COST INNOVATION: SCHUMPETER AND EQUILIBRIUM.
PART 2: INNOVATION AND THE MONEY SUPPLY

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Abstract

The control structure over money and real assets is considered in the process of cost innovation. The work here contrasts with the first part of this paper where the emphasis was on the physical aspects of innovation. Here the emphasis is primarily on the money supply aspects of innovation. We conclude with observations on evaluation and the locus of control in the process of innovation.

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1 Finance and Innovation

The specific "value-added" to the topics of innovation, control, and ownership attempted here is to start to bridge the mathematical gap between general equilibrium theory and Schumpeter's writings on innovation. In the past twenty to thirty years there have been considerable writing and empirical work on innovation and the economic and behavioral questions it raises, see for example Arthur [2] , Dosi et al. [8], Bechtel et al. [5], Baumol [4], Lamoreaux and Sololoff [10], Day [6], Eliasson and Wihlborg [7], Nelson [11], Nelson and Winter [12], Shubik [13] and in particular the essay of Day. The work here is aimed at being complementary with these but aimed specifically at trying to characterize mathematically via a dynamic programming formulation of strategic market games the monetary aspects of innovation eventually including ownership, financial control, and coordination features of a market economy.

1.1 Physical and financial assets, innovation and equilibrium?

This paper is addressed specifically to cost innovation and the breaking of the circular flow of funds. It considers some of the problems of the interaction between ownership and control. Although written to stand alone, it is based directly on two essays, one dealing with equilibrium in a closed monetary economy without innovation [9] and the other concerned with the physical good aspects of innovation in a Robinson Crusoe Economy [14]. In Section 3 the basic structure of the monetary economy and its dynamic equilibria are noted. In particular the role of the money interest rate as a control variable emerges in this setting. The earlier paper [14] is more or less a straightforward exercise in operations research where in a non-market, non-monetary setting Robinson Crusoe has to evaluate how to give up physical assets needed for use in a risky innovation.

In our deconstruction of the investment decision there are five features that merit individual analysis; they are:

1. Equilibrium in a closed monetary economy prior to the knowledge that innovation is feasible;

2. Innovation in a Robinson Crusoe setting, involving only physical assets;

3. Innovation in a closed monetary economy with only short term assets investigating the need for the expansion of money and credit;

4. The roles of long term capital assets, locus of control, evaluation and funding for innovation; and

5. The implications of continuing innovation for the distribution of firm size and investment.

As has been noted above the first two topics have been dealt with in separate essays. We limit our analysis here to the third item in order to make explicit the monetary flows and their control. We comment on the last two features of innovation in Sections 6 and 7, in the expectation that we and others will deal with these central aspects of control and valuation in a competitive innovating economy.

One of our goals is to provide some sufficiently tractable examples that can serve as a basis for experimental games.

2 Open and Closed Monetary Economies with Different Agents

Prior to constructing a fully closed model, an open model of competitive innovation is specified. For simplicity we keep the random component of the innovation process to a minimum. All individuals have an opportunity to innovate in the first period. Each individual's success depends on the size of her investment in innovation. After the first period there is no further opportunity to innovate.

2.1 A preliminary open economy model

A formal model of a large group of competitive firms in a partial equilibrium monetary economy with innovation is considered. After observations on this we turn to the basic structure of ownership and financial control in a closed monetary economy.

2.2 An open competitive economy with innovating individual agents

In our earlier essay [14] we studied Robinson Crusoe as an isolated single innovator with no financial or market system existing. We then considered the market analogue of a small individual, so small that he does not influence market input or output prices even if he innovates. In a monetary economy (unlike that of Crusoe) the firm can buy the desired inputs needed for innovation or production rather than have them in inventory. Furthermore, in general it does not produce for self-consumption, but for sales. This is tantamount to saying that the even the owner-controlled firm may maximize some function of expected profits. In this model we assume for simplicity that the input good for production is the same as the output good, and therefore has the same price. (Another interesting, but quite different model, would assume the input good to be different from the output good).

Consider a continuum of small firms $\phi \in [0,1]$ willing to innovate and consider that their actions as a whole influence market price. In each of a countable number of periods $n = 1, 2, \ldots$ each firm ϕ begins with a quantity q_n^ϕ of goods to be sold in a market. The total amount of goods for sale in period n is

$$Q_n = \int_0^1 q_n^\phi \, d\phi. \tag{1}$$

We assume that there is a demand function $\Phi(\cdot)$ so that the price of the goods in period n is

$$p_n = \Phi(Q_n). \tag{2}$$

The introduction of a demand function for the price of the produced good allows us to avoid modeling the consumers and owners, and the circular cash flows. However, the modeling of these features will be important in the closed models of later sections.

At the start of each period n each firm ϕ holds an amount 0 of cash and has as goods-in-process q_n^ϕ. The goods are sold in the market at the start of the period at the price $p_n = \Phi(Q_n)$. Each firm ϕ borrows an amount b_n^ϕ at the fixed interest rate $\rho \geq 0$ from a central bank to finance production. The loan b_n^ϕ enables the firm to buy an amount of input $i_n^\phi = b_n^\phi / p_n$. The loan is short-term and must be paid back with interest at the end of the period.

All firms begin in period 1 with the same production function $f_1 : [0, \infty) \mapsto [0, \infty)$, which is assumed to be concave, increasing, and to satisfy $f_1(0) = 0$. An input i_n^ϕ by a firm ϕ at period n with production function f_1 results in the production of goods $q_{n+1}^\phi = f_1(i_n^\phi)$ to be sold in the following period.

If a firm ϕ wishes to innovate it must seek out a long-term loan c^ϕ to purchase an amount of input goods $j = c^\phi / p_n$ to use in innovation. The firm must service the long-term loan at $\rho^* c^\phi$ per period where ρ^* is the long-term rate of interest. The servicing of the long-term loan is deducted from profits in each period. In the models of this paper, it is assumed that the decision to innovate is made at the beginning of period 1 and that there is no opportunity to innovate at later stages of the game. (In Part 3 we will consider models which allow for repeated attempts at innovation.)

A successful innovation attempt results in an improved production function $f_2 : [0,\infty) \mapsto [0,\infty)$ with the same properties as f_1 and such that $f_2(i) \geq f_1(i)$ for all inputs $i \geq 0$ and with strict inequality holding for some values of i. The probability of a successful innovation is an increasing function $\xi(j)$ of the amount j of goods invested. The probability of failure is $1-\xi(j)$ and, if failure occurs in the attempt, the firm must operate thereafter with the original production function f_1.

The (net) profit π_n^ϕ of firm ϕ in period n is the income from its sales in the period minus its interest payments:

$$\pi_n^\phi = p_n q_n^\phi - (1+\rho)b_n^\phi - \rho^* c^\phi. \tag{3}$$

The objective of the firm is to maximize the expected value of its total discounted profits, namely

$$\sum_{n=1}^{\infty} \left(\frac{1}{1+\rho}\right)^{n-1} \cdot \pi_n^\phi. \tag{4}$$

There are limits on the size of both the short-term and the long-term loans obtained by a firm ϕ. There is a fixed bound $E \geq 0$ on the size of the long-term loan c^ϕ. The limit on the short-term loan b_n^ϕ is set at $(p_n q_n^\phi - \rho^* c^\phi)/(1+\rho)$ in order to assure that the firm is able to pay its debts and avoid bankruptcy.

In seeking a type symmetric equilibrium for this economy, we assume that all firms begin in period 1 with the same quantity $q > 0$ of goods-in-process, make the same bid b for input goods for production, and invest the same amount c in attempting to innovate. At the beginning of period 2, there will be two types of firms, those called type 1 which have failed in the attempt and must continue with the production function f_1 and those called type 2 which have succeeded and henceforth have the improved production function f_2. There will be a fraction $\varepsilon = \xi(c/p_1)$ of firms of type 2 and $\bar{\varepsilon} = 1 - \varepsilon = 1 - \xi(c/p_1)$ of type 1 in all future periods.

In order to obtain a Bellman equation for the value of this game to a firm, we will first consider the values for the two types in a period $n \geq 2$ and then reason by backward induction to get the equation starting at the beginning.

So suppose that at the beginning of some period after the first, type 1 firms each have goods q_1 and type 2 firms have goods q_2. The total quantity of goods for sale is then

$$Q = \bar{\varepsilon} q_1 + \varepsilon q_2, \tag{5}$$

at the price $p = \Phi(Q)$. The Bellman equation for a firm of type k can be written, for $k = 1, 2$, as

$$W_k(q_k, p, c) = \sup_{0 \le b_k \le (pq_k - \rho^* c)/(1+\rho)} \left[pq_k - b_k(1+\rho) - \rho^* c + \frac{1}{1+\rho} W_k(\tilde{q}_k, \tilde{p}, c) \right], \tag{6}$$

where $\tilde{q}_k = f_k(b_k/p)$ is the quantity of goods held by the firm in the next period and $\tilde{p}$ is the price of goods in the next period. Thus $\tilde{Q} = \bar{\varepsilon}\tilde{q}_1 + \varepsilon \tilde{q}_2$ is the quantity of goods for sale in the next period and $\tilde{p} = \Phi(\tilde{Q})$.

The cost $\rho^* c$ to each firm for its long-term loan is the same in every period. So, if we set $W_k(q_k, p) = W_k(q_k, p, 0)$, it is easy to see that

$$W_k(q_k, p, c) = W_k(q_k, p) - (1 + \rho^*)c \tag{7}$$

and also that

$$W_k(q_k, p) = \sup_{0 \le b_k \le (pq_k - \rho^* c)/(1+\rho)} \left[pq_k - b_k(1+\rho) + \frac{1}{1+\rho} W_k(\tilde{q}_k, \tilde{p}) \right]. \tag{8}$$

Now let $W(q, p)$ be the value function for a firm starting in period 1 and facing the decision about how much to invest in innovation as well as in production. Then the Bellman equation is given by

$$W(q, p) = \sup_{\substack{0 \le b \le (pq - \rho^* c)/(1+\rho) \\ 0 \le c \le E}} \Big[pq - b(1+\rho) - \rho^* c \tag{9}$$

$$+ \frac{1}{1+\rho} \{ (1 - \xi(c/p)) W_1(\tilde{q}, \tilde{p}, c) + \xi(c/p) W_2(\tilde{q}, \tilde{p}, c) \} \Big], \tag{10}$$

where p is the price of goods in period 1, $\tilde{q} = f_1(b/p)$ is the amount of goods held by the firm at the beginning of period 2 and $\tilde{p}$ is the price of goods in period 2. In a type-symmetric equilibrium, all firms will begin period 2 with the same $\tilde{q}$. Thus $\tilde{Q} = \tilde{q}$ is also the total quantity of goods, and $\tilde{p} = \Phi(\tilde{Q})$.

For the open model of this section we do not describe consumer behavior beyond the implicit behavior given in the demand function. Furthermore we are not concerned with closure on the monetary flows that are required of a closed model. These are addressed in Sections 5 and 6. Here the dividends that the firm pays out disappear into a black box, as do the earnings of the central bank.

2.2.1 Convergence to stationary equilibrium after innovation

By a *stationary equilibrium* is meant a Nash equilibrium in which bids, prices and the quantity of goods produced remain constant. After the initial shock of innovation in the first period, the economy in our model always has a fixed fraction $\bar{\varepsilon}$ of firms with production function f_1 and the remaining fraction ε with production function f_2. All firms have the same long-term debt of $c \geq 0$, which requires a payment of $\rho^* c$ in every period. Under some additional assumptions, there is for such an economy a unique stationary equilibrium. It is rarely the case that the economy is in stationary equilibrium immediately or even soon after the innovation stage, but there is, in some generality, convergence to stationary equilibrium as the number of stages approaches infinity.

In this section we assume that the production functions are strictly concave, continuously differentiable, and that for $k = 1, 2$

$$f_k(0) = 0,\ f_k'(0) = \infty,\ \lim_{x \to \infty} f_k'(x) = 0. \tag{11}$$

We also assume that the demand function is continuous, decreasing with finite positive values, and that prices approach ∞ or 0 as the quantity of goods approaches 0 or ∞ respectively; that is,

$$\lim_{Q \to 0} \Phi(Q) = \infty,\ \lim_{Q \to \infty} \Phi(Q) = 0.$$

Finally, we also now allow for the possibility that the profit π^ϕ of a firm ϕ may be negative in some periods. This means that the bid b_k of a firm may exceed the limit $(pq_k - \rho^* c)/(1 + \rho)$ in some periods.

Consider the Bellman equation (8) above and, for $k = 1, 2$, let

$$\psi_k(b_k) = pq_k - (1 + \rho)b_k + \frac{1}{1 + \rho} W_k(f_k(b_k/p), \tilde{p}).$$

Recall that $\tilde{q}_k = f_k(b_k/p)$, so $\psi_k(b_k)$ is the expression inside the supremum in (8). Standard arguments show that

$$\frac{\partial W_k}{\partial q_k}(q_k, p) = p.$$

Consequently the Euler equations take the form

$$\psi_k'(b_k) = -(1 + \rho) + \frac{1}{1 + \rho} \cdot \frac{1}{p} \cdot f_k'(b_k/p) \cdot \tilde{p} = 0.$$

This holds if and only if

$$f_k'(b_k/p) = (1+\rho)^2 \cdot \frac{p}{\tilde{p}}. \tag{12}$$

In stationary equilibrium there will be a fixed price p^* for goods so that $p = \tilde{p} = p^*$ and

$$f_k'(b_k/p) = (1+\rho)^2, \ k = 1, 2.$$

The input of type k firms is $i_k^* = (f_k')^{-1}((1+\rho)^2)$ with output $q_k^* = f_k(i_k^*)$. The total quantity of goods is then $Q^* = \bar{\varepsilon} q_1^* + \varepsilon q_2^*$ and $p^* = \Phi(Q^*)$.

Theorem 1 *There is a unique stationary equilibrium with the constant price p^* and the constant quantity Q^* of goods produced. In every period, firms of type k, for $k = 1, 2$, bid $b_k^* = p^* i_k^*$, and produce q_k^*.*

Proof. The bids b_k^* are the unique solutions to the Euler equations, and an appropriate transversality condition is trivial because quantities and prices are constant by stationarity. ■

Suppose now that the economy begins in period 1 with the fraction $\bar{\varepsilon}$ of firms of type 1 each holding the quantity $q_1 > 0$ of goods and the fraction ε of firms holding the quantity $q_2 > 0$ of goods. So the initial quantity of goods in the economy is $Q = \bar{\varepsilon} q_1 + \varepsilon q_2$ and the initial price is $p = \Phi(Q)$.

Theorem 2 *If, in every period, every firm chooses its bids so that the Euler equation (12) is satisfied, then, as the number of periods approaches infinity, the total quantity of goods will approach Q^*, the price will approach p^*, and the bids of type k firms will approach q_k^* for $k = 1, 2$.*

More briefly, the economy converges to its stationary equilibrium as the number of periods converges to infinity. The proof is in an appendix.

2.2.2 A simple example for profit maximizing firms

In general, an analytic solution to the innovation model is not possible. This is, in part, because the innovation stage forces the economy out of stationary equilibrium. By Theorem 2 the economy will, under reasonable assumptions, converge to a new stationary equilibrium as the number of stages approaches infinity. In this section we consider a very simple example for which the convergence takes only one step and an analytic solution is easy.

Example 1 *Assume that every firm $\phi \in [0,1]$ begins with goods $q = q^\phi = 2$ and the production function*

$$f_1(i) = \begin{cases} 2i, & 0 \le i \le 1, \\ 2, & 1 < i. \end{cases} \tag{13}$$

The total quantity of goods is

$$Q = \int_0^1 q^\phi \, d\phi = 2.$$

The price of output is given by the demand function

$$\Phi(Q) = \begin{cases} 5 - Q, & 0 \le Q \le 5 \\ 0, & Q > 5. \end{cases}$$

So the initial price is $p = 5 - 2 = 3$.

Consider first the situation where there is no possibility of innovation. It is then easy to see that the optimal bid of every firm for input goods is $b = 3$, and each firm then produces

$$q = f_1(b/p) = f_1(1) = 2$$

and earns the profit

$$\pi = pq - (1+\rho)b = 3 \cdot 2 - (1.05) \cdot 3 = 2.85.$$

Indeed, total goods remain equal to 2 and the price of goods is again 3. The economy is in stationary equilibrium. Each firm earns the same profit in every period and receives a total discounted profit of

$$\sum_{n=1}^{\infty} \left(\frac{1}{1+\rho} \right)^{n-1} \cdot \pi = 21 \cdot 2.85 = 59.85. \tag{14}$$

Now suppose that there is the possibility of innovation at stage 1. Assume that a successful attempt at innovation results in the improved production function

$$f_2(i) = \begin{cases} 4i, & 0 \le i \le 1/2, \\ 2, & 1/2 < i. \end{cases} \tag{15}$$

Note that the maximum production level remains 2*, but efficiency is increased so that this maximum is attained with an input of* 1/2 *rather than* 1*.*

Assume that the firms can obtain long-term loans at interest rate $\rho^* = .05$ *in order to purchase goods to be used in the innovation process. Further assume that if* c *units of money are borrowed in order to obtain* $j = c/p = c/3$ *units of the input good, then the probability of a successful innovation is*

$$\xi(j) = \frac{j}{j+1} = \frac{c}{c+3}.$$

In order to find the optimal choice for c*, we will first calculate, as a function of* c*, the value of the game from stage* 2 *onwards for both the successful and the unsuccessful firms. Then we can use backward induction to find the optimal value of* c *at stage* 1*.*

Even with the possibility of innovation, it remains true that the optimal bid for goods to input for production is $b = p = 3$*. Thus every firm begins stage* 2 *with goods* $q = f_1(b/p) = f_1(1) = 2$*. Total goods for sale are* $Q = 2$ *with price* $p = \Phi(Q) = 3$*.*

However, the fraction $c/(c+3)$ *of the firms are successful and begin stage* 2 *with the improved production function* f_2*, while the remaining fraction* $3/(c+3)$ *are unsuccessful and still have the production function* f_1*. Call the unsuccessful firms type* 1*. These firms continue to have the optimal bid* $b_1 = 3$ *with output* $f_1(1) = 2$*. All firms have the same long-term debt of* c*. So the profit of type 1 firms at stage* 2 *is*

$$\pi_1 = pq - (1+\rho)b_1 - \rho^* c = 3 \cdot 2 - (1.05) \cdot 3 - .05c = 2.85 - .05c.$$

The successful firms, called type 2*, have the optimal bid* $b_2 = 3/2$ *with output* $f_2(b_2/p) = f_2(1/2) = 2$ *and profit*

$$\pi_2 = pq - (1+\rho)b_2 - \rho^* c = 3 \cdot 2 - (1.05) \cdot (3/2) - .05c = 4.425 - .05c.$$

Notice that total output remains $Q = 2$ *and thus the price is also constant at* $p = 3$*. The optimal bids remain the same in future periods, namely* $b_1 = 3$ *for type 1 firms and* $b_2 = 3/2$ *for type* 2 *firms. Thus profits also remain constant and the total discounted payoffs (from stage* 2 *on) to the two types as a function of* c *with* q *fixed at* 2 *are*

$$W_1(c) = \sum_{n=1}^{\infty} \left(\frac{1}{1+\rho}\right)^{n-1} \cdot \pi_1 = 21 \cdot \pi_1 = 59.85 - 1.05c$$

for type 1*, and*

$$W_2(c) = \sum_{n=1}^{\infty} \left(\frac{1}{1+\rho}\right)^{n-1} \cdot \pi_2 = 21 \cdot \pi_2 = 92.925 - 1.05c$$

for type 2*. Now we can calculate the value* $W(2)$ *to a firm from the beginning stage when all firms start at* $q = 2$*. This value is*

$$W = W(2) = \sup_{0 \le b \le \frac{pq - \rho^* c}{1+\rho},\, 0 \le c \le E} [\psi(b, c)]$$

where

$$\psi(b, c) = pq - (1+\rho)b - \rho^* c + \frac{1}{1+\rho}\left\{\frac{c}{c+3} W_2(c) + \frac{3}{c+3} W_1(c)\right\}.$$

Take $p = 3, q = 2, \rho = \rho^* = .05, b = 3$ *and the values calculated above for* $W_1(c), W_2(c)$ *to get*

$$\psi(b, c) = \psi(3, c) = 2.85 + \frac{1}{1.05}\left\{\frac{c}{c+3} \cdot 92.925 + \frac{3}{c+3} \cdot 59.85\right\} - 1.05c.$$

Differentiate with respect to c *to find the maximum at* $c = c^* = 6.49$*. (The bounds above in the formula for* W *are satisfied so long as the bound* E *imposed on long-term loans is at least* 6.49*.) So the probability of success is* $c^*/(c^* + 3) = .\ 68$ *and*

$$W = 2.85 + \frac{1}{1.05}\{(.68) \cdot 92.925 + (.32) \cdot 59.85\} - (1.05) \cdot (6.49) = 74.46.$$

Now $74.46 > 59.85$ *so that overall expected profits have increased due to innovation. However, the unsuccessful firms would have been better off had they not tried to innovate. Indeed the total discounted profit of a type* 2 *firm is just the same as in* (14) *minus the cost* $1.05c^*$ *of financing the attempt.*

2.3 A comment on open models

When studying a few firms or a single industry to answer questions such as the distribution of firm size, the need to consider the full feedbacks from a closed economy is for most purposes both unnecessary and more difficult than partial equilibrium analysis. However in order to appreciate the macroeconomic aspects of the influence of financial control and the money supply it is necessary to consider a closed economy. It is there that the separation among ownership, management and financing first appears with clarity and the meaning of the breaking of the equilibrium circular flow of capital may be illustrated.

3 The Closed Economy as a Sensing, Evaluating and Control Mechanism

Prior to considering the formal closed models with innovation, several general items that supply context are covered. A detailed sketch of the whole closed system is presented in Figure 1; it is somewhat simplified in Figure 2 prior to the formal analysis. Figure 1 shows differentiated economic units with some enforcement and evaluation included. Figure 1[1] provides an overall description describing how credit evaluation, clearing houses, the banks, central bank and courts fit into the information and enforcement structure. Institutional reality has many variations and it is easy to argue with the particular "wiring" presented here. but the purpose of this diagram is to give a fingerspitzengefühl or an intuitive feeling of what the many realities look like. Unlike Figure 2, three additional institutions appear. They are the clearinghouse, the credit evaluation agency (implicitly including the accountants) and the court house. In much of economic theory expertise is ignored primarily because it is too hard to deal with. In old fashioned securities analysis and accounting due diligence and expertise is central to applications, but it is often ignored in much of economic and finance theory. This is because it is subsumed in modeling the risky economic instruments and entities being dealt with as lottery tickets that have already been correctly evaluated. We follow this extreme approximation because for our prime purpose, which is consideration of the breaking of the circular flow of capital, even at this level of abstraction the phenomenon still occurs.

[1] Based on unpulished work of Shubik and Smith.

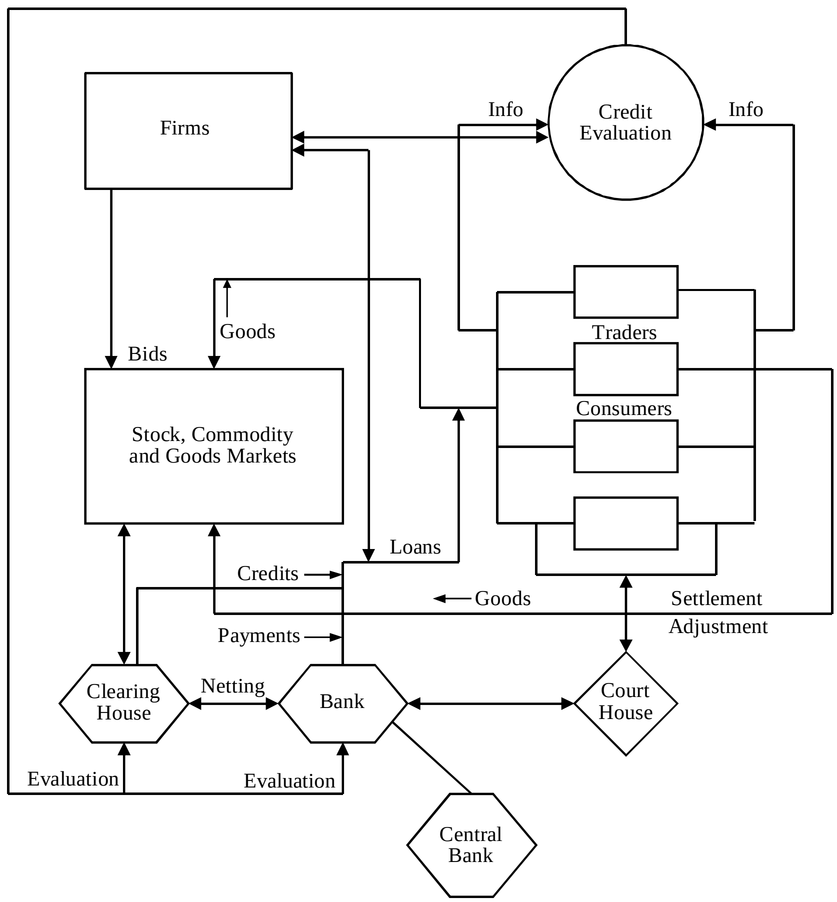

Figure 1: The economy with finance, clearing evaluation and enforcement

3.1 Individual or representative agents?

When there is no uncertainty, models utilizing representative agents and models with independent agents solved for type-symmetric noncooperative equilibria (TSNE) give the same equilibrium results. When there is any exogenous uncertainty present this is no longer generally true. With independent agents uncertainty is not necessarily correlated. However, with a representative agent, uncertainty is implicitly correlated for all members of the class. As is indicated below we consider a minimal amount of uncertainty.

3.2 On money, credit, banks, and central banks

In institutional fact the definition and measurement of the money supply is difficult at best. The distinctions between money and credit are not always clear. Here we utilize a ruthless simplification in order to highlight the distinction between money and credit and to be able to stress economic control. Consider money to be paper gold, or some form of blue chip in which payments are made. Credit is a contract between two entities A and B, in which individual A delivers money at time t_1 in return for an IOU or a promise from B to repay an amount of money to A at time t_2. An individual may be a natural person or a legal person such as a firm, a bill broker, a bank, a credit granting clearing house or a central bank.

We may consider two ways to vary the money supply. The first and simpler is that the central bank is permitted to print it. Another way to vary the money supply is to accept the IOU notes of commercial banks as money. Say they are red chips, in contrast with the central bank's blue chips. They are accepted in payment on a 1 : 1 basis with blue chips. A reserve ratio controls the amount a bank can issue, thus for any k units of red chips issued, a bank must hold one unit of blue chips.[2]

As we wish to maintain as high a level of simplification as possible in order to illustrate the breaking of the circular flow, we select the simpler structure. The banking system is considered as one and called the central bank. It has funds above its reserves[3] that it can lend and it can pay interest on deposits.[4]

4 The Separation of Management and Ownership

The next level of complexity above the single type of agent utilizes two types of agents: managers of the firms and stockholder-owners. (In the first model

[2]The justification for the acceptance of reserve ratio banking is in the dynamics along with acceptance of fiat (see for example, [3]).

[3]Central bank reserves in a fiat money economy are a creation of law and possibly economic theology. Mathematically they are just societal rules of the game or an algorithm stating how the central bank can create money. They specify its strategy set. In actuality the strategy set is also bounded by political pressures.

[4]In general, central banks do not accept deposits from natural persons, but for modeling simplicity here we permit them to do so.

below, there is also a class of *saver agents* who subsist on the returns from their bank deposits.) The economy can be interpreted as a fully defined game of strategy where there is a finite measure of firms and of stockholder-owners whose overall actions will influence prices. By assuming that we limit the solution to a type symmetric noncooperative equilibrium, all agents of each type, even though independent, will employ a strategy common to their type. In illustrating some of the basic aspects of financing and control of innovation, the independent agent models show microeconomic uncertainty at the innovation stage.

5 A Closed Economy Prior to Innovation: The Circular Flow of Money Illustrated

The model presented in this section is based on work of Karatzas et al. [9] without innovation. It will be extended in the next section to a model with innovation in order to consider the disequilibrium aspects of innovation on the money supply. Out stress so far has been on non-monetary models of Crusoe as an innovator, or on open microeconomic models. From here on the emphasis is on simple closed economies or macroeconomic models.

5.1 A closed economy with producers, consumers, monied individuals and a central bank

The underlying model is that of a "cash-in-advance" market economy with a continuum of firms $\phi \in J = [0, 1]$ that produce goods all of which must be put up for sale, and a continuum of stockholder agents $\alpha \in I = [0, 1]$ who own the firms and purchase these goods for consumption. The agents hold cash and bid for goods in each of a countable number of periods $n = 1, 2, \ldots$. The firms hold no cash[5] and must borrow from a single outside bank to purchase goods as input for production in every period. The bank is modeled as a strategic dummy that accepts deposits and offers loans at a fixed interest rate ρ. In addition to the owner agents, there may be a continuum of *saver agents* $\gamma \in K = [0, 1]$, each of whom holds cash, bids in every period to buy goods for consumption, and subsists entirely on her savings. These agents

[5]This reflects the payment of the 100% dividend, the timing of which is irrelevant in a perfect credit rating competitive economy.

can be thought of as "retirees" or private capitalists.[6] Figure 2 shows the structure of the economy with firms, owners, savers and a central bank.

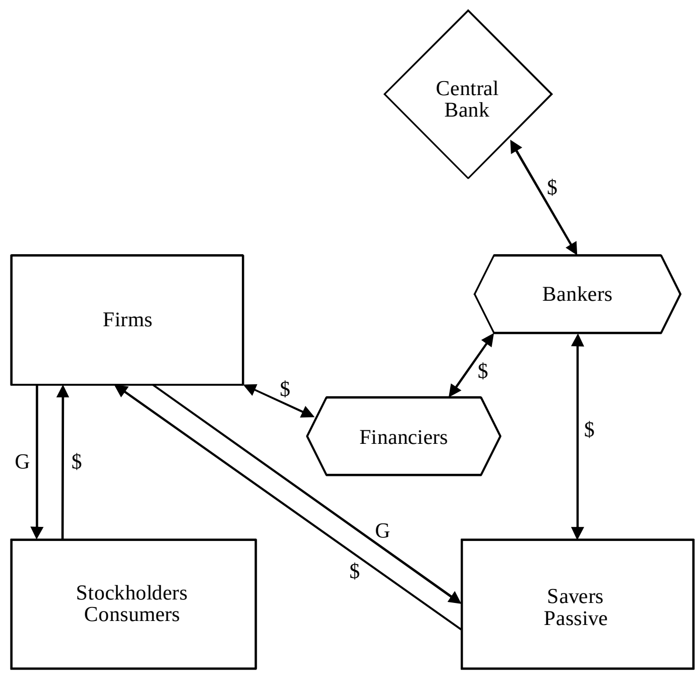

Figure 2: Who controls what?

The six boxes portray an economy somewhat more complex than our mathematics deals with, but give an intuitive insight into the spreading out of ownership and control in a modern enterprise economy. The firms are in general corporate, they do not own themselves. They have (at some ultimate level) natural person stockholders who are also consumers. Directly or indirectly they depend on at least four sets of decisionmakers for debt (and some equity or options) financing. They are the passive savers, the financiers, the commercial banks and the central bank. Without having to elaborate further it should be evident that in any dynamic setting the coordination problem is considerable. In the mathematical model below we grossly simplify the financial sector, ignoring the financiers, collapsing the commercial banks and

[6]In a less Draconian abstraction the difference between retirees and capitalists is not merely age, but expertise. The role of competent financing as a perception and evaluating device cannot be over stressed.

central bank into one and having the passive savers save in the aggregate bank, while the firms borrow only from this bank.

The situation of the firms in this model is similar to that of the firms in the open model of Section 2. However, the firms in this first closed model have no opportunity to innovate and carry no long-term debt. Each firm ϕ begins every period n with goods q_n^ϕ that are to be sold in the market. The total amount of goods offered for sale is defined as in equation (1) by

$$Q_n = \int q_n^\phi \, d\phi. \tag{16}$$

Each firm ϕ also borrows cash b_n^ϕ from a central bank, with $0 \leq b_n^\phi \leq (p_n q_n^\phi)/(1+\rho)$, where p_n is the price of the good in period n and $\rho > 0$ is the interest rate. There is no demand function in this model and the prices are formed endogenously as will be explained below.

The firm spends the cash b_n^ϕ to purchase the amount of goods $i_n^\phi = b_n^\phi / p_n$ as input for production, and begins the next period with an amount of goods

$$q_{n+1}^\phi = f(i_n^\phi).$$

Here $f(\cdot)$ is a production function, which satisfies the usual assumptions. During period n each firm ϕ earns the (net) profit

$$\pi_n^\phi = p_n q_n^\phi - (1+\rho) b_n^\phi,$$

since it must pay back its loan with interest. The goal of the firm is to maximize its total discounted profits[7]

$$\sum_{n=1}^{\infty} \left(\frac{1}{1+\rho} \right)^{n-1} \cdot \pi_n^\phi.$$

In a given period n, the total profits generated by all the firms, are

$$\Pi_n = \int \pi_n^\phi \, d\phi.$$

The profits Π_n are distributed to the owner agents in equal shares at the end of the period.

[7] In institutional fact the large firm has a considerable constituency of customers, employees, the government and others as well as the owners.

The owner agents are now considered. A typical owner agent α holds money m_n^α at the beginning of each period n. The agent bids an amount of money a_n^α with $0 \le a_n^\alpha \le m_n^\alpha + \Pi_n/(1+\rho)$, which buys him an amount $x_n^\alpha = a_n^\alpha/p_n$ of goods for immediate consumption. Any extra money an owner agent has is deposited and earns interest at rate ρ. The agent begins the next period with cash

$$m_{n+1}^\alpha = (1+\rho)\,(m_n^\alpha - a_n^\alpha) + \Pi_n.$$

Each agent α seeks to maximize his total discounted utility

$$\sum_{n=1}^{\infty} \beta^{n-1} u(x_n^\alpha),$$

where u is a concave increasing utility function and $0 < \beta < 1$ is a given discount factor.

Also considered is a typical saver agent γ, who holds m_n^γ in cash at the start of period n. The saver bids an amount c_n^γ of cash with $0 \le c_n^\gamma \le m_n^\gamma$, which buys him a quantity $y_n^\gamma = c_n^\gamma/p_n$ of goods, and starts the next period with

$$m_{n+1}^\gamma = (1+\rho)\,(m_n^\gamma - c_n^\gamma)$$

in cash. If $v(\cdot)$ is his utility function, with the same properties as $u(\cdot)$, the saver agent's objective is to maximize the total discounted utility

$$\sum_{n=1}^{\infty} \beta^{n-1} v(y_n^\gamma).$$

The total amounts of money bid in period n by the owner agents, the firms, and the saver agents, are

$$A_n = \int a_n^\alpha \, d\alpha, \qquad B_n = \int b_n^\phi \, d\phi \qquad \text{and} \qquad \Gamma_n = \int c_n^\gamma d\gamma,$$

respectively. The price p_n is formed as the total bid over the total production

$$p_n = \frac{A_n + B_n + \Gamma_n}{Q_n}.$$

An equilibrium is constructed as follows. Suppose that all owner agents begin with cash $M_1^A = m^A > 0$, all saver agents begin with cash $M_1^\Gamma = m^\Gamma \ge$

0, and all firms begin with goods $Q_1 = q > 0$. Thus, the total amount of cash $M_1 = M_1^A + M_1^\Gamma$ across agents, is equal to

$$m = m^A + m^\Gamma,$$

and the proportion of money held by the saver agents is

$$\nu = \frac{m^\Gamma}{m} = \frac{m^\Gamma}{m^A + m^\Gamma}, \qquad \text{with} \qquad 0 \le \nu < 1.$$

Suppose that the bids of the agents and firms are

$$a_1 = am, \qquad b_1 = bm, \qquad c_1 = cm,$$

that is, proportional to the total amount of cash, so that the price is also proportional to this amount:

$$p_1 = p(m) = \frac{(a+b+c)m}{q}.$$

Then the profit of each firm is

$$\Pi_1 = p_1 q - (1+\rho)b_1 = (a + c - \rho b)m,$$

the cash of each owner agent at the beginning of the next period is

$$M_2^A = (1+\rho)\left(m^A - am\right) + \Pi_1,$$

and the cash held by each saver agent is

$$M_2^\Gamma = (1+\rho)\left(m^\Gamma - cm\right).$$

Thus, the total amount of cash held by all agents at the beginning of the next period is

$$M_2 = M_2^A + M_2^\Gamma = (1 + \rho - \rho(a+b+c))m = \tau m,$$

where we have set

$$\tau = 1 + \rho - \rho(a+b+c).$$

Define

$$r = \frac{(1+\rho)(1-\beta)}{\rho}. \tag{17}$$

The following theorem was established in [9].

Theorem 3 *Suppose that there exists* i^* *with* $f'(i^*) = (1+\rho)/\beta$ *. Then there is an equilibrium for which, in every period: each firm inputs* i^**, produces* $q^* = f(i^*)$*, and bids the amount* $b_n = b^* M_n$ *; each owner agent bids* $a_n = a^* M_n$*; and each saver agent bids* $c_n = c^* M_n$ *. Here*

$$a^* + b^* + c^* = r\,, \qquad b^* = \frac{r}{q^*} \cdot i^*\ , \qquad c^* = (1-\beta)\nu \tag{18}$$

and $M_n = M_n^A + M_n^\Gamma$ *is the amount of cash held across agents in period* n*.*

Furthermore, in each period n*: every owner agent consumes the amount* $x^* = (1 - \frac{\rho\nu}{1+\rho})q^* - i^*$*; every saver agent consumes the amount* $y^* = (\frac{\rho\nu}{1+\rho})q^*$*; whereas every firm makes* $\pi^* M_n$ *in profits, with* $\pi^* = r - (1+\rho)b^*$*.*

It is shown in [9] that, in the equilibrium of Theorem 3, the consumption and total discounted utility of the owner agents are decreasing functions of ρ, such agents prefer as *low* an interest rate as possible. Similarly, the firms also prefer an interest rate as close to zero as possible, in order to maximize their profits. But the situation of the saver agents is subtler: under certain configurations of the various parameters of the model (discount factor, production function, utility function) they prefer as *high* an interest rate as possible, whereas under other configurations they settle on an interest rate $\rho^* \in (0, \infty)$ that uniquely maximizes their welfare. Let

$$\tau^* = 1 + \rho - \rho(a^* + b^* + c^*).$$

Then money and prices inflate (or deflate) at rate τ^* in the equilibrium of Theorem 3. We also have $a^* + b^* + c^* = r$, so that the Fisher equation $\tau^* = \beta(1+\rho)$ holds.

Remark 1 *By setting* $\nu = 0$ *in Theorem 3, we obtain an economy with only producer firms and owner-consumer agents.*[8] *We will similarly dispense with saver agents in the models below. This will be useful in illustrating the basic problems with the circular flow and money supply with innovation in a simple context. Also we will take* $\tau = \beta(1+\rho) = 1$ *so that there is no inflation.*

[8] Of course, the proportion ν has to be strictly less than one; for otherwise there is no one to engage in productive activity, own the firms or receive their profits, and the model unravels.

6 Innovation in an Asset Poor Economy: Breaking the Circular Flow

As in the previous models we aggregate all goods in the model of this section into a single perishable consumable that is utilized in consumption or production or consumed in innovation. There is no capital stock, such as steel mills. There is no "fat" in the economy, resources for innovation must come directly out of consumption resources.

6.1 The meaning of an asset poor economy

In actuality a modern economy is rich in real durable assets with a time profile of durables of many ages that are consumed only in production, not consumption. Gross Domestic Product may be split into consumption and investment. If we consider around 70% in consumption, then we note that at market prices the value of real assets such as steel mills, automobile factories, houses, automobiles, machinery, land and other consumer durables are priced probably between 5 to 10 times the value of consumption. None of these items are meaningfully placed directly in the utility functions of the individuals. Furthermore, it is the services of consumer durables that are ultimately valued and not the durables themselves. This is even truer of items such as steel mills. In the models considered so far we have not indicated that the presence of this large mass of assets owned by individuals may be such that the loss or exchange of a small percentage of these assets while pursuing innovation will hardly change the consumption of the owners of large amounts of real assets.

In a poor country the amount of available assets relative to consumption will be much smaller than in a rich one. We consider in this section the extreme simplifying case where innovation must come directly out of consumption. This makes it easier to be specific about the breaking of the circular flow of capital and the match between real assets and money.

In essence innovation is nothing other than the execution of an idea for a new process to rearrange and employ existing assets in a different manner.[9] It is a breaking of equilibrium that in a rich country calls for an alternative use for productive assets but does not directly cut down heavily on current

[9] Bankruptcy in a basic way is similar to innovation in the sense that it involves a nonequilibrium redeployment of assets.

consumption. In contrast, in an asset poor economy, an immediate sacrifice in consumption is called for.

6.2 Innovation in an asset poor economy

We consider a model with a class of identical manufacturers, a class of identical, individual consumers, who also own the firms, and an outside or central bank.

One could consider three variants:

1. The managers are in control, the owners are passive and the central bank is willing to create new money to make investment loans.
2. The managers are in control, the owners are passive and the central bank does not create new money. It is a flow-through institution
3. The stockholders are in control, they dictate corporate policy, thus the firms are operationally utility maximizing rather than profit maximizing. There are at least two possibilities here that need to be distinguished (a) the central bank is willing to create new money and the stockholders cannot create their own credit; and (b) the central bank is unwilling to create new money and the stockholders can create their own credit.

All three variants are found in a modern economy. The third is the most representative of fights for oligopolistic control of the firms by individual stockholders (in partnership or corporate structure) holding large blocks of stock, while the remaining stockholders are passive, riding coattails or selling. We do not construct a mathematical model of this case here. The first model serves adequately to illustrate the problems with financing and is now described in detail.

6.2.1 A model with managerial control and central bank lending

As in the model of Section 5.1, there is a continuum of firms $\phi \in J = [0,1]$. Each firm ϕ begins each period n with goods in process q_n^ϕ to be sold in the market, and borrows cash b_n^ϕ from the central bank to purchase goods $i_n^\phi = b_n^\phi / p_n$ as input for production. Each firm ϕ begins in period 1 with no long term debt, but may borrow an amount of money c^ϕ from the bank to purchase goods $j^\phi = c^\phi/p_1$ to be used in innovation. The interest on this long term debt must be paid in every period and the short term loan b_n^ϕ must be paid back with interest at the end of each period n. In general, the long term rate ρ^* might differ from the short term rate, but it is sufficient and simpler to assume that they are equal to a common value $\rho \geq 0$. In order that a firm be able to meet its debt obligations, the bid b_n^ϕ is restricted to lie in the interval $[0, (\hat{p}_n q_n^\phi - c^\phi \rho)/(1+\rho)]$, where $\hat{p}_n$ is the bank's estimate of the price p_n in period n. (In a rational expectations equilibrium, $\hat{p}_n = p_n$.) The bank may also impose an upper limit E on the long term loan c^ϕ.

As in the model of Section 2.2, all firms begin in period 1 with the same production function f_1 and thus a firm ϕ will begin period 2 with goods $q_2^\phi = f_1(i_1^\phi)$. However, a successful innovation results in the improved production function f_2. Thus in periods after the first, there are two types of firms - those of type 1, that failed in the attempt at innovation, and continue with production function f_1, and the type 2 firms, that succeeded, and have f_2.

The profit π_n^ϕ of a firm ϕ in period n is defined by formula (3) in Section 2.2, and each firm seeks to maximize its total discounted profits (4). The total profit in period n of all the firms is the integral

$$\Pi_n = \int \pi_n^\phi \, d\phi,$$

and is paid to the consumer-owners in equal shares at the end of the period, as is explained below.

Because we will again look for a type symmetric equilibrium, we will assume that all firms begin period 1 with the same quantity $q_1 > 0$ of goods, and we will often omit the superscript ϕ below. When all firms begin in the same state, make the same bids b_1 and c, and earn the same profit $\pi_1 = p_1 q_1 - (1+\rho) b_1 - \rho c$, the total profit and total goods in period 1 simplify to

$$\Pi_1 = \int \pi_1 \, d\phi = \pi_1, \qquad Q_1 = \int q_1 \, d\phi = q_1.$$

Suppose W is the overall value of the program to a firm. W_1 is the value after a failed investment, and W_2 is the value after a successful investment. Let $\xi(c/p_1) = \xi(j)$ be the probability of success when $c/p_1 = j$ is invested in innovation. Then the value functions satisfy the following optimality equations.

$$W(q) = \sup_{\substack{0\le b\le \frac{\hat{p}q-\rho c}{1+\rho} \\ 0\le c\le E}} \Bigg[pq - (1+\rho)b - \rho c + \frac{1}{1+\rho}\cdot\left\{\left(1-\xi\left(\frac{c}{p}\right)\right)\cdot W_1\left(f_1\left(\frac{b}{p}\right),c\right) + \xi\left(\frac{c}{p}\right)\cdot W_2\left(f_1\left(\frac{b}{p}\right),c\right)\right\}\Bigg] \tag{19}$$

where

$$W_k\left(q,c\right) = \sup_{0\le b\le \frac{\hat{p}q-\rho c}{(1+\rho)}} \left[pq - b(1+\rho) - \rho c + \frac{1}{1+\rho}\cdot W_k\left(f_k\left(\frac{b}{p}\right),c\right)\right] \tag{20}$$

for $k = 1, 2$.

For simplicity we have suppressed super and subscripts above and will often do so below as well. In both (19) and (20) the notation $\hat{p}$ is for the bank's estimate of the price for goods in the period, whereas p denotes the price actually formed as will be explained below.

As in Section 2.2 there will be after the first period the fraction $\varepsilon = \xi(c/p)$ of type 2 firms that succeeded at innovation and the fraction $\bar{\varepsilon} = 1 - \varepsilon$ of type 1 firms that failed.

In seeking a type symmetric solution, we will assume that at the beginning of periods $n \ge 2$ all firms of type 1(respectively type 2) will hold the same quantity of goods q_n^1 (respectively q_n^2) and earn the same profit π_n^1 (respectively π_n^2) in the period. Thus the total profit and totals goods in period n are given by

$$\Pi_n = \bar{\varepsilon}\pi_n^1 + \epsilon\pi_n^2, \qquad Q_n = \bar{\varepsilon}q_n^1 + \varepsilon q_n^2.$$

In addition to the firms there is also a continuum of consumer-stockholder agents $\alpha \in I = [0,1]$. As in the model of Section 5.1 each agent α begins every period n with cash m_n^α and bids $a_n^\alpha \in [0, m_n^\alpha]$ to purchase goods a_n^α/p_n for immediate consumption. The agent deposits the excess cash $m_n^\alpha - a_n^\alpha$ in the bank and gets back $(1+\rho)(m_n^\alpha - a_n^\alpha)$ at the end of the period.

The accounting profit D_n of the bank in period n consists of its earnings from the loans made to the firms less the interest paid on the deposits of the owners. Thus

$$D_n = \rho\cdot\left[\int b_n^\phi\, d\phi - \int (m_n^\alpha - a_n^\alpha)d\alpha + c\right]. \tag{21}$$

For this model we assume that the profit of the bank, like that of the firms, is paid to the owners in equal shares at the end of the period. (This assumption and a possible alternative are discussed in Section 6.2.3 below.) Thus an owner agent α begins period $n+1$ with cash

$$m_{n+1}^{\alpha} = (1+\rho)(m_n^{\alpha} - a_n^{\alpha}) + \Pi_n + D_n. \tag{22}$$

The value function V for an owner satisfies

$$V(m) = \sup_{0 \le a \le m} \left[u\left(\frac{a}{p}\right) + \beta V((1+\rho)(m-a) + D + \Pi) \right] \tag{23}$$

where u is a concave, nondecreasing utility function and we have again suppressed super and subscripts.

The price p_n in each period n is formed as the ratio of the total cash bid in the goods market to the total amount of goods for sale. In the type symmetric case, the prices are given by

$$p_1 = \frac{a_1 + b_1 + c}{q_1}, \qquad p_n = \frac{a_n + \bar{\varepsilon} b_n^1 + \varepsilon b_n^2}{\bar{\varepsilon} q_n^1 + \varepsilon q_n^2},\ n \ge 2.$$

If $m_1 = m$, then by (22)

$$\begin{aligned} m_2 &= (1+\rho)(m - a_1) + \Pi_1 + D_1 \\ &= (1+\rho)(m - a_1) + p_1 q_1 - (1+\rho) b_1 - \rho c + \rho \cdot [b_1 - (m - a_1) + c]. \end{aligned}$$

Now $p_1 q_1 = a_1 + b_1 + c$. Substitute this into the previous equation and simplify the result to see that $m_2 = m + c$. A similar calculation shows that $m_n = m + c$ for all $n \ge 2$. Thus in this model the money supply has an initial increase because of the long-term loan in the first period and then remains constant.

6.2.2 Stationary equilibrium and the question of convergence

A *stationary equilibrium* for the economy of the previous section is an equilibrium in which bids, prices, and the quantities of goods and money remain constant. The economy experiences a shock due to innovation in the first period after which there is always a fixed fraction $\bar{\varepsilon}$ of type 1 firms and ε of type 2 firms. We cannot expect to have a stationary equilibrium until sometime after the first period. Under some additional regularity assumptions, there does exist a type symmetric stationary equilibrium for the economy as it is configured after the initial shock.

Assume now that the production functions f_1, f_2 and the utility function u are strictly concave, continuously differentiable, and that the production functions satisfy the condition (11) of Section 2.2.1.

Suppose as above that there is a fraction $\bar{\varepsilon}$ of type 1 firms having production function f_1 and holding goods q_1, a fraction ε of type 2 firms having production function f_2 and holding goods q_2, and a continuum of consumer-owner agents $\alpha \in [0,1]$ each with cash m. The argument in Section 2.2.1 using Euler equations works here as well to show that in stationary equilibrium each type k firm will input the quantity $i_k^* = (f_k)^{-1}((1+\rho)^2)$ and produce $q_k^* = f_k(i_k^*)$ in every period.

The Euler equation for a consumer-owner takes the form

$$\frac{1}{p}u'\left(\frac{a}{p}\right) = \frac{\beta(1+\rho)}{\tilde{p}}u'(\frac{\tilde{a}}{\tilde{p}}) = \frac{1}{\tilde{p}}u'\left(\frac{\tilde{a}}{\tilde{p}}\right), \tag{24}$$

where $\beta(1+\rho) = 1$ by assumption, and $\tilde{a}$ and $\tilde{p}$ are the agent's bid and the price in the next period. But in stationary equilibrium $a = \tilde{a}$ and $p = \tilde{p}$. So the only condition on the optimal bid a^* is that $0 \leq a^* \leq m$.

Let $Q^* = \bar{\varepsilon}q_1^* + \varepsilon q_2^*$ be the total production when firms of type k input i_k^* for $k = 1, 2$. Now in order to purchase i_k^*, firms of type k must bid $b_k^* = pi_k^*$. Thus the price must satisfy

$$p = \frac{a^* + \bar{\varepsilon}b_1^* + \varepsilon b_2^*}{\bar{\varepsilon}q_1^* + \varepsilon q_2^*} = \frac{a^* + \bar{\varepsilon}pi_1^* + \varepsilon pi_2^*}{\bar{\varepsilon}q_1^* + \varepsilon q_2^*},$$

or equivalently

$$\frac{a^*}{p} = \bar{\varepsilon}(q_1^* - i_1^*) + \varepsilon(q_2^* - i_2^*),$$

which means that the owner agents consume all the goods produced by the firms that are not used by the firms as input for production of goods for the next period.

Let $p = m/Q^*$ so that

$$b_k^* = pi_k^* = \frac{m}{Q^*} \cdot i_k^*, \;\; k = 1, 2$$

and

$$a^* = \frac{m}{Q^*} \cdot [\bar{\varepsilon}(q_1^* - i_1^*) + \varepsilon(q_2^* - i_2^*)] < m.$$

Observe also that, for $k = 1, 2$,

$$q_k^* = f_k(i_k^*) = \int_0^{i_k^*} f_k'(x)\, dx \geq f_k'(i_k^*) \cdot i_k^* = (1+\rho)^2 \cdot i_k^* > (1+\rho) \cdot i_k^*.$$

Thus the quantities $q_k^* - (1+\rho)i_k^*, k = 1, 2$ are strictly positive. Now the conditions on the bids b_k^* that

$$b_k^* \leq \frac{pq_k^* - \rho c}{1+\rho}$$

can be rewritten as

$$\rho c \leq pq_k^* - (1+\rho)b_k^* = \frac{m}{Q^*} \cdot (q_k^* - (1+\rho)i_k^*).$$

By assumption, the long term debt c cannot exceed the bound E. Thus the inequality above will hold if

$$\frac{E}{m} \leq \frac{q_k^* - (1+\rho)i_k^*}{\rho Q^*}.$$

Theorem 4 *If the ratio E/m is sufficiently small, then there is a stationary equilibrium such that, in every period, each firm of type k inputs i_k^*, produces $q_k^* = f_k(i_k^*)$, and bids $b_k^* = \frac{m}{Q^*} \cdot i_k^*$; each owner consumer agent bids $a^* = \frac{m}{Q^*}[\bar{\varepsilon}(q_1^* - i_1^*) + \varepsilon(q_2^* - i_2^*)]$. Furthermore, in every period, every owner-consumer agent consumes the amount of goods $\bar{\varepsilon}(q_1^* - i_1^*) + \varepsilon(q_2^* - i_2^*)$ and every firm of type k makes the profit $\pi_k^* = \frac{m}{Q^*} \cdot (q_k^* - (1+\rho)i_k^*)$.*

Proof. The bids a^* and $b_k^*, k = 1, 2$ satisfy their Euler equations, and the appropriate transversality condition is trivial because, by stationarity, the payoffs are the same in every period. ■

Recall that Theorem 2 of Section 2.2.1 shows there is convergence to stationary equilibrium for the open model there. We suspect that an analogous result holds for the closed model of this section. Even if this is true, convergence may be slow and a general analytic solution to the model with innovation seems unlikely. Some simple examples for which convergence is fast are in Section 6.3 below.

6.2.3 The modeling of central bank profits

In the model of Section 6.2.1, it is assumed that the amount ρc of long-term interest is part of the accounting profit D_n (defined in (21)) of the central bank and is paid in each period to the consumer-owner agents (see (22)). This is one of several fairly natural models each with different financial, economic and political implications. One possibility is to neutralize the money as it comes in, leaving a deflationary trend in place. Other alternatives are for the bank to subsidize some group of agents with this income, or spend it to buy resources (such as foreign aid subsidies for purchases in the economy, or the destruction of government purchases of resources for a foreign war). As many institutional variants can be defined, the choice among them depends on the questions to be answered and their empirical relevance.

In order to define the minimal viable model we have collapsed five banking functions into a single institution. They are:

1. Financing circulating capital or goods in process;
2. Accepting consumer savings;
3. Making short term consumer loans;
4. Making long term investment banking loans;
5. Varying the money supply.

A more detailed model would use at least three institutions: a central bank, commercial banks, and investment bankers. Here we have chosen a model with only three types of agents: the firms, the consumer-owners, and a banking system. This seems to be the minimal number necessary to build a playable game that illustrates the phenomenon of breaking the circular flow of capital.

6.3 Two simple examples

In this section equilibria are calculated for two very simple examples. In both examples the production functions f_1 and f_2 are defined by the equations (13) and (15); that is, they are assumed to be the same as those that were used for the example of Section 2.2.2. Similarly we assume that $\rho = \rho^* = .05$ as in that example and take $\beta = 1/1.05$.

The first example treats a consumer-producer who labors in isolation to produce goods for his personal consumption and has the opportunity to innovate. The second example contrasts the first with the situation in a monetary economy with many firms and owner-consumers.

6.3.1 Robinson Crusoe revisited

Consider first the situation of Robinson Crusoe equipped with the production function f_1 and without the opportunity to innovate. Suppose that Crusoe begins with a quantity of goods $q > 0$, selects an amount i, $0 \leq i \leq q$ to put into production, and consumes the remaining $q - i$ resulting in a utility of $u(q - i)$. He then begins the next period with goods $\tilde{q} = f_1(i)$ and continues the game.

Let $V_1(q)$ be the value of this one-person game to Crusoe. It satisfies the Bellman equation

$$V_1(q) = \sup_{0 \leq i \leq q} [u(q - i) + \beta V_1(f_1(i))].$$

For simplicity we assume that Crusoe is risk neutral with utility function $u(q) = q$.

It is not difficult to check that a stationary equilibrium has $q = 2$ and $i = 1$ at every stage of the game. Thus

$$V_1(2) = \sum_{n=1}^{\infty} \beta^{n-1} u(1) = \frac{u(1)}{1 - \beta} = \frac{1}{1 - 1/1.05} = 21.$$

Similarly, if Crusoe begins with the production function f_2, a stationary equilibrium has $q = 2$ and $i = 1/2$ with value

$$V_2(2) = \sum_{n=1}^{\infty} \beta^{n-1} u(2 - 1/2) = \frac{3/2}{1 - 1/1.05} = 31.5.$$

Next assume that Crusoe begins with $q = 2$ and the production function f_1, but has the opportunity to invest a portion of his goods in an attempt at innovation. Suppose further that the opportunity to innovate can be represented by a binary lottery ticket that can be obtained by utilizing $j = 1/2$ units of input material. The ticket is such that with probability $1/2$ the innovation succeeds and Crusoe has the production function f_2 thereafter, but also with probability 1/2 it fails and Crusoe must continue with f_1. Let $V = V(2)$ be the value of this new game.

Now Crusoe can reject the investment opportunity and continue with his original production function f_1 thereby earning $V_1(2) = 21$, or make the investment and receive in expectation

$$\sup_{0 \leq i \leq 1.5} \left[u(1.5 - i) + \beta \left\{ \frac{1}{2} V_1(f_1(i)) + \frac{1}{2} V_2(f_1(i)) \right\} \right].$$

The optimal choice for the input is again $i = 1$ and the quantity above equals

$$u(1/2) + \frac{1}{1.05} \left\{ \frac{1}{2} V_1(2) + \frac{1}{2} V_2(2) \right\} = \frac{1}{2} + \frac{1}{2.1} \{21 + 31.5\} = 25.5.$$

Since $25.5 > 21$, it pays the non-monetary Crusoe to innovate. A smaller value for the discount factor β, say $\beta = .8$, would go against innovation.

We now split Crusoe into two and place him in a monetary economy. The resource base per capita remains the same but, prior to innovation, Crusoe is in an economy that uses fiat money but has no commercial bank and in a stationary equilibrium only implicitly needs the services of the central banks as no more money enters or leaves the economy. This changes with innovation.

6.3.2 A simple monetary economy

The following is an example of the model with many firms and consumer owners that was presented abstractly in Section 6.2.

Let $m = 1$ be the amount of money held initially by the consumers, and suppose that the firms begin with goods $q = 2$ and the production function f_1. Assume first that the firms do not attempt to innovate. The optimal input for the firms is 1 unit of goods. Thus, if the price of goods is p, the firms borrow and then bid $b = p$ thereby obtaining $i = b/p = 1$ as input in order to produce $\tilde{q} = f_1(1) = 2$ for the next period. The (short-term) loan to the firms is financed by the deposit of $m - a = b$ of the owner-consumers. So the owners bid $a = m - b = m - p$ and

$$p = \frac{a + b}{q} = \frac{m - p + p}{q} = \frac{m}{q} = \frac{1}{2}.$$

The economy is in stationary equilibrium and each period the firms earn the profit

$$\pi = pq - (1 + \rho)b = \frac{1}{2} \cdot 2 - 1.05 \cdot \frac{1}{2} = .475$$

with a total discounted return of

$$W_1(2) = \sum_{n=1}^{\infty} \left(\frac{1}{1+\rho}\right)^{n-1} \cdot \pi = 9.975. \tag{25}$$

The consumers, like Crusoe in the previous example, are assumed to be risk neutral with utility function $u(q) = q$. In each period they receive in utility $u(a/p) = u(1) = 1$ with a total discounted utility of

$$V_1(2) = \sum_{n=1}^{\infty} \beta^{n-1} u(1) = 21. \tag{26}$$

Now suppose that the firms have the opportunity to innovate. The physical aspects of the economy will be the same as for Crusoe in the previous example, but prices and money will now play a role.

By investing 1/2 unit of goods, each firm can, independently of the others, purchase a lottery that with probability 1/2 results in the improved production function f_2 for the firm, but also with probability 1/2 fails causing the firm to continue with f_1. The question for the managers of the firms is whether they can improve upon the return achievable without making the attempt at innovation.

To answer this question, assume that the firms do purchase the lottery. Suppose that the price of goods in the first period is p. The firms will need to bid $b + c = p + p/2 = 1.5p$ in order to purchase 1 unit of goods as input for production and 1/2 unit for the innovation attempt. The short-term loan of $b = p$ is again financed by the consumer-owners who bid a and deposit $m - a = b = p$ as before. However, the bid $c = p/2$ is financed by a long-term bank loan which must be repaid over the infinite future in payments of ρc in every period. The price of goods in the first period is then

$$p = \frac{a+b+c}{q} = \frac{m - p + p/2 + p}{2} = \frac{1 + p/2}{2}.$$

So the price is $p = 2/3$, and $b = 2/3, a = 1 - p = 1/3 = c$. The firms earn in the first period the profit

$$\pi = pq - (1+\rho)b - \rho c = \frac{2}{3} \cdot 2 - 1.05 \cdot \frac{2}{3} - .05 \cdot \frac{1}{3} = .6167.$$

The owner-consumers receive in the first period

$$u(a/p) = a/p = \frac{1/3}{2/3} = 1/2.$$

In all subsequent periods the unsuccessful firms called type 1 with production function f_1 bid $b_1 = p$ in order to input 1 unit of goods while the successful firms called type 2 with production function f_2 bid $b_2 = p/2$ in order to input 1/2. As before these short-term loans are financed by the owner-consumers, who now hold cash $m + c = 1 + 1/3 = 4/3$. So they deposit

$$4/3 - a = \frac{1}{2}b_1 + \frac{1}{2}b_2 = \frac{3}{4}p$$

Hence, the price in periods after the first satisfies

$$p = \frac{a + \frac{1}{2}b_1 + \frac{1}{2}b_2}{q} = \frac{\frac{4}{3} - \frac{3}{4}p + \frac{3}{4}p}{2} = 2/3,$$

that is, the price equals 2/3 in every period. (One should not expect constant prices in general. This example was constructed to make for a simple analysis.) Notice that because of the constant price and the constant derivative $u' = 1$, the Euler equation (24) is satisfied at every stage.

In periods after the first the type 1 firms have the profit

$$\pi_1 = pq - (1+\rho)b_1 - \rho c = \frac{2}{3} \cdot 2 - 1.05 \cdot \frac{2}{3} - .05 \cdot \frac{1}{3} = .6167,$$

type 2 firms make

$$\pi_2 = pq - (1+\rho)b_2 - \rho c = \frac{2}{3} \cdot 2 - 1.05 \cdot \frac{1}{3} - .05 \cdot \frac{1}{3} = .9667,$$

and owner-consumers receive

$$u(a/p) = a/p = \frac{5/6}{2/3} = 5/4.$$

The total expected value to a firm is

$$\begin{aligned} W = W(2) =& \pi + \frac{1}{1+\rho}\left\{\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{1+\rho}\right)^{n-1}\pi_1 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{1+\rho}\right)^{n-1}\pi_2\right\} \\ =& .6167 + \frac{1}{1.05}\left\{\frac{1}{2}\cdot 21 \cdot .6167 + \frac{1}{2}\cdot 21 \cdot .9667\right\} = 16.4507. \end{aligned}$$

Since $16.4507 > 9.975$, the innovation lottery is good for the firms.

The total expected utility for an owner-consumer is

$$1/2 + \beta\sum_{n=1}^{\infty}\beta^{n-1}\frac{5}{4} = 25.5,$$

which is greater than 21. So the lottery is good for consumers also.

6.3.3 Innovation financed by a money market

In the economy of Section 6.2.1 as in the example of Section 6.3.2, the attempt at innovation is financed by the bank with an an injection of additional money into the system. It is also possible to construct examples for which innovation is financed by a money market with loans made to the firms from the consumer-owners and the quantity of money remains fixed. We suspect that there are also examples where there is an equilibrium with innovation when there are additional funds available from a bank and that innovation will not occur without such additional funds.

6.4 A comment on monied individuals: Retirees or active capitalists?

In Section 5.1 we considered a model with a class of individuals whose only asset was government money. Because the solution supported the fiat as both a means of payment and a store of value these individuals were able to live off their money. In the model of Section 6.2, our main concern being central bank financing, we omitted them for simplicity.

The introduction of a class of agents living off money provides for a basic reconsideration of the role of finance in the economy. In particular their interest in influencing a government set rate of interest may be diametrically opposed to the desires of the producers.

Is a retired surgeon with $10,000,000 the economic equivalent of a professional money lender with $10,000,000? Almost always the answer is no. Information, evaluation, expertise, and specialization of the financial functions are in essence an evolutionary aspect of the overall body economic. The essential difference between a merely rich amateur investor and a professional is perception, expertise, knowledge and a network of professional connections. The professional investor is part of the general sensory system of the economy dealing in the perception and evaluation of risk in a dynamic economy. The rich retiree is better off investing indirectly though a professional investor be it a bank, investment bank, or other financial professional unless she has a network of connections of her own that enable her to invest directly in a family's or friend's business.

The remarks above imply that at least we should split the savers in the model of Section 5.1 into two parts, passive savers and active financiers. The savers deposit only in the commercial banks or pension funds, while the financiers are involved in evaluation and deal directly with the firms and the markets for firms and their stocks. The consideration of such a model is left for a future project.

7 Ongoing Innovation Opportunities

In this, Part 2 of our consideration of the financing of innovation, we confined our observations to models with randomness only at the initial stage. In essence we were able to utilize a modification of two dynamic programming models with independent agents each facing only one stochastic element at the start. Even with the piling up of gross simplifications the conditions needed to be able to obtain a stationary state involved an adjustment period of arbitrarily length. With a random variable each period the turbulence will increase considerably and the characterization of even the simplest market with innovation with a random element in each period will lead to a path dependent distribution of firm size and stochastically increasing returns of the variety indicated by Brian Arthur [1]. We intend to pursue the possibility of an ongoing innovation process in a separate essay.

8 Summary Remarks

Our basic goal was to produce an adequate mathematical model that could reflect mathematically the meaning of Schumpeter's breaking of the circular flow of capital in a closed economy. There are several other basic features that static or even dynamic conventional equilibrium models cannot capture:

1. Innovation and comparative statics: Innovation requires an extra process that utilizes existing resources. This is illustrated here by a comparative analysis of two economies, one with and the other without innovation in Sections 6 and 5, respectively.

2. Robinson Crusoe and the parallel worlds of goods and finance: An understanding of Robinson Crusoe's innovation opportunities provides a clear preliminary way to understand the roles of real resources and ownership control prior to seeing the strategic decoupling offered by money and the financial system in a complex economy [9].

3. The problem of convergence to stationary equilibrium even with only one random event: Under reasonable assumptions the open economy of Section 2 converges to stationary equilibrium after an initial shock due to innovation. It remains open whether the same is true for the closed economy of Section 6. If so, the rate of convergence will no doubt depend on the specific structures of production and consumption.

4. Financing and two way causality: The availability of extra goods may bring forth a demand for extra money; however the financing of innovation may be generated by the availability of extra money or credit. Thus causality may go in both directions.

5. Bankruptcy as the delimiter of risk: Bankruptcy laws are a logical necessity needed to account for the possibility of failure. If innovation fails and individuals are bankrupted their remaining resources may be redistributed to cover in part the contractual obligations. Thus from the viewpoint of society as a whole the bankruptcy laws are a public good delineating how much the economy as a whole shares in the outcome from the individual gamble.

   By selecting a bankruptcy penalty greater than or equal to the highest marginal utility of money on an equilibrium path and limiting the amounts that individuals borrow, we can avoid in our models solutions involving active bankruptcy.

6. The locus of innovation finance may be public or private: Historically both private and public resources have been involved in innovation. Global exploration and then space exploration were heavily government enterprises to start with and the private sector followed. This is also true for the internet.

This paper was basically aimed at understanding the nature of the cash flows in innovation. As such we purposely played down the distribution of power and wealth masking it by extreme aggregation.These factors require a separate treatment.

9 Appendix: The Proof of Theorem 2

Here the notation and assumptions are those of Section 2, and, in particular, Section 2.2.1. Consider an economy as in that section with two types of firm at the beginning of a period in which every type 1 firm holds the quantity of goods q_1 and every type 2 firm holds the quantity q_2 with the total quantity of goods being $Q = \bar{\varepsilon} q_1 + \varepsilon q_2$.

By equation (3) in Section 2.2.1,

$$f_1'(i_1) = f_2'(i_2) = (1+\rho)^2 \cdot \frac{\Phi(Q)}{\Phi(\tilde{Q})},$$

where $i_k = b_k/p$, $k = 1, 2$ and $\tilde{Q}$ is the total quantity of goods at the beginning of the next period. Moreover,

$$\tilde{Q} = \bar{\varepsilon} f_1(i_1) + \varepsilon f_2(i_2)$$

because $\tilde{q}_k = f_k(i_k)$ is the quantity of goods held by firms of type k at the beginning of the next period.

It follows from our assumptions on f_1 and f_2 that, for every positive value of i_1 there is a unique positive value of i_2 such that $f_1'(i_1) = f_2'(i_2)$. Also this value of i_2 is a continuous, increasing function of i_1 and approaches ∞ when i_1 does.

Consider now the function

$$\lambda(i_1) = f_1'(i_1) \cdot \Phi(\tilde{Q}) = f_1'(i_1) \cdot \Phi(\bar{\varepsilon} f_1(i_1) + \varepsilon f_2(i_2))$$

where i_2 has the value described in the previous paragraph. The function λ is continuous and decreases from $+\infty$ to 0 on $[0, +\infty)$. Hence, for each positive Q, there are unique positive numbers $i_1 = i_1(Q)$ and $i_2 = i_2(Q)$ such that $f_1'(i_1(Q)) = f_2'(i_2(Q))$ and

$$\lambda(i_1(Q)) = f_1'(i_1(Q)) \cdot \Phi(\bar{\varepsilon} f_1(i_1(Q)) + \varepsilon f_2(i_2(Q))) = (1+\rho)^2 \cdot \Phi(Q).$$

Thus we can define the mapping ψ from the current value for the total quantity of goods Q to the quantity $\tilde{Q}$ for the next period by

$$\psi(Q) = \bar{\varepsilon} f_1(i_1(Q)) + \varepsilon f_2(i_2(Q)) = \tilde{Q}.$$

The mapping ψ provides a law of motion for the economy when firms choose their bids in agreement with the Euler equations.

Lemma 5 *1. The functions $i_1(Q), i_2(Q)$ and $\psi(Q)$ are continuous, increasing functions of Q on $(0, \infty)$.*

2. The function ψ has a unique fixed point, namely

$$Q^* = \bar{\varepsilon} f_1(i_1^*) + \varepsilon f_2(i_2^*)$$

where $i_k^* = (f_k)^{-1}((1+\rho)^2)$, $k = 1, 2$.

3. If $0 < Q < Q^*$, then $Q \leq \psi(Q) \leq Q^*$; if $Q^* < Q$, then $Q^* \leq \psi(Q) \leq Q$.

4. For $Q > 0$, $\psi^n(Q) \to Q^*$ as $n \to \infty$, where ψ^n is the n-fold composition of ψ with itself.

Proof. 1. The continuity of i_1, i_2, ψ follows from the assumed continuity of f_1', f_2', Φ.

To see that i_1 is increasing, let $0 < Q_1 < Q_2$. Then

$$\lambda(i_1(Q_1)) = (1+\rho)^2 \Phi(Q_1) > (1+\rho)^2 \Phi(Q_2) = \lambda(i_1(Q_2)).$$

Since λ is decreasing, $i_1(Q_1) < i_1(Q_2)$. Also $i_2(Q)$ is an increasing function of $i_1(Q)$. So $i_2(Q_1) < i_2(Q_2)$.

Finally, because f_1, f_2 are increasing, we have

$$\psi(Q_1) = \bar{\varepsilon} f_1(i_1(Q_1)) + \varepsilon f_2(i_2(Q_1)) < \bar{\varepsilon} f_1(i_1(Q_2)) + \varepsilon f_2(i_2(Q_2)) = \psi(Q_2).$$

2. The quantity Q is a fixed point of ψ means that

$$\psi(Q) = \bar{\varepsilon} f_1(i_1(Q)) + \varepsilon f_2(i_2(Q)) = Q,$$

which is equivalent to

$$\lambda(i_1(Q)) = f_1'(i_1(Q)) \cdot \Phi(Q) = (1+\rho)^2 \cdot \Phi(Q).$$

This holds if and only if

$$f_1'(i_1(Q)) = (1+\rho)^2 = f_2'(i_2(Q)),$$

that is, $i_1(Q) = i_1^*$ and $i_2(Q) = i_2^*$ and so $Q = \bar{\varepsilon} f_1(i_1^*) + \varepsilon f_2(i_2^*)$.

3. Let $0 < Q < Q^*$. By parts 1 and 2, we have

$$\psi(Q) \leq \psi(Q^*) = Q^*.$$

Moreover,

$$\begin{aligned}\psi(Q) =& \bar{\varepsilon} f_1(i_1(Q)) + \varepsilon f_2(i_2(Q)) \\ =& \Phi^{-1}\left(\frac{(1+\rho)^2 \Phi(Q)}{f_1'(i_1(Q))}\right) \\ \geq& \Phi^{-1}(\Phi(Q)) = Q.\end{aligned}$$

The inequality above holds because $i_1(Q) \leq i_1(Q^*) = i_1^*$ by part 1 of the theorem and thus $f_1'(i_1(Q)) \geq f_1'(i_1^*) = (1+\rho)^2$.

The proof for the case when $Q^* < Q$ is similar.

4. It follows from part 3 that for $Q > 0$, whether $Q \leq Q^*$ or $Q \geq Q^*$, the limit

$$L = \lim_n \psi^n(Q)$$

exists and is finite. Also, by part 1, ψ is continuous and thus

$$\psi(L) = \psi(\lim_n \psi^n(Q)) = \lim_n \psi^{n+1}(Q) = L.$$

By part 2, $L = Q^*$. ■

To complete the proof of Theorem 2, suppose that the firms always make bids in agreement with the Euler equations. Let Q_n be the total quantity of goods in period n. Then by part 4 of the lemma, $Q_n = \psi^{n-1}(Q_1) \to Q^*$. Also, if p_n is the price in period n, then $p_n = \Phi(Q_n) \to \Phi(Q^*) = p^*$, because Φ is continuous by assumption. Now let $b_{k,n}$ be the bid of firms of type k in period n. Then

$$\begin{aligned}\frac{b_{k,n}}{p_n} &= \left(f_k'^{-1}\left((1+\rho)^2 \cdot \frac{p_n}{p_{n+1}}\right)\right) \to \left(f_k'^{-1}\left((1+\rho)^2 \cdot \frac{p^*}{p^*}\right)\right) \\ &= (f_k'^{-1}((1+\rho)^2)) = \frac{b_k^*}{p^*}.\end{aligned}$$

So $b_{k,n} \to b_k^*$, and the proof is complete.

References

[1] Arthur, W. B. 1994. *Increasing Returns and Path Dependence in the Economy.* Ann Arbor, University of Michigan Press.

[2] Arthur, W. B. 2009. *The Nature of Technology.* The Free Press (Simon & Schuster) in the US, Penguin Books in the UK.

[3] Bak, P.„ S. F. Nørrelykke and M. Shubik, 1999. "Dynamics of Money," *Physical Review E*, 60(3): 2528–2532.

[4] Baumol, W.J. 2002. *The Free-Market Innovation Machine,* Princeton University Press:Princeton.

[5] Bechtel, S.D. et al. 1996. "Managing Innovation," *Deadelus* 125: 2.

[6] Day, R. H. 1993. "Bounded Rationality and the Co-evolution of Market and State," in Day, Eliasson and Wilborg, eds., *The Markets for Innovation, Ownership and Control.* North-Holland:Amsterdam and New York

[7] Day, Eliasson and Wilborg. 1993. *The Markets for Innovation, Ownership and Control.* North-Holland:Amsterdam and New York

[8] Dosi, G., C. Freeman, R. Nelson, G. Silverberg and L. Soete. 1988. *Technical Change and Economic Theory.* Pinter: London and New York

[9] Karatzas, I., M. Shubik, and W. Sudderth. 2006. Production, interest, and saving in deterministic economies with additive endowments. Economic Theory. 29, 3: 525-548.

[10] Lamoreaux N and Sokoloff (editors) 2007. *Financing Innovation in the United States: 1870 to the Present.* MIT Press: Cambridge.

[11] Nelson, R. 1996. *The Sources of Economic Growth.* Harvard University Press: Cambridge.

[12] Nelson, R. R., and S. G. Winter. 1982. *An Evolutionary Theory of Economic Change.* Harvard, Belknap: Cambridge.

[13] Shubik, M. 2010. *Innovation and Equilibrium in Essays in Honor of Hyman Minsky. The Elgar Companion to Hyman Minsky* D. Papadimitriou and L.R. Wray (eds). Ch 8, pp.153-168

[14] Shubik,M. and W. Sudderth. 2011. "Cost Innovation, Schumpeter and Equilibrium. Part 1," Cowles Foundation Discussion Paper 1786.