

**FINANCIAL INNOVATION, COLLATERAL AND INVESTMENT**

**By**

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**July 2013**

**Revised March 2015**

**COWLES FOUNDATION DISCUSSION PAPER NO. 1903R**



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# Financial Innovation, Collateral and Investment.

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January, 2015.

## Abstract

Financial innovations that change how promises are collateralized can affect investment, even in the absence of any change in fundamentals. In *C*-models, the ability to leverage an asset always generates over-investment compared to Arrow Debreu. The introduction of CDS always leads to under-investment with respect to Arrow Debreu, and in some cases even robustly destroys competitive equilibrium. The need for collateral would seem to cause under-investment. Our analysis illustrates a countervailing force: goods that serve as collateral yield additional services and are therefore over-valued and over-produced. In models without cash flow problems there is never marginal under-investment on collateral.

**Keywords:** Financial Innovation, Collateral, Investment, Repayment Enforceability Problems, Cash Flow Problems, Leverage, CDS, Non-Existence, Marginal Efficiency.

**JEL Codes:** D52, D53, E44, G01, G10, G12.

## 1 Introduction

After the recent subprime crisis and the sovereign debt crisis in the euro zone, many observers have placed financial innovations such as leverage and credit default swaps

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(CDS) at the root of the problem.<sup>1</sup> Figure 1 shows how the financial crisis in the US was preceded by years in which leverage, prices and investment increased dramatically in the housing market and all collapsed together after the crisis. Figure 2 shows that CDS was a financial innovation introduced much later than leverage. Figure 3 shows how the peak in CDS volume coincides with the crisis and the crash in prices and investment.<sup>2</sup>

The goal of this paper is to study the effect of financial innovation on prices and investment. The main result is that financial innovation, such as leverage and CDS, can affect prices and investment, even in the absence of any changes in fundamentals such as preferences, production technologies or asset payoffs. Moreover, our results provide precise predictions on the direction of these changes.

The central element of our analysis is *repayment enforceability problems*: we suppose that agents cannot be coerced into honoring their promises except by seizing collateral agreed upon by contract in advance. Agents need to post collateral in order to issue promises. We define financial innovation as the use of new kinds of collateral, or new kinds of promises that can be backed by collateral. In the incomplete markets literature, financial innovations were modeled by securities with new kinds of payoffs. Financial innovations of this kind do have an effect on asset prices and real allocations, but the direction of the consequences is typically ambiguous and therefore has not been much explored. When we model financial innovation taking into account collateral, we can prove unambiguous results.

In the first part of our analysis we focus on a special class of models, which we call *C*-models, introduced by Geanakoplos (2003).<sup>3</sup> These economies are complex enough to allow for the possibility that financial innovation can have a big effect on prices and investment. But they are simple enough to be tractable and to generate unambiguous (as well as intuitive) results that we now describe.

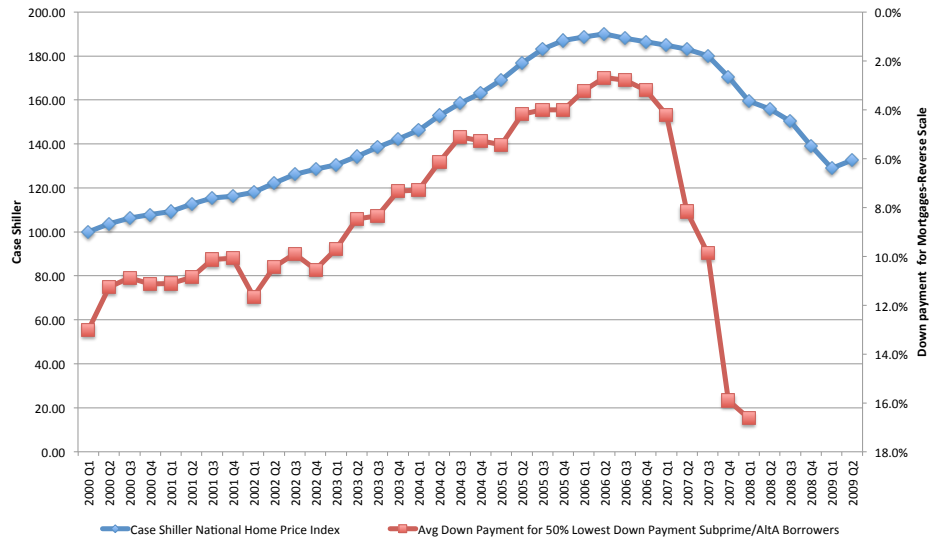
First we suppose that financial innovation has enabled agents to issue non-contingent promises using the risky asset as collateral, but not to sell short or to issue contingent

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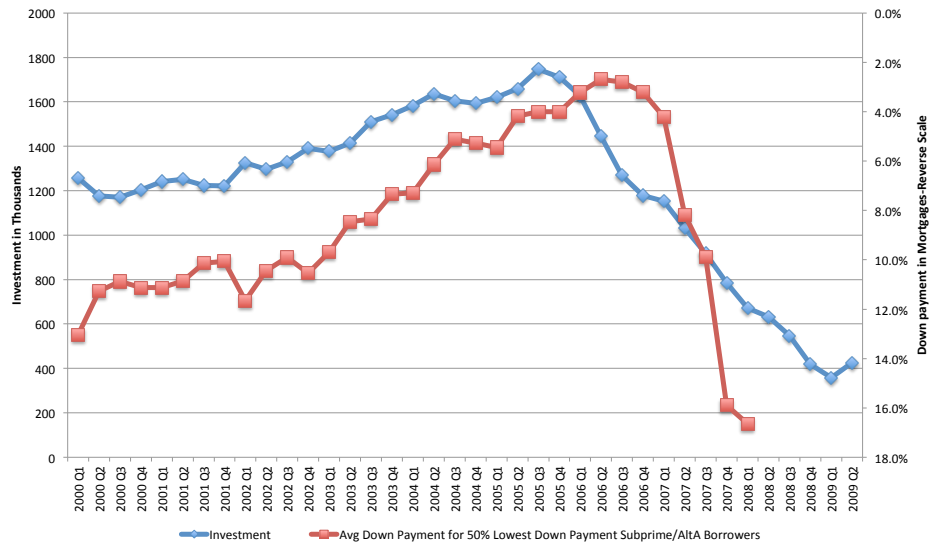
<sup>1</sup>See for example Brunneimeier (2009), Geanakoplos (2010), Gorton (2009) and Stultz (2009). Geanakoplos (2003) and Fostel and Geanakoplos (2008) wrote before the crisis.

<sup>2</sup>The available numbers on CDS volumes are not specific to mortgages, since most CDS were over the counter, but the fact that subprime CDS were not standardized until late 2005 suggests that the growth of mortgage CDS in 2006 is likely even sharper than Figure 3 suggests.

<sup>3</sup>*C*-economies have two states of nature and a continuum of risk neutral agents. Except for period 0, consumption is entirely derived from asset dividends.

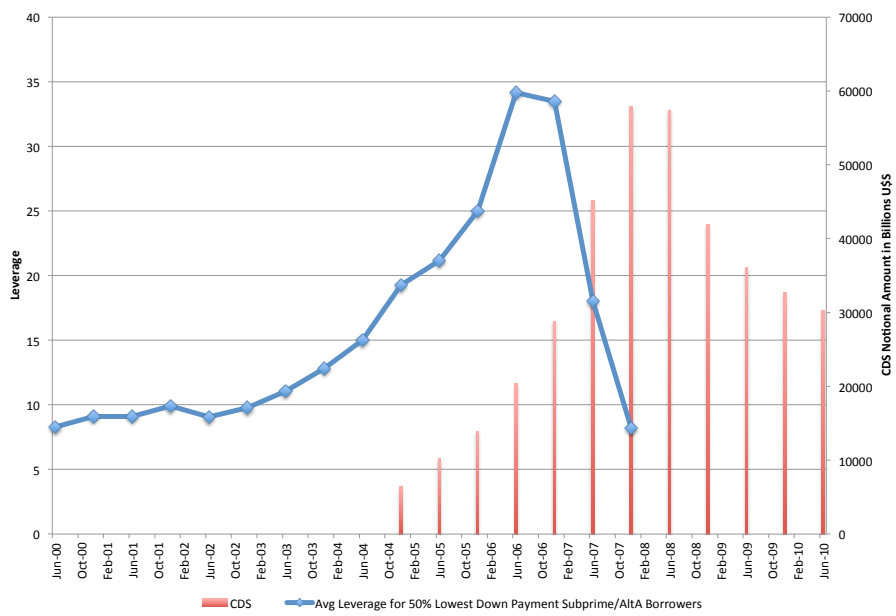


Note: Observe that the Down Payment axis has been reversed, because lower down payment requirements are correlated with higher home prices. For every AltA or Subprime first loan originated from Q1 2000 to Q1 2008, down payment percentage was calculated as appraised value (or sale price if available) minus total mortgage debt, divided by appraised value. For each quarter, the down payment percentages were ranked from highest to lowest, and the average of the bottom half of the list is shown in the diagram. This number is an indicator of down payment required: clearly many homeowners put down more than they had to, and that is why the top half is dropped from the average. A 13% down payment in Q1 2000 corresponds to leverage of about 7.7, and 2.7% down payment in Q2 2006 corresponds to leverage of about 37. Note Subprime/AltA Issuance Stopped in Q1 2008. Source: Geanakoplos (2010).



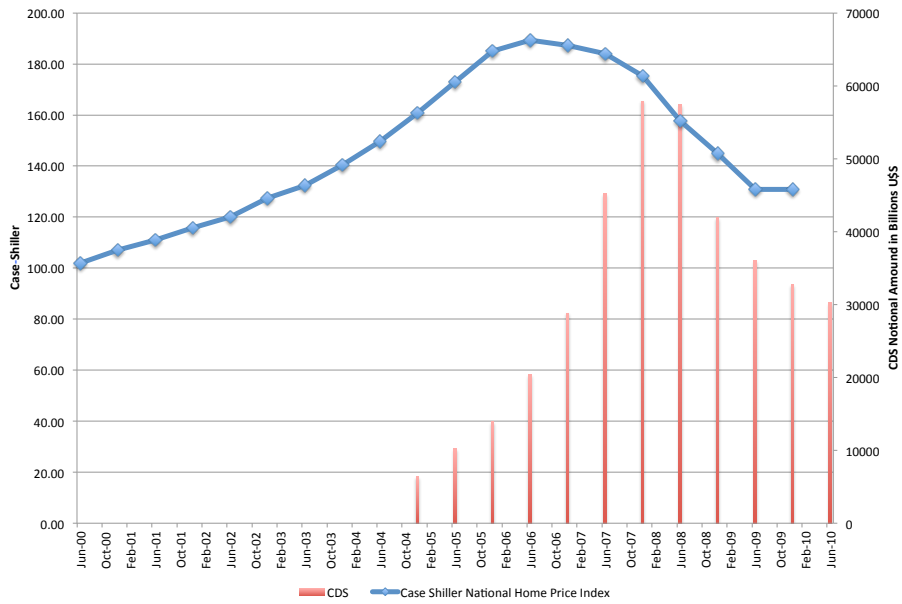
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Figure 1: Top Panel: Leverage and Prices. Bottom Panel: Leverage and Investment.

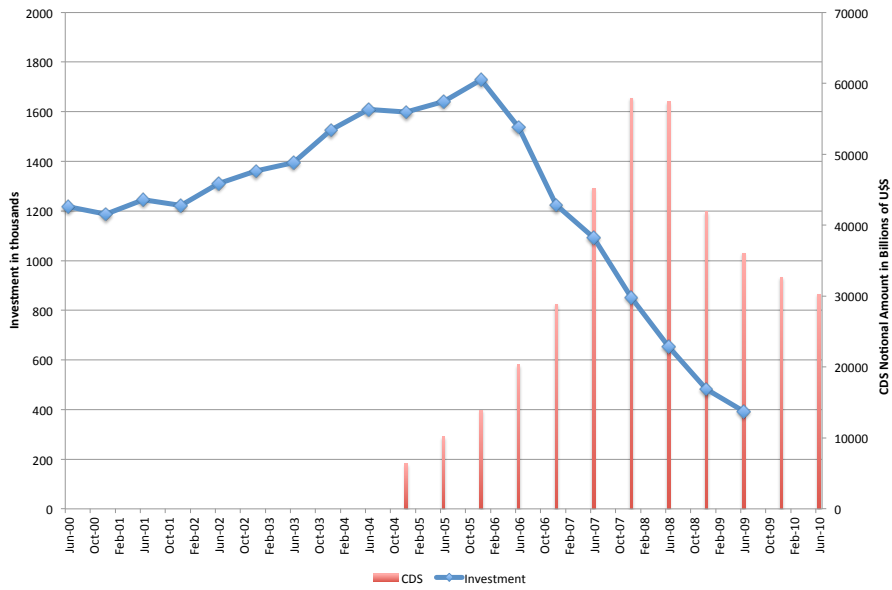


Source CDS: IBS OTC Derivatives Market Statistics

Figure 2: Leverage and Credit Default Swaps



Source CDS: IBS OTC Derivatives Market Statistics



Source CDS: IBS OTC Derivatives Market Statistics.  
 Source Investment: Construction new privately owned housing units completed.  
 Department of Commerce.

Figure 3: Top Panel: CDS and Prices. Bottom Panel: CDS and Investment.

promises. We show that this ability to leverage an asset generates over-investment compared to the Arrow-Debreu level. This over-investment result also holds with a finite number of risk averse agents ( $C^*$ -models), provided that production displays constant returns to scale. Under the same conditions, we show that the leverage economy is Pareto dominated by the Arrow Debreu allocation.

Second, into the previous leverage economy we introduce CDS on the risky asset collateralized by the riskless asset. We show that equilibrium aggregate investment dramatically falls not only below the initial leverage level but beneath the Arrow Debreu level. However, in this case we cannot establish unambiguous welfare results for the CDS economy.

Finally, taking our logic to the extreme, we show that the creation of CDS may in fact destroy equilibrium by choking off all production. CDS is a derivative, whose payoff depends on some underlying instrument. The quantity of CDS that can be traded is not limited by the market size of the underlying instrument.<sup>4</sup> If the volume of the underlying security diminishes, the CDS trading may continue at the same high levels. But when the volume of the underlying instrument falls to zero, CDS trading must come to an end by definition. This discontinuity can cause robust non-existence.

We prove all these results both algebraically and by way of a diagram. One novelty in the paper is an Edgeworth Box diagram for trade with a continuum of agents with heterogeneous but linear preferences.

Our over-investment result may seem surprising to the reader, since it stands in contrast with the traditional macroeconomic/corporate finance literature with financial frictions such as in Bernanke and Gertler (1989) and Kiyotaki and Moore (1997). In these papers financial frictions generate under-investment with respect to Arrow Debreu. Their result may appear intuitive since one would expect that the need for collateral would prevent some investors from borrowing the money to invest, thus reducing production. In our model borrowers may also find themselves constrained: they cannot borrow more at the same interest rate on the same collateral. Yet we show that in  $C$  and in  $C^*$ -models there is never under-investment with respect to Arrow Debreu. There are two reasons for the discrepancy. First, the traditional literature did not recognize (or at least did not sufficiently emphasize) the collateral value

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<sup>4</sup>Currently the outstanding notional value of CDS in the United States is far in excess of \$50 trillion, more than three times the value of their underlying asset.

of assets that can back loans. Precisely because agents are constrained in what they can borrow, they will overvalue commodities that can serve as collateral (compared to perishable consumption goods or other commodities that cannot), which might lead to over-production of these collateral goods. The second reason for the discrepancy is that in the macro/corporate finance models, it is assumed that borrowers cannot pledge the whole future value of the assets they produce. In other words, these papers are explicitly considering what we here call *cash flow problems*.<sup>5</sup> In our model we completely abstract from collateral cash flow problems and assume that all of the future value of investment can be pledged: every agent knows exactly how the future cash flow depends on the exogenous state of nature, independent of how the investment was financed. This eliminates any issues associated with hidden effort or unobservability. When we disentangle the *cash flow problems* from the *repayment enforceability problems* we get the opposite result: there can be over-investment even when agents are constrained in their borrowing. With our modeling strategy we expose a countervailing force in the incentives to produce: when only some assets can be used as collateral, they become relatively more valuable, and are therefore produced more.<sup>6</sup>

Needless to say, it is impossible to draw unambiguous conclusions about financial innovation across all general equilibrium models. But we indicate how our analysis exposes forces which push in the direction we describe. Leverage allows the purchase of the asset to be divided between two kinds of buyers, the optimists who hold the residual, which pays off exclusively in the good state, and the general public who holds the riskless piece that pays the same in both states. By dividing up the risky asset payoffs into two different kinds of assets, attractive to two different clienteles, demand is increased. To put the same idea differently, the buyers of the asset are willing to pay more for it (or buy more of it) because they can sell off a riskless piece of it for a price above their own valuation of the riskless payoffs. This gives the

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<sup>5</sup>In Kiyotaki and Moore (1997), the lender cannot confiscate the fruit growing on the land but just the land. Other examples of cash flow problems are to be found in corporate finance asymmetric information models such as Holmstrom and Tirole (1997), Adrian and Shin (2010), and Acharya and Viswanathan (2011). The idea in this literature is that collateral payoffs deteriorate if too much money is borrowed, because then the owner has less incentive to work hard to obtain good cash flows.

<sup>6</sup>It follows that one way to move from over-investment to under-investment is to suppose that some good could be fully collateralized at one point, and then becomes prohibited from being used as collateral at another. Many subprime mortgages went from being prominent collateral on Repo in 2006 to being not accepted as collateral in 2009.



risky asset an additional collateral value, beyond its payoff value. Agents have more incentive to produce goods that are better collateral.

CDS decreases investment in the risky asset because the seller of CDS is effectively making the same kind of investment as the buyer of the leveraged risky asset: she obtains a portfolio of the riskless asset as collateral and the CDS obligation, which on net pays off precisely when the asset does very well, just like the leveraged purchase. The creation of CDS thus lures away many potential leveraged purchasers of the risky asset. More generally, CDS can be thought of as a sophisticated tranching of the riskless asset, since cash is generally used as collateral for sellers of CDS. This tends to raise demand for the riskless asset, thereby reducing the production of risky asset.

When restricting ourselves to a special class of models ( $C$  and  $C^*$ -models) we can generate sharp results. However, results comparing collateral equilibrium with Arrow Debreu equilibrium are bound not to be general.<sup>7</sup> In the special case of two states we exploit the fact that for leverage economies there are always state prices that can value all the securities even though short selling is forbidden, and in CDS economies we exploit the fact that writing a CDS is tantamount to purchasing the asset with maximal leverage. With three or more states neither fact holds.<sup>8</sup> For this reason, in the second part of our analysis we identify a completely general phenomenon, which applies to any commodity that can serve as collateral for any kind of promise, provided there are no cash flow problems. We replace the Arrow Debreu benchmark with a local concept of efficiency. If agents are really under-investing because they are borrowing constrained, then if presented with a little bit of extra money to make a purely cash purchase, they should invest. Yet we prove in a general model with arbitrary preferences and states of nature that none of them would choose to produce more of any good that can be used as collateral, even if they were also given access to the best technology available in the economy. Thus without cash flow problems, repayment enforceability problems can lead to marginal over-investment, but never

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<sup>7</sup>For instance, with risk neutral agents, if we change endowments in the future, collateral equilibrium would not change, since future endowments cannot be used as collateral, but the Arrow Debreu equilibrium would. In  $C$ -models we suppose that all future consumption is derived from dividends of assets existing from the beginning.

<sup>8</sup>We conjecture that for a suitable extension of  $C$ -models to multiple states, leverage investment would also be greater than Arrow Debreu investment, and that the introduction of CDS would reduce investment. But the proof would have to be radically altered and is beyond the scope of this paper.

marginal under-investment. In  $C$  and  $C^*$ -models the marginal over-investment is big enough to exceed the Arrow Debreu level.

In this paper we follow the model of collateral equilibrium developed in Geanakoplos (1997, 2003, 2010), Fostel-Geanakoplos (2008, 2012a and 2012b, 2014a, 2014b), and Geanakoplos-Zame (2014). Geanakoplos (2003) showed that leverage can raise asset prices. Geanakoplos (2010) and Che and Sethi (2011) showed that in the kind of models studied by Geanakoplos (2003), CDS can lower risky asset prices. Fostel-Geanakoplos (2012b) showed more generally how different kinds of financial innovations can have big effects on asset prices. In this paper we move a step forward and show that financial innovation affects investment as well.

Our model is related to a literature on financial innovation pioneered by Allen and Gale (1994), though in our paper financial innovation is taken as given, and concerns collateral. There are other macroeconomic models with financial frictions such as Kilenthong and Townsend (2011) that produce over-investment in equilibrium. The underlying mechanism in these papers is very different from the one presented in our paper. In those papers the over-investment is due to an externality through changing relative prices in the future states. Our results do not rely on relative price changes in the future, and to make the point clear we restrict our  $C$  and  $C^*$ -models to a single consumption good in every future state. Our paper is also related to Polemarchakis and Ku (1990). They provide a robust example of non-existence in a general equilibrium model with incomplete markets due to the presence of derivatives. Existence was proved to be generic in the canonical general equilibrium model with incomplete markets and no derivatives by Duffie and Shaffer (86). Geanakoplos and Zame (1997, 2014) proved that equilibrium always exists in pure exchange economies even with derivatives if there is a finite number of potential contracts, with each requiring collateral. Thus the need for collateral to enforce deliveries on promises eliminates the non-existence problem in pure exchange economies with derivatives such as in Polemarchakis-Ku. Our paper gives a robust example of non-existence in a general equilibrium model with incomplete markets with collateral, production, and derivatives. Thus the non-existence problem emerges again with derivatives and production, despite the collateral.

The paper is organized as follows. Section 2 presents the collateral general equilibrium model and the special class of  $C$  and  $C^*$ -models. Section 3 presents numerical

examples and our propositions for the  $C$  and  $C^*$ -models. Section 4 characterizes the equilibrium for different financial innovations and uses Edgeworth boxes to give geometrical proofs of the propositions in Section 3. Section 5 discusses the non-existence result. Section 6 introduces the notion of marginal efficiency and presents the marginal over-investment result in the general model of Section 2. The Appendix presents algebraic proofs.

## 2 Collateral General Equilibrium Model

In this section we present the collateral general equilibrium model and a special class of collateral models introduced by Geanakoplos (2003), that we call the  $C$ -model and  $C^*$ -model, which will be extensively used in the paper.

### 2.1 Time and Commodities

We consider a two-period general equilibrium model, with time  $t = 0, 1$ . Uncertainty is represented by different states of nature  $s \in S$  including a root  $s = 0$ . We denote the time of  $s$  by  $t(s)$ , so  $t(0) = 0$  and  $t(s) = 1, \forall s \in S_T$ , the set of terminal nodes of  $S$ . Suppose there are  $L_s$  commodities in  $s \in S$ . Let  $p_s \in R_+^{L_s}$  the vector of commodity prices in each state  $s \in S$ .

### 2.2 Agents

Each investor  $h \in H$  is characterized by Bernoulli utilities,  $u_s^h, s \in S$ , a discount factor,  $\beta_h$ , and subjective probabilities,  $\gamma_s^h, s \in S_T$ . The utility function for commodities in  $s \in S$  is  $u_s^h : R_+^{L_s} \rightarrow R$ , and we assume that these state utilities are differentiable, concave, and weakly monotonic (more of every good in any state strictly improves utility). The expected utility to agent  $h$  is:

$$U^h = u_0^h(x_0) + \beta_h \sum_{s \in S_T} \gamma_s^h u_s^h(x_s). \quad (1)$$

Investor  $h$ 's endowment of the commodities is denoted by  $e_s^h \in R_+^{L_s}$  in each state  $s \in S$ .

## 2.3 Production

For each  $s \in S$  and  $h \in H$ , let  $Z_s^h \subset \mathbb{R}^{L_s}$  denote the set of feasible intra-period production for agent  $h$ . Commodities can enter as inputs and outputs of the intra-period production process; inputs appear as negative components  $z_l < 0$  of  $z \in Z_s^h$ , and outputs as positive components  $z_l > 0$  of  $z \in Z_s^h$ . We assume that  $Z_s^h$  is convex, compact and that  $0 \in Z_s^h$ .

We allow for inter-period production too. For each  $h \in H$ , let  $F^h : \mathbb{R}_+^{L_0} \rightarrow \mathbb{R}_+^{S_T L_s}$  be a linear inter-period production function connecting a vector of commodities  $x_0$  at state  $s = 0$  with the vector of commodities  $F_s^h(x_0)$  it becomes in each state  $s \in S_T$ .

Production enables our model to include many different kinds of commodities. Commodities could either be perishable consumption goods (like food), or durable consumption goods (like houses), or they could represent assets (like Lucas trees) that pay dividends. The holder of a durable consumption good can enjoy current utility as well as the prospect of the future realization of the goods (either by consuming them or selling them). The buyer of a durable asset can expect the income from future dividends.

## 2.4 Financial Contracts and Collateral

The heart of our analysis involves financial contracts and collateral. We explicitly incorporate *repayment enforceability problems*, but exclude *cash flow problems*. Agents cannot be coerced into honoring their promises except by seizing collateral agreed upon by contract in advance. Agents need to post collateral in the form of durable assets in order to issue promises. But there is no doubt what the collateral will pay, conditional on the future state of nature.

A *financial contract*  $j$  promises  $j_s \in R_+^{L_s}$  commodities in each final state  $s \in S_T$  backed by collateral  $c_j \in R_+^{L_0}$ . This allows for non contingent promises of different sizes, as well as contingent promises. The price of contract  $j$  is  $\pi_j$ . Let  $\theta_j^h$  be the number of contracts  $j$  traded by  $h$  at time 0. A positive  $\theta_j^h$  indicates agent  $h$  is buying contracts  $j$  or lending  $\theta_j^h \pi_j$ . A negative  $\theta_j^h$  indicates agent  $h$  is selling contracts  $j$  or borrowing  $|\theta_j^h| \pi_j$ .

We wish to exclude cash flow problems, stemming for example from adverse selection

or moral hazard, beyond repayment enforceability problems. Accordingly we eliminate adverse selection by restricting the sale of each contract  $j$  to a set  $H(j) \subset H$  of traders with the same durability functions,  $F^h(c_j) = F^{h'}(c_j)$  if  $h, h' \in H(j)$ . Since we assumed that the maximum borrowers can lose is their collateral if they do not honor their promise, the actual delivery of contract  $j$  in states  $s \in S_T$  is

$$\delta_s(j) = \min\{p_s \cdot j_s, p_s \cdot F_s^{H(j)}(c_j)\} \quad (2)$$

Notice that there are no cash flow problems: the value of the collateral in each future state does not depend on the size of the promise, or on what other choices the seller  $h \in H(j)$  makes, or on who owns the asset at the very end. This eliminates any issues associated with hidden effort or unobservability.

A final hypothesis we will make to eliminate cash flow problems is to suppose that promises are not artificially limited. We suppose that if  $c_j$  is the collateral for some contract  $j$ , then there is a “large” contract  $j'$  with  $c_{j'} = c_j$  and  $H(j') = H(j)$  and  $j'_s \geq F_s^{H(j)}(c_j)$  for all  $s \in S_T$ .

## 2.5 Budget Set

Given commodity and debt contract prices  $(p, (\pi_j)_{j \in J})$ , each agent  $h \in H$  chooses production,  $z_s$ , and commodities,  $x_s$ , for each  $s \in S$ , and contract trades,  $\theta_j$ , at time 0, to maximize utility (1) subject to the budget set defined by

$$B^h(p, \pi) = \{(z, x, \theta) \in R^{SL_s} \times R_+^{SL_s} \times (R^J) :$$

$$p_0 \cdot (x_0 - e_0^h - z_0) + \sum_{j \in J} \theta_j \pi_j \leq 0$$

$$p_s \cdot (x_s - e_s^h - z_s) \leq F_s^h(x_0) + \sum_{j \in J} \theta_j \min\{p_s \cdot j_s, p_s \cdot F_s^{H(j)}(c_j)\}, \forall s \in S_T$$

$$z_s \in Z_s^h, \forall s \in S$$

$$\theta_j < 0 \text{ only if } h \in H(j)$$

$$\sum_{j \in J} \max(0, -\theta_j) c_j \leq x_0, \forall l\}.$$

The first inequality requires that money spent on commodities beyond the revenue from endowments and production in state 0 be financed out of the sale of contracts. The second inequality requires that money spent on commodities beyond the revenue

from endowments and production in any state  $s \in S_T$  be financed out of net revenue from dividends from contracts bought or sold in state 0. The third constraint requires that production is feasible, the fourth constraint requires that only agents  $h \in H(j)$  can sell contract  $j$ , and the last constraint requires that agent  $h$  actually holds at least as much of each good as she is required to post as collateral.

## 2.6 Collateral Equilibrium

A *Collateral Equilibrium* is a set of commodity prices, contract prices, production and commodity holdings and contract trades  $((p, \pi), (z^h, x^h, \theta^h)_{h \in H}) \in R_+^{SL_s} \times R_+^J \times (R^{SL_s} \times R_+^{SL_s} \times R^J)^H$  such that

1.  $\sum_{h \in H} (x_0^h - e_0^h - z_0^h) = 0.$
2.  $\sum_{h \in H} (x_s^h - e_s^h - z_s^h - F_s^h(x_0^h)) = 0, \forall s \in S_T.$
3.  $\sum_{h \in H} \theta_j^h = 0, \forall j \in J.$
4.  $(z^h, x^h, \theta^h) \in B^h(p, \pi), \forall h$   
 $(z, x, \theta) \in B^h(p, \pi) \Rightarrow U^h(x) \leq U^h(x^h), \forall h.$

Markets for consumption in state 0 and in states  $s \in S_T$  clear, as do contract markets. Furthermore, agents optimize their utility in their budget set. Geanakoplos and Zame (1997) show that collateral equilibrium always exists.

## 2.7 Financing Investment

Let us pause for a moment to consider three possible interpretations of how investment is financed in our model.

In the first interpretation, a firm is defined by intra-period production. The firm sells its output in advance to the buyers, and then uses the proceeds to buy the inputs needed to produce the output, just like a home builder who lines up the owner before she begins construction. In this interpretation, we emphasize consumer durables and the collateral constraint affecting the consumer. The firm does not directly face any

financing restrictions, but the fact that the consumer does, indirectly affects the firm's investment decision.

In the second interpretation, production takes two periods, and  $F^h$  does not depend on  $h$ . A firm is characterized by  $F \circ Z_0^h$ . In this interpretation, the firm founder  $h$  finances her purchase of inputs  $\max(0, -z_0^h)$  by selling shares once her production plans  $\max(0, z_0^h) = \lambda c_j$  are irrevocably in place, where  $\lambda > 0$  and  $c_j$  is the collateral for a contract  $j$ . The buyers of shares can in turn finance their purchase with cash and by issuing financial contracts using the firm shares as collateral. If  $Z^h$  is strictly convex, the original owner can make a profit from her sale of shares.

The third interpretation is the same as the second, except that now we allow  $F^h$  to depend on  $h$ . Now  $h$  is the sole equity holder in the firm, so this interpretation requires  $z_0^+ \equiv \max(0, z_0^h) \leq x_0^h$ . The firm can issue debt by selling contracts. The intra-period output could be interpreted as intangible but irrevocable plans to produce. Once these plans are in place there is no doubt about the future output  $F^h(z_0^+)$ . Now the firm itself is the collateral for any borrowing.

## 2.8 $C$ -economies and $C^*$ -economies

The  $C$ -model is defined as follows. We consider a binary tree, so that  $S = \{0, U, D\}$ . In states  $U$  and  $D$  there is a single commodity, called the consumption good, and in state 0 there are two commodities, called assets  $X$  and  $Y$ . We take the price of the consumption good in each state  $U, D$  to be 1 and the price of  $X$  to be 1 at 0. We denote the price of asset  $Y$  at time 0 by  $p$ . The riskless asset  $X$  yields dividends  $d_U^X = d_D^X = 1$  unit of the consumption good in each state, and the risky asset  $Y$  pays  $d_U^Y$  units of the consumption good in state  $U$  and  $0 < d_D^Y < d_U^Y$  units of the consumption good in state  $D$ .

Inter-period production is defined as  $F_U^h(X, Y) = F_U(X, Y) = d_U^X X + d_U^Y Y = X + d_U^Y Y$  and  $F_D^h(X, Y) = F_D(X, Y) = d_D^X X + d_D^Y Y = X + d_D^Y Y$ . Since inter-period production is the same for each agent, we take  $H(j) = H$  for all contracts  $j \in J$ . The intra period technology at 0,  $Z_0^h = Z_0 \subset \mathbb{R}^2$ , is also the same for all agents, and allows each of them to invest the riskless asset  $X$  and produce the risky asset  $Y$ . Denote by  $\Pi^h = z_x + pz_y$  the profits associated to production plan  $(z_x, z_y)$ .

There is a continuum of agents  $h \in H = [0, 1]$ .<sup>9</sup> Each agent is risk neutral with subjective probabilities,  $(\gamma_U^h, \gamma_D^h = 1 - \gamma_U^h)$  and does not discount the future. The expected utility to agent  $h$  is  $U^h(X, Y, x_U, x_D) = \gamma_U^h x_U + \gamma_D^h x_D$ . Agents get no utility from holding the assets  $X$  and  $Y$ . We assume that  $\gamma_U^h$  is strictly increasing and continuous in  $h$ . If  $\gamma_U^h > \gamma_U^{h'}$  we shall say that agent  $h$  is more optimistic (about state  $U$ ) than agent  $h'$ . Finally, each agent  $h \in H$  has an endowment  $x_{0^*}$  of  $X$  at time 0, and no other endowment.<sup>10</sup>

Finally we define the  $C^*$ -model as a  $C$ -model where the number of agents can be finite or infinite, and utilities  $U^h(X, Y, x_U, x_D) = \gamma_U^h u^h(x_U) + \gamma_D^h u^h(x_D)$  allow for different attitudes toward risk in terminal consumption.

The set  $J$  of contracts is defined in the next section.

### 3 Investment and Welfare relative to First Best in $C$ and $C^*$ Models

In this section we present our propositions regarding investment and welfare in  $C$  and  $C^*$ -models. In Section 4 we analyze the equilibria corresponding with different financial innovations more closely and provide geometrical proofs of the results when possible.

#### 3.1 Financial Innovation and Collateral

A vitally important source of financial innovation involves the possibility of using assets and firms as collateral to back promises. Financial innovation in our model is described by a different set  $J$ . We shall always write  $J = J^X \cup J^Y$ , where  $J^X$  is the set of contracts backed by one unit of  $X$  and  $J^Y$  is the set of contracts backed by one unit of  $Y$ .

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<sup>9</sup>We suppose that agents are uniformly distributed in  $(0, 1)$ , that is they are described by Lebesgue measure.

<sup>10</sup>The hypothesis that agents have no endowment of  $Y$  is not needed for the five propositions on investment and welfare in Section 4, but it is required for the non-existence result in Section 5.



### 3.1.1 Leverage: L-economy

The first type of financial innovation we focus on is leverage. Consider an economy in which agents can leverage asset  $Y$ . That is, agents can issue non-contingent promises of the consumption good using the risky asset as collateral. In this case  $J = J^Y$ , and each contract  $j$  uses one unit of asset  $Y$  as collateral and promises  $(j, j)$  units of consumption in the two states  $U, D$ , for all  $j \in J = J^Y$ . We call this the  $L$ -economy.

Let us briefly describe the equilibrium. Since  $Z_0^h = Z_0$  is convex, without loss of generality we may suppose that every agent chooses the same production plan  $(z_x, z_y)$  and  $\Pi^h = \Pi$ . Since we have normalized the mass of agents to be 1,  $(z_x, z_y)$  is also the aggregate production.

In equilibrium, it turns out that the only contract actively traded is  $j^* = d_D^Y$ . Borrowers are constrained: if they wish to borrow more on the same collateral by selling  $j > j^*$ , they would have to promise sharply higher interest  $j/\pi_j$ .

In equilibrium, there is a marginal buyer  $h_1$  at state  $s = 0$  whose valuation  $\gamma_U^{h_1} d_U^Y + \gamma_D^{h_1} d_D^Y$  of the risky asset  $Y$  is equal to its price  $p$ .<sup>11</sup> The optimistic agents  $h > h_1$  collectively buy all the risky asset  $z_y$  produced in the economy, financing this with debt. The optimists leverage the risky asset, that is, they buy  $Y$  and sell the riskless contract  $j^*$ , at a price of  $\pi_{j^*}$ , using the asset as collateral. In doing so, they are effectively buying the Arrow security that pays in the  $U$  state (since at  $D$ , their net payoff after debt repayment is 0). The pessimistic agents  $h < h_1$  buy all the remaining safe asset and lend to the optimist agents. Figure 4 shows the equilibrium regime.

### 3.1.2 CDS-economy

The second type of financial innovation we consider is a Credit Default Swap on the risky asset  $Y$ . A Credit Default Swap (CDS) on the asset  $Y$  is a contract that promises to pay 0 when  $Y$  pays  $d_U^Y$ , and promises  $d_U^Y - d_D^Y$  when  $Y$  pays only  $d_D^Y$ . CDS is a derivative, since its payoffs depend on the payoff of the underlying asset  $Y$ . A seller of a CDS must post collateral, typically in the form of money. In a two-period model, buyers of the CDS would insist on at least  $d_U^Y - d_D^Y$  units of  $X$  as collateral. Thus, for every one unit of payment, one unit of  $X$  must be posted as collateral. We can

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<sup>11</sup>This is because of the linear utilities, the continuity of utility in  $h$  and the connectedness of the set of agents  $H$  at state  $s = 0$ .

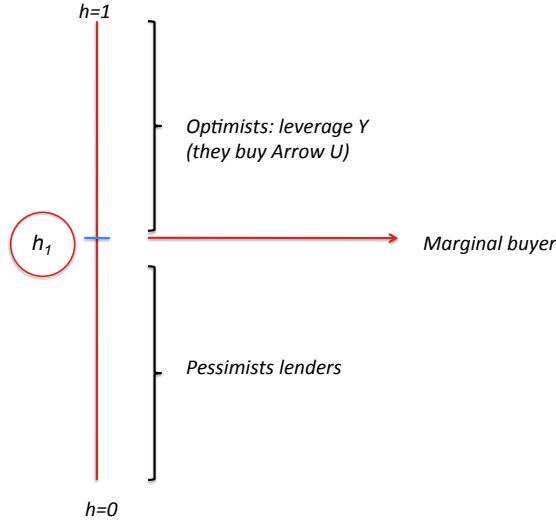


Figure 4: Equilibrium Regime in the  $L$ -economy.

therefore incorporate CDS into our economy by taking  $J^X$  to consist of one contract promising  $(0, 1)$ . A very important real world example is CDS on sovereign bonds or on corporate debt. The bonds themselves give a risky payoff and can be leveraged, but not tranching. The collateral for their CDS is generally cash, and not the bonds themselves.<sup>12</sup>

We introduce into the previous  $L$ -economy a CDS, which pays off in the bad state  $D$ , and is collateralized by  $X$ . Thus we take  $J = J^X \cup J^Y$  where  $J^X$  consists of contracts promising  $(0, 1)$  and  $J^Y$  consists of contracts  $(j, j)$  as described in the leverage economy above. We call this the  $CDS$ -economy. Selling a CDS using  $X$  as collateral is like “tranching” the riskless asset into Arrow securities. The holder of  $X$  can get the Arrow  $U$  security by selling the CDS using  $X$  as collateral. Selling a CDS is like selling an Arrow  $D$  security.

As before, we may suppose that every agent chooses the same production plan  $(z_x, z_y)$

<sup>12</sup>A CDS can be “covered” or “naked” depending on whether the buyer of the CDS needs to hold the underlying asset  $Y$ . Notice that holding the asset and buying a CDS is equivalent to holding the riskless bond, which was already available without CDS in the  $L$ -economy. Hence, introducing covered CDS has no effect on the equilibrium above. For this reason in what follows we will focus on the case of naked CDS.

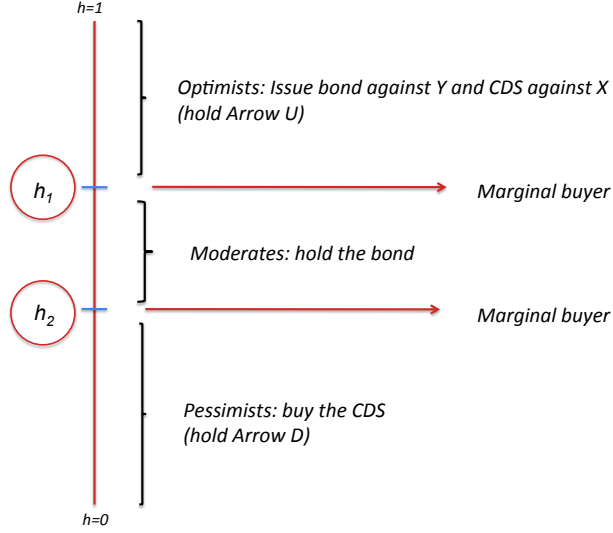


Figure 5: Equilibrium Regime in the *CDS*-economy.

and  $\Pi^h = \Pi$ , and  $(z_x, z_y)$  is also the aggregate production. The equilibrium, however, is more subtle in this case. There are two marginal buyers  $h_1 > h_2$ . Optimistic agents  $h > h_1$  hold all the  $X$  and all the  $Y$  produced in the economy, selling the bond  $j^* = d_D^Y$ , at a price of  $\pi_{j^*}$ , using  $Y$  as collateral and selling CDS, at a price of  $\pi_C$ , using  $X$  as collateral. Hence, they are effectively buying the Arrow  $U$  security (the net payoff net of debt and CDS payment at state  $D$  is zero). Moderate agents  $h_2 < h < h_1$  buy the riskless bonds sold by more optimistic agents. Finally, agents  $h < h_2$  buy the CDS security from the most optimistic investors (so they are effectively buying the Arrow  $D$ ). This regime is described in Figure 5.

### 3.1.3 Arrow Debreu

The Arrow Debreu equilibrium will be our benchmark in Sections 3 and 4. In equilibrium there is a marginal buyer  $h_1$ . All agents  $h > h_1$  use all their endowment and profits from production  $x_{0^*} + \Pi$  and buy all the Arrow  $U$  securities in the economy. Agents  $h < h_1$  instead buy all the Arrow  $D$  securities in the economy. Figure 6 describes the equilibrium regime.

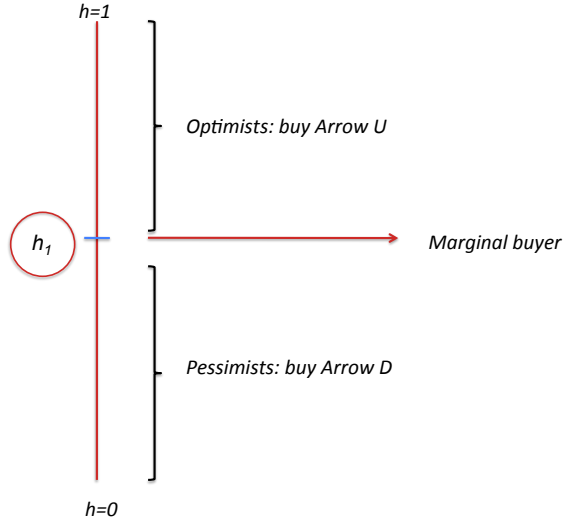


Figure 6: Equilibrium Regime in the Arrow-Debreu Economy.

Collateral equilibrium can implement the Arrow Debreu equilibrium. Consider the economy defined by the set of available financial contracts as follows. We take  $J = J^X \cup J^Y$  where  $J^X$  consists of the single contract promising  $(0, 1)$  and  $J^Y$  consists of a single contract  $(0, d_D^Y)$ . In this case both assets in the economy can be used as collateral to issue the Arrow  $D$  promise, that is, both assets  $X$  and  $Y$  can be perfectly tranced into Arrow securities. Since there are no endowments in the terminal states all the cash flows in the economy get tranced into Arrow  $U$  and  $D$  securities, and hence the collateral equilibrium in this economy is equivalent to the Arrow Debreu equilibrium.

In the remainder of this section we will compare the equilibrium prices, investment and welfare across these economies and present our main results. In Section 4 we will delve into the details of how these different equilibria are characterized and provide intuition as well as geometrical proofs for the results that follow.

## 3.2 Numerical Examples

We first present numerical examples in order to motivate the propositions that follow. Consider a constant returns to scale technology  $Z_0 = \{z = (z_x, z_y) \in R_- \times R_+ : z_y = -kz_x\}$ , where  $k \geq 0$ . Beliefs are given by  $\gamma_U^h = 1 - (1 - h)^2$ , and parameter values are  $x_{0^*} = 1$ ,  $d_U^Y = 1$ ,  $d_D^Y = .2$  and  $k = 1.5$ . Table 1 presents the equilibrium in the three economies we just described.

Table 1: Equilibrium for  $k = 1.5$ .

Arrow Debreu Economy	$L$ -economy	$CDS$ -economy
$q_Y$ 0.6667	$p$ 0.6667	$p$ 0.6667
$q_U$ 0.5833	$h_1$ 0.3545	$\pi_{j^*}$ 0.1904
$q_D$ 0.4167	$z_x$ -0.92	$\pi_C$ 0.4046
$h_1$ 0.3545	$z_y$ 1.38	$h_1$ 0.3880
$z_x$ -0.2131		$h_2$ 0.3480
$z_y$ 0.3197		$z_x$ -0.14
		$z_y$ 0.2

Notice that investment is the highest in the  $L$ -economy and is the lowest in the  $CDS$ -economy. Figure 7 reinforces the results showing total investment in  $Y$ ,  $-z_x$ , in each economy for different values of  $k$ .

The most important lesson coming from this numerical example is that financial innovation affects investment decisions, even without any change in fundamentals. Notice that across the three economies we do not change fundamentals such as asset payoffs or productivity parameters, utilities or endowments. The only variation is in the type of financial contracts available for trade using the assets as collateral, as described by the different sets  $J$ . In other words, financial innovation drives investment variations. We formalize these results in Sections 3.3 and 4.

It is also interesting to study the welfare implications of these financial innovations. Figure 8 shows the welfare corresponding to tail agents as well as the different equilibrium marginal buyers in each economy (calculated based on individual beliefs) when  $k = 1.5$ , across the three different economies. The Arrow Debreu equilibrium Pareto dominates the  $L$ -economy equilibrium. However, no such domination holds for the  $CDS$ -economy. In particular, moderate agents are better off in the  $CDS$ -economy than in Arrow Debreu. We will formally discuss these results in Sections 3.4 and 4.

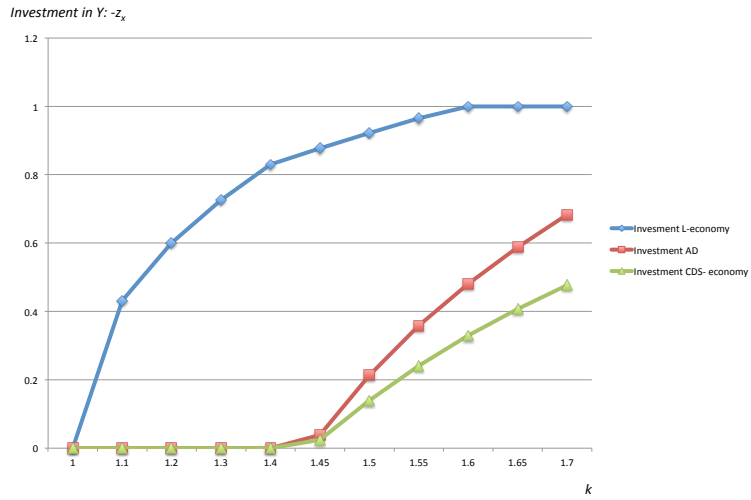


Figure 7: Total investment in  $Y$  in different economies for varying  $k$ .

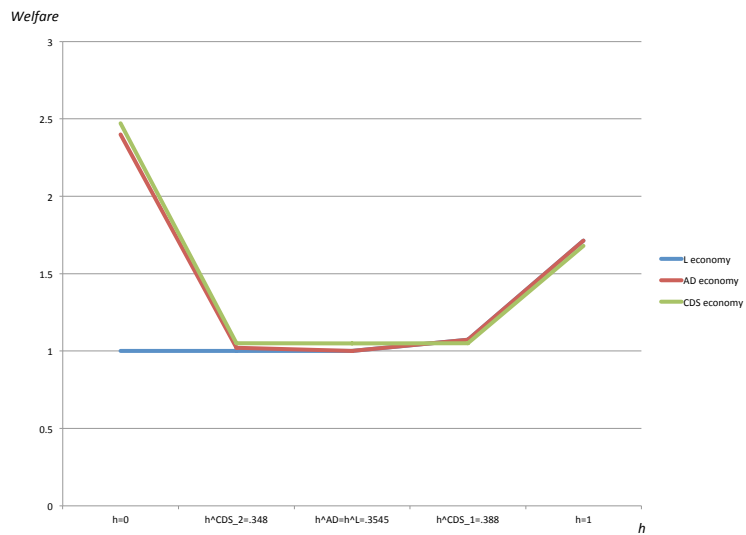


Figure 8: Financial Innovation and Welfare.

### 3.3 Over Investment and Welfare relative to the First Best

First we show that when agents can leverage the risky asset in the  $L$ -economy, investment levels are above those of the Arrow Debreu level. Hence, leverage generates *over*-investment with respect to the first best allocation. Our numerical example is consistent with a general property of the  $C$ -model as the following proposition shows.

**Proposition 1: Over-Investment compared to First Best in C-Models.**

Let  $(p^L, (z_x^L, z_y^L))$ , and  $(p^A, (z_x^A, z_y^A))$  denote the asset price and aggregate outputs for any equilibria in the  $L$ -economy and the Arrow Debreu Economy respectively. Then  $(p^L, z_y^L) \geq (p^A, z_y^A)$  and at least one of the two inequalities is strict, except possibly when  $z_x^L = -x_{0^*}$ , in which case all that can be said is that  $z_y^L \geq z_y^A$ .

**Proof:** See Section 4.3 and Appendix.

A first way to understand the result is the following. In the  $L$ -economy,  $Y$  is the only way of buying the Arrow  $U$  security. Leverage allows the purchase of the asset to be divided between two kinds of buyers, the optimists who hold the residual, which pays off exclusively in the good state, and the general public who holds the riskless piece that pays the same in both states. By dividing up the risky asset payoffs into two different kinds of assets, attractive to two different clienteles, demand is increased, and hence agents have more incentive to produce  $Y$ .

Another (and related) way to understand the result is in terms of the presence of collateral value as in Fostel and Geanakoplos (2008). In the  $L$ -economy the risky asset can be used as collateral to issue debt. This gives the risky asset an additional collateral value compared to the riskless asset. To illustrate this idea, consider the numerical example from Section 3.2.<sup>13</sup> To fix ideas consider the optimistic agent  $h = .9$ . The marginal utility of cash at time 0 for  $h = .9$ ,  $\mu^{h=.9}$ , is given by the optimal investment of one unit of  $X$ . As we saw, optimistic agents invest in the production of  $Y$  using  $Y$  as collateral to issue riskless debt, and hence, per dollar of downpayment the optimistic agent gets expected utility in state  $U$  of  $\mu^{h=.9} = \frac{\gamma_U(.9)(d_U^Y - d_D^Y)}{p - \pi_j^*} = \frac{.99(1-.2)}{.67-.2} = 1.70$  (see Table 1). The *Payoff Value* of  $Y$  for agent  $h = .9$  is given by the marginal utility of  $Y$  measured in dollar equivalents, or  $PV_Y^{h=.9} = \frac{.99(1)+.01(.2)}{\mu^{h=.9}} = .58 < p$ . Finally, the *Collateral Value* of  $Y$  for agent  $h = .9$  is given by  $CV_Y^{h=.9} = p - PV_Y^{h=.9} = .67 - .58 = .09$ . The utility from holding  $Y$  for its dividends

<sup>13</sup>We formally discuss these concepts in detail in Section 6.

alone is less than the utility that could be derived from  $p$  dollars; the difference is the utility derived from holding  $Y$  as collateral, measured in dollar equivalents. On the other hand,  $X$  cannot be used as collateral, so  $PV_X^{h=.9} = 1$  and hence  $CV_X^{h=.9} = 0$ . A similar analysis can be done for other agents as well.

Agents have more incentive to produce goods that are better collateral as measured by their collateral values. Investment migrates to better collateral.

It turns out that this result is valid for any type of preferences or space of agents, under constant return to scale technologies, as the following propositions shows.

**Proposition 2: Over-Investment with respect to the First Best in C\*-Models.**

*Let  $(p^L, (z_x^L, z_y^L))$ , and  $(p^A, (z_x^A, z_y^A))$  denote the asset price and aggregate outputs for any equilibria in the L-economy and the Arrow Debreu Economy respectively. Suppose that  $Z_0$  exhibits constant returns to scale and that  $z_y^A > 0$ . Then  $p^L = p^A$  and  $z_y^L > z_y^A$ , unless they are the same.*

**Proof:** See Section 4.3.

Under the same general conditions the following is true.

**Proposition 3: Arrow Debreu Pareto-dominates Leverage in C\*-Models.**

*Let  $(p^L, (z_x^L, z_y^L))$ , and  $(p^A, (z_x^A, z_y^A))$  denote the asset price and aggregate outputs for any equilibria in the L-economy and the Arrow Debreu Economy respectively. Suppose that  $Z_0$  exhibits constant returns to scale and that  $z_y^A > 0$ . Then the Arrow Debreu equilibrium Pareto-dominates the L-equilibrium.*

**Proof:** See Section 4.3.

### 3.4 Under-Investment relative to the First Best

We show that introducing a CDS using  $X$  as collateral generates *under*-investment with respect to the investment level in the  $L$ -economy. The result coming out of our numerical example is a general property of our  $C$ -model as the following proposition shows.



**Proposition 4: Under-Investment compared to Leverage in C-Models.**

Let  $(p^L, (z_x^L, z_y^L))$ , and  $(p^{CDS}, (z_x^{CDS}, z_y^{CDS}))$  denote the asset price and aggregate outputs for the  $L$ -economy and the  $CDS$ -economy respectively. Then  $(p^L, z_y^L) \geq (p^{CDS}, z_y^{CDS})$  and at least one of the two inequalities is strict, except possibly when  $z_x^L = -x_{0^*}$ , in which case all that can be said is that  $z_y^L \geq z_y^{CDS}$ .

**Proof:** See Section 4.5 and Appendix.

The basic intuition is along the same lines discussed in Proposition 1. Notice that selling a CDS using  $X$  as collateral is like “tranching” the riskless asset into Arrow securities. The holder of  $X$  can get the Arrow  $U$  security by selling the CDS using  $X$  as collateral. Hence, in the  $CDS$ -economy, the Arrow  $U$  security can be created through both,  $X$  and  $Y$ , whereas in the  $L$ -economy only thorough  $Y$ . This gives less incentive in the  $CDS$ -economy to invest in  $Y$ .

Finally, investment in the  $CDS$ -economy falls even below the investment level in the Arrow Debreu economy, provided that we make the additional assumption that  $\gamma_U(h)$  is concave. This concavity implies that there is more heterogeneity in beliefs among the pessimists than among the optimists.

**Proposition 5: Under-Investment compared to First Best in C-Models.**

Suppose  $\gamma_U(h)$  is concave in  $h$ , then  $(p^A, z_y^A) \geq (p^{CDS}, z_y^{CDS})$  and at least one of the two inequalities is strict, except possibly when  $z_x^A = -x_{0^*}$ , in which case all that can be said is that  $z_y^A \geq z_y^{CDS}$ , and when  $(z_y^A) = 0$ , in which case the  $CDS$  equilibrium might not exist.

**Proof:** See Section 5 and Appendix.

The intuition can also be seen in terms of the collateral values of the input  $X$  and the output  $Y$ . Using the same numerical example as before, the marginal utility of money at time 0 for  $h = .9$  is given by  $\mu^{h=.9} = \frac{\gamma_U(.9)(d_U^Y - d_D^Y)}{p - \pi_{j^*}} = \frac{.99(1-.2)}{.67-.1904} = \frac{\gamma_U(.9)(d_U^X)}{1-\pi_C} = \frac{.99(1)}{1-.40} = 1.66$  (optimists in the  $CDS$ -economy buy the Arrow  $U$  security using  $X$  and  $Y$  as collateral to sell  $CDS$  and the riskless bond). The payoff value of  $Y$  for agent  $h = .9$  is given by  $PV_Y^{h=.9} = \frac{.99(1)+.01(.2)}{\mu^{h=.9}} = .60 < p$  and the collateral value of  $Y$  for agent  $h = .9$  is given by  $CV_Y^{h=.9} = p - PV_Y^{h=.9} = .67 - .6 = .07$ . In the  $CDS$ -economy  $X$  can also be used as collateral. The payoff value of  $X$  for agent  $h = .9$  is given by  $PV_X^{h=.9} = \frac{.99(1)+.01(1)}{\mu^{h=.9}} = .60$  and the collateral value of  $X$  for agent  $h = .9$  is given by

$CV_X^{h=.9} = 1 - PV_X^{h=.9} = 1 - .60 = .40$ . So whereas the collateral value of  $Y$  accounts for 10.5% of its price, the collateral value of  $X$  accounts for 40% of its price.

In our numerical example the price of  $Y$  is the same across the different economies (given the constant return to scale technology), but financial innovation affects the collateral value of assets. Leverage increases the collateral value of  $Y$  relative to  $X$  and  $CDS$  has the opposite effect. Investment responds to these changes in collateral values, migrating to those assets with higher collateral values.

Propositions 4 and 5 cannot be generalized to the  $C^*$ -models, neither can we prove unambiguous welfare results. The reason is that in the  $L$ -economy and Arrow Debreu economy there are state prices that can be used to value every asset and contract.<sup>14</sup> In the  $CDS$ -economy this is not the case. We further discuss this in Section 4.

## 4 Financial Innovation and Collateral

In this section we fully characterize the equilibrium in the Arrow Debreu,  $L$  and  $CDS$ -economies presented before, each defined by a different set of feasible contracts  $J$ . We also use an Edgeworth box diagram to illustrate each case and to provide a geometrical proof of the results in Section 3, when possible. We start the section by characterizing the Arrow Debreu benchmark.

### 4.1 Arrow Debreu Equilibrium

Arrow Debreu equilibrium in the  $C$ -model is given by present value consumption prices  $(q_U, q_D)$ , which without loss of generality we can normalize to add up to 1, and by consumption  $(x_U^h, x_D^h)_{h \in H}$  and production  $(z_x^h, z_y^h)_{h \in H}$  satisfying

1.  $\int_0^1 x_s^h dh = \int_0^1 (x_{0^*} + z_x^h + d_s^Y z_y^h) dh$ ,  $s = U, D$ .
2.  $(x_U^h, x_D^h) \in B_W^h(q_U, q_D, \Pi^h) \equiv \{(x_U^h, x_D^h) \in R_+^2 : q_U x_U^h + q_D x_D^h \leq (q_U + q_D)x_{0^*} + \Pi^h\}$ .
3.  $(x_U, x_D) \in B_W^h(q_U, q_D, \Pi^h) \Rightarrow U^h(x_U, x_D) \leq U^h(x_U^h, x_D^h), \forall h$ .

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<sup>14</sup>See Fostel-Geanakoplos (2014a).

$$4. \Pi^h \equiv q_U(z_x^h + z_y^h d_U^Y) + q_D(z_x^h + z_y^h d_D^Y) \geq q_U(z_x + z_y d_U^Y) + q_D(z_x + z_y d_D^Y), \forall (z_x, z_y) \in Z_0^h.$$

Condition (1) says that supply equals demand for the consumption good at  $U$  and  $D$ . Conditions (2) and (3) state that each agent optimizes in her budget set, where income is the sum of the value of endowment  $x_{0^*}$  of  $X$  and the profit from her intra-period production. Condition (4) says that each agent maximizes profits, where the price of  $X$  and  $Y$  are implicitly defined by state prices  $q_U$  and  $q_D$  as  $q_X = q_U + q_D$  and  $q_Y = q_U d_U^Y + q_D d_D^Y$ .

We can easily compute Arrow-Debreu equilibrium. As mentioned in Section 3.1, since  $Z_0^h = Z_0$  does not depend on  $h$ , then profits  $\Pi^h = \Pi$ . Because  $Z_0$  is convex, without loss of generality we may suppose that every agent chooses the same production plan  $(z_x, z_y)$ . Since we have normalized the mass of agents to be 1,  $(z_x, z_y)$  is also the aggregate production. In Arrow-Debreu equilibrium there is a marginal buyer  $h_1$ .<sup>15</sup> All agents  $h > h_1$  use all their endowment and profits from production  $(q_U + q_D)x_{0^*} + \Pi = (x_{0^*} + \Pi)$  and buy all the Arrow  $U$  securities in the economy. Agents  $h < h_1$  instead buy all the Arrow  $D$  securities in the economy.

It is clarifying to describe the equilibrium using the Edgeworth box diagram in Figure 9. The axes are defined by the potential total amounts of  $x_U$  and  $x_D$  available from the economy final output as dividends from the stock of assets emerging at the end of period 0. Point  $Q$  represents the economy total final output from the actual equilibrium choice of aggregate intra-production  $(z_x, z_y)$ , so  $Q = (z_y d_U^Y + x_{0^*} + z_x, z_y d_D^Y + x_{0^*} + z_x)$ , where we take the vertical axis  $U$  as the first coordinate.

The 45-degree dotted line in the diagram is the set of consumption vectors that are collinear with the dividends of the aggregate endowment  $x_{0^*}$ . The steeper dotted line includes all consumption vectors collinear with the dividends of  $Y$ . The curve connecting the two dotted lines is the aggregate intra-period production possibility frontier, describing how the aggregate endowment of the riskless asset,  $x_{0^*}$ , can be transformed into  $Y$ . As more and more  $X$  is transformed into  $Y$ , the total output in  $U$  and  $D$  gets closer and closer to the  $Y$  dotted line.

The equilibrium prices  $q = (q_U, q_D)$  determine parallel price lines orthogonal to  $q$ . One of these price lines is tangent to the production possibility frontier at  $Q$ .

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<sup>15</sup>This is because of the linear utilities, the continuity of utility in  $h$  and the connectedness of the set of agents  $H$  at state  $s = 0$ .



In the classical Edgeworth Box there is room for only two agents. One agent takes the origin as her origin, while the second agent looks at the diagram in reverse from the point of view of the aggregate point  $Q$ , because she will end up consuming what is left from the aggregate production after the first agent consumes. The question is, how to put a whole continuum of heterogeneous agents into the same diagram? When the agents have linear preferences and the heterogeneity is one-dimensional and monotonic, this can be done. Suppose we put the origin of agent  $h = 0$  at  $Q$ . We can mark the aggregate endowment of all the agents between  $h = 0$  and any arbitrary  $h = h_1$  by its distance from  $Q$ . Since endowments are identical, and each agent makes the same profit, it is clear that this point will lie  $h_1$  of the way on the straight line from  $Q$  to the origin at 0, namely at  $(1 - h_1)Q = Q - h_1Q$ . The aggregate budget line of these agents is then simply the price line determined by  $q$  through their aggregate endowment, (their aggregate budget set is everything in the box between this line and  $Q$ ). Of course looked at from the point of view of the origin at 0, the same point represents the aggregate endowment of the agents between  $h = h_1$  and  $h = 1$ . (Since every agent has the same endowment, the fraction  $(1 - h_1)$  of the agents can afford to buy the fraction  $(1 - h_1)$  of  $Q$ .) Therefore the same price line represents the aggregate budget line of the agents between  $h_1$  and 1, as seen from their origin at 0, (and their aggregate budget set is everything between the budget line and the origin 0).

At this point we invoke the assumption that all agents have linear utilities, and that they are monotonic in the probability assigned to the  $U$  state. Suppose the prices  $q$  are equal to the probabilities  $(\gamma_U^{h_1}, \gamma_D^{h_1})$  of agent  $h_1$ . Agents  $h > h_1$ , who are more optimistic than  $h_1$ , have flatter indifference curves, illustrated in the diagram by the indifference curves near the origin 0. Agents  $h < h_1$ , who are more pessimistic than  $h_1$ , have indifference curves that are steeper, as shown by the steep indifference curves near the origin  $Q$ . The agents more optimistic than  $h_1$  collectively will buy at the point  $C$  where the budget line crosses the  $x_U$  axis above the origin, consuming exclusively in state  $U$ . The pessimists  $h < h_1$ , will collectively choose to consume at the point where the budget line crosses the  $x_D$  axis through their origin at  $Q$ , the same point  $C$ , consuming exclusively in state  $D$ . Clearly, total consumption of optimists and pessimists equals  $Q$ , i.e.  $(z_y d_U^Y + x_{0^*} + z_x, 0) + (0, z_y d_D^Y + x_{0^*} + z_x) = Q$ .

From the previous analysis it is clear that the equilibrium marginal buyer  $h_1$  must have two properties: (i) one of her indifference curves is tangent to the production possibility frontier at  $Q$ , and (ii) her indifference curve through the collective endow-

ment point  $(1 - h_1)Q$  cuts the top left point of the Edgeworth Box whose top right point is determined by  $Q$ .

Finally, the system of equations that characterizes the Arrow Debreu equilibrium is given by

$$(z_x, z_y) \in Z_0 \tag{3}$$

$$\Pi = z_x + q_Y z_y \geq \tilde{z}_x + q_Y \tilde{z}_y, \forall (\tilde{z}_x, \tilde{z}_y) \in Z_0. \tag{4}$$

$$q_U d_U^Y + q_D d_D^Y = q_Y \tag{5}$$

$$\gamma_U^{h_1} = q_U \tag{6}$$

$$\gamma_D^{h_1} = q_D \tag{7}$$

$$(1 - h_1)(x_{0^*} + \Pi) = q_U((x_{0^*} + z_x) + z_y d_U^Y) \tag{8}$$

Equations (3) and (4) state that intra-production plans should be feasible and should maximize profits. Equation (5) uses state prices to price the risky asset  $Y$ . Equations (6) and (7) state that the price of the Arrow  $U$  and Arrow  $D$  are given by the marginal buyer's state probabilities. Equation (8) states that all the money spent on buying the total amount of Arrow  $U$  securities in the economy (described by the RHS) should equal the total income of the buyers (described by the LHS).

## 4.2 The L-economy

In this case  $J = J^Y$ , and each contract  $j$  uses one unit of asset  $Y$  as collateral and promises  $(j, j)$  for all  $j \in J = J^Y$ . Agents can issue debt using any contract, in particular they could choose to sell contract  $(d_U^Y, d_U^Y)$ . But they do not. Geanakoplos (2003, 2010), Fostel and Geanakoplos (2012a) proved that in the  $C$ -model, there is a

unique equilibrium in which the *only* contract actively traded is  $j^* = d_D^Y$  (provided that  $j^* \in J$ ) and that the riskless interest rate equals zero. Hence,  $\pi_{j^*} = j^* = d_D^Y$  and there is no default in equilibrium. Even though agents are not restricted from selling bigger promises, the price  $\pi_j$  rises so slowly for  $j > j^*$  that they choose not to issue  $j > j^*$ . In other words, they cannot borrow more on the same collateral without raising the interest rate prohibitively fast: they are effectively constrained to  $j^*$ . Fostel and Geanakoplos (2014a) also showed that in every equilibrium in  $C$  and  $C^*$ -models there are unique state probabilities such that  $X$  and  $Y$  and all the contracts are priced by their expected payoffs.

As we saw in Section 3.1, in equilibrium there is a marginal buyer  $h_1$  at state  $s = 0$  whose valuation  $\gamma_U^{h_1} d_U^Y + \gamma_D^{h_1} d_D^Y$  of the risky asset  $Y$  is equal to its price  $p$ . The optimistic agents  $h > h_1$  collectively buy all the risky asset  $z_y$  produced in the economy, financing this with debt contracts  $j^*$ . The pessimistic agents  $h < h_1$  buy all the remaining safe asset and lend to the optimist agents.

The endogenous variables to solve for are the price of the risky asset  $p$ , the marginal buyer  $h_1$  and production plans  $(z_x, z_y)$ . The system of equations that characterizes the equilibrium in the  $L$ -economy is given by

$$(z_x, z_y) \in Z_0 \tag{9}$$

$$\Pi = z_x + pz_y \geq \tilde{z}_x + p\tilde{z}_y, \forall (\tilde{z}_x, \tilde{z}_y) \in Z_0. \tag{10}$$

$$(1 - h_1)(x_{0^*} + \Pi) + d_D^Y z_y = pz_y \tag{11}$$

$$\gamma_U^{h_1} d_U^Y + \gamma_D^{h_1} d_D^Y = p \tag{12}$$

Equations (9) and (10) describe profit maximization. Equation (11) equates money  $pz_y$  spent on the asset, with total income from optimistic buyers in equilibrium: all their endowment  $(1 - h_1)x_{0^*}$  and profits from production  $(1 - h_1)\Pi$ , plus all they can borrow  $d_D^Y z_y$  from pessimists using the risky asset as collateral. Equation (12) states that the marginal buyer prices the asset.

We can also describe the equilibrium using the Edgeworth box diagram in Figure 10. As in Figure 9, the axes are defined by the potential total amounts of  $x_U$  and  $x_D$  available as dividends from the stock of assets emerging at the end of period 0. The probabilities  $\gamma^{h_1} = (\gamma_U^{h_1}, \gamma_D^{h_1})$  of the marginal buyer  $h_1$  define state prices that are used to price  $x_U$  and  $x_D$ , and to determine the price lines orthogonal to  $\gamma^{h_1}$ . One of those price lines is tangent to the production possibility frontier at  $Q$ , representing the economy total final output,  $Q = (z_y d_U^Y + x_{0^*} + z_x, z_y d_D^Y + x_{0^*} + z_x)$ .

The dividend coming from the equilibrium choice of  $X$ ,  $x_{0^*} + z_x$ , lies at the intersection of the “ $X$ -dotted” line starting from 0 and the “ $Y$ -dotted” line starting at  $Q$ . The dividends coming for the equilibrium investment in  $Y$  (the firm total output),  $z_y(d_U^Y, d_D^Y)$ , lies at the intersection of the “ $Y$ -dotted” line starting at 0 and the “ $X$ -dotted” line starting at  $Q$ .

Again we put the origin of agent  $h = 0$  at  $Q$ . We can mark the aggregate endowment of all the agents between 0 and any arbitrary  $h_1$  by its distance from  $Q$ . Since endowments are identical, and each agent makes the same profit, it is clear that this point will lie  $h_1$  of the way on the line from  $Q$  to the origin, namely at  $(1 - h_1)Q = Q - h_1 Q$ . Similarly the same point describes the aggregate endowment of all the optimistic agents  $h > h_1$  looked at from the point of view of the origin at 0.

In equilibrium optimists  $h > h_1$  consume at point  $C$ . As in the Arrow Debreu equilibrium they only consume in the  $U$  state. They consume the total amount of Arrow  $U$  securities available in the economy,  $z_y(d_U^Y - d_D^Y)$ . Notice that when agents leverage asset  $Y$ , they are effectively creating and buying a “synthetic” Arrow  $U$  security that pays  $d_U^Y - d_D^Y$  and costs  $p - d_D^Y$ , namely at price  $\gamma_U^{h_1} = (d_U^Y - d_D^Y)/(p - d_D^Y)$ .

The total income of the pessimists between 0 and  $h_1$  is equal to  $h_1 Q$ . Hence looked at from the origin  $Q$ , the pessimists must also be consuming on the same budget line as the optimists. However, unlike the Arrow-Debreu economy, pessimists now must consume in the cone generated by the 45-degree line from  $Q$  and the vertical axis starting at  $Q$ . Since their indifference curves are steeper than the budget line, they will also choose consumption at  $C$ . However at  $C$ , unlike in the Arrow Debreu equilibrium, they consume the same amount,  $x_{0^*} + z_x + z_y d_D^Y$ , in both states. Clearly, total consumption of optimists and pessimists equals  $Q$ , i.e.  $(z_y(d_U^Y - d_D^Y), 0) + (x_{0^*} + z_x + z_y d_D^Y, x_{0^*} + z_x + z_y d_D^Y) = Q$ .

From the previous analysis we deduce that the marginal buyer  $h_1$  must satisfy two





properties: (i) one of her indifference curves must be tangent to the production possibility frontier at  $Q$ , and (ii) her indifference curve through the point  $(1 - h_1)Q$  must intersect the vertical axis at the level  $z_y(d_U^Y - d_D^Y)$ , which corresponds to point  $C$  and the total amount of Arrow  $U$  securities in equilibrium in the  $L$ -economy.

### 4.3 Over Investment and Welfare with respect to First Best: Proofs

#### 4.3.1 Geometrical Proof of Proposition 1

The Edgeworth Box diagrams in Figures 9 and 10 allow us to see why production is higher in the  $L$ -economy than in the Arrow Debreu economy. In the  $L$ -economy, optimists collectively consume  $z_y^L(d_U^Y - d_D^Y)$  in state  $U$  while in the Arrow Debreu economy they consume  $z_y^A d_U^Y + (x_0^* + z_x^A)$ . The latter is evidently much bigger, at least as long as  $z_y^A \geq z_y^L$ . So suppose, contrary to what we want to prove, that Arrow-Debreu output of  $Y$  were at least as high,  $z_y^A \geq z_y^L$ . Since the total economy output  $Q^L$  maximizes profits at the leverage equilibrium prices, at those leverage prices  $(1 - h_1^L)Q^A$  is worth no more than  $(1 - h_1^L)Q^L$ . Thus  $(1 - h_1^L)Q^A$  must lie on the origin side of the  $h_1^L$  indifference curve through  $(1 - h_1^L)Q^L$ . Suppose also that the Arrow Debreu price is higher than the leverage price:  $p^A \geq p^L$ . Then the Arrow Debreu marginal buyer is at least as optimistic,  $h_1^A \geq h_1^L$ . Then  $(1 - h_1^A)Q^A$  would also lie on the origin side of the  $h_1^L$  indifference curve through  $(1 - h_1^L)Q^L$ . Moreover, the indifference curve of  $h_1^A$  would be flatter than the indifference curve of  $h_1^L$  and hence cut the vertical axis at a lower point. By property (ii) of the marginal buyer in both economies, this means that optimists would collectively consume no more in the Arrow Debreu economy than they would in the leverage economy, a contradiction. It follows that either  $z_y^A < z_y^L$  or  $p^A < p^L$ . But a routine algebraic argument from profit maximization (given in the appendix) proves that if one of these strict inequalities holds, the other must also hold weakly in the same direction. (If the price of output is strictly higher, it cannot be optimal to produce strictly less.) This geometrical proof shows that in the Arrow Debreu economy there is more of the Arrow  $U$  security available (coming from the tranching of  $X$  as well as better tranching of  $Y$ ) and this extra supply lowers the price of the Arrow  $U$  security, and hence lowers the marginal buyer and therefore the production of  $Y$ . ■

### 4.3.2 Proof of Proposition 2

In case there is constant returns to scale in production of  $Y$  from  $X$ , and when  $Y$  is actually produced, the relative price of  $X$  and  $Y$  is determined by technology, and so is the same in the  $L$ -economy and in the Arrow Debreu economy. Therefore the state probabilities must also be the same in the two economies. The budget set for each agent  $h$  in the  $L$ -economy is equal to her budget set in the Arrow Debreu economy restricted to the cone between the vertical axis and the 45-degree line. Hence, demand by each agent  $h$  for consumption in the up state,  $x_U$ , is equal or higher in the  $L$ -economy than it is in the Arrow Debreu economy. It follows that if the  $L$ -equilibrium is different from the Arrow Debreu equilibrium, then the total supply of consumption at  $U$  must be greater in the  $L$ -economy. This means that production of  $Y$  is higher in the  $L$ -economy. ■

### 4.3.3 Proof of Proposition 3

Using the same argument as in the proof of Proposition 2, the budget set of each agent  $h$  is strictly bigger in the Arrow Debreu economy than in the  $L$ -economy. Hence, either the equilibria are identical or Arrow Debreu equilibrium allocation Pareto dominates the  $L$ -economy equilibrium allocation. ■

## 4.4 The CDS-economy

We introduce into the previous  $L$ -economy a CDS collateralized by  $X$ . Thus we take  $J = J^X \cup J^Y$  where  $J^X$  consists of contracts promising  $(0, 1)$  and  $J^Y$  consists of contracts  $(j, j)$  as described in the Leverage economy above. As in the  $L$ -economy, we know that the only contract in  $J^Y$  that will be traded is  $j^* = d_D^Y$ .

As we saw in Section 3.1, there are two marginal buyers  $h_1 > h_2$ . Optimistic agents  $h > h_1$  hold all the  $X$  and all the  $Y$  produced in the economy, selling the bond  $j^* = d_D^Y$  using  $Y$  as collateral and selling CDS using  $X$  as collateral. Hence, they are effectively buying the Arrow  $U$  security (the payoff net of debt and CDS payment at state  $D$  is zero). Moderate agents  $h_2 < h < h_1$  buy the riskless bonds sold by more optimistic agents. Finally, agents  $h < h_2$  buy the CDS security from the most optimistic investors (so they are effectively buying the Arrow  $D$ ).

The variables to solve for are the two marginal buyers,  $h_1$  and  $h_2$ , the asset price,  $p$ , the price of the riskless bond,  $\pi_{j^*}$ , the price of the CDS,  $\pi_C$ , and production plans,  $(z_x, z_y)$ . The system of equations that characterizes the equilibrium in the *CDS*-economy with positive production of  $Y$  is given by

$$(z_x, z_y) \in Z_0 \quad (13)$$

$$\Pi = z_x + pz_y \geq \tilde{z}_x + p\tilde{z}_y, \forall (\tilde{z}_x, \tilde{z}_y) \in Z_0. \quad (14)$$

$$\pi_U \equiv \frac{p - \pi_{j^*}}{d_U^Y - d_D^Y} = 1 - \pi_C \quad (15)$$

$$\frac{\gamma_U^{h_1}}{1 - \pi_C} = \frac{d_D^Y}{\pi_{j^*}} \quad (16)$$

$$\frac{\gamma_D^{h_2}}{\pi_C} = \frac{d_D^Y}{\pi_{j^*}} \quad (17)$$

$$(1 - h_1)(x_{0^*} + \Pi) + (x_{0^*} + z_x)\pi_C + \pi_{j^*}z_y = x_{0^*} + z_x + pz_y \quad (18)$$

$$h_2(x_{0^*} + \Pi) = \pi_C(x_{0^*} + z_x) \quad (19)$$

Equations (13) and (14) describe profit maximization. Equation (15) rules away arbitrage between buying the Arrow  $U$  through leveraging asset  $Y$  and through selling CDS while using asset  $X$  as collateral, assuming that the price of  $X$  is 1. Equation (16) states that  $h_1$  is indifferent between holding the Arrow  $U$  security (through asset  $X$ ) and holding the riskless bond. Equation (17) states that  $h_2$  is indifferent between holding the CDS security and the riskless bond. Equation (18) states that total money spent on buying the total available collateral in the economy should equal the optimistic buyers' income in equilibrium, which equals all their endowments and profits  $(1 - h_1)(x_{0^*} + \Pi)$ , plus all the revenues  $(x_{0^*} + z_x)\pi_C$  from selling CDS promises backed by their holdings  $(x_{0^*} + z_x)$  of  $X$ , plus all they can borrow  $\pi_{j^*}z_y$  using their

holdings  $z_y$  of  $Y$  as collateral. Finally, equation (19) states the analogous condition for the market of CDS, that is the total money spent on buying all the CDS in the economy,  $\pi_C(x_{0^*} + z_x)$ , should equal the income of the pessimistic buyers,  $h_2(x_{0^*} + \Pi)$ . By plugging the expressions  $p - \pi_{j^*} = \pi_U(d_U^Y - d_D^Y)$  and  $\pi_U + \pi_C = 1$  from equation (15) into equation (18), and rearranging terms, we get

$$(1 - h_1)(x_{0^*} + \Pi) = \pi_U(x_{0^*} + z_x + (d_U^Y - d_D^Y)z_y) \quad (20)$$

Dividing equation (17) into equation (18) yields

$$\frac{\gamma_U^{h_1}}{\gamma_D^{h_2}} = \frac{\pi_U}{\pi_C} \quad (21)$$

It might seem that  $\pi_U, \pi_C$  are the appropriate state prices that can be used to value all the securities, just as  $\gamma_U^{h_1}, \gamma_D^{h_2}$  did for the leverage economy. Unfortunately, this is not the case. There are no state prices in the CDS economy that will value all securities. In fact,  $\pi_U d_U^Y + \pi_D d_D^Y > p$ . Of course we can always define state prices  $q_U, q_D$  that will correctly price  $X$  and  $Y$ , but these will over-value  $j^*$ . The equilibrium price  $p$  of  $Y$  and the price 1 of  $X$  give two equations that uniquely determine these state prices.

$$p = q_U d_U^Y + q_D d_D^Y \quad (22)$$

$$p_X = 1 = q_U + q_D \quad (23)$$

Equations (22) and (23) define state prices that can be used to price  $X$  and  $Y$ , but not the other securities. From the fact that  $\pi_U, \pi_C$  over-value  $Y$  and that  $q_U, q_D$  over-value  $j^*$ , it is immediately apparent that

$$\frac{\gamma_U^{h_1}}{\gamma_D^{h_1}} > \frac{\gamma_U^{h_1}}{\gamma_D^{h_2}} = \frac{\pi_U}{\pi_C} > \frac{q_U}{q_D} > \frac{\gamma_U^{h_2}}{\gamma_D^{h_2}} \quad (24)$$

As before, we can describe the equilibrium using the Edgeworth box diagram in Figure 11. The complication with respect to the previous diagrams in Figures 9 and

10 is that now there are four state prices to keep in mind. So as not to clutter the diagram too much, we draw only three. The state prices  $q_U, q_D$  determine orthogonal price lines, one of which must be tangent to the production possibility frontier at  $Q$ . The optimistic agents  $h > h_1$  collectively own  $(1 - h_1)Q$ , indicated in the diagram. Consider the point  $x_1$  where the orthogonal price line with slope  $-q_D/q_U$  through  $(1 - h_1)Q$  intersects the  $X$  line. That is the amount of  $X$  the optimists could own by selling all their  $Y$ . Scale up  $x_1$  by the factor  $\gamma_U^{h_1} + \gamma_D^{h_2} > 1$ , giving the point  $x_1^*$ . That is how much riskless consumption those agents could afford by selling  $X$  (at a unit price) and buying the cheaper bond (at the price  $\pi_{j^*} < 1$ ). Now draw the indifference curve of agent  $h_1$  with slope  $-\gamma_D^{h_1}/\gamma_U^{h_1}$  from  $x_1^*$  until it hits the vertical axis. By equations (21) and (23), that is the budget trade-off between  $j^*$  and  $x_U$ . Similarly, draw the indifference curve of agent  $h_2$  with slope  $-\gamma_D^{h_2}/\gamma_U^{h_2}$  from  $x_1^*$  until it hits the horizontal axis of the optimistic agents. By equations (21) and (22), that is the budget trade-off between  $j^*$  and  $x_D$ . These two lines together form the collective budget constraint of the optimists. It is convex, but kinked at  $x_1^*$ . Notice that unlike before, the aggregate endowment is at the interior of the budget set (and not on the budget line). This is a consequence of lack of state prices that can price all securities. Because they have such flat indifference curves, optimists collectively will choose to consume at  $C_0$ , which gives  $x_U = (x_{0^*} + z_x) + z_y(d_U^Y - d_D^Y)$ .<sup>16</sup>

The pessimistic agents  $h < h_2$  collectively own  $h_2Q$ , which looked at from  $Q$  is indicated in the diagram by the point  $Q - h_2Q$ . Consider the point  $x_2$  where the orthogonal price line with slope  $-q_D/q_U$  through  $(1 - h_2)Q$  intersects the  $X$  line drawn from  $Q$ . Scale up that point by the factor  $\gamma_U^{h_1} + \gamma_D^{h_2} > 1$ , giving the point  $x_2^*$ . This represents how much riskless consumption those agents could afford by selling all their  $Y$  for  $X$ , and then selling  $X$  and buying the cheaper bond. The budget set for the pessimists can now be constructed as it was for the optimists, kinked at  $x_2^*$ . Their aggregate endowment is at the interior of their budget set for the same reason given above. Pessimists collectively will consume at  $C_P$ , which gives  $x_D = (x_{0^*} + z_x)$ .<sup>17</sup> Finally, the moderate agents  $h_1 < h < h_2$  collectively must consume  $z_y d_D^Y$ , which

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<sup>16</sup>If we were to connect the point  $x_1$  with  $C_0$ , this new line would describe the budget trade-off between  $x_U$  and  $x_D$ , obtained via tranching  $X$ , and would have a slope  $-\pi_C/\pi_U$ . By (26) the line would be flatter than the orthogonal price lines with slope  $-q_D/q_U$ .

<sup>17</sup>If we were to connect the point  $x_2$  with  $C_P$ , this new line would describe the budget trade-off between  $x_D$  and  $x_U$ , obtained via selling  $X$  and buying the down tranche, and would have a slope  $-\pi_C/\pi_U$ . By (26) the line would be flatter than the orthogonal price lines with slope  $-q_D/q_U$ .

collectively gives them the 45-degree line between  $C_0$  and  $C_P$ .

## 4.5 CDS and Under Investment: Proof

The geometrical proof of Proposition 4 using the Edgeworth boxes in Figures 10 and 11 is almost identical to that of Proposition 1. The optimists in the  $CDS$ -economy consume  $z_y^{CDS}(d_U^Y - d_D^Y) + (x_0^* + z_x^{CDS})$  which is strictly more than in the  $L$ -economy as long as production is at least as high in the  $CDS$ -economy, and not all of  $X$  is used in production. So suppose  $z_y^{CDS} \geq z_y^L$  and  $p^{CDS} \geq p^L$ . Then by (18),  $h_1^{CDS} \geq h_1^L$ . By the argument given in the geometrical proof of Proposition 1 in Section 4.3, consumption of the optimists in the  $CDS$ -economy cannot be higher than in the  $L$ -economy, which is a contradiction. Thus either  $z_y^{CDS} < z_y^L$  or  $p^{CDS} < p^L$ . But as we show in the Appendix, profit maximization implies that if one inequality is strict, the other holds weakly in the same direction. ■

Finally, the proof of Proposition 5 involves some irreducible algebra, so we do not try to give a purely geometric proof. But the diagram is helpful in following the algebra.

## 5 CDS and Non-Existence

A CDS is very similar to an Arrow  $D$  security. When  $Y$  exists, they both promise  $(0, 1)$  and both use  $X$  as collateral. The only difference between a CDS and an Arrow  $D$  is that when  $Y$  ceases to be produced a CDS is no longer well-defined. By definition, a derivative does not deliver when the underlying asset does not exist. It is precisely this difference that can bring about interesting non-existence properties as we now show.

Let us define the  $LT$ -economy  $J = J^X \cup J^Y$  where  $J^X$  consists of the single contract promising an Arrow  $D$ ,  $(0, 1)$  and  $J^Y$  consists of contracts  $(j, j)$  as described in the leverage economy above. Hence, the  $LT$ -economy is exactly the same as the  $CDS$ -economy, except that  $J^X$  consists of the single contract promising  $(0, 1)$  backed by  $X$  independent of the production of  $Y$ . The  $LT$ -economy always has an equilibrium, which may involve no production.

We now show how introducing CDS can robustly destroy competitive equilibrium in economies with production. The argument is the following. Equilibrium in the  $CDS$ -

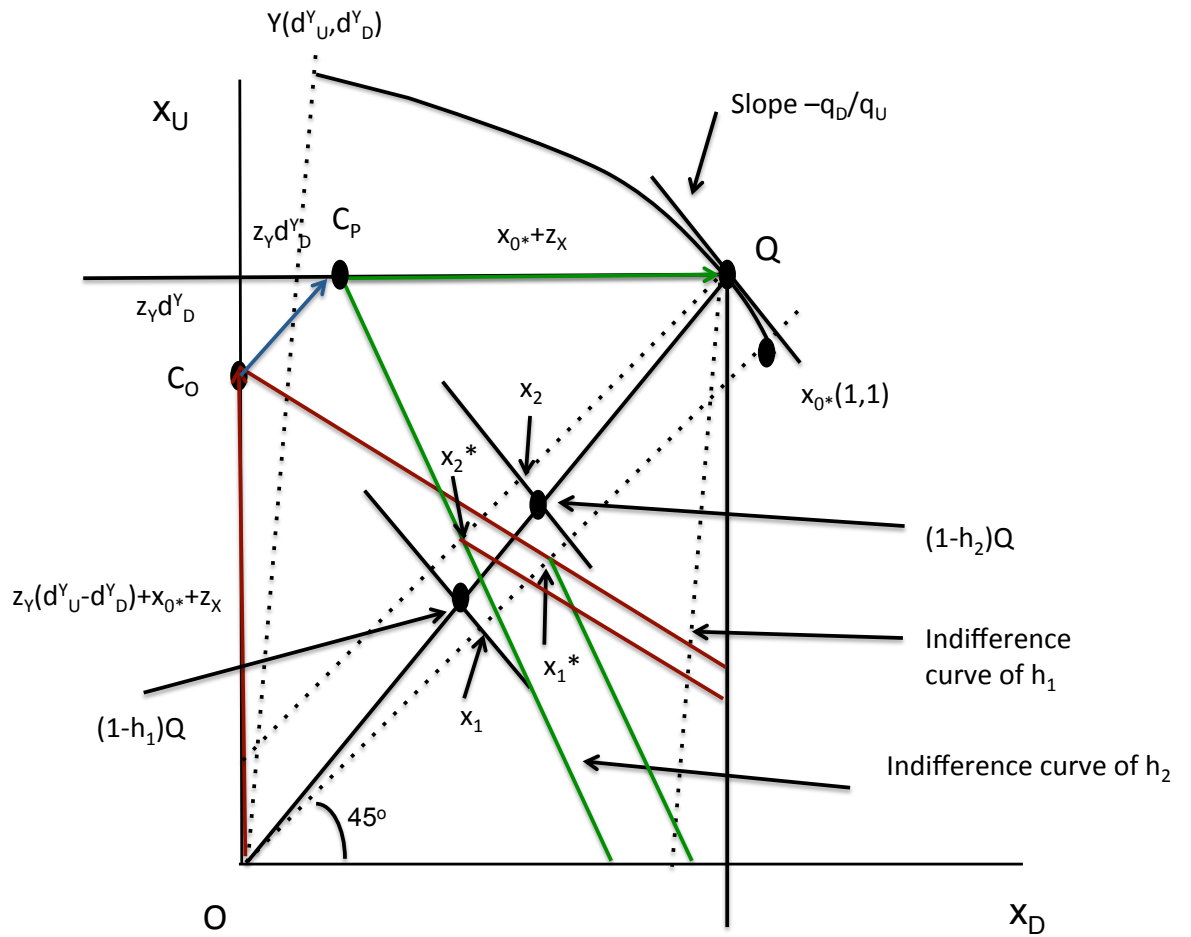


Figure 11: Equilibrium in the *CDS*-economy. Edgeworth Box.



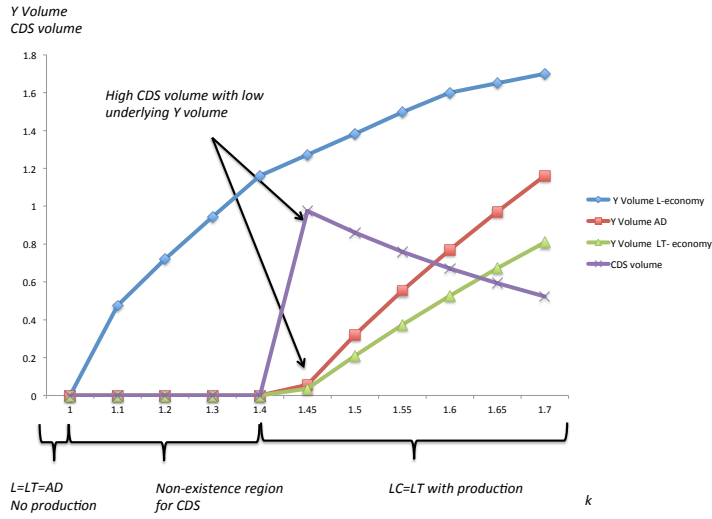


Figure 12: CDS and Non-Existence.

economy is equal to the equilibrium in the  $LT$ -economy if  $Y$  is produced, and is equal to the equilibrium in the  $L$ -economy if  $Y$  is not produced.<sup>18</sup> Thus, if all  $LT$ -equilibria involve no production of  $Y$  and all  $L$ -equilibria involve positive production of  $Y$ , then there cannot exist a  $CDS$ -equilibrium.

Recall our numerical example in Section 3.2. Observe that for all  $k$  such that  $k \in (1, 1.4)$ , the  $L$ -economy has positive production whereas the  $LT$ -economy has no production. For that entire range,  $CDS$ -equilibrium does not exist, as shown in Figure 12.<sup>19</sup>

$CDS$  is a derivative, whose payoff depends on some underlying instrument. The quantity of  $CDS$  that can be traded is not limited by the market size of the underlying instrument. If the value of the underlying security diminishes, the  $CDS$  trading may continue at the same high levels, as shown in the figure. But when the value of the underlying instrument falls to zero,  $CDS$  trading must come to an end by definition. This discontinuity can cause robust non-existence. The classical non-existence ob-

<sup>18</sup>This corresponds to an autarky equilibrium.

<sup>19</sup>We could also find non-existence example in economies with convex technologies, provided that Inada conditions (which prevent equilibrium production to be zero) are assumed away.

served in Hart (1975), Radner (1979) and Polemarchakis-Ku (1990) stemmed from the possibility that asset trades might tend to infinity when the payoffs of the assets tended toward collinear. A discontinuity arose when they became actually collinear. Collateral restores existence by (endogenously) bounding the asset trades. In our model CDS trades stay bounded away from zero and infinity even as production disappears. Collateral does not affect this, since the bounded promises can be covered by the same collateral. But the moment production disappears, the discontinuity arises, since then CDS sales must become zero.

## 6 Marginal Over-investment

Repayment enforceability problems restrict borrowing and thus naturally raise the specter of under-investment. But when a commodity can serve as collateral, it thereby acquires an additional usefulness, and an opposite force is created which tends to over-valuation and over-production of the commodity. In this section we show that under the general conditions of the model in Section 2, at the margin (i.e. with prices fixed), the over-production force always dominates, despite the fact that agents are constrained in what they can borrow.

Proposition 6 shows that when agents are constrained in equilibrium, if they were suddenly given a little extra money to make purely cash purchases, none of them would choose to produce more of any good that can be used as collateral.

In the general model of Section 2 we allow for heterogeneous productivity both at the intra-period and inter-period level. One type of agent  $h$ , with small wealth, might be very productive (good  $F^h$ ) at the inter-period level relative to everyone else. If  $h$  is limited in how much she can borrow by the need to post collateral, one might suspect that there could be under-investment: perhaps another type of agent  $h'$  is wealthy at time 0 and would like to consume the output of  $F^h$  but is not as productive as type  $h$ .<sup>20</sup> Proposition 7 shows that if the output of  $F^h$  is fully collateralizable, this could never happen. Our result thus stands in contrast to the situation which prevails when cash flow problems are layered on top of repayment enforceability problems. It shows

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<sup>20</sup>In Kiyotaki and Moore (1997),  $h'$  ends up holding land on which she is not productive when another agent  $h$  could have produced more with it, because by hypothesis the fruit growing on the land cannot be confiscated along with the land in case of default, preventing  $h$  from borrowing enough to buy more land.

that one way of generating a large swing from over-production to under-production would be to move from a situation in which a good can be fully collateralized to one in which it can't be used as collateral at all.

Our marginal over-investment proposition does not mean that there is necessarily more investment than in Arrow Debreu (even though were were able to prove that in  $C$  and  $C^*$  models) because not all goods can serve as collateral, as we said, and because prices might differ in Arrow Debreu. In Arrow Debreu, the output of the investment can be tranché, with one investor getting its dividends in the state  $U$  and another in state  $D$ , and that might raise the price of the output beyond its collateral price and thus incentivize greater production. Marginal over-investment is evaluated under the hypothesis that prices stay fixed.

We now make these ideas precise using the notions of collateral value and liquidity value from Fostel-Geanakoplos (2008) and Geanakoplos-Zame (2014). Let us assume that every agent has strictly positive “extended” endowments of commodities in every state, where we define the extended endowment in state  $s \in S_T$  to be  $e_s^h + F_s^h(e_0^h)$ . Given commodity and contract prices  $(p, \pi)$  define the indirect utility  $U^h((p, \pi), w_0, w_1, \dots, w_S)$  as the maximum utility agent  $h$  can get by trading at prices  $(p, \pi)$ , where the  $w_s \in (-\varepsilon, \varepsilon)$  represent small transfers of income, positive or negative. Since agents have strictly positive endowments, for small negative income transfers their starting endowment wealth will be positive in each state. Since utilities are concave, the indirect utility function must be concave in  $w$ , and hence differentiable from the right and the left at every point, including the point with equilibrium prices and  $w = 0$ . Let  $\mu_s^h$  be the derivative from the right for states  $s \in S_T$ , and let  $\mu_0^h$  be the derivative from the left for state  $s = 0$ .

To simplify the statement of our marginal over-investment propositions we shall assume differentiability of the utility functions for each agent. Given an equilibrium, it is evident that for any state  $s \in S_T$ ,

$$\mu_s^h = \frac{\partial u^h(x_{s\ell}^h)}{x_{s\ell}^h} \frac{1}{p_{s\ell}}$$

whenever  $x_{s\ell}^h > 0$ . Similarly, if  $0\ell$  is completely perishable, and  $x_{0\ell}^h > 0$ , then

$$\mu_0^h = \frac{\partial u^h(x_{0\ell}^h)}{x_{0\ell}^h} \frac{1}{p_{0\ell}}$$

For possibly non-perishable commodities, define the *Payoff Value* for each commodity  $\ell \in L_0$  to each agent  $h$  by

$$PV_{0\ell}^h = \frac{\frac{\partial u^h(x_0^h)}{\partial x_{0\ell}} + \sum_{s \in S_T} p_s \cdot F_s^h(1_{0\ell}) \mu_s^h}{\mu_0^h}$$

and the *Collateral Value* for each commodity  $\ell \in L_0$  to each agent  $h$  by

$$CV_{0\ell}^h = p_{0\ell} - PV_{0\ell}^h$$

Similarly we define the *Liquidity Value* of contract  $j$  to any (potential seller)  $h$  as

$$LV_j^h = \pi_j - \frac{\sum_{s \in S_T} \min(p_s \cdot F_s^h(c_j), p_s \cdot j_s) \mu_s^h}{\mu_0^h}$$

Agent  $h$  is liquidity constrained in equilibrium if and only if there is some contract  $j$  that has strictly positive liquidity value to him. In equilibrium we must have  $LV_j^h \geq 0$  for all  $h \in H, j \in J$ , otherwise agent  $h$  ought to have bought more  $j$ .

Fostel-Geanakoplos (2008) and Geanakoplos-Zame (2014) proved that  $CV_{0\ell}^h = LV_j^h$ , so that the liquidity value associated to any contract  $j$  that is actually issued using commodity  $\ell$  as collateral equals the collateral value of the commodity.

The next proposition shows that there is never marginal under-investment in goods that can be used as collateral.

**Proposition 6: No Direct Marginal Under-Investment**

*Consider a collateral equilibrium  $((p, \pi), (z^h, x^h, \theta^h)_{h \in H})$ . Then for every  $h \in H, \ell \in L_0$  we must have  $p_{0\ell} \geq PV_{0\ell}^h$ .*

*Moreover, if there is some contract  $j$ , with  $c_j = 1_{0\ell}$ , that has strictly positive liquidity value to  $h$ , then  $p_{0\ell} > PV_{0\ell}^h$ . In this case, if  $h$  were given an extra unit of cash to make a purely cash purchase, she would not purchase or produce more of good  $0\ell$ .*

**Proof:**

If  $p_{0\ell} < PV_{0\ell}^h$ , then agent  $h$  ought to have reduced a little of what she was doing in equilibrium, and instead bought a little of commodity  $0\ell$ , a contradiction. If on top of buying a little  $0\ell$ ,  $h$  could also use it to collateralize a little borrowing via

contract  $j$ , with positive liquidity value, then we would also contradict  $p_{0\ell} \leq PV_{0\ell}^h$ . The concluding statement follows immediately. ■

Next we prove a stronger result: there is never marginal under-investment even when agents are allowed to invest in technologies owned by other agents in the economy.

For each commodity  $\ell \in L_0$ , define its *Payoff Value to agent  $h'$  via agent  $h$*  by

$$PV_{0\ell}^{h,h'} = \frac{\frac{\partial u^h(x_0^h)}{\partial x_{0\ell}}}{\mu_0^h} + \frac{\sum_{s \in S_T} p_s \cdot F_s^{h'}(1_{0\ell}) \mu_s^{h'}}{\mu_0^{h'}}$$

**Proposition 7: No Indirect Marginal Under-Investment**

*Consider a collateral equilibrium  $((p, \pi), (z^h, x^h, \theta^h)_{h \in H})$ . Suppose an agent  $h$  can use  $1_{0\ell}$  as collateral, and in particular can issue a large contract  $j$  backed by  $1_{0\ell}$ . Then for every  $h' \in H, \ell \in L_0$  we must have  $p_{0\ell} \geq PV_{0\ell}^{h,h'}$ .*

*Moreover, if  $j$  has strictly positive liquidity value to  $h$ , then  $p_{0\ell} > PV_{0\ell}^{h,h'}$ . In this case, even if  $h'$  were given an extra unit of cash to make a purely cash purchases, she would not use it to buy  $0\ell$  or to pay  $h$  to use  $0\ell$  to produce dividends for her.*

**Proof:**

Agent  $h$  can always sell contract  $j$  to agent  $h'$  for at least  $\pi_j \geq \frac{\sum_{s \in S_T} p_s \cdot F_s^{h'}(1_{0\ell}) \mu_s^{h'}}{\mu_0^{h'}}$ . Hence if  $p_{0\ell} < PV_{0\ell}^{h,h'}$ , agent  $h$  ought to have reduced a little of what he was doing in equilibrium, and instead bought a little of commodity  $0\ell$ , used it to issue the large contract  $j$ , thus paying on net at most

$$p_{0\ell} - \pi_j < PV_{0\ell}^{h,h'} - \frac{\sum_{s \in S_T} p_s \cdot F_s^{h'}(1_{0\ell}) \mu_s^{h'}}{\mu_0^{h'}} = \frac{\frac{\partial u^h(x_0^h)}{\partial x_{0\ell}}}{\mu_0^h}$$

and being better off, a contradiction. The rest follows as in Proposition 6. ■

A simple special case is when each unit of a completely perishable good  $C$  gives positive utility from consumption but can be used instead intra period to produce one unit of a durable good  $Y$  that can be used as collateral. Suppose agents consume a positive amount of  $C$  in equilibrium. Suppose an agent  $h$  is liquidity constrained but could use  $Y$  to produce large future dividends. Then proposition 6 assures us that starting in equilibrium,  $h$  would not use an extra dollar to buy  $C$  to produce more  $Y$ . The agent would strictly prefer to consume more  $C$ . Furthermore, if  $h$  could issue a

large contract (promising the whole future output of  $Y$ ) backed by  $Y$ , then no other agent  $h'$  would wish to use an extra dollar to pay  $h$  to buy more  $Y$  and to give the future dividends to  $h'$ . We can further illustrate our results in a case where there is no consumption at 0 by considering the numerical example in the  $C$ -model in Section 3.2. As we saw the marginal utility cash for agent  $h = .9$  is  $\mu^{h=.9} = \frac{.99(1-.2)}{.67-.2} = 1.70$ . On the other hand the expected utility per dollar of investing in  $Y$  without issuing debt for the same agent is given by  $.99(1.5) + .01(.2).1.5 = 1.48$ . Hence, on the margin there is over-investment. No agent would use an extra unit of cash in producing the asset if he could not also borrow to do it. In fact, the agents do not borrow to buy the asset, they buy the asset because it allows them to borrow (and hence consume only in the up state).

## 7 References

- Allen F. and Gale D. 1994. "Financial Innovation: An Overview," Part I (Chapters 1-5) in *Financial Innovation and Risk Sharing*. Cambridge, MA: MIT Press.
- Bernanke, B. and Gertler M. 1989. "Agency Costs, Net Worth, and Business Fluctuations." *American Economic Review*, 79(1): 14-31.
- Che, Yeon-Koo, and Rajiv Sethi. 2010. "Credit Derivatives and the Cost of Capital." Unpublished.
- Duffie, D. and W. Shafer, 1986, Equilibrium in incomplete markets II: Generic existence in stochastic economies, *Journal of Mathematical Economics*, 15, no. 3, 199-218.
- Fostel A. and Geanakoplos J. 2008. "Leverage Cycles and the Anxious Economy." *American Economic Review*, 98:4, 1211-1244.
- Fostel A. and Geanakoplos J. 2012a. "Why Does Bad News Increase Volatility and Decrease Leverage." *Journal of Economic Theory*. 147(2): 501-525.
- Fostel A. and Geanakoplos J. 2012b. "Tranching, CDS and Asset Prices: How Financial Innovation can Cause Bubbles and Crashes." *American Economic Journal: Macroeconomics*, 4(1): 190-225.
- Fostel A. and Geanakoplos J. 2014a. "Leverage and Default in Binomial Economies: A Complete Characterization." Cowles Foundation Discussion Paper.

- Fostel A. and Geanakoplos J. 2014b. “Endogenous Collateral Constraints and the Leverage Cycle.” *Annual Review of Economics*. Vol. 6: 771–799
- Geanakoplos J. 2003. “Liquidity, Default, and Crashes: Endogenous Contracts in General Equilibrium.” In *Advances in Economics and Econometrics: Theory and Applications, Eighth World Conference*, Vol. 2, 170-205. Econometric Society Monographs.
- Geanakoplos J. 2010. “The Leverage Cycle.” In *NBER Macroeconomics Annual*, ed. Daron Acemoglu, Kenneth Rogoff, and Michael Woodford, 1-65. Chicago: University of Chicago Press.
- Geanakoplos J and Zame W. 2014. “Collateralized Security Markets.” *Economic Theory*, .
- Gorton, Gary. 2009. “The Subprime Panic.” *European Financial Management*, 15(1): 10–46.
- Hart, O. 1975. “On the Existence of Equilibrium when the Market Structure is Incomplete.” *Journal of Economic Theory*, 9, 293-311.
- Kilenthong Robert M. Townsend. 2011. “Segregated Security Exchanges with Ex Ante Rights to Trade A Market-Based Solution to Collateral-Constrained Externalities.” RIPPED WP Series.
- Kiyotaki, N. and Moore, J. 1997. “Credit Cycles.” *Journal of Political Economy*, 105(2): 211-248.
- Polemarchakis H. and Ku B. 1990. “Options and Equilibrium.” *Journal of Mathematical Economics*, 19, 107-112.
- Radner, R. 1979. “Rational Expectations Equilibrium: Generic Existence and the Information Revealed by Prices.” *Econometrica*, 47, 655-678.
- Stultz, Rene. 2009. “Credit Default Swaps and the Credit Crisis.” Charles A. Dice Center Working Paper 2009-16.

# Appendix

## Proof of Proposition 1

**Step 1:** In every equilibrium, each agent must be maximizing profit. Without loss of generality, we can suppose that every agent chooses the same production  $(z_x, z_y)$ . Since by hypothesis the mass of agents is normalized to 1, total holdings in the economy are then  $(x_{0^*} + z_x, z_y)$ . Consider two asset prices  $p, q$ , and production plans  $z^p = (z_x^p, z_y^p), z^q = (z_x^q, z_y^q)$  that maximize profits at the corresponding prices, so  $z_x^p + pz_y^p \geq z_x^q + pz_y^q$  and  $z_x^q + qz_y^q \geq z_x^p + qz_y^p$ . Adding the inequalities and rearranging,  $(p - q)(z_y^p - z_y^q) \geq 0$ .

So  $p > q$  implies  $z_y^p \geq z_y^q$ , and  $z_y^p > z_y^q$  implies  $p \geq q$ . Moreover, it is clear that maximizing profit  $\Pi(p)$  also maximizes total wealth  $W(p)$  (since wealth is profit plus  $x_{0^*}$ ). It is more convenient to think of maximizing wealth. It is obvious that increasing either the price of  $X$  or  $Y$  gives rise to higher value of maximal wealth, since choosing the same production plan gives at least the same wealth when prices are higher. Hence  $W(p)$  is weakly increasing in  $p$ , and strictly increasing if  $z_y > 0$ . Finally it is also clear that  $\frac{W(p)}{p}$  is weakly decreasing in  $p$ , and strictly decreasing if  $x_{0^*} + z_x > 0$ . The reason is that multiplying both prices by a common scalar does not change the profit maximizing production plan, so that wealth is therefore homogeneous of degree 1 in the price vector. Scaling up just the price of  $Y$ , holding the price of  $X$  fixed at 1, therefore does less than scale up the value of wealth.

**Step 2:** Step 1 shows that if the leverage asset price  $p^L > p^A$ , or if the leverage output  $z_y^L > z_y^A$ , we are done. Hence we only need to show that assuming  $p^L \leq p^A$  and  $z_y^L \leq z_y^A$  leads to a contradiction. With that assumption, individual profits  $\Pi^L$  in the leverage economy are no higher than in the Arrow-Debreu economy  $\Pi^A$ .

**Step 3:** Since  $(d_U^X, d_D^X, d_X^Y, d_D^Y) \gg 0$ , there will always be positive aggregate consumption in both states  $U$  and  $D$ . Thus for the Arrow-Debreu marginal buyer,  $0 < h^A < 1$ . Suppose that  $z_y^L = 0$ . Then every agent in the leverage economy consumes his initial endowment  $x_{0^*}$ . And no agent, including  $h = 1$ , prefers  $Y$  to  $X$  at price  $p^L$ . Hence  $p^L \geq \gamma_U(1)d_U^Y + \gamma_D(1)d_D^Y > \gamma_U(h^A)d_U^Y + \gamma_D(h^A)d_D^Y = p^A$  and we are done. Alternatively, suppose that  $x_{0^*} + z_x^L = 0$ . Then the leverage economy is producing the maximum possible  $y$ , so trivially  $z_y^L \geq z_y^A$ . Thus without loss of generality,



we suppose that  $x_{0^*} + z_x^L > 0$  and  $z_y^L > 0$ . That guarantees that there is a marginal buyer  $0 < h^L < 1$ .

**Step 4:** First, since the prices are set by the marginal buyer in both economies, under the maintained hypothesis  $p^L \leq p^A$ , we must have  $h^L \leq h^A$ . In equilibrium,  $\frac{W(p^A)}{p^A}(1 - h^A) = z_y^A d_U^Y + (x_{0^*} + z_x^A)1$  and  $\frac{W(p^L)}{p^L}(1 - h^L) = z_y^L(d_U^Y - d_D^Y)$ . By our second maintained hypothesis,  $z_y^A \geq z_y^L$ . Hence RHS of the first equation above is strictly more than the RHS of the second equation above. But by the maintained price hypothesis and Step 1  $\frac{W(p^A)}{p^A} \leq \frac{W(p^L)}{p^L}$ . It follows that  $(1 - h^A) > (1 - h^L)$ , and hence that  $h^A < h^L$ , a contradiction. ■

### Proof of Proposition 4

**Step 1:** Reasoning as in the last proof, we need only reach a contradiction from the hypothesis that  $p^L \leq p^{CDS}$  and  $z_y^L \leq z_y^{CDS}$ . From this hypothesis we deduce that  $\Pi^L \leq \Pi^{CDS}$  and  $W^L \leq W^{CDS}$ .

**Step 2:** Since  $(d_U^X, d_D^X, d_X^Y, d_D^Y) \gg 0$ , there will always be positive aggregate consumption in both states  $U$  and  $D$ . Thus for the Arrow-Down marginal buyer,  $0 < h_2^{CDS} \leq h_1^{CDS} < 1$ . Suppose that  $z_y^L = 0$ . Then every agent in the leverage economy consumes his initial endowment  $x_{0^*}$ . And no agent prefers  $Y$  to  $X$  at price  $p^L$ . Hence  $p^L \geq \gamma_U(1)d_U^Y + \gamma_D(1)d_D^Y > \gamma_U(h^{CDS})d_U^Y + \gamma_D(h^{CDS})d_D^Y \geq p^{CDS}$  and we are done. Alternatively, suppose that  $x_{0^*} + z_x^L = 0$ . Then the leverage economy is producing the maximum possible  $y$ , so trivially  $z_y^L \geq z_y^{CDS}$ . Thus without loss of generality, we suppose that  $x_{0^*} + z_x^L > 0$  and  $z_y^L > 0$ . That guarantees that there is a marginal buyer  $0 < h^L < 1$ .

**Step 3:** Under the maintained assumption that more resources are devoted to production in the  $CDS$ -economy, the remaining  $X$  must be at least as high in the leverage economy:  $(x_{0^*} + z_x^L) + (z_y^L)d_D^Y \geq (x_{0^*} + z_x^{CDS}) + (z_y^{CDS})d_D^Y$ . Recall that the wealth of each agent in the respective economies is  $W^L = x_{0^*} + \Pi^L = (x_{0^*} + z_x^L) + z_y^L p^L$  and  $W^{CDS} = x_{0^*} + \Pi^{CDS} = (x_{0^*} + z_x^{CDS}) + z_y^{CDS} p^{CDS}$ .

It then follows from  $\Pi^{CDS} \geq \Pi^L$  and  $p^{CDS} \geq p^L$  that  $W^{CDS} \geq W^L$  and therefore  $\frac{(x_{0^*} + z_x^L) + z_y^L d_D^Y}{(x_{0^*} + z_x^L) + z_y^L p^L} \geq \frac{(x_{0^*} + z_x^{CDS}) + z_y^{CDS} d_D^Y}{(x_{0^*} + z_x^{CDS}) + z_y^{CDS} p^{CDS}}$ . From the equilibrium conditions presented earlier for the  $CDS$ -economy, we know that  $p^{CDS} = \frac{\gamma_U(h_1^{CDS})d_U^Y + (1 - \gamma_U(h_1^{CDS}))d_D^Y}{\gamma_U(h_1^{CDS}) + (1 - \gamma_U(h_2^{CDS}))} \leq$

$\gamma_U(h_1^{CDS})d_U^Y + (1 - \gamma_U(h_1^{CDS}))d_D^Y$  and  $\pi_{j^*} = \frac{1}{\gamma_U(h_1^{CDS}) + (1 - \gamma_U(h_2^{CDS}))}d_D^Y \leq d_D^Y$ , with a strict inequality in both cases if  $z_y^{CDS} > 0$ , since then  $h_1^{CDS} > h_2^{CDS}$ .

For the  $L$ -economy,  $p^L = \gamma_U(h_1^L)d_U^Y + (1 - \gamma_U(h_1^L))d_D^Y$ . It follows from the strict monotonicity of  $\gamma_U(h)$  and from  $p^L \leq p^{CDS}$  that  $h_1^L \leq h_1^{CDS}$  with a strict inequality if  $z_y^{CDS} > 0$ .

In  $CDS$ -equilibrium we must have that the agents above  $h_1^{CDS}$  spend all their money to buy all the assets

$$\begin{aligned} (x_{0^*} + z_x^{CDS} + p^{CDS}z_y^{CDS})(1 - h_1^{CDS}) &= (x_{0^*} + z_x^{CDS} + p^{CDS}z_y^{CDS}) - z_y^{CDS}\pi_{j^*} - (x_{0^*} + z_x^{CDS})p_D^{CDS} \\ (x_{0^*} + z_x^{CDS} + p^{CDS}z_y^{CDS})h_1^{CDS} &= z_y^{CDS}\pi_{j^*} + (x_{0^*} + z_x^{CDS})p_D^{CDS} \\ &< z_y^{CDS}d_D^Y + (x_{0^*} + z_x^{CDS}) \end{aligned}$$

The last inequality is strict, because if  $z_y^{CDS} > 0$ , then  $z_y^{CDS}\pi_{j^*} < z_y^{CDS}d_U^Y$ , while if  $z_y^{CDS} = 0$ , then  $(x_{0^*} + z_x^{CDS}) = x_{0^*} > 0$  and  $p_D^{CDS} = \frac{(1 - \gamma_U(h_2^{CDS}))}{\gamma_U(h_1^{CDS}) + (1 - \gamma_U(h_2^{CDS}))} < 1$  because  $\gamma_U(h_1^{CDS}) > 0$  since  $h_1^{CDS} > 0$ . Similarly, in leverage equilibrium we must have that the agents above  $h_1^L$  spend all their money to buy all the  $Y$  assets

$$\begin{aligned} (x_{0^*} + z_x^L + p^Lz_y^L)(1 - h_1^L) &= (x_{0^*} + z_x^L + p^Lz_y^L) - z_y^Ld_D^Y - (x_{0^*} + z_x^L) \\ (x_{0^*} + z_x^L + p^Lz_y^L)h_1^L &= z_y^Ld_D^Y + (x_{0^*} + z_x^L) \end{aligned}$$

Putting these last two conclusions together we get  $h_1^{CDS} < \frac{z_y^{CDS}d_D^Y + (x_{0^*} + z_x^{CDS})}{(x_{0^*} + z_x^{CDS} + p^{CDS}z_y^{CDS})}$  and  $h_1^L = \frac{z_y^Ld_D^Y + (x_{0^*} + z_x^L)}{(x_{0^*} + z_x^L + p^Lz_y^L)}$ . But we showed at the outset of the proof that the upper RHS is no bigger than the lower RHS. This implies that  $h_1^{CDS} < h_1^L$ , which is the desired contradiction. ■

## Proof of Proposition 5

**Step 1:** Reasoning as in the last proofs, we need only reach a contradiction from the hypothesis that  $p^A \leq p^{CDS}$  and  $z_y^A \leq z_y^{CDS}$ . From this hypothesis we deduce that  $p_U^A \leq q_U$ ,  $\Pi^A \leq \Pi^{CDS}$ ,  $W^A \equiv x_{0^*} + \Pi^A \leq W^{CDS} \equiv x_{0^*} + \Pi^{CDS}$ , and that  $[z_y^{CDS}d_D^Y + (x_{0^*} + z_x^{CDS})] \leq [z_y^A d_D^Y + (x_{0^*} + z_x^A)]$ .

**Step 2:** Since  $(d_U^X, d_D^X, d_U^Y, d_D^Y) \gg 0$ , there will always be positive aggregate consumption in both states  $U$  and  $D$ . Thus for the Arrow Debreu marginal buyer,

$0 < h_1^A < 1$ , and for the  $CDS$ -economy marginal buyers,  $0 < h_2^{CDS} \leq h_1^{CDS} < 1$ . It is obvious that if there is no  $Y$  in a  $CDS$ -equilibrium, then the equilibrium must also be an Arrow-Debreu equilibrium. Hence we may assume that  $z_y^{CDS} > 0$ , and hence that  $h_2^{CDS} < h_1^{CDS}$ .

**Step 3:** From the inequalities (24) from the  $CDS$  economy, we know that  $\gamma_U(h_1^{CDS}) > q_U$  and  $\gamma_D(h_2^{CDS}) > q_D$  and more subtly<sup>21</sup>

$$\frac{\gamma_U(h_1^{CDS}) - q_U}{q_U - \gamma_U(h_2^{CDS})} = \frac{\gamma_U(h_1^{CDS}) - q_U}{\gamma_D(h_2^{CDS}) - q_D} > \frac{q_U}{q_D}$$

From the continuity and the strict monotonicity of  $\gamma_U(h)$ , we can define a unique  $h^*$  with  $\gamma_U(h^*) = q_U$ . From the concavity of  $\gamma_U(h)$ , we deduce that

$$\frac{h_1^{CDS} - h^*}{h^* - h_2^{CDS}} > \frac{q_U}{q_D}$$

**Step 4:** It is now convenient to define the fictitious agents  $h^{**}, h_1^{**}, h_2^{**}$  who act as if they could trade  $U$  and  $D$  goods at the state prices  $(q_U, q_D)$ . In terms of the diagram in Figure 11, call the points where the orthogonal price lines through  $C_0$  and  $C_P$  and the top left of the Edeworth box intersect the diagonal  $(1 - h_1^{**})Q$  and  $(1 - h_2^{**})Q$  and  $(1 - h^{**})Q$ , respectively. It is obvious from the picture that  $(1 - h_1^{**})Q > (1 - h^{**})Q$  and that  $(1 - h_2^{**})Q > (1 - h_2)Q$ . We show this algebraically. Define  $h^{**}$  to solve

$$\begin{aligned} W^{CDS}(1 - h^{**}) &= q_U[z_y^{CDS}d_U^Y + (x_{0^*} + z_x^{CDS})] \\ W^{CDS}h^{**} &= q_D[z_y^{CDS}d_D^Y + (x_{0^*} + z_x^{CDS})] \end{aligned}$$

From Step 1,  $h^{**} \leq h^A$ . Now define  $h_1^{**}, h_2^{**}$  by the following equations

$$\begin{aligned} W^{CDS}(1 - h_1^{**}) &= q_U[z_y^{CDS}(d_U^Y - d_D^Y) + (x_{0^*} + z_x^{CDS})] \\ W^{CDS}h_2^{**} &= q_D[(x_{0^*} + z_x^{CDS})] \end{aligned}$$

The  $CDS$  equations (20) and (19) are very similar, except with  $\pi_U > q_U, \pi_D < q_D$  replacing  $q_U, q_D$ . It follows that  $h_2^{CDS} < h_2^{**}$  and  $(1 - h_1^{CDS}) > (1 - h_1^{**})$ , that is also  $h_1^{CDS} < h_1^{**}$ .

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<sup>21</sup>For the inequality we rely on the algebraic fact that if  $\frac{a}{b} > \frac{c}{d}$ , then  $\frac{a-c}{b-d} > \frac{c}{d}$ , provided that  $a > c, b > d$ .

From the above equations we also have that

$$\begin{aligned}
 W^{CDS}(h_1^{**} - h^{**}) &= q_U z_y^{CDS} d_D^Y \\
 W^{CDS}(h^{**} - h_2^{**}) &= q_D z_y^{CDS} d_D^Y \\
 \frac{(h_1^{**} - h^{**})}{(h^{**} - h_2^{**})} &= \frac{q_U}{q_D}
 \end{aligned}$$

From step 3 it now follows that  $h^* < h^{**} \leq h^A$ , giving us the desired contradiction to our previous findings that  $q_U = \gamma_U(h^*) \geq p_U^A = \gamma_U(h^A)$ . ■