NOTES ON COMPUTATIONAL COMPLEXITY OF GE INEQUALITIES

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July 2012
Revised August 2012

COWLES FOUNDATION DISCUSSION PAPER NO. 1865R

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Polynomial Complexity of the General Equilibrium Inequalities

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Abstract

This paper is a revision of my paper, CFDP 1865. The principal innovation is an equivalent reformulation of the decision problem for weak feasibility of the GE inequalities, using polynomial time ellipsoid methods, as a semidefinite optimization problem, using polynomial time interior point methods. We minimize the maximum of the Euclidean distances between the aggregate endowment and the Minkowski sum of the sets of consumer's Marshallian demands in each observation. We show that this is an instance of the generic semidefinite optimization problem: $\inf_{x \in K} f(x) \equiv Opt(K, f)$, the optimal value of the program,where the convex feasible set K and the convex objective function $f(x)$ have semidefinite representations. This problem can be approximately solved in polynomial time. That is, if $p(K, x)$ is a convex measure of infeasibilty, where for all x, $p(K, x) \geqq 0$ and $p(K, z) = 0$ iff $z \in K$, then for every $\varepsilon > 0$ there exists an ε-optimal y such that $p(K, y) \leqq \varepsilon$ and $f(y) \leqq \varepsilon + Opt(K, f)$ where y is computable in polynomial time using interior point methods.
Keywords: GE Inequalities, Polynomial solvability, Semidefinite Programming

Introduction

We consider a pure exchange economy with N price-taking consumers, where in each of S periods we observe the market prices, the income distribution and the aggregate endowments of L goods and services. Consumer demands are unobservable. In these notes, we propose a polynomial time algorithm for deciding the weak feasibility of the general equilibrium inequalities introduced by Brown and Shannon (2000). This system of multivariate polynomial inequalities is feasible iff there exists an indirect utility function for each consumer and utility maximizing demands, subject to her budget constraints, that define a competitive equilibrium in each observation. That is, in each observation, the endogenous aggregate demand of goods and services of consumers at the observed market prices and income distribution equals the exogenous aggregate supply of goods and services.

Refutable Theories of Value

(1) Rationalization

(2) Weak Axiom of Revealed Preference (*WARP*)

(3) Afriat's Theorem

(4) Tarski–Seidenberg Theorem

Rationalization

If the Marshallian demands x_j are observed, then

$$D \equiv \{x_j, p_j\}_{j=1}^{j=S}, \text{ where } x_j, p_j \in R^L_{++}.$$

The non-satiated utility function $U(x)$ rationalizes D if for all j and for all $x \in R^L_{++}$:

$$p_j \cdot x \leq p_j \cdot x_j \Rightarrow U(x_j) \geq U(x).$$

Weak Axiom of Revealed Preference (WARP)

(1) If the Marshallian demands x_j are observed, then

$$D \equiv \{x_j, p_j\}_{j=1}^{j=S}, \text{ where } x_j, p_j \in R^L_{++}$$

and

$$p_1 \cdot x_2 \leq p_1 \cdot x_1 \Rightarrow p_2 \cdot x_1 > p_2 \cdot x_2.$$

(2) *WARP* is necessary but not sufficient for rationalizing D.

(3) The Generalized Axiom of Revealed Preference $(GARP)$ is necessary and sufficient for rationalizing D.

(4) *GARP*, due to Varian (1982), is a generalization of the Strong Axiom of Revealed Preference $(SARP)$.

Afriat's Theorem

If the Marshallian demands x_j are observed, then the following conditions are equivalent:

(a) D is rationalized by a non-satiated utility function u.

(b) The "Afriat Inequalities":

$$u_i \leq u_j + \lambda_j p_j \cdot (x_i - x_j)$$

for $i, j = 1, ..., S$ are a solvable family of linear inequalities in the unobserved utility levels u_j and marginal utilities of income λ_j.

(c) D satisfies *GARP*.

(d) D is rationalized by a non-satiated, **concave** utility function U.

Tarski–Seidenberg Theorem

(1) Semi-algebraic sets are solutions of a finite family of multivariate polynomial inequalities.

(2) The projection of a semi-algebraic set is a semi-algebraic set.

(3) Example:

$$ax^2 + bx + c = 0$$

has a real solution iff

$$b^2 - 4ac \geq 0.$$

(4) Example: In Afriat's Theorem

$$(b) \text{ iff } (c).$$

The General Equilibrium (GE) Manifold

(1) If 0 is a regular value of the smooth market excess demand function

$$F(p, \omega_1, \omega_2, ..., \omega_L),$$

then

$$M \equiv \{(p, \omega_1, \omega_2, .., \omega_L) : F(p, \omega_1, \omega_2, ..., \omega_L) = 0\}$$

is a smooth manifold.

(2) 0 is a regular value of the smooth market excess demand function for almost all $(\omega_1, \omega_2, ..., \omega_L)$.

(3) The GE manifold M rationalizes the market data D, where

$$D = \{(p_r, \omega_{r,1}, \omega_{r,2}, ..., \omega_{r,L})_{r=1}^{r=N}\}$$

if $D \subseteq M$.

The Marshallian GE Inequalities

(1) The Afriat inequalities for each consumer, where the Marshallian demands of consumers are unobserved, the budget constraints for each consumer in each observation and the market clearing conditions in each observation.

(2) Brown and Matzkin (1996) prove that a finite family of observations of aggregate endowments, market prices and the income distribution can be rationalized with a GE manifold, iff the Marshallian GE inequalities are feasible.

(3) Recently, Cherchye et al. (2011) proved the important negative result that deciding the **strong** feasibility of the Marshallian GE inequalities is a NP-complete problem. That is, if $P \neq NP$ then there is no "efficient" (polynomial time) method for deciding the **strong** feasibility of the Marshallian GE inequalities.

NP-Completeness of the Marshallian GE Inequalities

Cherchye et al. propose a non-polynomial time (inefficient) algorithm to decide strong feasibility of the Marshallian GE inequalities, where they convert the problem of deciding the strong feasibility of the Marshallian equilibrium inequalities to solving an equivalent mixed-integer linear programming problem. Note that mixed-integer linear programming problems are also NP-hard.

A Local Characterization of the GE Manifold

(1) Ekeland and Chiappori (1999) proposed a local characterization of the GE manifold, using the exterior calculus and the consumer's smooth, convex indirect utility function $V(p, I)$.

(2) If p are the market prices and I is the consumer's income, then they express the consumer's Marshallian demand as:

$$\frac{I[\nabla_p V(p, I)]}{[p \cdot \nabla_p V(p, I)]}.$$

The Dual Afriat Inequalities

Theorem

Let $(p^r, x^r), r = 1, \ldots, S$ be given and let $I^r = p^r \cdot x^r$ for each r. There exists a utility function rationalizing this data that is strictly quasiconcave and monotone if and only if there exist numbers V^i, λ^i, and vectors $q^i \in \mathbb{R}^\ell$, $i = 1, \ldots, N$ such that:

(a) for $i \neq j$,

$$V^i - V^j \geq q^j \cdot \left(\frac{p^i}{I^i} - \frac{p^j}{I^j}\right) \text{ for } i, j = 1, \ldots, S.$$

(b) $\lambda^j > 0$, $q^j \ll 0$, $j = 1, \ldots, S$ (c) $\frac{q^j}{I^j} = -\lambda^j x^j$, $j = 1, \ldots, S$.

Remark: Solutions of (a) define a convex indirect utility function $V(\frac{P}{I})$.

The Dual Afriat Inequalities (continued)

Conditions (a) and (b) constitute the "dual Afriat inequalities." Condition (c) is an expression of Roy's identity in this context. To see this, note that if (c) holds for some $\lambda^j > 0$, then $\frac{p^j \cdot q^j}{I^j} = -\lambda^j (p^j \cdot x^j) = -\lambda^j I^j$, i.e., $\lambda^j = -\frac{p^j \cdot q^j}{(I^j)^2}$, which implies that the vector $(\frac{q^j}{I^j}, \lambda^j)$ corresponds to the gradient of the rationalizing indirect utility function V evaluated at (p^j, I^j). This is essentially the content of (a). More precisely, (a) says that q^j is the derivative of V with respect to the income normalized price vector $\frac{p}{I}$ evaluated at (p^j, I^j). Thus $\frac{\partial V}{\partial p}(p^j, I^j) = \frac{q^j}{I^j}$ and $\frac{\partial V}{\partial I}(p^j, I^j) = -\frac{p^j \cdot q^j}{(I^j)^2}$. If $\frac{q^j}{I^j} = -\lambda^j x^j$, then x^j is the demand at the price-income pair (p^j, I^j) by Roy's identity.

A Global Characterization of the GE Manifold

(1) Brown and Shannon proposed a global characterization of the GE manifold, using the theory of revealed preference and the consumer's smooth, convex indirect utility function, $V(\frac{p}{I})$.

(2) That is, they introduced the Hicksian GE inequalities, consisting of the first-order conditions for minimizing a smooth convex indirect utility function, $V(\frac{p}{I})$, subject to a budget constraint, i.e., the dual Afriat inequalities for each consumer, the budget constraints for each consumer and the market clearing conditions in each observation.

(3) Brown and Shannon proved that a finite family of observations of aggregate endowments, market prices and the income distribution can be rationalized with a GE manifold iff the Hicksian GE inequalities are feasible.

Complexity of GE Inequalities

(1) Brown and Shannon proved that the Hicksian GE inequalities are feasible iff the Marshallian GE inequalities are feasible.

(2) The Cherchye et al. result suggests that the Hicksian equilibrium inequalities are also *NP*-complete. That is, there does not exist a polynomial time algorithm for deciding the **strong** feasibility of the Hicksian GE inequalities.

(3) In fact, we show there exits an polynomial time algorithm for deciding the **weak** feasibility of the Hicksian GE inequalities. That is, there exists a polynomial time algorithm for deciding if the aggregate endowment in each observation is "ε-near" the aggregate demand.

(4) Hence there exists a polynomial time algorithm for deciding the **weak** feasibility of the Marshallian GE inequalities.

Weak Membership Oracle for Convex Bodies

(1) A convex body is a compact convex set with nonempty interior. A strong membership oracle for a convex body K asserts for any rational $y \in R^L$ that $y \in K$ or $y \notin K$. See Proposition 2.1.5 in in Grotschel, Lovasz and Schrijver $[GLS]$.

(2) A convex body K is centered if there explicitly exists $\alpha_0 \in K$ and $r \in R$ such that $B_r(\alpha_0) \subset K$.

(3) A weak membership oracle for a centered, convex body K asserts for any positive rational δ and any rational $y \in R^L$ that y is "ε-near" K or $y \notin K$ — see Lemma 4.3.3 in $[GLS]$.

(4) There exists a polynomial time algorithm for weak membership in $K_1 + K_2$, given polynomial time algorithms for weak membership in the centered convex bodies K_1 and K_2. See section 4.5 in $[GLS]$.

The Perspective Map

(1) The perspective map

$$P : R^K \times R_{++} \to R^K,$$

where

$$P(x, x_{K+1}) \equiv \frac{x}{x_{K+1}}.$$

(2) If

$$(x, x_{K+1}), (y, y_{K+1}) \in R^K \times R_{++}$$

then the perspective image of the interval.

$$[(x, x_{K+1}), (y, y_{K+1})] \subset R^K \times R_{++}$$

is the interval

$$[P(x, x_{K+1}), P(y, y_{K+1})] \subset R^K.$$

(3) Hence, the image and pre-image of a convex set is a convex set under the perspective map. See section 2.3.3 in Boyd and Vandenberghe (2004) for proof.

Centered Convex Bodies of Marshallian Demands

(1) A sequence of S non-negative vectors in R^L_+, bounded by the sequence of S aggregate endowments, are the images of the perspective map, i.e., Marshallian demands for some indirect utility function, iff the linear Afriat inequalities for the given S non-negative vectors and observed market prices and income distribution are solvable for the unobserved utility levels and marginal utilities of income.

(2) Solutions of the strict Afriat inequalities constitute the interior of the convex body of Marshallian demands defined as the intersection of the **closure** of the perspective image of the convex set of marginal indirect utilities and marginal utilities of income and the convex interval in the positive orthant defined by the origin and the aggregate endowments.

(3) Since any strict smooth concave utility function satisfies the strict Afriat inequalities, we can center each convex body of Marshallian demands.

A Weak Feasibility Oracle for the GE Inequalities

(1) It follows from a theorem of Yudin and Nemirovskii (1976) — see section 4.3 in $[GLS]$ — that there exists a polynomial time algorithm for weak membership of the aggregate endowment in the sum of the centered convex bodies of Marshallian demands in each observation.

(2) That is, there exists a polynomial time algorithm for the weak feasibility of the Hicksian (Marshallian) GE inequalities.

Strong and Weak Feasibility of the GE Inequalities

(1) The apparent contradiction between the negative result of Cherchye et al. and our positive result on deciding feasibility of the GE inequalities in polynomial time derives from two different notions of feasibility.

(2) We use the notion of **weak** feasibility, common in convex optimization and Cherchye et al. use the notion of **strong** feasibility, common in combinatorial optimization.

Market Clearing

(1) Infeasibility of the aggregate endowment in the Hicksian GE inequalities is a measure of the lack of market clearing. We propose the minimization of a real-valued convex measure of the lack of market clearing over each consumer's family of piece-wise linear indirect utility functions.

(2) We minimize the maximum Euclidean distance between the aggregate endowment and the Minkowski sum of the centered convex bodies of Marshallian demands in each observation. It follows from Theorem 4.3.13 in $[GLS]$ that this optimization problem can be solved in polynomial time, using the ellipsoid method The optimal value of this problem is a measure of the lack of market clearing. In practice, polynomial time interior point methods are more efficient then the ellipsoid methods.

Polynomial Time Solvability of Convex Programs

Sufficient conditions for polynomial time solvability of a family of convex optimization problems using the ellipsoid method are:

(1) Polynomial computability,

(2) Polynomial growth,

(3) Polynomial boundness of feasible sets,

(4) See Theorem 5.3.1 in Ben-Tal and Nemirovski (2001).

Polynomial Time Solvability: Some Examples

Subfamilies of linear, conic quadratic and semidefinite convex optimization problems with box constraints on the feasible sets are polynomial solvable. That is, they satisfy the conditions of polynomial computability polynomial boundness of feasible sets and polynomial growth.
See section 5.3 in Ben-Tal and Nemirovski.

Properties of Spectrahedra

(1) A spectrahedron is the solution set of a linear matrix inequality (LMI). That is,

$$\left\{ x \in R^L : A_0 + \sum_{i=1}^{i=K} x_i A_i \succcurlyeq 0 \right\}$$

where the A_i are qxq symmetric matrices, for $i = 0, 1, 2, ..., K$.

(2) Every polyhedron is a spectrahedron, where the A_i are diagonal matrices.

Spectrahedral Shadows of Convex Sets

(1) A convex subset $C \subseteq R^L$ is called a spectrahedral shadow,If $C = L(S)$, where S is a spectrahedron in R^J and $L : R^J \longrightarrow R^L$ is an affine linear map.
(2) Spectrahedral shadows of convex sets are called semidefinite representations of convex sets in Ben-Tal and Nemirovski. A convex function is semidefinite representable if its epigraph is semidefinite representable.
(3) A polyhedron is a spectrahedral shadow.
(4) The linear image of a spectrahedral shadow is a spectrahedral shadow.
(5) The perspective image of a spectrahedral shadow is a spectrahedral shadow.
(6) The Minkowski sum of two spectrahedral shadows is a spectrahedral shadow.
(7) The max of two semidefinite representable convex functions is a semidefinite representable convex function.

Spectrahedral Shadows of Convex Sets (Continued)

(8) See the chapter on spectrahedral shadows in Nitzer (2012) and section 4.2 in Ben-Tal and Nemirovski.
(9) If the spectrahedral shadow C has a nonempty interior,then C is also the canonical projection of the spectrahedron $\overline{C}$, where

$$\overline{C} = \left\{ (x, y) \in R^K \times R^H : A_0 + \sum_{i=1}^{i=K} x_i A_i + \sum_{j=1}^{j=H} y_j B_j \succcurlyeq 0 \right\}$$

and

$$C = \{\pi_x(x, y) : (x, y) \in \overline{C}\}.$$

See Lemma 4.1.6 in Nitzer (10) Optimization problems, where the feasible set is a semidefinite representable convex set and the objective function is a semidefinite representable convex function are solvable in polynomial time, using the ellipsoid method.

Perspective Images of Spectrahedral Shadows

Theorem

The perspective image of a spectrahedral shadow with nonempty interior is a spectrahedral shadow. If

$$\overline{C} = \left\{ ((x, x_{K+1}), y) \in R^{K+1} \times R^H : A_0 + \sum_{i=1}^{i=K+1} x_i A_i + \sum_{j=1}^{j=H} y_j B_j \succcurlyeq 0 \right\}$$

then

$$\overline{C} = \left\{ ((x, x_{K+1}), y) \in R^K \times R_{++} \times R^H : \left[x_{K+1} \frac{1}{x_{K+1}} A_0 + \sum_{i=1}^{i=K+1} \frac{x_i}{x_{K+1}} A_i \sum_{j=1}^{j=H} \frac{y_j}{x_{K+1}} B_j \right] \succcurlyeq 0 \right\}.$$

Proof

$$C = \{\pi_{(x, x_{K+1})}((x, x_{K+1}), y) : ((x, x_{K+1}), y) \in \overline{C}\}.$$

For all $((x, x_{K+1}), y) \in \overline{C}$

$$\left[\frac{1}{x_{K+1}} A_0 + \sum_{i=1}^{i=K+1} \frac{x_i}{x_{K+1}} A_i + \sum_{j=1}^{j=H} \frac{y_j}{x_{K+1}} B_j\right] \succcurlyeq 0$$

$$\Longleftrightarrow \quad x_{K+1}\left[\frac{1}{x_{K+1}} A_0 + \sum_{i=1}^{i=K+1} \frac{x_i}{x_{K+1}} A_i + \sum_{j=1}^{j=L} \frac{y_j}{x_{K+1}} B_j\right] \succcurlyeq 0 \, .$$

Proof (Continued)

$$E \equiv \left\{\left(\frac{x}{x_{K+1}}, y\right) \in R^K \times R_{++} \times R^L : \left[\frac{1}{x_{K+1}} A_0 + \sum_{i=1}^{i=K+1} \frac{x_i}{x_{K+1}} A_i + \sum_{j=1}^{j=H} \frac{y_j}{x_{K+1}} B_j\right] \succcurlyeq 0\right\}$$

The canonical projection of E is the perspective image of the canonical projection of $\overline{C}$.

Corollary

The perspective image of a polyhedron is a spectrahedral shadow.

Spectrahedral Shadows of Marshallian Demands

(1) We project the polyhedron of solutions of the dual Afriat inequalities onto the set of gradients of indirect utility functions in each observation. This is a polyhedron, denoted X.
(2) We define a linear map of X into the 2-tuples of the gradient of the indirect utility function and the marginal utility of income in that observation,denoted $\overline{X}$. That is,

$$q_{t,j} \longrightarrow \left(q_{t,j}, \frac{p_j}{I_{t,j}} \cdot q_{t,j} \right).$$

(3) $\overline{X}$ is a polyhedron, hence a spectrahedron or more generally a spectrahedral shadow.
(4) The perspective image of $\overline{X}$,denoted $P[\overline{X}]$, is the set of Marshallian demands at the observed market prices and incomes for each indirect utility function. That is,

$$P[\overline{X}] \equiv \left\{ P\left(q_{t,j}, \frac{p_j}{I_{t,j}} \cdot q_{t,j} \right) = \frac{I_{t,j} q_{t,j}}{p_j \cdot q_{t,j}} : \left(q_{t,j}, \frac{p_j}{I_{t,j}} \cdot q_{t,j} \right) \in \overline{X} \right\}$$

is a spectrahedral shadow.

Polynomial Time Interior Point Methods

(1) If the feasible set K of a convex optimization problem is a spectrahedral shadow, i.e., has a semidefinite representation and the epigraph of the convex objective function is a spectrahedral shadow, then there exists a polynomial time interior point method that solves the following generic semidefinite optimization problem.
(2) Given a positive number $\varepsilon > 0$, and a real-valued convex measure of infeasibility for the feasible set K , denoted $p(x, \varepsilon)$, where for all $x \in R^N$

$$[p(x, \varepsilon) \geqq 0] \text{ and } [p(x, \varepsilon) = 0 \Leftrightarrow x \in K]$$

find a y such that $p(y, \varepsilon) \leqq \varepsilon$ and

$$f(y) \leq Opt(p) + \varepsilon$$

where f is a semidefinite representable convex function on K. and $Opt(f; K)$ is the infimum of f on K. See section 6.6.3 in Ben-Tal and Nemirovski for the complexity analysis of semidefinite programming.

Epsilon Feasibility of General Equilibrium Inequalities

Minimizing the maximum of the Euclidean distances between the aggregate Minkowski sum of each agent's Marshallian demands and the aggregate endowment in each observation is an instance of the generic semidefinite optimization problem. Hence ε-feasibility of the Hicksian (Marshallian) GE inequalities can be decided in polynomial time using interior point methods.

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Acknowledgements

As stated in the abstract of the paper, these notes are a revision of my paper, CFDP 1865. This revision is, in part, a response to the helpful comments and questions of Rahul Deb, Chris Shannon and Dan Spielman I thank them all. Also the typesetting suggestions of Glena Ames are much appreciated.