KEYNESIAN UTILITIES: BULLS AND BEARS

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April 2013

COWLES FOUNDATION DISCUSSION PAPER NO. 1891

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
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http://cowles.econ.yale.edu/

Keynesian Utilities: Bulls and Bears

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April 3, 2013

Abstract

We propose Keynesian utilities as a new class of non-expected utility functions representing the preferences of investors for optimism, defined as the composition of the investor's preferences for risk and her preferences for ambiguity. The optimism or pessimism of Keynesian utilities is determined by empirical proxies for risk and ambiguity. Bulls and bears are defined respectively as optimistic and pessimistic investors. The resulting family of Afriat inequalities are necessary and sufficient for rationalizing the asset demands of bulls and bears with Keynesian utilities.

JEL Classification: D81, G02, G11
Keywords: Uncertainty, Optimism, Afriat inequalities

1 Introduction

Financial markets unlike roulette or craps are not games of chance. There are no well defined relative frequencies of future payoffs of financial assets. More generally, the returns in financial markets are not generated by Brownian motion. Random walks down Wall Street have no ex post empirical foundation — see Mandelbroit (2004). The modern theory of finance, also termed stochastic finance by some authors, is a subfield of applied probability theory that at best is a normative theory of gambling in idealized casinos. It is not a descriptive theory of an investor's behavior in financial markets.

Bracha and Brown (2012) extended the theory of variational preferences introduced by F. Maccheroni, M. Marinacci, and A. Rustichini [MMR] (2006) to a theory of revealed preferences for ambiguity, where investors are ambiguity-seeking iff they believe today that tomorrow large state-utility payoffs are more likely than small state-utility payoffs and investors are ambiguity-averse iff they believe today that tomorrow small state-utility payoffs are more likely than large state-utility payoffs. That is, Bracha and Brown conflate the notions of ambiguity and optimism. This

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equivalence of optimism and ambiguity is a mathematical implication not a substantive consequence of their economic model. Optimism or pessimism is the defining characteristic of the investor's unobservable beliefs on the likelihood of future payoffs, but ambiguity-aversion is a fundamental aspect of the investor's revealed preferences for risk and ambiguity, i.e., observable choice. Recall that the domain of the utility functions representing preferences for ambiguity in MMR and in Bracha and Brown are state-utility vectors, not state-contingent claims, as in the Arrow- Debreu general equilibrium analysis of markets with uncertainty, see chapter 7 in Debreu (1959).

MMR show that utility functions concave in state-utility vectors are ambiguity-averse. Bracha and Brown show that utility functions convex in state-utility vectors are ambiguity-seeking. Using the Legendre–Fenchel biconjugate representation of convex functions and the envelope theorem they also show that the unobserved beliefs of investors are the gradient, with respect to state-utility vectors, of the investor's utility function. As is well known in convex analysis, the gradient of a function is a monotone increasing map iff the function is a convex function. It follows that investors are ambiguity-seeking iff they are optimistic. In a similar argument, Bracha and Brown show that investors are ambiguity-averse iff they are pessimistic. Their characterization of investor's behavior in financial markets, where investor's preferences are not over state-contingent claims but over unobservable state-utility vectors, makes their econometric models difficult to identify and estimate.

The subtitle of this paper is bulls and bears. Here's why. Keynes (1936): "The market price will be fixed at the point at which the sales of the bears and the purchases of the bulls are balanced." Sargent (2008) remarks: "The use of expectations in economic theory is not new. Many earlier economists, including A. C. Pigou, John Maynard Keynes, and John R. Hicks, assigned a central role in the determination of the business cycle to people's expectations about the future. Keynes referred to this as 'waves of optimism and pessimism' that helped determine the level of economic activity." Consequently**,** we introduce Keynesian utilities as a new class of non-expected utilities, representing the investor's preferences for optimism. This is an empirically tractable and descriptive characterization of an investor's preferences in financial markets, where she is either a bull or a bear. Simply put, bulls are optimistic and believe that market prices will go up, but bears are pessimistic and believe that market prices will go down. Hence bulls buy long and bears sell short.

Keynesian utilities are defined as the composition of the investor's preferences for risk and her preferences for ambiguity, where we assume preferences for risk and preferences for ambiguity are independent. If $U(x)$ denotes preferences for risk, and $J(y)$ denotes preferences for ambiguity then

$$U : X \subseteq R^N_{++} \to Y \subseteq R^N_{++}$$

and

$$J : Y \subseteq R^N_{++} \to R$$

where

$$x \to J \circ U(x)$$

is the composition of U and J, denoted $J \circ U(x)$.This specification is a special case of amenable functions, introduced by Rockafellar (1988). We make the additional assumption that $U(x)$ is a concave or convex diagonal map. See Rockafellar and Wets (1998) for a discussion of chain rules for amenable functions. Here we use a chain rule originally proposed by Bentler and Lee (1978) and proved in Magnus and Neudecker (1985). In our model, bulls are investors endowed with convex Keynesian utilities and bears are investors endowed with concave Keynesian utilities. It follows from convex analysis that these specifications are equivalent to assuming that investors are bulls iff they have optimistic beliefs about the future payoffs of state-contingent claims and that investors are bears iff they have pessimistic beliefs about the future payoffs of state-contingent claims.

In the following 2×2 contingency table on the types of Keynesian utilities, the rows are ambiguity-averse and ambiguity-seeking preferences and the columns are risk-averse and risk-seeking preferences. The cells are the investor's preferences for optimism and pessimism. The diagonal cells of the table are the symmetric Keynesian utilities and the off-diagonal cells of the table are the asymmetric Keynesian utilities.

Table 1

Keynesian preferences	Risk-averse	Risk-seeking
Ambiguity-averse	Bears	Asymmetric
Ambiguity-seeking	Asymmetric	Bulls

It is not surprising that bears have Keynesian utilities that are the composition of ambiguity-averse preferences and risk-averse preferences or that bulls have Keynesian utilities that are the composition of ambiguity-seeking preferences and risk-seeking preferences. This observation follows from the theorems in convex analysis on the convexity or concavity of the composition of monotone convex or concave functions. See section 3.2 in Boyd and Vandenberghe (2004).

For asymmetric Keynesian utilities, given α, the proxy for risk, and β, the proxy for ambiguity, we show there exists a state-contingent claim $\hat{x}$, "the reference point" where $J \circ U(x)$ is concave or pessimistic on

$$[\hat{x}, +\infty] \equiv \{x \in R_+^N : x \geq \hat{x}\}$$

and $J \circ U(x)$ is convex or optimistic on

$$(0, \hat{x}] \equiv \{x \in R_+^N : x \leq \hat{x}\}.$$

Here is an example of a 2×2 contingency table for an investor endowed with asymmetric Keynesian utilities. We divide R_+^N into the standard 4 quadrants with the reference point, $\hat{x}$, as the origin:

Table 2

Quadrant II	Quadrant I
Quadrant III	Quadrant IV

Define $(0, \hat{x}] \equiv$ quadrant III and $[\hat{x}, +\infty] \equiv$ quadrant I. The cells of Table 2 is a partition of the domain of $J \circ U(x)$: In Table 3, losses relative to the reference point $\hat{x}$ i.e., the state-contingent claims south-west of $\hat{x}$,are evaluated with a convex utility function and gains relative relative to the reference point $\hat{x}$, i.e., the state-contingent claims north-east of $\hat{x}$, are evaluated with a concave utility function.

Table 3

$\nabla_x^2 J \circ U(x)$ is indefinite on Quadrant II	$J \circ U(x)$ is concave on Quadrant I
$J \circ U(x)$ is convex on Quadrant III	$\nabla_x^2 J \circ U(x)$ is indefinite on Quadrant IV

That is, the investor is a bull for "losses," quadrant III, but a bear for "gains," quadrant I. In prospect theory, preferences for risk have a similar "shape," see figure 10 in Kahneman (2011). Despite this apparent similarity, prospect theory, inspired in part by the Allais paradox, is a critique and elaboration of expected utility theory and Keynesian utility theory, inspired in part by the Ellsberg paradox, is a critique and elaboration of subjective expected utility theory. In Keynesian utility theory preferences for risk, as well as preferences for ambiguity, are the same for losses and gains, and in prospect theory the investor is risk-averse for gains, but risk-seeking for losses..

In the next section,we briefly review the theory of optimism-bias as proposed by Bracha and Brown. In sections 3 and 4, we propose parametric and semiparametric specifications of preferences for risk, preferences for ambiguity and their composition, preferences for optimism. In the final section of the paper, we derive the Keynesian Afriat inequalities for rationalizing the asset demands of investors with Keynesian utilities.

Finally, a few words about the notions of risk, uncertainty, ambiguity and optimism, as they are used in this paper. Risk means we know the probabilities of tomorrow's state of the world. For Keynes (1937) uncertainty means we do not know the probabilities, in fact the notion of probability of states of the world tomorrow may be meaningless. Ellsberg (1961) introduced the notion of ambiguity as the alternative notion to risk, where we are ignorant of the probability of states of the world tomorrow. For Ellsberg there are two kinds of uncertainty: risk and ambiguity. These are the conventions we follow. Optimism (pessimism) refers to the investor's subjective beliefs today regarding the relative likelihood of large versus small payoffs of a state-contingent claim tomorrow.

2 Preferences for Ambiguity

The Bracha and Brown model of preferences for ambiguity has its origins in the following quote of Keynes (1937): "By uncertain knowledge, let me explain, I do not mean merely to distinguish what is known for certain from what is only probable. The game of roulette is not subject, in this sense, to uncertainty; nor is the prospect of a Victory bond being drawn. Or, again, the expectation of life is only slightly uncertain. Even the weather is only moderately uncertain. The sense in which I am using the term is that in which the prospect of a European war is uncertain, or the price of copper and the rate of interest twenty years hence, or the obsolescence of a new invention, or the position of private wealth owners in the social system in 1970. About these matters there is no scientific basis on which to form any calculable probability whatever. We simply do not know."[1]

By preferences for ambiguity we mean the variational preferences introduced by MMR, and the ADM preferences introduced by Bracha and Brown. There is a third class of preferences for ambiguity: ambiguity-neutral or subjective expected utility functions, originally proposed by Savage (1954), as the foundation of Bayesian statistics. As remarked by Aumann (1987) "His (Savage's) postulate $P4$ (roughly speaking, that the probability of an event is independent of the prize offered contingent on that event) can only be understood in terms of a probability concept that has an existence of its own in the decision maker's mind, quite apart from preferences on acts. He (Savage) wrote that '... the ... view sponsored here does not leave room for optimism or pessimism... to play any role in the person's judgement' (1954, p. 68)." This is not the view in the Keynesian aphorism: "The market price will be fixed at the point at which the sales of the bears and the purchases of the bulls are balanced," where as we previously noted: bulls are optimistic investors and bears are pessimistic investors.

We are persuaded by Ellsberg's (1961) critique of Savage's theory of subjective beliefs, where postulate $P4$ is refuted in Ellsberg's thought experiments with two urns containing black and white balls, that subjective expected utility is not a behavioral theory of investment in financial markets, where investors may be either bulls or bears. In Ellsberg's thought experiments, each urn contains 100 balls. The risky urn is known to contain 50 black balls. The distribution of black and white balls in the other urn is unknown, this is the ambiguous urn. There are two experiments. In the first experiment the investor is asked to choose an urn , if a black ball is drawn then the investor receives \$10.00, otherwise \$0.00. In the second experiment the payoffs are reversed. That is, if a white ball is drawn from the selected urn then the investor receives \$10.00, otherwise \$0.00. If the investor chooses the risky urn in both experiments, then she is ambiguity-averse If she chooses the ambiguous urn in both experiments, then she is ambiguity-seeking. The investor is ambiguity-neutral iff she chooses different urns in each experiment. Ellsberg predicts that most investors will be ambiguity-seeking or ambiguity-averse. This is the so-called Ellsberg paradox. These experiments have been performed many times in many classrooms, and Ellsberg's prediction has been confirmed. Here is Ellsberg's explanation of the Ellsberg paradox in his (1961) article: "...we would have to regard the subject's subjective probabilities as being dependent upon his payoffs, his evaluation of the outcomes ... it is impossible to infer from the resulting behavior a set of probabilities for events independent of his payoffs." His assertion contradicts Savage's postulate $P4$.

Bracha and Brown proposed formal definitions of optimism-bias and pessimism-bias, where they implicitly identify ambiguity and optimism. These definitions derive from the representations of ambiguity-seeking and ambiguity-averse utility functions

[1] Uncertainty in this quote means ambiguity in Ellsberg.

as the Legendre–Fenchel biconjugates of convex and concave functions, where $J^*(\pi)$ is the Legendre–Fenchel conjugate of $J(U(x))$. For optimistic utility functions, we invoke the Legendre–Fenchel biconjugate-conjugate for convex functions, where

$$J(U(x)) \equiv \max_{\pi \in R^N_{++}} [\sum \pi \cdot U(x) + J^*(\pi)]$$

and $J^*(\pi)$ is a smooth convex function on $R^N_{++}$where

$$J^*(\pi) \equiv \max_{U(x)} [\sum \pi \cdot U(x) + J(U(x))].$$

For pessimistic utility functions, we invoke the Legendre–Fenchel biconjugate- conjugate for concave functions, where

$$J(U(x)) \equiv \min_{\pi \in R^N_{++}} [\sum \pi \cdot U(x) + J^*(\pi)]$$

and $J^*(\pi)$ is a smooth concave function on $R^N_{++}$where

$$J^*(\pi) \equiv \min_{U(x)} [\sum \pi \cdot U(x) + J(U(x))].$$

If $F(y)$ is a vector-valued map from R^N into R^N,then F is strictly, monotone increasing if for all x and $y \in R^N$:

$$[x - y] \cdot [F(x) - F(y)] > 0$$

If $F(y)$ is a vector-valued map from R^N into R^N,then F is strictly, monotone decreasing if for all x and $y \in R^N$:

$$[x - y] \cdot [F(x) - F(y)] < 0$$

As Bracha and Brown observed $J(z)$ is strictly convex in z where $z = U(x)$ iff $\nabla_z J(z)$ is a strictly, monotone increasing map of z.See section 5.4.3 in Ortega and Rheinboldt (1970) for proof. It follows from the envelope theorem,

$$\nabla_z J(z) = \arg \max_{\pi \in R^N_{++}} [\sum \pi \cdot U(x) + J^*(\pi)] = \widehat{\pi}, \text{ where}$$

$$J(U(x)) = \max_{\pi \in R^N_{++}} [\sum \pi \cdot U(x) + J^*(\pi)] = \sum \widehat{\pi} \cdot U(x) + J^*(\widehat{\pi})]$$

and it follows from the envelope theorem,

$$\nabla_z J(z) = \arg \min_{\pi \in R^N_{++}} [\sum \pi \cdot U(x) + J^*(\pi)] = \widehat{\pi}, \text{ where}$$

$$J(U(x)) = \min_{\pi \in R^N_{++}} [\sum \pi \cdot U(x) + J^*(\pi)] = \sum \widehat{\pi} \cdot U(x) + J^*(\widehat{\pi})].$$

That is,

$$\frac{\nabla_z J(z)}{\|\nabla_z J(z)\|_1} = \frac{\widehat{\pi}}{\|\widehat{\pi}\|_1} \in \Delta^0,$$

the interior of the probability simplex. Hence $\nabla_z J(z)$ and $\nabla_z J(z)/\|\nabla_z J(z)\|_1$ define the same subjective betting odds that a given payoff z will be realized. If $J(U(x))$ is ambiguity-seeking and $U(y)$and $U(z)$ differ in only state t of the world, where $u(v_t) > u(z_t)$, the optimistic investor "believes" that

$$\frac{\Pr(u(v_t)}{1 - \Pr(u(v_t))} > \frac{\Pr(u(z_t))}{1 - \Pr(u(z_t))}$$

and $J(U(v)) > J(U(z))$ consistent with Ellsberg's explanation of ambiguity-seeking behavior. Hence ambiguity-seeking investors are bulls. If $J(U(x))$ is ambiguity-averse and $U(y)$and $U(z)$ differ in only state t of the world, where $u(v_t) > u(z_t)$. The pessimistic investor "believes" that

$$\frac{\Pr(u(v_t))}{1 - \Pr(u(v_t))} < \frac{\Pr(u(z_t))}{1 - \Pr(u(z_t))}$$

and $J(U(v)) < J(U(z))$ consistent with Ellsberg's explanation of ambiguity-averse behavior. Hence ambiguity-averse investors are bears.

3 Separable Keynesian Utilities

We begin our analysis with a parametric example of Keynesian utilities. Consider the following additively separable utility function on the space of state-contingent claims:

$$J \circ U(x) \equiv \sum_{s=1}^{s=N} j \circ u(x_s)$$

where

$$x \equiv (x_1, x_2, .., x_N), \;\; U(x) \equiv (u(x_1), u(x_2), ..., u(x_N)) \text{ and } j \circ u(x_s) \equiv x_s^{\alpha\beta}$$

If

$$u(x_s) \equiv x_s^{\beta} \text{ then } j \circ u(x_s) = [u(x_s)]^{\alpha}.$$

If $\beta \leqslant 2$, then $u(x_s)$ is concave in x_s. If $\alpha\beta \leqslant 2$, then $\alpha \leqslant 2/\beta$ and $j \circ u(x_s)$ is concave in x_s. If $\alpha \geqslant 2$, then $j \circ u(x_s)$is convex in $u(x_k)$. Hence, the composite utility function $j \circ u(x_s)$ is optimistic i.e., convex in the utilities of the payoffs. Moreover, $j \circ u(x_s)$ is concave in x_s. In this case, $J \circ U(x)$ is concave in x and convex in $U(x)$

If $\beta \leqslant 2$, then $u(x_s)$ is concave in x_s. If $\alpha\beta \geqslant 2$, then $\alpha \geqslant 2/\beta$ and $j \circ u(x_s)$ is convex in x_s. If $\alpha \geqslant 2$, then $j \circ u(x_s)$ is convex in $u(x_s)$. Hence, the composite utility function $j \circ u(x_s)$ is optimistic, i.e., convex in the utilities of the payoffs. Moreover, $j \circ u(x_s)$ is convex in x_s. In this case, $J \circ U(x)$ is convex in x and convex in $U(x)$. Hence the value of $\alpha\beta$ determines if the investor is endowed with pessimistic Keynesian utility functions or endowed with optimistic Keynesian utility functions.

We now present a nonparametric example of Keynesian utilities, where we again consider additive utility functions of the form

$$J \circ U(x) \equiv \sum_{s=1}^{s=N} j \circ u(x_s)$$

where $u : R_+ \to R_+$ and $j : R_+ \to R_+$.

Here is a family of nonparametric examples. If $J \circ U(x)$ is additively separable, then

$$\frac{\partial J(y)}{\partial y_s} = \frac{dj(y_s)}{dy_s}$$

and

$$\frac{\partial J \circ U(x)}{\partial x_s} = \frac{dj(y_s)}{dy_s}\frac{dy_s}{dx_s}$$

where $y_s = u(x_s)$; $x = (x_1, x_2, .., x_N)$ and $y = (y_1, y_2, .., y_N)$.To check if $J \circ U(x)$ is strictly concave in x, we compute the Hessian of $J \circ U(x)$, where we use the chain rule:

$$\frac{\partial^2 J \circ U(x)}{\partial x_s \partial x_r} = \frac{\partial\left(\frac{\partial J \circ U(x)}{\partial x_r}\right)}{\partial x_s} = \frac{\partial\left(\frac{dj(y_r)}{dy_r}\frac{dy_r}{dx_r}\right)}{\partial x_s} = \frac{\partial\left(\frac{dj(y_r)}{dy_r}\frac{dy_r}{dx_r}\right)}{\partial x_s} = 0 \text{ if } s \neq r$$

The diagonal of the Hessian

$$\frac{\partial^2 J \circ U(x)}{\partial x_s^2} = \left(\frac{d^2 j(y_s)}{dy_s^2}\right)\left(\frac{dy_s}{dx_s}\right)^2 + \left(\frac{dj(y_s)}{dy_s}\right)\left(\frac{d^2 y_s}{dx_s^2}\right) \text{ for } 1 \leq s \leq N$$

where

$$\frac{d^2 j(y_s)}{dy_s^2} > 0 \text{ and } \frac{d^2 y_s}{dx_s^2} < 0 \text{ for } 1 \leq s \leq N.$$

Hence the Hessian matrix

$$\frac{\partial^2 J \circ U(x)}{\partial x_s \partial x_r}$$

is negative definite at x iff

$$\left(\frac{d^2 j(y_s)}{dy_s^2}\right)\left(\frac{dy_s}{dx_s}\right)^2 + \left(\frac{dj(y_s)}{dy_s}\right)\left(\frac{d^2 y_s}{dx_s^2}\right) < 0 \text{ for } 1 \leq s \leq N.$$

Using

$$R(w) = -\frac{u''(w)}{u'(w)}$$

the Arrow–Pratt local measure of absolute risk-aversion for $u(x_s)$ and

$$-R(w) = \frac{u''(w)}{u'(w)}$$

the Arrow–Pratt local measure of absolute risk-seeking for $j(y_s)$, where $y_s = u(x_s)$, we derive a sufficient condition for the Hessian of the additively separable $J \circ U(x)$ to be negative definite. That is,

$$(\frac{dy_s}{dx_s}) < -\frac{\left(\frac{d^2y_s}{dx_s^2}\right)}{\left(\frac{dy_s}{dx_s}\right)} \Bigg/ \frac{\left(\frac{d^2j(y_s)}{dy_k^2}\right)}{\left(\frac{dj(y_s)}{dy_s}\right)} \text{ for } 1 \leq s \leq N.$$

If

$$-\frac{\left(\frac{d^2y_s}{dx_s^2}\right)}{\left(\frac{dy_s}{dx_s}\right)} \equiv \text{Risk}_{\text{aver}}(x_s) > 0$$

and

$$\frac{\left(\frac{d^2j(y_s)}{dy_s^2}\right)}{\left(\frac{dj(y_s)}{dy_s}\right)} \equiv -\text{Ambig}_{\text{aver}}(y_s) > 0$$

then

$$\frac{dy_s}{dx_s}(x) < \frac{\text{Risk}_{\text{aver}}(x_s)}{-\text{Ambig}_{\text{aver}}(y_s)}(x)$$

is sufficient for additively separable $J \circ U(x)$ to be concave at x. Constant relative risk-aversion ($CARA$) utility functions are positive affine transformations of negative exponential, i.e.,

$$u(w) = -[\exp -Aw]$$

where

$$A = \frac{d^2(-[\exp -Aw])}{dw^2} / \frac{d(-[\exp -Aw])}{dw}$$

is the coefficient of absolute risk- aversion. If $u(x_s)$ is a $CARA$ concave utility function and $j(y_s)$ is a $CARA$ convex utility function, then a sufficient condition for additively separable $J \circ U(x)$ to be concave on Ω, a compact, convex subset of R^N_{++} is:

$$\frac{dy_s}{dx_s}(x) < \frac{A_{\text{Risk}}}{A_{\text{Ambig}}} \text{ for } 1 \leq s \leq N, \text{ for all } x \in \Omega$$

where

$$A_{\text{Risk}} \equiv -\frac{\left(\frac{d^2y_s}{dx_s^2}\right)}{\left(\frac{dy_s}{dx_s}\right)}(x) \text{ and } -A_{\text{Ambig}} \equiv \frac{\left(\frac{d^2j(y_s)}{dy_s^2}\right)}{\left(\frac{dj(y_s)}{dy_s}\right)}(x)$$

4 Amenable Keynesian Utilities

In this section, we propose semiparametric specifications of preferences for risk and preferences for ambiguity, defined in part by scalar proxies for risk and ambiguity: β and α. Piecewise linear-quadratic functions and (fully) amenable functions were introduced by Rockafellar (1988). A function $f : R^N \to \bar{R}$ is called piecewise linear-quadratic if dom f can be represented as the union of finitely many polyhedral sets, where relative to each set $f(x)$ is of the form $\frac{1}{2}xDx + d \cdot x + \delta$, where $\delta \in R$, $d \in R^N$ and $D \in R^{N \times N}$ is a symmetric matrix. A special case is where dom f consists of a single set. A function $f : R^n \to \bar{R}$ is (fully) amenable if $f = g \circ F$, where F is a C^2 mapping and g is a piecewise linear-quadratic function.[2] Concave quadratic utility functions were introduced by Shannon and Zame (2002) in their analysis of indeterminacy in infinite dimension general equilibrium models. $f(x)$ is a concave quadratic function if for all y and z:

$$f(y) < f(z) + \nabla f(z) \cdot (y - z) - \tfrac{1}{2}K \left\| y - z \right\|^2 \text{ ,where } K > 0.$$

We begin with necessary and sufficient conditions to rationalize the demands for assets of investors endowed with amenable Keynesian utilities, where $J \circ U(x)$ is the composition of a smooth, concave quadratic map $U(x)$,where $U(x)$ is a diagonal NxN matrix for each $x \in R^N_{++}$ and a smooth, convex quadratic function $J(y)$. That is, we derive the quadratic Afriat inequalities for $U(x)$ and $J(y)$. If $f(x)$ is a smooth, concave quadratic utility function, then the quadratic Afriat inequalities for $f(x)$ are:

$$f_i < f_j + \nabla f_j \cdot (x_i - x_j) - \beta \tfrac{1}{2} \left\| x_i - x_j \right\|^2 \text{ for } 1 \leq i, j \leq N, \text{ where } \beta > 0.$$

If we define the linear concave quadratic functions

$$l_j(x) \equiv f_j + \nabla f_j \cdot (x - x_j) - \beta \tfrac{1}{2} \left\| x - x_j \right\|^2 \text{ for } 1 \leq j \leq N$$

then $l(x) = \wedge_{j=1}^N l_j(x)$ is a strictly concave function, where for all $x, \nabla^2_x l(x) = -\beta$. If $f(x)$ is a smooth, convex quadratic utility function, then the quadratic Afriat inequalities for $f(x)$ are:

$$f_i > f_j + \nabla f_j \cdot (x_i - x_j) + \alpha \tfrac{1}{2} \left\| x_i - x_j \right\|^2 \text{ for } 1 \leq i, j \leq N \text{ where } \alpha > 0.$$

If we define convex quadratic functions

$$h_j(x) \equiv f_j + \nabla f_j \cdot (x - x_j) + \alpha \tfrac{1}{2} \left\| x - x_j \right\|^2 \text{ for } 1 \leq j \leq N$$

then $h(x) = \vee_{j=1}^N h_j(x)$ is a strictly convex function, where for all x, $\nabla^2_x h(x) = \alpha$. If $u : R_+ \to R_+$, then

$$U(x) \equiv (u(x_1), u(x_2), ..., u(x_N)).$$

is the state-utility vector for the state-contingent claim

$$x = (x_1, x_2, ..., x_N).$$

If $z = [z_1, z_2, ..., z_N]$ and $w = [w_1, w_2, ..., w_N]$, then

$$z \cdot w \equiv [z_1 w_1, z_2 w_2, ..., z_N w_N]$$

[2] There is a constraint qualification that is trivially satisfied in our case, where F is a diffeomorphism from R^N_{++} onto $R^N_{++}$and dom g is R^N_+.

is the Hadamard or pointwise product of z and w. If we define the gradient of state-utility vector $U(x)$ as the vector

$$\nabla_x U(x) \equiv [\partial u(x_1), \partial u(x_2), ..., \partial u(x_N)]$$

then by the chain rule

$$\nabla_x J \circ U(x) = [\nabla_x U(x)] \cdot [\nabla_{U(x)} J(U(x))].$$

If

$$G(x) = z(x) \cdot w(x),$$

where $z(x)$ and $w(x) \in R^N_{++}$, then Bentler and Lee (1978) state and Magnus and Neudecker (1985) prove that

$$\nabla_x G(x) = \nabla_x z(x) \text{diag}(w(x)) + \nabla_x w(x) \text{diag}(z(x))$$

All of analysis in this section derives from the following representation of $\nabla^2_x J \circ U(x)$:

$$\begin{aligned} \nabla^2_x J \circ U(x) &= \nabla_x([\nabla_x U(x)] \cdot [\nabla_{U(x)} J(U(x))]) \\ &= [\nabla^2_{U(x)} J(U(x))](\text{diag}[\nabla_x U(x)])^2 + [\nabla^2_x U(x)] \text{diag}[\nabla_{U(x)} J(U(x))]. \end{aligned}$$

If $U(x)$ is a concave quadratic map and $J(y)$ is a convex quadratic function, then

$$\nabla^2_x U(x) = -\text{diag}(\beta) < 0$$

and

$$\nabla^2_y J(y) = \text{diag}(\alpha) > 0$$

If A and B are $N \times N$ symmetric matrices then $A \precsim B$ iff $A - B$ is negative semi-definite, denoted: $[A - B] \lesssim 0$, where

$$\nabla^2_x J \circ U(x) \lesssim 0 \text{ iff } \text{diag}(\alpha)\text{diag}[\nabla_x U(x)]^2 - \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(U(x))] \lesssim 0$$

See matrix inequalities in section A.5.2 in Boyd and Vandenberghe for a discussion of the partial ordering $\lesssim$ on the linear vector space of $N \times N$ symmetric matrices. For diagonal $N \times N$ matrices E and F :

$$E \lesssim F \Leftrightarrow E \leq F$$

In the literature on expected utility theory, a Bernoulli utility function of wealth, $u(w)$, is said to be prudent if the marginal utility, du/dw, is a convex function of wealth, i.e., the third derivative of $u(w)$ is positive. Prudence is often associated with a precautionary motive for saving in the standard two period investment model. See Leland (1968), who showed that the precautionary motive for saving is equivalent to prudence. Also see Keynes (1930) on why investors hold money. There is a second equivalent interpretation of prudence as observed by Tarazona-Gomez: "If we consider the expected-utility framework, a prudent agent can be thought either as one who increases his savings when uncertainty affects his future income, or even simpler, as someone who prefers to face a risk attached to a good state (the best outcome of a lottery), rather than to a bad one (to the worst outcome). If symmetric Keynesian utilities are the composition of quadratic utilities for risk and quadratic utilities for ambiguity, then the third derivative of Keynesian utilities is positive. We define this family of Keynesian utilities as prudent Keynesian utilities. We define asymmetric Keynesian utilities,where the preferences for risk and preferences for ambiguity are quadratic utilities, hence the third derivative is negative, as imprudent Keynesian utilities. In both cases, all higher order derivatives, i.e., greater than four, are zero. Hence prudent and imprudent Keynesian utilities have representations as fourth order multivariate Taylor polynomials. These results follow from repeated application of the chain rule to the derivatives of the Keynesian utilities. As in the literature on expected utility theory — see Tarazona-Gomez (2004) and Roitman (2011) — risk and prudence are uncorrelated in prudent Keynesian utilities. That is, the Keynesian utilities for bulls are risk-seeking and the Keynesian utilities for bears are risk-averse, but both bulls and bears are prudent, although we suspect that the bulls are less prudent, since they expect high future incomes and the bears expect low future incomes.

Theorem 1 *If $J \circ U(x)$ is the composition of quadratic utilities for risk and quadratic utilities for ambiguity,where*

$$\text{diag}(\beta) = \text{diag}[\nabla_x^2 U(x)] \text{ and } \text{diag}(\alpha) = \text{diag}[\nabla_{U(x)}^2 J(U(x)]$$

then symmetric Keynesian utilities are prudent and asymmetric Keynesian utilities are imprudent. That is, if $J \circ U(x)$ is symmetric then $\nabla_x^3 J \circ U(x) > 0$ and if $J \circ U(x)$ is asymmetric $\nabla_x^3 J \circ U(x) < 0$. Moreover, $\nabla_x^k J \circ U(x) = 0$ for $k \geq 5$.

Proof. If

$$\nabla_x J \circ U(x) = [\nabla_x U(x)] \cdot [\nabla_{U(x)} J(U(x))]$$

then

$$\nabla_x^2 J \circ U(x) = \text{diag}(\alpha)(\text{diag}[\nabla_x U(x)])^2 + \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(U(x))]$$

$$\nabla_x^3 J \circ U(x) = 3\text{diag}(\beta)\text{diag}(\alpha)\text{diag}[\nabla_x U(x)]$$

$$\nabla_x^4 J \circ U(x) = 3\text{diag}(\beta)[\text{diag}(\alpha)]^2$$

$$\nabla_x^k J \circ U(x) = 0 \text{ for } k \geq 5$$

$\nabla_x^3 J \circ U(x) < 0$ iff α and β have different signs. That is, iff $J \circ U(x)$ is asymmetric. $\nabla_x^3 J \circ U(x) > 0$ iff α and β have the same sign. That is, iff $J \circ U(x)$ is symmetric. ■

5 Keynesian Afriat Inequalities

In our model, financial assets are limited liability, state-contingent claims on a finite state-space. Investors are price-taking, utility maximizers subject to a budget constraint defined by the investor's income and the market prices of assets. We begin by recalling, Afriat's (1967) celebrated revealed preferences theorem on rationalizing a finite number of a consumer's utility maximizing demands, subject to a budget constraint. The theorem states that these demands are rationalized by a concave non-satiated utility function iff the Afriat inequalities, a finite family of multivariate polynomial (linear) inequalities, where the unknowns are utility levels and marginal utilities of income and the parameters are the market data, are feasible. The concave non-satiated rationalizing utility function constructed by Afriat is a polyhedral function. That is, the minimum of a finite number of affine functions on R^N, derived from solutions to the Afriat inequalities. If we restrict attention to systems of strict Afriat inequalities, then this polyhedral function is differentiable at each of the consumer's utility maximizing demands. We extend Afriat's theorem to the utility maximizing demands for assets of investors endowed with uncertainty-aversion. The market data is denoted

$$D \equiv \{p_l, x_l\}_{l=1}^{l=T},$$

p_l and x_l are in R^N_{++}. The utility function $u : R^N_+ \to R$ rationalizes D if

$$u(x_l) = \max_{p_l \cdot y \leqslant p_l \cdot x_l} u(y) \text{ for } 1 \leqslant l \leqslant T$$

The strict Afriat inequalities:

$$u(x_r) < u(x_s) + \lambda_s p_s \cdot (x_r - x_s) \text{ for } 1 \leqslant r, s \leqslant T$$

Afriat's polyhedral function:

$$\widehat{u}(y) \equiv \wedge_{l=1}^{l=T} [u(x_l) + \lambda_l p_l \cdot (y - x_l)]$$

It follows from Danksin's Theorem on directional derivatives of polyhedral functions — see Proposition 4.5.1 in Bertsekas et al. (2003) that

$$\nabla_y \widehat{u}(x_l) = \lambda_l p_l \text{ for } 1 \leqslant l \leqslant T.$$

The logic of our proofs is based on the chain rule for the composition of a smooth, convex quadratic function $J(y)$from R^N_{++} into R_+ with a smooth, concave quadratic map $U(x)$ from R^N_{++} onto R^N_{++} and the Kuhn–Tucker Theorem (KTT). In particular, we assume that U is differentiable at the data points x_k and $J(y_k)$ is differentiable at $y_k \equiv U(x_k)$, for $1 \leqslant k \leqslant T$.

In Theorems 2 and 3, we derive the quadratic Afriat inequalities for asymmetric Keynesian utilities, where we prove the existence of a reference point $\hat{x}$ that partitions R^N_+ into the standard four quadrants, with the reference point $\hat{x}$ as the origin. $J \circ U(x)$ is concave in quadrant I, where quadrant $I \equiv \{x \in R^N_+ : x \geq \hat{x}\}$ and convex in quadrant III, where quadrant $III \equiv \{x \in R^N_+ : x \leq \hat{x}\}$. The Hessian of $J \circ U(x)$ is indefinite in quadrants II and IV. That is, $\nabla^2_x J \circ U(x)$ is indefinite on $R^N_+ / \{(\hat{x}, +\infty] \cup (0, \hat{x}]\}$. $J \circ U(x)$ is optimistic for "losses," i.e., $x \leq \hat{x}$ and pessimistic for "gains," i.e., $x \geq \hat{x}$, analogous with the shape of the utility of risk in prospect theory — see figure 10 in Kahneman (2011).

Theorem 2 *If $J \circ U(x)$, is the composition of $U(x)$ and $J(y)$, where (a) $(y_1, y_2, ..., y_N) \equiv y = U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$ is a monotone, smooth, concave quadratic map from R^N_{++} onto R^N_{++}, with the proxy for risk, $-\beta < 0$, (b) $J(y)$ is a monotone, smooth, convex quadratic function from R^N_+ into R, with the proxy for ambiguity, $\alpha > 0$, (c)*

$$\nabla^2_x J \circ \hat{U}(x) = \text{diag}(\alpha)(\text{diag}[\nabla_x \hat{U}(x)])^2 - \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(\hat{U}(x))]\text{: Chain Rule}$$

then there exists a reference point $\hat{x}$ such that the financial market data D is rationalized by the composite function $J(U(x))$ with two domains of convexity: $(\hat{x}, +\infty]$ and $(0, \hat{x}]$, where $J \circ U(x)$ is concave on $(\hat{x}, +\infty]$ and $J \circ U(x)$ is convex on $(0, \hat{x}]$ iff the quadratic Afriat inequalities are feasible for risk-averse $U : R^N_{++} \rightarrow R^N_{++}$ and the quadratic Afriat inequalities are feasible for ambiguity-seeking $J : R^N_+ \rightarrow R$.

Proof. Necessity is obvious. For sufficiency, we consider the following family of quadratic Afriat inequalities:

$$x_j = (x_{j,1}, x_{j.2}, .x_{j,s}.., x_{j,N}) \in \text{interior } (\hat{x}, +\infty] \text{ or } \in \text{interior } (0,\hat{x}]$$

$$u(x_{j,s}) - u(x_{k,s}) < \partial u(x_{k,s}) x (x_{j,s} - x_{k,s}) - \beta[x_{j,s} - x_{k,s}] x [x_{j,s} - x_{k,s}]$$
$$\text{for } 1 \leqslant j, k \leqslant L; 1 \leqslant s \leqslant N$$

$$U(x_j) \equiv (u(x_{j,1}), u(x_{j,2}), ..., u(x_{j,N})) \text{ for } 1 \leqslant j \leqslant L$$

$$\nabla_x U(x_j) \equiv (\partial u(x_{j,1}), \partial u(x_{j,2}), ..., \partial u(x_{j,N})) \text{ for } 1 \leqslant j \leqslant L$$

$$U(x_j) - U(x_k) < [\nabla_x U(x_k)] \cdot [x_j - x_k] - \tfrac{\beta}{2}[x_j - x_k] \circ [x_j - x_k] \text{ for } 1 \leqslant j, k \leqslant L$$

$$J(U(x_j)) - J(U(x_k)) > [\nabla_{U(x)} J(U(x_k))] \cdot [U(x_j) - U(x_k)] + \tfrac{\alpha}{2} \|[U(x_j)) - (U(x_k)]\|^2$$
$$\text{for } 1 \leqslant j, k \leqslant L.$$

If

$$\widehat{u}(x_{j,s}) \equiv \wedge_{k=1}^{k=L} [u(x_{k,s}) + \partial u(x_{k,s})(x_{j,s} - x_{k,s}) - \tfrac{\beta}{2}(x_{j,s} - x_{k,s})^2 \text{ for } 1 \leqslant s \leqslant N \text{ and } \delta > 0$$

$$\hat{U}(x_j) \equiv (\widehat{u}(x_{j,1}), \widehat{u}(x_{j,2}), ..., \widehat{u}(x_{j,N})) \text{ for } 1 \leqslant j \leqslant L$$

$$\nabla_x \hat{U}(x_j) \equiv (\partial \widehat{u}(x_{j,1}), \partial \widehat{u}(x_{j,2}), ..., \partial \widehat{u}(x_{j,N})) \text{ for } 1 \leqslant j \leqslant L$$

$$\hat{U}(x) \equiv \wedge_{k=1}^{k=N} [\hat{U}(x_k) + \nabla_x \hat{U}(x_k) \circ (x - x_k) - \tfrac{\text{diag}(\beta)}{2}(x - x_k) \circ (x - x_k)]$$

$$\nabla^2_x \hat{U}(x) = -\text{diag}(\beta) \text{ where } \beta > 0\text{: Risk-Averse}$$

$$J(\hat{U}(x)) \equiv \vee_{k=1}^{k=T} [J(\hat{U}(x_k)) + \nabla_{U(x)} J(\hat{U}(x_k)) \cdot (\hat{U}(x) - \hat{U}(x_k)) + \tfrac{\text{diag}(\alpha)}{2} \left\| (\hat{U}(x) - \hat{U}(x_k)) \right\|^2$$

$$\nabla^2_{\hat{U}(x)} J(\hat{U}(x)) = \text{diag}(\alpha) \text{ where } \alpha > 0\text{: Ambiguity-Seeking}$$

$$\nabla^2_x J \circ \hat{U}(x) = \text{diag}(\alpha)(\text{diag}[\nabla_x \hat{U}(x)])^2 - \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(\hat{U}(x))]\text{: Chain Rule}$$

$$\lim_{\|x\|_\infty \to \infty} \left\| \text{diag}[\nabla_{U(x)} J(\hat{U}(x))]^{-1} \text{diag}[\nabla_x \hat{U}(x)]^2 \right\|_\infty = 0$$

$$\begin{aligned} \text{diag}[\nabla_{U(x)} J(\hat{U}(x))]^{-1} \text{diag}[\nabla_x \hat{U}(x)]^2 &\leq \text{diag}[\nabla_{U(x)} J(\hat{U}(\hat{x}))]^{-1} \text{diag}[\nabla_x \hat{U}(\hat{x})]^2 \\ &\leq \text{diag}[\tfrac{\beta}{\alpha}]\text{: Bears} \end{aligned}$$

$$\lim_{x \to 0} \left\| \text{diag}[\nabla_{U(x)} J(\hat{U}(x))] \text{diag}[\nabla_x \hat{U}(x)]^{-2} \right\|_\infty = 0$$

$$\begin{aligned} \text{diag}[\nabla_{U(x)} J(\hat{U}(x))]^{-1} \text{diag}[\nabla_x \hat{U}(x)]^2 &\leq \text{diag}[\nabla_{U(x)} J(\hat{U}(\hat{x}))] \text{diag}[\nabla_x \hat{U}(\hat{x})]^{-2} \\ &\leq \text{diag}[\tfrac{\alpha}{\beta}]\text{: Bulls} \end{aligned}$$

■

Theorem 3 *If $J \circ U(x)$, is the composition of $U(x)$ and $J(y)$,where (a) $(y_1, y_2, ..., y_N) \equiv y = U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$ is a monotone, smooth, convex quadratic map from R^N_{++} onto R^N_{++} with the proxy for risk, $\beta > 0$, (b) $J(y)$ is a monotone, smooth, concave quadratic function from R^N_+ into R with the proxy for risk,$-\alpha < 0$, (c)*

$$\nabla^2_x J \circ \hat{U}(x) = -\text{diag}(\alpha)(\text{diag}[\nabla_x \hat{U}(x)])^2 + \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(\hat{U}(x))]\text{: } \textit{Chain Rule}$$

then there exists a reference point $\hat{x}$ such that the financial market data D is rationalized by the composite function $J(U(x))$ with two domains of convexity: $(\hat{x}, +\infty]$ and $(0, \hat{x}]$,where $J(U(x))$ is concave on $(\hat{x}, +\infty]$ and $J(U(x))$ is convex on $(0, \hat{x}]$ iff the quadratic Afriat inequalities are feasible for risk-seeking $U : R^N_{++} \to R^N_{++}$ and the quadratic Afriat inequalities are feasible for ambiguity-averse $J : R^N_+ \to R$.
Proof. *Necessity is obvious. For sufficiency, we consider the following family of quadratic Afriat inequalities:*

$$x_j = (x_{j,1}, x_{j.2}, .x_{j,s}.., x_{j,N}) \in \textit{interior } (\hat{x}, +\infty] \textit{ or } \in \textit{interior } (0, \hat{x}]$$

$$\begin{aligned} u(x_{j,s}) - u(x_{k,s}) &< \partial u(x_{k,s}) x (x_{j,s} - x_{k,s}) - \beta[x_{j,s} - x_{k,s}] x [x_{j,s} - x_{k,s}] \\ &\quad \textit{for } 1 \leqslant j, k \leqslant L; 1 \leqslant s \leqslant N \end{aligned}$$

$$U(x_j) \equiv (u(x_{j,1}), u(x_{j,2}), ..., u(x_{j,N})) \textit{ for } 1 \leqslant j \leqslant L$$

$$\nabla_x U(x_j) \equiv (\partial u(x_{j,1}), \partial u(x_{j,2}), ..., \partial u(x_{j,N})) \textit{ for } 1 \leqslant j \leqslant L$$

$$U(x_j) - U(x_k) > [\nabla_x U(x_k)] \circ [x_j - x_k] + \tfrac{\beta}{2}[x_j - x_k] \circ [x_j - x_k] \textit{ for } 1 \leqslant j, k \leqslant L$$

$$\begin{aligned} J(U(x_j)) - J(U(x_k)) &< [\nabla_{U(x)} J(U(x_k))] \cdot [U(x_j) - U(x_k)] - \tfrac{\alpha}{2} \|[U(x_j)) - (U(x_k)]\|^2 \\ &\quad \textit{f or } 1 \leqslant j, k \leqslant L. \end{aligned}$$

If

$$\widehat{u}(x_{j,s}) \equiv \wedge_{k=1}^{k=L} [u(x_{k,s}) + \partial u(x_{k,s}) \cdot (x_{j,s} - x_{k,s}) - \tfrac{\beta}{2}(x_{j,s} - x_{k,s})^2 \textit{ for } 1 \leqslant s \leqslant N$$

$$\hat{U}(x_j) \equiv (\widehat{u}(x_{j,1}), \widehat{u}(x_{j,2}), ..., \widehat{u}(x_{j,N})) \textit{ for } 1 \leqslant j \leqslant L$$

$$\nabla_x \hat{U}(x_j) \equiv (\partial\widehat{u}(x_{j,1}), \partial\widehat{u}(x_{j,2}), ..., \partial\widehat{u}(x_{j,N})) \textit{ for } 1 \leqslant j \leqslant L$$

$$\hat{U}(x) \equiv \wedge_{k=1}^{k=N}[\hat{U}(x_k) + \nabla_x \hat{U}(x_k) \circ (x - x_k) + \tfrac{\mathrm{diag}(\beta)}{2}(x - x_k) \circ (x - x_k)]$$

$$\nabla_x^2 \hat{U}(x) = \mathrm{diag}(\beta) \textit{ where } \beta > 0\textit{: Risk-Seeking}$$

$$J(\hat{U}(x)) \equiv \vee_{k=1}^{k=T}[J(\hat{U}(x_k)) + \nabla_{U(x)} J(\hat{U}(x_k)) \cdot (\hat{U}(x) - \hat{U}(x_k)) - \tfrac{\mathrm{diag}(\alpha)}{2} \left\|(\hat{U}(x) - \hat{U}(x_k))\right\|^2$$

$$\nabla_{\hat{U}(x)}^2 J(\hat{U}(x)) = -\mathrm{diag}(\alpha) \textit{ where } \alpha > 0\textit{: Ambiguity-Averse}$$

$$\nabla_x^2 J \circ \hat{U}(x) = -\mathrm{diag}(\alpha)(\mathrm{diag}[\nabla_x \hat{U}(x)])^2 + \mathrm{diag}(\beta)\mathrm{diag}[\nabla_{U(x)} J(\hat{U}(x))]\textit{: Chain Rule}$$

$$\lim_{\|x\|_\infty \to \infty} \left\|\mathrm{diag}[\nabla_{U(x)} J(\hat{U}(x))]\mathrm{diag}[\nabla_x \hat{U}(x)]^{-2}\right\|_\infty = 0$$

$$\begin{aligned} \mathrm{diag}[\nabla_{U(x)} J(\hat{U}(x))]\mathrm{diag}[\nabla_x \hat{U}(x)]^{-2} &\leq \mathrm{diag}[\nabla_{U(x)} J(\hat{U}(\hat{x}))]\mathrm{diag}[\nabla_x \hat{U}(\hat{x})]^{-2} \\ &\leq \mathrm{diag}[\tfrac{\alpha}{\beta}]\textit{: Bears} \end{aligned}$$

$$\lim_{\|x\| \to 0} \left\|\mathrm{diag}[\nabla_{U(x)} J(\hat{U}(x))]^{-1}\mathrm{diag}[\nabla_x \hat{U}(x)]^{2}\right\|_\infty = 0$$

$$\begin{aligned} \mathrm{diag}[\nabla_{U(x)} J(\hat{U}(x))]\mathrm{diag}[\nabla_x \hat{U}(x)]^{-2} &\leq \mathrm{diag}[\nabla_{U(x)} J(\hat{U}(\hat{x}))]\mathrm{diag}[\nabla_x \hat{U}(\hat{x})]^{-2} \\ &\leq \mathrm{diag}[\tfrac{\beta}{\alpha}]\textit{: Bulls} \end{aligned}$$

■

The symmetric Keynesian utilities are the bulls and the bears, where for all $x \in R_{++}^N$

Bulls: $\nabla_x^2 J \circ \hat{U}(x) = \mathrm{diag}(\alpha)(\mathrm{diag}[\nabla_x \hat{U}(x)])^2 + \mathrm{diag}(\beta)\mathrm{diag}[\nabla_{U(x)} J(\hat{U}(x))] \ > 0$

Bears: $\nabla_x^2 J \circ \hat{U}(x) = -\mathrm{diag}(\alpha)(\mathrm{diag}[\nabla_x \hat{U}(x)])^2 - \mathrm{diag}(\beta)\mathrm{diag}[\nabla_{U(x)} J(\hat{U}(x))] \ < 0$

The quadratic Afriat inequalities for symmetric Keynesian utilities follow easily from the arguments in Theorems 2 and 3.

Theorem 4 *If $J \circ U(x)$, is the composition of $U(x)$ and $J(y)$, where (a) $(y_1, y_2, ..., y_N) \equiv y = U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$ is a monotone, smooth, concave quadratic map from R_{++}^N onto R_{++}^N, with the proxy for risk, $-\beta < 0$, (b) $J(y)$ is a monotone, smooth, concave quadratic function from R_{++}^N into R, with the proxy for ambiguity, $-\alpha < 0$, (c)*

$$\nabla_x^2 J \circ \hat{U}(x) = -\mathrm{diag}(\alpha)(\mathrm{diag}[\nabla_x \hat{U}(x)])^2 - \mathrm{diag}(\beta)\mathrm{diag}[\nabla_{U(x)} J(\hat{U}(x))]\textit{: Chain Rule}$$

then $J \circ U(x)$ is concave on R_{++}^N iff the quadratic Afriat inequalities are feasible for risk-averse $U : R_{++}^N \to R_{++}^N$ and the quadratic Afriat inequalities are feasible for ambiguity-averse $J : R_+^N \to R$.

Proof. Necessity is obvious. For sufficiency, we consider the following family of Afriat quadratic inequalities:

$$x_j = (x_{j,1}, x_{j,2}, x_{_{j,s}}, ..., x_{j,N}) \in \text{interior } (\hat{x}, +\infty] \text{ or } \in \text{interior } (0,\hat{x}]$$

$$\begin{aligned} u(x_{j,s}) - u(x_{k,s}) &< \partial u(x_{k,s}) \cdot (x_{j,s} - x_{k,s}) - \beta[x_{j,s} - x_{k,s}]x[x_{j,s} - x_{k,s}] \\ &\text{for } 1 \leqslant j,k \leqslant L;\ 1 \leqslant s \leqslant N \end{aligned}$$

$$U(x_j) \equiv (u(x_{j,1}), u(x_{j,2}), ..., u(x_{j,N})) \text{ for } 1 \leqslant j \leqslant L$$

$$\nabla_x U(x_j) \equiv (\partial u(x_{j,1}), \partial u(x_{j,2}), ..., \partial u(x_{j,N})) \text{ for } 1 \leqslant j \leqslant L$$

$$U(x_j) - U(x_k) < [\nabla_x U(x_k)] \circ [x_j - x_k] - \tfrac{\beta}{2}[x_j - x_k] \circ [x_j - x_k] \text{ for } 1 \leqslant j,k \leqslant L$$

$$\begin{aligned} J(U(x_j)) - J(U(x_k)) &< [\nabla_{U(x)} J(U(x_k))] \cdot [U(x_j) - U(x_k)] - \tfrac{\alpha}{2} \left\| [U(x_j)) - (U(x_k)] \right\|^2 \\ &\text{for } 1 \leqslant j,k \leqslant L. \end{aligned}$$

If

$$\widehat{u}(x_{j,s}) \equiv \wedge_{k=1}^{k=L}[u(x_{k,s}) + \partial u(x_{k,s})(x_{j,s} - x_{k,s}) - \tfrac{\beta}{2}(x_{j,s} - x_{k,s})^2 \text{ for } 1 \leqslant s \leqslant N$$

$$\hat{U}(x_j) \equiv (\widehat{u}(x_{j,1}), \widehat{u}(x_{j,2}), ..., \widehat{u}(x_{j,N})) \text{ for } 1 \leqslant j \leqslant L$$

$$\nabla_x \hat{U}(x_j) \equiv (\partial\widehat{u}(x_{j,1}), \partial\widehat{u}(x_{j,2}), ..., \partial\widehat{u}(x_{j,N})) \text{ for } 1 \leqslant j \leqslant L$$

$$\hat{U}(x) \equiv \wedge_{k=1}^{k=N}[\hat{U}(x_k) + \nabla_x \hat{U}(x_k) \circ (x - x_k) - \tfrac{\text{diag}(\beta)}{2}(x - x_k) \circ (x - x_k)]$$

$$\nabla_x^2 \hat{U}(x) = \text{diag}(\beta) \text{ where } -\beta < 0\text{: Risk-Averse}$$

$$J(\hat{U}(x)) \equiv \vee_{k=1}^{k=T}[J(\hat{U}(x_k)) + \nabla_{U(x)} J(\hat{U}(x_k)) \cdot (\hat{U}(x) - \hat{U}(x_k)) - \tfrac{\text{diag}(\alpha)}{2} \left\| (\hat{U}(x) - \hat{U}(x_k)) \right\|^2$$

$$\nabla_{\hat{U}(x)}^2 J(\hat{U}(x)) = -\text{diag}(\alpha) \text{ where } -\alpha < 0\text{: Ambiguity-Averse}$$

$$\text{Bears: } \nabla_x^2 J \circ \hat{U}(x) = -\text{diag}(\alpha)(\text{diag}[\nabla_x \hat{U}(x)])^2 - \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(\hat{U}(x))] < 0$$

■

Theorem 5 *If $J \circ U(x)$, is the composition of $U(x)$ and $J(y)$, where (a) $(y_1, y_2, ..., y_N) \equiv y = U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$ is a monotone, smooth, convex quadratic map from R_{++}^N onto R_{++}^N, with the proxy for risk, $\beta > 0$, (b) $J(y)$ is a monotone, smooth, convex quadratic function from R_{++}^N into R, with the proxy for ambiguity, $\alpha > 0$, (c)*

$$\nabla_x^2 J \circ \hat{U}(x) = \text{diag}(\alpha)(\text{diag}[\nabla_x \hat{U}(x)])^2 + \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(\hat{U}(x))]\text{: Chain Rule}$$

then $J \circ U(x)$ is convex on R_{++}^N iff the quadratic Afriat inequalities are feasible for risk-seeking $U : R_{++}^N \to R_{++}^N$ and the quadratic Afriat inequalities are feasible for ambiguity-seeking $J : R_+^N \to R$.

Proof. Necessity is obvious. For sufficiency, we consider the following family of Afriat quadratic inequalities:

$$x_j = (x_{j,1}, x_{j.2}, .x_{j,s}.., x_{j,N}) \in \text{interior } (\hat{x}, +\infty] \text{ or } \in \text{interior } (0,\hat{x}]$$

$$\begin{aligned} u(x_{j,s}) - u(x_{k,s}) &> \partial u(x_{k,s})(x_{j,s} - x_{k,s}) + \beta[x_{j,s} - x_{k,s}]x[x_{j,s} - x_{k,s}] \\ &\quad \text{for } 1 \leqslant j,k \leqslant L; 1 \leqslant s \leqslant N \end{aligned}$$

$$U(x_j) \equiv (u(x_{j,1}), u(x_{j,2}), ..., u(x_{j,N})) \text{ for } 1 \leqslant j \leqslant L$$

$$\nabla_x U(x_j) \equiv (\partial u(x_{j,1}), \partial u(x_{j,2}), ..., \partial u(x_{j,N})) \text{ for } 1 \leqslant j \leqslant L$$

$$U(x_j) - U(x_k) > [\nabla_x U(x_k)] \circ [x_j - x_k] + \tfrac{\beta}{2}[x_j - x_k] \circ [x_j - x_k] \text{ for } 1 \leqslant j,k \leqslant L$$

$$\begin{aligned} J(U(x_j)) - J(U(x_k)) &> [\nabla_{U(x)} J(U(x_k))] \cdot [U(x_j) - U(x_k)] + \tfrac{\alpha}{2} \left\| [U(x_j)) - (U(x_k)] \right\|^2 \\ &\quad \text{for } 1 \leqslant j,k \leqslant L. \end{aligned}$$

If

$$\widehat{u}(x_{j,s}) \equiv_{k=1}^{k=L} \vee [u(x_{k,s}) + \partial u(x_{k,s})(x_{j,s} - x_{k,s}) + \tfrac{\beta}{2}(x_{j,s} - x_{k,s})^2 \text{ for } 1 \leqslant s \leqslant N$$

$$\hat{U}(x_j) \equiv (\widehat{u}(x_{j,1}), \widehat{u}(x_{j,2}), ..., \widehat{u}(x_{j,N})) \text{ for } 1 \leqslant j \leqslant L$$

$$\nabla_x \hat{U}(x_j) \equiv (\partial\widehat{u}(x_{j,1}), \partial\widehat{u}(x_{j,2}), ..., \partial\widehat{u}(x_{j,N})) \text{ for } 1 \leqslant j \leqslant L$$

$$\hat{U}(x) \equiv \vee_{k=1}^{k=N} [\hat{U}(x_k) + \nabla_x \hat{U}(x_k) \circ (x - x_k) + \tfrac{\text{diag}(\beta)}{2}(x - x_k) \circ (x - x_k)]$$

$$\nabla_x^2 \hat{U}(x) = \text{diag}(\beta) \text{ where } \beta > 0\text{: Risk-Seeking}$$

$$J(\hat{U}(x)) \equiv \vee_{k=1}^{k=T} [J(\hat{U}(x_k)) + \nabla_{U(x)} J(\hat{U}(x_k)) \cdot (\hat{U}(x) - \hat{U}(x_k)) + \tfrac{\text{diag}(\alpha)}{2} \left\| (\hat{U}(x) - \hat{U}(x_k)) \right\|^2$$

$$\nabla_{\hat{U}(x)}^2 J(\hat{U}(x)) = -\text{diag}(\alpha) \text{ where } \alpha > 0\text{: Ambiguity-Seeking}$$

$$\text{Bulls: } \nabla_x^2 J \circ \hat{U}(x) = \text{diag}(\alpha)(\text{diag}[\nabla_x \hat{U}(x)])^2 + \text{diag}(\beta)\text{diag}[\nabla_{U(x)} J(\hat{U}(x))] > 0$$

■

6 Acknowledgments

The views expressed in this paper are solely those of the authors and not those of the Federal Reserve Bank of Boston or the Federal Reserve System.

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