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Specification Testing for Nonlinear Cointegrating Regression *

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Abstract

We provide a limit theory for a general class of kernel smoothed U statistics that may be used for specification testing in time series regression with nonstationary data. The framework allows for linear and nonlinear models of cointegration and regressors that have autoregressive unit roots or near unit roots. The limit theory for the specification test depends on the self intersection local time of a Gaussian process. A new weak convergence result is developed for certain partial sums of functions involving nonstationary time series that converges to the intersection local time process. This result is of independent interest and useful in other applications.

Key words and phrases: Intersection local time, Kernel regression, Nonlinear nonparametric model, Ornstein-Uhlenbeck process, Specification tests, Weak convergence.
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1 Introduction

One of the advantages of nonparametric modeling is the opportunity for specification testing of particular parametric models against general alternatives. The past three decades have witnessed many developments in such specification tests involving nonparametric and semiparametric techniques that allow for independent, short memory, and long range dependent data. Recent research on the nonparametric modeling of nonstationary data

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opens up some new possibilities that seem relevant to applications in many fields including nonlinear diffusion models in continuous time (Bandi and Phillips, 2003; 2007) and cointegration models in economics and finance.

Cointegration models were originally developed in a linear parametric framework (Engle and Granger, 1987) in econometrics that has been widely used in applications. That framework was extended in Park and Phillips (1999, 2001) to allow for nonlinear parametric formulations under certain restrictions on the function nonlinearity. While considerably broadening the class of allowable nonstationary models, the potential for parametric misspecification in these models is still present and is important to test in applied work.

The hypothesis of linear cointegration is of particular interest in this context given the vast empirical literature. Recent papers by Karlsen, Mykelbust and Tjøstheim (2007), Wang and Phillips (2009a, 2009b, 2010), and Schienle (2008) have developed asymptotic theory for nonparametric kernel regression of nonlinear cointegrated systems. This work facilitates the comparison of various parametric specifications against a more general nonparametric nonlinear alternative. Such comparisons may be based on weighted sums of squared differences between the parametric and nonparametric estimates of the system or on a kernel-based U statistic test which uses a smoothed version of the parametric estimator in its construction (e.g, Gao, 2007, chapter 3).

A major obstacle in the development of such specification tests is the technical difficulty of developing a limit theory for these weighted sums which typically involve kernel functions with multiple nonstationary regressor arguments. Few results are currently available and because of this shortage, attempts to develop specification tests for nonlinear models with nonstationarity have been highly specific and do not involve nonparametric alternatives or kernel methods. Some examples of recent work in parametric models include Choi and Saikonnen (2004, 2009), Marmer (2008), Hong and Phillips (2010) and Kasparis and Phillips (2009). An exception is the recent work for testing linearity in autoregression and parametric time series regression by Gao et al. (2009a, 2009 b) who obtained a limit distribution theory for a kernel based specification test in a setting that involves martingale difference errors and random walk regressors.

The present paper makes a similar contribution and seeks to provide a general theory of specification tests that is applicable for a wider class of nonstationary regressors that includes both unit root and near unit root processes. The paper contributes to this emerging literature in two ways. First, we provide a limit theory for a general class of kernel based specification tests of parametric nonlinear models that allows for near unit root processes driven by short memory (linear process) errors. This limit theory should be widely applicable to specification testing in nonlinear cointegrated systems.

Second, the limit theory of the specification test involves the self-intersection local time of a Gaussian limit process. The result requires establishing weak convergence to this self-intersection local time process, which is of independent interest, and a feasible central limit theorem involving an empirical estimator of the intersection local time that can be used to construct the test statistic. Thus, the results provide some new theory for intersection local time, weak convergence, and specification test asymptotics that are relevant in applications.

The paper is organized as follows. Section 2 lays out the nonparametric and parametric models and assumptions. Section 3 gives the main results on specification test limit theory. Section 4 provides the weak convergence theory for intersection local time. Section 5 gives proofs of the local time limit theory, Section 6 gives proofs of the main results of the paper, and Section 7 provides some supplemental technical results which are needed for the development.

2 Model and Assumptions

We consider the nonlinear cointegrating regression model

$$y_{t+1} = f(x_t) + u_{t+1}, \quad t = 1, 2, ..., n, \tag{2.1}$$

where u_t is a stationary error process and x_t is a nonstationary regressor. We are interested in testing the null hypothesis:

$$H_0 : \ f(x) = f(x, \theta), \quad \theta \in \Omega_0,$$

for $x \in R$, where $f(x, \theta)$ is a given real function indexed by a vector θ of unknown parameters which lie in the parameter space Ω_0.

To test H_0 we make use of the following kernel-smoothed test statistic

$$S_n = \sum_{s,t=1, s\neq t}^{n} \hat{u}_{t+1}\hat{u}_{s+1}\, K\big[(x_t - x_s)/h\big], \tag{2.2}$$

involving the parametric regression residuals $\hat{u}_{t+1} = y_{t+1} - f(x_t, \hat{\theta})$, where $K(x)$ is a non-negative real kernel function, h is a bandwidth satisfying $h \equiv h_n \to 0$ as the sample size $n \to \infty$ and $\hat{\theta}$ is a parametric estimator of θ that is consistent under the null.

The statistic S_n in (2.2) has commonly been applied to test parametric specifications in stationary time series regression (see Gao, 2007) and was used by Gao et al (2009a, 2009b) to test for linearity in autoregression and a parametric conditional mean function in time series regression involving a random walk regressor.

S_n is a weighted U statistic with kernel weights that depend on standardized differentials $(x_t - x_s)/h$ of the regressor. The weights focus attention in the statistic on those components in the sum where the nonstationary regressor x_t nearly intersects itself. This smoothing scheme gives prominence to product components $\hat{u}_{t+1}\hat{u}_{s+1}$ in the sum where s and t may differ considerably but for which the corresponding regressor process takes similar values (that is, $x_t, x_s \simeq x$ for some x), thereby enabling a test of H_0.

The difficulty in the development of an asymptotic theory for S_n stems from the presence of the kernel weights $K((x_t - x_s)/h)$ which focus on the self intersection properties of x_t in the sample. As $n \to \infty$, this translates into the corresponding self intersection properties of the stochastic process to which a standardized version of x_t converges. To establish asymptotics for S_n, we need to account for the limit behavior of this self intersection factor, which leads to a limit theory involving the self intersection local time of a Gaussian process.

We use the following assumptions in our development.

Assumption 1. (i) $\{\epsilon_t\}_{t\in\mathbf{Z}}$ *is a sequence of independent and identically distributed (iid) continuous random variables with* $E\epsilon_0 = 0$, $E\epsilon_0^2 = 1$, *and with the characteristic function* $\varphi(t)$ *of* ϵ_0 *satisfying* $|t||\varphi(t)| \to 0$, *as* $|t| \to \infty$. *(ii)*

$$x_t = \rho\, x_{t-1} + \eta_t, \quad x_0 = 0, \quad \rho = 1 + \kappa/n, \quad 1 \le t \le n, \tag{2.3}$$

where κ *is a constant and* $\eta_t = \sum_{k=0}^{\infty} \phi_k \epsilon_{t-k}$ *with* $\phi \equiv \sum_{k=0}^{\infty} \phi_k \neq 0$ *and* $\sum_{k=0}^{\infty} k^{1+\delta}|\phi_k| < \infty$ *for some* $\delta > 0$.

Assumption 2. (i) $\{u_t, \mathcal{F}_t\}_{t\ge 1}$, *where* $\mathcal{F}_t$ *is a sequence of increasing* σ*-fields which is independent of* $\epsilon_k, k \ge t+1$, *forms a martingale difference satisfying* $E(u_{t+1}^2|\mathcal{F}_t) \to_{a.s.} \sigma^2 > 0$ *as* $t \to \infty$ *and* $\sup_{t\ge 1} E(|u_{t+1}|^4 \mid \mathcal{F}_t) < \infty$. (ii) x_t *is adapted to* $\mathcal{F}_t$ *and there exists a correlated vector Brownian motion* (W, V) *such that*

$$\Big(\frac{1}{\sqrt{n}}\sum_{j=1}^{[nt]} \epsilon_j,\ \frac{1}{\sqrt{n}\sigma}\sum_{j=1}^{[nt]} u_{j+1}\Big) \Rightarrow_D (W, V) \tag{2.4}$$

on $D[0,1]^2$ *as* $n \to \infty$.

Assumption 3. *$K(x)$ is a nonnegative real function satisfying $\sup_x K(x) < \infty$ and $\int |x|^{\max\{[\beta]+1,k+1\}} K(x)dx < \infty$, where $\beta \geq 0$ and $k \geq 0$ appear below in Assumption 4.*

Assumption 4. (i) *Under the null hypothesis H_0, there is a sequence of positive real numbers δ_n satisfying $\delta_n \to 0$ as $n \to \infty$ such that $\sup_{\theta_0 \in \Omega_0} ||\widehat{\theta} - \theta_0|| = o_P(\delta_n)$, where $||\cdot||$ denotes the Euclidean norm.* (ii) *There exists some $\varepsilon_0 > 0$ such that $\frac{\partial^2 f(x,\theta)}{\partial \theta^2}$ is continuous in both $x \in R$ and $\theta \in \Theta_0$, where $\Theta_0 = \{\theta : ||\theta - \theta_0|| \leq \varepsilon_0, \theta_0 \in \Omega_0\}$. (iii) Uniformly for $\theta_0 \in \Omega_0$,*

$$\left|\left|\frac{\partial f(x,\theta)}{\partial \theta}|_{\theta=\theta_0}\right|\right| + \left|\left|\frac{\partial^2 f(x,\theta)}{\partial \theta^2}|_{\theta=\theta_0}\right|\right| \leq C(1+|x|^\beta),$$

for some constants $\beta \geq 0$ and $C > 0$. (iv) Uniformly for $\theta_0 \in \Omega_0$, there exist $0 < \gamma \leq 1$ and an integer $k \geq 0$ such that

$$\|g(x+y,\theta_0) - g(x,\theta_0)\| \leq C|y| \begin{cases} 1+|x|^{\beta-1}+|y|^k, & \text{if } \beta > 0, \\ 1+|x|^{\gamma-1}+|y|^k, & \text{if } \beta = 0, \end{cases}$$

for any $x, y \in R$, where $g(x,\theta) = \frac{\partial f(x,\theta)}{\partial \theta}$.

Assumption 5. *$nh^2 \to \infty$, $\delta_n^2 n^{1+\beta}\sqrt{h} \to 0$ and $nh^4 \log^2 n \to 0$, where β and δ_n^2 are defined as in Assumption 4. Also $E|\epsilon_0|^\nu < \infty$, where $\nu = \max\{[4\beta]+1, 2[\beta]+2\}$ with $[\beta]$ denoting the integer part of β.*

Assumption 1 allows for both a unit root ($\kappa = 0$) and a near unit root ($\kappa \neq 0$) regressor by virtue of the localizing coefficient κ and is standard in the near integrated regression framework (Phillips, 1987, 1988; Chan and Wei, 1987). Compared to the estimation theory developed in Wang and Phillips (2009a, b) and for technical convenience in the present work we impose the stronger summability condition $\sum_{k=0}^{\infty} k^{1+\delta}|\phi_k| < \infty$ for some $\delta > 0$ on the coefficients of the linear process $\eta_t = \sum_{k=0}^{\infty} \phi_k \epsilon_{t-k}$ driving the regressor x_t. Under these conditions, it is well known that the standardized process $x_{[nt],n} = x_{[nt]}/\sqrt{n}\phi$ converges weakly to the Gaussian process $G(t) = \int_0^t e^{\kappa(t-s)}dW(s)$, where $W(t)$ is a standard Brownian motion. See (4.2) below.

Assumption 2 (i) is a standard martingale difference condition on the equation innovations u_t, so that $cov(u_{t+1}, x_t) = E[x_t E(u_{t+1} \mid \mathcal{F}_t)] = 0$. Wang and Phillips (2009b) allowed for endogeneity in their nonparametric structure, so the equation error could be serially dependent and cross-correlated with x_t for $|t-s| \leq m_0$ for some finite m_0. It is not clear at the moment if the results of the present paper on testing extend to the more general error structure considered in Wang and Phillips (2009b). Assumption 2 (ii) is a standard functional law for partial sums of linear processes (e.g. Phillips and Solo, 1992).

Assumption 3 is a standard condition on $K(x)$ as in the stationary situation. The integrability condition is weaker than the common alternative requirement that $K(x)$ has compact support.

As seen in Assumption 5, the sequence δ_n in Assumption 4(i) may be chosen as $\delta_n^2 = n^{-(1+\beta)/2}h^{-1/8}$. As $h \to 0$, Assumption 4(i) holds under very general conditions, such as those of Theorem 5.2 in Park and Phillips (2001). Roughly speaking, we may choose $\hat{\theta}$ under the null such that $\sup_{\theta_0 \in \Omega_0} ||\widehat{\theta} - \theta_0|| = O_P(n^{-(1+\beta)/2})$, under our Assumption 4(ii)-(iv). Assumptions 4(ii)-(iv) are quite weak and include a wide class of functions. Typical examples include polynomial forms like $f(x, \theta) = \theta_1 + \theta_2 x + ... + \theta_k x^{k-1}$, where $\theta = (\theta_1, ..., \theta_k)$, power functions like $f(x, \alpha, \beta, \gamma) = \alpha + \beta\, x^\gamma$, shift functions like $f(x, \theta) = x(1 + \theta x)I(x \geq 0)$, and weighted exponentials such as $f(x, \alpha, \beta) = (\alpha + \beta\, e^x)/(1 + e^x)$. However, Assumption 4 excludes models where $f(x, \theta)$ is integrable, because parametric rates of convergence are known to be $O\left(n^{1/4}\right)$ in this case (see Park and Phillips, 2001). It seems that cases with integrable $f(x, \theta)$ require different techniques and these are left for future investigation.

As in estimation theory, the condition in Assumption 5 that the bandwidth h satisfy $nh^2 \to \infty$ is necessary. The further condition that $nh^4 \log^2 n \to 0$ restricts the choice of h and, at least with the techniques used here, it seems difficult to relax. We also impose a higher moment condition on the innovation ϵ_0 in Assumption 5 which helps in the development of the limit theory.

3 Main Results on Specification

Our main result shows that S_n has a mixture normal distribution with a (variance) mixing variate that depends on the intersection local time process of $G\,(t) = \int_0^t e^{\kappa(t-s)} dW(s)$. This process is defined as

$$\begin{aligned} L_G(t, u) &= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \int_0^t \mathbf{1}[|(G(x) - G(y)) - u| < \varepsilon] dx dy \\ &= \int_0^t \int_0^t \delta_u [G(x) - G(y)] dx dy, \end{aligned} \tag{3.1}$$

where δ_u is the dirac function. $L_G(t, u)$ characterizes the amount of time over the interval $[0, t]$ that the process $G\,(t)$ spends at a distance u from itself, and is well defined as shown in Section 5. When $u = 0$, $L_G(t, 0)$ describes the self-intersection time of the process $G\,(t)$. Using the definition of the dirac function, the extended occupation times formula (e.g. Revuz and Yor, 1999, p.232), and integration by parts with the local time measure, we may write

$$
\begin{aligned}
L_G(t,0) &= 2\int_0^t\int_0^y \delta_0[G(x)-G(y)]dxdy \\
&= 2\int_0^t \ell_G\left(s,G\left(s\right)\right)ds \\
&= 2\int_{-\infty}^{\infty}\int_0^t \ell_G\left(s,a\right)d\ell_G\left(s,a\right)da \\
&= \int_{-\infty}^{\infty}\ell_G\left(t,a\right)^2 da, \qquad (3.2)
\end{aligned}
$$

where $\ell_G\left(t,a\right)$ is the local time spent by the process G at a over the time interval $[0,t]$, viz.

$$\ell_G\left(t,a\right)=\int_0^t \delta_a[G(s)]ds=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t \mathbf{1}[|G(s)-a|<\varepsilon]ds.$$

The process $\ell_G\left(s,G\left(s\right)\right)$ is the local time that the process G has spent at its current position $G\left(s\right)$ over the time interval $[0,s]$. It appears in the limit theory for nonparametric nonstationary spurious regression (Phillips, 2009). Aldous (1986) gave (3.2) for the case of Brownian motion.

Our main result follows.

THEOREM 3.1. *Under Assumptions 1-5, we have*

$$\frac{S_n}{\tau_n}\to_D \eta\, N$$

where $\tau_n^2=(8\phi)^{-1}\,\sigma^4\,n^{3/2}h\int_{-\infty}^{\infty}K^2(x)dx$, $\eta^2=L_G(1,0)$ *is defined as in (3.2) denoting the self intersection local time generated by the process* $G(t)=\int_0^t e^{\kappa(t-s)}dW(s)$, *and* N *is a standard normal variate which is independent of* η^2.

It is readily seen that S_n may be decomposed as

$$
\begin{aligned}
S_n &= \sum_{\substack{i,t=1\\ i\neq t}}^{n}[u_{i+1}+f(x_i,\theta)-f(x_i,\hat\theta)]\,[u_{t+1}+f(x_t,\theta)-f(x_t,\hat\theta)]K\left[(x_t-x_i)/h\right] \\
&= \sum_{\substack{i,t=1\\ i\neq t}}^{n}u_{i+1}u_{t+1}K\left[(x_t-x_i)/h\right]+2\sum_{\substack{i,t=1\\ i\neq t}}^{n}u_{i+1}\left[f(x_t,\theta)-f(x_t,\hat\theta)\right]K\left[(x_t-x_i)/h\right] \\
&\quad +\sum_{\substack{i,t=1\\ i\neq t}}^{n}\left[f(x_i,\theta)-f(x_i,\hat\theta)\right]\left[f(x_t,\theta)-f(x_t,\hat\theta)\right]K\left[(x_t-x_i)/h\right] \\
&= S_{1n}+2S_{2n}+S_{3n}, \quad \text{say.} \qquad (3.3)
\end{aligned}
$$

It will be proved in Section 6.2 that terms S_{2n} and S_{3n} are negligible in comparison with S_{1n}. Therefore the limit theory of S_{1n} plays a key role in the proof of Theorem 3.1. The following theorem gives a joint convergence result for S_{1n} and its conditional variance, which is of some independent interest.

THEOREM 3.2. *Under Assumptions 1-3 and when $nh^2 \to \infty$ and $nh^4 \log^2 n \to 0$, we have*

$$\Big(\frac{1}{\sigma\, d_n}\sum_{t=2}^{n} u_{t+1}\, Y_{nt},\ \frac{1}{d_n^2}\sum_{t=2}^{n} Y_{nt}^2\Big) \to_D (\eta\, N, \eta^2)$$

where $Y_{nt} = \sum_{i=1}^{t-1} u_{i+1} K\big[(x_t - x_i)/h\big]$, $\eta^2 = L_G(1,0)$ is the self intersection local time generated by the process $G = \int_0^t e^{\kappa(t-s)} dW(s)$, N is a standard normal variate which is independent of η^2, and $d_n^2 = (2\,\phi)^{-1}\sigma^2 n^{3/2} h \int_{-\infty}^{\infty} K^2(x)dx$..

It is interesting to note that S_{1n} is a martingale sequence with conditional variance $\sum_{t=2}^n Y_{nt}^2$, suggesting that some version of the martingale central limit theorem (e.g. Hall and Heyde, 1980, chapter 3) may be applicable. However, the problem is complicated by the U statistic structure and the weak convergence of the conditional variance and use of existing limit theory seems difficult. To investigate the asymtotics of S_{1n}, we therefore develop our own approach.

As part of this development, the next section provides a general weak convergence theory to intersection local time. The condition required for this development is weaker than the Assumption 1 used in establishing Theorem 3.1 and that section may be read separately.

The result in Theorem 3.1 involves a standardization that depends on σ, which is the limit of Eu_t^2 as $t \to \infty$. While convenient, this formulation obviously restricts direct use of Theorem 3.1 in applications. However, the dependence on σ^2 can be simply removed by self-normalization. Define

$$V_n^2 = \sum_{s,t=1, s\neq t}^{n} \hat{u}_{t+1}^2 \hat{u}_{s+1}^2\, K^2\big[(x_t - x_s)/h\big].$$

We may prove, under Assumptions 1-5 and the null hypothesis, that

$$\begin{aligned} \frac{V_n^2}{d_n^2} &= \frac{2\sigma^2}{d_n^2}\sum_{t=2}^{n} Y_{nt}^2 + o_P(1) \\ &= \frac{\sigma^4}{d_n^2}\sum_{\substack{t,s=1\\ t\neq s}}^{n} K^2\big[(x_t - x_s)/h\big] + o_P(1). \qquad (3.4) \end{aligned}$$

See Section 6.3. This result, together with Theorem 3.2, leads to the following feasible central limit theorem that is useful in practical work.

THEOREM 3.3. *Under Assumptions 1-5 and the null hypothesis, we have*

$$\frac{S_n}{\sqrt{2}\,V_n} \to_D N \tag{3.5}$$

It is interesting to note that the limit in Theorem 3.3 is normal and does not depend on either of the parameters σ or ϕ. As a test statistic, $Z_n = S_n/\sqrt{2}\,V_n$ therefore has a big advantage in applications. We mention that, under very strict restrictions (namely that x_t is a random walk and x_t is independent of u_t), the result has been established in Gao et al (2009a). We also mention that σ^2 can be estimated by a localized version of the usual residual based method, in particular

$$\hat{\sigma}_n^2 = \frac{\sum_{t=1}^n \left[y_t - \hat{f}(x)\right]^2 K[(x_t - x)/h]}{\sum_{t=1}^n K[(x_t - x)/h]},$$

where $\hat{f}(x)$ is a kernel estimate of $f(x)$ defined by

$$\hat{f}(x) = \frac{\sum_{t=1}^n y_t K[(x_t - x)/h]}{\sum_{t=1}^n K[(x_t - x)/h]}.$$

Under certain condition on $f(x)$ and u_t, we may prove $\sigma_n^2 \to_P \sigma^2$. For more details in this regard, we refer to Wang and Phillips (2009b).

4 Convergence to Intersection Local Time

Consider a linear process $\{\eta_j, j \geq 1\}$ defined by $\eta_j = \sum_{k=0}^{\infty} \phi_k\, \epsilon_{j-k}$, where $\{\epsilon_j, j \in Z\}$ is a sequence of iid random variables with $E\epsilon_0 = 0$ and $E\epsilon_0^2 = 1$, and the coefficients $\phi_k, k \geq 0$, are assumed to satisfy $\sum_{k=0}^{\infty} |\phi_k| < \infty$ and $\phi \equiv \sum_{k=0}^{\infty} \phi_k \neq 0$. Let

$$y_{k,n} = \rho\, y_{k-1,n} + \eta_k, \quad y_{0,n} = 0, \quad \rho = 1 + \kappa/n, \tag{4.1}$$

where κ is a constant. The array $y_{k,n}$, $k \geq 0$, is known as a nearly unstable process or, in the econometric literature, as a near-integrated time series. Write $x_{k,n} = y_{k,n}/\sqrt{n}\phi$. The classical invariance principle gives

$$x_{[nt],n} \Rightarrow G(t) := \int_0^t e^{\kappa(t-s)} dW(s) = W(t) + \kappa \int_0^t e^{\kappa(t-s)} W(s) ds, \tag{4.2}$$

on $D[0,1]$, where $W(t)$ is a standard Brownian motion (e.g. Phillips, 1987; Buchmann and Chan, 2007; Wang and Phillips, 2009b). Furthermore, $\{\epsilon_j, j \in Z\}$ can be redefined on a richer probability space which also contains a standard Brownian motion $W_1(t)$ such that

$$\sup_{0\le t\le 1} |x_{[nt],n} - G_1(t)| = o_P(1). \tag{4.3}$$

where $G_1(t) = W_1(t) + \kappa \int_0^t e^{\kappa(t-s)} W_1(s)ds$. Indeed, by noting on the richer space that

$$\sup_{0\le t\le 1} \Big|\frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} \epsilon_j - W_1(t)\Big| = o_P(1) \tag{4.4}$$

[see, e.g., Csörgö and Révész (1981)], and using this result in place of the fact that $\frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} \epsilon_j \Rightarrow W(t)$ on $D[0,1]$, the same technique as in the proof of Phillips (1987) [see also Chan and Wei (1987)] yields

$$\sup_{0\le t\le 1} \Big|\frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} \rho^{[nt]-j}\epsilon_j - G_1(t)\Big| = o_P(1).$$

The result (4.3) can now be obtained by the same argument, with minor modifications, as in the proof of Proposition 7.1 in Wang and Phillips (2009b).

The aim of this section is to investigate the asymptotic behavior of a functional $S_{[nr]}$ of the $x_{k,n}$, defined by

$$S_{[nr]} = \frac{c_n}{n^2} \sum_{k,j=1}^{[nr]} g\big[c_n\,(x_{k,n} - x_{j,n})\big], \tag{4.5}$$

where g is a real function on R, and c_n is a certain sequence of positive constants. Under certain conditions on $g(x)$, ϵ_0 and c_n, it is established that, for each fixed $0 < r < 1$, $S_{[nr]}$ converges to an intersection local time process of $G(t)$, Explicitly, we have the following main result.

THEOREM 4.1. *Suppose that $\int_{-\infty}^{\infty} |g(x)|dx < \infty$, $\omega \equiv \int_{-\infty}^{\infty} g(x)dx \neq 0$ and $\int_{-\infty}^{\infty} |Ee^{it\epsilon_0}|dt < \infty$. Then, for any $c_n \to \infty$, $n/c_n \to \infty$ and fixed $r \in [0,1]$,*

$$S_{[nr]} \to_D \omega\, L_G(r,0), \tag{4.6}$$

where $L_G(t,u)$ is the intersection local time of $G(t)$ defined in (3.1) Furthermore, under the same probability probability space for which (4.3) holds, we have that, for any $c_n \to \infty$ and $n/c_n \to \infty$,

$$\sup_{0\le r\le 1} \Big|S_{[nr]} - \omega\, L_{G_1}(r,0)\Big| \to_P 0. \tag{4.7}$$

The integrability condition on the characteristic function of ϵ_0 can be weakened if we place further restrictions on $g(x)$. Indeed we have the following theorem.

THEOREM 4.2. *Theorem 4.1 still holds if $\int_{-\infty}^{\infty} |Ee^{it\epsilon_0}|dt < \infty$ is replaced by the Cramér condition, i.e., $\limsup_{|t|\to\infty} |Ee^{it\epsilon_0}| < 1$, and in addition to the stated conditions already on $g(x)$ we have $|g(x)| \le M/(1+|x|^{1+b})$ for some $b > 0$, where M is a constant.*

It is interesting to notice that the additional condition on $g(x)$ in Theorem 4.2 cannot be reduced without further restriction on ϵ_0 like that in Theorem 4.1. This claim can be explained as in Example 4.2.2 of Borodin and Ibragimov (1995) with some minor modifications. On the other hand, the asymptotic behavior of $S_{[nr]}$ when $c_n = 1$ is quite different, as seen in the following theorem.

THEOREM 4.3. *Suppose that $g(x)$ is Borel measurable function satisfying*

$$\lim_{h\to 0} \int_K^K |x|^{\alpha-1} \sup_{|u|\le h} |g(x+u) - g(x)|dx = 0, \tag{4.8}$$

for all $K > 0$ and some $0 < \alpha \le 1$. Then, under the same probability space for which (4.3) holds, we have

$$\sup_{0\le r\le 1} \Big| \frac{1}{n^2} \sum_{k,j=1}^{[nr]} g\big(x_{k,n} - x_{j,n}\big) - \int_0^r \int_0^r g[G_1(u) - G_1(v)]dudv \Big| = o_P(1). \tag{4.9}$$

We mention that the condition (4.8) is quite weak. Indeed, the same example as in Berkes and Horváth (2006) shows that (4.8) cannot be replaced by

$$\lim_{h\to 0} \int_K^K |x|^{\alpha-1} |g(x+u) - g(x)|dx = 0,$$

for all $K > 0$ and some $0 < \alpha \le 1$.

Local time has figured in much recent work on parametric and nonparametric estimation with nonstationary data. Motivated by nonlinear regression with integrated time series (Park and Phillips, 1999, 2001) and nonparametric estimation of nonlinear cointegration models, many authors (Phillips and Park, 1998; Karlsen and Tjøstheim, 2001; Karlsen et al., 2007); Wang and Phillips, 2009a) have used or proved weak convergence to the local time of a stochastic process, including results of the following type: under certain conditions on the function g, the limiting stochastic process $G(t)$, a sequence $c_n \to \infty$, and normalized data $x_{k,n}$

$$\frac{c_n}{n} \sum_{k=1}^{[nr]} g(c_n\, x_{k,n}) \to_D \omega\, \ell_G(1, 0), \tag{4.10}$$

where $\ell_G(t,s)$ is the local time of the process $G(t)$ at the spatial point s. We refer to Borodin and Ibragimov (1995) (and its references for related work) for the particular situation where $c_n x_{k,n}$ is a partial sum of iid random variables, and to Akonom (1993), Phillips and Park (1998), Jeganathan (2004), and De Jong and Wang (2005) for the case where $c_n x_{k,n}$ is a partial sum of a linear process. Wang and Phillips (2009a, theorem 2.1) generalized these results to include not only linear process partial sums but also cases where $c_n x_{k,n}$ is a partial sum of a Gaussian process, including fractionally integrated time series.

Our present research on the statistic $S_{[nr]}$ in (4.5) has a similar motivation to this earlier work on convergence to a local time process. However, the statistic $S_{[nr]}$ has a much more complex U statistic form and the technical difficulties of establishing weak convergence are greater. The approach of Wang and Phillips (2009a, theorem 2.1) remains useful, however, and is implemented in the proofs of Theorems 3.1-3.3.

Finally we mention some earlier work investigating the intersection local time process and weak convergence for certain specialized situations. This work restricts the function g in (4.5) to the indicator function and the discrete process $y_{k,n}$ in (4.1) to a lattice random walk taking integer values. See, for instance, Aldous (1986), van der Hofstad, et al. (1997), van der Hofstad and Wolfgang (2001), and van der Hofstad, et al. (2003). The present paper seems to the first to consider weak convergence to intersection local time for a general linear process and a general function g.

5 Proofs of Local Time Convergence

Here and elsewhere in the paper we let $C, C_1, C_2, ...$ be constants that may differ at each appearance. In order to prove Theorems 4.1-4.3, we first claim that, for any finite t, x, $L_G(t,x)$ can be defined as an L_2 limit of $L_G^N(t,x) := L_G^{-N,N}(t,x)$ as $N \to \infty$, where

$$L_G^{M,N}(t,x) = \frac{1}{2\pi} \int_M^N e^{-iux} \int_0^t \int_0^t e^{iu[G(s)-G(s_1)]} ds\, ds_1\, du. \tag{5.1}$$

In this regard, it suffices to show that, for any fixed t, x,

$$E|L_G^{M,N}(t,x)|^2 \ \to\ 0, \tag{5.2}$$

as $M, N \to \infty$. Indeed, the result (5.2) implies the L_2 convergence of $L_G^N(t,x)$. Hence, in the framework of L_2-theory, we have

$$\begin{aligned}
\lim_{N\to\infty} L_G^N(t,x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\,ux}\int_0^t\int_0^t e^{iu(G(s)-G(s_1))}ds\,ds_1\,du \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\,ux}\int_{-\infty}^{\infty} e^{iuy}L_G(t,y)dy\,du \\
&= \int_{-\infty}^{\infty} L_G(t,y)dy\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iu(y-x)}du \\
&= L_G(t,x),
\end{aligned}$$

which implies the claim. To prove (5.2), note that

$$\begin{aligned}
E|L_G^{M,N}(t,x)|^2 &= \frac{1}{(2\pi)^2}\int_M^N\int_M^N e^{i\,(u_1-u)x}\int_0^t\int_0^t\int_0^t\int_0^t Ee^{iu[G(s)-G(s_1)]-iu_1[G(s_2)-G(s_3)]} \\
&\qquad ds\,ds_1\,ds_2\,ds_3\,du\,du_1 \\
&\le C\sum_{\substack{i_j\in\{1,2,3,4\}\\ i_1\neq i_2\neq i_3\neq i_4}}\int_M^N\int_M^N \Pi_{i_1,i_2,i_3,i_4}(u,\,u_1)\,dudu_1, \qquad (5.3)
\end{aligned}$$

where, with $f_{i_1,i_2,i_3,i_4}(u,u_1) = Ee^{iu[G(x_{i_1})-G(x_{i_2})]-iu_1[G(x_{i_3})-G(x_{i_4})]}$,

$$\Pi_{i_1,i_2,i_3,i_4}(u,u_1) = \int\ldots\int_{0<x_1<x_2<x_3<x_4<t}\big|\,f_{i_1,i_2,i_3,i_4}(u,u_1)\big|\,dx_1\,dx_2\,dx_3\,dx_4\,.$$

The result (5.2) will follow if each term in the right hand of (5.3) converges to 0, as $M, N\to\infty$. Only consider $i_1 = 4, i_3 = 3, i_2 = 2, i_4 = 1$, and without loss of generality, assume $t = 1$. All others are similar except more simpler. Note that, for any $s_1 \le s_2$,

$$G(s_2) = \int_0^{s_2} e^{\kappa(s_2-u)}dW(u) = e^{\kappa(s_2-s_1)}\,G(s_1) + \int_{s_1}^{s_2} e^{\kappa(s_2-u)}dW(u).$$

Write $G^*(s_1,s_2) = \int_{s_1}^{s_2} e^{\kappa(s_2-u)}dW(u)$. Simple calculations show that $G^*(s_1,s_2)\sim N(0,\sigma^2_{s_1s_2})$ where $\sigma^2_{s_1s_2} = \int_{s_1}^{s_2} e^{2\kappa(s_2-u)}du \ge \gamma_0(s_2-s_1)$ for some $\gamma_0 > 0$. This, together with the independence between $G(s_1)$ and $G^*(s_1,s_2)$, yields that, for $0 < x_1 < x_2 < x_3 < x_4 \le 1$

$$\begin{aligned}
|f_{4,2,3,1}(u,u_1)| &= \big|E\exp\big\{iuG(x_4) - iu_1G(x_3) - iuG(x_2) + iu_1G(x_1)\big\}\big| \\
&= \big|E\exp\big\{iuG^*(x_3,x_4) + i(ue^{\kappa(x_4-x_3)}-u_1)G(x_3) - iuG(x_2) + iu_1G(x_1)\big\}\big| \\
&\le \big|E\exp\big\{iuG^*(x_3,x_4) + i(ue^{\kappa(x_4-x_3)}-u_1)G^*(x_2,x_3)\big\}\big| \\
&\le \exp\big\{-\gamma_0u^2(x_4-x_3)/2 - \gamma_0(ue^{\kappa(x_4-x_3)}-u_1)^2(x_3-x_2)/2\big\}.
\end{aligned}$$

Hence it is readily seen that

$$\begin{aligned}
& \int_M^N \int_M^N \Pi_{4,2,3,1}(u,u_1)\, dudu_1 \\
\leq\; & \int_M^N \int_M^N \int_0^1 \int_0^1 \exp\big\{-\gamma_0 u^2 x/2 - \gamma_0 (ue^{\kappa x} - u_1)^2 y/2\big\}\, dxdydudu_1 \\
\leq\; & \int_M^N \int_0^1 \exp\big\{-\gamma_0 u^2 x/2\big\}\, dxdu \int_{-\infty}^{\infty} \int_0^1 \exp\big\{-\gamma_0 u_1^2 y/2\big\}\, dydu_1 \\
\leq\; & C \int_M^N u^{-2} du \to 0,
\end{aligned}$$

as $M, N \to \infty$, as required.

Proof of Theorem 4.1. We first prove the results (4.6) and (4.7) under an additional condition:

$$\begin{aligned}
\textbf{Con:}\quad & g(x) \text{ is continuous and } \hat{g}(t) \text{ has a compact support,} \\
& \text{where } \hat{g}(x) = \textstyle\int_{-\infty}^{\infty} e^{ixt} g(t) dt. \qquad (5.4)
\end{aligned}$$

To start, noting that $g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itx}\hat{g}(-t)dt$, we have

$$\begin{aligned}
\frac{c_n}{n^2}\sum_{k,j=1}^{[nr]} g\big[c_n\,(x_{k,n} - x_{j,n})\big] &= \frac{1}{2\pi n^2}\sum_{k,j=1}^{[nr]}\int_{-\infty}^{\infty}\hat{g}(-s/c_n)\, e^{is\,(x_{k,n}-x_{j,n})} ds \\
&= R_{1n}(r) + R_{2n}(r), \qquad (5.5)
\end{aligned}$$

where, for some $A > 0$,

$$\begin{aligned}
R_{1n}(r) &= \frac{1}{2\pi n^2}\sum_{k,j=1}^{[nr]}\int_{|s|\leq A}\hat{g}(-s/c_n)\, e^{is\,(x_{k,n}-x_{j,n})} ds, \\
R_{2n}(r) &= \frac{1}{2\pi n^2}\sum_{k,j=1}^{[nr]}\int_{|s|> A}\hat{g}(-s/c_n)\, e^{is\,(x_{k,n}-x_{j,n})} ds.
\end{aligned}$$

Furthermore, $R_{1n}(r)$ can be written as

$$R_{1n}(r) = \frac{1}{2\pi}\int_{|s|\leq A}\hat{g}(-s/c_n)\int_0^r\int_0^r e^{is\,(x_{[nu],n}-x_{[nv],n})} du\, dv\, ds + o_P(1).$$

Recall $c_n \to \infty$. It is readily seen that $\sup_{|s|\leq A}|\hat{g}(-s/c_n) - \hat{g}(0)| \to 0$ for any fixed $A > 0$. Hence, by recalling (4.2), it follows from the continuous mapping theorem that, for any $A > 0$ and fixed r,

$$R_{1n}(r) \to_D \frac{\hat{g}(0)}{2\pi}\int_{|s|\leq A}\int_0^r\int_0^r e^{is[G(u)-G(v)]} dudv\, ds, \qquad (5.6)$$

as $n \to \infty$. On the other hand, under the same probability space used in (4.3), we have

$$\begin{aligned}
&\sup_{0\le r\le 1} \big|R_{1n}(r) - \frac{\hat g(0)}{2\pi} \int_{|s|\le A} \int_0^r\int_0^r e^{is[G_1(u)-G_1(v)]} du dv\, ds\big| \\
&\le \frac{1}{2\pi}\int_{|s|\le A} \big|\hat g(-s/c_n) - \hat g(0)\big| ds \\
&\quad + \frac{|\hat g(0)|}{2\pi} \int_{|s|\le A}\int_0^1\int_0^1 \big|e^{is(x_{[nu],n}-G_1(u)+x_{[nv],n}-G_1(v))} - 1\big| du dv ds \\
&\le o(1) + O(1) \sup_{0\le u\le 1} |x_{[nu],n} - G_1(u)| = o_P(1). \qquad (5.7)
\end{aligned}$$

By (5.6) and (5.7), noting that $\hat g(0) = \omega$ and, as $A\to\infty$, $\frac{1}{2\pi}\int_{|s|\le A}\int_0^r\int_0^r e^{is[G(u)-G(v)]}du dv\, ds$ converges to $L_G(r,0)$ in L_2 uniformly on $r\in[0,1]$, the results (4.6) and (4.7) will follow under the additional condition (5.4), if we prove

$$\sup_{0\le r\le 1} E|R_{2n}(r)|^2 \;\to\; 0, \qquad (5.8)$$

as $n\to\infty$ first and then $A\to\infty$.

In order to prove (5.8), we need some preliminaries. Write $\eta_r' = \sum_{q=1}^r \epsilon_q\phi_{r-q}$ and

$$y_{\lambda,n}' = \sum_{r=1}^{\lambda} \rho^{\lambda-r}\eta_r' = \sum_{q=1}^{\lambda}\epsilon_q \sum_{r=0}^{\lambda-q}\rho^{\lambda-q-r}\phi_r = \sum_{q=1}^{\lambda}\epsilon_q\, a(\lambda-q),$$

where $a(v) = \sum_{r=0}^{v}\rho^{v-r}\phi_r$. Simple calculations show that, whenever $k\ge j\ge l\ge m$,

$$\begin{aligned}
&s\,(y_{k,n}' - y_{j,n}') - t\,(y_{l,n}' - y_{m,n}') \\
&= s\sum_{q=j+1}^{k}\epsilon_q\, a(k-q) + s\sum_{q=l+1}^{j}\epsilon_q\,[a(k-q)-a(j-q)] \\
&\quad + \sum_{q=m+1}^{l}\epsilon_q\big\{s\,[a(k-q)-a(j-q)] - t\,a(l-q)\big\} \\
&\quad + \sum_{q=1}^{m}\epsilon_q\big\{s\,[a(k-q)-a(j-q)] - t\,[a(l-q)-a(m-q)]\big\}.
\end{aligned}$$

By virtue of the independence between ϵ_q, it follows that, whenever $k\ge j\ge l\ge m$,

$$\Big|E\exp\big\{\big[is\,(y_{k,n}'-y_{j,n}') - it\,(y_{l,n}'-y_{m,n}')\big]\big\}\Big| \;\le\; I_1(s)\, I_2(s,t),$$

where

$$\begin{aligned}
I_1(s) &= \Big|E\exp\big\{is\sum_{q=j+1}^{k}\epsilon_q\, a(k-q)\big\}\Big|, \\
I_2(s,t) &= \Big|E\exp\Big(i\sum_{q=m+1}^{l}\epsilon_q\big\{s\,[a(k-q)-a(j-q)] - t\,a(l-q)\big\}\Big)\Big|.
\end{aligned}$$

We may claim that there exist constants $\gamma_1 > 0$ and $\gamma_2 > 0$ such that

$$\int_{|s|\geq A} I_1\big(\frac{s}{\sqrt{n}\phi}\big)\, |\hat{g}(-s/c_n)| ds \;\leq\; C \Big[c_n\, e^{-\gamma_1 \sqrt{k-j}} + \int_{|s|\geq A} e^{-\gamma_2\,(k-j+1)s^2/n} ds\Big], \quad (5.9)$$

and for each $s \in R$,

$$\int_{|t|\geq A} I_2\big(\frac{s}{\sqrt{n}\phi}, \frac{t}{\sqrt{n}\phi}\big)\, |\hat{g}(-t/c_n)| dt \;\leq\; C \Big[c_n\, e^{-\gamma_1 \sqrt{l-m}} + \int e^{-\gamma_2 (l-m+1)t^2/n}\, dt\Big] (5.10)$$

In order to prove (5.10), write Ω_1 (Ω_2, respectively) for the set of $m \leq q \leq (l+m)/2$ such that

$$|s\,[a(k-q) - a(j-q)] - t\,a(l-q)| \geq \sqrt{n}|\phi|$$

($|s\,[a(k-q) - a(j-q)] - t\,a(l-q)| < \sqrt{n}|\phi|$, respectively). Also let

$$B_1 = \sum_{q\in\Omega_2} a(l-q)^2, \quad B_2 = \sum_{q\in\Omega_2} a(l-q)[a(k-q) - a(j-q)]$$

and $B_3 = \sum_{q\in\Omega_2}[a(k-q) - a(j-q)]^2$. By Hölder's inequality, $B_2^2 \leq B_3 B_1$ and hence

$$\begin{aligned} & \sum_{q\in\Omega_2} \big\{s\,[a(k-q) - a(j-q)] - t\,a(l-q)\big\}^2 \\ = \;& s^2 B_3 - 2s\,t\,B_2 + t^2\,B_1 = B_1(t - s\,B_2/B_1)^2 + s^2(B_3 - B_2^2/B_1) \\ \geq \;& B_1(t - sB_2/B_1)^2. \end{aligned}$$

Next, using the fact that there exist constants $\gamma_1' > 0$ and $\gamma_2' > 0$ such that

$$\big|Ee^{i\,\epsilon_1\,t}\big| \;\leq\; \begin{cases} e^{-\gamma_1'} & \text{if } |t| \geq 1, \\ e^{-\gamma_2' t^2} & \text{if } |t| \leq 1, \end{cases} \quad (5.11)$$

since $E\epsilon_1 = 0$, $E\epsilon_1^2 = 1$ and ϵ_1 satisfies the Cramér condition $\limsup_{|t|\to\infty} |Ee^{it\epsilon_1}| < 1$, it follows from the independence of ϵ_t that, for all $l \geq m$,

$$\begin{aligned} & I_2\big(\frac{s}{\sqrt{n}\phi}, \frac{t}{\sqrt{n}\phi}\big) \\ \leq \;& \exp\big\{-\gamma_1' \#(\Omega_1) - \gamma_2'\,(n\phi^2)^{-1} \sum_{q\in\Omega_2} \big\{s\,[a(k-q) - a(j-q)] - t\,a(l-q)\big\}^2\big\} \\ \leq \;& \exp\big\{-\gamma_1' \#(\Omega_1) - \gamma_2'\,B_1\,(n\phi^2)^{-1}\,(t - sB_2/B_1)^2\big\}, \end{aligned} \quad (5.12)$$

where $\#(A)$ denotes the number of elements in A. Note that $\Omega_1 + \Omega_2 = (l-m)/2$ and $|a(v)| \geq e^{-|\kappa|}|\phi|/4$ for all sufficiently large v [see (7.14) in Wang and Phillips (2009b)]. It is readily seen that, there exists a n_0 such that for all $l - m \geq n_0$, $B_1 \geq C\,(l-m)$, whenever

$\#(\Omega_1) \le \sqrt{l-m}$. Using these facts in (5.12), we find that, whenever $l-m \ge n_0$,

$$\begin{aligned}
&\int_{|t|\ge A} I_2\big(\frac{s}{\sqrt{n}\phi}, \frac{t}{\sqrt{n}\phi}\big)\, |\hat{g}(-t/c_n)|dt \\
\le\ & \int_{\#(\Omega_1)\ge\sqrt{l-m}} e^{-\gamma_1'\#(\Omega_1)}|\ \hat{g}(-t/c_n)|dt + C\int_{\#(\Omega_1)\le\sqrt{l-m}} e^{-\gamma_2'\, B_1\, (n\phi^2)^{-1}\, (t-sB_2/B_1)^2}\, dt \\
\le\ & C\, c_n\, e^{-\gamma_1'\, \sqrt{l-m}} \int |\hat{g}(t)|\, dt + C_1 \int e^{-C\, \gamma_2'\, (l-m)t^2/n}\, dt\big) \\
\le\ & C_2 \Big[c_n\, e^{-\gamma_1\, \sqrt{l-m}} + \int e^{-\gamma_2(l-m+1)t^2/n}\, dt\Big],
\end{aligned}$$

for some $\gamma_1 > 0$ and $\gamma_2 > 0$. This prove (5.10), since (5.10) is obvious if $0 \le l-m \le n_0$.

Similarly, it follows from (5.11) and the fact that $|a(v)| \ge e^{-|\kappa|}|\phi|/4$ for all sufficiently large v that, there exist $\epsilon > 0$ and n_1 such that for all $k-j \ge n_1$, if $|s| \ge \epsilon\sqrt{n}$, then

$$I_1\big(\frac{s}{\sqrt{n}\phi}\big) \quad \le \quad e^{-\gamma_1'(k-j)/2},$$

and if $|s| \le \epsilon\sqrt{n}$, then

$$I_1\big(\frac{s}{\sqrt{n}\phi}\big) \quad \le \quad e^{-C\, \gamma_2' s^2\, \sum_{q=j+1}^{k} a(k-q)^2/n} \le e^{-\gamma_1\, (k-j+1)s^2/n},$$

for some $\gamma_2 > 0$. By virtue of these facts, it is readily seen that

$$\begin{aligned}
&\int_{|s|\ge A} I_1\big(\frac{s}{\sqrt{n}\phi}\big)\, |\hat{g}(-s/c_n)|ds \\
\le\ & e^{-\gamma_1(k-j)/2} \int_{|s|\ge\epsilon\,\sqrt{n}} |\hat{g}(-s/c_n)|ds + \int_{A\le|s|\le\epsilon\,\sqrt{n}} e^{-\gamma_2\, (k-j+1)s^2/n} ds \\
\le\ & C\, \Big[c_n\, e^{-\gamma_1\, \sqrt{k-j}} + \int_{|s|\ge A} e^{-\gamma_2\, (k-j+1)s^2/n} ds\Big].
\end{aligned}$$

which yields (5.9).

We are now ready to prove (5.8). Write $\eta_r'' = \sum_{q=-\infty}^{0} \epsilon_q \phi_{r-q}$. Note that $\eta_r = \eta_r' + \eta_r''$. Simple calculations show that

$$s\,(x_{k,n} - x_{j,n}) - t\,(x_{l,n} - x_{m,n}) = s\,(y_{k,n}' - y_{j,n}') - t\,(y_{l,n}' - y_{m,n}') + F(\epsilon_0, \epsilon_{-1}, ...),$$

where $F(\epsilon_0, \epsilon_{-1}, ...)$ depends only on $\epsilon_j, j \le 0$, which is independent of $s\,(y_{k,n}' - y_{j,n}') -$

$t\,(y'_{l,n} - y'_{m,n})$. It now follows from (5.9) and (5.10) that

$$\begin{aligned}
E|R_{2n}(r)|^2 &\leq \frac{C}{n^4} \sum_{k,j,l,m=1}^{n} \int_{|s|\geq A} \int_{|t|\geq A} |\hat{g}(-s/c_n)|\,|\hat{g}(-t/c_n)| \\
&\qquad\qquad \Big|E\Big[e^{is\,(x_{k,n}-x_{j,n})}\, e^{-it\,(x_{l,n}-x_{m,n})}\Big]\Big|\, ds\, dt \\
&\leq \frac{C}{n^4} \sum_{k,j,l,m=1}^{n} \int_{|s|\geq A} \int_{|t|\geq A} |\hat{g}(-s/c_n)|\,|\hat{g}(-t/c_n)| \\
&\qquad\qquad \Big|E\exp\big\{\big[is\,(y'_{k,n}-y'_{j,n}) - it\,(y'_{l,n}-y'_{m,n})\big]\big\}\Big|\, ds\, dt \\
&\leq \frac{C_1}{n^4} \sum_{k\geq j\geq l\geq m} \int_{|s|\geq A} I_1\big(\frac{s}{\sqrt{n\phi}}\big)\,|\hat{g}(-s/c_n)|ds \int_{|t|\geq A} I_2\big(\frac{s}{\sqrt{n\phi}}, \frac{t}{\sqrt{n\phi}}\big)\,|\hat{g}(-t/c_n)|dt \\
&\leq \frac{C_2}{n^4} \sum_{1\leq j\leq k\leq n} \Big[c_n\, e^{-\gamma_1\sqrt{k-j}} + \int_{|s|\geq A} e^{-\gamma_2\,(k-j+1)s^2/n} ds\Big] \\
&\qquad\qquad \times \sum_{1\leq m\leq l\leq n} \Big[c_n\, e^{-\gamma_1\sqrt{l-m}} + \int e^{-\gamma_2(l-m+1)t^2/n}\, dt\Big] \\
&\leq C_2\Big[\frac{c_n}{n^2} \sum_{1\leq j\leq k\leq n} e^{-\gamma_1\sqrt{k-j}} + \int_{|s|\geq A} \frac{1}{n^2} \sum_{1\leq j\leq k\leq n} e^{-\gamma_2\,(k-j+1)s^2/n} ds\Big] \\
&\qquad\qquad \times\Big[\frac{c_n}{n^2} \sum_{1\leq m\leq l\leq n} e^{-\gamma_1\sqrt{l-m}} + \int \frac{1}{n^2} \sum_{1\leq m\leq l\leq n} e^{-\gamma_2\,(l-m+1)s^2/n} ds\Big] \\
&\leq C_3\Big[\frac{c_n}{n} + \int_{|s|\geq A} \int_0^1 \int_0^u e^{-\gamma_2\,(u-v)s^2} dv du\, ds\Big] \\
&\qquad\qquad \times\Big[\frac{c_n}{n} + \int \int_0^1 \int_0^u e^{-\gamma_2\,(u-v)s^2} dv du\, ds\Big] \\
&\to 0,
\end{aligned}$$

when $n \to \infty$ first and then $A \to \infty$. This proves (5.8), and hence the results (4.6) and (4.7) under the additional condition (5.4).

We next remove the additional condition (5.4). Recall $\int_{-\infty}^{\infty} |g(x)|dx < \infty$. We first claim that, for any $\epsilon > 0$, we may construct a $g_{\delta_0}(x)$ satisfying (5.4), $\int_{-\infty}^{\infty} |g_{\delta_0}(x)|dx < \infty$ and

$$\int_{-\infty}^{\infty} |g(x) - g_{\delta_0}(x)|dx < \epsilon. \tag{5.13}$$

Since Theorem 4.1 holds true for $g_{\delta_0}(x)$, it remains to show that

$$\frac{c_n}{n^2} \sum_{k,j=1}^{n} \big|g\big[c_n\,(x_{k,n} - x_{j,n})\big] - g_{\delta_0}\big[c_n\,(x_{k,n} - x_{j,n})\big]\big| = O_P(\epsilon). \tag{5.14}$$

This follows from (5.13) and the assumption that $\int_{-\infty}^{\infty}|Ee^{it\epsilon_0}|dt<\infty$. Indeed, under this assumption, similar arguments to those in the proof of Corollary 3.2 in Wang and Phillips (2009a) show that $\sqrt{n}(x_{k,n}-x_{j,n})/\sqrt{k-j}$, for all $1\le j<k\le n$, has a density $h_{k,j}(x)$ which is uniformly bounded by a constant K. This fact implies that

$$\begin{aligned}
&\frac{c_n}{n^2}\sum_{k,j=1}^{n} E\big|g\big[c_n\,(x_{k,n}-x_{j,n})\big]-g_{\delta_0}\big[c_n\,(x_{k,n}-x_{j,n})\big]\big| \\
=\;& \frac{2c_n}{n^2}\sum_{1\le j<k\le n}\int_{-\infty}^{\infty}\Big|g\Big(\frac{c_n\sqrt{n}\,x}{\sqrt{k-j}}\Big)-g_{\delta_0}\Big(\frac{c_n\sqrt{n}\,x}{\sqrt{k-j}}\Big)\Big|h_{k,j}(x)dx+\frac{c_n\,|g(0)|}{n} \\
\le\;& 2K\int_{-\infty}^{\infty}|g(x)-g_{\delta_0}(x)|dx\,\frac{1}{n^{5/2}}\sum_{1\le j<k\le n}\sqrt{k-j}+\frac{c_n\,|g(0)|}{n}=O(\epsilon),
\end{aligned}$$

which yields (5.14).

It remains only to construct a $g_{\delta_0}(x)$ satisfying (5.4), (5.13) and $\int_{-\infty}^{\infty}|g_{\delta_0}(x)|dx<\infty$. This can be done according to the similar idea as in the proof of Theorem 4.2.1 in Borodin and Ibragimov (1995). For the sake of completeness, we describe this process as follows.

First of all, for given $\epsilon>0$, there exists a continuous function $f_\epsilon(x)$ having a compact support such that

$$\int_{-\infty}^{\infty}|g(x)-g_\epsilon(x)|dx \;\le\; \epsilon/2. \tag{5.15}$$

See Proposition 15.3.3 of Gasquet and Witomski (1999). Next, set

$$g_\delta(x) \;=\; \frac{\delta}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2[(x-y)/\delta]}{(x-y)^2}g_\epsilon(y)dy. \tag{5.16}$$

Since

$$\int_{-\infty}^{\infty}\frac{\sin^2(x)}{x^2}e^{itx}dx=\begin{cases}\pi(1-|t|/2), & \text{if } |t|<2\\ 0, & \text{otherwise,}\end{cases}$$

it is readily seen that, for any $\delta>0$, $g_\delta(x)$ is continuous and

$$\begin{aligned}
\hat{g}_\delta(t) \;&=\; \int_{-\infty}^{\infty}e^{itx}g_\delta(x)dx \\
&=\; \frac{1}{\pi}\int_{-\infty}^{\infty}g_\epsilon(y)e^{ity}dy\int_{-\infty}^{\infty}\frac{\sin^2(x)}{x^2}e^{it\delta\,x}dx,
\end{aligned}$$

has a compact support. That is, $g_\delta(x)$ satisfies the condition (5.4). Furthermore, by noting $\int_{-\infty}^{\infty}\frac{\sin^2(x)}{x^2}dx=\pi$ and

$$g_\delta(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2(y)}{y^2}g_\epsilon(x+\delta y)dy,$$

simple calculations show that, as $\delta \to 0$,

$$\int_{-\infty}^{\infty} |g_\delta(x) - g_\epsilon(x)| dx \le \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sin^2(y)}{y^2} \big| g_\epsilon(x + \delta y) - g_\epsilon(x) \big| dx dy \to 0. \tag{5.17}$$

Consequently, for any $\epsilon > 0$, there exists a $g_{\delta_0}(x)$ satisfying (5.4), $\int_{-\infty}^{\infty} |g_{\delta_0}(x)| dx < \infty$ and

$$\begin{aligned} \int_{-\infty}^{\infty} |g(x) - g_{\delta_0}(x)| dx &\le \int_{-\infty}^{\infty} |g(x) - g_\epsilon(x)| dx + \int_{-\infty}^{\infty} |g_{\delta_0}(x) - g_\epsilon(x)| dx \\ &\le \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

This completes the construction of $g_\delta(x)$. The proof of Theorem 4.1 is now complete. □

Proof of Theorem 4.2. By checking the proof of Theorem 4.1, the integrability condition $\int_{-\infty}^{\infty} |Ee^{it\epsilon_0}| dt < \infty$, which is stronger than the Cramér condition $\limsup_{|t| \to \infty} |Ee^{it\epsilon_0}| < 1$, is only used to remove the additional condition (5.4). So, to prove Theorem 4.2, it is sufficient to show that this can be done by an alternative method under the additional condition on $g(x)$. Explicitly, we only need to prove that, for any $\epsilon > 0$, there exist $g_{\delta_0}^+(x)$ and $g_{\delta_0}^-(x)$ such that $g_{\delta_0}^-(x) \le g(x) \le g_{\delta_0}^+(x)$, both $g_{\delta_0}^+(x)$ and $g_{\delta_0}^-(x)$ satisfy (5.4), $\int_{-\infty}^{\infty} (|g_{\delta_0}^+(x)| + |g_{\delta_0}^-(x)|) dx < \infty$ and

$$\int_{-\infty}^{\infty} \big[g_{\delta_0}^+(x) - g_{\delta_0}^-(x) \big] dx < \epsilon. \tag{5.18}$$

Indeed it follows from these facts that

$$\begin{aligned} \frac{c_n}{n} \sum_{k,j=1}^{[nr]} g\big[c_n (x_{k,n} - x_{j,n})\big] &\le \frac{c_n}{n} \sum_{k,j=1}^{[nr]} g_{\delta_0}^+ \big[c_n (x_{k,n} - x_{j,n})\big] \\ &\to_D \int_{-\infty}^{\infty} g_{\delta_0}^+(x) dx \, L_G(r, 0) \le (\int_{-\infty}^{\infty} g(x) dx + \epsilon) \, L_G(r, 0), \\ \frac{c_n}{n} \sum_{k,j=1}^{[nr]} g\big[c_n (x_{k,n} - x_{j,n})\big] &\ge \frac{c_n}{n} \sum_{k,j=1}^{[nr]} g_{\delta_0}^- \big[c_n (x_{k,n} - x_{j,n})\big] \\ &\to_D \int_{-\infty}^{\infty} g_{\delta_0}^-(x) dx \, L_G(r, 0) \ge (\int_{-\infty}^{\infty} g(x) dx - \epsilon) \, L_G(r, 0). \end{aligned}$$

This yields (4.6) since ϵ is arbitrary. In a similar way we may prove (4.7).

The constructions of $g_{\delta_0}^+(x)$ and $g_{\delta_0}^-(x)$ again are similar to those in the proof of Theorem 4.2.1 in Borodin and Ibragimov (1995). To start with, we notice that, for $\forall \epsilon > 0$, there exist two continuous functions $g_\epsilon^+(x)$ and $g_\epsilon^-(x)$ such that $g_\epsilon^-(x) \le g(x) \le g_\epsilon^+(x)$ and

$$\int_{-\infty}^{\infty} \big[g_\epsilon^+(x) - g_\epsilon^-(x) \big] dx < \epsilon. \tag{5.19}$$

See, e.g., part (b) and (c) in the proof of Theorem 4.2.1 in Borodin and Ibragimov (1995). Here $g_\epsilon^{\pm}(x)$ can be chosen such that $|g_\epsilon^{\pm}(x)| \le M/(1+|x|^{1+b})$ for some $b>0$, under the same conditions on $|g(x)|$. Using $g_\epsilon^+(x)$ and $g_\epsilon^-(x)$, as in (5.16), define

$$\begin{aligned} f_\delta^+(x) &= \frac{\delta}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2[(x-y)/\delta]}{(x-y)^2} g_\epsilon^+(y)dy, \\ f_\delta^-(x) &= \frac{\delta}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2[(x-y)/\delta]}{(x-y)^2} g_\epsilon^-(y)dy. \end{aligned}$$

Also write (defining $\sin(y)/y = 1$ for $y=0$)

$$f(x) = \sum_{n=1}^{\infty} n^{-1-b/2} \frac{\sin^2(x-n)}{(x-n)^2}.$$

Simple calculation shows that

$$c_1\,(1+|x|^{-1-b/2}) \le f(x) \le c_2\,(1+|x|^{-1-b/2}), \tag{5.20}$$

where $c_1>0$ and $c_2>0$ are constants. On the other hand, as in the proof of (5.17), for any $\epsilon>0$, there exists a δ_1 such that for all $0<\delta\le\delta_1$,

$$\int_{-\infty}^{\infty} \big|f_\delta^+(x) - g_\epsilon^+(x)\big|dx \le \epsilon, \qquad \int_{-\infty}^{\infty} \big|f_\delta^-(x) - g_\epsilon^-(x)\big|dx \le \epsilon. \tag{5.21}$$

It can also be proved (see below) that, for any $\epsilon>0$, there exists a δ_2 such that for all $0<\delta\le\delta_2$,

$$\big|f_\delta^+(x) - g_\epsilon^+(x)\big|dx \le \epsilon\, f(x), \qquad \big|f_\delta^-(x) - g_\epsilon^-(x)\big|dx \le \epsilon\, f(x). \tag{5.22}$$

Now the required $g_{\delta_0}^+(x)$ and $g_{\delta_0}^-(x)$ can be defined by

$$g_{\delta_0}^+(x) \;=\; f_{\delta_0}^+(x) + \epsilon\, f(x) \quad \text{and} \quad g_{\delta_0}^-(x) = f_{\delta_0}^-(x) - \epsilon\, f(x),$$

where $\delta_0 = \min\{\delta_1,\delta_2\}$. Indeed, as in the proof of Theorem 4.1, both $g_{\delta_0}^+(x)$ and $g_{\delta_0}^+(x)$ satisfy the additional condition (5.4). By virtue of (5.22),

$$g_{\delta_0}^-(x) \le g_\epsilon^-(x) \le g(x) \le g_\epsilon^+(x) \le g_{\delta_0}^+(x),$$

and by (5.19)–(5.21), we have that $\int_{-\infty}^{\infty}(|g_{\delta_0}^+(x)| + |g_{\delta_0}^-(x)|)dx < \infty$ and

$$\begin{aligned} \int_{-\infty}^{\infty}\big[g_{\delta_0}^+(x) - g_{\delta_0}^-(x)\big]dx &\le \int_{-\infty}^{\infty}\big[g_\epsilon^+(x) - g_\epsilon^-(x)\big]dx + \int_{-\infty}^{\infty}\big|f_{\delta_0}^+(x) - g_\epsilon^+(x)\big|dx \\ &\quad + \int_{-\infty}^{\infty}\big|f_{\delta_0}^-(x) - g_\epsilon^-(x)\big|dx + 2\epsilon\int_{-\infty}^{\infty} f(x)dx \\ &\le C\,\epsilon, \end{aligned}$$

where C is a constant.

We next prove (5.22). We have

$$\begin{aligned}
\big|f_\delta^+(x) - g_\epsilon^+(x)\big| &\le \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2(y)}{y^2}\big|g_\epsilon^+(x+\delta y) - g_\epsilon^+(x)\big|dy \\
&\le \frac{1}{\pi}\Big(\int_{\substack{|x|\le A\\ y\in R}} + \int_{\substack{|x|>A\\ |\delta y|\ge |x|/2}} + \int_{\substack{|x|>A\\ |\delta y|<|x|/2}}\Big)\frac{\sin^2(y)}{y^2}\big|g_\epsilon^+(x+\delta y) - g_\epsilon^+(x)\big|dy \\
&:= R_{1A} + R_{2A} + R_{3A}, \qquad (5.23)
\end{aligned}$$

where A is chosen later. Recall that (5.20) and $|g_\epsilon^\pm(x)| \le M/(1+|x|^{1+b})$ for some $b>0$. For any $\epsilon > 0$, there exists an $A_0 > 0$ such that, whenever $A \ge A_0$,

$$\begin{aligned}
R_{3A} &\le 2M\,(A/2)^{-b/2}\,[1 + (|x|/2)^{-1-b/2}]I(|x|\ge A) \\
&\le \epsilon\, f(x) I(|x|\ge A)/3. \qquad (5.24)
\end{aligned}$$

For this A_0, routine calculations yield that there exists a δ_2 such that, for all $0<\delta\le\delta_2$,

$$R_{1A_0} \le \epsilon\, f(x) I(|x|\le A_0)/3, \qquad (5.25)$$

since $g_\epsilon^+(x)$ is continuous, and

$$\begin{aligned}
R_{2A_0} &\le \frac{M\delta}{\pi}\int_{|y|\ge|x|/2} y^{-2}\Big[|x|^{-1-b} + (1+|x+y|^{1+b})^{-1}\Big]dy\, I(|x|\ge A) \\
&\le \epsilon\, f(x) I(|x|\ge A)/3. \qquad (5.26)
\end{aligned}$$

Taking $A = A_0$ in (5.23), it follows from (5.23)-(5.26) that, for any $\epsilon > 0$, there exists a δ_2 such that for all $0<\delta\le\delta_2$, $\big|f_\delta^+(x) - g_\epsilon^+(x)\big| \le \epsilon\, f(x)$. Similarly we have $\big|f_\delta^-(x) - g_\epsilon^-(x)\big| \le \epsilon\, f(x)$. This proves (5.22) and hence completes the proof of Theorem 4.2. □

Proof of Theorem 4.3. The idea for the proof of this theorem is similar to Berkes and Horváth (2006). First notice that, for any $\epsilon > 0$, there exists N_0 such that for all $N \ge N_0$,

$$P\Big(\sup_{0\le u,v\le 1}|G_1(u) - G_1(v)| \ge N/2\Big) \le 2P\Big(\sup_{0\le u\le 1}|G_1(u)| \ge N/4\Big) \le \epsilon.$$

This, together with (4.3), also implies that, for all $N \ge N_0$,

$$\begin{aligned}
&P\Big(\sup_{1\le k,j\le n}|x_{k,n} - x_{j,n}| \ge N\Big) \\
\le\ & 2P\Big(\sup_{0\le u\le 1}|x_{[nu],n} - G_1(u)| \ge N/2\Big) + P\Big(\sup_{0\le u,v\le 1}|G_1(u) - G_1(v)| \ge N/2\Big) \\
\le\ & 2\epsilon.
\end{aligned}$$

Write

$$g_N(x) = \begin{cases} g(x) & \text{if } |x| \le N, \\ 0 & \text{if } |x| > N. \end{cases}$$

By virtue of the above facts and noting that

$$\frac{1}{n^2} \sum_{k,j=1}^{[nr]} g_N\big(x_{k,n} - x_{j,n}\big) = \int_0^r \int_0^r g_N\big(x_{[nu],n} - x_{[nv],n}\big) du dv + o_P(1),$$

(4.9) will follow if we prove, for $\forall \epsilon > 0$ and $N \ge N_0$,

$$P(\Lambda_N \ge \epsilon) \quad \to \quad 0, \quad \text{as } n \to \infty, \tag{5.27}$$

where

$$\Lambda_N = \int_0^1 \int_0^1 \big| g_N\big(x_{[nu],n} - x_{[nv],n}\big) - g_N[G_1(u) - G_1(v)]\big| du dv.$$

In fact, for any $\epsilon > 0$ and $h > 0$, we have

$$\begin{aligned} P(\Lambda_N \ge \epsilon) \le\ & 2P(\sup_{0 \le u \le 1} |x_{[nu],n} - G_1(u)| \ge h/2) \\ & + P\Big(\int_0^1 \int_0^1 \sup_{|t| \le h} \big| g_N[G_1(u) - G_1(v) + t] - g_N[G_1(u) - G_1(v)]\big| du dv \ge \epsilon \Big). \end{aligned} \tag{5.28}$$

By (4.3), we may choose $h = h_n \to 0$ such that

$$P(\sup_{0 \le u \le 1} |x_{[nu],n} - G_1(u)| \ge h/2) \to 0, \quad \text{as } n \to \infty. \tag{5.29}$$

Recall (**??**) and note that $G_1(u) - G_1(v) \sim N(0, var[G_1(u) - G_1(v)])$. For this chosen $h = h_n \to 0$, it is readily seen that, for some $\alpha > 0$,

$$\begin{aligned} & P\Big(\int_0^1 \int_0^1 \sup_{|t| \le h} \big| g_N[G_1(u) - G_1(v) + t] - g_N[G_1(u) - G_1(v)]\big| du dv \ge \epsilon \Big) \\ \le\ & \epsilon^{-1} \int_0^1 \int_0^1 E \sup_{|t| \le h} \big| g_N[G_1(u) - G_1(v) + t] - g_N[G_1(u) - G_1(v)]\big| du dv \\ \le\ & C\, \epsilon^{-1} \int_0^1 \int_0^1 \frac{1}{\sqrt{|u-v|}} \int_{-\infty}^{\infty} \sup_{|t| \le h} |g_N(x+t) - g_N(x)| e^{-C\, x^2/|u-v|} dx\, du dv \\ \le\ & C_1\, \epsilon^{-1} \int_0^1 \int_0^1 \frac{1}{|u-v|^{\alpha}} du dv \int_{-N}^{N} |x|^{\alpha-1} \sup_{|t| \le h} |g(x+t) - g(x)| dx \\ \le\ & C_2\, \epsilon^{-1} \int_{-N}^{N} |x|^{\alpha-1} \sup_{|t| \le h} |g(x+t) - g(x)| dx \\ \to\ & 0, \quad \text{as } n \to \infty, \end{aligned} \tag{5.30}$$

where we have used (4.8) and the fact that $t^{1-\alpha} e^{-t^2/2} \le C$ for all $0 \le t < \infty$. Taking (5.29) and (5.30) into (5.28), we prove (5.27), and also complete the proof of Theorem 4.3.
□

6 Proofs of main results

This section provides the proofs of Theorems 3.1-3.3. Some technical propositions needed here will be postponed to the next section. Again, we let $C, C_1, C_2, ...$ be constants which may differ at each appearance. We start with Theorem 3.2.

6.1 Proof of Theorem 3.2. We first assume $|u_t| \le A$, where A is a constant. This restriction will be removed later. Write $G_n(t) = x_{[nt]}/\sqrt{n}\phi$ and $V_n(t) = \sum_{j=1}^{[nt]} u_{j+1}/\sqrt{n}\sigma$. Under Assumptions 1-2, the same arguments as in Buchmann and Chan (2007) or Wang and Phillips (2009b) with minor modifications show that

$$(G_n, V_n) \Rightarrow_D (G, V), \tag{6.1}$$

on $D[0,1]^2$, where $G(t) = W(t) + \kappa \int_0^t e^{\kappa(t-s)} W(s) ds$. By virtue of (6.1), it follows from the so-called Skorohod-Dudley-Wichura representation theorem that there is a common probability space $(\Omega, \mathcal{F}, P)$ supporting (G_n^0, V_n^0) and (G, V) such that

$$(G_n, V_n) =_d (G_n^0, V_n^0) \quad \text{and} \quad (G_n^0, V_n^0) \to_{a.s.} (G, V) \tag{6.2}$$

in $D[0,1]^2$ with the uniform topology. Moreover, as in the proof of Lemma 2.1 in Park and Phillips (2001), V_n^0 can be chosen such that, for each $n \ge 1$

$$V_n^0(k/n) = V(\tau_{nk}/n), \qquad k = 1, 2, ..., n, \tag{6.3}$$

where $\tau_{n,k}, 1 \le k \le n$, are stopping times with respect to $\mathcal{F}_{n,k}^0$ in $(\Omega, \mathcal{F}, P)$ with

$$\mathcal{F}_{n,k}^0 = \sigma\big\{V(r), r \le \tau_{n,k}/n;\ G_n^0(s/n),\ s = 1, ..., k+1\big\},$$

satisfying $\tau_{n,0} = 0$,

$$\sup_{1\le k\le n} \Big|\frac{\tau_{n,k} - k}{n^\delta}\Big| \to_{a.s} 0 \tag{6.4}$$

as $n \to \infty$ for any $1/2 < \delta < 1$, and

$$\begin{aligned} &E\big[(\tau_{n,k} - \tau_{n,k-1}) \mid \mathcal{F}_{n,k-1}^0\big] = \sigma^{-2} E\big[u_{k+1}^2 \mid \mathcal{F}_k\big] \quad \text{and} \\ &E\big[(\tau_{n,k} - \tau_{n,k-1})^{2m} \mid \mathcal{F}_{n,k-1}^0\big] \le C\,\sigma^{-4m} E\big[u_{k+1}^{4m} \mid \mathcal{F}_k\big], \quad m \ge 1, \quad a.s. \end{aligned} \tag{6.5}$$

for some constant $C > 0$. We mention that result (6.5) does not explicitly appear in Lemma 2.1 of Park and Phillips (2001). However it can be obtained by a construction along the same lines as Theorem A1 of Hall and Heyde (1980).

It follows from (6.3) that, under the extended probability space,

$$\begin{aligned}&\Big(\frac{1}{\sigma\, d_n}\sum_{t=2}^{n} u_{t+1}\, Y_{nt},\quad \frac{1}{d_n^2}\sum_{t=2}^{n} Y_{nt}^2\Big)\\&=_d\ \Big(\sum_{t=2}^{n}\big[V(\tau_{n,t}/n)-V(\tau_{n,t-1}/n)\big]\, Y_{n,t}^*,\quad \frac{1}{n}\sum_{t=2}^{n} Y_{nt}^{*2}\Big)\end{aligned}\tag{6.6}$$

where, with $c_n=\sqrt{n}\phi/h$,

$$Y_{nt}^*=\frac{n\sigma}{d_n}\sum_{i=1}^{t-1}\big[V(\tau_{n,i}/n)-V(\tau_{n,i-1}/n)\big]\, K\big\{c_n\,[G_n^0(t/n)-G_n^0(i/n)]\big\}.$$

To establish our main result, we extend $\sum_{i=2}^{n}\big[V(\tau_{n,t}/n)-V(\tau_{n,t-1}/n)\big]\, Y_{n,t}^*$ to a continuous martingale. This can be done by defining

$$M_n(r)=\sum_{t=2}^{j-1} Y_{nt}^*\big[V(\frac{\tau_{n,t}}{n})-V(\frac{\tau_{n,t-1}}{n})\big]+Y_{n,j}^*\big[V(r)-V(\frac{\tau_{n,j-1}}{n})\big],\tag{6.7}$$

for $\tau_{n,j-1}/n<r\le\tau_{n,j}/n, j=1,2,...,n$, and

$$M_n(r)=\sum_{t=2}^{n} Y_{nt}^*\big[V(\frac{\tau_{n,t}}{n})-V(\frac{\tau_{n,t-1}}{n})\big]+\frac{1}{\sqrt{n}}\big[V(r)-V(\frac{\tau_{n,n}}{n})\big],\tag{6.8}$$

for $r\ge\tau_{n,n}/n$. It is readily seen that M_n is a continuous martingale with quadratic variation process $[M_n]$ given by

$$[M_n]_r\ =\ \sum_{t=2}^{j-1} Y_{nt}^{*2}\Big(\frac{\tau_{n,t}}{n}-\frac{\tau_{n,t-1}}{n}\Big)+Y_{n,j}^{*2}\Big(r-\frac{\tau_{n,j-1}}{n}\Big)\tag{6.9}$$

for $\tau_{n,j-1}/n<r\le\tau_{n,j}/n, j=1,2,...,n$, and

$$[M_n]_r\ =\ \sum_{t=2}^{n} Y_{nt}^{*2}\Big(\frac{\tau_{n,t}}{n}-\frac{\tau_{n,t-1}}{n}\Big)+\frac{1}{n}\Big(r-\frac{\tau_{n,n}}{n}\Big)\tag{6.10}$$

for $r\ge\tau_{n,n}/n$. Similarly, the covariance process $[M_n,V]$ of M_n and V is given by

$$[M_n,V]_r\ =\ \sum_{t=2}^{j-1} Y_{nt}^{*}\Big(\frac{\tau_{n,t}}{n}-\frac{\tau_{n,t-1}}{n}\Big)+Y_{n,j}^{*}\Big(r-\frac{\tau_{n,j-1}}{n}\Big),\tag{6.11}$$

for $\tau_{n,j-1}/n<r\le\tau_{n,j}/n, j=1,2,...,n$, and

$$[M_n,V]_r\ =\ \sum_{t=2}^{n} Y_{nt}^{*}\Big(\frac{\tau_{n,t}}{n}-\frac{\tau_{n,t-1}}{n}\Big)+\frac{1}{\sqrt{n}}\Big(r-\frac{\tau_{n,n}}{n}\Big),\tag{6.12}$$

for $r\ge\tau_{n,n}/n$.

Write $\rho_n(t) = \inf\{s : [M_n]_s > t\}$, a sequence of time changes. Note that $[M_n]_\infty = \infty$ for every $n \geq 1$ and

$$[M_n, V]_{\rho_n(t)} \to_P 0, \quad \text{as } n \to \infty, \tag{6.13}$$

for every $t \in R$, by (6.15) in Proposition 6.1 below. Theorem 2.3 of Revuz and Yor (1999, page 524) yields that, if we call B^n (i.e., $B^n(r) = M_n\{\rho_n(r)\}$) the DDS Brownian motion (see, for example, Revuz and Yor (1999), page 181) of the continuous martingale M_n defined by (6.7) and (6.8), then B^n converges in distribution to a Wiener process W. Since the law of the processes B^n are all given by Wiener measure, it is plain that $B^n(r) \Rightarrow W(r)$ (mixing), where the concept of mixing can be found in Hall and Heyde (1980, page 56). This, together with (6.16) in Proposition 6.2 below, yields that $(B^n(r), [M_n]_1) \Rightarrow (W(r), \eta^2)$, where W is independent of $\eta^2 = L_G(1, 0)$, defined as in (3.1). Now, by noting that $M_n(1)$ is equal to $B^n([M_n]_1)$, the continuous mapping theorem implies that

$$(M_n(1), [M_n]_1) \to_D (\eta\, N, \eta^2), \tag{6.14}$$

where N is a normal variate independent of η.

By virtue of (6.6) and (6.14), the required result of the theorem follows (6.17) and (6.18) in Proposition 6.2 and Proposition 6.3 below.

It remains to show the following Propositions 6.1-6.3, which will be given in Section 7.2-7.4, respectively. The proof of Theorem 3.2 under $|u_j| \leq A$ is now complete.

PROPOSITION 6.1. *In addition to Assumptions 1-3, assume that $|u_j| \leq A$, $nh^2 \to \infty$ and $h \log^2 n \to 0$. Then, as $n \to \infty$,*

$$[M_n, V]_r \quad \to \quad 0, \quad \textit{in Probab.} \tag{6.15}$$

uniformly on $r \in [0, T]$, where T is an arbitrary given constant.

PROPOSITION 6.2. *In addition to Assumptions 1-3, assume that $|u_j| \leq A$, $nh^2 \to \infty$ and $nh^4 \log^2 n \to 0$. Under the extended probability space used in (6.2), we have*

$$[M_n]_1 \to_P \eta^2, \tag{6.16}$$

where $\eta^2 = L_G(1, 0)$ is defined as in (3.1), and

$$[M_n]_1 - \frac{1}{n}\sum_{t=1}^{n} Y_{nt}^{*2} = o_P(1). \tag{6.17}$$

PROPOSITION 6.3. *In addition to Assumptions 1-3, assume that* $|u_j| \le A$, $nh^2 \to \infty$ *and* $nh^4 \log^2 n \to 0$. *Then,*

$$M_n(1) - \sum_{t=2}^n Y_{nt}^* \big[V(\frac{\mathcal{T}_{n,t}}{n}) - V(\frac{\mathcal{T}_{n,t-1}}{n})\big] \;=\; o_P(1). \tag{6.18}$$

We next remove the restriction $|u_j| \le A$. To this end, let

$$\begin{aligned} u_{1j} &= u_j I(|u_j| \le A/2) - E\big[u_j I(|u_j| \le A/2) \mid \mathcal{F}_{j-1}\big], \\ u_{2j} &= u_j I(|u_j| > A/2) - E\big[u_j I(|u_j| > A/2) \mid \mathcal{F}_{j-1}\big], \end{aligned}$$

and

$$Y_{1nt} \;=\; \sum_{i=1}^{t-1} u_{1,i+1}\, K[(x_t - x_i)/h], \quad Y_{2nt} = \sum_{i=1}^{t-1} u_{2,i+1}\, K[(x_t - x_i)/h].$$

With these notation, we may write

$$\begin{aligned} \frac{1}{d_n}\sum_{t=2}^n u_{t+1}\, Y_{nt} &= \frac{1}{d_n}\sum_{t=2}^n u_{1,t+1}\, Y_{1nt} + \frac{1}{d_n}\sum_{t=2}^n u_{1,t+1}\, Y_{2nt} + \frac{1}{d_n}\sum_{t=2}^n u_{2,t+1}\, Y_{nt} \\ &:= \frac{1}{d_n}\sum_{t=2}^n u_{1,t+1}\, Y_{1nt} + \Lambda_{1n} + \Lambda_{2n}, \end{aligned} \tag{6.19}$$

$$\begin{aligned} \frac{1}{d_n^2}\sum_{t=2}^n Y_{nt}^2 &= \frac{1}{d_n^2}\sum_{t=2}^n Y_{1nt}^2 + \frac{2}{d_n^2}\sum_{t=2}^n Y_{1nt}\, Y_{2nt} + \frac{1}{d_n^2}\sum_{t=2}^n Y_{2nt}^2 \\ &:= \frac{1}{d_n^2}\sum_{t=2}^n Y_{1nt}^2 + \Lambda_{3n} + \Lambda_{4n}. \end{aligned} \tag{6.20}$$

Recall that $|u_{1j}| \le A$ and u_{1j} is a martingale difference satisfying

$$E(u_{1t}^2 \mid \mathcal{F}_{t-1}) \;=\; E(u_t^2 I(|u_t| \le A) \mid \mathcal{F}_{t-1}) - \big[E(u_t I(|u_t| \le A) \mid \mathcal{F}_{t-1})\big]^2 \to \sigma^2, \quad a.s.$$

as $j, A \to \infty$. It follows from the previous proof that, when $n \to \infty$ first and then $A \to \infty$,

$$\Big(\frac{1}{\sigma\, d_n}\sum_{t=2}^n u_{1,t+1}\, Y_{1nt}, \;\; \frac{1}{d_n^2}\sum_{t=2}^n Y_{1nt}^2\Big) \to_D (\eta\, N, \; \eta^2). \tag{6.21}$$

Now it is readily seen that the required result will follow if we prove

$$\Lambda_{in} \to_P 0, \qquad i = 1, 2, 3, 4, \tag{6.22}$$

as $n \to \infty$ first and then $A \to \infty$. In fact, by virtue of (7.26) in Lemma 7.6 below,

$$\sup_{1 \le i \le n} Eu_i^2 \le \sup_{1 \le i \le n} (Eu_i^4)^{1/4} < \infty$$

and $\sup_x K(x) < \infty$, we have, for $1 \le t \le n$,

$$\begin{aligned} EY_{nt}^2 &\le 2\sup_x K(x)\, Eu_t^2 + 2E\Big(\sum_{i=1}^{t-2} u_{i+1}\, K[(x_t - x_i)/h]\Big)^2 \\ &\le C \sup_{1\le i\le n} Eu_i^2\big(1 + h^2\sqrt{t}\log t + h\sqrt{t}\big) \le C_1 h\sqrt{n}, \end{aligned}$$

since $h\log n \to 0$ and $nh^2 \to \infty$. Similarly

$$\begin{aligned} EY_{1nt}^2 &\le C \sup_{1\le i\le n} Eu_i^2 I(|u_i| \le A)\big(1 + h^2\sqrt{t}\log t + h\sqrt{t}\big) \le C_1 h\sqrt{n}, \\ EY_{2nt}^2 &\le C \sup_{1\le i\le n} Eu_i^2 I(|u_i| > A)\big(1 + h^2\sqrt{t}\log t + h\sqrt{t}\big) \le C_1 A^{-2} h\sqrt{n}. \end{aligned}$$

These results, together with the fact that u_{1j} and u_{2j} both are martingale difference satisfying

$$\begin{aligned} \sup_j E(u_{1,j+1}^2 \mid \mathcal{F}_j) &\le \sup_j [E(u_j^4 \mid \mathcal{F}_j)]^{1/2} \le C, \\ \sup_j E(u_{2,j+1}^2 \mid \mathcal{F}_j) &\le \sup_j E(u_j^2 I_{|u_j|>A} \mid \mathcal{F}_j) \le A^{-2} \sup_j E(u_j^4 \mid \mathcal{F}_j) \le CA^{-2}, \end{aligned}$$

yield that, as $n \to \infty$ first and then $A \to \infty$,

$$\begin{aligned} E\Lambda_{1n}^2 &\le \frac{C}{n^{3/2}h} \sum_{t=2}^{n} EY_{2nt}^2 \le CA^{-2} \to 0, \\ E\Lambda_{2n}^2 &\le \frac{CA^{-2}}{n^{3/2}h} \sum_{t=2}^{n} EY_{nt}^2 \le CA^{-2} \to 0, \\ E\Lambda_{4n} &\le \frac{C}{n^{3/2}h} \sum_{t=2}^{n} EY_{2nt}^2 \le CA^{-2} \to 0, \\ E|\Lambda_{3n}| &\le \frac{C}{n^{3/2}h} \sum_{t=2}^{n} (EY_{1nt}^2)^{1/2}(EY_{2nt}^2)^{1/2} \le CA^{-1} \to 0. \end{aligned}$$

This proves (6.22) and hence the proof of Theorem 3.2 is complete. □

6.2 Proof of Theorem 3.1. By virtue of (3.3) and Theorem 3.2, it suffices to show that

$$S_{2n} = o_P(n^{3/4}\sqrt{h}) \quad \text{and} \quad S_{3n} = o_P(n^{3/4}\sqrt{h}). \tag{6.23}$$

To prove (6.23), we require the following propositions. Their proofs will be given in Sections 7.5 and 7.6 respectively.

PROPOSITION 6.4. *Suppose that Assumptions 1-2 hold, $E|\epsilon_0|^{\max\{[\alpha],2\}} < \infty$ and $p(x)$ satisfies $\int(|p(x)| + p^2(x))dx < \infty$. Then, for any $\alpha > 0$, $h > 0$ and $1 \le s \le n-1$,*

$$E\Big\{|u_{s+1}|\,(1+|x_s|^{\alpha-1})\,|p[(x_{s+1}-x_s)/h]|\Big\} \;\le\; Ch^{1/2}(1+s^{(\alpha-1)/2}); \tag{6.24}$$

for any $\alpha > 0$, $h > 0$ and $t \ge s+2$,

$$E\Big\{(1+|u_{s+1}|)(1+|x_s|^{\alpha-1})\,|p[(x_t-x_s)/h]|\Big\} \;\le\; Ch(1+s^{(\alpha-1)/2})/\sqrt{t-s}. \tag{6.25}$$

If in addition $\int |x|^{\max\{[\alpha_1],[\alpha_2]\}+1}|p(x)|dx < \infty$ and $E|\epsilon_0|^{[\alpha_1]+[\alpha_2]+2} < \infty$, then for any $\alpha_1, \alpha_2 \ge 0$, $h > 0$ and $0 \le s < t < i$,

$$E\Big\{(1+|x_s|^{\alpha_1})\,(1+|x_t|^{\alpha_2})|p[(x_i-x_t)/h]|\,|p[(x_i-x_s)/h]|\Big\} \le \frac{Ch^2\, s^{\alpha_1/2}t^{\alpha_2/2}}{\sqrt{t-s}\sqrt{i-t}}; \tag{6.26}$$

for any $\alpha_1, \alpha_2 \ge 0$, $h > 0$ and $t \ne s, s+1$,

$$\begin{aligned} &E\Big\{g(u_{s+1})g_1(u_{t+1})(1+|x_s|^{\alpha_1})(1+|x_t|^{\alpha_2})K[(x_t-x_s)/h]\Big\} \\ &\qquad \le \frac{Ch\, s^{\alpha_1/2}t^{\alpha_2/2}}{\sqrt{t-s}}, \end{aligned} \tag{6.27}$$

where $g(x)$ and $g_1(x)$ are positive real functions such that

$$\sup_{s\ge1} E\big\{[g(u_{s+1}) + g_1(u_{s+1})] \mid \mathcal{F}_s\big\} < \infty.$$

PROPOSITION 6.5. *Write $Z_{2i} = u_{i+1}\, g(x_i) \sum_{t=i+1}^n K\big[(x_t-x_i)/h\big]$, where $|g(x)| \le C(1+|x|^\beta)$ for some $\beta \ge 0$. Then, under Assumptions 1-3 and $E|\epsilon_0|^{\max\{[4\beta]+1,\ 2[\beta]+2\}} < \infty$, we have*

$$E\Big(\sum_{i=1}^n Z_{2i}\Big)^2 \;\le\; C\, n^{5/2+\beta}h^{3/2}(h^{1/2}\,\log n + 1). \tag{6.28}$$

We now turn back to the proof of (6.23). To this end, for $\delta > 0$, let $\Omega_n = \{\widehat{\theta} : ||\widehat{\theta}-\theta|| \le \delta\,\delta_n, \theta \in \Omega_0\}$, where δ_n is given in Assumption 4 (i). Recall Assumption 4 and note that $\Omega_n \subset \Theta_0$ for all n sufficiently large. It follows by Taylor expansion that, whenever n is sufficiently large and $\hat\theta \in \Omega_n$,

$$\begin{aligned} S_{2n} &= (\theta-\hat\theta)\sum_{i=1}^n u_{i+1}\sum_{t=i+1}^n \frac{\partial f(x_t,\theta)}{\partial\theta}\,K\big[(x_t-x_i)/h\big] + \Delta_{1n} + \Delta_{2n} \\ &= \Delta_n + \Delta_{3n} + \Delta_{1n} + \Delta_{2n}, \end{aligned} \tag{6.29}$$

where,

$$\begin{aligned}
\Delta_{1n} &= (\theta-\hat\theta)\sum_{i=1}^n u_{i+1}\sum_{t=1}^{i-1}\frac{\partial f(x_t,\theta)}{\partial\theta}K\big[(x_t-x_i)/h\big],\\
\Delta_{2n} &\le C\,|\hat\theta-\theta|^2\sum_{\substack{i,t=1\\ i\neq t}}^n |u_{i+1}|(1+\,|x_t|^\beta)\,K\big[(x_t-x_i)/h\big],\\
\Delta_n &= (\theta-\hat\theta)\sum_{i=1}^n u_{i+1}\,\frac{\partial f(x_i,\theta)}{\partial\theta}\,\sum_{t=i+1}^n K\big[(x_t-x_i)/h\big],
\end{aligned}$$

and with $K_s(x)=|x|^sK(x)$ and $\nu(x)=\begin{cases}1+|x|^{\beta-1}, & \text{if } \beta>0,\\ 1+|x|^{\gamma-1}, & \text{if } \beta=0;\end{cases}$,

$$\Delta_{3n} \;\le\; C\,h\,|\hat\theta-\theta|\sum_{i=1}^n|u_{i+1}|\,\nu(x_i|)\,\sum_{t=i+1}^n\Big(K_1\big[(x_t-x_i)/h\big]+K_{k+1}\big[(x_t-x_i)/h\big]\Big).$$

Recall $\delta_n^2 n^{1+\beta}\sqrt{h}\to 0$ by Assumption 5. It follows from (6.28) in Proposition 6.5 with $g(x)=\frac{\partial f(x,\theta)}{\partial\theta}$ that, whenever $\hat\theta\in\Omega_n$,

$$E\Delta_n^2\le C\delta^2\,\delta_n^2\,n^{5/2+\beta}h^{3/2}(h^{1/2}\,\log n+1)=o(n^{3/2}h). \tag{6.30}$$

Similarly, using (6.24)-(6.25) and (6.27) in Proposition 6.4, simple calculations show that

$$\begin{aligned}
E(|\Delta_{2n}|+|\Delta_{3n}|) &\le C\delta^2\,\delta_n^2\Big(h^{1/2}\sum_{s=1}^n s^{\beta/2}+h\sum_{\substack{s,t=1\\ s\neq t}}^n\frac{s^{\beta/2}}{\sqrt{|t-s|}}\Big)\\
&\quad +C\delta\,h\,\delta_n\Big(h^{1/2}\sum_{s=1}^n(1+s^{(\beta-1)/2})+h\sum_{s=1}^n\sum_{t=s+2}^n\frac{(1+s^{(\beta-1)/2})}{\sqrt{t-s}}\Big)\\
&\le C\delta\,\big(h\delta_n^2n^{(3+\beta)/2}+h^2\delta_n n^{\max\{3/2,1+\beta/2\}}\big)\\
&= o(n^{3/4}\sqrt{h}),
\end{aligned}\tag{6.31}$$

since $\delta_n^2n^{1+\beta}\sqrt{h}\to 0$ and $nh^4\log^2 n\to 0$. As for Δ_{1n}, by recalling $x_1,...,x_t$ are $\mathcal{F}_t$-measurable by Assumption 2, it follows from Assumptions 2-5 and (6.26)-(6.27) with

$p(x) = K(-x)$ that

$$\begin{aligned}
E\Delta_{1n}^2 &\le C\delta^2\,\delta_n^2 \sum_{i=1}^n E\Big\{\big(\sum_{t=1}^{i-1}\frac{\partial f(x_t,\theta)}{\partial\theta}K\big[(x_t-x_i)/h\big]\big)^2 E(u_{t+1}^2\mid\mathcal{F}_t)\Big\}\\
&\le C_1\,\delta^2\,\delta_n^2\sum_{i=1}^n\sum_{1\le s<t\le i-1} E\Big\{(1+|x_s|^\beta)(1+|x_t|^\beta)K[(x_t-x_i)/h]|\,K[(x_s-x_i)/h]|\Big\}\\
&\qquad\qquad +C_1\,\delta^2\,\delta_n^2\sum_{i=1}^n\sum_{t=1}^{i-1}E\Big\{(1+|x_t|^{2\beta})K^2[(x_t-x_i)/h]|\Big\}\\
&\le C_2\,\delta^2\,\delta_n^2\,h^2\sum_{i=1}^n\sum_{1\le s<t\le i-1}\frac{s^{\beta/2}t^{\beta/2}}{\sqrt{i-t}\sqrt{t-s}}+C_2\,\delta^2\,\delta_n^2\,h\sum_{i=1}^n\sum_{t=1}^{i-1}\frac{t^\beta}{\sqrt{i-t}}\\
&\le C_3\delta^2\,\delta_n^2\,\big(h^2n^{2+\beta}+hn^{3/2+\beta}\big)=o(n^{3/2}h). \qquad (6.32)
\end{aligned}$$

Combining (6.29)–(6.32), we obtain, for any $\delta > 0$,

$$\begin{aligned}
&P(|S_{2n}|\ge \delta\,n^{3/4}\sqrt{h})\\
\le\;& P(|S_{2n}|\ge\delta\,n^{3/4}\sqrt{h},\hat\theta\in\Omega_n)+P(||\hat\theta-\theta||\ge\delta\,\delta_n)\\
\le\;& \frac{1}{\delta^2n^{3/2}h}E\big(\Delta_n^2+\Delta_{1n}^2\big)+\frac{1}{\delta n^{3/4}\sqrt{h}}E\big(|\Delta_{2n}|+|\Delta_{3n}|\big)+P(||\hat\theta-\theta||\ge\delta\,\delta_n)\\
\to\;& 0,\quad \text{as } n\to\infty.
\end{aligned}$$

This proves the first part of (6.23). Similarly, by noting

$$\begin{aligned}
E|S_{3n}| &\le C\,E\Big\{(\hat\theta-\theta)^2\sum_{\substack{i,t=1\\ i\ne t}}^n\big|\frac{\partial f(x_i,\theta)}{\partial\theta}\big|\,\big|\frac{\partial f(x_t,\theta)}{\partial\theta}\big|\,K\big[(x_t-x_i)/h\big]\Big\}\\
&\le C\,\delta^2\,\delta_n^2\sum_{\substack{i,t=1\\ i\ne t}}^n E\Big\{(1+|x_i|^\beta)\,(1+|x_t|^\beta)\,K\big[(x_t-x_i)/h\big]\Big\}\\
&\le C\,\delta^2\,\delta_n^2\sum_{\substack{i,t=1\\ i\ne t}}^n\frac{h\,i^\beta}{\sqrt{|t-i|}}\le C\delta^2\,h\,\delta_n^2 n^{3/2+\beta}=o(n^{3/4}\sqrt{h}), \qquad (6.33)
\end{aligned}$$

whenever $\hat\theta\in\Omega_n$ and n is sufficiently large, we obtain, for any $\delta>0$,

$$\begin{aligned}
&P(|S_{3n}|\ge\delta\,n^{3/4}\sqrt{h})\\
\le\;& P(|S_{3n}|\ge\delta\,n^{3/4}\sqrt{h},\hat\theta\in\Omega_n)+P(||\hat\theta-\theta||\ge\delta\,\delta_n)\\
\le\;& \frac{1}{\delta n^{3/4}\sqrt{h}}E|S_{3n}|+P(||\hat\theta-\theta||\ge\delta\,\delta_n)\\
\to\;& 0,\quad \text{as } n\to\infty.
\end{aligned}$$

This proves the second part of (6.23), and hence the proof of Theorem 3.1 is complete. □

6.3 Proof of Theorem 3.3. It suffices to show (3.4). This is similar to the proof of Theorem 3.1. So we only give a outline. We may write

$$V_n^2 \;=\; \Big(\sum_{\substack{s,t=1\\ s\neq t,|s-t|\leq 1}}^n + \sum_{\substack{s,t=1\\ |s-t|\geq 2}}^n\Big)\hat{u}_{t+1}^2\hat{u}_{s+1}^2\,K^2\big[(x_t-x_s)/h\big] := V_{1n}+V_{2n},$$

where we further have

$$\begin{aligned} V_{2n} &= \sum_{\substack{s,t=1\\ |s-t|\geq 2}}^n u_{s+1}^2u_{t+1}^2K^2\big[(x_t-x_s)/h\big] + \sum_{\substack{s,t=1\\ |s-t|\geq 2}}^n (\hat{u}_{s+1}^2-u_{s+1}^2)\,\hat{u}_{t+1}^2\,K^2\big[(x_t-x_s)/h\big] \\ &\quad + \sum_{\substack{s,t=1\\ |s-t|\geq 2}}^n u_{s+1}^2\,(\hat{u}_{t+1}^2-u_{t+1}^2)\,K^2\big[(x_t-x_s)/h\big] \\ &:= V_{3n}+V_{4n}+V_{5n}. \end{aligned}$$

As in the proof of Theorem 3.1, the result (3.4) will follow if we prove

$$V_{3n} = \sigma^4 \sum_{\substack{s,t=1\\ s\neq t}}^n K^2\big[(x_t-x_s)/h\big] + o_P(n^{3/2}h), \tag{6.34}$$

and

$$E|V_{kn}| \;=\; o_P(n^{3/2}h), \quad k=1,4,5, \tag{6.35}$$

whenever $\hat{\theta}\in\Omega_n$, where $\Omega_n=\{\widehat{\theta}:||\widehat{\theta}-\theta||\leq \delta\,\delta_n, \theta\in\Omega_0\}$.

The result (6.35) is simple for $k=1$. Indeed, by recalling $|f(x_s,\theta)-f(x_s,\hat{\theta})|\leq C\delta\,\delta_n(1+|x_s|^{\beta})$ when $\hat{\theta}\in\Omega_n$, $\hat{u}_{s+1}^2\leq 2(u_{s+1}^2+|f(x_s,\theta)-f(x_s,\hat{\theta})|^2)$ and $\sup_x K(x)<\infty$, simple calculations show that

$$\begin{aligned} E|V_{1n}| &\leq C \sum_{\substack{s,t=1\\ s\neq t,|s-t|\leq 1}}^n (E\hat{u}_{s+1}^4)^{1/2}\,(E\hat{u}_{4+1}^4)^{1/2} \\ &\leq C \sum_{\substack{s,t=1\\ s\neq t,|s-t|\leq 1}}^n \big[Eu_{s+1}^4+\delta\,\delta_n^4\,E(1+|x_s|^{4\beta})\big]^{1/2}\big[Eu_{t+1}^4+\delta\,\delta_n^4\,E(1+|x_t|^{4\beta})\big]^{1/2} \\ &\leq C\,n(1+\delta\,\delta_n^4n^{2\beta}) = o(n^{3/2}h), \end{aligned}$$

since $nh^2\to\infty$ and $\delta_n^2n^{1+\beta}\sqrt{h}\to 0$, where we have used (7.11) in Lemma 7.2 below.

Similarly, it follows from (6.27) in Proposition 6.4 that

$$\begin{aligned}
E|V_{4n}| + E|V_{5n}| &\le C\delta_n \sum_{\substack{s,t=1\\|s-t|\ge 2}}^{n} E\big\{|u_{s+1}|\, u_{t+1}^2\, (1+|x_s|^{\beta})\, K\big[(x_t-x_s)/h\big]\big\} \\
&\quad + C\delta_n^2 \sum_{\substack{s,t=1\\|s-t|\ge 2}}^{n} E\big\{u_{t+1}^2\, (1+|x_s|^{2\beta})\, K\big[(x_t-x_s)/h\big]\big\} \\
&\quad + C\delta_n^3 \sum_{\substack{s,t=1\\|s-t|\ge 2}}^{n} E\big\{|u_{s+1}|\, (1+|x_s|^{\beta})\, (1+|x_t|^{2\beta})\, K\big[(x_t-x_s)/h\big]\big\} \\
&\quad + C\delta_n^4 \sum_{\substack{s,t=1\\|s-t|\ge 2}}^{n} E\big\{(1+|x_s|^{2\beta})\, (1+|x_t|^{2\beta})\, K\big[(x_t-x_s)/h\big]\big\} \\
&\le \sum_{\substack{s,t=1\\|s-t|\ge 2}}^{n} \frac{Ch}{\sqrt{t-s}} \big(\delta_n n^{\beta/2} + \delta_n^2 n^{\beta} + \delta_n^3 n^{3\beta/2} + \delta_n^4 n^{2\beta}\big) \\
&= o(n^{3/2}h),
\end{aligned}$$

since $nh^2 \to \infty$ and $\delta_n^2 n^{1+\beta}\sqrt{h} \to 0$. This proves (6.35) for $k = 4$ and 5.

In order to prove (6.34), let $K^*(x) = \frac{1}{2}[K(x) + K(-x)]$. It is readily seen that $K^*(x)$ is symmetric and $K^*(x)$ still satisfies Assumption 3. Simple calculations show that

$$\begin{aligned}
V_{3n} &= \sum_{\substack{s,t=1\\|s-t|\ge 2}}^{n} u_{s+1}^2 u_{t+1}^2 K^{*2}\big[(x_t-x_s)/h\big] \\
&= \sigma^4 \sum_{\substack{s,t=1\\|s-t|\ge 2}}^{n} K^2\big[(x_t-x_s)/h\big] + 2 \sum_{\substack{s,t=1\\t\ge s+2}}^{n} (u_{t+1}^2+\sigma^2)\, \big[u_{s+1}^2 - E(u_{s+1}^2 \mid \mathcal{F}_s)\big] K^{*2}\big[(x_t-x_s)/h\big] \\
&\quad + 2 \sum_{\substack{s,t=1\\t\ge s+2}}^{n} (u_{t+1}^2+\sigma^2)\, \big[E(u_{s+1}^2 \mid \mathcal{F}_s) - \sigma^2\big] K^{*2}\big[(x_t-x_s)/h\big] \\
&:= V_{3n1} + V_{3n2} + V_{3n3}.
\end{aligned} \tag{6.36}$$

Note that $\sum_{s\ne t,|s-t|\le 1}^{n} K^2\big[(x_t-x_s)/h\big] \le 2\, n\, \sup_x K^2(x) = o(n^{3/2}h)$, since $nh^2 \to \infty$. It follows that

$$V_{3n1} = \sigma^4 \sum_{\substack{s,t=1\\s\ne t}}^{n} K^2\big[(x_t-x_s)/h\big] + o_P(n^{3/2}h). \tag{6.37}$$

The conditional arguments, together with (7.27) in lemma 7.6 below, implies that

$$
\begin{aligned}
E|V_{3n2}| &\le 2\sum_{t=3}^{n} E\Big\{E\big[(u_{t+1}^2+\sigma^2)\mid \mathcal{F}_t\big]\,\big|\sum_{s=1}^{t-2}\big[u_{s+1}^2-E(u_{s+1}^2\mid\mathcal{F}_s)\big]K^{*2}\big[(x_t-x_s)/h\big]\big|\Big\} \\
&\le C\sum_{t=3}^{n}\Big\{E\big|\sum_{s=1}^{t-2}\big[u_{s+1}^2-E(u_{s+1}^2\mid\mathcal{F}_s)\big]K^{*2}\big[(x_t-x_s)/h\big]\big|^2\Big\}^{1/2} \\
&\le C\sum_{t=3}^{n}(h^2\sqrt{t}\log t+h\sqrt{t})^{1/2} \\
&\le Cn^{5/4}\sqrt{h}(h\log n+1)^{1/2}=o(n^{3/2}h), \qquad (6.38)
\end{aligned}
$$

since $nh^2\to 0$ and $nh^4\log^2 n\to 0$. To estimate V_{3n3}, we rewrite V_{3n3} as

$$
V_{3n3} = \sum_{s=1}^{n-3}\big[E(u_{s+1}^2\mid\mathcal{F}_s)-\sigma^2\big]\sum_{t=s+2}^{n}(u_{t+1}^2+\sigma^2)\,K^{*2}\big[(x_t-x_s)/h\big].
$$

Recall that, by (6.27) in Proposition 6.4,

$$
E\big\{(u_{t+1}^2+\sigma^2)\,K^{*2}\big[(x_t-x_s)/h\big]\big\}\le \frac{Ch}{\sqrt{t-s}},
$$

whenever $t\ge s+1$. It is readily seen that

$$
\sum_{s=1}^{n-3}\sum_{t=s+2}^{n}E(u_{t+1}^2+\sigma^2)\,K^{*2}\big[(x_t-x_s)/h\big] \le C\,n^{3/2}h,
$$

and for any $c_n/n\to 0$,

$$
\sum_{s=1}^{c_n}\sum_{t=s+2}^{n}E(u_{t+1}^2+\sigma^2)\,K^{*2}\big[(x_t-x_s)/h\big] \le C\,c_n\sqrt{n}\,h=o(n^{3/2}h).
$$

By virtue of these facts and $E(u_{s+1}^2\mid\mathcal{F}_s)\to\sigma^2, a.s.$, it follows from Lemma 7.3 below that

$$
E|V_{3n3}|^{1/2} = o(n^{3/4}\sqrt{h}). \qquad (6.39)
$$

The results (6.38) and (6.39) imply that $V_{3n2}+V_{3n3}=o_P(n^{3/2}h)$. This, together with (6.36)-(6.37), yields (6.34). The proof of Theorem 3.3 is now complete. □

7 Proofs of propositions

Except where mentioned explicitly, the notation in this section is the same as in previous sections. In Section 7.1, we introduce some previous lemmas. The proofs of Propositions 4.1-4.5 will be given in Sections 7.2-7.6.

7.1 Preliminaries. First note that

$$\begin{aligned} x_t &= \sum_{j=1}^t \rho^{t-j}\,\eta_j = \sum_{j=1}^t \rho^{t-j} \sum_{i=-\infty}^{j} \epsilon_i\phi_{j-i} \\ &= \rho^{t-s}\,x_s + \sum_{j=s+1}^t \rho^{t-j} \sum_{i=-\infty}^{s} \epsilon_i\phi_{j-i} + \sum_{j=s+1}^t \rho^{t-j} \sum_{i=s+1}^{j} \epsilon_i\phi_{j-i} \\ &:= \rho^{t-s}\,x_s + \Delta_{s,t} + x'_{s,t}, \qquad (7.1) \end{aligned}$$

where

$$x'_{s,t} = \sum_{j=1}^{t-s} \rho^{t-j-s} \sum_{i=1}^{j} \epsilon_{i+s}\phi_{j-i} = \sum_{i=s+1}^{t} \epsilon_i \sum_{j=0}^{t-i} \rho^{t-j-i}\,\phi_j.$$

Write $d_{s,t}^2 = \sum_{i=s+1}^t \rho^{2(t-i)}(\sum_{j=0}^{t-i}\rho^{-j}\phi_j)^2 = E(x'_{s,t})^2$. Without loss of generality, assume $d_{s,t}\neq 0$ for all $0\le s<t\le n$. Otherwise, $x'_{s,t}=0, a.s.$ This occurs only in the situation such as $\phi_0=\phi_1=...=\phi_{k_0}=0$ with a finite k_0. Hence $t-s$ must be small and the main results can be obtained by a routine modification. By virtue of this fact, for all $0\le s<t\le n$, we have that $C_1(t-s)\le d_{s,t}^2\le C_2(t-s)$ and

$$\tfrac{1}{\sqrt{t-s}}x'_{s,t} \text{ has a density } h_{s,t}(x), \qquad (7.2)$$

which is uniformly bounded by a constant C_0. See (7.14) and Proposition 7.2 (page 1934 there) of Wang and Phillips (2009b) with a minor modification. Furthermore we may prove that,

$$\text{conditional on } \mathcal{F}_s,\ \ x_t/\sqrt{t} \text{ has a density } h_{s,t}(x - x_s^*/\sqrt{t-s}), \qquad (7.3)$$

where $x_s^* = \rho^{t-s}\,x_s + \Delta_{s,t}$, and under Assumption 1,

$$\sup_x |h_{s,t}(x+y)-h_{s,t}(x)| \le C\,|y|, \qquad (7.4)$$

$$|\Delta_{s,t}|\ \le\ e\sum_{i=-\infty}^{s}|\epsilon_i|\sum_{j=s+1}^{t}|\phi_{j-i}| \le C\sum_{k=0}^{\infty}(k+1)^{-1-\delta}|\epsilon_{s-k}|. \qquad (7.5)$$

Indeed (7.3) follows from (7.2) because of the independence between $\mathcal{F}_s$ and $\epsilon_k, k\ge s+1$ and (7.5) is obvious by recalling $\sum_{k=0}^\infty k^{1+\delta}|\phi_k|<\infty$. If we write $\varphi_{s,t}(u) = Ee^{iux'_{s,t}/\sqrt{t-s}}$, arguments similar to those in the proof of Corollary 2.2 in Wang and Phillips (2009a) yield that, uniformly for $0\le s<t\le n$, $\int_{-\infty}^\infty(1+|u|)|\varphi_{s,t}(u)|du<\infty$. It follows by inversion of the characteristic function $\varphi_{s,t}(u)$ that

$$\begin{aligned} \sup_x |h_{s,t}(x+y)-h_{s,t}(x)| &= \frac{1}{\sqrt{2\pi}}\Big|\int_{-\infty}^{\infty}\big(e^{-iu(x+y)}-e^{-iux}\big)\varphi_{s,t}(u)du\Big| \\ &\le C\,|y|\int_{-\infty}^{\infty}|u||\varphi_{s,t}(u)|du \le C_1\,|y|, \end{aligned}$$

which implies (7.4).

By making use of (7.2)-(7.5), we may establish the following lemmas which play a key part in the proofs of our main results.

LEMMA 7.1. *Assume that $p(x)$ satisfies $\int |p(x)|dx < \infty$. Let Assumptions 1-2 hold. Then,*

(i) for any $0 \le s < t \le n$ and $h > 0$,

$$E\{p(x_t/h) \,|\mathcal{F}_s\} \;=\; \frac{h}{\sqrt{t-s}} \int_{-\infty}^{\infty} p\big(\rho^{t-s}\, x_s/h + x\big) h_{s,t}\big(\frac{hx}{\sqrt{t-s}}\big)dx \,+ \mathcal{L}_n \qquad (7.6)$$

where

$$|\mathcal{L}_n| \;\le\; \frac{Ch}{t-s}\Big(1+\sum_{k=0}^{\infty}(k+1)^{-1-\delta}|\epsilon_{s-k}|\Big)\int_{-\infty}^{\infty}|p(x)|dx\,;$$

(ii) for any $2 \le s+2 \le t \le n$ and $h > 0$,

$$\Big|\, E\Big\{g(u_{s+1})p(x_t/h)\,\big|\mathcal{F}_s\Big\}\,\Big| \;\le\; \frac{Ch}{t-s}\int_{-\infty}^{\infty}|p(x)|dx\,, \qquad (7.7)$$

where $g(x)$ satisfies that $E\big[g(u_{s+1}) \mid \mathcal{F}_s\big] = 0, s \ge 1$ and $\sup_{s\ge 1} E\big[g(u_{s+1})^2 \mid \mathcal{F}_s\big] \le C$;

(iii) for any $0 \le s < t \le n$,

$$E\big(|p(x_t/h)|\,\big|\mathcal{F}_s\big) \;\le\; \frac{Ch}{\sqrt{t-s}}\int_{-\infty}^{\infty}|p(x)|dx\,; \qquad (7.8)$$

(iv) for any $2 \le s+2 \le t \le n$ and $h > 0$,

$$E\Big\{|u_{s+1}|\,|p(x_t/h)|\,\big|\mathcal{F}_s\Big\} \;\le\; \frac{C\,h\,E(|u_{s+1}| \mid \mathcal{F}_s)}{\sqrt{t-s}}\int_{-\infty}^{\infty}|p(x)|dx\,; \qquad (7.9)$$

(v) for any $s+1 \le k_1, ..., k_m \le t$ and $2 \le s+m+2 \le t \le n$ and $h > 0$,

$$E\Big\{(1+|u_{s+1}|)\prod_{u=1}^{m}|\epsilon_{k_u}|\,|p(x_t/h)|\,\big|\mathcal{F}_s\Big\} \;\le\; \frac{Ch}{\sqrt{t-s-m}}\int_{-\infty}^{\infty}|p(x)|dx\,. \qquad (7.10)$$

Proof. By virtue of the independence between $\mathcal{F}_s$ and $\epsilon_k, k \ge s+1$, it follows from (7.1)–(7.2) that

$$\begin{aligned}
& E\Big\{p(x_t/h) \mid \mathcal{F}_s\Big\} \\
=\;& \frac{h}{\sqrt{t-s}}\int_{-\infty}^{\infty} p\big(\rho^{t-s}\, x_s/h + \Delta_{s,t}/h + x\big) h_{s,t}\big(\frac{hx}{\sqrt{t-s}}\big)dx \\
=\;& \frac{h}{\sqrt{t-s}}\int_{-\infty}^{\infty} p\big(\rho^{t-s}\, x_s/h + x\big) h_{s,t}\big[\frac{h}{\sqrt{t-s}}(x-\Delta_{s,t}/h)\big]dx \\
=\;& \frac{h}{\sqrt{t-s}}\int_{-\infty}^{\infty} p\big(\rho^{t-s}\, x_s/h + x\big) h_{s,t}\big(\frac{hx}{\sqrt{t-s}}\big)dx \,+ \mathcal{L}_n
\end{aligned}$$

where, by (7.4)–(7.5),

$$\begin{aligned}\mathcal{L}_n &\le \frac{h}{\sqrt{t-s}} \sup_x \big|h_{s,t}\big(x - \frac{\Delta_{s,t}}{\sqrt{t-s}}\big) - h_{s,t}(x)\big| \int_{-\infty}^{\infty} |p\big(\rho^{t-s}\, x_s/h + x\big)| dx \\ &\le \frac{Ch}{t-s} |\Delta_{s,t}| \int_{-\infty}^{\infty} |p(x)| dx \\ &\le \frac{Ch}{t-s} \Big(1 + \sum_{k=0}^{\infty} (k+1)^{-1-\delta} |\epsilon_{s-k}|\Big) \int_{-\infty}^{\infty} |p(x)| dx\,.\end{aligned}$$

This proves (7.6). Similarly, it follows from (7.1) and (7.2) that

$$\begin{aligned}& E\Big\{g(u_{s+1}) p(x_t/h) \,\big| \mathcal{F}_s\Big\} \\ =\; & \frac{h}{\sqrt{t-s-1}} \int_{-\infty}^{\infty} E\Big\{g(u_{s+1}) p\Big(\frac{\rho^{t-s} x_s}{h} + \frac{\Delta_{s,t}}{h} + \frac{(x'_{s,t} - x'_{s+1,t})}{h} + y\Big) \big| \mathcal{F}_s\Big\} \\ & \qquad h_{s+1,t}\big(\frac{hy}{\sqrt{t-s-1}}\big) dy \\ =\; & \frac{h}{\sqrt{t-s-1}} \int_{-\infty}^{\infty} p\Big(\frac{\rho^{t-s} x_s}{h} + \frac{\Delta_{s,t}}{h} + y\Big) \\ & \qquad E\Big\{g(u_{s+1}) h_{s+1,t}\big[\frac{h}{\sqrt{t-s-1}} \{y - (x'_{s,t} - x'_{s+1,t})/h\}\big] \big| \mathcal{F}_s\Big\}\, dy.\end{aligned}$$

This, together with $E(g(u_{s+1}) \mid \mathcal{F}_s) = 0$ and (7.4), yields

$$\begin{aligned}& \Big|E\Big\{u_{s+1} p(x_t/h) \,\big| \mathcal{F}_s\Big\}\Big| \\ \le\; & \frac{Ch}{t-s} \int_{-\infty}^{\infty} |p\Big(\frac{\rho^{t-s} x_s}{h} + \frac{\Delta_{s,t}}{h} + y\Big)|\, dy\, E\Big\{|g(u_{s+1})|\, |x'_{s,t} - x'_{s+1,t}| \mid \mathcal{F}_s\Big\} \\ \le\; & \frac{Ch}{t-s} \int_{-\infty}^{\infty} |p(y)|\, dy,\end{aligned}$$

where we have used the fact:

$$E\Big\{|g(u_{s+1})|\, |x'_{s,t} - x'_{s+1,t}| \mid \mathcal{F}_s\Big\} \le C\, \big(E\{g^2(u_{s+1}) \mid \mathcal{F}_s\}\big)^{1/2} \big(E\epsilon_1^2\big)^{1/2} \le C.$$

This proves (7.7). The proofs of (7.8) and (7.9) are simple. Note that, whenever $2 \le s + m + 2 \le t \le n$, similar to (7.2) we find that

$$\tfrac{1}{\sqrt{t-s-m}} \big(x'_{s+1,t} - \textstyle\sum_{u=1}^{m} \epsilon_{k_u} \sum_{j=0}^{t-k_u} \rho^{t-k_u-j} \phi_j\big) \text{ has a density, say } h'_{s,t}(x),$$

which is uniformly bounded by a constant C_0. It follows from (7.2) and independence between ϵ_j that, for $k_1, ..., k_m \le t$ and $2 \le s+m+2 \le t \le n$,

$$
\begin{aligned}
& E\Big\{(1+|u_{s+1}|) \prod_{u=1}^m |\epsilon_{k_u}|\, |p(x_t/h)|\, \big|\mathcal{F}_s\Big\} \\
= \;& \frac{h}{\sqrt{t-s}}\, E\Big\{(1+|u_{s+1}|) \prod_{u=1}^m |\epsilon_{k_u}| \\
& \int_{-\infty}^{\infty} \Big|p\Big(\frac{\rho^{t-s-1}x_{s+1}}{h} + \frac{\Delta_{s+1,t}}{h} + \frac{\sum_{u=1}^m \epsilon_{k_u} \sum_{j=0}^{t-k_u} \rho^{t-k_u-j}\phi_j}{h} + y\Big)\Big|\, |h'_{s,t}(\frac{yh}{\sqrt{t-s}})|\, dy \;\mid \mathcal{F}_s\Big\} \\
\le \;& \frac{C_0 h}{\sqrt{t-s}} \int_{-\infty}^{\infty} |p(y)| dy\, E\Big[(1+|u_{s+1}|) \prod_{u=1}^m |\epsilon_{k_u}|\, \big|\, \mathcal{F}_s\Big] \\
\le \;& \frac{Ch}{\sqrt{t-s}} \int_{-\infty}^{\infty} |p(y)| dy,
\end{aligned}
$$

which yields (7.10). The proof of Lemma 7.1 is now complete. □

LEMMA 7.2. *Suppose Assumption 1 holds. For any $1 \le s < t$ and $\alpha > 0$, we have*

$$
E\big(|x_t|^{\alpha-1}\big) \;\le\; C\, t^{(\alpha-1)/2}, \tag{7.11}
$$

provided $E|\epsilon_0|^{\max\{[\alpha],2\}} < \infty$;

$$
E\big\{|x_t|^{\alpha} \mid \mathcal{F}_s\big\} \;\le\; C\big[(t-s)^{\alpha/2} + |x_s|^{\alpha} + \big(\sum_{k=0}^{\infty}(k+1)^{-1-\delta}|\epsilon_{s-k}|\big)^{\alpha}\big], \tag{7.12}
$$

provided $E|\epsilon_0|^{\max\{[\alpha]+1,2\}} < \infty$.

Proof. If $\alpha - 1$ is an integer greater than 2, (7.11) is well-known. If $\alpha \ge 1$, it follows from $|x_t|^{\alpha-1} \le t^{(\alpha-1)/2}(1 + |x_t/\sqrt{t}|^{\max\{[\alpha],2\}})$ that

$$
E\big(|x_t|^{\alpha-1}\big) \;\le\; t^{(\alpha-1)/2}(1 + E|x_t/\sqrt{t}|^{\max\{[\alpha],2\}}) \le C\, t^{(\alpha-1)/2}.
$$

If $1 \ge \alpha > 0$, by recalling (7.3), we have

$$
\begin{aligned}
E\big(|x_t/\sqrt{t}|^{\alpha-1} \mid \mathcal{F}_0\big) \;&=\; \int_{-\infty}^{\infty} |x|^{\alpha-1} p_{0,t}(x - x_0^*/\sqrt{t}) dx \\
&\le\; C_0 \int_{|x|\le 1} |x|^{\alpha-1} dx + \int_{|x|\ge 1} p_{0,t}(x - x_0^*/\sqrt{t}) dx \le C,
\end{aligned}
$$

and hence $E\big(|x_t|^{\alpha-1}\big) \le C\, t^{(\alpha-1)/2}$. Combining all these fact, we obtain (7.11).

Recall $|\rho| \le C$ and $\sum_{j=1}^{\infty} |\phi_j| < \infty$. It is readily seen that $E|x'_{s,t}|^{\alpha} \le C(t-s)^{\alpha/2}$, where $x'_{s,t}$ is defined as in (7.1). Now the result (7.12) follows from (7.1), (7.2) and the fact that, whenever $\alpha \ge 0$,

$$
(|x| + |y| + |z|)^{\alpha} \le C_{\alpha}(|x|^{\alpha} + |y|^{\alpha} + |z|^{\alpha}),
$$

where C_{α} is a constant depending only on α. □

LEMMA 7.3. *Let A_k and B_k be two sequence of random variables satisfying $A_k \to 0, a.s.$ and $|A_k| \le C$, $\sum_{k=1}^n E|B_n| \le \Lambda_n^2$ and for any $c_n/n \to 0$, $\sum_{k=1}^{c_n} E|B_n| = o\big(\Lambda_n^2\big)$. Then,*

$$E\Big|\sum_{k=1}^n A_n\, B_n\Big|^{1/2} \;=\; o\big(\Lambda_n\big). \tag{7.13}$$

Proof. Note that $(|x|+|y|)^{1/2} \le |x|^{1/2} + |y|^{1/2}$ and

$$|\sum_{k=1}^n A_n\, B_n| \le C \sum_{k=1}^{\sqrt{n}} |B_n| + \max_{\sqrt{n}\le k\le n} |A_k| \sum_{k=1}^n |B_n|.$$

It follows that

$$\begin{aligned} E\Big|\sum_{k=1}^n A_n\, B_n\Big|^{1/2} &\le C\,(\sum_{k=1}^{\sqrt{n}} E|B_n|)^{1/2} + (E \max_{\sqrt{n}\le k\le n} |A_k|)^{1/2} \big(\sum_{k=1}^n E|B_n|\big)^{1/2} \\ &= o\big(\Lambda_n\big), \end{aligned}$$

as, by the dominated convergence theorem,

$$\lim_{n\to\infty} E \max_{\sqrt{n}\le k\le n} |A_k| = E \lim_{n\to\infty} \max_{\sqrt{n}\le k\le n} |A_k| = 0.$$

This proves Lemma 7.3. □

LEMMA 7.4. *Let $p_j(x), j = 1, 2, 3,$ be positive functions satisfying $\int_{-\infty}^{\infty} p_j(x)dx < \infty$ and $sup_x p_j(x) < \infty$. Then, under Assumption 1, we have*

$$E\, p_1[(x_t - x_i)/h] \le \frac{Ch}{\sqrt{t-i}} \int_{-\infty}^{\infty} p_1(x)dx, \tag{7.14}$$

$$E\, p_1[(x_t - x_i)/h]\, p_2[(x_t - x_j)/h] \le \frac{Ch^2}{\sqrt{t-j}\sqrt{j-i}} \int_{-\infty}^{\infty} p_1(x)dx \int_{-\infty}^{\infty} p_2(y)dy, \quad \text{for } i<j, \tag{7.15}$$

$$E\, p_1[(x_t - x_i)/h]\, p_2[(x_t - x_j)/h]\, p_3[(x_t - x_k)/h] \le \frac{C\, h^3}{\sqrt{t-k}\,\sqrt{k-j}\,\sqrt{j-i}} \int_{-\infty}^{\infty} p_1(y)\, dy \int_{-\infty}^{\infty} p_2(y)\, dy \int_{-\infty}^{\infty} p_3(y)\, dy, \tag{7.16}$$

for $i<j<k$. For the $K(x)$ defined in Assumption 3, write

$$\chi_{s,t}(i,j,k,l) \;=\; E\Big\{K\big[(x_s - x_i)/h\big]\, K\big[(x_s - x_j)/h\big] K\big[(x_t - x_k)/h\big]\, K\big[(x_t - x_l)/h\big]\Big\}.$$

Similar to (7.14)-(7.16), we also have

(i) if $t > s, s-2 \ge i, j,\ i \ne j,\ k = i$ and $l = j$ or $l = s-1$ or $l = s$ then

$$\chi_{s,t}(i,j,k,l) \;\le\; \frac{Ch^3}{\sqrt{t-s}} \frac{1}{\sqrt{s-i_1}}, \tag{7.17}$$

where $i_1 = \max\{i, j\}$*;*

(ii) if $t > s, s-2 \ge i, j, k,\ i \ne j \ne k$ *and* $l = i$ *or* $l = s-1$ *or* $l = s$*, then*

$$\chi_{s,t}(i,j,k,l) \;\le\; \frac{Ch^4}{\sqrt{t-s}} \frac{1}{\sqrt{s-i_1}} \frac{1}{\sqrt{i_2-i_3}}, \tag{7.18}$$

where $i_1 = \max\{i, j, k\}$*,* $i_3 = \min\{i, j, k\}$ *and* i_2 *is the median value of* i, j, k*.*

Proof. We only prove (7.16) and (7.18) for $i < j < k$ and $l = s-1$. The other results are similar but simpler. In order to prove (7.16), let $p^*(x) = p_1(x - x_i/h)p_2(x - x_j/h)p_3(x - x_k/h)$. Note that $p^*(x)$ is $\mathcal{F}_k$-measurable and

$$\begin{aligned}\int_{-\infty}^{\infty} p^*(x)dx &= \int_{-\infty}^{\infty} p_1[x + (x_k - x_i)/h]p_2[x + (x_k - x_j)/h]p_3(x)dx \\ &\le C\int_{-\infty}^{\infty} p_3(x)dx < \infty. \end{aligned} \tag{7.19}$$

It follows from (7.8) with $p(x) = p^*(x)$ that

$$\begin{aligned} Ep^*(x) &= E\big\{E\big[p^*(x) \mid \mathcal{F}_k\big]\big\} \\ &\le \frac{ch}{\sqrt{t-s}} \int_{-\infty}^{\infty} E\big\{p_1[x + (x_k - x_i/h)]p_2[x + (x_k - x_j)/h]\big\}\, p_3(x)dx \\ &\le \frac{C\, h^3}{\sqrt{t-k}\,\sqrt{k-j}\,\sqrt{j-i}} \int_{-\infty}^{\infty} p_1(y)\, dy \int_{-\infty}^{\infty} p_2(y)\, dy \int_{-\infty}^{\infty} p_3(y)\, dy, \end{aligned}$$

where, in the last inequality, we have used the result (7.15). This proves (7.16).

The idea to prove (7.18) for $i < j < k$ and $l = s-1$ is similar. Indeed, by using (7.8), we have

$$\begin{aligned} M_s &:= E\Big\{K\big[(x_t - x_k)/h\big]\, K\big[(x_t - x_{s-1})/h\big] \mid \mathcal{F}_s\Big\} \\ &\le \frac{Ch}{\sqrt{t-s}} \int_{-\infty}^{\infty} E\big\{K[x + (x_{s-1} - x_k/h)]\big\}\, K(x)dx, \\ N_s &:= E\Big\{K\big[(x_s - x_i)/h\big]\, K\big[(x_s - x_j)/h\big] \mid \mathcal{F}_{s-1}\Big\} \\ &\le Ch\int_{-\infty}^{\infty} E\big\{K[x + (x_j - x_i)/h]\big\}\, K(x)dx, \end{aligned}$$

Similarly, uniformly on $x, y \in R$, it follows from (7.8) that

$$\begin{aligned} &E\big\{K[x + (x_j - x_i)/h]K[y + (x_{s-1} - x_k)/h]\big\} \\ &\quad= E\big\{K[x + (x_j - x_i)/h]E\big(K[y + (x_{s-1} - x_k)/h] \mid \mathcal{F}_k\big)\big\} \\ &\quad\le \frac{Ch}{\sqrt{s-k}} E\big\{K[x + (x_j - x_i)/h] \int_{-\infty}^{\infty} K(y + z - x_k/h)dx\big\} \\ &\quad\le \frac{C_1h}{\sqrt{s-k}} EK[x + (x_j - x_i)/h] \\ &\quad\le \frac{Ch^2}{\sqrt{s-k}} \frac{1}{\sqrt{j-i}}. \end{aligned}$$

These facts, together with conditional arguments, yield that

$$\begin{aligned}
& \chi_{s,t}(i,j,k,s-1) = E\Big\{K\big[(x_s-x_i)/h\big]\,K\big[(x_s-x_j)/h\big]\,M_s\Big\} \\
\leq\ & \frac{Ch}{\sqrt{t-s}}\int_{-\infty}^{\infty} E\big\{K\big[(x_s-x_i)/h\big]\,K\big[(x_s-x_j)/h\big]\,K[x+(x_{s-1}-x_k/h)]\big\}\,K(x)dx \\
=\ & \frac{Ch}{\sqrt{t-s}}\int_{-\infty}^{\infty} E\big\{K[x+(x_{s-1}-x_k/h)]\,N_s\big\}\,K(x)dx \\
\leq\ & \frac{Ch^2}{\sqrt{t-s}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\big\{K[x+(x_j-x_i)/h]K[y+(x_{s-1}-x_k)/h]\big\}K(x)K(y)\,dxdy \\
\leq\ & \frac{Ch^4}{\sqrt{t-s}}\frac{1}{\sqrt{s-k}}\,\frac{1}{\sqrt{j-i}}
\end{aligned}$$

This proves (7.16) for $i<j<k$ and $l=s-1$. The proof of Lemma 7.4 is now complete. $\square$

LEMMA 7.5. *Write*

$$\begin{aligned}
I(i,j,k,l,s,t) \;=\; & E\Big\{u_{i+1}u_{j+1}u_{k+1}u_{l+1}K\big[(x_s-x_i)/h\big]\,K\big[(x_s-x_j)/h\big] \\
& K\big[(x_t-x_k)/h\big]\,K\big[(x_t-x_l)/h\big]\Big\}.
\end{aligned}$$

In addition to Assumptions 1-3, assume $|u_j|\leq A$. Then,
(i) for $t>s, s-2\geq i,j,k,l$ and $i<j<k<l$,

$$|I(i,j,k,l,s,t)| \;\leq\; Ch^4\Big(\frac{1}{t-s}\frac{1}{\sqrt{s-l}}+\frac{1}{\sqrt{t-s}}\frac{1}{s-l}\Big)\,\frac{1}{\sqrt{l-k}}\,\frac{1}{\sqrt{j-i}}; \qquad (7.20)$$

(ii) for $t\geq l+2, 1\leq i<j\leq s-2$, $l\geq s+1$ and $k<l$

$$|I(i,j,k,l,s,t)| \;\leq\; \begin{cases} \frac{Ch^4}{t-l}\,\frac{1}{\sqrt{l-k}}\frac{1}{\sqrt{s-j}}\frac{1}{\sqrt{j-i}}, & \text{if } k\geq s, \\ \frac{Ch^4}{t-l}\,\frac{1}{\sqrt{l-s}}\frac{1}{\sqrt{s-j}}\frac{1}{\sqrt{j-i}}, & \text{if } k<s. \end{cases} \qquad (7.21)$$

(iii) for $t=s$ and $s-2\geq i,j,k,l$ and $i<j<k<l$,

$$|I(i,j,k,l,t,t)| \;\leq\; \frac{C_1h^4}{t-l}\frac{1}{\sqrt{l-k}}\frac{1}{\sqrt{k-j}}\frac{1}{\sqrt{j-i}}. \qquad (7.22)$$

Proof. First for (7.20). It follows (7.6) of Lemma 7.1 with $p(x)=K(x-x_k/h)K(x-x_l/h)$ that

$$\begin{aligned}
T_s \;:=\; & E\Big\{\,K\big[(x_t-x_k)/h\big]\,K\big[(x_t-x_l)/h\big]\mid \mathcal{F}_s\Big\} \\
=\; & \frac{h}{\sqrt{t-s}}\int_{-\infty}^{\infty} p\big(\rho^{t-s}\,x_s/h+x\big)h_{s,t}\big(\frac{hx}{\sqrt{t-s}}\big)dx+\mathcal{L}_n
\end{aligned}$$

where

$$|\mathcal{L}_n| \;\; \leq \;\; \frac{Ch}{t-s}\Big(1+\sum_{u=0}^{\infty}(u+1)^{-1-\delta}|\epsilon_{s-u}|\Big)\;\int_{-\infty}^{\infty}|p(x)|dx\,.$$

Now conditional arguments, together with $|h_{st}(y)| \leq C_0$ and some simple calculations, yield that

$$\begin{aligned}
& |I(i,j,k,l,s,t)| \\
= \;\; & \Big|E\Big\{u_{i+1}u_{j+1}u_{k+1}u_{l+1}K\big[(x_s-x_i)/h\big]\,K\big[(x_s-x_j)/h\big]\,T_s\Big\}\Big| \\
\leq \;\; & \frac{C_0 h}{\sqrt{t-s}}\int_{-\infty}^{\infty}\Big|E\Big\{u_{i+1}u_{j+1}u_{k+1}u_{l+1}K\big[(x_s-x_i)/h\big]\,K\big[(x_s-x_j)/h\big] \\
& \qquad p\big(\rho^{t-s}\,x_s/h+x\big)\Big\}\Big|dx + A^4\,E\Big\{K\big[(x_s-x_i)/h\big]\,K\big[(x_s-x_j)/h\big]\,|\mathcal{L}_n|\Big\} \\
\leq \;\; & \frac{C_0 h}{\sqrt{t-s}}\int_{-\infty}^{\infty}|EI_1(x)|\,K(x)\,dx + \frac{Ch}{t-s}\Big(\int_{-\infty}^{\infty}E|I_2(x)|\,K(x)\,dx \\
& \qquad +\sum_{u=0}^{\infty}(u+1)^{-1-\delta}\int_{-\infty}^{\infty}E\big\{|\epsilon_{s-u}||I_2(x)|\big\}\,K(x)\,dx\Big),
\end{aligned} \tag{7.23}$$

where

$$\begin{aligned}
I_1(x) &= u_{i+1}u_{j+1}u_{k+1}u_{l+1}K\big[(x_s-x_i)/h\big]\,K\big[(x_s-x_j)/h\big]\,K\big[x+(x_l-x_k)/h\big], \\
I_2(x) &= K\big[(x_s-x_i)/h\big]\,K\big[(x_s-x_j)/h\big]\,K\big[x+(x_l-x_k)/h\big].
\end{aligned}$$

It follows from (7.7) with $p(y)=K(y-x_i/h)K(y-x_j/h)$ that

$$\begin{aligned}
T_{1s} \;\; := \;\; & \big|E\Big\{\,u_{l+1}\,K\big[(x_s-x_i)/h\big]\,K\big[(x_s-x_j)/h\big]\mid \mathcal{F}_l\Big\}\big| \\
\leq \;\; & \frac{Ch}{s-l}\int_{-\infty}^{\infty}K(y-x_i/h)K(y-x_j/h)dy \\
= \;\; & \frac{Ch}{s-l}\int_{-\infty}^{\infty}K(y)K[y+(x_j-x_i)/h]dy.
\end{aligned}$$

Hence, using conditional arguments and (7.8) repeatedly, we have

$$\begin{aligned}
|EI_1(x)| \;\; \leq \;\; & A^3\,E\Big(\,K\big[x+(x_l-x_k)/h\big]\,\big|T_{1s}\big|\Big) \\
\leq \;\; & \frac{Ch}{s-l}\int_{-\infty}^{\infty}E\Big\{K[x+(x_l-x_k)/h]\,K[y+(x_j-x_i)/h]\Big\}\,K(y)dy \\
\leq \;\; & \frac{C_1 h}{s-l}\,\frac{h}{\sqrt{l-k}}\,\frac{h}{\sqrt{j-l}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(x+z)\,K(y+u)\,K(y)dydzdu \\
\leq \;\; & \frac{C_2 h^3}{s-l}\,\frac{1}{\sqrt{l-k}}\,\frac{1}{\sqrt{j-i}}.
\end{aligned} \tag{7.24}$$

The same idea as in the proof of (7.24), but with (7.10) instead of (7.8), yields that

$$E|I_2(x)| + E\big\{|\epsilon_{s-u}||I_2(x)|\big\} \quad \leq \quad \frac{Ch^3}{\sqrt{s-l}}\,\frac{1}{\sqrt{l-k}}\,\frac{1}{\sqrt{j-i}}, \tag{7.25}$$

for all $0 \leq u < \infty$. Note that $\sum_{u=0}^{\infty}(u+1)^{-1-\delta} < \infty$. Taking (7.24) and (7.25) into (7.23), we obtain

$$|I(i,j,k,l,s,t)| \quad \leq \quad Ch^4\Big(\frac{1}{t-s}\frac{1}{\sqrt{s-l}} + \frac{1}{\sqrt{t-s}}\frac{1}{s-l}\Big)\,\frac{1}{\sqrt{l-k}}\,\frac{1}{\sqrt{j-i}},$$

which yields (7.20).

Next for (7.22). It follows from (7.7) with $p(y) = K(y - x_i/h)K(y - x_j/h)K(y - x_k/h)K(y - x_l/h)$ that

$$\begin{aligned}
&\big|E\Big\{\,u_{l+1}\,K\big[(x_s - x_i)/h\big]\,K\big[(x_s - x_j)/h\big]\,K\big[(x_s - x_k)/h\big]\,K\big[(x_s - x_l)/h\big]\mid \mathcal{F}_l\Big\}\big| \\
&\quad\leq \frac{Ch}{s-l}\int_{-\infty}^{\infty} K(y - x_i/h)K(y - x_j/h)K(y - x_k/h)K(y - x_l/h)dy \\
&\quad= \frac{Ch}{s-l}\int_{-\infty}^{\infty} K[y + (x_l - x_i)/h]K[y + (x_l - x_j)/h]K[y + (x_l - x_k)/h]K(y)dy.
\end{aligned}$$

This, together with (7.16) in Lemma 7.4, yields that

$$\begin{aligned}
&|I(i,j,k,l,s,s)| \\
\leq\;& \frac{CA^3h}{s-l}\int_{-\infty}^{\infty} E\big\{K[y + (x_l - x_i)/h]K[y + (x_l - x_j)/h]K[y + (x_l - x_k)/h]\big\}K(y)dy \\
\leq\;& \frac{C_1h^4}{s-l}\frac{1}{\sqrt{l-k}}\frac{1}{\sqrt{k-j}}\frac{1}{\sqrt{j-i}}\Big(\int_{-\infty}^{\infty}K(y)dy\Big)^4 \\
\leq\;& \frac{C_1h^4}{s-l}\frac{1}{\sqrt{l-k}}\frac{1}{\sqrt{k-j}}\frac{1}{\sqrt{j-i}},
\end{aligned}$$

which implies (7.22).

Finally for(7.21). The same idea as above, together with (7.16) in Lemma 7.4, yields that

$$\begin{aligned}
&|I(i,j,k,l,s,t)| \\
\leq\;& A^3\,E\Big(K\big[(x_s - x_i)/h\big]\,K\big[(x_s - x_j)/h\big] \\
&\qquad \big|E\big\{u_{l+1}K\big[(x_t - x_k)/h\big]\,K\big[(x_t - x_l)/h\big]\mid \mathcal{F}_l\big\}\big|\Big) \\
\leq\;& \frac{Ch}{t-l}\int_{-\infty}^{\infty} E\Big\{K\big[(x_s - x_i)/h\big]\,K\big[(x_s - x_j)/h\big]K[x + (x_l - x_k)/h]\Big\}K(x)dx \\
\leq\;& \begin{cases} \frac{Ch^2}{t-l}\,\frac{1}{\sqrt{l-k}}E\Big\{K\big[(x_s - x_i)/h\big]\,K\big[(x_s - x_j)/h\big]\Big\}, & \text{if } k \geq s, \\ \frac{Ch^2}{t-l}\,\frac{1}{\sqrt{l-s}}E\Big\{K\big[(x_s - x_i)/h\big]\,K\big[(x_s - x_j)/h\big]\Big\}, & \text{if } k < s, \end{cases} \\
\leq\;& \begin{cases} \frac{Ch^4}{t-l}\,\frac{1}{\sqrt{l-k}}\frac{1}{\sqrt{s-j}}\frac{1}{\sqrt{j-i}}, & \text{if } k \geq s, \\ \frac{Ch^4}{t-l}\,\frac{1}{\sqrt{l-s}}\frac{1}{\sqrt{s-j}}\frac{1}{\sqrt{j-i}}, & \text{if } k < s. \end{cases}
\end{aligned}$$

The proof of Lemma 7.5 is complete. □

LEMMA 7.6. *Write* $Z_{tkr} = \sum_{i=k}^{t-r} u_{i+1} K\big[(x_t - x_i)/h\big]$. *Under Assumptions 1-3, for* $1 \le k \le t-r$ *and* $r \ge 2$, *we have,*

$$EZ_{tkr}^2 \;\le\; C \max_{1\le i,j\le n} E[|u_i|(1+|u_j|)](h\log t + 1)\, h\sqrt{t-r-k}. \tag{7.26}$$

Similarly, we have

$$E\Big\{\sum_{i=1}^{t-2}\big[u_{i+1}^2 - E(u_{i+1}^2 \mid \mathcal{F}_j)\big]K^2\big[(x_t-x_i)/h\big]\Big\}^2 \;\le\; C\left(h^2\sqrt{t}\log t + h\sqrt{t}\right). \tag{7.27}$$

If in addition $|u_j| \le A$, *where* A *is a constant, then*

$$EZ_{t12}^4 \;\le\; Ch^4 t^{3/2}\log t + Ch^3 t^{3/2}, \tag{7.28}$$

and for any $1 \le m \le t/2$,

$$EZ_{tm}^{*2} \;\le\; \frac{Ch^2t^2}{m^{3/2}} + \frac{Ch^2 t\log(t-m)}{\sqrt{m}} + \frac{Ch^2 t}{m}, \tag{7.29}$$

where $Z_{tm}^* = \sum_{i=1}^{t-m-1} u_{i+1} E\big(K\big[(x_t-x_i)/h\big] \mid \mathcal{F}_{t-m}\big)$.

Proof. First consider (7.26). For $i = j-1$ or j, it follows from (7.8) with $p(x) = K(x - x_i/h)K(x - x_j/h)$ that

$$\begin{aligned} &\Big|E\Big\{u_{i+1}u_{j+1}K\big[(x_t-x_i)/h\big]K[(x_t-x_j)/h]\Big\}\Big| \\ \le\;& E\Big\{|u_{i+1}||u_{j+1}|E\big(K\big[(x_t-x_i)/h\big]K[(x_t-x_j)/h] \mid \mathcal{F}_{j+1}\big)\Big\} \\ \le\;& \frac{Ch}{\sqrt{t-j}}\, E(|u_{i+1}||u_{j+1}|). \end{aligned} \tag{7.30}$$

Similarly, for $i \le j-2$, we have

$$\begin{aligned} E\big\{|u_{i+1}|K\big[x + (x_j-x_i)/h\big]\big\} \;&\le\; E\big\{|u_{i+1}|E\big(K\big[x+(x_j-x_i)/h\big] \mid \mathcal{F}_{i+1}\big)\big\} \\ &\le\; \frac{Ch}{\sqrt{t-j}}\, E(|u_{i+1}|), \end{aligned} \tag{7.31}$$

uniformly for $x \in R$. The result (7.31), together with the usage of (7.7) with $p(x) = K(x - x_i/h)K(x - x_j/h)$, yields that

$$\begin{aligned} &\Big|E\Big\{u_{i+1}u_{j+1}K\big[(x_t-x_i)/h\big]K[(x_t-x_j)/h]\Big\}\Big| \\ \le\;& E\Big\{|u_{i+1}|\,\big|E\big(u_{j+1}K\big[(x_t-x_i)/h\big]K[(x_t-x_j)/h] \mid \mathcal{F}_j\big)\,\big|\Big\} \\ \le\;& \frac{Ch}{t-j}\int_{-\infty}^{\infty} E\big\{|u_{i+1}|K\big[x+(x_j-x_i)/h\big]\big\}K(x)dx \\ \le\;& \frac{Ch^2}{t-j}\,\frac{1}{\sqrt{j-i}}\, E|u_{i+1}|. \end{aligned} \tag{7.32}$$

Combining (7.30) and (7.32), we obtain that, for $1 \le k \le t-r$ and $r \ge 2$,

$$
\begin{aligned}
EZ_{tkr}^2 &= 2\Big(\sum_{\substack{i,j=k\\ j-i\ge 2}}^{t-r} + \sum_{\substack{i,j=k\\ 0\le j-i\le 1}}^{t-r}\Big)E\Big\{u_{i+1}u_{j+1}K\big[(x_t-x_i)/h\big]K\big[(x_t-x_j)/h\big]\Big\} \\
&\le C \max_{1\le i,j\le n} E[|u_i|(1+|u_j|)]\,\Big(\sum_{\substack{i,j=k\\ j-i\ge 2}}^{t-r}\frac{h^2}{t-j}\frac{1}{\sqrt{j-i}} + \sum_{\substack{i,j=k\\ 0\le j-i\le 1}}^{t-r}\frac{h}{\sqrt{t-j}}\Big) \\
&\le C \max_{1\le i,j\le n} E[|u_i|(1+|u_j|)](h\log t+1)\,h\sqrt{t-r-k},
\end{aligned}
$$

which yields (7.26).

The proof of (7.27) is similar. We omit the details.

We next prove (7.28). Recalling $\sup_x K(x) < \infty$, $|u_j| \le A$ and the notation defined in Lemma 7.5, we have

$$
\begin{aligned}
EZ_{1t}^4 &\le \sum_{\substack{i,j,k,l=1\\ i\ne j\ne k\ne l}}^{t-2}\big|I(i,j,k,l,t,t)\big| + C\sum_{\substack{i,j,k=1\\ i\ne j\ne k}}^{t-2}E\Big\{K\big[(x_t-x_i)/h\big]\,K\big[(x_t-x_j)/h\big]K\big[(x_t-x_k)/h\big]\Big\} \\
&\quad + C\sum_{\substack{i,j=1\\ i\ne j}}^{t-2}E\Big\{K\big[(x_t-x_i)/h\big]\,K\big[(x_t-x_j)/h\big]\Big\} + C\sum_{i=1}^{t-2}EK\big[(x_t-x_i)/h\big] \\
&\le V_{1t}+V_{2t}+V_{3t}+V_{4t}. \qquad (7.33)
\end{aligned}
$$

Using (7.22), we have

$$
\begin{aligned}
V_{1t} &\le Ch^4\sum_{l=4}^{t-2}\sum_{k=3}^{l-1}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\frac{1}{t-l}\frac{1}{\sqrt{l-k}}\frac{1}{\sqrt{k-j}}\frac{1}{\sqrt{j-i}} \\
&\le Ch^4t^{3/2}\log t. \qquad (7.34)
\end{aligned}
$$

Using (7.14)-(7.16) with $p_1(x)=p_2(x)=p_3(x)=K(x)$, we obtain

$$
\begin{aligned}
V_{2t}+V_{3t}+V_{4t} &\le \sum_{k=3}^{t-2}\sum_{j=2}^{k-1}\sum_{i=1}^{j-1}\frac{Ch^3}{\sqrt{t-k}}\frac{1}{\sqrt{k-j}}\frac{1}{\sqrt{j-i}} \\
&\quad + \sum_{j=2}^{t-2}\sum_{i=1}^{j-1}\frac{Ch^2}{\sqrt{t-j}}\frac{1}{\sqrt{j-i}} + \sum_{j=1}^{t-2}\frac{Ch}{\sqrt{t-j}} \\
&\le Ch^3t^{3/2} + Ch^2t + Ch\sqrt{t} \le Ch^3t^{3/2}, \qquad (7.35)
\end{aligned}
$$

since $t \le n$ and $nh^2 \to \infty$. By virtue of (7.33)-(7.35), we obtain (7.28).

Finally for (7.29). For $i<j$, write

$$
II_{i,j} = E\Big\{u_{i+1}u_{j+1}E\big(K\big[(x_t-x_i)/h\big] \mid \mathcal{F}_{t-m}\big)E\big(K\big[(x_t-x_j)/h\big] \mid \mathcal{F}_{t-m}\big)\Big\}.
$$

Using (7.6), (7.8) and $|u_j| \le A$, simple calculations show that

$$\begin{aligned} II_{i,j} &= \frac{h}{\sqrt{m}} \int_{-\infty}^{\infty} E\Big\{ u_{i+1} u_{j+1} E\big(K\big[(x_t - x_i)/h\big] \mid \mathcal{F}_{t-m}\big) \\ &\qquad\qquad K\big[(\rho^m x_{t-m} - x_j)/h + x\big]\Big\}\, h_{t-m,t}(\frac{hx}{\sqrt{m}})\, dx + V_{5n} \\ &= \frac{h^2}{m} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\Big\{ u_{i+1}u_{j+1} K\big[(\rho^m x_{t-m} - x_i)/h + y\big] K\big[(\rho^m x_{t-m} - x_j)/h + x\big]\Big\} \\ &\qquad\qquad h_{t-m,t}(\frac{hx}{\sqrt{m}})\, h_{t-m,t}(\frac{hy}{\sqrt{m}})\, dxdy + V_{5n} + V_{6n}, \end{aligned} \tag{7.36}$$

where $h_{s,t}(z)$ is the density given in (7.2),

$$\begin{aligned} |V_{5n}| &\le \frac{Ch}{m} E\Big\{ \big(1 + \sum_{k=0}^{\infty}(k+1)^{-1-\delta}|\epsilon_{t-m-k}|\big)\, |E\big(K\big[(x_t - x_i)/h\big] \mid \mathcal{F}_{t-m}\big)|\Big\} \\ &\le \frac{Ch^2}{m^{3/2}}, \\ |V_{6n}| &\le \frac{Ch^2}{m^{3/2}} \int_{-\infty}^{\infty} E\Big\{\big(1 + \sum_{k=0}^{\infty}(k+1)^{-1-\delta}|\epsilon_{t-m-k}|\big)\, K\big[(\rho^m x_{t-m} - x_j)/h + x\big]\Big\}\, dx \\ &\le \frac{Ch^2}{m^{3/2}}. \end{aligned}$$

On the other hand, using (7.7) with $p(z) = K\big[(\rho^m z - x_i/h + y\big] K\big[(\rho^m z - x_j/h + x\big]$,

$$\begin{aligned} &\Big| E\Big\{ u_{j+1} K\big[(\rho^m x_{t-m} - x_i)/h + y\big] K\big[(\rho^m x_{t-m} - x_j)/h + x\big] \mid \mathcal{F}_j\Big\}\Big| \\ \le\ & \frac{Ch}{t-m-j} \int_{-\infty}^{\infty} K\big[(\rho^m z - x_i/h + y\big] K\big[(\rho^m z - x_j/h + x\big]\, dz \\ \le\ & \frac{Ch}{t-j} \int_{-\infty}^{\infty} K(\rho^m z) K\big[\rho^m z + (x_j - x_i)/h + y - x\big]\, dz. \end{aligned}$$

Taking these estimates into (7.36) and recalling $h_{s,t}(z) \le C_0$ and $\int_{-\infty}^{\infty} h_{s,t}(z)dz = 1$, we obtain

$$\begin{aligned} II_{i,j} &\le \frac{C_1 h^3}{m\,(t-j)} E\Big\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\rho^m z) K\big[\rho^m z + (x_j - x_i)/h + y - x\big] \\ &\qquad\qquad h_{t-m,t}(\frac{hx}{\sqrt{m}})\, dxdy\, dz\Big\} + \frac{Ch^2}{m^{3/2}} \\ &\le \frac{C_3 h^2}{\sqrt{m}\,(t-j)} + \frac{Ch^2}{m^{3/2}}. \end{aligned}$$

This, together with (7.8), yields that

$$
\begin{aligned}
& E\Big|\sum_{i=1}^{t-m-1} u_{i+1} E\big(K\big[(x_t-x_i)/h\big] \mid \mathcal{F}_{t-m}\big)\Big|^2 \\
\le\ & 2 \sum_{1\le i<j\le t-m-1} II_{i,j} + A^2 \sum_{i=1}^{t-m-1} E\big|E\big(K\big[(x_t-x_i)/h\big] \mid \mathcal{F}_{t-m}\big)\big|^2 \\
\le\ & 2 \sum_{1\le i<j\le t-m-1} \Big(\frac{C_3 h^2}{\sqrt{m}\,(t-j)} + \frac{Ch^2}{m^{3/2}}\Big) + C \sum_{i=1}^{t-m-1} \frac{h^2}{m} \\
\le\ & \frac{Ch^2 t^2}{m^{3/2}} + \frac{Ch^2 t \log(t-m)}{\sqrt{m}} + \frac{Ch^2 t}{m}.
\end{aligned}
$$

This proves (7.29) and also completes the proof of Lemma 7.6. □

7.2 Proof of Proposition 6.1. Write $\tau^*_{n,t} = \tau_{n,t} - \tau_{n,t-1}$ and recall $E(\tau^*_{n,t} \mid \mathcal{F}^0_{n,t-1}) = \sigma^{-2} E[u^2_{t+1} \mid \mathcal{F}_t]$ by (6.5). We have, for $0 \le r \le T$,

$$
\begin{aligned}
[M_n, V]_r \;=\; & \frac{1}{n}\sum_{t=1}^{[nr]\wedge n} Y^*_{nt} + \frac{1}{n\sigma^2}\sum_{t=1}^{[nr]\wedge n} Y^*_{nt}\big[\tau^*_{n,t} - E(\tau^*_{n,t} \mid \mathcal{F}^0_{n,t-1})\big] \\
& + \frac{1}{n\sigma^2}\sum_{t=1}^{[nr]\wedge n} Y^*_{nt}\big[E(u^2_{t+1} \mid \mathcal{F}_t) - \sigma^2\big] + R_n(r) \\
:=\; & Z_{1n}(r) + Z_{2n}(r) + Z_{3n}(r) + R_n(r), \qquad (7.37)
\end{aligned}
$$

where, by recalling $\tau_{n,j-1} < nr \le \tau_{n,j}$ and (6.4),

$$
\begin{aligned}
\sup_{0\le r\le T} |R_n(r)| \;\le\; & \sup_{0\le r\le 1} |R_n(r)| + \sup_{1\le r\le T} |R_n(r)| \\
\le\; & n^{-1} \max_{1\le k\le n} \big[|[nr]-(j-1)|\,|Y^*_{nk}|\,\tau^*_{n,k}\big] + (T + |1-\tau_{n,n}/n|)/\sqrt{n} \\
\le\; & C\, n^{\delta-1} \max_{1\le k\le n} [|Y^*_{nk}|\,\tau^*_{n,k}] + C/\sqrt{n}, \qquad (7.38)
\end{aligned}
$$

for any $\delta > 1/2$. Since $|u_k| \le C$, it follows from (6.5), (7.28) in Lemma 7.6 and

$$
Y^*_{nt} \;=_d\; \frac{\sqrt{n}}{d_n} Z_{t11} \qquad (7.39)
$$

(where $Z_{tkr} = \sum_{i=k}^{t-r} u_{i+1} K\big[(x_t - x_i)/h\big]$ is defined as in Lemma 7.6) that

$$
\begin{aligned}
E \max_{1\le k\le n} [|Y^*_{nk}|\,\tau^*_{n,k}] \;\le\; & \Big\{ \sum_{k=1}^{n} E\Big[|Y^*_{nk}|^4\, E\big(\tau^{*4}_{n,k} \mid \mathcal{F}^0_{n,t-1}\big)\Big]\Big\}^{1/4} \\
\le\; & C\,(nh^2)^{-1/4} \big(EZ^4_{t11}\big)^{1/4} \le C\,(nh^2)^{-1/4} \big(C + EZ^4_{t12}\big)^{1/4} \\
\le\; & C\,(nh^2)^{-1/4} \Big(1 + h^4 \sum_{t=2}^{n} t^{3/2}\log t + h^3 \sum_{t=1}^{n} t^{3/2}\Big)^{1/4} \\
\le\; & C\, n^{3/8}\, h^{1/4}, \qquad (7.40)
\end{aligned}
$$

whenever $nh^2 \to \infty$ and $h\log^2 n \to 0$. Taking this estimate into (7.38), we obtain $\sup_{0\le r\le T}|R_n(r)| = o_P(1)$ by choosing $\delta < 1/8$. Note that $\{Y_{nt}^*\big[\tau_{n,t}^* - E(\tau_{n,t}^* \mid \mathcal{F}_{n,t-1}^0)\big],\, \mathcal{F}_{n,t}^0\}_{1\le t\le n}$ forms a martingale difference. It follows from the maximal inequality for martingales, (6.5) and (7.26) in Lemma 7.6 that

$$\begin{aligned} E\sup_{0\le r\le T}|Z_{2n}(r)|^2 &\le \frac{C}{n^2}\sum_{t=1}^n E\big\{Y_{nt}^{*2}\, E(\tau_{n,t}^{*2} \mid \mathcal{F}_{n,t-1}^0)\big\} \\ &\le \frac{C_1}{nd_n^2}\sum_{t=1}^n E\big(Z_{t11}\big)^2 \le \frac{C_2}{n^{5/2}h}\sum_{t=1}^n (1 + h\sqrt{t}) \le C/n, \end{aligned}$$

which yields that $\sup_{0\le r\le T}|Z_{2n}(r)| = o_P(1)$. To estimate $Z_{3n}(r)$, recall

$$E|Y_{nt}^*| \le \frac{C}{n^{1/4}\sqrt{h}}(E|Z_{t11}|^2)^{1/2} \le C_1(t/n)^{1/4}.$$

It is readily seen that $\sum_{t=1}^n E|Y_{nt}^*| \le Cn$ and for any $c_n/n \to 0$, $\sum_{t=1}^{c_n} E|Y_{nt}^*| \le C c_n^{5/4} n^{-1/4} = o(n)$. By virtue of these facts and $E[u_{t+1}^2 \mid \mathcal{F}_t] \to_{a.s.} \sigma^2$, it follows from Lemma 7.3 that

$$E\sup_{0\le r\le T}|Z_{3n}(r)|^{1/2} \le \frac{C}{\sqrt{n}}E\Big(\sum_{t=1}^n |Y_{nt}^*|\big|\big[E(u_{t+1}^2 \mid \mathcal{F}_t) - \sigma^2\big]\big|\Big)^{1/2} = o(1),$$

which yields that $\sup_{0\le r\le T}|Z_{3n}(r)| = o_P(1)$.

Hence, it suffices to show that

$$\sup_{0\le r\le T}|Z_{1n}(r)| =_d \frac{1}{\sqrt{n}\,d_n}\max_{1\le k\le n}\Big|\sum_{t=1}^k\sum_{i=1}^{t-1} u_{i+1}K[(x_t - x_i)/h]\Big| = o_P(1). \tag{7.41}$$

Note that, with $\eta_n = \epsilon_n n$ and $m = \eta_n/2$ where ϵ_n is chosen such that $\epsilon_n \to 0$ and $\epsilon_n^{-1/2} h\log^2 n \to 0$,

$$\begin{aligned} &\max_{1\le k\le n}\Big|\sum_{t=1}^k\sum_{i=1}^{t-1} u_{i+1}K[(x_t - x_i)/h]\Big| \le C\,n + \max_{1\le k\le n}\Big|\sum_{t=1}^k Z_{t12}\Big| \\ \le\ & C\,n + 2\sum_{t=1}^{\eta_n}|Z_{t12}| + \sum_{\eta_n+1\le t\le n}\Big|\sum_{i=t-m+1}^{t-2} u_{i+1}K\big[(x_t - x_i)/h\big]\Big| \\ &+ \max_{\eta_n+1\le k\le n}\Big|\sum_{t=\eta_n+1}^k\sum_{i=1}^{t-m} u_{i+1}K\big[(x_t - x_i)/h\big]\Big| \\ :=\ & C\,n + 2\,R_{1n} + R_{2n} + R_{3n}. \end{aligned} \tag{7.42}$$

By (7.26) in Lemma 7.6, it follows that

$$ER_{1n} \le \sum_{t=1}^{\eta_n}(E|Z_{t12}|^2)^{1/2} \le C\,h^{1/2}\sum_{t=1}^{\eta_n} t^{1/4} \le C_1\,\epsilon_n\,n^{5/4}h^{1/2}, \tag{7.43}$$

$$\begin{aligned} ER_{2n} &\le \sum_{\eta_n+1\le t\le n}(E\big|Z_{t(m-1)2}\big|^2)^{1/2} \\ &\le Cn\,m^{1/4}\sqrt{h} \le C_1\,\epsilon_n^{1/4}\,n^{5/4}h^{1/2}. \end{aligned} \tag{7.44}$$

By virtue of (7.43) and (7.44), we have

$$C\,n + 2\,R_{1n} + R_{2n} \le C\,n^{5/4}h^{1/2}\left[1/(nh^2)^{1/4} + O_P(\epsilon_n^{1/4})\right] = o_P(n^{5/4}h^{1/2}).$$

Taking this fact into (7.42), the result (7.41) will follow if we prove

$$R_{3n} = \max_{\eta_n+1\le k\le n}\Big|\sum_{t=\eta_n+1}^{k} Z_{t1m}\Big| = o_P(n^{5/4}\sqrt{h}). \tag{7.45}$$

It is readily seen that

$$\begin{aligned} R_{3n} &\le \max_{\eta_n+1\le k\le n}\Big|\sum_{t=\eta_n+1}^{k}\left\{Z_{t1m} - E(Z_{t1m}\mid\mathcal{F}_{t-m})\right\}\Big| + \sum_{t=\eta_n+1}^{n}\big|E(Z_{t1m}\mid\mathcal{F}_{t-m})\big| \\ &:= R_{3n1} + R_{3n2}. \end{aligned}$$

Note that, for fixed m, $\{Z_{t1m} - E(Z_{t1m}\mid\mathcal{F}_{t-m},\ \mathcal{F}_{t-m}\}_{t\ge1}$ forms a martingale difference. It follows that

$$ER_{3n1}^2 \;\le\; C\sum_{t=1}^{n} EZ_{t1m}^2 \le C\,h\sum_{t=1}^{n} t^{1/2} \le C_1\,hn^{3/2}, \tag{7.46}$$

which yields that $R_{3n1} = o_P(n^{5/4}\sqrt{h})$.

As for R_{3n2}, by noting that u_{i+1} is $\mathcal{F}_{t-m}$ measurable when $i \le t-m-1$, it follows from (7.29) in Lemma 7.6 that

$$\begin{aligned} ER_{3n2} &\le Cn + \sum_{t=m+1}^{n} E\Big|\sum_{i=1}^{t-m-1} u_{i+1}E\big(K\big[(x_t - x_i)/h\big]\mid\mathcal{F}_{t-m}\big)\Big| \\ &\le Cn + \sum_{t=m+1}^{n}\Big(E\Big|\sum_{i=1}^{t-m-1} u_{i+1}E\big(K\big[(x_t - x_i)/h\big]\mid\mathcal{F}_{t-m}\big)\Big|^2\Big)^{1/2} \\ &\le Cn + C\sum_{t=m+1}^{n}\Big(\frac{h^2t^2}{m^{3/2}} + \frac{h^2t\log(t-m)}{\sqrt{m}} + \frac{h^2t}{m}\Big)^{1/2} \\ &\le Cn + Ch\Big(\frac{n^2}{m^{3/4}} + \frac{n^{3/2}\log n}{m^{1/4}} + \frac{n^{3/2}}{m^{1/2}}\Big) \\ &\le Cn + C\epsilon_n^{-1/4}n^{5/4}h\log n \\ &= o_P(n^{5/4}\sqrt{h}), \end{aligned}$$

since $\epsilon_n^{-1/4}\sqrt{h}\log n = o(1)$. This proves (7.45), and hence the proof of Proposition 6.1 is complete. □

7.3 Proof of Proposition 6.2. As in the proof of Proposition 6.1, write $\tau^*_{n,t} = \tau_{n,t} - \tau_{n,t-1}$. Recall $E(\tau^*_{n,t} \mid \mathcal{F}^0_{n,t-1}) = \sigma^{-2} E[u^2_{t+1} \mid \mathcal{F}_t]$ by (6.5). We have,

$$\begin{aligned} [M_n]_1 &= \frac{1}{n}\sum_{t=1}^n Y^{*2}_{nt} + \frac{1}{n\sigma^2}\sum_{t=1}^n Y^{*2}_{nt}\big[\tau^*_{n,t} - E(\tau^*_{n,t} \mid \mathcal{F}^0_{n,t-1})\big] \\ &\qquad \frac{1}{n\sigma^2}\sum_{t=1}^n Y^{*2}_{nt}\big[E(u^2_{t+1} \mid \mathcal{F}_t) - \sigma^2\big] + R^*_n \\ &:= \frac{1}{n}\sum_{t=1}^n Y^{*2}_{nt} + Z_{1n} + Z_{2n} + R_{nj}, \qquad (7.47) \end{aligned}$$

where, for some $1 \le j \le n$ satisfying $\tau_{n,j-1} < n \le \tau_{n,j}$,

$$|R_{nj}| \;\le\; \frac{1}{n}\Big(1 - \frac{\tau_{n,n}}{n}\Big) + \frac{1}{n}\sum_{t=j}^n Y^{*2}_{nt}\tau^*_{n,t}\,.$$

Note that j is a random variable satisfying

$$n - j \le \max_{1\le j\le n} |\tau_{n,j} - j| + 1/n = o(n^\delta), \quad a.s. \qquad (7.48)$$

for any $\delta > 0$, due to (6.4). It is readily seen that

$$|R_{nj}| \le o_P(1) + \frac{1}{n}\sum_{t=n-n^\delta}^n Y^{*2}_{nt}\,\tau^*_{nt} = o_P(1), \qquad (7.49)$$

where we have used the fact that, for any $\delta > 0$,

$$\begin{aligned} \frac{1}{n}\sum_{t=n-n^\delta}^n E\big[Y^{*2}_{nt}\,\tau^*_{nt}\big] &\le \frac{1}{n}\sum_{t=n-n^\delta}^n E\Big(|Y^{*2}_{tn}|\,E\big[(\tau_{n,t} - \tau_{n,t-1})^2 \mid \mathcal{F}^0_{n,t-1}\big]\Big) \\ &\le \frac{C}{n^{3/2}h}\sum_{t=n-n^\delta}^n EZ^2_{t11} \le \frac{C}{n^{3/2}}\sum_{t=n-n^\delta}^n (1 + h\log t)\sqrt{t} \\ &\le \frac{C}{n^{3/2}}\big[n^{3/2} - (n - n^\delta)^{3/2}\big] = o(1), \qquad (7.50) \end{aligned}$$

due to (6.4), (7.39), (7.26) in Lemma 7.6 and $h\log^2 n \to 0$.

By noting that $\{Y^{*2}_{nt}\big[\tau^*_{n,t} - E(\tau^*_{n,t} \mid \mathcal{F}^0_{n,t-1})\big], \mathcal{F}^0_{n,t-1}\}_{t\ge1}$ forms a martingale difference, it follows from (6.5) and (7.28) in Lemma 7.6 that

$$\begin{aligned} EZ^2_{1n} &\le \frac{C}{n^2}\sum_{t=1}^n E\Big\{Y^{*4}_{nt}E(\tau^{*2}_{n,t} \mid \mathcal{F}^0_{n,t-1})\Big\} \\ &\le \frac{C}{nd_n^2}\sum_{t=1}^n E\Big|\sum_{i=1}^n u_{i+1}K\big(\frac{x_t - x_i}{h}\big)\Big|^4 \\ &\le \frac{C}{n^{5/2}h}\sum_{t=1}^n \big(C + h^4 t^{3/2}\log t + h^3 t^{3/2}\big) \\ &\le Ch^2(1 + h\log n) = o(1), \end{aligned}$$

since $h \log^2 n \to 0$, which implies that $Z_{1n} = o_P(1)$. To estimate Z_{2n}, recall that, by (7.26) in Lemma 7.6,

$$EY_{nt}^{*2} \leq \frac{Cn}{d_n^2} E\Big|\sum_{i=1}^{t-1} u_{i+1} K\big(\frac{x_t - x_i}{h}\big)\Big|^2 \leq C\,(t/n)^{1/2}.$$

It is readily seen that $\sum_{t=1}^n EY_{nt}^{*2} \leq Cn$ and for any $c_n/n \to 0$, $\sum_{t=1}^{c_n} EY_{nt}^{*2} \leq C c_n^{3/2} n^{-1/2} = o(n)$. By virtue of these facts and $E[u_{t+1}^2 \mid \mathcal{F}_t] \to_{a.s.} \sigma^2$, it follows from Lemma 7.3 that

$$E|Z_{2n}|^{1/2} \leq \frac{C}{\sqrt{n}} E\Big(\sum_{t=1}^n Y_{nt}^{*2} \big|\big[E(u_{t+1}^2 \mid \mathcal{F}_t) - \sigma^2\big]\big|\Big)^{1/2} = o(1),$$

which yields that $Z_{2n} = o_P(1)$.

By virtue of all these facts, namely $Z_{1n} + Z_{2n} + R_{nj} = o_P(1)$, and (7.47), we obtain (6.17). We next prove (6.16), which will follow if we prove that, under the extended probability space,

$$\frac{1}{n}\sum_{t=1}^n Y_{nt}^{*2} \to_P \eta^2. \tag{7.51}$$

Recall (6.2)-(6.3). Simple calculations show that

$$\frac{1}{n}\sum_{t=1}^n Y_{nt}^{*2} = \frac{\sigma^2}{d_n^2}\sum_{t=1}^{[nr]}\sum_{i=1}^{t-1} K^2\big\{c_n\,[G_n^0(t/n) - G_n^0(i/n)]\big\} + R_{1n}^* + R_{2n}^*, \tag{7.52}$$

where

$$R_{1n}^* =_d \frac{1}{d_n^2}\sum_{t=1}^n\sum_{i=1}^{t-1}(u_{i+1}^2 - \sigma^2) K^2\big[(x_t - x_i)/h\big]\big\},$$

$$R_{2n}^* =_d \frac{2}{d_n^2}\sum_{t=1}^n\sum_{\substack{i,j=1\\ i\neq j}}^{t-1} u_{i+1}u_{j+1} K\big[(x_t - x_i)/h\big]\, K\big[(x_t - x_j)/h\big].$$

By virtue of (6.2)-(6.3) and Theorem 4.1, under the extended probability space,

$$\Big|\frac{\sigma^2}{d_n^2}\sum_{t=1}^n\sum_{i=1}^{t-1} K^2\big\{c_n\,[G_n^0(t/n) - G_n^0(i/n)]\big\} - \eta^2\Big| \to_P 0. \tag{7.53}$$

So, it remains to show that

$$|R_{in}^*| = o_P(1), \quad i = 1,\ 2. \tag{7.54}$$

The proof of (7.54) for $i = 1$ follows from the same arguments as in the establishment of $V_{3n2} + V_{3n3} = o_P(n^{3/2}h)$, given in the proof of Theorem 3.3. To prove (7.54) with $i = 2$, we write

$$\begin{aligned} R^*_{2n} &=_d \frac{4}{d_n^2} \sum_{t=2}^{n} \sum_{i=2}^{t-1} \sum_{j=1}^{i-1} u_{i+1} u_{j+1} K\big[(x_t - x_i)/h\big]\, K\big[(x_t - x_j)/h\big] \\ &= R^*_{3n} + R^*_{4n}, \end{aligned} \tag{7.55}$$

where, by using the convention that $\sum_{i=s}^{t} = 0$ if $t < s$,

$$\begin{aligned} R^*_{3n} &= \frac{4}{d_n^2} \sum_{t=1}^{n} \sum_{j=1}^{t-2} u_t u_{j+1} K\big[(x_t - x_{t-1})/h\big]\, K\big[(x_t - x_j)/h\big] \\ R^*_{4n} &= \frac{4}{d_n^2} \sum_{t=1}^{n} Z_t, \end{aligned}$$

where

$$Z_t = \sum_{i=1}^{t-2} \sum_{j=1}^{i-1} u_{i+1} u_{j+1} K\big[(x_t - x_i)/h\big]\, K\big[(x_t - x_j)/h\big].$$

Recall $|u_t| \le A$. It follows easily from (7.15) with $p_1(x) = p_2(x) = K(x)$ in Lemma 7.4 that

$$\begin{aligned} E|R^*_{3n}| &\le \frac{C}{n^{3/2} h} \sum_{t=1}^{n} \sum_{j=1}^{t-2} E\Big\{K\big[(x_t - x_{t-1})/h\big]\, K\big[(x_t - x_j)/h\big]\Big\} \\ &\le \frac{Ch}{n^{3/2}} \sum_{t=1}^{n} \sum_{j=1}^{t-2} \frac{1}{\sqrt{t-j}} \le C_1\, h \to 0. \end{aligned} \tag{7.56}$$

As for R^*_{4n}, we have

$$\begin{aligned} ER^{*2}_{4n} &\le \frac{C}{n^3 h^2} \sum_{t,s=1}^{n} E(Z_t\, Z_s) \\ &= \frac{2C}{n^3 h^2} \sum_{t=2}^{n} \sum_{s=1}^{t-1} E(Z_t\, Z_s) + \frac{C}{n^3 h^2} \sum_{t=1}^{n} EZ_t^2. \end{aligned} \tag{7.57}$$

In the following, we will show that

$$EZ_t^2 \le C\, t^{3/2} h^3, \tag{7.58}$$

and for $s < t$,

$$\begin{aligned} E(Z_s\, Z_t) \le\ & \frac{Ch^4}{\sqrt{t-s}} \Big\{ s^2 [\log(t-s) + \log s] + \frac{s^{5/2}}{\sqrt{t-s}} + s(t-s)\log(t-s) \Big\} \\ & + \frac{Ch^3 s}{\sqrt{t-s}}. \end{aligned} \tag{7.59}$$

Taking (7.58) and (7.59) into (7.57), we obtain

$$
\begin{aligned}
ER_{4n}^{*2} &\le \frac{C}{n^3h^2}\sum_{t=1}^{n}\sum_{s=1}^{t-2}E(Z_t\,Z_s)+\frac{C}{n^3h^2}\sum_{t=1}^{n}EZ_t^2 \\
&\le C\,h^2\sqrt{n}\log n + Cn^{-1/2}h \to 0,
\end{aligned}
$$

since $nh^4\log^2 n\to 0$. This, together with (7.55) and (7.56), yields that

$$
E|R_{2n}^*| \;\le\; E|R_{3n}^*| + (ER_{4n}^{*2})^{1/2}\to 0, \quad \text{as } n\to\infty,
$$

which implies (7.54) with $i=2$.

We next prove (7.58) and (7.59). The result (7.58) is simple. Indeed, by letting $Z_{t12}=\sum_{i=1}^{t-2}u_{i+1}K\big[(x_t-x_i)/h\big]$ as before, it is readily seen that

$$
2Z_t = Z_{t12}^2-\sum_{i=1}^{t-2}u_{i+1}^2K^2\big[(x_t-x_i)/h\big].
$$

This, together with (7.28) and (7.14)-(7.15) in Lemma 7.4, implies that

$$
\begin{aligned}
EZ_t^2 &\le \frac{1}{4}EZ_{t12}^4+\frac{A^4}{4}E\Big(\sum_{i=1}^{t-2}K^2\big[(x_t-x_i)/h\big]\Big)^2 \\
&\le C\,t^{3/2}h^3+C\,h^2\sum_{1\le i<j\le t-2}\frac{1}{\sqrt{t-j}}\frac{1}{\sqrt{j-i}}+C\,h\sum_{i=1}^{t-2}\frac{1}{\sqrt{t-i}} \\
&\le C_1(t^{3/2}h^3+h^2t+h\sqrt{t})\le C_2\,t^{3/2}h^3,
\end{aligned}
$$

since $t\le n$ and $nh^2\to\infty$. This proves (7.58).

To prove (7.59), by noting that, for $s<t$,

$$
\sum_{\substack{i,j=1\\ i<j}}^{s-2}\sum_{\substack{k,l=1\\ k<l}}^{t-2}=\sum_{\substack{i,j=1\\ i<j}}^{s-2}\Big(\sum_{\substack{k,l=1\\ k<l}}^{s-2}+\sum_{l=s-1}^{s}\sum_{k=1}^{l-1}+\sum_{l=s+1}^{t-2}\sum_{k=1}^{l-1}\Big)
$$

and

$$
4\sum_{\substack{i,j=1\\ i<j}}^{s-2}\sum_{\substack{k,l=1\\ k<l}}^{s-2} = \sum_{\substack{i,j=1\\ i\ne j}}^{s-2}\sum_{\substack{k,l=1\\ k\ne l}}^{s-2}=\sum_{i\ne j\ne k\ne l}^{s-2}+4\sum_{\substack{i\ne j\ne k\\ l=i}}^{s-2}+2\sum_{\substack{i\ne j\\ k=i,l=j}}^{s-2},
$$

we may write, with the notation $I(...)$ as in Lemma 7.5,

$$
\begin{aligned}
Z_sZ_t &= \sum_{\substack{i,j=1\\ i<j}}^{s-2}\sum_{\substack{k,l=1\\ k<l}}^{t-2}I(i,j,k,l,s,t) \\
&= I_{1st}+I_{2st}+I_{3st}+I_{4st}+I_{5st},
\end{aligned} \tag{7.60}
$$

where, by symmetry and (7.20),

$$
\begin{aligned}
|EI_{1st}| &= \Big|\sum_{i\neq j\neq k\neq l}^{s-2} EI(......)\Big| \\
&\le \frac{Ch^4}{t-s}\sum_{1\le i<j<k<l\le s-2}\frac{1}{\sqrt{s-l}}\frac{1}{\sqrt{l-k}}\frac{1}{\sqrt{j-i}} \\
&\quad+\frac{Ch^4}{\sqrt{t-s}}\sum_{1\le i<j<k<l\le s-2}\frac{1}{s-l}\frac{1}{\sqrt{l-k}}\frac{1}{\sqrt{j-i}} \\
&\le Ch^4\Big(\frac{s^{5/2}}{t-s}+\frac{s^2\log s}{\sqrt{t-s}}\Big); \qquad (7.61)
\end{aligned}
$$

by $|u_j|\le A$ and (7.18) with $l=i$ in Lemma 7.4,

$$
\begin{aligned}
&|EI_{2st}| = 4\Big|\sum_{\substack{i\neq j\neq k\\ l=i}}^{s-2} EI(......)\Big| \\
&\le 4A^4\sum_{i\neq j\neq k}^{s-2} E\Big\{K\big[(x_s-x_i)/h\big]\,K\big[(x_t-x_i)/h\big]\,K\big[(x_s-x_j)/h\big]K\big[(x_t-x_k)/h\big]\Big\} \\
&\le \frac{Ch^4}{\sqrt{t-s}}\sum_{1\le i<j<k\le s-2}\frac{1}{\sqrt{s-k}}\,\frac{1}{\sqrt{j-i}} \\
&\le \frac{Ch^4s^2}{\sqrt{t-s}}; \qquad (7.62)
\end{aligned}
$$

by $|u_j|\le A$ and (7.17) with $k=i$ and $l=j$ in Lemma 7.4,

$$
\begin{aligned}
&|EI_{3st}| = 2\Big|\sum_{\substack{i\neq j\\ k=i,l=j}}^{s-2} EI(......)\Big| \\
&\le 4A^4\sum_{i\neq j}^{s-2} E\Big\{K\big[(x_s-x_i)/h\big]\,K\big[(x_t-x_i)/h\big]\,K\big[(x_s-x_j)/h\big]K\big[(x_t-x_j)/h\big]\Big\} \\
&\le \frac{Ch^3}{\sqrt{t-s}}\sum_{1\le i<j\le s-2}\frac{1}{\sqrt{s-j}} \\
&\le \frac{Ch^3s^{3/2}}{\sqrt{t-s}}; \qquad (7.63)
\end{aligned}
$$

by (7.21) in Lemma 7.5,

$$\begin{aligned}
|EI_{5st}| &= \sum_{\substack{i,j=1\\ i<j}}^{s-2}\sum_{l=s+1}^{t-2}\Big(\sum_{k=1}^{s-1}+\sum_{k=s}^{l-1}\Big)\big|EI(......)\big| \\
&\le \sum_{\substack{i,j=1\\ i<j}}^{s-2}\sum_{l=s+1}^{t-2}\sum_{k=1}^{s-1}\frac{Ch^4}{t-l}\frac{1}{\sqrt{l-k}}\frac{1}{\sqrt{s-j}}\frac{1}{\sqrt{j-i}} \\
&\quad+\sum_{\substack{i,j=1\\ i<j}}^{s-2}\sum_{l=s+1}^{t-2}\sum_{k=s}^{l-1}\frac{Ch^4}{t-l}\frac{1}{\sqrt{l-s}}\frac{1}{\sqrt{s-j}}\frac{1}{\sqrt{j-i}} \\
&\le Ch^4 s\,\Big(\sum_{l=s+1}^{t-2}\sum_{k=1}^{s-1}\frac{1}{t-l}\frac{1}{\sqrt{l-k}}+\sum_{l=s+1}^{t-2}\sum_{k=s}^{l-1}\frac{1}{t-l}\frac{1}{\sqrt{l-s}}\Big) \\
&\le Ch^4 s\Big[\sqrt{t-s}\,\log(t-s)+\frac{s\log(t-s)}{\sqrt{t-s}}\Big]. \qquad (7.64)
\end{aligned}$$

As for I_{4st}, by noting

$$\begin{aligned}
\sum_{\substack{i,j=1\\ i<j}}^{s-2}\sum_{l=s-1}^{s}\sum_{k=1}^{l-1} &= \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{s-2}\Big(\sum_{l=s-1}^{s}\sum_{k=1}^{s-2}+\sum_{l=s-1}^{s}\sum_{k=s-1}^{l-1}\Big) \\
&= \frac{1}{2}\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{s-2}\sum_{l=s-1}^{s}+\sum_{\substack{i,j\\ i\neq j,k=i}}^{s-2}\sum_{l=s-1}^{s}+\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{s-2}\sum_{l=s-1}^{s}\sum_{k=s-1}^{l-1},
\end{aligned}$$

it follows from $|u_j|\le A$, (7.17)-(7.18) in Lemma 7.4,

$$\begin{aligned}
|EI_{4st}| &\le \sum_{\substack{i,j=1\\ i<j}}^{s-2}\sum_{l=s-1}^{s}\sum_{k=1}^{l-1}\big|EI(......)\big| \\
&\le A^4\Big(\frac{1}{2}\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{s-2}\sum_{l=s-1}^{s}+\sum_{\substack{i,j\\ i\neq j,k=i}}^{s-2}\sum_{l=s-1}^{s}+\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{s-2}\sum_{l=s-1}^{s}\sum_{k=s-1}^{l-1}\Big)\chi_{st}(i,j,k,l) \\
&\le \frac{Ch^4}{\sqrt{t-s}}\sum_{1\le i<j<k\le s-2}\frac{1}{\sqrt{s-k}}\frac{1}{\sqrt{j-i}} \\
&\quad+\frac{Ch^3}{\sqrt{t-s}}\sum_{1\le i<j\le s-2}\frac{1}{\sqrt{s-j}} \\
&\le \frac{Ch^4 s^2}{\sqrt{t-s}}, \qquad (7.65)
\end{aligned}$$

since $s\le n$ and $nh^2\to\infty$.

By virtue of (7.60)-(7.65), the result (7.59) follows from a simple calculation. The proof of Proposition 6.2 is now complete. □

7.4 Proof of Proposition 6.3. As in (7.47), we have

$$M_n(1) \;=\; \sum_{t=2}^{n}\big[V(\tau_{n,t}/n)-V(\tau_{n,t-1}/n)\big]\,Y_{n,t}^{*}+R_{nj}, \tag{7.66}$$

where, for some $1\le j\le n$ satisfying $\tau_{n,j-1}<n\le\tau_{n,j}$,

$$\begin{aligned}|R_{nj}| \;&\le\; \frac{1}{\sqrt{n}}|V(1)-V(\tau_{n,n}/n)|+|Y_{n,j}^{*}|\,|V(1)-V(\tau_{n,j-1}/n)| \\ &\quad +\Big|\sum_{t=j}^{n}\big[V(\tau_{n,t}/n)-V(\tau_{n,t-1}/n)\big]\,Y_{n,t}^{*}\Big| \\ &:=\; Z_{3n}+Z_{4n}+Z_{5n}.\end{aligned} \tag{7.67}$$

Write $\Omega_j=\{j: n-j\le n^{\delta},\text{where } \tau_{n,j-1}<n\le\tau_{n,j}\}$. It follows from (6.4) that

$$P(j\notin\Omega_j)\to 0.$$

This yields that, for any $\epsilon>0$,

$$\begin{aligned}P(Z_{5n}\ge\epsilon) \;&\le\; P(j\notin\Omega_j)+P(j\in\Omega_j, Z_{5n}\ge\epsilon) \\ &\le\; o(1)+\epsilon^{-2}\,E\big(I_{(j\in\Omega_j)}Z_{5n}^2\big).\end{aligned} \tag{7.68}$$

Note that Ω_j is $\mathcal{F}_{n,t-1}^{0}$-measurable for $t\ge j$ and

$$E\Big\{\big[V(\tau_{n,t}/n)-V(\tau_{n,t-1}/n)\big]\mid\mathcal{F}_{n,t-1}\Big\}=0.$$

It follows from the conditional arguments and (7.50) that

$$\begin{aligned}E\big(I_{(j\in\Omega_j)}Z_{5n}^2\big) \;&\le\; E\Big(I_{(j\in\Omega_j)}\sum_{t=j}^{n}\big[V(\tau_{n,t}/n)-V(\tau_{n,t-1}/n)\big]^2\,Y_{n,t}^{*2}\Big) \\ &\le\; \sum_{t=n-n^{\delta}}^{n}E\Big(\big[V(\tau_{n,t}/n)-V(\tau_{n,t-1}/n)\big]^2\,Y_{n,t}^{*2}\Big) \\ &\le\; \sum_{t=n-n^{\delta}}^{n}E\Big(Y_{n,t}^{*2}\,E\big[\big[V(\tau_{n,t}/n)-V(\tau_{n,t-1}/n)\big]^2\mid\mathcal{F}_{n,t-1}^{0}\big]\Big) \\ &\le\; \frac{1}{n}\sum_{t=n-n^{\delta}}^{n}E\Big(Y_{n,t}^{*2}\,\big[\tau_{n,t}-\tau_{n,t-1}\big]\Big)\to 0.\end{aligned}$$

This yields that $Z_{5n}=o(1)$. By virtue of (6.4), $Z_{3n}=o_p(1)$ is obvious. As for Z_{4n}, similar to the proof of (7.68), we have

$$\begin{aligned} P(Z_{4n}\geq \epsilon) &\leq P(j\notin\Omega_j)+P(j\in\Omega_j, Z_{4n}\geq\epsilon)\\ &\leq o(1)+\epsilon^{-2}\,E\big(I_{(j\in\Omega_j)}Z_{4n}^2\big)\\ &\leq o(1)+\sum_{t=n-n^\delta}^{n} E\Big(Y_{n,t}^{*2}\sup_{\tau_{n,t-1}<n\leq\tau_{n,t}}\big[V(1)-V(\tau_{n,t-1}/n)\big]^2\Big)\\ &\leq o(1)+\sum_{t=n-n^\delta}^{n} E\Big\{Y_{n,t}^{*2}\,E\big(\sup_{\tau_{n,t-1}<n\leq\tau_{n,t}}\big[V(1)-V(\tau_{n,t-1}/n)\big]^2\mid\mathcal{F}_{n,t-1}^0\big)\Big\}\\ &\leq o(1)+\frac{1}{n}\sum_{t=n-n^\delta}^{n} E\Big(Y_{n,t}^{*2}\big[\tau_{n,t}-\tau_{n,t-1}\big]\Big)\to 0. \qquad (7.69)\end{aligned}$$

Taking all these estimates into (7.67), we obtain the required (6.18). □

7.5 Proof of Proposition 6.4. We only prove (6.26). By using conditional arguments and Lemmas 7.1-7.2, the other results are similar but simpler. Note that

$$\begin{aligned} &1+|x_t|^{\alpha_2}\leq t^{\alpha_2/2}\big[1+(x_t/\sqrt{t})^{\alpha_2}\big]\\ \leq\ & C_\alpha\,t^{\alpha_2/2}\big[1+(|x_t-x_i|/\sqrt{t})^{[\alpha_2]+1}+(|x_s-x_i|/\sqrt{t})^{[\alpha_2]+1}+(|x_s|/\sqrt{t})^{[\alpha_2]+1}\big].\end{aligned}$$

We have that

$$\begin{aligned} &E\Big\{(1+|x_s|^{\alpha_1})\,(1+|x_t|^{\alpha_2})|p[(x_i-x_t)/h]|\,|p[(x_i-x_s)/h]|\Big\}\\ \leq\ & C_\alpha\,t^{\alpha_2/2}E\Big\{(1+|x_s|^{\alpha_1})\,\big[1+(|x_s|/\sqrt{t})^{[\alpha_2]+1}\big]|p[(x_i-x_t)/h]|\,|p[(x_i-x_s)/h]|\Big\}\\ &+C_\alpha\,t^{\alpha_2/2}E\Big\{(1+|x_s|^{\alpha_1})p_{[\alpha_2]+1}[(x_i-x_t)/h]|\,|p[(x_i-x_s)/h]|\Big\}\\ &+C_\alpha\,t^{\alpha_2/2}E\Big\{(1+|x_s|^{\alpha_1})\,|p[(x_i-x_t)/h]|\,|p_{[\alpha_2]+1}[(x_i-x_s)/h]|\Big\}\\ :=\ & I_{1n}+I_{2n}+I_{3n}. \qquad (7.70)\end{aligned}$$

It follows from (7.8) with $p(x)=|p(x-x_t/h)|\,|p(x-x_s/h)|$ that

$$\begin{aligned}\Xi_t &:= E\big(|p[(x_i-x_t)/h]|\,|p[(x_i-x_s)/h]|\mid\mathcal{F}_t\big)\\ &\leq \frac{Ch}{\sqrt{i-t}}\int_{-\infty}^{\infty}|p(x-x_t/h)|\,|p(x-x_s/h)|\,dx\\ &= \frac{Ch}{\sqrt{i-t}}\int_{-\infty}^{\infty}|p(x)|\,|p[x+(x_t-x_s)/h]|\,dx.\end{aligned}$$

Similarly, uniformly for $x \in R$, it follows from (7.8) first and then (7.11) that

$$\begin{aligned} & E\Big\{(1+|x_s|^{\alpha_1})\,\big[1+(|x_s|/\sqrt{t})^{[\alpha_2]+1}\big]\,|p(x+(x_t-x_s)/h)|\Big\} \\ \leq\ & \frac{Ch}{\sqrt{t-s}}\int_{-\infty}^{\infty}|p(x+y)|dy\,E\Big\{(1+|x_s|^{\alpha_1})\,\big[1+(|x_s|/\sqrt{t})^{[\alpha_2]+1}\big]\Big\} \\ \leq\ & \frac{C_1 h s^{\alpha/2}}{\sqrt{t-s}}. \end{aligned}$$

By virtue of these facts, it is readily seen that

$$\begin{aligned} I_{1n} &= C_\alpha\, t^{\alpha_2/2}\, E\Big\{(1+|x_s|^{\alpha_1})\,\big[1+(|x_s|/\sqrt{t})^{[\alpha_2]+1}\big]\,\Xi_t\Big\} \\ &\leq \frac{Ch\, t^{\alpha_2/2}}{\sqrt{i-t}}\int_{-\infty}^{\infty}|p(x)|\,E\Big\{(1+|x_s|^{\alpha_1})\,\big[1+(|x_s|/\sqrt{t})^{[\alpha_2]+1}\big]\,|p(x+(x_t-x_s)/h)|\Big\}dx \\ &\leq \frac{C_1\, h^2}{\sqrt{i-t}}\,\frac{s^{\alpha_1/2}t^{\alpha_2/2}}{\sqrt{t-s}}. \end{aligned}$$

Similarly, we have

$$I_{2n}+I_{3n} \leq \frac{C\,h^2}{\sqrt{i-t}}\,\frac{s^{\alpha_1/2}t^{\alpha_2/2}}{\sqrt{t-s}}.$$

Taking these estimates into (7.70), we obtain the required (6.26). The proof of Proposition 6.4 is now complete. □

7.6 Proof of Proposition 6.5. We will repeatedly use the following fact:

$$E(|u_{s+1}| \mid \mathcal{F}_s) \leq \big[E(|u_{s+1}|^2 \mid \mathcal{F}_s)\big]^{1/2} \leq \big[E(|u_{s+1}|^4 \mid \mathcal{F}_s)\big]^{1/4} \leq C.$$

Let $Z_{2i}^* = u_{i+1}\,g(x_i)\,\sum_{t=i+2}^n K\big[(x_t-x_i)/h\big]$. We may write

$$Z_{2i} = u_{i+1}\,g(x_i)K\big[(x_{i+1}-x_i)/h\big] + Z_{2i}^*. \tag{7.71}$$

The result (6.28) will follow if we prove

$$EZ_{2i}^{*2} \leq Cnh^2 i^\beta, \tag{7.72}$$

$$E(Z_{2i}^*\,Z_{2j}^*) \leq C\,\sqrt{n}h^{3/2}\,(1+h^{1/2}\log n)\,i^{\beta/2}\,j^{\beta/2}, \tag{7.73}$$

for $i<j$, and

$$E\Big(\sum_{i=1}^n u_{i+1}\,g(x_i)K\big[(x_{i+1}-x_i)/h\big]\Big)^2 \leq C\,n^{2+\beta}\,\sqrt{h}, \tag{7.74}$$

Indeed, by (7.71)-(7.74), it is readily seen that

$$\begin{aligned} E\Big(\sum_{i=1}^n Z_{2i}\Big)^2 &\leq C\,n^{2+\beta}\,\sqrt{h} + Cnh^2\sum_{i=1}^n i^\beta + C\sqrt{n}h^{3/2}\,(1+h^{1/2}\log n)\sum_{1\leq i<j\leq n-1} i^{\beta/2}\,j^{\beta/2} \\ &\leq C\,n^{5/2+\beta}h^{3/2}(h^{1/2}\,\log n+1), \end{aligned}$$

since $nh^2 \to \infty$, which yields (6.28).

We next prove (7.72)-(7.74). To this end, write

$$J(i,j,s,t) \;=\; E\Big[u_{i+1}u_{j+1}g(x_i)g(x_j)K\big[(x_s-x_i)/h\big]K\big[(x_t-x_j)/h\big]\Big].$$

If $i=j$ and $s=t\geq i+2$, by recalling $\sup_{i\geq 1}E(u_{i+1}^4\mid \mathcal{F}_i)<\infty$ and $\sup_x K(x)<\infty$, it follows from (6.27) in Proposition 6.4 that

$$\begin{aligned} J(i,i,t,t) &\leq \sup_x K(x)\,E\Big\{u_{i+1}^2(1+|x_i|^{2\beta})K\big[(x_t-x_i)/h\big]\Big\} \\ &\leq \frac{Chi^\beta}{\sqrt{t-i}}. \end{aligned}$$

Similarly, if $i=j$ and $i+2\leq s<t$, then by (7.8) and (6.27)

$$\begin{aligned} J(i,i,s,t) &= E\Big\{u_{i+1}^2 g(x_i)^2K[(x_s-x_i)/h]E\big(K\big[(x_t-x_i)/h\big]\mid \mathcal{F}_s\big)\Big\} \\ &\leq \frac{Ch}{\sqrt{t-s}}E\Big\{u_{i+1}^2(1+|x_i|^{2\beta})\,K\big[(x_s-x_i)/h\big]\Big\} \\ &\leq \frac{Ch^2 i^\beta}{\sqrt{t-s}}\frac{1}{\sqrt{s-i}}. \end{aligned}$$

By virtue of these estimates, it is readily seen that

$$\begin{aligned} EZ_{2i}^{*2} &= 2\sum_{i+2\leq s<t\leq n} J(i,i,s,t)+\sum_{t=i+2}^n J(i,i,t,t) \\ &\leq Ch^2\sum_{i+2\leq s<t\leq n}\frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s-i}}\,i^\beta + Ch\sum_{t=i+2}^n\frac{1}{\sqrt{t-i}}i^\beta \\ &\leq C(nh^2+\sqrt{n}h)\,i^\beta \leq C\,nh^2 i^\beta, \end{aligned}$$

since $nh^2\to\infty$, which yields (7.72).

To prove (7.73), for $i<j$ and $m_0=1+[\beta]$, write $\Omega=\{s: s=j+1 \text{ or } i+1\leq s\leq i+m_0+1\}$ and assume $\sum_{i=k}^s=0$ if $s<k$. We have

$$\begin{aligned} E\big(Z_{2i}^*\,Z_{2j}^*\big) &= \Big(\sum_{s\in\Omega}\sum_{t=j+2}^n+\sum_{s=i+m_0+2}^j\sum_{t=j+2}^n+\sum_{s=j+2}^n\sum_{t=j+2}^n\Big)J(i,j,s,t) \\ &:= A_{1i}+A_{2i}+A_{3i}. \end{aligned} \tag{7.75}$$

First calculate A_{2i}. For $i<j$, $i+m_0+2\leq s\leq j$ and $t\geq j+2$, it follows from (7.7) with $p(x)=K(x-x_j/h)$ that

$$\begin{aligned} J(i,j,s,t) &\leq E\Big\{|u_{i+1}||g(x_i)||g(x_j)|K\big[(x_s-x_i)/h\big]\,\big|E\big(u_{j+1}K[(x_t-x_j)/h]\mid \mathcal{F}_j\big)\big|\Big\} \\ &\leq \frac{Ch}{t-j}E\Big\{|u_{i+1}||g(x_i)||g(x_j)|K\big[(x_s-x_i)/h\big]\Big\}. \end{aligned} \tag{7.76}$$

Furthermore, by recalling $|g(x)| \le C(1+|x|^\beta)$ for some $\beta \ge 0$, the result (7.12) implies that

$$\begin{aligned}
& E\Big\{|u_{i+1}||g(x_i)||g(x_j)|K\big[(x_s-x_i)/h\big]\Big\} \\
\le\ & CE\Big[|u_{i+1}|(1+|x_i|^\beta)K\big[(x_s-x_i)/h\big]\,E\big\{(1+|x_j|^\beta)\mid \mathcal{F}_s\big\}\Big] \\
\le\ & C_1(j-s)^{\beta/2}]E\Big\{|u_{i+1}|(1+|x_i|^\beta)K\big[(x_s-x_i)/h\big]\Big\} \\
& +C_1\,E\Big\{|u_{i+1}|(1+|x_i|^\beta|)\,|x_s|^\beta\,K\big[(x_s-x_i)/h\big]\Big\} \\
& +C_1\,E\Big\{|u_{i+1}|(1+|x_i|^\beta|)\,\big(\sum_{k=0}^\infty (k+1)^{-1-\delta}|\epsilon_{s-k}|\big)^\beta\,K\big[(x_s-x_i)/h\big]\Big\} \\
:=\ & B_{i1}+B_{i2}+B_{i3}. \qquad (7.77)
\end{aligned}$$

It follows from (6.27) in Proposition 6.4 that

$$B_{i1}+B_{i2} \;\le\; \frac{Ch}{\sqrt{s-i}}i^{\beta/2}j^{\beta/2}. \qquad (7.78)$$

To calculate B_{i3}, first notice that

$$\begin{aligned}
\big(\sum_{k=0}^\infty (k+1)^{-1-\delta}|\epsilon_{s-k}|\big)^\beta \;&\le\; 1+\big(\sum_{k=0}^\infty (k+1)^{-1-\delta}|\epsilon_{s-k}|\big)^{[\beta]+1} \\
&=\; 1+\sum_{k_1,\dots,k_{m_0}=0}^\infty \prod_{u=1}^{m_0}(k_u+1)^{-1-\delta}\prod_{u=1}^{m_0}|\epsilon_{t-k_u}|,
\end{aligned}$$

and by using (7.10) in Lemma 7.1 with $p(x)=K(x-x_i/h)$,

$$\begin{aligned}
& E\Big\{|u_{i+1}|(1+|x_i|^\beta|)\prod_{u=1}^{m_0}|\epsilon_{s-k_u}|K\big[(x_s-x_i)/h\big]\Big\} \\
\le\ & E\Big\{(1+|x_i|^\beta|)\prod_{\substack{u=1\\ s-k_u\le i}}^{m_0}|\epsilon_{s-k_u}|E\Big(|u_{i+1}|\prod_{\substack{u=1\\ s\ge s-k_u\ge i+1}}^{m_0}|\epsilon_{s-k_u}|K\big[(x_s-x_i)/h\big]\mid \mathcal{F}_i\Big)\Big\} \\
\le\ & \frac{Ch}{\sqrt{s-i-m_0}}E\Big\{(1+|x_i|^\beta|)\prod_{\substack{u=1\\ s-k_u\le i}}^{m_0}|\epsilon_{s-k_u}|\Big\} \\
\le\ & \frac{Ch}{\sqrt{s-i}}\big[E(1+|x_i|^\beta|)^2\big]^{1/2}\Big(E\prod_{\substack{u=1\\ s-k_u\le i}}^{m_0}|\epsilon_{s-k_u}|^2\Big)^{1/2} \\
\le\ & \frac{Ch}{\sqrt{s-i}}i^{\beta/2}.
\end{aligned}$$

These facts imply that

$$\begin{aligned}
B_{i3} &\le CE\Big[|u_{i+1}|(1+|x_i|^\beta)\,\big\{1+\big(\sum_{k=0}^{\infty}(k+1)^{-1-\delta}|\epsilon_{t-k}|\big)^{1+[\beta]}\big\}\,K\big[(x_t-x_i)/h\big]\Big] \\
&\le C\,E\Big\{|u_{i+1}|(1+|x_i|^\beta)\,K\big[(x_s-x_i)/h\big]\Big\} \\
&\quad + \sum_{k_1,\dots,k_{m_0}=0}^{\infty}\prod_{u=1}^{m_0}(k_u+1)^{-1-\delta}E\Big\{|u_{i+1}|(1+|x_i|^\beta)\prod_{u=1}^{m_0}|\epsilon_{s-k_u}|K\big[(x_s-x_i)/h\big]\Big\} \\
&\le \frac{Ch}{\sqrt{s-i}}i^{\beta/2}. \qquad (7.79)
\end{aligned}$$

Combining (7.76)-(7.79), we get

$$\begin{aligned}
A_{2i} &\le \sum_{s=i+m+2}^{j}\sum_{t=j+2}^{n}\frac{Ch}{t-j}\frac{C_1 h}{\sqrt{s-i}}i^{\beta/2}j^{\beta/2} \\
&\le C\sqrt{n}h^2\,\log n\, i^{\beta/2}j^{\beta/2}. \qquad (7.80)
\end{aligned}$$

Next calculate A_{1i}. For $i<j$, $s\in\Omega$ and $t\ge j+2$, similar arguments to those above show that

$$\begin{aligned}
&\big|J(i,j,s,t)\big| \\
&\le E\Big[|u_{i+1}|\,|u_{j+1}|\,|g(x_i)||g(x_j)|K\big[(x_s-x_i)/h\big]\,\big|E\big(K[(x_t-x_j)/h]\mid\mathcal{F}_{j+1}\big)\big|\Big] \\
&\le \frac{Ch}{\sqrt{t-j}}E\Big[|u_{i+1}|\,|u_{j+1}|\,|g(x_i)||g(x_j)|K\big[(x_s-x_i)/h\big]\Big] \\
&\le \frac{Ch}{\sqrt{t-j}}\Big\{E\big(|u_{i+1}|^2\,|g(x_i)|^2K^2\big[(x_s-x_i)/h\big]\big)\Big\}^{1/2}\Big\{E\big[|u_{j+1}^2g^2(x_j)|\big]\Big\}^{1/2} \\
&\le \frac{Ch}{\sqrt{t-j}}\frac{h^{1/2}}{(s-i)^{1/4}}[E(1+|x_i|^{2\beta}]^{1/2}[E(1+|x_j|^{2\beta})]^{1/2} \\
&\le \frac{Ch}{\sqrt{t-j}}\frac{h^{1/2}}{(s-i)^{1/4}}i^{\beta/2}j^{\beta/2}.
\end{aligned}$$

This yields

$$\begin{aligned}
A_{i1} &\le Ch^{3/2}i^{\beta/2}j^{\beta/2}\sum_{t=j+2}^{n}\frac{1}{\sqrt{t-j}}\sum_{s=i+1}^{i+m_0+1}\frac{1}{(s-i)^{1/4}} \\
&\le C\,m_0\,\sqrt{n}h^{3/2}\,i^{\beta/2}j^{\beta/2}. \qquad (7.81)
\end{aligned}$$

Finally, we calculate A_{3i}. It follows from (7.6) that, for $i<j$, $j+2\le s<t\le n$,

$$\begin{aligned}
& J(i,j,s,t) \\
= \ & E\Big[u_{i+1}u_{j+1}g(x_i)g(x_j)K\big[(x_s-x_i)/h\big]\,E\big\{K\big[(x_t-x_j)/h\big]\mid \mathcal{F}_s\big\}\Big] \\
= \ & \frac{h}{\sqrt{t-s}}\int_{-\infty}^{\infty}E\Big\{u_{i+1}u_{j+1}g(x_i)g(x_j)K\big[(x_s-x_i)/h\big]K\big[(\rho^{t-s}\,x_s-x_j)/h+x\big]\Big\} \\
& \qquad h_{s,t}\big(\frac{hx}{\sqrt{t-s}}\big)dx+C_{1n} \\
:= \ & C_n+C_{1n}, \qquad (7.82)
\end{aligned}$$

where

$$|C_{1n}| \ \le \ \frac{Ch}{t-s}\,E\Big\{|u_{i+1}||u_{j+1}||g(x_i)g(x_j)|K\big[(x_s-x_j)/h\big]\,\Big(1+\sum_{k=0}^{\infty}(k+1)^{-1-\delta}|\epsilon_{s-k}|\Big)\Big\}.$$

Note that, by (7.10) in Lemma 7.1 and Proposition 6.4,

$$\begin{aligned}
& J_k^*(i,j,s,t) \\
:= \ & E\Big\{|u_{i+1}||u_{j+1}||g(x_i)g(x_j)|K\big[(x_s-x_j)/h\big]\,(1+|\epsilon_{s-k}|)\Big\} \\
= \ & E\Big\{|u_{i+1}||g(x_i)g(x_j)|E\big[|u_{j+1}|K\big[(x_s-x_j)/h\big]\,(1+|\epsilon_{s-k}|)\mid \mathcal{F}_j\big]\Big\} \\
\le \ & \frac{Ch}{\sqrt{s-j}}E\big\{|u_{i+1}||g(x_i)g(x_j)|\big\} \\
\le \ & \frac{Ch}{\sqrt{s-j}}\big[E\big\{u_{i+1}^2(1+|x_i|^{2\beta})\big\}\big]^{1/2}\,\big[E(1+|x_j|^{2\beta})\big]^{1/2} \\
\le \ & \frac{Ch}{\sqrt{s-j}}i^{\beta}j^{\beta/2}, \quad \text{for } s-k\ge j+1;
\end{aligned}$$

and similarly,

$$\begin{aligned}
J_k^*(i,j,s,t) \ = \ & E\Big\{|u_{i+1}||g(x_i)g(x_j)|\,(1+|\epsilon_{s-k}|)\,E\big[|u_{j+1}|K\big[(x_s-x_j)/h\big]\ \mid \mathcal{F}_j\big]\Big\} \\
\le \ & \frac{Ch}{\sqrt{s-j}}E\big\{|u_{i+1}||g(x_i)g(x_j)|\,(1+|\epsilon_{s-k}|)\,\big\} \\
\le \ & \frac{Ch}{\sqrt{s-j}}\big[E\big\{u_{i+1}^2(1+|x_i|^{2\beta})\big\}\big]^{1/2}\,\big[E(1+|x_j|^{4\beta})\big]^{1/4}\,\big\{E(1+|\epsilon_{s-k}|)^4\big\}^{1/4} \\
\le \ & \frac{Ch}{\sqrt{s-j}}i^{\beta}j^{\beta/2}, \quad \text{for } s-k\le j
\end{aligned}$$

By virtue of these estimates, it is readily seen that

$$|C_{1n}| \ \le \ \frac{Ch}{t-s}\sum_{k=0}^{\infty}(k+1)^{-1-\delta}\,J_k^*(i,j,s,t)\le\frac{Ch^2}{t-s}\,\frac{1}{\sqrt{s-j}}i^{\beta}j^{\beta/2}. \qquad (7.83)$$

On the other hand, by noting that (7.7) with $p(y) = K(y - x_j/h)K(\rho^{t-s}y - x_i/h + x)$ yields that

$$\begin{aligned} &\Big|E\Big\{u_{j+1}K\big[(x_s - x_j)/h\big]\,K\big[(\rho^{t-s}\,x_s - x_i)/h + x\big] \mid \mathcal{F}_j\Big\}\Big| \\ &\quad \le \frac{Ch}{s-j}\int_{-\infty}^{\infty} K(y - x_j/h)K(\rho^{t-s}y - x_i/h + x)dy \le \frac{C_1 h}{s-j}\,, \end{aligned}$$

the conditional argument implies that

$$\begin{aligned} |C_n| &\le \frac{Ch^2}{\sqrt{t-s}}\,\frac{1}{s-j}E\big\{|u_{i+1}||g(x_i)g(x_j)|\big\} \\ &\le \frac{Ch^2}{\sqrt{t-s}}\,\frac{1}{s-j}\big[E\big\{u_{i+1}^2(1+|x_i|^{2\beta})\big\}\big]^{1/2}\,\big[E(1+|x_j|^{2\beta})\big]^{1/2} \\ &\le \frac{Ch^2}{\sqrt{t-s}}\,\frac{1}{s-j}i^{\beta/2}j^{\beta/2}. \end{aligned} \tag{7.84}$$

It follows from (7.82)-(7.84) that, for $i < j$ and $j+2 \le s < t \le n$,

$$|J(i,j,s,t)| \le ch^2\Big(\frac{1}{\sqrt{t-s}}\,\frac{1}{s-j} + \frac{1}{t-s}\,\frac{1}{\sqrt{s-j}}\Big)\,i^{\beta/2}j^{\beta/2}.$$

Similarly, for $i < j$ and $j+2 \le s = t \le n$, we have

$$\begin{aligned} |J(i,j,t,t)| &= E\Big\{|u_{i+1}|\,|g(x_i)g(x_j)|\,\big|E\big(u_{j+1}K\big[(x_t - x_i)/h\big]K[(x_t - x_j)/h] \mid \mathcal{F}_j\big)\big|\Big\} \\ &\le \frac{Ch}{t-j}\,E\big\{|u_{i+1}|\,|g(x_i)g(x_j)|\big\} \\ &\le \frac{Ch}{t-j}\,i^{\beta/2}j^{\beta/2}. \end{aligned}$$

It is now readily seen that

$$\begin{aligned} |A_{3i}| &\le \sum_{t=j+2}^{n}|J(i,j,t,t)| + 2\sum_{j+2\le s<t\le n}|J(i,j,s,t)| \\ &\le Ci^{\beta/2}j^{\beta/2}\Big[\sum_{t=j+2}^{n}\frac{h}{t-j} + \sum_{j+2\le s<t\le n}\Big(\frac{h^2}{\sqrt{t-s}}\,\frac{1}{s-j} + \frac{1}{t-s}\,\frac{1}{\sqrt{s-j}}\Big)\Big] \\ &\le C\sqrt{n}\,h^2\,\log n\,i^{\beta/2}j^{\beta/2}, \end{aligned} \tag{7.85}$$

since $nh^2 \to \infty$. The result (7.73) follows from (7.75), (7.80), (7.81) and (7.85).

Finally, we prove (7.74). Using similar arguments to those above, we have

$$\begin{aligned} J(i,i,i+1,i+1) &= E\Big(u_{i+1}^2\,g^2(x_i)K^2\big[(x_{i+1} - x_i)/h\big]\Big) \\ &\le \Big\{E\Big[\,(1+|x_i|^{4\beta})E(u_{i+1}^4 \mid \mathcal{F}_i)\Big]\Big\}^{1/2}\,\Big\{EK^4\big[(x_{i+1} - x_i)/h\big]\Big\}^{1/2} \\ &\le C\sqrt{h}\,i^{\beta}, \end{aligned}$$

and Hölder's inequality implies that

$$
\begin{aligned}
J(i,j,i+1,j+1) &\le J^{1/2}(i,i,i+1,i+1)\,J^{1/2}(j,j,j+1,j+1)\\
&\le C\sqrt{h}\,i^{\beta/2}j^{\beta/2}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& E\Big(\sum_{i=1}^{n} u_{i+1}\, g(x_i)K\big[(x_{i+1}-x_i)/h\big]\Big)^2\\
\le\ & \sum_{i=1}^{n} J(i,i,i+1,i+1) + 2\sum_{1\le i<j\le n-1} J(i,j,i+1,j+1)\\
\le\ & C\, n^{2+\beta}\sqrt{h},
\end{aligned}
$$

which yields (7.74). The proof of Proposition 6.5 is now complete. □

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