AFFECTIVE UTILITIES:
A RATIONAL THEORY OF OPTIMISTIC BIAS IN ASSET MARKETS

By

Anat Bracha and Donald J. Brown

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YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/

Affective Utilities: A Rational Theory of Optimistic Bias in Asset Markets

Anat Bracha* and Donald J. Brown†

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Abstract

The equilibrium prices in asset markets, as stated by Keynes (1930): "...will be fixed at the point at which the sales of the bears and the purchases of the bulls are balanced." We propose a descriptive theory of finance explicating Keynes' claim that the prices of assets today equilibrate the optimism and pessimism of bulls and bears regarding the payoffs of assets tomorrow.

This equilibration of optimistic and pessimistic beliefs of investors is a consequence of investors maximizing affective utilities subject to budget constraints defined by market prices and investor's income. The set of affective utilities is a new class of non-expected utility functions representing the attitudes of investors for optimism or pessimism, defined as the composition of the investor's attutudes for risk and her attitudes for ambiguity. Bulls and bears are defined respectively as optimistic and pessimistic investors. .

JEL Classification: D81, G02, G11

Keywords: Risk, Ambiguity, Irrational Exhuberance

1 Introduction

Subjective expected utility theory, originally proposed by Savage as the foundation of Bayesian statistics, is a theory of decision-making under uncertainty that "... does not leave room for optimism or pessimism to play any role in the person's judgment" (Savage, 1954, p. 68). This perspective is inconsistent with the view of Keynes who thought of the market price as a balance of the sales of bears, the pessimists, and the purchases of bulls, the optimists. That is, "equilibrium prices in asset markets will be fixed at the point at which the sales of the bears and the purchases of the bulls are balanced" (Keynes, 1930). In Keynes, the equilibrium in asset markets is an affective notion. That is,the optimism and pessimism of investors not the risk and return of different asset classes determine the equilibrium asset prices. In addition to economists, there are also psychologists who acknowledge the presence

*Research Department, The Federal Reserve Bank of Boston, 600 Atlantic Avenue, Boston, MA 02210, USA.

†Department of Economics, Yale University, Box 208268, New Haven, CT 06520-8268, USA.

of optimistic bias: [2002 Nobel Laureate] Kahneman "Most of us view the world as more benign than it really is.... We also tend to exaggerate our ability to forecast the future. In terms of its consequences for decisions, the optimistic bias may well be the most significant cognitive bias." Cognitive biases, such as the optimistic bias, are thought by some economists and psychologists to be inconsistent with the economists' conception of rational choice.

In this paper, we propose a rational theory of optimistic (pessimistic) bias in asset markets. That is, the behavior of bulls and bears is rational in the standard economic sense of agents maximizing (non-expected) utility subject to a budget constraint, defined by asset prices and the agent's income. This new class of non-expected utilities, defined as affective utilities in this paper, are utility representations of the investor's attitudes for optimism. Affective utility is an empirically tractable and descriptive representation of an investor's attitudes in financial markets, if she is either a bull or a bear. Simply put, bulls are optimists who believe that tomorrow asset prices will go up, while bears are pessimists who believe that tomorrow asset prices will go down.

Affective utilities are defined as the composition of the utility representation of investor's attitudes for risk and the utility representation of her attitudes for ambiguity, where attitudes for risk and attitudes for ambiguity are assumed to be independent. If $U(x)$ represents the investor's utility attitudes for risk, where $x \equiv \langle x_1, x_2, ..., x_N \rangle$ is a limited liability state-contingent claim, then $U(x) \equiv \langle u(x_1), u(x_2), ..., u(x_N) \rangle$ is the corresponding state-utility vector for x, where $u(x_j)$ is the utility of the payoff x_j, if state j occurs. If $J(y)$ represents her utility attitudes for ambiguity, where y is the state-utility vector $U(x)$, then

$$U : X \subseteq R^N_{++} \longrightarrow Y \subseteq R^N_{++}$$

and

$$J : Y \subseteq R^N_{++} \longrightarrow R$$

$$x \longrightarrow J \circ U(x)$$

is the composition of U and J, where $J \circ U(x)$ represents the investor's utility attitudes for optimism(pessimism)

We follow the asset pricing literature where an investor is said to be risk-averse if her utility of wealth $u(w)$ is a concave, monotone function of wealth w and risk-seeking if her utility of wealth $u(w)$ is a convex, monotone function of wealth w. To represent attitudes for ambiguity, we follow the decision- theoretic literature,where a decision-maker is said to be ambiguity-averse if $J(U(x))$ is a concave function of state-utility vectors $U(x)$ — for details, see Maccheroni, F., Marinacci, M., Rustichini, A., (2006) — and a decision-maker is said to be ambiguity-seeking if $J(U(x))$ is a convex function of state-utility vectors $U(x)$ – for details, see Bracha and Brown (2012). We define bulls as investors endowed with affective utilities $J \circ U(x)$ convex in x and bears as investors endowed with affective utilities $J \circ U(x)$ concave in x. We show in the next section that these specifications are equivalent to investors being bulls if and only if they have optimistic beliefs about the future payoffs of state-contingent claims and investors are bears if and only if they have pessimistic beliefs about the future payoffs of state-contingent claims.

Table 1 below summarizes the four types of affective utilities, where the cells are investors' attitudes for optimism and pessimism. An investor who is both risk -averse and ambiguity- averse is a bear, i.e., a pessimist. Similarly, an investor who is both risk- seeking and ambiguity- seeking is a bull, i.e., an optimist. These cases, the diagonal cells of the table, are the symmetric affective utilities and the off-diagonal cells of the table are the asymmetric affective utilities.

Table 1: Affective Utilities

Utilities	Risk-averse	Risk-seeking
Ambiguity-averse	Bears	Asymmetric
Ambiguity-seeking	Asymmetric	Bulls

Economists are willing to believe that investors endowed with the composition of ambiguity -averse utilities and risk -averse utilities are bears and that investors endowed with the composition of ambiguity-seeking utilities and risk -seeking utilities are bulls, where we assume that both U and J are monotone. Their intuition follows from the convexity or concavity of the composition of monotone convex or concave functions, see section 3.2 in Boyd and Vandenberghe (2004). They may be surprised that for asymmetric quadratic affective utilities, given a scalar proxy for risk, and a scalar proxy for ambiguity, that there exists a state-contingent claim $\widehat{x}$, "the reference point," where for quadratic utilities of ambiguity and risk, $J \circ U(x)$ is concave or pessimistic on

$$[\widehat{x}, +\infty] \equiv \{x \in R_+^N : x \geq \widehat{x}\}$$

and $J \circ U(x)$ is convex or optimistic on

$$(0, \widehat{x}] \equiv \{x \in R_+^N : x \leq \widehat{x}\}$$

That is, an investor with quadratic utilities of ambiguity and quadratic utilities of risk is a bull for "losses," and a bear for "gains," reminiscent of the shape and rationale of risk preferences in prospect theory — see Kahneman (2011).

To prove Keynes' claim that the prices of assets today equilibrate the optimism and pessimism of bulls and bears regarding the payoffs of assets tomorrow, we consider the existence and optimality of competitive equilibria in a two period, limited liability, state-contingent claim model of exchange, with a finite number of states and a continuum of bulls and bears. It follows from Aumann's (1969) existence and (1966) core equivalence theorems that equilibria exist and the third welfare theorem holds, i.e., every core allocation can be supported as a competitive allocation. In the final section of the paper, the equilibration of the optimism of bulls and the pessimism of bears, as claimed by Keynes, is discussed in the context of the patterns of trade between bulls and bears at the equilibrium prices.

In the next section we present formal definitions of attitudes for risk and ambiguity. We illustrate these notions with quadratic utility representations of attitudes for risk and ambiguity.

2 Notions of Risk, Ambiguity and Optimism

First, a few words about the notions of risk and ambiguity as they are used in this paper. For von Neumann and Morgenstern (1944) risk means we know the probabilities of tomorrow's state of the world. Risk-seeking investors prefer risky lotteries to certain lotteries with payoffs equal to the expected values of the risky lotteries. Risk-averse agents prefer lotteries with certain payoffs to lotteries where the expected values are equal to the certain payoffs. Ellsberg (1961) introduced the notion of ambiguity as the alternative notion to risk, when we are ignorant of the probability of states of the world tomorrow. In Ellsberg's celebrated two-color paradox, subjects who choose the ambiguous urn in both trials are ambiguity-seeking and subjects who choose the risky urn in both trials are ambiguity-averse.

If the affective utility function is a strictly, convex (concave) smooth function of limited liability state-contingent claims x then the investor is optimistic (pessimistic). The value of the gradient of the affective utility function at x is the investor's perceived unnormalized probability distribution for x. That is, the investor's perceived odds that state k will occur tomorrow with payoff x_k. It follows from the envelope theorem applied to the Legendre–Fenchel biconjugate representation of strictly, convex (concave) smooth functions, that the gradient of the investor's utility function with respect to x is a strictly monotone increasing (decreasing) map. In fact, these conditions are both necessary and sufficient for $J \circ U(x)$ to be strictly, convex (concave). This formal definition of optimism or pessimism with respect to state-contingent claims depends on both the investor's attitudes for risk and her attitudes for ambiguity. This is an immediate consequence of the chain rule in computing the gradient of the composite affective utility function. For bulls, the Legendre–Fenchel biconjugate of $J \circ U(x)$ is denoted $[J \circ U(x)]^{**}$, where R^N_{++} is the effective domain of $J \circ U(x)$, $[J \circ U(x)]^{**}$, $[J(\pi)]$ and $[J(\pi)]^*$

$$[J \circ U(x)]^{**} \equiv \max_{\pi \in R^N_{++}} \left[\sum \pi \cdot x - [J(\pi)]^* \right]$$

where $[J(\pi)]^*$, the Legendre -Fenchel conjugate of $J \circ U(x)$ is a smooth, strictly convex function on R^N_{++} and

$$[J(\pi)]^* \equiv \max_{x \in R^N_{++}} \left[\sum \pi \cdot x - J \circ U(x) \right]$$

For bears,

$$[J \circ U(x)]^{**} \equiv \min_{\pi \in R^N_{++}} \left[\sum \pi \cdot x - [J(\pi)]^* \right]$$

the Legendre–Fenchel conjugate of $J \circ U(x)$, where $[J(\pi)]^*$ is a smooth, strictly concave function on R^N_{++}, and

$$[J(\pi)]^* \equiv \min_{x \in R^N_{++}} \left[\sum \pi \cdot x - J \circ U(x) \right]$$

Hence by the envelope theorem, if

$$[J \circ U(x)]^{**} = \max_{\pi \in R^N_{++}} \left[\sum \pi \cdot x - [J(\pi)]^*\right] = \left[\sum \widehat{\pi} \cdot x - [J(\widehat{\pi})]^*\right]$$

then

$$\nabla_x [J \circ U(x)]^{**} = \arg \max_{\pi \in R^N_{++}} \left[\sum \pi \cdot x - [J(\pi)]^*\right] = \widehat{\pi}$$

and if

$$[J \circ U(x)]^{**} = \min_{\pi \in R^N_{++}} \left[\sum \pi \cdot x - [J(\pi)]^*\right] = \left[\sum \widehat{\pi} \cdot x - [J(\widehat{\pi})]^*\right].$$

then

$$\nabla_x [J \circ U(x)]^{**} = \arg \min_{\pi \in R^N_{++}} \left[\sum \pi \cdot x - [J(\pi)]^*\right] = \widehat{\pi}$$

It follows from the biconjugate theorem that

$$[J \circ U(x)]^{**} \equiv J \circ U(x)$$

Hence the beliefs of bulls are monotone increasing maps of asset payoffs and the beliefs of bears are monotone decreasing maps of asset payoffs. See our working paper, CFDP 1898 for additional details.

3 Quadratic Utilities for Risk and Ambiguity

In working paper CFDP 1898 we examine the relationship between attitudes for optimism (pessimism), risk and ambiguity for additively-separable affective utilities, where in that paper, affective utilities are called Keynesian utilities. This family of examples is intended to illustrate the concepts – see CFDP 1898 for additional details. These examples do suggest the richer class of quadratic representations of investor's attitudes for risk, and ambiguity that allow econometric estimation of two scalar proxies for optimistic (pessimistic) bias in asset markets, denoted respectively as β and α.

In this section we analyze the case of quadratic utilities and present conditions on β and α, such that the investor is optimistic or pessimistic. Here we show that for a class of asymmetric affective utilities where the investor is risk averse and ambiguity seeking or risk seeking and ambiguity averse, that the space of state-contingent claims can be partitioned into quadrants, relative to some reference state-contingent claim, where the the investor's affective utility function is convex in the first quadrant and concave in the third quadrant or concave in the first quadrant ad convex in the third quadrant – see CFDP 1898 for additional details.

Definition 1 *Quadratic utilities for risk:*$U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$ *is a monotone, smooth, strictly concave (convex), diagonal quadratic map from* R^N_{++} *onto* R^N_{++} *,with* β *the proxy for risk. That is,* $U(x)$ *is an* $N \times N$ *diagonal matrix, where for* $k = 1, 2, ..., N$*:* $U_{k,k}(x) = u(x_k) \equiv \beta_0 + \beta_1 x_k + \frac{\beta}{2} x_k^2$*.* $u(x_k)$ *is strictly concave iff* $\beta < 0$ *and* $u(x_k)$ *is strictly convex iff* $\beta > 0$

Definition 2 *If $\eta \in R$, then diag$(\eta) \in R^{N\times N}$ is a symmetric diagonal matrix with eigenvalues equal to η. Quadratic utilities for ambiguity: $J(y)$ is a monotone, smooth, strictly concave (convex) quadratic function from R^N_+ into R,with α the proxy for ambiguity. That is, $J(y) = \frac{1}{2} y diag(\alpha) y + \alpha_1 \cdot y + \alpha_0$, where $\alpha_0 \in R$, $\alpha_1 \in R^N_{++}$. $J(y)$ is strictly concave iff $\alpha < 0$ and $J(y)$ is strictly convex iff $\alpha > 0$.*

Theorem 3 *If $J \circ U(x)$ is the composition of quadratic utilities for risk and quadratic utilities for ambiguity,where*

$$\text{diag}(\beta) \equiv \text{diag}[\nabla^2_x U(x)] \text{ and } \text{diag}(\alpha) \equiv \text{diag}[\nabla^2_{U(x)} J(U(x))]$$

then $\nabla^K_x J \circ U(x) \equiv 0$ for $K \geq 5$.

Proof. If

$$\nabla_x J \circ U(x) = [\nabla_x U(x)] \cdot [\nabla_{U(x)} J(U(x))].$$

the Hadamard or pointwise product of $[\nabla_x U(x)]$ and $[\nabla_{U(x)} J(U(x))]$, then by application of the chain rule for Hadamard products proposed by Bentler and Lee (1978) and proved by Magnus and Neudecker (1985) — see CFDP 1898 for details, we obtain:

$$\nabla^2_x J \circ U(x) = \text{diag}(\alpha)(\text{diag}[\nabla_x U(x)])^2 + \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(U(x))]$$

$$\nabla^3_x J \circ U(x) = 3\text{diag}(\beta)\text{diag}(\alpha)\text{diag}[\nabla_x U(x)]$$

$$\nabla^4_x J \circ U(x) = 3[\text{diag}(\beta)]^2[\text{diag}(\alpha)]$$

$$\nabla^K_x J \circ U(x) = 0 \text{ for } K \geq 5.$$

■

In Theorems 2 and 3, we characterize asymmetric affective utilities, where we prove the existence of a reference point $\widehat{x}$ that partitions R^N_+ into the standard four quadrants, with the reference point $\widehat{x}$ as the origin. $J \circ U(x)$ is concave in quadrant I,where quadrant $I \equiv \{x \in R^N_+ : x \geq \widehat{x}\}$ and convex in quadrant III, where quadrant $III \equiv \{x \in R^N_+ : x \leq \widehat{x}\}$ The Hessian of $J \circ U(x)$ is indefinite in quadrants II and IV. That is, $\nabla^2_x J \circ U(x)$ is indefinite on $R^N_+/\{(\widehat{x}, +\infty] \cup (0, \widehat{x}]\}$. $J \circ U(x)$ is optimistic for "losses," i.e., $x \leq \widehat{x}$ and pessimistic for "gains," i.e., $x \geq \widehat{x}$, analogous with the shape of the utility of risk in prospect theory — see figure 10 in Khaneman (2011). In Theorems 4 and 5, we characterize symmetric affective utilities or optimistic and pessimistic investors.

Theorem 4 *If $J \circ U(x)$, is the composition of $U(x)$ and $J(y)$,where (a) $(y_1, y_2, ..., y_N) \equiv y = U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$ is a monotone, smooth,strictly concave, diagonal quadratic map from R^N_{++} onto R^N_{++} ,with the proxy for risk, $-\beta < 0$, (b) $J(y)$ is a monotone, smooth,strictly convex quadratic map from R^N_+ into R,with the proxy for ambiguity, $\alpha > 0$, (c)*

$$\nabla^2_x J \circ \widehat{U}(x) = \text{diag}(\alpha)(\text{diag}[\nabla_x \widehat{U}(x)])^2 - \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(\widehat{U}(x))] : \textit{Chain Rule}$$

then there exists a reference point $\widehat{x}$ such that the financial market data D is rationalized by the composite function $J \circ U(x)$ with two domains of convexity: $(\widehat{x}, +\infty]$ and $(0, \widehat{x}]$,where $J \circ U(x)$ is concave on $(\widehat{x}, +\infty]$ and $J \circ U(x)$ is convex on $(0, \widehat{x}]$

Proof.

$$\nabla_x^2 U(x) = -\text{ diag}(\beta) \text{ where } -\beta < 0\text{: Risk Averse}$$

$$\nabla_{U(x)}^2 J(U(x)) = \text{diag}(\alpha) \text{ where } \alpha > 0\text{: Ambiguity-Seeking}$$

$$\nabla_x^2 J \circ U(x) = \text{diag}(\alpha)(\text{diag}[\nabla_x U(x)])^2 - \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(U(x))]\text{: Chain Rule}$$

$$\lim_{\|x\|_\infty \to \infty} \left\| diag[\nabla_{U(x)} J(U(x))]^{-1} diag[\nabla_x U(x)]^2 \right\|_\infty = 0$$

$$\text{diag}[\nabla_{U(x)} J(U(x))]^{-1}\text{diag}[\nabla_x U(x)]^2 \leq \text{diag}[\nabla_{U(x)} J(U(\widehat{x}))]^{-1}\text{diag}[\nabla_x U(\widehat{x})]^2 \leq \text{diag}[\tfrac{\beta}{\alpha}]\text{: Bears}$$

$$\lim_{x \to 0} \left\| \text{diag}[\nabla_{U(x)} J(U(x))]\text{diag}[\nabla_x U(x)]^{-2} \right\|_\infty = 0$$

$$\text{diag}[\nabla_{U(x)} J(U(x))]^{-1}\text{diag}[\nabla_x U(x)]^2 \leq \text{diag}[\nabla_{U(x)} J(U(\widehat{x}))]\text{diag}[\nabla_x U(\widehat{x})]^{-2} \leq \text{diag}[\tfrac{\alpha}{\beta}]\text{: Bulls}$$

■

Theorem 5 *If* $J \circ U(x)$*, is the composition of* $U(x)$ *and* $J(y)$*,where (a)* $(y_1, y_2, ..., y_N) \equiv y = U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$ *is a monotone, smooth, convex, diagonal quadratic map from* R_{++}^N *onto* R_{++}^N *with the proxy for risk,* $\beta > 0$*, (b)* $J(y)$ *is a monotone, smooth, concave quadratic map from* R_+^N *into* R *with the proxy for ambiguity,* $-\alpha < 0$*, (c)*

$$\nabla_x^2 J \circ \widehat{U}(x) = -\text{diag}(\alpha)(\text{diag}[\nabla_x \widehat{U}(x)])^2 + \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(\widehat{U}(x))]\text{: } \textit{Chain Rule}$$

then there exists a reference point $\widehat{x}$ *such that the financial market data* D *is rationalized by the composite function* $J(U(x))$ *with two domains of convexity:* $(\widehat{x}, +\infty]$ *and* $(0, \widehat{x}]$*, where* $J \circ U(x)$ *is concave on* $(\widehat{x}, +\infty]$ *and* $J \circ U(x)$ *is convex on* $(0, \widehat{x}]$*.*

Proof.

$$\nabla_x^2 U(x) = \text{diag}(\beta) \text{ where } \beta > 0\text{: Risk-Seeking}$$

$$\nabla_{\widehat{U}(x)}^2 J(U(x)) = -\text{diag}(\alpha) \text{ where } -\alpha < 0\text{: Ambiguity-Averse}$$

$$\nabla_x^2 J \circ \widehat{U}(x) = -\text{diag}(\alpha)(\text{diag}[\nabla_x \widehat{U}(x)])^2 + \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(\widehat{U}(x))]\text{: Chain Rule}$$

$$\lim_{\|x\|_\infty \to \infty} \left\| \text{diag}[\nabla_{U(x)} J(\widehat{U}(x))]\text{diag}[\nabla_x \widehat{U}(x)]^{-2} \right\|_\infty = 0$$

$$\text{diag}\nabla_{U(x)} J(\widehat{U}(x))]\text{diag}[\nabla_x \widehat{U}(x)]^{-2} \leq \text{diag}[\nabla_{U(x)} J(\widehat{U}(\widehat{x}))]\text{diag}[\nabla_x \widehat{U}(\widehat{x})]^{-2} \leq \text{diag}[\tfrac{\alpha}{\beta}]\text{: Bears}$$

$$\lim_{\|x\| \to 0} \left\| \text{diag}[\nabla_{U(x)} J(\widehat{U}(x))]^{-1}\text{diag}[\nabla_x \widehat{U}(x)]^2 \right\|_\infty = 0$$

$$\text{diag}[\nabla_{U(x)} J(\widehat{U}(x))]\text{diag}[\nabla_x \widehat{U}(x)]^{-2} \leq \text{diag}[\nabla_{U(x)} J(\widehat{U}(\widehat{x}))]\text{diag}[\nabla_x \widehat{U}(\widehat{x})]^{-2} \leq \text{diag}[\tfrac{\beta}{\alpha}]\text{: Bulls.}$$

Theorem 6 *If* $J \circ U(x)$, *is the composition of* $U(x)$ *and* $J(y)$,*where (a)* $(y_1, y_2, ..., y_N) \equiv y = U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$ *is a monotone, smooth, concave, diagonal quadratic map from* R^N_{++} *onto* R^N_{++},*with the proxy for risk,*$-\beta < 0$ *(b)* $J(y)$ *is a monotone, smooth,concave quadratic function from* R^N_{++} *into* R, *with the proxy for ambiguity,* $-\ \alpha < 0$ *(c)*

$$\nabla^2_x J \circ \widehat{U}(x) = -\mathrm{diag}(\alpha)(\mathrm{diag}[\nabla_x \widehat{U}(x)])^2 - \mathrm{diag}(\beta)\mathrm{diag}[\nabla_{U(x)} J(\widehat{U}(x))]: \textit{ Chain Rule}$$

then $J \circ U(x)$ *is concave on* R^N_{++}

■

Proof.

$$\nabla^2_x U(x) = -\mathrm{diag}(\beta) \text{ where } -\beta < 0\text{: Risk-Averse}$$

$$\nabla^2_{\widehat{U}(x)} J(U(x)) = -\mathrm{diag}(\alpha) \text{ where } -\alpha < 0\text{: Ambiguity-Averse}$$

$$\text{Bears: } \nabla^2_x J \circ U(x) = -\mathrm{diag}(\alpha)(\mathrm{diag}[\nabla_x U(x)])^2 - \mathrm{diag}(\beta)\mathrm{diag}[\nabla_{U(x)} J(U(x))] < 0.$$

■

Theorem 7 *If* $J \circ U(x)$, *is the composition of* $U(x)$ *and* $J(y)$,*where (a)* $(y_1, y_2, ..., y_N) \equiv y = U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$ *is a monotone, smooth, convex, diagonal quadratic map from* R^N_{++} *onto* R^N_{++} *,with the proxy for risk,* $\beta > 0$, *(b)* $J(y)$ *is a monotone, smooth, convex quadratic function from* R^N_{++} *into* R, *with the proxy for ambiguity,* $\alpha > 0$, *(c)*

$$\nabla^2_x J \circ \widehat{U}(x) = \mathrm{diag}(\alpha)(\mathrm{diag}[\nabla_x \widehat{U}(x)])^2 + \mathrm{diag}(\beta)\mathrm{diag}[\nabla_{U(x)} J(\widehat{U}(x))]: \textit{ Chain Rule}$$

then $J \circ U(x)$ *is convex on* R^N_{++}.

Proof.

$$\nabla^2_x U(x) = \mathrm{diag}(\beta) \text{ where } \beta > 0\text{: Risk-Seeking}$$

$$\nabla^2_{\widehat{U}(x)} J(U(x)) = \mathrm{diag}(\alpha) \text{ where } \alpha > 0\text{: Ambiguity-Seeking}$$

$$\text{Bulls: } \nabla^2_x J \circ U(x) = \mathrm{diag}(\alpha)(\mathrm{diag}[\nabla_x U(x)])^2 + \mathrm{diag}(\beta)\mathrm{diag}[\nabla_{U(x)} J(U(x))] > 0.$$

■

4 Patterns of Trade between Bulls and Bears

Recall the Keynesian aphorism: "The equilibrium prices in asset markets will be fixed at the point at which the sales of the bears and the purchases of the bulls are balanced." In this final section, we explicate Keynes' claim that the prices of assets today equilibrate the optimism and pessimism of bulls and bears regarding the payoffs of assets tomorrow. We assume, that the consumption sets of investors are convex, open subsets of R^N containing the positive orthant and that investors maximize smooth, monotone concave or smooth, monotone convex affective utility function subject to a budget constraint. The budget constraint is defined by market prices and the investor's income.

Bears maximize a smooth, monotone, concave (pessimistic) affective utility function; deriving the asset demand of bears is therefore a standard application of the Karush–Kuhn–Tucker (KKT) Theorem, where in our case the Slater constraint qualification is trivially satisfied (see Boyd and Vandenberghe (2004). For this reason, the first order conditions for a saddle-point of the Lagrangian are necessary and sufficient for optimality. For bears, the utility maximizing optimum may be in the interior of the positive orthant. where the expected odds today, by bears, of tomorrow's market prices are equal to the odds determined by today's market prices.

Bulls maximize a monotone, convex (optimistic) affective utility function subject to a budget constraint. This is quite a different problem: the optimum in this case is achieved at an extreme point of the budget set — for details, see chapter 32 in Rockafellar (1970). If there are only two states of the world, where the market prices are (1,1) and the investor's income is 1, then the extreme points of the budget set are (0,0), (0,1) and (1,0). For a monotone convex utility function, the optimum is achieved at (0,1) or (1,0), the corners of the budget line. More generally, the utility maximizing optimum for bulls is always on the boundary of the positive orthant, i.e., bulls speculate on the most optimistic outcomes.

We consider a two period investment model with two states of the world, where $x = (x_1, x_2)$ is a state-contingent claim and today's state prices are (p_1, p_2). If the investor's income today is I and she is endowed with a convex affective utility function, $U_{Bulls}(x)$, then her optimal investment problem is (P):

$$\max\{U_{Bulls}(x) \mid -x_1 \leq 0, \ \ -x_2 \leq 0, \ p \cdot x - I \leq 0\}$$

where the Fritz John Lagrangian for constrained maximization

$$L(x_1, x_2, \lambda_0, \lambda_1, \lambda_2, \lambda_3) \equiv \lambda_0 U_{Bulls}(x) - \lambda_1[-x_1] - \lambda_2[-x_2] - \lambda_3[p \cdot x - I].$$

Theorem 8 *[Fritz John]: If x^* is a local maximizer of (P) then there exists multipliers $\lambda^* \equiv (\lambda_0^*, \lambda_1^*, \lambda_2^*, \lambda_3^*) \gneqq 0$ such that:*

$$\lambda_0^*(\partial_{x_1} U_{Bulls}(x^*), \partial_{x_2} U_{Bulls}(x^*)) = (-\lambda_1^* + \lambda_3^* p_1, -\lambda_2^* + \lambda_3^* p_2),$$

where $\lambda_0^* = 1$, by Theorems 19.12 in Simon and Blume (1994) (a)If $x^* = (0, x_2^*)$,then $\lambda_2^* = 0$.Hence

$$(\partial_{x_1} U_{Bulls}((0, x_2^*)), \partial_{x_2} U_{Bulls}((0, x_2^*)) = (-\lambda_1^* + \lambda_3^* p_1, \lambda_3^* p_2)$$

It follows that some bulls are more optimistic than the market that tomorrow's state of the world is state 2. That is,

$$\frac{\partial_{x_2} U_{Bulls}((0, x_2^*))}{\partial_{x1} U_{Bulls}((0, x_2^*))} = \frac{\lambda_3^* p_2}{-\lambda_1^* + \lambda_3^* p_1} > \frac{p_2}{p_1}$$

(b)If $x^* = (x_1^*, 0)$, then $\lambda_1^* = 0$.Hence

$$(\partial_{x_1} U_{Bulls}((x_1^*, 0)), \partial_{x_2} U_{Bulls}((x_1^*, 0)) = (\lambda_3^* p_1, -\lambda_2^* + \lambda_3^* p_2).$$

It follows that other bulls are more optimistic than the market that tomorrow's state of the world is state 1. That is,

$$\frac{\partial_{x_1} U_{Bulls}((x_1^*, 0))}{\partial_{x_2} U_{Bulls}((x_1^*, 0))} = \frac{\lambda_3^* p_1}{-\lambda_2^* + \lambda_3^* p_2} > \frac{p_1}{p_2}.$$

If the investor's income today is I and she is endowed with concave affective utilities $U_{Bears}(x)$, then her optimal investment problem is (P) :

$$\max\{U_{Bears}(x) \mid -x_1 \leq 0, \ -x_2 \leq 0, \ p \cdot x - I \leq 0\}$$

where the KKT Lagrangian for constrained maximization

$$L(x_1, x_2, \eta) \equiv U_{Bears}(x) - \eta_1[-x_1] - \eta_2[-x_2] - \eta_3[p \cdot x - I].$$

Theorem 9 *[Karush-Kuhn-Tucker] If Slater's constraint qualification is satisfied then x^* is a maximizer of (P), where $x^* \in R_+^N$, iff there exists a multipliers $\eta^* \equiv (\eta_1^*, \eta_2^*, \eta_3^*) \gneqq 0$ such that:*

$$(\partial_{x_1} U_{Bears}(x^*), \partial_{x_2} U_{Bears}(x^*)) = (\eta_3^* p_1 - \eta_1^*, \eta_3^* p_2 - \eta_2^*).$$

(a)If $x^* = (0, x_2^*)$, then $\eta_2^* = 0$ and

$$(\partial_{x_1} U_{Bears}((0, x_2^*)), \partial_{x_2} U_{Bears}((0, x_2^*)) = (\eta_3^* p_1 - \eta_1^*, \eta_3^* p_2).$$

It follows that some bears are more pessimistic than the market that tomorrow's state of the world is state 1. That is,

$$\frac{\partial_{x_1} U_{Bears}((0, x_2^*))}{\partial_{x_2} U_{Bears}((0, x_2^*))} = \frac{\eta_3^* p_1 - \eta_1^*}{\eta_3^* p_2} < \frac{p_1}{p_2}$$

(b)If $x^* = (x_1^*, 0)$,then $\eta_1^* = 0$ and

$$(\partial_{x_1} U_{Bears}(x^*), \partial_{x_2} U_{Bears}(x^*)) = (\eta_3^* p_1 - \eta_1^*, \eta_3^* p_2).$$

It follows that other bears are more pessimistic than the market that tomorrow's state of the world is state 2. That is,

$$\frac{\partial_{x_2} U_{Bears}((x_1^*, 0))}{\partial_{x_1} U_{Bears}((x_1^*, 0))} = \frac{\eta_3^* p_2 - \eta_2^*}{\eta_3^* p_1} < \frac{p_2}{p_1}.$$

Theorem 10 *(a) At the market prices (p_1, p_2), some bulls trade Arrow–Debreu state-contingent claims for state 2 with bears for Arrow–Debreu state-contingent claims for state 1.That is,*

$$\frac{\partial_{x_2} U_{Bulls}((0, x_2^*))}{\partial_{x_1} U_{Bulls}((0, x_2^*))} > \frac{p_2}{p_1} \geq \frac{\partial_{x_2} U_{Bears}((x_1^*, 0))}{\partial_{x_1} U_{Bears}((x_1^*, 0))}.$$

(b) At the market prices (p_1, p_2), other bulls trade Arrow–Debreu state-contingent claims for state 2 *with other bulls for Arrow–Debreu state-contingent claims for state* 1*. That is,*

$$\frac{\partial_{x_2} U_{Bulls}((0, x_2^*))}{\partial_{x_1} U_{Bulls}((0, x_2^*))} > \frac{p_2}{p_1} > \frac{\partial_{x_2} U_{Bulls}((x_1^*, 0))}{\partial_{x_1} U_{Bulls}((x_1^*, 0))}.$$

(c) At the market prices (p_1, p_2), some bulls trade Arrow-Debreu state-contingent claims for state 1 *with bears for Arrow–Debreu state-contingent claims for state* 2*. That is,*

$$\frac{\partial_{x_1} U_{Bulls}((x_1^*, 0))}{\partial_{x_2} U_{Bulls}((x_1^*, 0))} > \frac{p_1}{p_2} \geq \frac{\partial_{x_1} U_{Bears}((0, x_2^*))}{\partial_{x_2} U_{Bears}((0, x_2^*))}.$$

(d) At the market prices (p_1, p_2), other bulls trade Arrow-Debreu state-contingent claims for state 1 *with other bulls for Arrow–Debreu state-contingent claims for state* 2*. That is,*

$$\frac{\partial_{x_1} U_{Bulls}((x_1^*, 0))}{\partial_{x_2} U_{Bulls}((x_1^*, 0))} > \frac{p_1}{p_2} > \frac{\partial_{x_1} U_{Bulls}((0, x_2^*))}{\partial_{x_2} U_{Bulls}((0, x_2^*))}.$$

In our model, the fundamental difference between bears and bulls is that bulls always speculate, by purchasing only the Arrow–Debreu security that pays 1 if state 1 occurs or purchasing only the Arrow–Debreu security that pays 1 if state 2 occurs. Bulls never diversify by purchasing a portfolio of the two Arrow-Debreu security, suggesting to some economists that these investors are "irrationally exuberant." In contrast, bears may speculate or diversify, depending on the equilibrium prices and the shape of their indifference curves. That is, if indifference curves don't cut the coordinate axes, say the indifference curves of a Cobb–Douglas utility function, then bears only diversify and never speculate. Examples of equilibrium for this special case are presented in CFDP 1898.

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