomes.
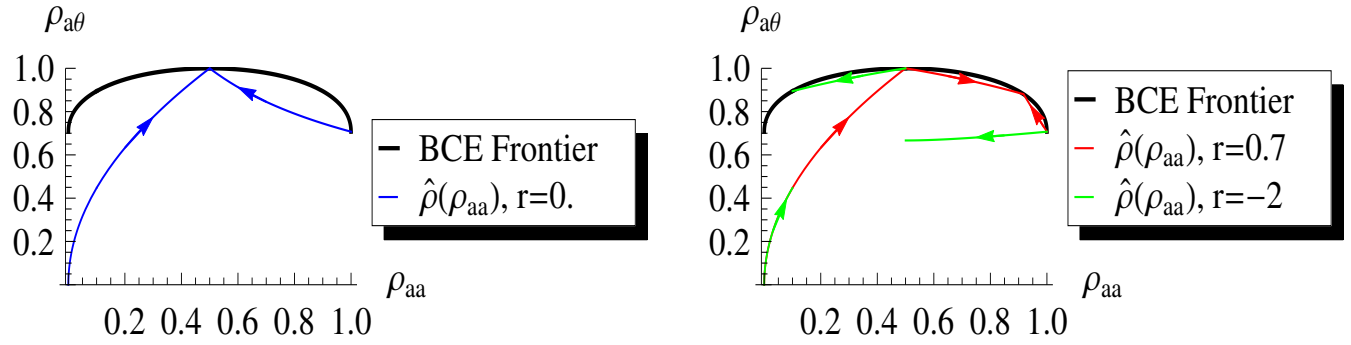

Figure 7: Boundary of the set of feasible correlations for information structures $\mathbf{s}_i$ ($\rho_{\theta\theta} = 1/2$)

Appendix C:

Proofs for Section 7 and the Additional Results of Appendix B

Proof of Proposition 13.   The action of agent i is given by $a_i = \nu s_i^1 = \nu(\theta_i + \varepsilon_i)$, for some $\nu \in \mathbb{R}$. Thus, we can calculate $\rho_{a\theta}$ and ρ_{aa} in terms of σ_ε. We find that:

$$\rho_{a\theta} = \frac{\text{cov}(a_i, \theta_i)}{\sigma_a \sigma_\theta} = \frac{\nu \sigma_\theta^2}{\sigma_\theta \nu \sqrt{\sigma_\theta^2 + \sigma_\varepsilon^2}} = \frac{\sigma_\theta}{\sqrt{\sigma_\theta^2 + \sigma_\varepsilon^2}},$$

$$\rho_{aa} = \frac{\text{cov}(a_i, a_j)}{\sigma_a^2} = \frac{\nu^2 \sigma_\theta^2}{\nu^2 (\sigma_\theta^2 + \sigma_\varepsilon^2)} = \frac{\rho_{\theta\theta} \sigma_\theta^2}{\sigma_\theta^2 + \sigma_\varepsilon^2},$$

$$\rho_{a\phi} = \frac{\text{cov}(a_i, \bar{\theta})}{\sigma_a \sigma_\theta} = \frac{\nu \sigma_\theta^2}{\sigma_\theta \nu \sqrt{\sigma_\theta^2 + \sigma_\varepsilon^2}} = \frac{\rho_{\theta\theta} \sigma_\theta}{\sqrt{\sigma_\theta^2 + \sigma_\varepsilon^2}}.$$

From the above equalities it follows directly that (40) is satisfied. ∎

Proof of Proposition 14.   From the best response conditions, we have that, $a_i = \theta_i + r\mathbb{E}[A|\mathcal{I}_i]$, and multiplying by θ_i and taking expectations (note that because θ_i is in $\mathcal{I}_i$, we have that $\theta_i \mathbb{E}[A|\mathcal{I}_i] = \mathbb{E}[\theta_i A|\mathcal{I}_i]$), we find that $\rho_{a\theta} \sigma_a = \sigma_\theta + r\rho_{a\phi}\sigma_a$. We use the fact that $\sigma_a = \rho_{a\theta}\sigma_\theta + r\rho_{aa}\sigma_a$, and hence inserting into the former we obtain:

$$\rho_{a\phi} = \frac{1}{r}\left(\rho_{a\theta} - \frac{1 - r\rho_{aa}}{\rho_{a\theta}}\right). \tag{57}$$

Thus, the inequalities in (34) can be written as follows:

$$(1 - \rho_{aa})(1 - \rho_{\theta\theta}) \geq \frac{1}{r^2}\left((1 - r)\rho_{a\theta} - \frac{1 - r\rho_{aa}}{\rho_{a\theta}}\right)^2, \tag{58}$$

$$\rho_{aa}\rho_{\theta\theta} \geq \frac{1}{r^2}\left(\rho_{a\theta} - \frac{1 - r\rho_{aa}}{\rho_{a\theta}}\right)^2. \tag{59}$$

For both of the previous inequalities the right hand side is a convex function of $\rho_{a\theta}$. Thus, for a fixed ρ_{aa}, inequalities (58) and (59) independently constrain the set of feasible $\rho_{a\theta}$ to be in a convex interval with non-empty interior for all $\rho_{aa} \in [0, 1]$. Thus, if we impose both inequalities jointly, we find the intersection of both intervals which is also a convex interval. We first find the set of feasible ρ_{aa} and prove that it is always case that one of the inequalities provides the lower bound and the other inequality provides the upper bound on the set of feasible $\rho_{a\theta}$ for each feasible ρ_{aa}. For this we make several observations.

First, when agents know their own payoff state there are only two noise free equilibria, one in which agents know only their state and the complete information equilibria. This implies that

there are only two pairs of values for $(\rho_{aa}, \rho_{a\theta})$ such that inequalities (58) and (59) both hold with equality. Second, the previous point implies that there are only two pairs of values for $(\rho_{aa}, \rho_{a\theta})$ such that the bound of the intervals imposed by inequalities (58) and (59) are the same. Third, the previous two points imply that there are only two possible ρ_{aa} such that the set of feasible $\rho_{a\theta}$ is a singleton. These values are $\rho_{aa} \in \{\rho_{\theta\theta}, \widehat{\rho}_{aa}\}$, which corresponds to the ρ_{aa} of the complete information equilibria and the equilibria in which agents only know their state. Fourth, it is clear that there are no feasible BCE with $\rho_{aa} \in \{0, 1\}$. Fifth, the upper and lower bound on the feasible $\rho_{a\theta}$ that are imposed by inequalities (58) and (59) move smoothly with ρ_{aa}. Sixth, this implies that the set of feasible ρ_{aa} is bounded by the values of ρ_{aa} in which the set of feasible $\rho_{a\theta}$ is a singleton. Thus, the set of feasible ρ_{aa} is in $[\min\{\rho_{\theta\theta}, \widehat{\rho}_{aa}\}, \max\{\rho_{\theta\theta}, \widehat{\rho}_{aa}\}]$. Moreover, it is easy to see that for all ρ_{aa} in the interior of this interval one of the inequalities provides the upper bound for $\rho_{a\theta}$ while the other inequality will provide the lower bound.

We now provide the explicit functional forms for the upper and lower bounds. To check which of the inequalities provides the upper and lower bound respectively we can just look at the equilibria in which agents know only their own state, and thus $\rho_{aa} = \rho_{\theta\theta}$. In this case we have the following inequalities for $\rho_{a\theta}$:

$$(1 - \rho_{\theta\theta})^2 \geq \frac{1}{r^2}((1 - r)\rho_{a\theta} - \frac{1 - r\rho_{\theta\theta}}{\rho_{a\theta}})^2, \quad \rho_{\theta\theta}^2 \geq \frac{1}{r^2}(\rho_{a\theta} - \frac{1 - r\rho_{\theta\theta}}{\rho_{a\theta}})^2.$$

It is easy to check that $\rho_{a\theta} = 1$ satisfies both inequalities. Moreover, it is easy to see that if $r > 0$ then $\rho_{a\theta} = 1$ provides a lower bound on the set of $\rho_{a\theta}$ that satisfies the first inequality while $\rho_{a\theta} = 1$ provides an upper bound on the set of $\rho_{a\theta}$ that satisfies the second inequality. If $r < 0$ we get the opposite, $\rho_{a\theta} = 1$ provides a upper bound on the set of $\rho_{a\theta}$ that satisfies the first inequality while $\rho_{a\theta} = 1$ provides a lower bound on the set of $\rho_{a\theta}$ that satisfies the second inequality. Thus, if $r > 0$, then the inequality (58) provides the lower bound for $\rho_{a\theta}$ and the inequality (59) provides an upper bound on the set of feasible $\rho_{a\theta}$ for all ρ_{aa}. If $r < 0$ we get the opposite result. Note that the conclusions about the bounds when $\rho_{aa} = \rho_{\theta\theta}$ can be extended for all feasible ρ_{aa} because we know that the bounds of the intervals are different for all $\rho_{aa} \notin \{\rho_{\theta\theta}, \widehat{\rho}_{aa}\}$, and they move continuously, thus the relative order is preserved. Finally, we define implicitly the functions $\rho_{a\theta}^i$ and $\rho_{a\theta}^c$: For a given ρ_{aa}, $\rho_{a\theta}^c$ and $\rho_{a\theta}^i$ represent the solutions of the following equations:

$$\frac{1}{r}(-(1 - r)(\rho_{a\theta}^c)^2 + 1 - r\rho_{aa}) - \rho_{a\theta}^c \sqrt{(1 - \rho_{aa})(1 - \rho_{\theta\theta})} = 0, \tag{60}$$

and

$$\frac{1}{r}((\rho_{a\theta}^i)^2 - (1 - r\rho_{aa})) - \rho_{a\theta}^i \sqrt{\rho_{aa}\rho_{\theta\theta}} = 0. \tag{61}$$

Therefore, for all feasible BCE in which agents know their own state $\rho_{aa} \in [\min\{\rho_{\theta\theta}, \widehat{\rho}_{aa}\}, \max\{\rho_{\theta\theta}, \widehat{\rho}_{aa}\}]$, while the set of $\rho_{a\theta}$ is bounded by the functions $\rho_{a\theta}^i$ and $\rho_{a\theta}^c$. If $r > 0$ then function $\rho_{a\theta}^i$ provides the upper bound while $\rho_{a\theta}^c$ provides the lower bound. If $r < 0$, then the function $\rho_{a\theta}^c$ provides the upper bound while $\rho_{a\theta}^i$ provides the lower bound. ∎

Proof of Proposition 15. In equilibrium, the best response of players is given by:

$$a_i = \frac{\bar{\theta}}{1-r} + \mathbb{E}[\Delta\theta_i | \Delta s_i^1] = \frac{\bar{\theta}}{1-r} + \frac{\sigma_{\Delta\theta_i}^2}{\sigma_{\Delta\theta_i}^2 + \sigma_{\varepsilon^1}^2}(s_i^1 - \bar{\theta}) = \frac{\bar{\theta}}{1-r} + b(s_i^1 - \bar{\theta}),$$

where

$$\Delta s_i^1 \triangleq s_i^1 - \bar{\theta} = \Delta\theta_i + \varepsilon_i^1 \text{ and } b \triangleq \frac{\sigma_{\Delta\theta_i}^2}{\sigma_{\Delta\theta_i}^2 + \sigma_{\varepsilon^1}^2}.$$

We can now calculate the correlations $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ in terms of σ_ε. We get:

$$\rho_{a\theta} = \frac{\mathrm{cov}(a_i, \theta_i)}{\sigma_a \sigma_\theta} = \frac{\frac{\sigma_{\bar{\theta}}^2}{1-r} + b\sigma_{\Delta\theta_i}^2}{\sigma_\theta \sqrt{\sigma_{\bar{\theta}}^2/(1-r)^2 + b\sigma_{\Delta\theta_i}^2}};$$

$$\rho_{aa} = \frac{\mathrm{cov}(a_i, a_j)}{\sigma_a^2} = \frac{\frac{\sigma_{\bar{\theta}}^2}{(1-r)^2}}{\sigma_{\bar{\theta}}^2/(1-r)^2 + b\sigma_{\Delta\theta_i}^2};$$

$$\rho_{a\phi} = \frac{\mathrm{cov}(a_i, \bar{\theta})}{\sigma_a \sigma_\theta} = \frac{\frac{\sigma_{\bar{\theta}}^2}{1-r}}{\sigma_\theta \sqrt{\sigma_{\bar{\theta}}^2/(1-r)^2 + b\sigma_{\Delta\theta_i}^2}}.$$

Note that by definition $b \in [0, 1]$, and thus we must have that $\rho_{aa} \in [\widehat{\rho}_{aa}, 1]$. By solving for $\rho_{a\theta}$ and $\rho_{a\phi}$ we obtain (56). ∎

To establish Proposition 16, we first provide an auxiliary result, namely Lemma 3, that describes the set of correlations that can be achieved by the class of information structures (39). We define the following information to noise ratio:

$$b \triangleq \frac{\sigma_{\Delta\theta_i}^2}{\sigma_{\Delta\theta_i}^2 + \sigma_{\varepsilon^1}^2} \in [0, 1]. \tag{62}$$

Lemma 3 (Feasible Outcomes with $\mathbf{s}_i$).
A set of correlations $(\rho_{aa}, \rho_{a\theta}, \rho_{a\phi})$ can be achieved by a linear equilibrium in which agents receive a information structure of the form $\mathbf{s}_i$, if and only if, there exists $b \in [0, 1]$ such that:

1. *The following equalities are satisfied:*

$$\sigma_a = \frac{\rho_{a\theta}\sigma_a}{1 - \rho_{aa}r}. \tag{63}$$

$$(1 - r)\rho_{a\phi}\frac{\rho_{a\theta}}{1 - r\rho_{aa}} + \frac{1}{b}(\rho_{a\theta} - \rho_{a\phi})\frac{\rho_{a\theta}}{1 - r\rho_{aa}} = \sigma_\theta. \tag{64}$$

2. *The following inequalities are satisfied:*

$$\rho_{aa}\rho_{\theta\theta} \geq \rho_{a\phi}^2, \ (1 - \rho_{aa})(1 - \rho_{\theta\theta}) - \frac{1}{b}(\rho_{a\theta} - \rho_{a\phi})^2 \geq 0. \tag{65}$$

Proof. (If) Consider a linear Bayes Nash equilibrium in which agents get signals $\{s_i^1, s_i^2, s^3\}$. First, we note that (63) and the first inequality in (65) must be satisfied trivially, as this must be satisfied for any Bayes correlated equilibrium. We now prove that the second inequality in (65) must be satisfied.

In any linear Bayes Nash equilibrium the action of players can be written as follows, $a_i = \alpha_1 s_i^1 + \alpha_2 s_i^1 + \alpha_3 s_i^1$. Therefore, for any constants $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, we know that $\Delta a_i = \alpha_1(\Delta\theta_i + \varepsilon_i^1) + \alpha_2\varepsilon_i^2$, and thus: $\text{var}(\Delta a_i | \Delta\theta_i) \geq \alpha_1^2\sigma_{\varepsilon^1}^2$. Yet, note that we can write α_1 as follows: $\alpha_1 = \text{cov}(\Delta\theta_i, \Delta a_i)/\sigma_{\Delta\theta_i}^2$, and $\text{var}(\Delta a_i | \Delta\theta_i)$ is given by:

$$\text{var}(\Delta a_i | \Delta\theta_i) = \sigma_{\Delta a}^2 - \left(\frac{\text{cov}(\Delta\theta_i, \Delta a_i)}{\sigma_{\Delta\theta_i}^2}\right)^2 \sigma_{\Delta\theta_i}^2 = \sigma_a^2\left(1 - \rho_{aa}) - \frac{(\rho_{a\theta} - \rho_{a\phi})^2}{(1 - \rho_{\theta\theta})}\right).$$

Thus, we have that,

$$\sigma_a^2\left((1 - \rho_{aa}) - \frac{(\rho_{a\theta} - \rho_{a\phi})^2}{(1 - \rho_{\theta\theta})}\right) \geq \left(\frac{\text{cov}(\Delta\theta_i, \Delta a_i)}{\sigma_{\Delta\theta_i}^2}\right)^2 \sigma_{\varepsilon^1}^2 = \sigma_a^2\frac{(\rho_{a\theta} - \rho_{a\phi})^2}{(1 - \rho_{\theta\theta})}\frac{\sigma_{\varepsilon^1}^2}{\sigma_{\Delta\theta_i}^2}.$$

Finally, we note that by definition: $\sigma_{\varepsilon^1}^2 = \sigma_{\Delta\theta_i}^2(1/b - 1)$. Thus, we get,

$$\left((1 - \rho_{aa}) - \frac{1}{b}\frac{(\rho_{a\theta} - \rho_{a\phi})^2}{(1 - \rho_{\theta\theta})}\right) \geq 0,$$

which is the second inequality in (65).

Finally, we prove that condition (64) must be satisfied. For this, note that in any Bayes Nash equilibrium, we must have that $a_i = \mathbb{E}[\theta_i + rA|s_i^1, s_i^2, s^3]$, and multiplying the equation by s_i^1, we get:

$$s_i^1 a_i = \mathbb{E}[\theta_i \cdot s_i^1 + r \cdot A \cdot s_i^1|s_i^1, s_i^2, s^3].$$

Taking expectations and using the law of iterated expectations, we get: $\text{cov}(a_i, s_i^1) = \text{cov}(\theta_i, s_i^1) + r \cdot \text{cov}(A, s_i^1)$, and thus:

$$\text{cov}(\theta_i, s_i^1) = \text{cov}(\theta_i, \theta_i) = \sigma_\theta^2, \ \text{cov}(A, s_i^1) = \text{cov}(A, \theta_i) = \rho_{a\phi}\sigma_a\sigma_\theta,$$

and:

$$\begin{aligned}
\text{cov}(a_i, s_i^1) &= \text{cov}(a_i, \varepsilon_i^1) + \text{cov}(\theta_i, a_i) = \rho_{a\theta}\sigma_a\sigma_\theta + \alpha_1\sigma_{\varepsilon^1}^2 = \rho_{a\theta}\sigma_a\sigma_\theta + \frac{(\rho_{a\theta} - \rho_{a\phi})\sigma_a\sigma_\theta}{\sigma_{\Delta\theta_i}^2}\sigma_{\varepsilon^1}^2 \\
&= \rho_{a\theta}\sigma_a\sigma_\theta + (\rho_{a\theta} - \rho_{a\phi})\sigma_a\sigma_\theta(\frac{1}{b} - 1).
\end{aligned}$$

Thus, we get:

$$\rho_{a\theta}\sigma_a\sigma_\theta + (\rho_{a\theta} - \rho_{a\phi})\sigma_a\sigma_\theta(\frac{1}{b} - 1) = \sigma_\theta^2 + r\rho_{a\phi}\sigma_a\sigma_\theta,$$

and re-arranging terms we get (64).

(Only If) Let $(\theta_i, \bar{\theta}, a_i, A)$ be a Bayes correlated equilibrium with correlations $(\rho_{a\theta}, \rho_{a\phi}, \rho_{aa})$ satisfying (63)-(65). We will show that there exists a information structure $\mathbf{s}_i$, under which the Bayes Nash equilibrium induces the random variables $(\theta_i, \bar{\theta}, a_i, A)$.

Let ε_i^1 be a random variable that is uncorrelated with $\Delta\theta_i$ and $\bar{\theta}$ (and thus it is a noise term) with variance $\sigma_{\varepsilon^1}^2 = (1 - b)\sigma_{\Delta\theta_i}^2/b$ and such that:

$$\operatorname{cov}(\varepsilon_i^1, a_i) = \frac{\operatorname{cov}(\Delta\theta_i, a_i)}{\sigma_{\Delta\theta_i}^2}\sigma_{\varepsilon^1}^2.$$

Define signals s_i^1 and $\tilde{s}_i$ as follows, $s_i^1 \triangleq \theta_i + \varepsilon_i^1$,

$$\tilde{s}_i \triangleq a_i - \frac{\operatorname{cov}(a_i, \Delta\theta_i)}{\sigma_{\Delta\theta_i}^2}s_1.$$

Thus, by definition s_i^1 and $\tilde{s}_i$ are informationally equivalent to s_i^1 and a_i. Note that by definition $\operatorname{cov}(\tilde{s}_i, \Delta\theta_i) = 0$. Thus, we can define:

$$\tilde{\varepsilon}_i \triangleq \tilde{s}_i - \frac{\operatorname{cov}(\tilde{s}_i, \bar{\theta})}{\sigma_{\bar{\theta}}^2}\bar{\theta},$$

and write signal $\tilde{s}_i$ as follows: $\tilde{s}_i = \bar{\theta} + \tilde{\varepsilon}_i$, where $\tilde{\varepsilon}_i$ is a noise term (thus independent of $\bar{\theta}$ and $\Delta\theta_i$) with a correlation of $\rho_{\tilde{\varepsilon}\tilde{\varepsilon}}$ across agents. Note that $a_i = \mathbb{E}[\theta_i + rA|s_i^1, a_i]$ holds if and only if:

$$\mu_a = \mu_\theta + r\mu_A, \;\; \sigma_a = \rho_{a\theta}\sigma_\theta + r\rho_{aa}\sigma_a, \;\; \operatorname{cov}(a_i, s_i^1) = \operatorname{cov}(\theta_i, s_i^1) + r\operatorname{cov}(A, s_i^1). \tag{66}$$

To show this, just note that we can define a random variable z as follows: $z_i \triangleq \mathbb{E}[\theta_i + rA|s_i^1, a_i]$, and impose $a_i = z_i$. By definition:

$$\mathbb{E}[a_i] = \mathbb{E}[z_i]; \;\; \operatorname{var}(z_i) = \operatorname{cov}(a_i, z_i); \;\; \operatorname{cov}(a_i, z_i) = \operatorname{var}(z_i); \;\; \operatorname{var}(a_i, s_i^1) = \operatorname{cov}(z_i, s_i^1). \tag{67}$$

These corresponds to conditions (66) (obviously, since $a_i = z_i$, $\operatorname{cov}(a_i, z_i) = \operatorname{var}(z_i)$ or $\operatorname{cov}(a_i, z_i) = \operatorname{var}(a_i)$ are redundant). Moreover, any random variable z' that satisfies (67) must also satisfy $a = z'$. This just comes from the fact that these are normal random variables, thus the joint distribution of (z_i', a_i, s_i^1) is completely defined by its first and second moments. Thus, any random variable z_i' that has the same second moments as z_i must satisfy that $a_i - z_i = 0$.

Note that, conditions (66) hold since $(\theta_i, \bar{\theta}, a_i, A)$ form a Bayes correlated equilibrium, while $\mathrm{cov}(a_i, s_i^1) = \mathrm{cov}(\theta_i, s_i^1) + \mathrm{cov}(A, s_i^1)$ holds by the assumption (64) (where the calculation is the same as before). Thus, we have that, $(\theta_i, \bar{\theta}, a_i, A)$ is induced by the linear Bayes Nash equilibrium when players receive information structure $\{s_i^1, \tilde{s}_i\}$.

On the other hand, note that the signal $\Delta s_i^1 = \Delta\theta_i + \varepsilon_i^1$ is a sufficient statistic for $\Delta\theta_i$ given that agents only receive signals $(s_i^1, \tilde{s}_i)$. Thus, we have that,

$$\mathbb{E}[\Delta\theta_i | s_i^1, \tilde{s}_i] = \mathbb{E}[\mathbb{E}[\Delta\theta_i | \Delta s_i^1] | s_i^1, \tilde{s}_i] = \mathbb{E}[b\Delta s_i^1 | s_i^1, \tilde{s}_i].$$

From the Bayes correlated equilibrium condition, we know that:

$$a_i = \mathbb{E}[\theta_i + rA | s_i^1, a_i] = \mathbb{E}[b\Delta s_i + \bar{\theta} + rA | s_i^1, \tilde{s}_i].$$

Subtracting bs_i^1 from both sides, we get:

$$\tilde{a}_i \triangleq (a_i - bs_i^1) = \mathbb{E}[(1 - b)\bar{\theta} + rA | s_i^1, a_i] = \mathbb{E}[(1 - (1 - r)b)\bar{\theta} + r\tilde{A} | s_i^1, \tilde{s}_i]. \tag{68}$$

Thus, $(\theta_i, \bar{\theta}, \tilde{a}_i, \tilde{A})$ is induced by a linear Bayes Nash equilibrium with common values, in which players get signals $(s_i^1, \tilde{s}_i)$.

We now show that there exists signals (s_i^2, s^3), such that: $s_i^2 = \alpha\bar{\theta} + \varepsilon_i^2$, $s_i^3 = (1 - \alpha)\bar{\theta} + \bar{\varepsilon}$, and

$$\tilde{a}_i = \mathbb{E}[(1 - (1 - r)b)\bar{\theta} + r\tilde{A} | s_i^1, s_i^2, s^3],$$

with $\alpha \in [0, 1]$. If we show that such signals (s_i^2, s^3) exists, this directly implies that we must also have that $a_i = \mathbb{E}[\theta_i + rA | s_i^1, s_i^2, s^3]$. Thus, $(\theta_i, \bar{\theta}, a_i, A)$ is induced by the linear Bayes Nash equilibrium when players receive information structure $\{s_i^1, s_i^2, s^3\}$. For this we define ε_i^2 and $\bar{\varepsilon}$ in terms of $\tilde{\varepsilon}_i$ as follows:

$$\bar{\varepsilon} = \mathbb{E}_i[\tilde{\varepsilon}_i], \quad \varepsilon_i^2 = \Delta\tilde{\varepsilon}_i = (\tilde{\varepsilon}_i - \mathbb{E}_i[\tilde{\varepsilon}_i]).$$

Note that by (68) we know that:

$$\mathrm{cov}(\tilde{a}_i, \tilde{s}_i) = (1 - (1 - r)b)\,\mathrm{cov}(\bar{\theta}, \tilde{s}_i) + r\,\mathrm{cov}(\tilde{A}, \tilde{s}_i).$$

Yet, we can re-write the previous equality using the definition of $\tilde{s}_i$:

$$\mathrm{cov}(\tilde{a}_i, \bar{\theta} + \varepsilon_i^2 + \bar{\varepsilon}) = (1 - (1 - r)b)\,\mathrm{cov}(\bar{\theta}, \bar{\theta} + \varepsilon_i^2 + \bar{\varepsilon}) + r\,\mathrm{cov}(\tilde{A}, \bar{\theta} + \varepsilon_i^2 + \bar{\varepsilon}).$$

Yet, we can find α such that the following conditions hold:

$$\mathrm{cov}(\tilde{a}_i, \alpha\bar{\theta} + \varepsilon_i^2) = (1 - (1 - r)b)\,\mathrm{cov}(\bar{\theta}, \alpha\bar{\theta} + \varepsilon_i^2) + r\,\mathrm{cov}(\tilde{A}, \alpha\bar{\theta} + \varepsilon_i^2) \ ;$$

$$\text{cov}(\tilde{a}_i, (1-\alpha)\bar{\theta} + \bar{\varepsilon}) = (1-(1-r)b)\,\text{cov}(\bar{\theta}, (1-\alpha)\bar{\theta} + \bar{\varepsilon}) + r\,\text{cov}(\tilde{A}, (1-\alpha)\bar{\theta} + \bar{\varepsilon}).$$

Thus, the following conditions must hold:

$$\text{cov}(\tilde{a}_i, s_i^2) = (1-(1-r)b)\,\text{cov}(\bar{\theta}, s_i^2) + r\,\text{cov}(\tilde{A}, s_i^2)\ ; \tag{69}$$

$$\text{cov}(\tilde{a}_i, s_i^3) = (1-(1-r)b)\,\text{cov}(\bar{\theta}, s_i^3) + r\,\text{cov}(\tilde{A}, s_i^3). \tag{70}$$

Since (68) holds, we must have that:

$$\text{cov}(\tilde{a}_i, s_i^1) = (1-(1-r)b)\,\text{cov}(\bar{\theta}, s_i^1) + r\,\text{cov}(\tilde{A}, s_i^1), \tag{71}$$

thus, using the same argument as before, we know that (69)-(71) imply that:

$$\tilde{a}_i = \mathbb{E}[(1-(1-r)b)\bar{\theta} + r\tilde{A}|s_i^1, s_i^2, s^3].$$

Thus, we get the result. ∎

Proof of Proposition 16. We first prove that in any Bayes correlated equilibrium that achieves $(\bar{\rho}(\rho_{aa}), \rho_{aa})$ subject to (63)-(65) must satisfy the following two conditions. If $b < 1$ then both inequalities in (65) must be satisfied with equality. If $b = 1$ then at least one of the inequalities in (65) must be satisfied with equality.

From condition (64), we know that:

$$\rho_{a\phi} = \frac{b}{1 - b(1-r)}\Big(\frac{\rho_{a\theta}}{b} - \frac{1 - r\rho_{aa}}{\rho_{a\theta}}\Big).$$

Replacing $\rho_{a\phi}$ in both inequalities in (65), we get the following two inequalities:

$$\rho_{aa}\rho_{\theta\theta} \geq \Big(\frac{b}{1 - b(1-r)}\Big)^2\Big(\frac{\rho_{a\theta}}{b} - \frac{1 - r\rho_{aa}}{\rho_{a\theta}}\Big)^2. \tag{72}$$

$$(1 - \rho_{aa})(1 - \rho_{\theta\theta}) \geq \frac{b^2}{(1 - b(1-r))^2}\Big((1-r)\rho_{a\theta} - \frac{1 - r\rho_{aa}}{\rho_{a\theta}}\Big)^2. \tag{73}$$

Note that we need to maximize $\rho_{a\theta}$ given the previous two inequalities. Thus, it is clear that at least one of these inequalities must be binding. Thus, inequality (72) or (73) must be binding (it is easy to check that there exists values of $\rho_{a\theta}$ such that either inequality is strict). Thus, at least one of the restrictions must always be satisfied with equality. We now show that, if $b \in (0, 1)$, then both inequalities must be satisfied with equality. To show this just note that the derivative of the right hand side of the inequalities with respect to b is different than 0. Thus, if just one of the

constraints is binding and the other one is not, then one can change b and relax both constraints, which allow to get a higher $\rho_{a\theta}$. Note that the argument also holds for $b = 0$, as in this case the second inequality will be satisfied with slack, while the right hand side of the first inequality will be decreasing with respect to b. Thus, we must have that at the maximum $\rho_{a\theta}$ both constraints are binding or $b = 1$.

By the previous argument, $\overline{\rho}_{a\theta}$ is achieved by $b = 1$ or both (72) and (73) are satisfied with equality. If $b = 1$ then agents know their own payoffs. By Proposition 3, if both (72) and (73) are satisfied with equality, then the information structure that achieves this correlations must satisfy that $\sigma_{\varepsilon 2}, \sigma_{\varepsilon 3} \in \{0, \infty\}$. Thus, we can calculate $\overline{\rho}_{a\theta}$ using Propositions 13-14. ∎

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