SECOND-ORDER REFINEMENT OF EMPIRICAL LIKELIHOOD FOR TESTING OVERIDENTIFYING RESTRICTIONS

By

Yukitoshi Matsushita and Taisuke Otsu

April 2011
Revised January 2012

COWLES FOUNDATION DISCUSSION PAPER NO. 1791

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/

Second-order refinement of empirical likelihood for testing overidentifying restrictions*

Yukitoshi Matsushita†
Graduate School of Humanities and Social Science
University of Tsukuba

Taisuke Otsu‡
Cowles Foundation & Department of Economics
Yale University

January 2012

Abstract

This paper studies second-order properties of the empirical likelihood overidentifying restriction test to check the validity of moment condition models. We show that the empirical likelihood test is Bartlett correctable and suggest second-order refinement methods for the test based on the empirical Bartlett correction and adjusted empirical likelihood. Our second-order analysis supplements the one in Chen and Cui (2007) who considered parameter hypothesis testing for overidentified models. In simulation studies we find that the empirical Bartlett correction and adjusted empirical likelihood assisted by bootstrapping provide reasonable improvements for the properties of the null rejection probabilities.

1 Introduction

The generalized method of moments (GMM) by Hansen (1982) has been a standard tool for empirical economic analysis. GMM provides a unified framework for conducting statistical inference when economic models are specified by some moment conditions. However, the literature indicates that there are considerable problems with GMM particularly in its finite sample performance, such as the bias in point estimation and distortions of null rejection probabilities in hypothesis testing (see, e.g., the special issue of the *Journal of Business and Economic Statistics*, vol. 14).

One well-known problem of GMM-based inference is that the (first-order) asymptotic null distribution of the overidentifying restriction test based on the minimized GMM criterion function, often called the J-test, can be a poor approximation in finite samples. In order to overcome this problem, several alternative inference

*The authors would like to thank a co-editor, two anonymous referees, and the seminar participants at Hiroshima University and University of Tokyo and Tsukuba for helpful comments.

†E-mail: matsushita@dpipe.tsukuba.ac.jp. Address: 1-1-1, Tennoudai, Tsukuba, Ibaraki 305-8571, Japan. Phone: +81-29-853-4075.

‡E-mail: taisuke.otsu@yale.edu. Website: http://cowles.econ.yale.edu/faculty/otsu.htm. Address: P.O. Box 208281, New Haven, CT 06520-8281, USA. Phone: +1-203-432-9771. Fax: +1-203-432-6167.

methods have been developed. Hall and Horowitz (1996) proposed a uniform weight bootstrap method by using recentered moment restrictions. Brown and Newey (2002) proposed a weighted bootstrap method based on the implied probabilities obtained from the moment conditions. These bootstrap methods provide higher-order refinements for the property of null rejection probabilities of overidentifying restriction tests.

Another approach to tackle this finite sample problem of the GMM-based overidentifying restriction test is to employ an alternative criterion function to derive a test statistic, such as continuous updating GMM, exponential tilting, and empirical likelihood (see, Kitamura, 2007, for a survey). Among them, empirical likelihood is an attractive candidate to deal with the distortion problem of the null rejection probabilities because of its Bartlett correctability, a second-order refinement based on the Edgeworth expansion. The Bartlett correctability of the empirical likelihood-based test is reported in several contexts, such as smooth functions of means (DiCiccio, Hall and Romano, 1991) and quantiles (Chen and Hall, 1993). Also Baggerly (1998) focused on testing for the mean parameter (i.e., $E[X] = 0$) and showed that only empirical likelihood is Bartlett correctable in the power divergence family. Bravo (2004) showed that a bootstrap version of the empirical likelihood test achieves the same higher order accuracy as the Bartlett corrected test. Although the parameters of interest are different, these papers studied Bartlett correctability of empirical likelihood in just-identified moment restrictions (i.e., the number of moment restrictions equals the number of parameters). This paper is concerned with overidentified moment restrictions (i.e., the number of moment restrictions exceeds the number of parameters) which are common in economic analysis. Although the second-order analysis becomes substantially more complicated due to extra systems and terms brought by overidentifying restrictions, Chen and Cui (2007) tackled this problem and showed that the empirical likelihood test for parameter hypotheses is Bartlett correctable even if the models are overidentified.

This paper extends Chen and Cui's (2007) analysis to the overidentifying restriction testing problem and studies second-order properties of the empirical likelihood overidentifying restriction test. Although the basic idea of the second-order analysis follows from Chen and Cui (2007), the technical detail is case-by-case and specific to our test statistic. Indeed Chen and Cui (2007, p. 504) indicated the possibility of this extension, and this paper formally studies this issue. In particular, we show that the empirical likelihood test for overidentifying restrictions is also Bartlett correctable and propose second-order refinement methods for the test based on the empirical Bartlett correction and adjusted empirical likelihood. Adjusted empirical likelihood (Liu and Chen, 2010) is a modification of empirical likelihood to avoid non-existence of solutions for the likelihood maximization problem by introducing auxiliary observations. Our refinement methods are illustrated by simulation studies based on a linear instrumental variable regression model and asset pricing model. We find that (i) the GMM and empirical likelihood tests based on the asymptotic critical values show severe over-rejections particularly when the number of moment restrictions is large, and (ii) an empirical Bartlett correction and adjusted empirical likelihood assisted by bootstrapping provide reasonable improvements for the properties of null rejection probabilities. Since testing for overidentifying restrictions is a fundamental problem to assess the validity of economic theory which precedes to parameter estimation and inference, our refinement methods contribute to enhance the reliability of empirical economic analysis based on moment condition models.

In the context of hypothesis testing for overidentifying restrictions, there are several papers which derive global optimal properties of empirical likelihood-based tests. Kitamura (2001) focused on the generalized Neyman-Pearson criterion (i.e., comparison of decay rates of the type II errors under fixed alternatives subject to a restriction on the decay rate of the type I errors), and showed that the empirical likelihood test is generalized Neyman-Pearson optimal. Otsu (2010) focused on the Bahadur criterion (i.e., comparison of decay rates of p-values under fixed alternatives), and showed that the empirical likelihood test is Bahadur optimal. Canay and Otsu (2011) focused on the Hodges-Lehmann criterion (i.e., comparison of decay rates of the type II errors under fixed alternatives subject to a size constraint), and showed that not only the empirical likelihood test but also the GMM and generalized empirical likelihood tests are Hodges-Lehmann optimal. These studies concentrate on global (or fixed alternative) and first-order power properties and show that the empirical likelihood test satisfies these optimality criteria. On the other hand, this paper concentrates on second-order null rejection properties under the null hypothesis and show that the empirical likelihood test is Bartlett correctable (i.e., accepts a second-order refinement for the null rejection property). If the Bartlett correction factor B_c for the empirical likelihood test T_n is known, then the corrected test statistic $T_n\left(1+n^{-1}B_c\right)^{-1}$ shows the same first-order global power properties to the original statistic T_n. Thus, the Bartlett corrected empirical likelihood test also enjoys the above global optimal properties. However, if B_c is unknown (as always the case in practice) and is estimated by $\hat{B}_c$ based on data, then we need to incorporate the large deviation properties of the estimation error $\hat{B}_c - B_c$ and the first-order global power properties of the corrected test statistic $T_n\left(1+n^{-1}\hat{B}_c\right)^{-1}$ require further investigation.

Based on these considerations, we recommend applied researchers to employ the Bartlett corrected empirical likelihood test (with estimated B_c) when from previous studies the distortion in the null rejection probability is a major concern for their applications of interest, and to employ the uncorrected empirical likelihood test when the distortion in the null rejection probability is not a serious problem and better power property is desired. For example, our simulation results in Section 4 indicate that when the sample size is small and/or the number of moment conditions is large, the uncorrected empirical likelihood and GMM-based overidentification tests tend to over-reject the null hypothesis. Therefore, in such situations, we recommend the use of the Bartlett corrected empirical likelihood test suggested in this paper.

The rest of the paper is organized as follows. Section 2 introduces our setup and notation. Section 3 presents the main theoretical results: second-order properties of the empirical likelihood test statistic and refinements by the Bartlett correction and adjusted empirical likelihood. Section 4 conducts simulation studies based on a linear instrumental variable regression model and asset pricing model. Section 5 concludes. All technical details are contained in the Appendix.

2 Setup

Our notation closely follows that of Chen and Cui (2007). Suppose we observe a random sample $\{X_i\}_{i=1}^n$ from $X \in \mathcal{X} \subseteq \mathbb{R}^d$. Let $g : \mathcal{X} \times \Theta \to \mathbb{R}^r$ be a vector of moment functions, where $\Theta \subset \mathbb{R}^p$ is the parameter space and $r > p$ (overidentified). We wish to test the validity of the overidentifying restrictions:

$$\begin{aligned} H_0 : E\left[g\left(X,\theta\right)\right] &= 0 \quad \text{for some } \theta \in \Theta, \\ H_1 : E\left[g\left(X,\theta\right)\right] &\neq 0 \quad \text{for any } \theta \in \Theta. \end{aligned} \tag{1}$$

If the null hypothesis is uniquely satisfied at some $\theta_0 \in \Theta$ (i.e., the model is correctly specified and the parameter is point identified at θ_0), then we can estimate the true parameter value θ_0 by GMM or generalized empirical likelihood and also conduct hypothesis testing on θ_0 by the Wald, Lagrange multiplier, or likelihood ratio type tests. In contrast to Chen and Cui (2007) who focused on parameter hypothesis testing (i.e., $H_0^P : \theta_0 = c$ against $H_1^P : \theta_0 \neq c$), this paper studies second-order properties of the empirical likelihood test for the overidentifying restrictions H_0 against H_1. We consider the following setup adopted by Chen and Cui (2007). Let $g\left(X,\theta\right) = \left(g^1\left(X,\theta\right), \ldots, g^r\left(X,\theta\right)\right)'$ and $|\cdot|$ be the Euclidean norm.

Assumption.

1. *$\{X_i\}_{i=1}^n$ is i.i.d.*

2. *$E[g(X,\theta)] = 0$ is uniquely satisfied at $\theta_0 \in \Theta$, Θ is compact, $V = Var(g(X,\theta_0))$ is positive definite, and $G = E\left[\frac{\partial g(X,\theta_0)}{\partial \theta'}\right]$ has the full column rank.*

3. *There exists a neighborhood $\mathcal{N}$ of θ_0 such that for each $j = 1, \ldots, r$, $g^j(x,\theta)$ is continuously third-order differentiable in $\theta \in \mathcal{N}$ almost surely and the derivatives are bounded by integrable functions over $\mathcal{N}$.*

4. *$E\left[|g(X,\theta_0)|^{15}\right] < \infty$ and $\limsup_{|t|\to\infty} |E[\exp(it'g(X,\theta_0))]| < 1$.*

The same comments in Chen and Cui (2007) apply here. Assumption 1 excludes dependent data. An extension to time series data is beyond the scope of this paper. The first condition in Assumption 2 says that the overidentification null hypothesis H_0 is satisfied and the true parameter value θ_0 is point identified. The last condition in Assumption 2 excludes weak identification (or weak instruments) in the sense of Stock and Wright (2000). If Assumption 2 does not hold and G drifts to the zero matrix at the $\sqrt{n}$-rate (so-called weak identification asymptotics), the first-order asymptotic null distribution of the overidentification test statistic typically becomes non-standard. Assumption 3 is on smoothness of the moment functions. This assumption excludes, for example, quantile regression models. Assumption 4 imposes bounded moments and a Cramér condition used to establish the validity of the Edgeworth expansion. It is known that the Cramér condition is satisfied when the distribution of $g(X,\theta_0)$ has a non-degenerate and absolutely continuous component (see, e.g., Hall, 1992, pp. 65-67). This requirement is typically satisfied when X is continuous and $g(x,\theta_0)$ is smooth in x. For example, Assumption 4 can be verified for the simulation design in Section 4 motivated by an asset pricing model, and linear instrumental variable regression models with normal errors, regressors, and instruments. However, when the distribution of $g(X,\theta_0)$ has no absolutely continuous component, a conventional argument to verify the Cramér condition is not applicable in general, and the validity of the Cramér condition becomes questionable or at least hard to verify.[1]

We now introduce the empirical likelihood test statistic for H_0. Let T be an $r \times r$ orthogonal matrix satisfying

$$TV^{-1/2}GU = \left(\Lambda, 0_{p\times(r-p)}\right)',$$

where U is a $p \times p$ orthogonal matrix and Λ is a $p \times p$ non-singular diagonal matrix. We orthogonalize the moment functions as $w_i(\theta) = TV^{-1/2}g(X_i,\theta)$ so that $E\left[w_i(\theta_0)w_i(\theta_0)'\right] = I$. The empirical likelihood overidentifying restriction test statistic, proposed by Qin and Lawless (1994), can be defined as

$$T_n = \min_{\theta\in\Theta} \ell(\theta) = \min_{\theta\in\Theta} 2\sum_{i=1}^n \log\left(1 + \lambda(\theta)' w_i(\theta)\right),$$

where $\lambda(\theta)$ solves $\sum_{i=1}^n \frac{w_i(\theta)}{1+\lambda' w_i(\theta)} = 0$ with respect to λ for a given value of θ. From Qin and Lawless (1994, Corollary 3), we can see that $T_n \xrightarrow{d} \chi^2(r-p)$ under H_0.

The rest of this section presents an expansion formula for T_n derived by Chen and Cui (2007). Let $\hat{\theta} = \arg\min_{\theta\in\Theta} \ell(\theta)$ and $\hat{\lambda} = \lambda\left(\hat{\theta}\right)$. The first-order conditions for $\left(\hat{\lambda}', \hat{\theta}'\right)$ are written as $Q\left(\hat{\lambda}, \hat{\theta}\right) = 0$, where

$$Q(\lambda,\theta) = \left(\frac{1}{n}\sum_{i=1}^n \frac{w_i(\theta)'}{1+\lambda' w_i(\theta)}, \ \frac{1}{n}\sum_{i=1}^n \frac{\lambda'(\partial w_i(\theta)/\partial\theta')}{1+\lambda' w_i(\theta)}\right)'.$$

[1]For example, Hall (1992, Section 5.5.1) argued that a random variable $\frac{1}{2}\mathbb{I}\{|X-c| \le h\}$ used for the uniform kernel density estimation does not satisfy the Cramér condition. See also Horowitz (1998) and Whang (2006) on this issue in the context of quantile regression, where $g(X,\theta_0) = Z(\tau - \mathbb{I}\{Y \le Z'\theta_0\})$ for $X = (Y, Z')'$ and $\tau \in (0,1)$. In particular, these papers verified the Cramér condition for a kernel smoothed version of the moment function $g(X,\theta_0)$.

Thus, the fourth-order Taylor expansion of $Q\left(\hat{\lambda},\hat{\theta}\right)=0$ around $\left(\hat{\lambda}',\hat{\theta}'\right)=\left(0'_{r\times 1},\theta'_0\right)$ and inversions yield expansion formulae for $\hat{\lambda}$ and $\hat{\theta}-\theta_0$. By inserting those formulae to the fourth-order Taylor expansion of $T_n=2\sum_{i=1}^n \log\left(1+\hat{\lambda}'w_i\left(\hat{\theta}\right)\right)$ around $\hat{\lambda}'w_i\left(\hat{\theta}\right)=0$, Chen and Cui (2007) obtained an expansion formula for T_n. To present the formula, define $\eta=(\lambda',\theta')'$, $S=E\left[\frac{\partial Q(0,\theta_0)}{\partial\eta'}\right]$, $a^j=j$-th element of a vector a,

$$
\begin{aligned}
\alpha^{j_1\cdots j_k} &= E\left[w_i^{j_1}(\theta_0)\cdots w_i^{j_k}(\theta_0)\right], \qquad A^{j_1\cdots j_k}=\frac{1}{n}\sum_{i=1}^n w_i^{j_1}(\theta_0)\cdots w_i^{j_k}(\theta_0)-\alpha^{j_1\cdots j_k},\\
\beta^{j,j_1\cdots j_k} &= S^{-1}E\left[\frac{\partial^k Q^j(0,\theta_0)}{\partial\eta_{j_1}\cdots\partial\eta_{j_k}}\right], \qquad B^{j,j_1\cdots j_k}=S^{-1}\frac{\partial^k Q^j(0,\theta_0)}{\partial\eta_{j_1}\cdots\partial\eta_{j_k}}-\beta^{j,j_1\cdots j_k},\\
\gamma^{j,j_1\dots j_l;k,k_1\dots k_m;\dots p,p_1,\dots,p_n} &= E\left[\frac{\partial^l w^j(\theta_0)}{\partial\theta_{j_1}\cdots\partial\theta_{j_l}}\frac{\partial^m w^k(\theta_0)}{\partial\theta_{k_1}\cdots\partial\theta_{k_m}}\cdots\frac{\partial^n w^p(\theta_0)}{\partial\theta_{p_1}\cdots\partial\theta_{p_n}}\right],\\
C^{j,j_1\dots j_l;k,k_1\dots k_m;\dots p,p_1,\dots,p_n} &= \frac{1}{n}\sum_{i=1}^n\frac{\partial^l w^j(\theta_0)}{\partial\theta_{j_1}\cdots\partial\theta_{j_l}}\frac{\partial^m w^k(\theta_0)}{\partial\theta_{k_1}\cdots\partial\theta_{k_m}}\cdots\frac{\partial^n w^p(\theta_0)}{\partial\theta_{p_1}\cdots\partial\theta_{p_n}}-\gamma^{j,j_1\dots j_l;k,k_1\dots k_m;\dots p,p_1,\dots,p_n}.
\end{aligned}
$$

Hereafter, the ranges of the superscripts are fixed as $g,h,i,j\in\{1,\dots,r\}$, $k,l,m,n\in\{1,\dots,p\}$, and $q,s,t,u\in\{1,\dots,r+p\}$. Also, by the convention, repeated superscripts are summed over (e.g., $B^jA^j=\sum_{j=1}^r B^jA^j$). Based on this notation, Chen and Cui's (2007) expansion formula for T_n is presented as

$$
\begin{aligned}
n^{-1}T_n &= -2B^jA^j-B^jB^j+2C^{i,k}B^iB^{r+k,q}B^q+\frac{1}{2}\beta^{j,uq}\beta^{r+k,st}\gamma^{j,k}B^uB^qB^sB^t\\
&\quad -\beta^{j,uq}B^uB^qB^{r+k,s}B^s\gamma^{j,k}-\beta^{r+k,uq}B^uB^qC^{i,k}B^i-B^jB^iA^{ji}-\frac{2}{3}\alpha^{jih}B^jB^iB^h\\
&\quad +2C^{j,k}\left\{B^jB^{r+k}-B^{j,q}B^qB^{r+k}[2,j,r+k]+\frac{1}{2}\beta^{j,uq}B^uB^qB^{r+k}[2,j,r+k]\right\}\\
&\quad +\gamma^{j,kl}\left\{-B^jB^{r+k}B^{r+l}+B^jB^{r+k}B^{r+l,q}B^q[3,j,r+k,r+l]\right.\\
&\quad \left.-\frac{1}{2}\beta^{j,uq}B^{r+k}B^{r+l}B^uB^q[3,j,r+k,r+l]\right\}-C^{j,kl}B^jB^{r+k}B^{r+l}-\frac{2}{3}A^{jih}B^jB^iB^h\\
&\quad -B^{j,u}B^uB^{j,q}B^q-\frac{1}{4}\beta^{j,uq}\beta^{j,st}B^uB^qB^sB^t+\beta^{j,uq}B^uB^qB^{j,s}B^s+2\gamma^{j;i;h,k}B^jB^iB^hB^{r+k}\\
&\quad +B^jB^{i,q}B^qA^{ji}[2,j,i]-\frac{1}{2}\beta^{j,uq}B^uB^qB^iA^{ji}[2,j,i]+\frac{1}{3}\gamma^{j;k,lm}B^jB^{r+k}B^{r+l}B^{r+m}\\
&\quad +2\gamma^{j;i,l}\left\{B^jB^iB^{r+l}-B^jB^iB^{r+l,q}B^q+\frac{1}{2}\beta^{r+l,uq}B^jB^iB^uB^q-B^{r+l}B^iB^{j,q}B^q[2,j,i]\right.\\
&\quad \left.+\frac{1}{2}\beta^{j,uq}B^uB^qB^iB^{r+l}[2,j,i]\right\}+2B^jB^iB^{r+l}C^{j;i,l}-\left(\gamma^{j;i,lk}+\gamma^{j,l;i,k}\right)B^jB^iB^{r+l}B^{r+k}\\
&\quad +2\alpha^{jih}B^jB^iB^{h,q}B^q-\alpha^{jih}\beta^{j,uq}B^uB^qB^iB^h-\frac{1}{2}\alpha^{jihg}B^jB^iB^hB^g+O_p\left(n^{-5/2}\right)\\
&= L_1+\cdots+L_{33}+O_p\left(n^{-5/2}\right), \qquad (2)
\end{aligned}
$$

where $[2,j,i]$ means the sum of two terms by exchanging the superscripts i and j, and $[3,j,r+k,r+l]$ means the sum of three terms by exchanging the superscripts j, $r+k$, and $r+l$ (e.g., $B^jB^{r+k}B^{r+l,q}B^q[3,j,r+k,r+l]=B^jB^{r+k}B^{r+l,q}B^q+B^{r+k}B^{r+l}B^{j,q}B^q+B^jB^{r+l}B^{r+k,q}B^q$). Compared to Chen and Cui (2007) who investigated the second-order properties of the empirical likelihood ratio test statistic $\ell(c)-\ell\left(\hat{\theta}\right)$ for the parameter hypothesis $H_0^P:\theta_0=c$, this paper studies second-order properties of $T_n=\ell\left(\hat{\theta}\right)$. Except for the basic ideas, the second-order analysis below is specific to our setup and different from Chen and Cui (2007).

3 Main Results

3.1 Signed Root Expansion and Cumulants

Hereafter, the ranges of the superscripts are fixed as $a, b, c, d \in \{1, \ldots, r-p\}$. To study the second-order properties of T_n based on the expansion in (2), we first find a signed root expansion in the form of

$$n^{-1}T_n = (R_1 + R_2 + R_3)^{p+a} (R_1 + R_2 + R_3)^{p+a} + O_p\left(n^{-5/2}\right),$$

where $R_1 = O_p\left(n^{-1/2}\right)$, $R_2 = O_p\left(n^{-1}\right)$, and $R_3 = O_p\left(n^{-3/2}\right)$. By collecting the terms of order $O_p\left(n^{-1}\right)$ in (2), we have $R_1^{p+a}R_1^{p+a} = L_1 + L_2$. Using the formulae in Appendix A.1, R_1^{p+a} is obtained as

$$R_1^{p+a} = A^{p+a}. \tag{3}$$

By collecting the terms of order $O_p\left(n^{-3/2}\right)$ in (2), we have $2R_1^{p+a}R_2^{p+a} = L_7 + L_8 + L_9 + L_{12} + L_{24}$. Let $U\Lambda^{-1} = \left(\omega^{kl}\right)_{p\times p}$. Using the formulae in Appendix A.1, R_2^{p+a} is obtained as

$$\begin{aligned} R_2^{p+a} &= -\frac{1}{2}A^{p+b}A^{p+a\ p+b} + \frac{1}{3}\alpha^{p+a\ p+b\ p+c}A^{p+b}A^{p+c} - \omega^{kl}C^{p+a,k}A^l \\ &\quad + \frac{1}{2}\omega^{km}\omega^{ln}\gamma^{p+a,kl}A^mA^n + \omega^{lm}\gamma^{p+a;p+b,l}A^{p+b}A^m. \end{aligned} \tag{4}$$

Also, by collecting the terms of order $O_p\left(n^{-2}\right)$ in (2), we have $2R_1^{p+a}R_3^{p+a} + R_2^{p+a}R_2^{p+a} = \sum_{j=3}^{6} L_j + \sum_{j=10}^{11} L_j + \sum_{j=13}^{23} L_j + \sum_{j=25}^{33} L_j$. Thus, after tedious calculations in Appendix A.3, R_3^{p+a} is obtained as in Appendix A.2. Based on the signed root expansion obtained above, we compute cumulants of $R = R_1 + R_2 + R_3$. Observe that $E\left[R_1^{p+a}\right] = 0$ and $E\left[R_2^{p+a}\right] = n^{-1}\mu^{p+a}$, where

$$\mu^{p+a} = -\frac{1}{6}\alpha^{p+a\ p+b\ p+b} - \omega^{kl}\gamma^{l;p+a,k} + \frac{1}{2}\gamma^{p+a,kl}\omega^{km}\omega^{lm}. \tag{5}$$

Since all terms in R_3^{p+a} are product of three zero mean averages, it holds $E\left[R_3^{p+a}\right] = O\left(n^{-2}\right)$. Thus, the first-order cumulant is

$$cum\left(R^{p+a}\right) = E\left[R^{p+a}\right] = n^{-1}\mu^{p+a} + O\left(n^{-2}\right). \tag{6}$$

In Appendix A.4, we show that the second-order cumulant is

$$cum\left(R^{p+a}, R^{p+f}\right) = n^{-1}\delta^{p+a\ p+f} + n^{-2}\Delta^{p+a\ p+f} + O\left(n^{-3}\right). \tag{7}$$

Appendices A.5 and A.6 show that the third and fourth cumulants satisfy

$$cum\left(R^{p+a}, R^{p+b}, R^{p+d}\right) = O\left(n^{-3}\right), \qquad cum\left(R^{p+a}, R^{p+b}, R^{p+c}, R^{p+d}\right) = O\left(n^{-4}\right). \tag{8}$$

3.2 Second-order Properties and Bartlett Correction

Based on the cumulants for the signed root expansion obtained in the previous subsection, we can apply a conventional argument to derive the Edgeworth expansion and Bartlett correction for the empirical likelihood test statistic T_n (e.g., DiCiccio, Hall and Romano, 1991). Let c_α and $f_{r-p}(\cdot)$ be the $(1-\alpha)$-th quantile and probability density function of the $\chi^2(r-p)$ distribution, respectively. Also define the Bartlett factor as

$$B_c = \frac{\mu^{p+a}\mu^{p+a} + \Delta^{p+a\ p+a}}{r-p}, \tag{9}$$

where μ^{p+a} and $\Delta^{p+a\ p+a}$ are defined in (5) and (7), respectively. Let $\hat{B}_c$ be a $\sqrt{n}$-consistent estimator of B_c. The main results are summarized as follows.

Theorem 3.1. *Under Assumptions 1-4,*

$$
\begin{aligned}
&(i) \quad \Pr\{T_n \le c_\alpha\} = 1 - \alpha - n^{-1} c_\alpha f_{r-p}(c_\alpha) B_c + O\left(n^{-2}\right), \\
&(ii) \quad \Pr\left\{T_n \le c_\alpha\left(1 + n^{-1} B_c\right)\right\} = 1 - \alpha + O\left(n^{-2}\right), \\
&(iii) \quad \Pr\left\{T_n \le c_\alpha\left(1 + n^{-1} \hat{B}_c\right)\right\} = 1 - \alpha + O\left(n^{-2}\right).
\end{aligned}
$$

Theorem 3.1 says that (i) the error in the null rejection probability of the empirical likelihood test using the asymptotic critical value c_α is of order $O\left(n^{-1}\right)$, (ii) the error can be reduced to order $O\left(n^{-2}\right)$ by the Bartlett correction, and (iii) replacing the Bartlett factor B_c by a $\sqrt{n}$-consistent estimator $\hat{B}_c$ has no effect at the order of n^{-2} (see DiCiccio, Hall and Romano (1991), for instance).

In practice, B_c has to be estimated. The method of moments estimator of B_c can be obtained by substituting all the population moments involved by their corresponding sample moments. However, particularly when the moment function $g(X,\theta)$ is nonlinear in θ, the Bartlett factor B_c takes a complex form and the method of moments estimator can be less practical and precise. Chen and Cui (2007) employed a uniform weight bootstrap method using recentered moments (Hall and Horowitz, 1996) to estimate the normalized factor $\beta_c = 1 + n^{-1} B_c$ in the case of parameter hypothesis testing for overidentified models. We suggest a slightly different procedure to estimate β_c based on the implied probability bootstrap (Brown and Newey, 2002) which resamples from a distribution that imposes the moment restrictions instead of the empirical distribution. The procedure to estimate β_c is as follows.

1. Using $\hat{\theta}$ and $\hat{\lambda}$, calculate the implied probabilities

$$
\hat{p}_i = \frac{1}{n\left(1 + \hat{\lambda}' g\left(X_i, \hat{\theta}\right)\right)}, \tag{10}
$$

   for $i = 1, \ldots, n$.

2. Draw n i.i.d. observations $\left\{X_i^{*b}\right\}_{i=1}^n$ with replacement from the multinomial distribution with $\Pr\{X = x_i\} = \hat{p}_i$ and calculate the empirical likelihood test statistic T_n^{*b} based on $\left\{X_i^{*b}\right\}_{i=1}^n$.[2]

3. Repeat Step 2 B times to obtain $T_{n,}^{*1}, \ldots, T_n^{*B}$. Estimate β_c by

$$
\hat{\beta}_c = \frac{1}{B(r-p)} \sum_{b=1}^{B} T_n^{*b}. \tag{11}
$$

   The critical value for T_n is set as $c_\alpha \hat{\beta}_c$.

Brown and Newey (2002) argued that this version of bootstrap can provide an asymptotically efficient estimator of the distribution of overidentification test statistics. The asymptotic property of this procedure is presented as follows.

Theorem 3.2. *Under Assumptions 1-4,*

$$
\Pr\left\{T_n \le c_\alpha \hat{\beta}_c\right\} = 1 - \alpha + O\left(n^{-3/2}\right).
$$

[2]Since this multinomial distribution satisfies the overidentified moment conditions (i.e., $\sum_{i=1}^n \hat{p}_i g\left(X_i, \hat{\theta}\right) = 0$), we can use the original moment functions without recentering.

Compared to Theorem 3.1 (i), this theorem says that the error in the null rejection probability of the empirical likelihood test can be reduced to order $O\left(n^{-3/2}\right)$ by the bootstrap approximation to $\beta_c = 1 + n^{-1}B_c$. Compared to Theorem 3.1 (ii) and (iii), the asymptotic error increases from $O\left(n^{-2}\right)$ to $O\left(n^{-3/2}\right)$. This is due to the use of $\sqrt{n}$-consistent estimator $\hat{\beta}_c$ of β_c. The proof of this theorem is similar to that of Chen and Cui (2007, Theorem 3), which employs the uniform weight bootstrap with recentering.

3.3 Refinement by Adjusted Empirical Likelihood

Liu and Chen (2010) proposed an adjustment for the construction of empirical likelihood to avoid non-existence for the solution of the likelihood maximization problem (i.e., the case where the linear space spanned by $\left\{g\left(X_i, \hat{\theta}\right)\right\}_{i=1}^{n}$ may not contain the origin in finite samples). In our context, the adjusted empirical likelihood test statistic can be defined as

$$T_n^A = \min_{\theta \in \Theta} 2 \sum_{i=1}^{n+1} \log\left(1 + \lambda^A(\theta)' w_i(\theta)\right),$$

where $w_{n+1}(\theta) = -\frac{a_n}{n}\sum_{i=1}^{n} w_i(\theta)$ is a pseudo observation and $\lambda^A(\theta)$ solves $\sum_{i=1}^{n+1} \frac{w_i(\theta)}{1+\lambda' w_i(\theta)} = 0$ with respect to λ. If $a_n > 0$, the linear space spanned by $\{w_i(\theta)\}_{i=1}^{n+1}$ always contains the origin and thus the test statistic T_n^A always exists. By a similar argument to Liu and Chen (2010) combined with the results in Section 3.1, the signed root expansion of T_n^A is obtained as

$$n^{-1}T_n^A = \left(R_1 + R_2 + R_3^A\right)^{p+a}\left(R_1 + R_2 + R_3^A\right)^{p+a} + O_p\left(n^{-5/2}\right),$$

where $R_3^A = R_3 - \frac{a}{n}R_1$ with $a_n = a + O_p\left(n^{-1/2}\right)$. By setting $a = \frac{B_c}{2}$, the same calculations in Sections 3.1 with R_3^A imply that the Bartlett correction factor in (9) will be zero. This result is summarized in the following theorem.

Theorem 3.3. *Under Assumptions 1-4,*

$$(i) \quad \Pr\left\{T_n^A \le c_\alpha\right\} = 1 - \alpha + O\left(n^{-2}\right) \quad \text{if } a_n = \frac{B_c}{2},$$

$$(ii) \quad \Pr\left\{T_n^A \le c_\alpha\right\} = 1 - \alpha + O\left(n^{-2}\right) \quad \text{if } a_n = \frac{\hat{B}_c}{2} + O_p\left(n^{-1/2}\right).$$

Theorem 3.3 says that (i) by setting $a_n = \frac{B_c}{2}$, the adjusted empirical likelihood test with the chi-square critical value achieves the same higher-order precision as the Bartlett correction in Theorem 3.1 (ii); and (ii) estimation of B_c by a $\sqrt{n}$-consistent estimator has no effect on the error in the null rejection probability. Similar to the case of the Bartlett correction in Section 3.2, the correction factor B_c can be estimated by the method of moments or bootstrapping. If $\hat{B}_c$ is obtained by the method of moments, then mild conditions guarantee the $\sqrt{n}$-consistency for B_c. However, if we employ a bootstrap approximation for β_c based on either the uniform weight bootstrap (Chen and Cui, 2007) or implied probability bootstrap in Section 3.2 and estimate B_c by $\tilde{B}_c = n\left(\hat{\beta}_c - 1\right)$, then $\tilde{B}_c$ is not $\sqrt{n}$-consistent in general (even though $\hat{\beta}_c$ is $\sqrt{n}$-consistent for β_c). In a simulation study below, we find that the value of $\tilde{B}_c$ varies in a wide range across simulations compared to the value of $\hat{\beta}_c$.[3]

[3]For the uniform bootstrap approximation, Liu and Chen (2010, pp. 1355-1356) estimated B_c by using the median of bootstrap resamples of T_n (with recentered moments), while they reported that the estimates for B_c are unstable even after this modification.

4 Simulation

This section conducts simulation studies in order to evaluate finite sample properties of the second-order refinements proposed in the last section. We consider two simulation designs: a linear instrumental variable regression model (Section 4.1) and nonlinear moment restriction model (Section 4.2). Under the null and alternative hypotheses, we compare rejection frequencies of four overidentifying restriction tests: (i) the J-test based on the generalized method of moments (GMM),[4] (ii) usual empirical likelihood test (EL),[5] (iii) Bartlett corrected empirical likelihood test (BEL), and (iv) adjusted empirical likelihood test (AEL). To implement BEL, we obtain an estimator $\hat{\beta}_c$ for the correction factor $\beta_c = 1 + n^{-1}B_c$ by using the implied probability bootstrap method suggested in Section 3.2. To implement AEL, we estimate B_c by $\tilde{B}_c = n\left(\hat{\beta}_c - 1\right)$. The number of bootstrap replications is 199. All results are based on 1,000 Monte Carlo replications. All tables and figures are contained in Appendix B.

4.1 Linear Instrumental Variable Regression

4.1.1 Performance under the Null Hypothesis

We first consider the linear instrumental variable regression model:

$$\begin{aligned} Y_i &= W_i\theta_0 + U_i, \\ W_i &= Z_i'\pi + V_i, \end{aligned} \tag{12}$$

for $i = 1, \ldots, n$, where $\pi = (c, \ldots, c)'$ and $Z_i \sim N(0, I_r)$. The error terms are generated as $(U_i, V_i) = \left(\epsilon_{1i}, \rho\epsilon_{1i} + \sqrt{1-\rho^2}\epsilon_{2i}\right)$, where ϵ_{1i} and ϵ_{2i} are independent and drawn from three distributions: for $j = 1$ and 2, $\epsilon_{ji} \sim N(0,1)$ (normal case), $t(5)/\sqrt{5/3}$ (standardized $t(5)$ case), and $\left\{\chi^2(3) - 3\right\}/\sqrt{6}$ (standardized $\chi^2(3)$ case). The moment restrictions to estimate θ_0 are written as $E[g(X_i, \theta_0)] = E[Z_i(Y_i - W_i\theta_0)] = 0$. We set $\theta_0 = 0$ for the true parameter value of interest. For each Monte Carlo replication, we set the value of c to fix the value of the concentration parameter $\delta^2 = \pi'\left(\sum_{i=1}^n Z_iZ_i'\right)\pi$ (given the realized values of Z_i).

First, Tables 1-3 report the rejection frequencies of four tests at the 5% nominal significance level for the cases of normal, standardized $t(5)$, and standardized $\chi^2(3)$, respectively. We set $n = 200$ for the sample size, r =2, 5, and 10 for the number of instruments, ρ =0.2 and 0.8 for the degree of endogeneity, and δ^2 =20 and 100 for the concentration parameter. Our findings are summarized as follows. First, compared to the nominal level, the rejection frequencies of GMM and EL can be large when the number of moment restrictions r is large. Therefore, in this example the first-order asymptotic approximations for the J-test and its empirical likelihood analog are less precise. Second, improvements by BEL and AEL in the null rejection frequencies are reasonable. For example, in the normal case (Table 1), the rejection frequency varies between .034 and .125 for GMM and between .042 and

[4] The version of the J-test statistic considered here is $J = \min_\theta \left\{\sum_{i=1}^n g(X_i, \theta)\right\}' \left[\sum_{i=1}^n g\left(X_i, \tilde{\theta}\right) g\left(X_i, \tilde{\theta}\right)'\right]^{-1} \left\{\sum_{i=1}^n g(X_i, \theta)\right\}$, where $\tilde{\theta} = \arg\min_\theta \left\{\sum_{i=1}^n g(X_i, \theta)\right\}' \left\{\sum_{i=1}^n g(X_i, \theta)\right\}$ is the GMM estimator with the identity weight matrix (thus $\tilde{\theta}$ is consistent to estimate θ_0 and asymptotically normal under Assumptions 1-4). For the linear instrumental variable regression model, $\tilde{\theta}$ corresponds to the two-stage least square estimator.

[5] To compute the empirical likelihood statistic, $T_n = \min_{\theta\in\Theta} 2\sum_{i=1}^n \log\left(1 + \gamma(\theta)' g(X_i, \theta)\right)$ where $\gamma(\theta)$ solves $\sum_{i=1}^n \frac{g(X_i,\theta)}{1+\gamma' g(X_i,\theta)} = 0$ with respect to γ, we adopted a nested algorithm. For each θ, the computation of $\gamma(\theta)$ (called the inner loop) is implemented by a quasi-Newton method based on Bruce Hansen's MATLAB code (available at http://www.ssc.wisc.edu/~bhansen/progs/elikem.zip). For the minimization with respect to θ (called the outer loop), we employed a derivative free optimization algorithm based on the `fminsearch` function in MATLAB (because θ is scalar for both simulation designs).

.099 for EL, while it varies between .047 and .061 for BEL and between .020 and .053 for AEL. Third, comparing BEL and AEL, BEL shows slightly better performance in the null rejection frequencies particularly when r is large. Based on an inspection of simulation outputs, we conjecture this difference is partly due to the lack of stability of the estimates of B_c to implement AEL (compared to the estimates of β_c to implement BEL). Finally, in general the results are similar for the different distributions of the error terms. For the non-normal cases, all tests generally rejects the null hypothesis slightly more than the normal case.

Second, we examine how the rejection frequencies of these tests vary with the sample size. Our theoretical results in Section 3 indicate that the discrepancies between the actual rejection frequencies and the nominal level of BEL and AEL will decay faster than those of GMM and EL as the sample size increases. Figure 1 reports the plots of the rejection frequencies of four tests with the 5% nominal level for sample sizes n =30, 50, 70, 100, 200, 500, 700, and 1000 (with $r = 5$, $\rho = 0.8$, and $\delta^2 = 20$). We can see that as predicted by the theoretical results, the convergence speeds of the rejection frequencies of BEL and AEL to the nominal 5% level are faster than those of GMM and EL. In particular, the convergence speed of the rejection frequency of GMM is slow.

Third, we investigate the null rejection properties of these tests when the concentration parameter δ^2 is close to (or equal to) zero, i.e., weak instruments. Although our theoretical analysis focuses on the case of strong identification (i.e., G is full column rank, imposed in Assumption 2), it is important to examine finite sample behaviors of the proposed BEL and AEL tests when the strong identification assumption is questionable. Figure 2 reports the plots of the rejection frequencies of four tests with the 5% nominal level for δ^2 =0, 3, 5, 10, 20, 30, 50, 70, 100, and 200. It is remarkable that the rejection frequency of BEL and AEL are very robust against small non-zero values of δ^2 (ranges between 0.038 and 0.066 for BEL and between 0.034 and 0.061 for AEL). When $\delta^2 = 0$, all tests under-reject the null hypothesis. Although it is beyond the scope of this paper, it is interesting to provide some theoretical explanation on this phenomenon.

Finally, we examine the properties of these overidentifying restriction tests as pre-tests for parameter hypothesis testing. We consider a two-stage strategy to test the parameter null hypothesis $H_0^P : \theta_0 = 0$. In the first stage, we test the overidentifying restriction H_0. If the null hypothesis H_0 is not rejected, we proceed to the second stage and test the parameter null hypothesis H_0^P. Guggenberger and Kumar (2011) provided theoretical and simulation evidences for the size distortion of this two stage approach in linear instrumental variable regression models. In particular, they derived a lower bound for the asymptotic size of the two stage test and showed that surprisingly the lower bound can be as large as $1 - \alpha$, where α is the nominal size for the first stage test. Although formal analysis is beyond the scope of this paper, it is interesting to investigate finite sample behaviors of this two stage approach when we employ BEL or AEL in the first stage. We compare (i) the J-test followed by the t-test based on the two-step GMM estimator, (ii) the empirical likelihood overidentification test followed by the empirical likelihood ratio test for the parameter hypothesis,[6] (iii) the BEL overidentification test followed by the empirical likelihood ratio test for the parameter hypothesis, and (iv) the AEL overidentification test followed by the empirical likelihood ratio test for the parameter hypothesis.

In order to evaluate the asymptotic size of a test for H_0^P, we need to analyze the null rejection probabilities of the test for all possible values of nuisance parameters and find the worst one. It is not easy and beyond the scope of this paper to characterize the asymptotic size property for the two stage test in our simulation design. Thus, after some preliminary simulation studies, we replace the data generating process for Y_i in (12) with

$$Y_i = W_i\theta_0 + cZ_{1i} + U_i,$$

[6] The empirical likelihood ratio statistic for $H_0^P : \theta_0 = a$ is defined as $T_n^P = \ell(a) - \min_{\theta\in\Theta} \ell(\theta)$. Based on the first-order asymptotic approximation (Qin and Lawless, 1994), we use the χ^2 critical value.

where Z_{1i} is the first element of Z_i, $r = 5$, $\rho = 0.8$, $\delta^2 = 20$, and $n = 200$. By perturbing c from 0, we allow small deviations from the overidentification null hypothesis H_0, which corresponds to a nuisance parameter for testing H_0^P. Figure B reports the frequencies of the event "not rejecting H_0 in the first stage but rejecting H_0^P in the second stage". For both stages, the nominal level is 5%. In this particular setup (which does not necessarily characterize the finite sample size of the two stage tests), we can see that the frequencies for this event can be higher than 0.05 for all the two stage tests. The difference among EL, BEL, and AEL-based tests is small.

4.1.2 Power Property

In order to investigate the power properties of the proposes tests, we consider the following data generation process as alternative hypotheses:

$$Y_i = W_i\theta_0 + 0.1Z_{1i} + U_i,$$

where Z_{1i} is the first element of Z_i, $r = 5$, $\rho = 0.8$, $\delta^2 = 20$, and $n = 200$. We can see that there is no θ which satisfies $E\left[g\left(X_i, \theta\right)\right] = E\left[Z_i\left(Y_i - W_i\theta\right)\right] = 0$. We investigate the calibrated powers of GMM, EL, BEL, and AEL (i.e., the rejection frequencies of these overidentification tests where the critical values are given by the Monte Carlo 95% percentiles of these test statistics under the data generation process in (12)). Figure 4 reports the calibrated powers for the tests with sample sizes $n = 50$, 70, 100, 200, 300, 400, 500, and 600 under the normal case. In this setting, all tests show similar calibrated power properties.

Overall, the simulation results for the linear instrumental variable regression indicate that BEL and AEL have more attractive null properties than EL and GMM and have comparable power properties to EL and GMM.

4.2 Nonlinear Moment Restriction

4.2.1 Performance under the Null Hypothesis

We next consider a simulation design in Liu and Chen (2010) motivated by an asset pricing model, which is a multivariate version of Hall and Horowitz's (1996) simulation design. Let $X = (X_1, X_2, \ldots, X_r)'$ be a vector of mutually independent random variables, where $X_1, X_2 \sim N\left(0, \sigma^2\right)$ and $X_3, \ldots, X_r \sim \chi^2(1)$. The moment restrictions are written as

$$E\left[g\left(X, \theta_0\right)\right] = E\begin{bmatrix} m\left(X, \theta_0\right) \\ X_2 m\left(X, \theta_0\right) \\ \left(X_3 - 1\right) m\left(X, \theta_0\right) \\ \vdots \\ \left(X_r - 1\right) m\left(X, \theta_0\right) \end{bmatrix} = 0 \tag{13}$$

where $m(X, \theta) = \exp\left(-4.5\sigma^2 - \theta\left(X_1 + X_2\right) + 3X_2\right) - 1$. We treat σ as a given normalizing constant and treat θ as an unknown parameter to be estimated from (13). These restrictions are satisfied at $\theta_0 = 3$ for any $\sigma > 0$.[7]

First, Table 4 reports the rejection frequencies of the four tests at the 5% nominal significance level. We set $\sigma = 0.2$ for the standard deviation of X_1 and X_2, $n = 100$ and 200 for the sample size, and r =2, 3, 5, and 7 for the number of moment restrictions. Our findings are summarized as follows. First, compared to the nominal level, the rejection frequencies of GMM and EL can be quite large particularly when the number of moment

[7] Note that $E\left[m(X, 3)\right] = \frac{1}{\sqrt{2\pi\sigma^2}} \int \exp\left(-\frac{\left(x_1 + 3\sigma^2\right)^2}{2\sigma^2}\right) dx_1 - 1 = 0$ for any $\sigma > 0$, and X_1 is independent from other variables.

restrictions r is large. It should be noted that, GMM shows serious distortions in the null rejection frequencies even when r is as small as 3. Therefore, in this example the first-order asymptotic approximation for the J-test is less precise. Second, improvements by BEL and AEL in the null rejection frequencies are reasonable. The rejection frequency varies between .040 and .370 for GMM and between .054 and .255 for EL, while it varies between .054 and .089 for BEL and between .000 and .052 for AEL. Finally, comparing BEL and AEL, BEL shows better performance in the null rejection frequencies particularly when r is large. Based on an inspection of simulation outputs, we conjecture this difference is partly due to the lack of stability of the estimates of B_c to implement AEL (compared to the estimates of β_c to implement BEL).

Second, we examine how the rejection frequencies of these tests vary with the sample size. Our theoretical results in Section 3 indicate that the discrepancies between the actual rejection frequencies and the nominal level of BEL and AEL will decay faster than those of GMM and EL as the sample size increases. Figure B reports the plots of the rejection frequencies of four tests with the 5% nominal level for sample sizes $n =$30, 50, 70, 100, 200, 500, 700, and 1000 (with $\sigma = 0.2$ and $r = 3$). We can see that as predicted by the theoretical results, the convergence speeds of the rejection frequencies of BEL and AEL to the nominal 5% level are faster than those of GMM and EL. In particular, the convergence speed of the rejection frequency of GMM is slow.

Third, we investigate the null rejection properties of these tests when the matrix $G = E\left[\frac{\partial g(X,\theta_0)}{\partial \theta'}\right]$ is close to the zero matrix, i.e., weak identification (Stock and Wright, 2000). Although our theoretical analysis focuses on the case of strong identification (i.e., G is full column rank, imposed in Assumption 2), it is important to examine finite sample behaviors of the proposed BEL and AEL tests when the strong identification assumption is questionable. In order to characterize weak identification in our simulation design, Figure 6 reports the relationship between the constant σ and the scalar $\mu = nG'V^{-1}G$ computed by Monte Carlo integration. We call this μ as the degree of concentration since it is analogous to the so-called concentration parameter in the linear instrumental variable regression model. From Figure 6, we can see that μ gets smaller as σ increases. Thus, in our setup, large values of σ can be associated with weak identification for the parameter θ_0. Figure 7 reports the plots of the rejection frequencies of four tests with the 5% nominal level for $\sigma =$0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, and 0.8 (with $n = 200$ and $r = 3$). Note that all tests over-reject the null hypothesis when σ is large (i.e., the degree of concentration μ is small). The rejection frequencies of BEL and AEL are closer to the nominal size than those of GMM and EL. In particular, it is remarkable that the rejection frequency of BEL is very robust against large values of σ (ranges between 0.05 and 0.15). Although it is beyond the scope of this paper, it is interesting to provide some theoretical explanation on this phenomenon.

Finally, we examine the properties of these overidentifying restriction tests as pre-tests for parameter hypothesis testing. We consider a two-stage strategy to test the parameter null hypothesis $H_0^P : \theta_0 = 3$. In the first stage, we test the overidentifying restriction H_0 in (1). If the null hypothesis H_0 is not rejected, we proceed to the second stage and test the parameter null hypothesis H_0^P. Similarly to the previous section, we compare (i) the J-test followed by the t-test based on the two-step GMM estimator, (ii) the empirical likelihood overidentification test followed by the empirical likelihood ratio test for the parameter hypothesis, (iii) the BEL overidentification test followed by the empirical likelihood ratio test for the parameter hypothesis, and (iv) the AEL overidentification test followed by the empirical likelihood ratio test for the parameter hypothesis.

In order to evaluate the asymptotic size of a test for H_0^P, we need to analyze the null rejection probabilities of the test for all possible values of nuisance parameters and find the worst one. It is not easy and beyond the scope of this paper to characterize the asymptotic size property for the two stage test in our simulation design. Thus, after some preliminary simulation studies, we fix the data generating process as $X_1, X_2 \sim N\left(0, (c\sigma)^2\right)$ and

$X_3, \ldots, X_r \sim \chi^2(1)$ with $n = 200$, $\sigma = 0.2$, and $r = 3$.[8] By perturbing c from 1, we allow small deviations from the overidentification null hypothesis H_0, which corresponds to a nuisance parameter for testing H_0^P. Figure B reports the frequencies of the event "not rejecting H_0 in the first stage but rejecting H_0^P in the second stage". For both stages, the nominal level is 5%. In this particular setup (which does not necessarily characterize the finite sample size of the two stage tests), we can see that the frequencies for this event are typically higher than 0.05 for the GMM-based two stage test and lower than 0.05 for the other tests. Also the difference among EL, BEL, and AEL-based tests is small.

4.2.2 Power Property

In order to investigate the power properties of the proposes tests, we consider the two data generation processes as alternative hypotheses:

$$\text{Case 1:} X_1, X_2 \sim N\left(0.05, (0.2)^2\right), \quad X_3 \sim \chi^2(1),$$
$$\text{Case 2:} X_1, X_2 \sim N\left(0, (0.3)^2\right), \quad X_3 \sim \chi^2(1).$$

Under these data generation processes, we specify the moment functions as
$g(X, \theta) = (m(X, \theta), X_2 m(X, \theta), (X_3 - 1) m(X, \theta))'$, where

$$m(X, \theta) = \exp\left(-4.5(0.2)^2 - \theta(X_1 + X_2) + 3X_2\right) - 1.$$

We can see that for both cases, there is no θ which satisfies $E[g(X, \theta)] = 0$. For both cases, we investigate the calibrated powers of GMM, EL, BEL, and AEL (i.e., the rejection frequencies of these overidentification tests where the critical values are given by the Monte Carlo 95% percentiles of these test statistics under the data generation process $X_1, X_2 \sim N\left(0, (0.2)^2\right)$ and $X_3 \sim \chi^2(1)$ satisfying H_0). Figures 9 and 10 report the calibrated powers with sample sizes $n =$ 100, 200, 300, 400, 500, and 600 for each case. For Case 1, EL, BEL, and AEL show superior calibrated power properties than GMM. For Case 2, EL and BEL have better power than AEL and GMM. Lower calibrated power of GMM is partly due to the over-rejection properties of GMM under H_0 as illustrated in Figure B (which typically yield large critical values to compute calibrated power). For both cases, BEL is slightly less powerful than EL. Since BEL has better null rejection properties than EL (see Figure B), these power properties characterize a trade-off between the null rejection and power properties of EL and BEL. For Case 2, AEL tends to have lower calibrated power than BEL and EL, and shows similar properties to GMM. We find that this decay of power in AEL is partly due to the lack of stability of the estimates of B_c to implement AEL.[9]

Overall, our simulation results are encouraging. BEL and AEL have more attractive null properties than EL and GMM. Based on the power properties, we particularly recommend to use BEL.

5 Conclusion

In this paper, we show that the empirical likelihood test for overidentifying restrictions is Bartlett correctable and propose second-order refinement methods based on the empirical Bartlett correction and adjusted empirical

[8]In preliminary analysis, we tried the cases of $X_1, X_2 \sim N(c, \sigma^2)$ for different values of c, different values of σ (but not too large to avoid weak identification for θ_0), and different number of moments r, for example. The results are basically similar to the one in Figure B.

[9]For example, in Case 2 with $n = 200$, the bootstrap estimates of β_c and B_c range from 0.86 to 3.71 and from -13.73 to 271.18, respectively. In Case 2 with $n = 500$, they range from 0.79 to 2.87 and -104.81 from 938.26, respectively.

likelihood. Simulation results suggest that the empirical Bartlett correction and adjusted empirical likelihood assisted by bootstrapping exhibit better null rejection properties than the conventional GMM and empirical likelihood tests using the first-order asymptotic approximation. It is interesting to extend this research to a time series context and non-smooth moment functions (e.g. quantile instrumental variable regressions).

A Mathematical Appendix

A.1 Basic Formulae

Let $U\Lambda^{-1} = \left(\omega^{kl}\right)_{p\times p}$. From Chen and Cui (2007), we have the following formulae:

$$
\begin{aligned}
B^k &= 0, \qquad B^{p+a} = -A^{p+a}, \qquad B^{r+k} = \omega^{kl} A^l, \qquad \gamma^{j,k} B^{r+k} = A^j I\left\{j \le p\right\}, \qquad \gamma^{j,k} B^j = 0, \\
\gamma^{j,k} B^{j,l} &= C^{l,k}, \qquad \gamma^{j,k} B^{j,p+a} = 0,
\end{aligned}
$$

$$
\begin{pmatrix} B^{k,l} & B^{k,p+b} & B^{k,r+l} \\ B^{p+a,l} & B^{p+a,p+b} & B^{p+a,r+l} \\ B^{r+k,l} & B^{r+k,p+b} & B^{r+k,r+l} \end{pmatrix} = \begin{pmatrix} \omega^{mk} C^{l,m} & \omega^{mk} C^{p+b,m} & 0 \\ A^{lp+a} & A^{p+a\ p+b} & -C^{p+a,l} \\ \omega^{km}\left(\omega^{nm} C^{l,n} - A^{ml}\right) & \omega^{km}\left(\omega^{nm} C^{p+b,n} - A^{mp+b}\right) & \omega^{km} C^{m,l} \end{pmatrix},
$$

$$
\begin{aligned}
\beta^{l,p+a\ p+c} &= -\omega^{ol}\left(\gamma^{p+c;p+a,o} + \gamma^{p+a;p+c,o}\right), \qquad \beta^{l,r+m\ p+c} = \omega^{ol}\gamma^{p+c,om}, \qquad \beta^{p+a,p+b\ p+c} = -2\alpha^{p+a\ p+b\ p+c}, \\
\beta^{p+a,r+m\ p+c} &= \gamma^{p+c;p+a,m} + \gamma^{p+a;p+c,m}, \qquad \beta^{l,p+a\ r+n} = \omega^{ol}\gamma^{p+a,on}, \qquad \beta^{l,r+m\ r+n} = 0, \\
\beta^{p+a,p+b\ r+n} &= \gamma^{p+a,n;p+b} + \gamma^{p+a;p+b,n}, \qquad \beta^{p+a,r+m\ r+n} = -\gamma^{p+a,mn}, \qquad \beta^{r+k,r+m\ r+n} = \omega^{ko}\gamma^{o,mn}, \\
\beta^{r+k,p+a\ p+c} &= 2\omega^{ko}\alpha^{op+a\ p+c} - \omega^{ko}\omega^{no}\left(\gamma^{p+c;p+a,n} + \gamma^{p+a;p+c,n}\right), \\
\beta^{r+k,p+a\ r+n} &= \omega^{ko}\omega^{mo}\gamma^{p+a,mn} - \omega^{ko}\left(\gamma^{o,n;p+a} + \gamma^{o;p+a,n}\right), \\
\beta^{r+k,r+m\ p+c} &= \omega^{ko}\omega^{no}\gamma^{p+c,nm} - \omega^{ko}\left(\gamma^{p+c;o,m} + \gamma^{o,p+c;m}\right).
\end{aligned}
$$

A.2 Expression of R_3

R_3^{p+a} is written as

$$\begin{aligned}
R_3^{p+a} \;=\;& \omega^{kl}\omega^{mn}C^{n,k}C^{p+a,m}A^l + \frac{1}{2}\omega^{kl}C^{p+b,k}A^{p+a\ p+b}A^l - \frac{1}{2}\omega^{kl}\omega^{ml}C^{p+a,k}C^{p+b,m}A^{p+b} \\
&+\frac{3}{8}A^{p+a\ p+c}A^{p+b\ p+c}A^{p+b} + \omega^{kl}C^{p+b,k}A^{lp+a}A^{p+b} \\
&+\left\{ \begin{array}{c} \omega^{kl}\gamma^{p+c;p+d,k}\alpha^{lp+a\ p+b} - \frac{1}{2}\omega^{kl}\omega^{ml}\gamma^{p+a;p+b,k}\gamma^{p+c;p+d,m} \\ +\frac{4}{9}\alpha^{p+e\ p+a\ p+b}\alpha^{p+e\ p+c\ p+d} - \frac{1}{4}\alpha^{p+a\ p+b\ p+c\ p+d} \end{array} \right\} A^{p+b}A^{p+c}A^{p+d} \\
&+\omega^{mv}\left\{ \begin{array}{c} \omega^{kl}\left(\gamma^{p+c;l,m}+\gamma^{l;p+c,m}\right)\gamma^{p+a;p+b,k} + 3\omega^{kl}\gamma^{p+a,km}\alpha^{lp+b\ p+c} \\ +\omega^{lo}\gamma^{p+a;p+b,l}\left(\gamma^{p+c;o,m}+\gamma^{o;p+c,m}-3\omega^{no}\gamma^{p+a,nm}\right) \\ +\alpha^{p+d\ p+a\ p+b}\left(\frac{2}{3}\gamma^{p+d;p+c,m}+\gamma^{p+c;p+d,m}\right) - \gamma^{p+a;p+b;p+c,m} \end{array} \right\} A^vA^{p+b}A^{p+c} \\
&+\omega^{mv}\omega^{nv'}\left\{ \begin{array}{c} -\frac{1}{2}\omega^{kl}\omega^{ol}\gamma^{p+a,km}\gamma^{p+b,on} + \omega^{kl}\gamma^{p+b,km}\left(\gamma^{l,n;p+a}+\gamma^{l;p+a,n}\right) \\ +\frac{1}{2}\omega^{kl}\gamma^{l,mn}\gamma^{p+a;p+b,k} + \frac{1}{3}\gamma^{p+c,mn}\alpha^{p+a\ p+b\ p+c} - \frac{1}{2}\gamma^{p+a;p+b,mn} - \frac{1}{2}\gamma^{p+a,m;p+b,n} \\ +\frac{1}{2}\gamma^{p+a;p+c,m}\left(\gamma^{p+b;p+c,n}+\gamma^{p+c;p+b,n}\right) + \frac{1}{2}\gamma^{p+c;p+a,m}\gamma^{p+b;p+c,n} \end{array} \right\} A^vA^{v'}A^{p+b} \\
&+\frac{1}{2}\omega^{mv}\omega^{nv'}\omega^{ov''}\left\{\omega^{kl}\omega^{p+a,kl}\omega^{l,mn} + \gamma^{p+b,mn}\gamma^{p+a;p+b,o} - \frac{1}{3}\gamma^{p+a;m,no}\right\} A^vA^{v'}A^{v''} \\
&-\frac{1}{2}\omega^{kn}\omega^{lo}\omega^{vm}\gamma^{m,kl}C^{p+a,v}A^nA^o - \frac{1}{4}\omega^{kn}\omega^{lo}\gamma^{p+b,kl}A^{p+b\ p+a}A^nA^o + \omega^{km}\omega^{nm}\omega^{lo}\gamma^{p+a,kl}C^{p+b,n}A^oA^{p+b} \\
&-\omega^{km}\omega^{lo}\gamma^{p+a,kl}A^{mp+b}A^oA^{p+b} - \omega^{kv}\omega^{ln}\omega^{mo}\gamma^{p+a,kl}C^{v,n}A^nA^o + \frac{1}{2}\omega^{kn}\omega^{lo}C^{p+a,kl}A^nA^o \\
&+\frac{1}{3}A^{p+a\ p+b\ p+c}A^{p+b}A^{p+c} + \omega^{lm}\omega^{nm}\gamma^{p+a;p+b,l}C^{p+c,n}A^{p+b}A^{p+c} - \omega^{lm}\gamma^{p+a;p+b,l}A^{mp+c}A^{p+b}A^{p+c} \\
&-\omega^{ln}\omega^{mo}\gamma^{p+a;p+b,l}C^{n,m}A^oA^{p+b} - \omega^{ln}\omega^{mk}\left(\gamma^{k;p+a,l}+\gamma^{p+a;k,l}\right)C^{p+b,m}A^nA^{p+b} \\
&-\omega^{ln}\omega^{mo}\gamma^{p+a;p+b,l}C^{p+b,m}A^nA^o - \left(\frac{1}{2}\gamma^{p+c;p+a,l}+\gamma^{p+a;p+c,l}\right)\omega^{ln}A^{p+b\ p+c}A^nA^{p+b} \\
&+\omega^{lm}C^{p+a;p+b,l}A^mA^{p+b} - \omega^{mk}\alpha^{kp+a\ p+b}C^{p+c,m}A^{p+b}A^{p+c} \\
&-\frac{2}{3}\omega^{lm}\alpha^{p+a\ p+b\ p+c}C^{p+c,l}A^mA^{p+b} - \frac{5}{6}\alpha^{p+a\ p+b\ p+c}A^{p+c\ p+d}A^{p+b}A^{p+d}.
\end{aligned} \tag{14}$$

A.3 Derivation of R_3

We first evaluate the terms in $\sum_{j=3}^{6} L_j + \sum_{j=10}^{11} L_j + \sum_{j=13}^{23} L_j + \sum_{j=25}^{33} L_j$. Note that the terms L_5, L_6, L_{11}, L_{19}, and L_{22} cancel each other since

$$\beta^{j,uq}\left\{-B^{r+k,s}B^s\gamma^{j,k} + C^{j,k}B^{r+k} + B^{j,s}B^s - A^{ji}B^i\right\}B^uB^q = 0,$$

from the formulae in Appendix A.1. The other terms are written as follows.

$$\begin{aligned}
& L_3 + L_{10} + L_{17} + L_{21} \\
=\;& 2\omega^{kl}\omega^{mn}C^{n,k}C^{p+a,m}A^lA^{p+a} + 2\omega^{kl}C^{p+b,k}A^{p+a\ p+b}A^lA^{p+a} + \omega^{kl}\omega^{mn}C^{p+a,k}C^{p+a,m}A^lA^n \\
& -\omega^{kl}\omega^{ml}C^{p+a,k}C^{p+b,m}A^{p+a}A^{p+b} + A^{p+a\ p+c}A^{p+b\ p+c}A^{p+a}A^{p+b} + 2\omega^{kl}C^{p+b,k}A^{p+a,l}A^{p+a}A^{p+b}.
\end{aligned}$$

$$
\begin{aligned}
L_4 \quad = \quad & -\frac{1}{2}\omega^{kl}\left(\gamma^{p+b;p+a,k}+\gamma^{p+a;p+b,k}\right)\left\{2\alpha^{lp+cp+d}-\omega^{ml}\left(\gamma^{p+d;p+c,m}+\gamma^{p+c;p+d,m}\right)\right\}A^{p+a}A^{p+b}A^{p+c}A^{p+d} \\
& +\omega^{kl}\omega^{mo}\left(\gamma^{p+a;p+c,k}+\gamma^{p+c;p+a,k}\right)\left\{\omega^{nl}\gamma^{p+b,nm}-\left(\gamma^{l,m;p+b}+\gamma^{l;p+b,m}\right)\right\}A^{o}A^{p+a}A^{p+b}A^{p+c} \\
& +\omega^{kl}\omega^{mo}\gamma^{p+a,km}\left\{2\alpha^{lp+b\ p+c}-\omega^{nl}\left(\gamma^{p+c;p+b,n}+\gamma^{p+b;p+c,n}\right)\right\}A^{o}A^{p+a}A^{p+b}A^{p+c} \\
& +2\omega^{kl}\omega^{mv}\omega^{nv'}\gamma^{p+a,km}\left\{\omega^{ol}\gamma^{p+b,on}-\left(\gamma^{l,n;p+b}+\gamma^{l;p+b,n}\right)\right\}A^{v}A^{v'}A^{p+a}A^{p+b} \\
& -\frac{1}{2}\omega^{kl}\omega^{mo}\omega^{nv}\left(\gamma^{p+b;p+a,k}+\gamma^{p+a;p+b,k}\right)\gamma^{l,mn}A^{o}A^{v}A^{p+a}A^{p+b}-\omega^{kl}\gamma^{p+a,km}\gamma^{l,mn}\omega^{mv}A^{v}\omega^{nv'}A^{v'}\omega^{ov''}A^{v''}A^{p+a}.
\end{aligned}
$$

$$
\begin{aligned}
L_{13} \quad = \quad & -\omega^{kn}\omega^{lo}\omega^{vm}\gamma^{m,kl}C^{p+a,v}A^{n}A^{o}A^{p+a}-\omega^{kn}\omega^{lo}\gamma^{p+b,kl}A^{p+b\ p+a}A^{n}A^{o}A^{p+a} \\
& -\omega^{km}\omega^{lo}\omega^{nv}\gamma^{p+a,kl}C^{p+b,n}A^{m}A^{o}A^{v}+2\omega^{km}\omega^{lo}\left(\omega^{nm}C^{p+b,n}-A^{mp+b}\right)\gamma^{p+a,kl}A^{o}A^{p+a}A^{p+b} \\
& -2\omega^{kv}\omega^{ln}\omega^{mo}\gamma^{p+a,kl}C^{v,m}A^{n}A^{o}A^{p+a}.
\end{aligned}
$$

$$
\begin{aligned}
L_{14} \quad = \quad & \frac{1}{2}\omega^{om}\omega^{kn}\omega^{lv}\gamma^{m,kl}\left(\gamma^{p+b;p+a,o}+\gamma^{p+a;p+b,o}\right)A^{n}A^{v}A^{p+a}A^{p+b}+\omega^{km}\omega^{ln}\gamma^{p+c,kl}\alpha^{p+a\ p+b\ p+c}A^{m}A^{n}A^{p+a}A^{p+b} \\
& +\omega^{om}\omega^{kv}\omega^{lv'}\omega^{nv''}\gamma^{m,kl}\gamma^{p+a,on}A^{v}A^{v'}A^{v''}A^{p+a}+\omega^{km}\omega^{lo}\omega^{nv}\gamma^{p+b,kl}\left(\gamma^{p+b,n;p+a}+\gamma^{p+b;p+a,n}\right)A^{m}A^{o}A^{v}A^{p+a} \\
& +\frac{1}{2}\omega^{ko}\omega^{lv}\omega^{mv'}\omega^{nv''}\gamma^{p+a,kl}\gamma^{p+a,mn}A^{o}A^{v}A^{v'}A^{v''}+\omega^{ko}\omega^{lv}\omega^{mv'}\omega^{nv''}\gamma^{p+a,kl}\gamma^{o,mn}A^{v}A^{v'}A^{v''}A^{p+a} \\
& +\omega^{ko}\omega^{lv}\gamma^{p+a,kl}\{2\alpha^{op+b\ p+c}-\omega^{no}\left(\gamma^{p+c;p+b,n}+\gamma^{p+b;p+c,n}\right)\}A^{v}A^{p+a}A^{p+b}A^{p+c} \\
& -2\omega^{ko}\omega^{lv}\omega^{nv'}\gamma^{p+a,kl}\left\{\omega^{mo}\gamma^{p+b,mn}-\left(\gamma^{o,n;p+b}+\gamma^{o;p+b,n}\right)\right\}A^{v}A^{v'}A^{p+a}A^{p+b}.
\end{aligned}
$$

$$
L_{15}+L_{16}=\omega^{kn}\omega^{lo}C^{p+a,kl}A^{n}A^{o}A^{p+a}+\frac{2}{3}A^{p+a\ p+b\ p+c}A^{p+a}A^{p+b}A^{p+c}.
$$

$$
\begin{aligned}
L_{18} \quad = \quad & -\frac{1}{4}\omega^{ok}\omega^{o'k}\left(\gamma^{p+b;p+a,o}+\gamma^{p+a;p+b,o}\right)\left(\gamma^{p+d;p+c,o'}+\gamma^{p+c;p+d,o'}\right)A^{p+a}A^{p+b}A^{p+c}A^{p+d} \\
& -\omega^{ok}\omega^{o'k}\omega^{lm}\left(\gamma^{p+b;p+a,o}+\gamma^{p+a;p+b,o}\right)\gamma^{p+c,o'l}A^{m}A^{p+a}A^{p+b}A^{p+c} \\
& -\omega^{ok}\omega^{o'k}\omega^{ln}\omega^{mv}\gamma^{p+a,ol}\gamma^{p+b,o'm}A^{n}A^{v}A^{p+a}A^{p+b}-\alpha^{p+e\ p+a\ p+b}\alpha^{p+e\ p+c\ p+d}A^{p+a}A^{p+b}A^{p+c}A^{p+d} \\
& -2\omega^{lm}\left(\gamma^{p+d,l;p+c}+\gamma^{p+d;p+c,l}\right)\alpha^{p+d\ p+a\ p+b}A^{m}A^{p+a}A^{p+b}A^{p+c} \\
& -\alpha^{p+c\ p+a\ p+b}\gamma^{p+c,lm}\omega^{ln}\omega^{mo}A^{n}A^{o}A^{p+a}A^{p+b} \\
& -\omega^{ln}\omega^{mo}\left(\gamma^{p+c,l;p+a}+\gamma^{p+c;p+a,l}\right)\left(\gamma^{p+c,m;p+b}+\gamma^{p+c;p+b,m}\right)A^{n}A^{o}A^{p+a}A^{p+b} \\
& -\omega^{lo}\omega^{mv}\omega^{nv'}\left(\gamma^{p+b,l;p+a}+\gamma^{p+b;p+a,l}\right)\gamma^{p+b,mn}A^{o}A^{v}A^{v'}A^{p+a} \\
& -\frac{1}{4}\omega^{lv}\omega^{mv'}\omega^{nv''}\omega^{ov'''}\gamma^{p+a,lm}\gamma^{p+a,no}A^{v}A^{v'}A^{v''}A^{v'''}.
\end{aligned}
$$

$$
L_{20}+L_{23}=-2\omega^{kl}\gamma^{p+a;p+b;p+c,k}A^{l}A^{p+a}A^{p+b}A^{p+c}-\frac{1}{3}\omega^{kn}\omega^{lo}\omega^{mv}\gamma^{p+a;k,lm}A^{n}A^{o}A^{v}.
$$

$$
L_{25} \quad = \quad 2\omega^{lm}\gamma^{p+a;p+b,l}\left(\omega^{nm}C^{p+c,n}-A^{m,p+c}\right)A^{p+a}A^{p+b}A^{p+c}-2\omega^{ln}\omega^{mo}\gamma^{p+a;p+b,l}C^{n,m}A^{o}A^{p+a}A^{p+b}.
$$

$$L_{26} = \omega^{lo}\gamma^{p+a;p+b,l}\left\{2\alpha^{op+c\ p+d} - \omega^{no}\left(\gamma^{p+d;p+c,n} + \gamma^{p+c;p+d,n}\right)\right\} A^{p+a}A^{p+b}A^{p+c}A^{p+d}$$
$$-2\omega^{lo}\omega^{mv}\gamma^{p+a;p+b,l}\left\{\omega^{no}\gamma^{p+c,nm} - \left(\gamma^{p+c;o,m} + \gamma^{o;p+c,m}\right)\right\} A^{v}A^{p+a}A^{p+b}A^{p+c}$$
$$+\omega^{lo}\omega^{mv}\omega^{nv'}\gamma^{p+a;p+b,l}\gamma^{o,mn}A^{v}A^{v'}A^{p+a}A^{p+b}.$$

$$L_{27} = -2\omega^{ln}\omega^{mk}\gamma^{k;p+a,l}C^{p+b,m}A^{n}A^{p+a}A^{p+b} - 2\omega^{ln}\gamma^{p+c;p+a,l}A^{p+c\ p+b}A^{n}A^{p+a}A^{p+b}$$
$$-2\omega^{ln}\omega^{mo}\gamma^{p+b;p+a,l}C^{p+b,m}A^{n}A^{o}A^{p+a} - 2\omega^{ln}\omega^{mk}\gamma^{p+a;k,l}C^{p+b,m}A^{n}A^{p+a}A^{p+b}$$
$$-2\omega^{ln}\gamma^{p+a;p+b,l}A^{p+b\ p+c}A^{n}A^{p+a}A^{p+c} - 2\omega^{ln}\omega^{mo}\gamma^{p+a;p+b,l}A^{n}A^{o}C^{p+b,m}A^{p+a}.$$

$$L_{28}$$
$$= \omega^{ok}\omega^{lm}\gamma^{k;p+a,l}\left(\gamma^{p+c;p+b,o} + \gamma^{p+b;p+c,o}\right)A^{m}A^{p+a}A^{p+b}A^{p+c} + 2\omega^{ok}\omega^{mn}\omega^{lo}\gamma^{k;p+d,l}\gamma^{p+a,om}A^{n}A^{o}A^{p+d}A^{p+a}$$
$$+2\omega^{lm}\gamma^{p+d;p+a,l}\alpha^{p+d\ p+b\ p+c}A^{m}A^{p+a}A^{p+b}A^{p+c} + 2\omega^{mn}\omega^{lo}\gamma^{p+c;p+a,l}\left(\gamma^{p+c,m;p+b} + \gamma^{p+c;p+b,m}\right)A^{n}A^{o}A^{p+a}A^{p+b}$$
$$+\gamma^{p+b;p+a,l}\gamma^{p+b,mn}\omega^{mo}\omega^{nv}\omega^{lv'}A^{o}A^{v}A^{v'}A^{p+a} + \gamma^{p+a;k,l}\omega^{ok}\omega^{lm}\left(\gamma^{p+c;p+b,o} + \gamma^{p+b;p+c,o}\right)A^{m}A^{p+a}A^{p+b}A^{p+c}$$
$$+2\omega^{ok}\omega^{mn}\omega^{lo}\gamma^{p+a;k,l}\gamma^{p+b,om}A^{n}A^{o}A^{p+a}A^{p+b} + 2\omega^{lm}\gamma^{p+a;p+b,l}\alpha^{p+b\ p+c\ p+d}A^{m}A^{p+a}A^{p+c}A^{p+d}$$
$$+2\omega^{mn}\omega^{lo}\gamma^{p+a;p+b,l}\left(\gamma^{p+b,m;p+c} + \gamma^{p+b;p+c,m}\right)A^{n}A^{o}A^{p+a}A^{p+c} + \omega^{mo}\omega^{nv}\omega^{lv'}\gamma^{p+a;p+b,l}\gamma^{p+b,mn}A^{o}A^{v}A^{v'}A^{p+a}.$$

$$L_{29} + L_{30} = 2\omega^{lm}C^{p+a;p+b,l}A^{m}A^{p+a}A^{p+b} - \omega^{lm}\omega^{kn}\left(\gamma^{p+a;p+b,lk} + \gamma^{p+a,l;p+b,k}\right)A^{m}A^{n}A^{p+a}A^{p+b}.$$

$$L_{31} = -2\omega^{mk}\alpha^{kp+a\ p+b}C^{p+c,m}A^{p+a}A^{p+b}A^{p+c} - 2\alpha^{p+a\ p+b\ p+c}A^{p+c\ p+d}A^{p+a}A^{p+b}A^{p+d}$$
$$-2\omega^{lm}\alpha^{p+a\ p+b\ p+c}C^{p+c,l}A^{m}A^{p+a}A^{p+b}.$$

$$L_{32} = \omega^{ok}\alpha^{kp+a\ p+b}\left(\gamma^{p+d;p+c,o} + \gamma^{p+c;p+d,o}\right)A^{p+a}A^{p+b}A^{p+c}A^{p+d} + 2\omega^{ok}\omega^{lm}\alpha^{kp+a\ p+b}\gamma^{p+c,ol}A^{m}A^{p+a}A^{p+b}A^{p+c}$$
$$+2\alpha^{p+e\ p+a\ p+b}\alpha^{p+b\ p+c\ p+d}A^{p+a}A^{p+b}A^{p+c}A^{p+d} + \omega^{ln}\omega^{mo}\gamma^{p+c,lm}\alpha^{p+c\ p+a\ p+b}A^{n}A^{o}A^{p+a}A^{p+b}$$
$$+2\omega^{lm}\left(\gamma^{p+d,l;p+c} + \gamma^{p+d;p+c,l}\right)\alpha^{p+d\ p+a\ p+b}A^{m}A^{p+a}A^{p+b}A^{p+c}.$$

$$L_{33} = -\frac{1}{2}\alpha^{p+a\ p+b\ p+c\ p+d}A^{p+a}A^{p+b}A^{p+c}A^{p+d}.$$

Combining these results, we obtain the expression for $\sum_{j=3}^{6} L_j + \sum_{j=10}^{11} L_j + \sum_{j=13}^{23} L_j + \sum_{j=25}^{33} L_j$. By subtracting $R_2^{p+a}R_2^{p+a}$ from this expression, we obtain $2R_3^{p+a}R_1^{p+a}$, which yields the expression of R_3^{p+a} in (14).

A.4 Second-order Cumulant of R

In this subsection, let "[2]" mean "[2, a, f]" for $a, f \in \{1, \ldots, r-p\}$. Observe that

$$cum\left(R^{p+a}, R^{p+f}\right) = E\left[R^{p+a}R^{p+f}\right] - E\left[R^{p+a}\right]E\left[R^{p+f}\right]$$
$$= n^{-1}\delta^{p+a\ p+f} + E\left[R_2^{p+a}R_2^{p+f}\right] + E\left[R_2^{p+a}R_1^{p+f}\right][2] + E\left[R_3^{p+a}R_1^{p+f}\right][2] - n^{-2}\mu^{p+a}\mu^{p+f} + O\left(n^{-3}\right) \quad (15)$$

The second term of (15) is

$$
\begin{aligned}
& n^2 E\left[R_2^{p+a} R_2^{p+f}\right] \\
= \; & \frac{1}{4}\alpha^{p+a\ p+f\ p+b\ p+b} - \frac{7}{36}\alpha^{p+a\ p+b\ p+c}\alpha^{p+f\ p+b\ p+c} + \frac{1}{36}\alpha^{p+a\ p+b\ p+b}\alpha^{p+f\ p+c\ p+c} - \frac{1}{4}\delta^{p+a\ p+f} \\
& +\omega^{kl}\left(\frac{1}{6}\gamma^{p+a,k;l}\alpha^{p+f\ p+b\ p+b}[2] + \frac{1}{2}\gamma^{p+a,k;p+b}\alpha^{lp+f\ p+b}[2] - \frac{1}{2}\gamma^{p+b,k;p+a}\alpha^{lp+f\ p+b}[2]\right) \\
& +\omega^{km}\omega^{lm}\left(\gamma^{p+a,k;p+f,l} - \gamma^{p+b,l;p+a}\gamma^{p+f,k;p+b}[2] + \gamma^{p+b,l;p+a}\gamma^{p+b,k;p+f} - \frac{1}{12}\gamma^{p+a,kl}\alpha^{p+f\ p+b\ p+b}[2]\right) \\
& +\omega^{kl}\omega^{mn}\left(\gamma^{p+a,k;l}\gamma^{p+f,m;n} + \gamma^{p+a,k;n}\gamma^{p+d,m;l}\right) \\
& +\omega^{kl}\omega^{mn}\omega^{k_1n}\left(-\frac{1}{2}\gamma^{p+a,k;l}\gamma^{p+f,mk_1}[2] - \gamma^{p+a,kk_1}\gamma^{p+f,m;l}[2]\right) \\
& +\frac{1}{4}\left(\omega^{km}\omega^{lm}\omega^{k_1m_1}\omega^{l_1m_1} + \omega^{km}\omega^{ln}\omega^{k_1m}\omega^{l_1n} + \omega^{km}\omega^{ln}\omega^{k_1n}\omega^{l_1m}\right)\gamma^{p+a,kl}\gamma^{p+f,k_1l_1} + O\left(n^{-1}\right).
\end{aligned}
$$

The third term of (15) is

$$
\begin{aligned}
n^2 E\left[R_2^{p+a} R_1^{p+f}\right] = \; & -\frac{1}{2}\alpha^{p+a\ p+b\ p+b\ p+f} + \frac{1}{2}\delta^{p+a\ p+b}\delta^{p+b\ p+f} + \frac{1}{3}\alpha^{p+a\ p+b\ p+c}\alpha^{p+b\ p+c\ p+f} \\
& -\gamma^{l;p+f;p+a,k}\omega^{kl} + \frac{1}{2}\gamma^{p+a,kl}\omega^{km}\omega^{ln}\alpha^{mnp+f} + \gamma^{p+a;p+b,l}\omega^{lm}\alpha^{mp+b\ p+f} + O\left(n^{-1}\right).
\end{aligned}
$$

The fourth term of (15) is

$$
\begin{aligned}
& n^2 E\left[R_3^{p+a} R_1^{p+f}\right] \\
= \; & \omega^{kl}\omega^{mn}\left(\gamma^{n,k;p+f}\gamma^{p+a,m;l}+\gamma^{n,k;l}\gamma^{p+a,m;p+f}\right)+\frac{1}{2}\left(\gamma^{p+b,k;l}\alpha^{p+a\ p+b\ p+f}+\gamma^{p+b,k;p+f}\alpha^{p+a\ p+b\ l}\right) \\
& -\frac{1}{2}\omega^{kl}\omega^{ml}\left(\gamma^{p+a,k;p+f,m}+\gamma^{p+a,k;p+b}\gamma^{p+b,m;p+f}+\gamma^{p+a,k;p+f}\gamma^{p+b,m;p+b}\right) \\
& +\frac{3}{8}\left(\alpha^{p+a\ p+c\ p+c\ p+f}+\alpha^{p+a\ p+b\ p+c}\alpha^{p+b\ p+c\ p+f}+\alpha^{p+a\ p+c\ p+f}\alpha^{p+b\ p+b\ p+c}-\delta^{p+a\ p+f}\right) \\
& +\omega^{kl}\left(\gamma^{p+f,k;p+a,l}+\gamma^{p+b,k;p+b}\alpha^{lp+a\ p+b}+\gamma^{p+b,k;p+f}\alpha^{lp+a\ p+b}\right) \\
& +\omega^{kl}\left(\gamma^{p+b;p+f,k}+\gamma^{p+f;p+b,k}\right)\alpha^{lp+a\ p+b}+\omega^{kl}\gamma^{p+c;p+c,k}\alpha^{lp+a\ p+f} \\
& -\frac{1}{2}\omega^{kl}\omega^{ml}\gamma^{p+a;p+b,k}\left(\gamma^{p+b;p+f,m}+\gamma^{p+f;p+b,m}\right)-\frac{1}{2}\omega^{kl}\omega^{ml}\gamma^{p+a;p+f,k}\gamma^{p+c;p+c,m} \\
& +\frac{8}{9}\alpha^{p+a\ p+b\ p+e}\alpha^{p+b\ p+e\ p+f}+\frac{4}{9}\alpha^{p+a\ p+e\ p+f}\alpha^{p+c\ p+c\ p+e}-\frac{3}{4}\alpha^{p+a\ p+b\ p+b\ p+f} \\
& +\omega^{mv}\omega^{nv}\left\{\begin{array}{c}
-\frac{1}{2}\omega^{kl}\omega^{nl}\gamma^{p+a,km}\gamma^{p+f,on}+\omega^{kl}\gamma^{p+f,km}\left(\gamma^{l,m;p+a}+\gamma^{l;p+a,m}\right) \\
+\frac{1}{2}\omega^{kl}\gamma^{l,mn}\gamma^{p+a;p+f,k}+\frac{1}{3}\alpha^{p+a\ p+f\ p+c}\gamma^{p+c,mn}-\frac{1}{2}\gamma^{p+a;p+f,mn}-\frac{1}{2}\gamma^{p+a,m;p+f,n} \\
+\frac{1}{2}\gamma^{p+a;p+c,m}\left(\gamma^{p+f;p+c,n}+\gamma^{p+c;p+f,n}\right)+\frac{1}{2}\gamma^{p+c;p+a,m}\gamma^{p+f;p+c,n}
\end{array}\right\} \\
& -\frac{1}{2}\omega^{kn}\omega^{ln}\omega^{vm}\gamma^{m,kl}\gamma^{p+a,v;p+f}+\frac{1}{4}\omega^{kn}\omega^{ln}\gamma^{p+b,kl}\alpha^{p+a\ p+b\ p+f}+\omega^{km}\omega^{nm}\omega^{lo}\gamma^{p+a,kl}\gamma^{p+f,n;o} \\
& -\omega^{kn}\omega^{lo}\gamma^{p+a,kl}\alpha^{mop+f}-\omega^{kv}\omega^{ln}\omega^{mn}\gamma^{p+a,kl}\gamma^{v,n;p+f}+\frac{1}{2}\omega^{kn}\omega^{ln}\gamma^{p+a,kl;p+f}+\alpha^{p+a\ p+b\ p+b\ p+f} \\
& +\omega^{lm}\omega^{nm}\left\{\gamma^{p+a;p+b,l}\left(\gamma^{p+f,n;p+b}+\gamma^{p+b,n;p+f}\right)+\gamma^{p+a;p+f,l}\gamma^{p+c,n;p+c}\right\} \\
& -2\omega^{lm}\left(\gamma^{p+a;p+b,l}\alpha^{mp+b\ p+f}+\gamma^{p+a;p+f,l}\alpha^{mp+c\ p+c}\right)-\omega^{ln}\omega^{mo}\gamma^{p+a,p+f,l}\gamma^{n,m;o} \\
& -\omega^{ln}\omega^{mk}\left(\gamma^{k;p+a,l}+\gamma^{p+a;k,l}\right)\gamma^{p+f,m;n}-\omega^{ln}\omega^{mn}\gamma^{p+a;p+b,l}\gamma^{p+b,m;p+f} \\
& -\frac{1}{2}\omega^{ln}\left(\gamma^{p+c;p+a,l}+\gamma^{p+a;p+c,l}\right)\alpha^{np+f\ p+c}+\omega^{lm}\gamma^{p+a;p+f,l;m} \\
& -\omega^{mk}\left\{\alpha^{kp+a\ p+b}\left(\gamma^{p+f,m;p+b}+\gamma^{p+b,m;p+f}\right)+\alpha^{kp+a\ p+f}\gamma^{p+c,m;p+c}\right\}-\frac{2}{3}\omega^{lm}\gamma^{p+c,l;m}\alpha^{p+a\ p+f\ p+c} \\
& -\frac{5}{6}\left(2\alpha^{p+a\ p+b\ p+c}\alpha^{p+b\ p+c\ p+f}+\alpha^{p+a\ p+f\ p+c}\alpha^{p+c\ p+d\ p+d}\right)+O\left(n^{-1}\right).
\end{aligned}
$$

Combining these results,

$$
cum\left(R^{p+a}, R^{p+f}\right)=n^{-1}\delta^{p+a\ p+f}+n^{-2}\Delta^{p+a\ p+f}+O\left(n^{-3}\right),
$$

where

$$
\begin{aligned}
\Delta^{p+a\ p+f} = \; & \frac{1}{2}\alpha^{p+a\ p+f\ p+b\ p+b}-\frac{1}{3}\alpha^{p+a\ p+b\ p+c}\alpha^{p+f\ p+b\ p+c}-\frac{1}{36}\alpha^{p+a\ p+f\ p+c}\alpha^{p+b\ p+b\ p+c} \\
& +\omega^{kl}\gamma^{l;p+f;p+a,k}[2]-\frac{1}{2}\omega^{km}\omega^{ln}\gamma^{p+a,kl}\alpha^{m\ n\ p+f}[2] \\
& +\omega^{lm}\left(\gamma^{p+b;p+b,l}\alpha^{m\ p+a\ p+f}-\frac{1}{2}\gamma^{p+f;p+b,l}\alpha^{m\ p+a\ p+b}\right)[2] \\
& +\omega^{km}\omega^{lm}\left(-\gamma^{p+a,k;p+f,l}+\frac{1}{6}\gamma^{p+b,kl}\alpha^{p+a\ p+f\ p+b}\right) \\
& -\frac{1}{3}\omega^{kl}\gamma^{p+b,k;l}\alpha^{p+a\ p+f\ p+b}-\omega^{kl}\gamma^{p+a;p+f,k}\alpha^{l\ p+b\ p+b}[2]+\omega^{kl}\omega^{mn}\omega^{vn}\gamma^{p+a,v;l}\gamma^{p+f,km}[2] \\
& -\frac{1}{2}\omega^{km}\omega^{ln}\omega^{k'm}\omega^{l'n}\gamma^{p+a,kl}\gamma^{p+f,k'l'}+\omega^{kl}\omega^{ml}\gamma^{p+a;p+b,k}\gamma^{p+b,m;p+f}.
\end{aligned}
$$

A.5 Third-order Cumulant of R

Using the results to derive the first and second-order cumulants, the third-order cumulant is written as

$$\begin{aligned} & cum\left(R^{p+a}, R^{p+b}, R^{p+d}\right) \\ = \; & E\left[R^{p+a}R^{p+b}R^{p+d}\right] - E\left[R^{p+a}\right]E\left[R^{p+b}R^{p+d}\right][3] + 2E\left[R^{p+a}\right]E\left[R^{p+b}\right]E\left[R^{p+d}\right] \\ = \; & E\left[R_1^{p+a}R_1^{p+b}R_1^{p+d}\right] - E\left[R_2^{p+a}\right]E\left[R_1^{p+b}R_1^{p+d}\right][3] + E\left[R_2^{p+a}R_1^{p+b}R_1^{p+d}\right][3] + O\left(n^{-3}\right), \end{aligned} \tag{16}$$

where $E\left[R^{p+a}\right]E\left[R^{p+b}R^{p+d}\right][3] = E\left[R^{p+a}\right]E\left[R^{p+b}R^{p+d}\right] + E\left[R^{p+b}\right]E\left[R^{p+a}R^{p+d}\right] + E\left[R^{p+d}\right]E\left[R^{p+a}R^{p+b}\right]$ and other terms are similarly defined. The first term of (16) is

$$n^2 E\left[R_1^{p+a}R_1^{p+b}R_1^{p+d}\right] = \alpha^{p+a\ p+b\ p+d} + O\left(n^{-1}\right).$$

The second term of (16) is

$$n^2 E\left[R_2^{p+a}\right]E\left[R_1^{p+b}R_1^{p+d}\right] = -\left\{\frac{1}{6}\alpha^{p+a\ p+b_1\ p+b_1} + \omega^{kl}\gamma^{l;p+a,k} - \frac{1}{2}\gamma^{p+a,kl}\omega^{km}\omega^{lm}\right\}\delta^{p+b\ p+d} + O\left(n^{-1}\right).$$

The third term of (16) is

$$\begin{aligned} n^2 E\left[R_2^{p+a}R_1^{p+b}R_1^{p+d}\right] = \; & -\frac{1}{3}\alpha^{p+a\ p+b\ p+d} - \frac{1}{6}\alpha^{p+a\ p+b_1\ p+b_1}\delta^{p+b\ p+d} \\ & -\omega^{kl}\gamma^{p+a,k;l}\delta^{p+b\ p+d} + \frac{1}{2}\gamma^{p+a,kl}\omega^{km}\omega^{ln}\delta^{mn}\delta^{p+b\ p+d} + O\left(n^{-1}\right). \end{aligned}$$

Combining these results, we obtain $cum\left(R^{p+a}, R^{p+b}, R^{p+d}\right) = O\left(n^{-3}\right)$.

A.6 Fourth-order Cumulant of R

In this subsection, let

$$\begin{aligned} t_1 &= \alpha^{p+a\ p+b\ p+c\ p+d} \\ t_2 &= \delta^{p+a\ p+b}\delta^{p+c\ p+d} + \delta^{p+a\ p+c}\delta^{p+b\ p+d} + \delta^{p+a\ p+d}\delta^{p+b\ p+c}, \\ t_3 &= \alpha^{p+a\ p+b_1\ p+b_1}\alpha^{p+b\ p+c\ p+d} + \alpha^{p+b\ p+b_1\ p+b_1}\alpha^{p+a\ p+c\ p+d}, \\ & \quad +\alpha^{p+c\ p+b_1\ p+b_1}\alpha^{p+a\ p+b\ p+d} + \alpha^{p+d\ p+b_1\ p+b_1}\alpha^{p+a\ p+b\ p+c}, \\ t_4 &= \alpha^{p+a\ p+b\ p+b_1}\alpha^{p+c\ p+d\ p+b_1} + \alpha^{p+a\ p+c\ p+b_1}\alpha^{p+b\ p+d\ p+b_1} + \alpha^{p+a\ p+d\ p+b_1}\alpha^{p+b\ p+c\ p+b_1}. \end{aligned}$$

Using the results to obtain the first, second, and third-order cumulants,

$$\begin{aligned} & cum\left(R^{p+a}, R^{p+b}, R^{p+c}, R^{p+d}\right) \\ = \; & E\left[R^{p+a}R^{p+b}R^{p+c}R^{p+d}\right] - E\left[R^{p+a}R^{p+b}\right]E\left[R^{p+c}R^{p+d}\right][3] - E\left[R^{p+a}\right]E\left[R^{p+b}R^{p+c}R^{p+d}\right][4] \\ & +2E\left[R^{p+a}\right]E\left[R^{p+b}\right]E\left[R^{p+c}R^{p+d}\right][6] - 6E\left[R^{p+a}\right]E\left[R^{p+b}\right]E\left[R^{p+c}\right]E\left[R^{p+d}\right] \\ = \; & E\left[R_1^{p+a}R_1^{p+b}R_1^{p+c}R_1^{p+d}\right] - E\left[R_1^{p+a}R_1^{p+b}\right]E\left[R_1^{p+c}R_1^{p+d}\right][3] + E\left[R_2^{p+a}R_1^{p+b}R_1^{p+c}R_1^{p+d}\right][4] \\ & -E\left[R_2^{p+a}R_1^{p+b}\right]E\left[R_1^{p+c}R_1^{p+d}\right][12] + E\left[R_2^{p+a}R_2^{p+b}R_1^{p+c}R_1^{p+d}\right][6] - E\left[R_2^{p+a}R_2^{p+b}\right]E\left[R_1^{p+c}R_1^{p+d}\right][6] \\ & +E\left[R_3^{p+a}R_1^{p+b}R_1^{p+c}R_1^{p+d}\right][4] - E\left[R_3^{p+a}R_1^{p+b}\right]E\left[R_1^{p+c}R_1^{p+d}\right][12] - E\left[R_2^{p+a}\right]E\left[R_1^{p+b}R_1^{p+c}R_1^{p+d}\right][4] \\ & -E\left[R_2^{p+a}\right]E\left[R_2^{p+b}R_1^{p+c}R_1^{p+d}\right][12] + 2E\left[R_2^{p+a}\right]E\left[R_2^{p+b}\right]E\left[R_1^{p+c}R_1^{p+d}\right][6] + O\left(n^{-4}\right). \end{aligned} \tag{17}$$

The first term of (17) is

$$n^3 E\left[R_1^{p+a} R_1^{p+b} R_1^{p+c} R_1^{p+d}\right] = \alpha^{p+a\ p+b\ p+c\ p+d} + O\left(n^{-1}\right).$$

The second term of (17) is

$$n^3 E\left[R_1^{p+a} R_1^{p+b}\right] E\left[R_1^{p+c} R_1^{p+d}\right][3] = \delta^{p+a\ p+b}\delta^{p+c\ p+d} + \delta^{p+a\ p+c}\delta^{p+b\ p+d} + \delta^{p+a\ p+d}\delta^{p+b\ p+c} + O\left(n^{-1}\right).$$

The third and fourth terms of (17) are

$$\begin{aligned}
& n^3 E\left[R_2^{p+a} R_1^{p+b} R_1^{p+c} R_1^{p+d}\right][4] - n^3 E\left[R_2^{p+a} R_1^{p+b}\right] E\left[R_1^{p+c} R_1^{p+d}\right][12] \\
= \quad & -6t_1 + 2t_2 - \frac{1}{6}t_3 + \frac{2}{3}t_4 + \frac{1}{2}\gamma^{p+a,kl}\omega^{km}\omega^{lm}\alpha^{p+b\ p+c\ p+d}[4] \\
& -\omega^{kl}\left\{\gamma^{p+a,k;l}\alpha^{p+b\ p+c\ p+d} + \gamma^{p+a,k;p+b}\alpha^{lp+c\ p+d} + \gamma^{p+a,k;p+c}\alpha^{lp+b\ p+d} + \gamma^{p+a,k;p+d}\alpha^{lp+b\ p+c}\right\}[4] \\
& + \left\{\gamma^{p+a;p+d,l}\omega^{lm}\alpha^{mp+b\ p+c} + \gamma^{p+a;p+c,l}\omega^{lm}\alpha^{mp+b\ p+d} + \gamma^{p+a;p+b,l}\omega^{lm}\alpha^{mp+c\ p+d}\right\}[4] + O\left(n^{-1}\right).
\end{aligned}$$

The fifth and sixth terms of (17) are

$$\begin{aligned}
& n^3 E\left[R_2^{p+a} R_2^{p+b} R_1^{p+c} R_1^{p+d}\right][6] - n^3 E\left[R_2^{p+a} R_2^{p+b}\right] E\left[R_1^{p+c} R_1^{p+d}\right][6] \\
= \quad & 3t_1 - t_2 + \frac{1}{6}t_3 - \frac{5}{9}t_4 + \frac{1}{3}\omega^{kl}\left(\gamma^{p+a,k;l}\alpha^{p+b\ p+c\ p+d} + \gamma^{p+b,k;l}\alpha^{p+a\ p+c\ p+d}\right)[6] \\
& + \frac{1}{2}\omega^{kl}\left(\gamma^{p+a,k;p+c}\alpha^{lp+b\ p+d} + \gamma^{p+a,k;p+d}\alpha^{lp+b\ p+c} + \gamma^{p+b,k;p+c}\alpha^{lp+a\ p+d} + \gamma^{p+b,k;p+d}\alpha^{lp+a\ p+c} + \right)[6] \\
& - \frac{1}{2}\omega^{kl}\left(\gamma^{p+c,k;p+a}\alpha^{lp+b\ p+d} + \gamma^{p+c,k;p+b}\alpha^{lp+a\ p+d} + \gamma^{p+d,k;p+a}\alpha^{lp+b\ p+c} + \gamma^{p+d,k;p+b}\alpha^{lp+a\ p+c}\right)[6] \\
& + \omega^{kl}\omega^{ml}\left(\gamma^{p+a,k;p+c}\gamma^{p+b,m;p+d} + \gamma^{p+a,k;p+d}\gamma^{p+b,m;p+c} + \gamma^{p+c,k;p+a}\gamma^{p+d,m;p+b} + \gamma^{p+d,k;p+a}\gamma^{p+c,m;p+b}\right)[6] \\
& - \omega^{kl}\omega^{ml}\left(\gamma^{p+a,k;p+c}\gamma^{p+d,m;p+b} + \gamma^{p+a,k;p+d}\gamma^{p+c,m;p+b} + \gamma^{p+c,k;p+a}\gamma^{p+b,m;p+d} + \gamma^{p+d,k;p+a}\gamma^{p+b,m;p+c}\right)[6] \\
& - \frac{1}{6}\omega^{km}\omega^{lm}\left(\gamma^{p+a,kl}\alpha^{p+b\ p+c\ p+d} + \gamma^{p+b,kl}\alpha^{p+a\ p+c\ p+d}\right)[6] + O\left(n^{-1}\right).
\end{aligned}$$

The seventh and eighth terms of (17) are

$$n^3 E\left[R_3^{p+a} R_1^{p+b} R_1^{p+c} R_1^{p+d}\right][4] - n^3 E\left[R_3^{p+a} R_1^{p+b}\right] E\left[R_1^{p+c} R_1^{p+d}\right][12] = 2t_1 - \frac{1}{9}t_4 + O\left(n^{-1}\right).$$

Using the results to derive the first, second, and third cumulants, the last three terms of (17) are of order $O\left(n^{-4}\right)$. Combining these results, we obtain $cum\left(R^{p+a}, R^{p+b}, R^{p+c}, R^{p+d}\right) = O\left(n^{-4}\right)$.

A.7 Proof of Theorem 3.1

In Section 3.1, we have

$$n^{-1}T_n = \left(R_1 + R_2 + R_3\right)^{p+a}\left(R_1 + R_2 + R_3\right)^{p+a} + O_p\left(n^{-5/2}\right),$$

where R_1, R_2 and R_3 are given by (3), (4) and (14), respectively. The first four cumulants of $R = R_1 + R_2 + R_3$ are given by (6), (7) and (8), respectively.

Once we expand $n^{-1}T_n$ in (2) and compute its cumulants, the derivation of an Edgeworth expansion for the distribution of T_n is exactly the same as that of Chen and Cui (2007, Theorems 1 and 2).

A.8 Proof of Theorem 3.2

The proof is similar to that of Chen and Cui (2007, Theorem 3). Pick any $t \in \mathbb{R}$. Theorem 3.1 (i) implies

$$\Pr\left\{T_n \le t\right\} = F_{r-p}\left(t\right) - n^{-1} t f_{r-p}\left(t\right) B_c + O\left(n^{-2}\right),$$

where F_{r-p} is the cumulative distribution function of the $\chi^2\left(r-p\right)$ distribution. Let T_n^* be a bootstrap resample of T_n using the implied probabilities $\{\hat{p}_i\}_{i=1}^n$ in (10). By applying the same argument in Brown and Newey (2002, pp. 510-511) (i.e., applying the same argument for Theorem 3.1 (i) to T_n^* given the original sample $\mathbf{X}_n = (X_1, \ldots, X_n)$), we can obtain

$$\Pr\left\{\left. T_n^* < t \right| \mathbf{X}_n\right\} = F_{r-p}\left(t\right) - n^{-1} \hat{B}_c^* t f_{r-p}\left(t\right) + O_p\left(n^{-2}\right), \tag{18}$$

where $\hat{B}_c^*$ is a bootstrap counterpart of B_c obtained by replacing all population moments with the weighted averages based on $\{\hat{p}_i\}_{i=1}^n$. Since (i) $\hat{\beta}_c$ is a simulation estimator of $E\left[\left. T_n^* \right| \mathbf{X}_n\right]$ (where the error by simulation is asymptotically negligible for suitably chosen B), (ii) $\left(r-p\right)^{-1} E\left[\left. T_n^* \right| \mathbf{X}_n\right] = 1 + n^{-1}\hat{B}_c^* + O_p\left(n^{-3/2}\right)$ by (18), and (iii) $\hat{B}_c^* = B_c + O_p\left(n^{-1/2}\right)$ by Brown and Newey (2002, Theorem 1), we obtain

$$\hat{\beta}_c = 1 + n^{-1} B_c + O_p\left(n^{-3/2}\right).$$

Therefore, an application of the delta method (Hall, 1992, Section 2.7) yields

$$\Pr\left\{T_n < c_\alpha \hat{\beta}_c\right\} = \Pr\left\{T_n < c_\alpha\left(1 + n^{-1} B_c\right)\right\} + O\left(n^{-3/2}\right) = 1 - \alpha + O\left(n^{-3/2}\right).$$

B Tables and Figures

ρ	δ^2	r	GMM	EL	BEL	AEL
0.2	100	2	0.049	0.057	0.053	0.045
	100	5	0.041	0.054	0.048	0.040
	100	10	0.037	0.099	0.058	0.031
	20	2	0.039	0.042	0.049	0.041
	20	5	0.040	0.051	0.056	0.047
	20	10	0.034	0.075	0.049	0.031
0.8	100	2	0.045	0.047	0.053	0.048
	100	5	0.051	0.068	0.053	0.047
	100	10	0.047	0.092	0.055	0.027
	20	2	0.062	0.048	0.047	0.037
	20	5	0.095	0.061	0.061	0.053
	20	10	0.125	0.080	0.050	0.020

Table 1: Rejection frequencies of tests at 5% level with $n = 200$ (normal case)

ρ	δ^2	r	GMM	EL	BEL	AEL
0.2	100	2	0.041	0.047	0.046	0.042
	100	5	0.047	0.071	0.056	0.046
	100	10	0.039	0.110	0.060	0.012
	20	2	0.050	0.049	0.058	0.055
	20	5	0.040	0.056	0.052	0.042
	20	10	0.026	0.082	0.054	0.015
0.8	100	2	0.057	0.060	0.048	0.046
	100	5	0.045	0.071	0.057	0.043
	100	10	0.054	0.127	0.064	0.009
	20	2	0.049	0.047	0.050	0.040
	20	5	0.077	0.059	0.051	0.041
	20	10	0.129	0.124	0.066	0.016

Table 2: Rejection frequencies of tests at 5% level with $n = 200$ (standardized $t\,(5)$ case)

ρ	δ^2	r	GMM	EL	BEL	AEL
0.2	100	2	0.048	0.056	0.058	0.052
	100	5	0.042	0.076	0.062	0.044
	100	10	0.032	0.126	0.060	0.013
	20	2	0.046	0.047	0.056	0.053
	20	5	0.035	0.058	0.048	0.044
	20	10	0.039	0.092	0.057	0.019
0.8	100	2	0.055	0.061	0.057	0.052
	100	5	0.049	0.077	0.054	0.041
	100	10	0.048	0.125	0.063	0.011
	20	2	0.058	0.051	0.051	0.044
	20	5	0.082	0.084	0.064	0.053
	20	10	0.139	0.129	0.066	0.015

Table 3: Rejection frequencies of tests at 5% level with $n = 200$ (standardized $\chi^2(3)$ case)

n	r	GMM	EL	BEL	AEL
100	2	0.040	0.063	0.056	0.052
	3	0.135	0.105	0.074	0.047
	5	0.244	0.161	0.077	0.016
	7	0.370	0.255	0.087	0.000
200	2	0.048	0.054	0.054	0.052
	3	0.106	0.069	0.057	0.048
	5	0.243	0.114	0.072	0.035
	7	0.326	0.167	0.089	0.008

Table 4: Rejection frequencies of tests at 5% level with $\sigma = 0.2$

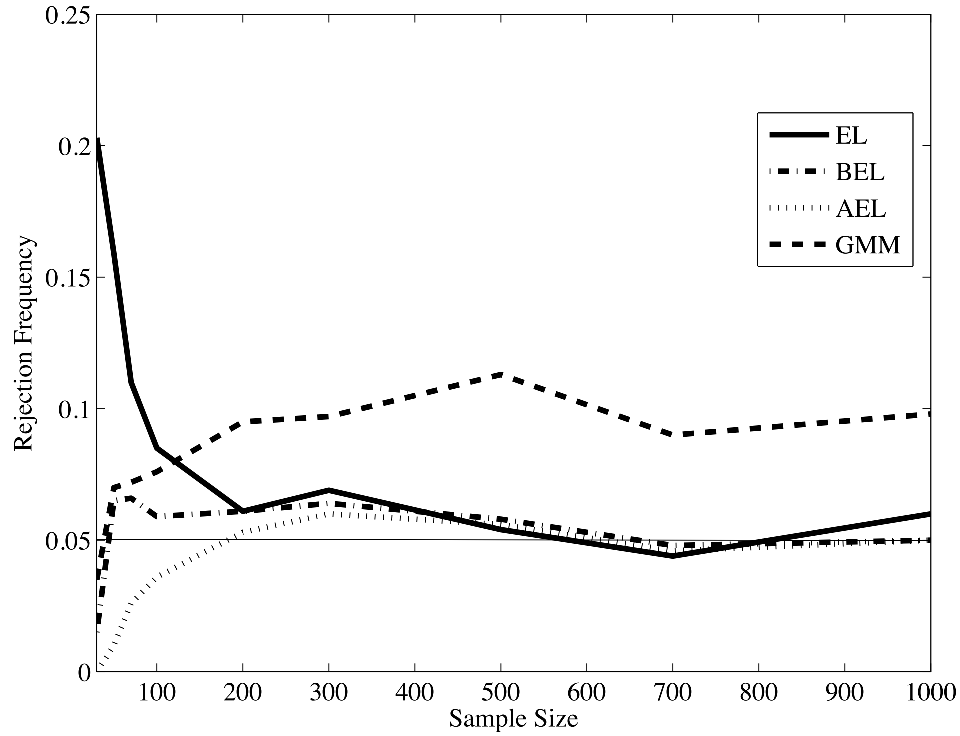

Figure 1: Rejection frequencies for different values of sample sizes with $r = 5$, $\rho = 0.8$, and $\delta^2 = 20$

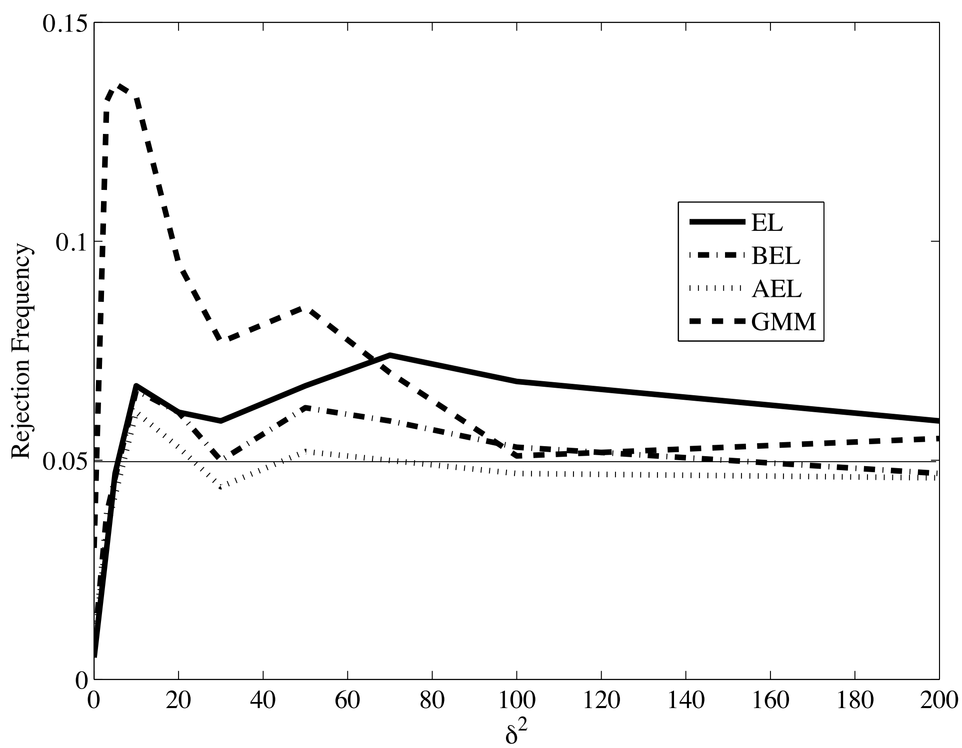

Figure 2: Rejection frequencies for different values of δ^2 with $r = 5$, $\rho = 0.8$, and $n = 200$

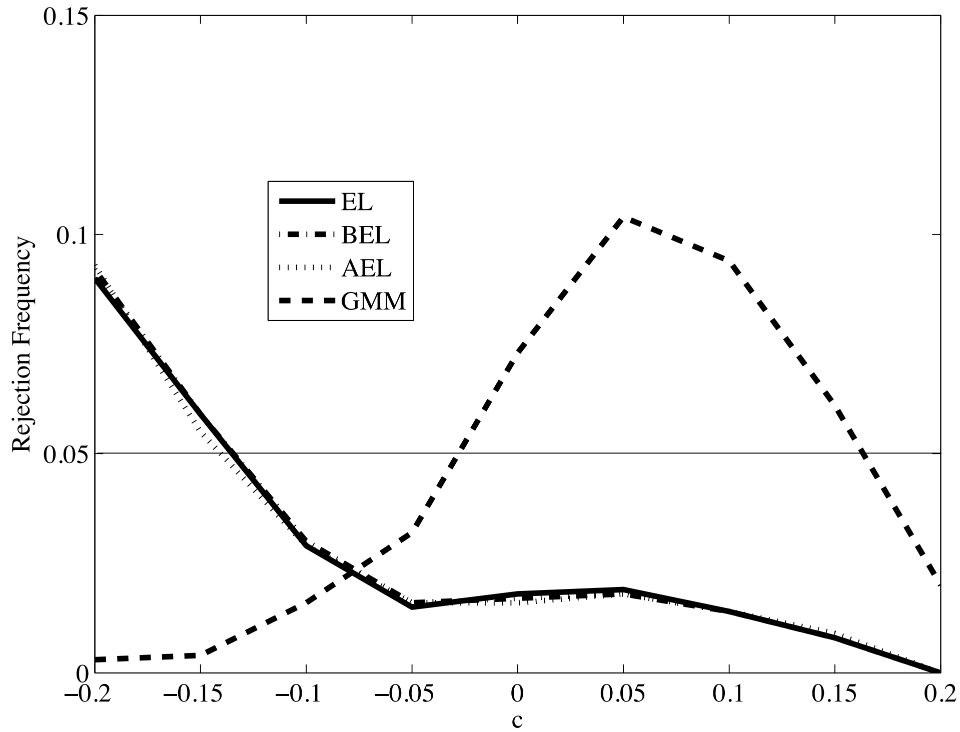

Figure 3: Frequencies of the event "not reject in the first stage but reject in the second stage" with $r = 5$, $\rho = 0.8$, $\delta^2 = 20$, and $n = 200$

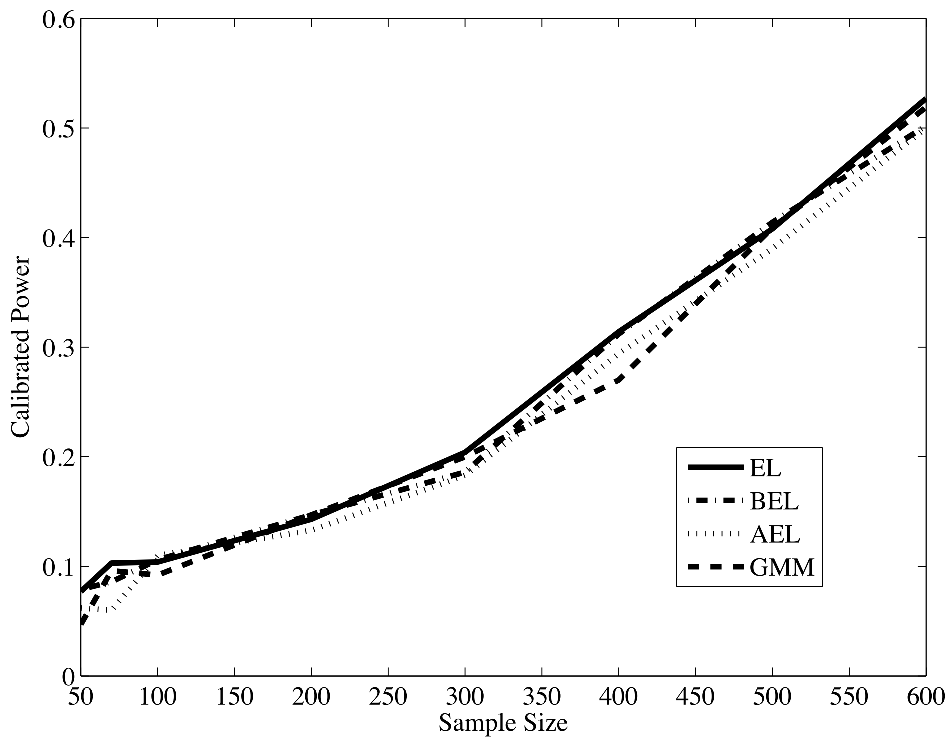

Figure 4: Calibrated power with $r = 5$, $\rho = 0.8$, $\delta^2 = 20$, and $n = 200$

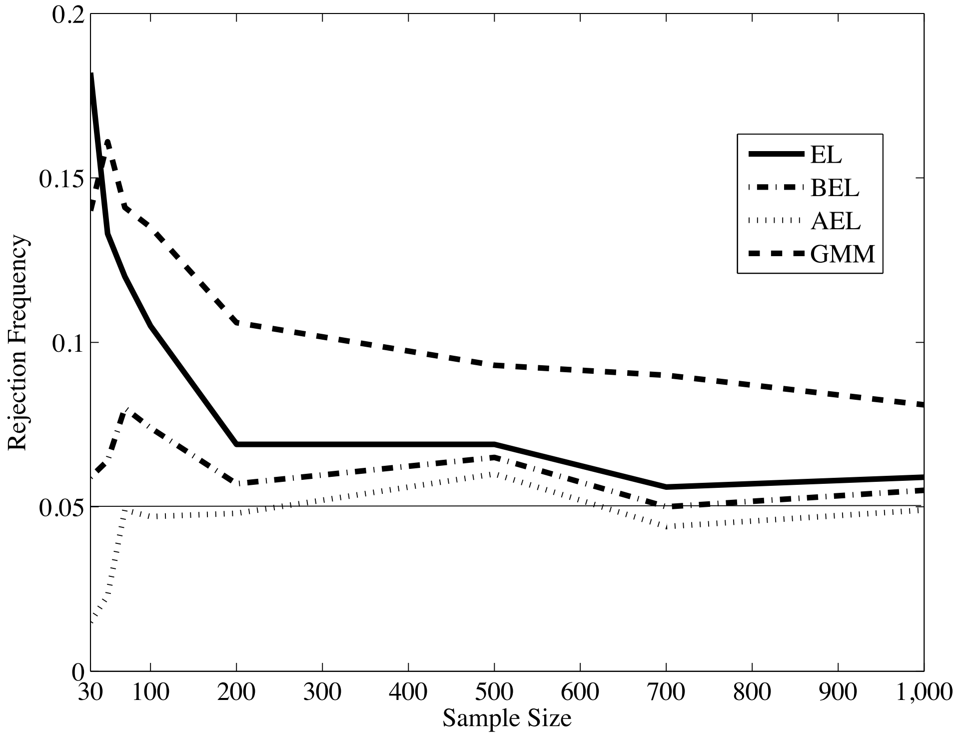

Figure 5: Rejection frequencies for different sample sizes with $\sigma = 0.2$ and $r = 3$

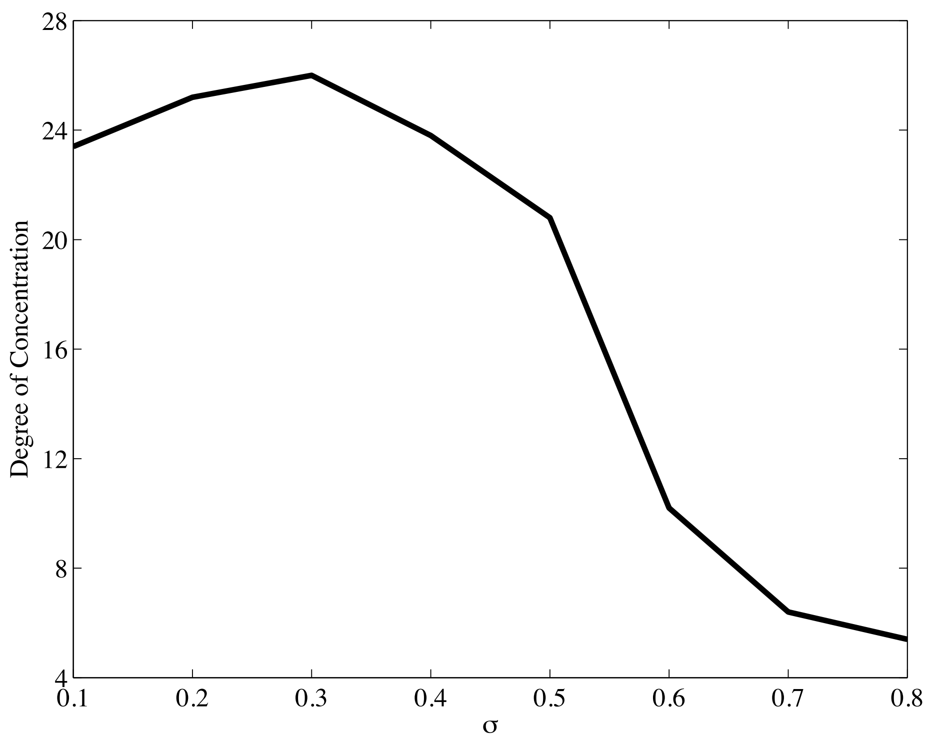

Figure 6: Degree of concentration $nG'V^{-1}G$ for different values of σ with $r = 3$

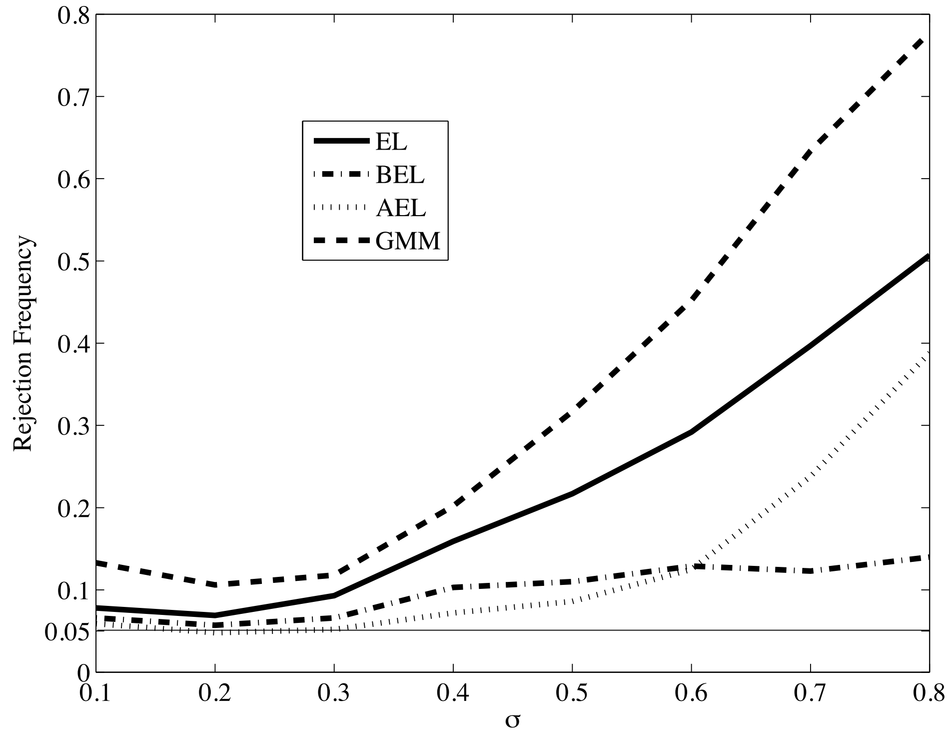

Figure 7: Rejection frequencies for different values of σ with $n = 200$ and $r = 3$

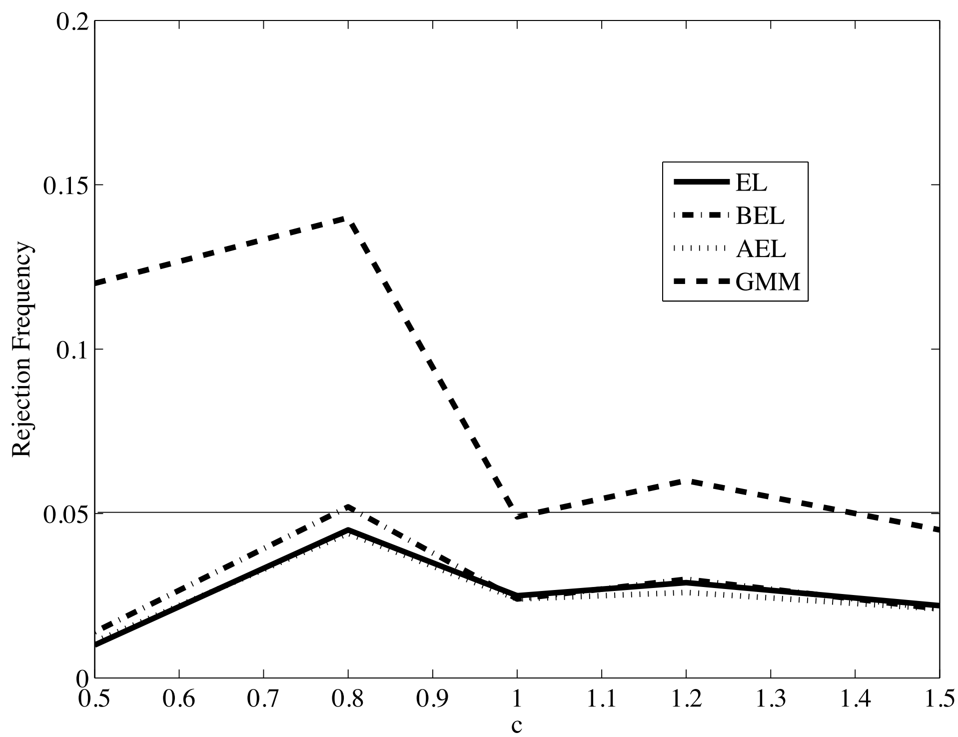

Figure 8: Frequencies of the event “not reject in the first stage but reject in the second stage” with $n = 200$, $\sigma = 0.2$, and $r = 3$

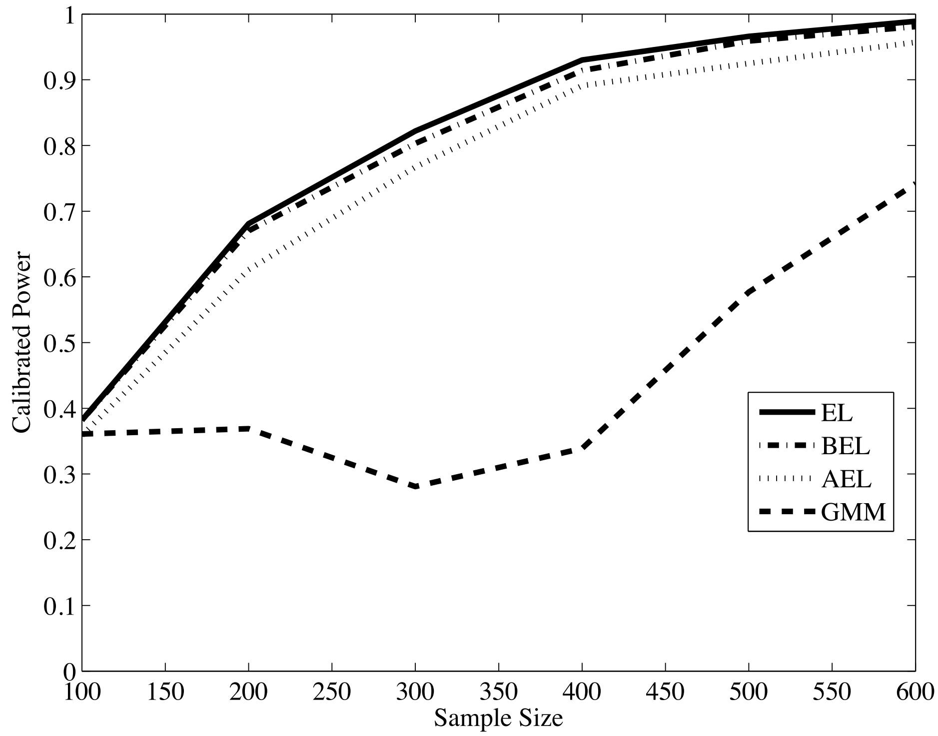

Figure 9: Calibrated power for Case 1

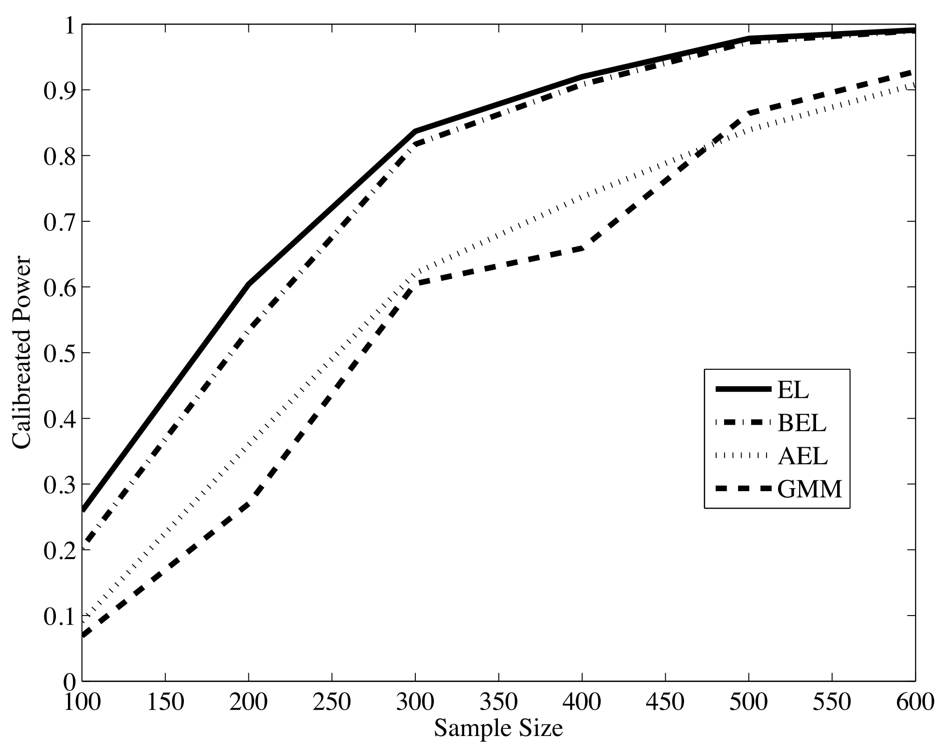

Figure 10: Calibrated power for Case 2

References

[1] Baggerly, K. A. (1998) Empirical likelihood as a goodness-of-fit measure, *Biometrika*, 85, 535-547.

[2] Bravo, F. (2004) Empirical likelihood based inference with applications to some econometric models, *Econometric Theory*, 20, 231-264.

[3] Brown, B. W. and W. K. Newey (2002) Generalized method of moments, efficient bootstrapping, and improved inference, *Journal of Business and Economic Statistics*, 20, 507-517.

[4] Canay, I. A. and T. Otsu (2011) Hodges-Lehmann optimality for testing moment condition models, Working paper.

[5] Chen, S. X. and H. Cui (2007) On the second-order properties of empirical likelihood with moment restrictions, *Journal of Econometrics*, 141, 492-516.

[6] Chen, S. X. and P. Hall (1993) Smoothed empirical likelihood confidence intervals for quantiles, *Annals of Statistics*, 21, 1166-1181.

[7] DiCiccio, T. J., Hall, P. and J. Romano (1991) Empirical likelihood is Bartlett-correctable, *Annals of Statistics*, 19, 1053-1061.

[8] Guggenberger, P. and G. Kumar (2011) On the size distortion of tests after an overidentifying restrictions pretest, forthcoming in *Journal of Applied Econometrics*.

[9] Hall, P. (1992) *The Bootstrap and Edgeworth Expansion*, Springer.

[10] Hall, P. and J. L. Horowitz (1996) Bootstrap critical values for tests based on generalized-method-of-moments estimators, *Econometrica*, 64, 891-916.

[11] Hansen, L. P. (1982) Large sample properties of generalized method of moments estimators, *Econometrica*, 50, 1029-1054.

[12] Horowitz, J. L. (1998) Bootstrap methods for median regression models, *Econometrica*, 66, 1327-1351.

[13] Kitamura, Y. (2001) Asymptotic optimality of empirical likelihood for testing moment restrictions, *Econometrica*, 69, 1661-1672.

[14] Kitamura, Y. (2007) Empirical likelihood methods in econometrics: theory and practice, in: Blundell, R., Newey, W. K. and T. Persson (eds.), *Advances in Economics and Econometrics: Theory and Applications, Ninth World Congress*, volume III, Cambridge University Press, Cambridge.

[15] Liu, Y. and J. Chen (2010) Adjusted empirical likelihood with high-order precision, *Annals of Statistics*, 38, 1341-1362.

[16] Otsu, T. (2010) On Bahadur efficiency of empirical likelihood, *Journal of Econometrics*, 157, 248-256.

[17] Qin, J. and J. Lawless (1994) Empirical likelihood and general estimating equations, *Annals of Statistics*, 22, 300-325.

[18] Stock, J. H. and J. H. Wright (2000) GMM with weak identification, *Econometrica*, 68, 1055-1096.

[19] Whang, Y.-J. (2006) Smoothed empirical likelihood methods for quantile regression models, *Econometric Theory*, 22, 173-205.