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The Role of Commitment in Bilateral Trade*

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Abstract

We examine the buyer-seller problem under different levels of commitment. The seller is informed of the quality of the good, which affects both his cost and the buyer's valuation, but the buyer is not. We characterize the allocations that can be achieved through mechanisms in which, unlike with full commitment, the buyer has the option to "walk away" after observing a given offer. We further characterize the equilibrium payoffs that can be achieved in the bargaining game in which the seller makes all the offers, as the discount factor goes to one. This allows us to identify how different levels of commitment affect outcomes, and which constraints, if any, preclude efficiency.

Keywords: bargaining, mechanism design, market for lemons.

JEL codes: C70, C78, D82

1 Introduction

Mechanism design and non-cooperative bargaining theory have until recently evolved quite independently. The former gives us a powerful tool and a compact language for describing what

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outcomes can be implemented. But the agents' commitment that it typically assumes makes it unsuitable for the study of many actual institutions. Bargaining theory, on the other hand, starts with a precise description of the rules that direct the negotiations. Unfortunately, this also means that it becomes difficult to disentangle what in the environment is driving the different properties of the equilibrium.

In this paper, we consider the standard buyer-seller trading problem with an informed seller and interdependent values, and provide a characterization of implementable allocations and achievable payoffs under different degrees of commitment. In doing so, we break down the role played by commitment, from the full-commitment mechanism to the bargaining game with vanishing frictions.

Our setting is that of the lemon problem, as introduced by Akerlof (1970). Values are interdependent, and the seller knows both the value and cost of the unit, while the buyer does not.[1] There is common knowledge of gains from trade. The (full commitment) problem has been throughly investigated by Samuelson (1984) and Myerson (1985), and their findings provide the starting point for our analysis. As already pointed out by Myerson (1981), optimal mechanisms with interdependent values can exhibit strange properties. In particular, in our problem, the optimal mechanism need not satisfy posterior individual rationality. That is, the buyer may lose from participating in the mechanism given the information that this mechanism conveys: if the buyer were to reconsider his willingness-to-trade in light of the offer that he is supposed to accept, given that this offer leads him to re-evaluate his expected value for the unit, he may very well prefer to pass. Allowing him to do so is a feature of most real-world institutions, as Gresik (1991a) has pointed out: buyers can rarely be coerced into accepting offers that make them worse off. And this is certainly the case if the buyer and seller are engaged in bargaining.

[1] By a simple change of variable, all our results apply to the case in which it is the buyer who is informed and who makes offers, and the seller is uninformed.

Therefore, we turn our attention to the optimal mechanism when the buyer cannot be forced to buy. This property, which we call *veto-incentive compatibility*, following Forges (1999), imposes restrictions on the mapping from reported types to the distribution over offers that the mechanism specifies. Veto-incentive compatibility, then, is a restriction on the graph of this map: conditional on any given offer, the posterior belief of the buyer should be such that he is willing to accept this offer. Note that we are not imposing *ex post* individual rationality, a stronger requirement that posits that the buyer gains given the actual state of nature (i.e., his true valuation). The difference matters here, since values are interdependent (see Gresik, 1991b, Forges, 1994, and Matthews and Postlewaite, 1989). Note also that we are not relaxing commitment on the seller's side. We choose not to do so both because it appears slightly less problematic from a practical point of view, but also because, as we shall see in Section 5.3, commitment by the seller bears little role in our main results.

As one immediately suspects, restricting attention to deterministic offers would entail a loss of generality. Nevertheless, we show that, if there are finitely many types (an assumption our model does not impose), it is enough to consider as many offers as types. Moreover, we may assume that the k-th highest offer only comes from the seller's k-th highest type and above. This result is somewhat reminiscent of Bester and Strausz (2001), although our environment does not fit their model.[2] More importantly, we show that whether a given allocation can be implemented in a veto-incentive compatible way or not is a property of (the map from reports to) the probability of trade and expected price alone. Unlike the mechanism itself, which is a joint distribution over the reported types and offers, these are simply functions of reported types. Therefore, the problem reduces then to a standard optimal control problem, to which variational techniques can be applied. The interesting feature is the property imposed by veto-incentive compatibility:

[2]There are some technical differences (notably, a continuum of types vs. finitely many types), but most importantly, we are not in their single-agent environment.

the necessary and sufficient condition is that the buyer's *ex ante* payoff, conditional on trading with all types above a given threshold be nonnegative, for all possible values of this threshold.

We then relax commitment even further, and turn to the infinite-horizon bargaining game in which the seller makes all the offers. Clearly, the resulting outcome must satisfy veto-incentive compatibility, since the buyer can reject any given offer. Clearly also, the temporal monopoly of the seller provides him with a lower bound on his payoff, a security payoff, that he can guarantee no matter how patient players are. More precisely, the seller can always secure a price equal to the buyer's lowest possible valuation. Sequential rationality imposes further constraints, since for instance, both the seller's, and the buyer's payoffs must be individually rational, not only from the *ex ante* point of view, but from any history onward. Yet we prove that this *ex ante* lower bound on the seller's payoff, along with veto-incentive compatibility, is the only further constraint imposed by bargaining: every payoff vector that can be achieved by a veto-incentive compatible allocation and that gives the seller this security payoff is an equilibrium payoff vector if the two players are patient enough. This might sound like a folk theorem, but this only holds in terms of payoffs: there are allocations that are veto-incentive compatible, and give the seller his security payoff, and yet cannot be implemented in the bargaining game.

Our contribution, therefore, is threefold:

- *Methodological*: we develop tools to deal with constraints imposed by a lack of commitment, and show how to reduce a multidimensional variational problem to a pair of one-dimensional ones. We believe that these tools might be useful for more general environments with private information and limited commitment.

- *Conceptual*: we clarify what it is about equilibrium in the bargaining game that restricts the set of attainable payoffs: non-commitment on one hand, as captured by veto-incentive compatibility, and individual rationality on the other, as captured by the lowest type's reservation payoff. We derive a type of "folk theorem" subject to those restrictions, though this folk theorem is in terms of *ex ante* payoffs. As we show through an example, this folk theorem does not extend to interim payoffs, and might fail to include the equilibrium payoff in the equilibrium in the game in which the buyer makes all the offers (Deneckere and Liang, 2006).

- With respect to the bargaining literature, our paper, alongside with Deneckere and Liang's, provides an understanding of the role of who makes the offers. For instance, the most efficient equilibrium outcome when the seller makes all the offers is strictly more efficient than the equilibrium outcome when the buyer makes the offers. Sometimes, even the most inefficient equilibrium does better. We provide sufficient conditions for bargaining to achieve constrained efficiency. In those circumstances, our result implies that as little commitment as bargaining suffices. Conversely, if bargaining fails to achieve efficiency, then trading institutions will only be successful in promoting efficiency if they manage to weaken the veto-incentive compatibility constraint, as is the case, for instance, when the uninformed party is asked to commit to a screening contract.

Among related papers, Ausubel and Deneckere (1989) analyze the link between mechanism design and and bargaining in the special case of private values (with one-sided incomplete information). They show that, when the uninformed party makes all the offers, a folk theorem holds. That is, every incentive compatible, individually rational, direct bargaining mechanism is implementable by sequential equilibria, if the frequency of offers is high enough. On the other hand, if the informed party makes the offer, a unique equilibrium outcome gets singled out as the frequency of offers increases. Our paper establishes that, as one would suspect, lack of commitment imposes more constraints with interdependent values than with private values.

Interestingly though, the set of equilibrium payoffs that can be achieved remains fairly easy to characterize, as the feasible set of some programming problem. The paper by Deneckere and Liang that was already mentioned provides a careful analysis of the bargaining game in which the (uninformed) buyer makes all the offers, and they prove that the equilibrium outcome is then unique. We comment further on the relationship with Deneckere and Liang, as well as with other papers, as we proceed.

Section 2 develops the set-up. The main results are stated in Section 3, and their proofs are provided in Section 4, with auxiliary results relegated to appendices. In Section 5, we discuss extensions. In particular, we analyze veto-incentive compatibility on the seller's side, and argue that it is less stringent a requirement than on the buyer's side. In particular, adding veto-incentive compatibility for the seller does not affect the set of achievable payoffs, whether one insists on veto-incentive compatibility for the buyer or not.

2 The Set-Up

2.1 The Trading Problem

Consider a trading problem in which player 1, the *seller*, owns an indivisible object that player 2, the *buyer*, wants to purchase. The players' valuations are determined by the realization of a random variable that is uniformly distributed over the unit interval, $t \sim \mathcal{U}[0,1]$. That is, given t, the seller's cost and the buyer's value for the object are given by $c(t)$ and $v(t)$ respectively. The functions $c : [0,1] \to \mathbb{R}_+$ and $v : [0,1] \to \mathbb{R}_+$ are assumed to be non-decreasing, piecewise continuous and right-continuous (and hence, a.e. differentiable). For convenience, in the statements of all results, we further make the assumption that c is piecewise differentiable on $(0,1)$. Because v need not be a constant function, this environment displays interdependent values, of which private values is a special case. Observe that the assumption that t is uniformly distributed is made with no loss of generality, given the restrictions imposed on the functions v and c. In Section 5, we extend some of the results to the case in which c and v are not both monotonic.

Information is asymmetric. The seller is informed of the realization of the random variable, and knows therefore both his cost and the buyer's value for the object. We refer to this realization as the seller's *type* $t \in T := [0, 1]$. The buyer, on the other hand, does not observe this realization. However, he knows the distribution of the random variable, and the functions v and c are common knowledge.

In particular, it is common knowledge that there are gains from trade. That is, we assume that

$$\inf_t v(t) - c(t) > 0.$$

Such a "gap" rules out the special case in which $v(0) = c(0) = 0$, as in Akerlof (1970), but it does not imply that the first-best allocation (which requires trade to take place with probability one) is attainable if individual rationality is imposed. Such a first-best mechanism is individually rational if and only if the buyer's expected value exceeds the seller's highest cost (see Lemma 1 of Deneckere and Liang, 2006). While our results can be adapted to this case, the trading problem becomes then rather uninteresting, and we rule it out in the sequel.

Our purpose is to characterize the allocations that can be achieved when there is limited commitment. More precisely, we wish to compare the set of allocations that are achievable under full commitment with those that can be obtained under weaker forms of commitment. First, we shall consider the case in which the buyer cannot be forced to trade if the actual offer that is being made leads to a negative expected payoff. Following Forges (1999), we refer to this assumption as *veto-incentive compatibility*. Given the mechanism, and for any outstanding offer, the buyer updates his expected value for the object. Veto-incentive compatibility requires this conditional expectation to exceed the offer, whenever the mechanism specifies trade in this event. This captures the notion that, in most trading environments, buyers can always reject an offer for which they anticipate a loss. In the words of Gresik (1991a), "in most markets each trader has the ability to refuse to trade when the "best" negotiated terms give him negative utility." For instance, a seller who puts up an object for sale in an auction house commits to the eventual outcome, given the auction mechanism, but potential buyers can drop out at any stage of the auction process. Note that, with interdependent values, this does not ensure that the buyer will not experience regret, that is, that his realized value will exceed the price that he paid. In many markets, there is not much a buyer can do to renege on a purchase for which his experienced utility falls short of the price that he paid. In this sense, the trade need not be *ex post* individually rational. (Note that the two notions coincide in the case of private values.) At the time of purchase, however, the potential buyer cannot be forced to accept an outstanding offer, if he anticipates a loss, simply because he chose to participate in the trading process.

The set of payoffs that can be achieved under these two mechanisms will then be compared to the set of payoffs in the infinite-horizon bargaining game with discounting, in which the seller makes all the offers.

2.2 Mechanisms

Direct mechanisms, that require the seller to report his type, provide a way for setting the terms of trade. To be more formal, a *direct mechanism* is a probability transition from T to $\{0,1\} \times \mathbb{R}_+$.[3,4] A direct mechanism, then, specifies whether trade occurs (the outcome "1" is

[3]That is, for each $t \in T$, $\mu(t)$ is a probability distribution on $\{0,1\} \times \mathbb{R}_+$, and the probability $\mu(\cdot)[A]$ assigned to any Borel set $A \subset \{0,1\} \times \mathbb{R}_+$ is a measurable function of $t \in T$.

[4]It is not hard to see that the restriction to offers in $\mathbb{R}_+$ rather than $\mathbb{R}$ is without loss of generality in this environment.

interpreted as trade, while the outcome "0" means no trade), and at what price, according to some joint distribution, and given the announcement of the seller. We let $x(t)$ denote the probability of trade, given the announcement t. That is,

$$x(t) := \mu(t)[1, \mathbb{R}_+]. \tag{1}$$

Without loss of generality, we assume that no payment is made if no trade occurs, that is, we assume that $\mu(t)[0, \{0\}] = 1 - x(t)$. If $x(t) > 0$, we let $p(t)$ denote the expected price, given the announcement t, i.e.

$$p(t) := \int_{\mathbb{R}_+} p\mu(t)[1, dp]/x(t), \tag{2}$$

and set $p(t) := 0$ otherwise. Given $x : T \to [0, 1]$ and $p : T \to \mathbb{R}_+$, the *allocation* (x, p) is *implementable* if there exists a mechanism μ (which *implements* (x, p)) such that x and p coincide everywhere with the functions that are defined by (1) and (2).

It follows from the revelation principle that attention can be restricted to direct mechanisms in which the seller announces his type truthfully. Furthermore, under commitment, attention can be restricted to mechanisms in which prices are deterministic, i.e. $p(t)$ is the only price assigned positive probability by $\mu(t)[1, \cdot]$, for all t.

Given some direct mechanism μ, the payoff to the seller of type t that reports s is given by

$$\pi^S(s|t) := x(s)[p(s) - c(t)].$$

The mechanism μ is *incentive compatible* if, for all $s, t \in T$, $\pi^S(t) := \pi^S(t|t) \geq \pi^S(s|t)$. We shall also be interested in the *ex ante* payoff of the seller before his type is determined, that is, given some incentive compatible mechanism μ,

$$\pi^S = \mathbb{E}_t[\pi^S(t)] = \int_T \pi^S(t)dt = \int_T x(t)[p(t) - c(t)]dt. \tag{3}$$

Fix some incentive compatible mechanism μ. Suppose that the buyer is offered to trade at some price p in the support of $\mu(t)[1, \cdot]$ for some $t \in T$. What is his expected payoff, conditional on this outcome $(1, p)$? Given the mechanism μ, fix a version of the conditional distribution $\nu : (\{0, 1\} \times \mathbb{R}_+) \times \mathcal{B} \to [0, 1]$, where $\mathcal{B}$ is the Borel field on T. Given $\mathcal{T} \in \mathcal{B}$, we write $\nu(\mathcal{T}|p)$ for $\nu((1, p), \mathcal{T})$, the conditional probability assigned to the seller's type being in the set $\mathcal{T}$, given the event $(1, p)$ (with an abuse of notation, we also write $\nu(t|p)$ for $\nu(\{t\}|p)$). The buyer's expected payoff, given p, is then

$$\pi^B(p) := \int_T v(t)d\nu(t|p) - p.$$

The *ex ante* payoff of the buyer is given by

$$\pi^B := \int_T x(t)[v(t) - p(t)]dt. \tag{4}$$

An incentive compatible mechanism μ is *individually rational* if $\pi^S(t) \geq 0$ for all $t \in T$, and $\pi^B \geq 0$. Further, it is *veto-incentive compatible* if $\pi^B(p) \geq 0$ for all prices in the support of μ. Because the buyer must break even given his conditional expectation, there is *a priori* no reason to expect that it is sufficient to consider mechanisms that specify deterministic prices, when considering veto-incentive compatible mechanisms.

To summarize, we shall be interested in determining the allocations (x, p) that can be implemented by incentive compatible, individually rational and veto-incentive compatible mechanisms, and in the set of *ex ante* payoffs $\pi = (\pi^B, \pi^S)$ spanned by such allocations.[5] For short, we refer

[5]A set of allocations $\{(x, p)\}$ *spans* the payoff set $A \subset \mathbb{R}^2$ if the image of that set, by the mappings defined by

to this problem as the veto-incentive compatible program, and these allocations as the veto-incentive compatible allocations, to be compared with the *full commitment allocations*, in which the requirement of veto-incentive compatibility is dropped. The problem of determining the latter set is well-known (see, in particular, Samuelson, 1984, and Myerson, 1985), and is referred to in the sequel as the full commitment program.

Of particular interest is the (constrained) *efficient* allocation for each program, that is, any allocation (x, p) that maximizes the overall gains from trade $\int_T x(t)[v(t) - c(t)]dt$, or equivalently, that maximizes the sum of *ex ante* payoffs $\pi^S + \pi^B$.

2.3 The Bargaining Game

In the next, we drop the assumption of commitment on both sides, and consider the infinite-horizon bargaining game. Trivially, this further reduces the set of implementable allocations. Deneckere and Liang (2006) have provided a comprehensive analysis of the game in which the uninformed party, the buyer, makes all the offers. Doing so allows to abstract from signaling issues, since after any history there is only one action that the informed party can take that does not terminate the game. Therefore, the analysis becomes tractable, although far from trivial, and the equilibrium outcome turns out to be unique. We shall consider the opposite case, in which the seller makes all the offers, and show that, in this case as well, it is possible to provide a simple characterization of the equilibrium payoffs as bargaining frictions vanish. Furthermore, the best equilibrium improves upon the equilibrium in the game in which the buyer makes the offers (in terms of efficiency).

Let us define the bargaining game more formally. Time is discrete, and indexed by $n = 1, \ldots, \infty$. At each time or period n, the seller asks a price for the unit. After observing the price,

(3) and (4), is equal to A.

the buyer either accepts or rejects the price. If the price is accepted, the game ends. If the offer is rejected, a period elapses and the seller asks for a price again. We shall allow for a public randomization device in the initial period (for concreteness, think of a draw from the uniform distribution on the unit interval), before the seller sets the first price. This allows us to focus on the extreme points of the equilibrium payoff set, and we shall not refer to this randomization device in the sequel.

The seller's asking price can take any real value. An outcome of the game is a triple (t, n, p_n), with the interpretation that the realized type is t, and that the buyer accepts the seller's price p_n in period n (which implies that all previous prices were rejected). The case $n = \infty$ corresponds to the outcome in which the buyer rejects all the prices (as a convention, set p_∞ equal to 0). Buyer and seller discount future payoffs at the common discount factor $\delta \in (0, 1)$. The seller's von Neumann-Morgenstern utility function over outcomes is his net surplus $\delta^{n-1}(p_n - c(t))$ when $n < \infty$, and zero otherwise. This suggests the interpretation of the cost as an actual production cost incurred at the time of the transaction, but an alternative and equivalent formulation is that the seller derives a flow utility of $(1 - \delta)c(t)$ in every period in which he holds on to the unit.

The buyer's realized utility is $\delta^{n-1}(v(t) - p_n)$ when the outcome is (t, n, p_n), $n < \infty$, and zero if $n = \infty$.[6] The players' expected utilities over lotteries of outcomes, or payoffs, are defined as usual.

A history (of prices) $h^{n-1} \in H^{n-1}$ in case trade has not occurred by time n is a sequence $(p_1, \ldots, p_{n-1})$ of asking prices that the seller set and the buyer rejected (set $H_0 := \varnothing$). A behavior strategy σ^S for the seller is a sequence $\{\sigma_n^S\}$, where σ_n^S is a probability transition from $T \times H^{n-1}$ into $\mathbb{R}$, mapping the seller's type, the history h^{n-1} into a (possibly random) asking price. A

[6]Discounting plays no role in the optimality of the buyer's strategy. Results would also apply to the case of a sequence of short-run buyers, as long as we interpret the buyer's payoff as the discounted sum of these short-run buyers' payoffs.

behavior strategy σ^B for the buyer is a sequence $\{\sigma^B_n\}$, where σ^B_n is a probability transition from $H^{n-1} \times \mathbb{R}$ into $\{0,1\}$, mapping the history h^{n-1} and the outstanding price into a probability of acceptance (as before, "1" denotes acceptance, and "0" rejection). We use the perfect Bayesian equilibrium (PBE) concept as defined in Fudenberg and Tirole (1991, Definition 8.2).[7] Given some (perfect Bayesian) equilibrium, we follow standard terminology in calling a buyer's offer *serious* if it is accepted by the seller with positive probability. An offer is *losing* if it is not serious. Clearly, the specification of losing offers in an equilibrium is, to a large extent, arbitrary.

Given some equilibrium $\sigma = (\sigma^B, \sigma^S)$, we denote by $\pi^S(\sigma)$ and $\pi^B(\sigma)$ the *ex ante* payoff of the seller and the buyer, respectively. Note that this involves taking expectations with respect to the seller's type. Given δ, the payoff vector $\pi = (\pi^B, \pi^S)$ can be *achieved* in the bargaining game if there exists an equilibrium σ of the bargaining game such that $\pi = (\pi^B(\sigma), \pi^S(\sigma))$.

Let $E(\delta)$ denote the set of equilibria in the bargaining game with discount factor δ, and $\Pi(\delta) \subset \mathbb{R}^2$ the set of payoff vectors given discount factor δ. Further, define $\underline{\Pi} := \liminf_{\delta\to 1} \Pi(\delta)$ and $\overline{\Pi} := \limsup_{\delta\to 1} \Pi(\delta)$ as the inner and outer limits of the equilibrium payoff set as frictions vanish. We shall show that those two sets are equal, and provide a simple characterization of this set.

3 Main Results

3.1 Preliminaries: The Full Commitment Program

We start by recalling the characterizations obtained by Samuelson (1984) and Myerson (1985) for the set of *ex ante* payoffs that can be achieved through mechanisms that satisfy incentive

[7] Fudenberg and Tirole define perfect Bayesian equilibria for finite games of incomplete information only. The suitable generalization of their definition to infinite games is straightforward and omitted.

compatibility and individual rationality.

For later purposes, it is useful to define the following. Given a mechanism μ, define the expected payment $\bar{p}(t)$ received by type $t \in T$ as

$$\bar{p}(t) := x(t)p(t).$$

Note that specifying the function $\bar{p} : T \to [0,1]$ is equivalent to specifying the function p, given our convention that $p(t) = 0$ whenever $x(t) = 0$. Incentive compatibility is the requirement that

$$\pi^S(t) = \bar{p}(t) - x(t)c(t) \geq \bar{p}(s) - x(s)c(t),$$

for all $s, t \in T$. This implies, in particular, that

$$\pi^S(t) \geq \lim_{s \downarrow t} \pi^S(s|t),$$

for all $t \in T$. We refer to this set of constraints as the set of local incentive compatibility constraints.

Suppose that the local incentive compatibility constraints are binding for all $t \in T$.[8] It is then standard to show that π^S is absolutely continuous and equal to, for all t, [9]

$$\pi^S(t) = \pi^S(1) + \int_t^1 x(s)dc(s).$$

In this case, all expected payments are uniquely determined by the probabilities of trade (and

[8]Because the cost function need not be continuous, there are allocations that are implementable in the full commitment program for which some local incentive compatibility constraints are not binding.

[9]Here and in what follows, $\int_T x(s)dc(s) := \int_{(0,1)} x(t)c'(t)dt + \sum_{t \in D^c} x(t)(c(t) - \lim_{s \uparrow t} c(s))$, where c' is the derivative of c on each interval, D^c is the set of discontinuities of c, and x is assumed to be right-continuous (since c and v are, this is without loss of generality).

the price $\bar{p}(1)$) through

$$\bar{p}(t) = \bar{p}(1) - x(1)c(1) + x(t)c(t) + \int_t^1 x(s)dc(s).$$

Let us also define the buyer's payoff $B(t)$ accruing from all seller's types above t, given some allocation (x, p), as

$$B(t) := \int_t^1 (x(s)v(s) - \bar{p}(s))ds. \tag{5}$$

Note that $B(0) = \pi^B$. Further, if all local incentive compatibility constraints are binding, we can express $B(t)$ as a function of x (and $\bar{p}(1)$) only. Explicitly,

$$\begin{aligned} B(t) &= \int_t^1 [x(s)v(s) - (\bar{p}(1) - x(1)c(1) + x(s)c(s) + \int_s^1 x(u)dc(u))]ds \\ &= \int_t^1 [x(s)(v(s) - c(s)) - \int_s^1 x(u)dc(u)]ds - (1-t)(\bar{p}(1) - x(1)c(1)). \end{aligned}$$

Trivially, given the revelation principle, the set of implementable allocations in the full commitment program is characterized by incentive compatibility and individual rationality. A sharper characterization can be obtained for the set of payoff vectors that can be achieved. The following theorem follows from the results of Samuelson (1984) and Myerson (1985).

Theorem 1 *(Samuelson, 1984, Myerson, 1985) Suppose that $c(1) \geq \int_T v(t)dt$. In the full commitment program:*

1. *The payoff set can be obtained, without loss of generality, by assuming that all local incentive compatibility constraints bind, and that the highest seller type's payoff is zero: $\pi^S(1) = 0$;*

2. *The payoff set is spanned by the set of non-increasing functions $x : T \to [0,1]$ subject to*

$$\int_0^1 [x(s)(v(s) - c(s)) - \int_s^1 x(u)dc(u)]ds \geq 0,$$

given expected payments, for all $t \in T$,

$$\bar{p}(t) = x(t)c(t) + \int_t^1 x(s)dc(s).$$

3. *The payoff set is a convex polygon whose extreme points are achieved by functions $x : T \to [0,1]$ that are step functions with either two or three steps; the origin is an extreme point, and for all other extreme points, it can be assumed that $x(0) = 1$.*

Note that the constraint in the second part of the theorem is simply the requirement that $B(0) \geq 0$, given the definition of $\bar{p}$. The requirement that x be non-increasing ensures incentive compatibility, given the definition of $\bar{p}$. Theorem 1.2. states that any non-increasing function $x \in [0,1]$ satisfying $B(0) \geq 0$ (a constraint that only involves the function x) is part of an allocation that is implementable in the full commitment program, along with the expected payments defined in the theorem, and that these allocations are a sufficient class to generate all the payoffs that can be achieved in this program. As mentioned, one mechanism implementing any such allocation is a mechanism with deterministic prices. Of course, there are other mechanisms implementing this allocation, and there are other allocations that are implementable, but they do not lead to any additional payoff vectors.

In light of this characterization, the payoff set of the full commitment program can be obtained by considering a family of continuous linear programs, in which one maximizes $\lambda \cdot \pi$ over functions x satisfying (2), where $\lambda \in \mathbb{R}^2$ are the (possibly negative) weights on the buyer and seller's payoffs.

The maxima of these programs determine the extreme points of the payoff set, and it is then a standard result that such extreme points are themselves achieved by extreme points of the admissible set, i.e., by step functions.

The (constrained) efficient allocation takes a very simple form, given that it is the solution of a maximization problem in which both the objective and the single constraint are linear. Namely, as Samuelson and Myerson show, the *ex ante* efficient mechanism takes the following form: there exist $0 < t_1 < t_2 \leq 1$ such that

$$x(t) = \begin{cases} 1 & t \in [0, t_1), \\ x & t \in [t_1, t_2], \\ 0 & t > t_2, \end{cases}$$

where

$$x := \frac{t_1 \left(v_0^{t_1} - c(t_1) \right)}{t_2 c(t_2) - (t_2 - t_1) v_{t_1}^{t_2} - t_1 c(t_1)}, \tag{6}$$

and $v_0^{t_1}$, $v_{t_1}^{t_2}$ are the conditional expectations of the buyer's value over the relevant intervals, namely

$$v_0^{t_1} := \tfrac{1}{t_1} \int_0^{t_1} v(t)\, dt,$$

$$v_{t_1}^{t_2} := \tfrac{1}{t_2 - t_1} \int_{t_1}^{t_2} v(t)\, dt.$$

As can be verified, the threshold t_1 (resp., t_2) minimizes (resp., maximizes) the ratio

$$\frac{\int_{t_1}^{t_2} (v(t) - c(t)) dt}{\int_{t_1}^{t_2} t c'(t) dt},$$

given t_2 (resp., t_1). The numerator measures the gains from trade with the types in the interval $[t_1, t_2]$, while the denominator measures the information rents of the seller's types in that interval.[10] Indeed, if the buyer were to trade with, and only with, the seller's types $[0, t]$, his expected gains would be at most

$$Y(t) := \int_0^t (v(s) - c(t))ds = \int_0^t (v(s) - c(s) - sc'(s))ds, \tag{7}$$

a function that plays an important role in Samuelson and Myerson's analysis, as in ours.

3.2 The Veto-Incentive Compatible Program

Recall that the veto-incentive compatible program is obtained by adding to the full commitment program the requirement that, for any outstanding offer, the buyer's payoff is always non-negative, conditional on the outstanding offer, given his updated beliefs. At first sight, these constraints appear rather intractable, since these are restrictions on the marginal distributions over types derived from the joint distribution over types and offers that a mechanism defines. The main result of this subsection establishes that, in fact, these constraints can be formulated in terms of the probabilities of trade alone. Therefore, as in the full commitment problem, it is enough to consider functions x, rather than distributions defined by μ, to determine the payoff set, so that standard variational techniques can be applied.

We first characterize implementable allocations, and then achievable payoffs. The following proposition, proved in Section 4, characterizes the set of allocations that can be implemented in the veto-incentive compatible program. Recall that, trivially, incentive compatibility and individual rationality are minimal requirements.

Proposition 1 *An incentive compatible, individually rational allocation* (x, p) *is implementable*

[10] To see this, note that, from the formula for Y given by (7), $\int_{t_1}^{t_2} sc'(s)ds$ is the difference between the gains from trade and the buyer's additional profit accruing from the types $[t_1, t_2)$.

in the veto-incentive compatible program if and only if, for all $t \in T$,

$$B(t) = \int_t^1 x(s)[v(s) - p(s)]ds \geq 0.$$

Equipped with Proposition 1, it is then straightforward to characterize the set of payoffs that can be achieved in the veto-incentive compatible program.

Theorem 2 *Suppose that $c(1) \geq \int_T v(t)dt$. In the veto-incentive compatible program:*

1. *The payoff set can be obtained, without loss of generality, by assuming that all local incentive compatibility constraints bind, and that the highest seller type's payoff is zero: $\pi^S(1) = 0$;*

2. *The payoff set is spanned by the set of non-increasing functions $x : T \to [0,1]$ subject to, for all $t \in T$,*

$$\int_t^1 [x(s)(v(s) - c(s)) - \int_s^1 x(u)dc(u)]ds \geq 0, \tag{8}$$

given expected payments, for all $t \in T$,

$$\bar{p}(t) = x(t)c(t) + \int_t^1 x(s)dc(s).$$

Note that the constraint in the second part of the theorem is simply the requirement that $B(t) \geq 0$ for all $t \in T$, given the definition of $\bar{p}$. Theorem 2 .2. states that any non-increasing function $x \in [0,1]$ satisfying $B(t) \geq 0$ for all t (a constraint that only involves the function x) is part of an allocation that is implementable in the veto-incentive compatible program, along with the expected payments defined in the theorem, and that these allocations are a sufficient class to generate all the payoffs that can be achieved in this program. Because of the veto-incentive compatibility constraint, the mechanism that is constructed in the proof of this theorem is not, however, a mechanism with deterministic prices.

The constraints $B(t) \geq 0$ (as stated in Theorem 2.2. in terms of the probabilities $x(t)$ only) are linear (in x) as well. It follows that the payoff set can be once again determined by using continuous linear programming. There is, however, one difficulty that is common to incentive problems with hidden characteristics and a continuum of types, namely the requirement that the function x be non-increasing. Fortunately, tools exist for such constraints. See, in particular, Hellwig (2009). What is the structure of the solution for boundary points of the payoff set? It depends, of course, on the specific boundary point and the underlying functions c and v. Nevertheless, for such a point, it is standard to show that the set of types can be divided into a finite partition of intervals T_k such that, on each interval, either the probability x is constant, or B is identically 0 (i.e. (8) is binding). Note that, by differentiating twice (8), we obtain that the probability x must satisfy the ordinary differential equation

$$x'(t)(v(t) - c(t)) + x(t)v'(t) = 0$$

on any such interval. The problem then reduces to identifying this finite partition. Indeed, examples can be constructed for which B is identically zero over some interval, and therefore, the allocation need not be a step function, nor the payoff set a convex polygon (the set of extreme points need not be finite).

It is an easy consequence of this theorem that the payoff vector maximizing the buyer's payoff in the veto-incentive compatible program coincides with the payoff vector that maximizes the buyer's payoff in the full commitment program.[11] The seller's highest payoff is either equal to, or smaller than the corresponding payoff in the full commitment program. Sufficient conditions

[11] In fact, this follows from Proposition 1 in Samuelson (1984), as he shows that the buyer's favorite outcome is a take-it-or-leave it offer, so that veto-incentive compatibility does not bind at this allocation.

for equality will be provided in the next section.

3.3 Bargaining Game

We finally consider the bargaining game. Clearly, for any history, given any outstanding offer that is accepted with positive probability, sequential rationality requires that the buyer's conditional payoff from accepting it must be non-negative. Therefore, the *ex ante* payoffs that can be achieved via bargaining must form a subset of the payoff set of the veto-incentive compatible program. But bargaining imposes additional constraints. For instance, since v is non-decreasing, it is common knowledge that the object is worth at least $v(0)$ to the buyer. Therefore, the seller of type t can secure a payoff of $v(0) - c(t)$, since he can always insist on such an offer. (The formal argument is standard and omitted. See, for instance, Fudenberg, Levine and Tirole (1985), Lemma 2, which establishes that no lower offer is ever submitted in equilibrium, so that any such offer is necessarily accepted.) It is worth pointing out here that, if (x, p) is incentive compatible, then $\pi^S(0) \geq v(0) - c(0)$ implies that $\pi^S(t) \geq v(0) - c(t)$ for all $t \geq 0$, so that the aforementioned requirement reduces to $\pi^S(0) \geq v(0) - c(0)$. Since this provides a lower bound on the seller's payoff, we may think of this as the seller's *reservation payoff* in the bargaining game, a strengthening of individual rationality.

One might wonder whether bargaining imposes additional restrictions on achievable payoffs. The main result of this subsection shows that this is not the case, as the discount factor tends to one.

Before stating this result, note that, to every equilibrium σ, and for each seller's type t, one can associate a probability of trade $x(t)$, namely the discounted total probability with which trade occurs under σ, given t, or

$$x(t) = \mathbb{E}_\sigma \left[\sum_n \delta^{n-1} \mathbf{1}_{\sigma_n^B(h^{n-1}, p_n)=1} \right],$$

where $\mathbf{1}_A$ is the indicator function of the event A. Similarly, given some equilibrium σ, we let $\bar{p}(t) \in \mathbb{R}$ denote the expected discounted payment received by type t in this equilibrium. References to local incentive compatibility, or individual rationality, can be understood in terms of the pair $(x, \bar{p})$.

Theorem 3 *Suppose that $c(1) \geq \int_T v(t)dt$. Then $\underline{\Pi} = \overline{\Pi} =: \Pi$. Further, this set of payoff is equal to the set of payoffs that can be achieved by veto-incentive compatible allocations for which*

$$\pi^S(0) \geq v(0) - c(0).$$

Which constraints bind depends on the vertex that is considered. On the upper boundary of this set, it can be assumed, without loss of generality, that all local incentive compatibility constraints are binding, and that the highest type's payoff of the seller trading with positive probability is zero: $\pi^S(1) = 0$; on the other hand, for those vertices that minimize some convex combination of the seller's and buyer's payoff, the incentive compatibility constraints bind "downward," that is, for all $t \in T$,

$$\pi^S(t) = \lim_{s \uparrow t} \pi^S(s|t),$$

with the boundary condition that the trading price of the highest seller's type t is given by the minimum of $v(t)$ and either $\lim_{s \downarrow t} c(s)$, $t < 1$, or $c(1)$ if $t = 1$.

Given that the bargaining game imposes only one additional linear constraint to the veto-incentive compatible program, it can be analyzed via linear programming as well. Depending on c and v, this additional constraint can create a discontinuity (i.e., a step) in the function x which has no counterpart in the previous (veto-incentive compatible) program, and arises before the first binding constraint $B(t) = 0$. Notice also that the constraint that $\pi^S(0) \geq v(0) - c(0)$ implies that the seller secures the *ex ante* payoff $\mathbb{E}[(v(0)-c(t))^+]$ (because, as already mentioned, it implies that $\pi^S(t) \geq v(0) - c(t)$ for all t). However, the two requirements are not equivalent, as the example in the next subsection illustrates.

Theorem 3 is proved in the next section. In doing so, we shall show that the payoff vector maximizing the seller's payoff, which is also the efficient payoff vector in this set, coincides with the payoff vector maximizing the seller's payoff in the veto-incentive compatible program. That is, as far as efficiency is concerned, bargaining imposes no constraint beyond veto-incentive compatibility. In all three programs, the *ex ante* payoff of the buyer must be zero in any efficient allocation.

The proof is by construction. This requires us to specify beliefs after out-of-equilibrium offers. While sequential equilibrium is not well-defined in this game (the action space being infinite), our equilibrium can be made sequential by restricting this action set to a sufficiently rich but finite set of values. In this sense, our choice of off-path beliefs, while dictated by convenience, is not particularly fragile. It would be of interest, of course, to examine how the use of more restrictive concepts (such as perfect sequential equilibrium) would alter our results, but this is beyond the scope of this paper.

As mentioned, this result establishes that the only additional constraint on payoffs imposed by the bargaining game is that the lowest seller's type must secure his reservation payoff. However, it is not true that any individually rational, incentive compatible allocation satisfying veto-incentive compatibility, and giving the lowest seller's type his reservation payoff can be necessarily implemented in the bargaining game. In Section 5.2, we provide an example of such an allocation, and explain why it cannot be implemented. For any such allocation, our result implies that there exists a payoff-equivalent allocation (in terms of *ex ante* payoffs for the seller and the buyer) that can be implemented. Therefore, bargaining imposes restrictions on implementable allocations that go beyond veto-incentive compatibility (and the restriction imposed by the security payoff), but not on payoffs.

3.4 Examples and Economic Implications

To illustrate the results, we consider here an example with three equiprobable types.

Example 4 *The functions v and c are step functions with three steps, and the two discontinuities occur for both functions at $t = 1/3$ and $2/3$. To simplify, we refer to those three types as 1, 2, and 3. Values and costs are given by*

$$(c_1, c_2, c_3) = (0, 4, 9), \text{ and } (v_1, v_2, v_3) = (2, 5, 12),$$

so that a higher index means a higher value, but also a higher cost. The left panel of Figure 1 represents the three payoff sets. The largest area is the set of payoffs in the full commitment case, while the smaller area is the payoff set for the veto-incentive compatible program. The smallest payoff set is the equilibrium payoff set in the bargaining game as $\delta \to 1$. By changing only one parameter, namely, by increasing v_2 from 5 to 10, the payoff sets change considerably. See right panel. The two points $(440/1323, 20/63)$ on the left, and $(56/243, 2/9)$ on the right, represent the unique equilibrium payoff vectors in the bargaining game in which the (uninformed) buyer makes the offers in every period, as characterized in Deneckere and Liang (2006) for $\delta \to 1$.

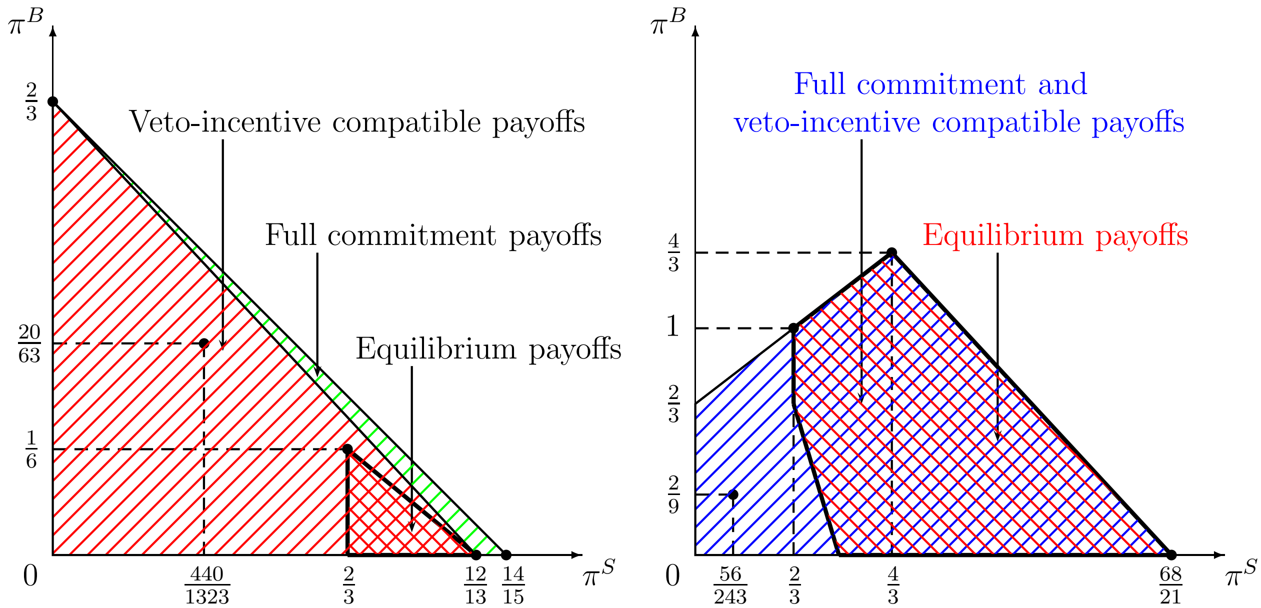

Figure 1: Full commitment, Veto-Incentive Compatible, and Limiting Equilibrium Payoff Sets.

This example illustrates several points that hold more generally. First, as mentioned, the buyer's highest payoff coincides in the veto-incentive compatible and the full commitment programs, but clearly, it might be lower in the equilibrium of the bargaining game. More importantly, the seller's highest payoff coincides in the bargaining game and the veto-incentive compatible program. This highest payoff, however, might fall short of the highest payoff in the commitment program.[12]

When is (constrained) efficiency possible under bargaining, i.e., when is veto-incentive compatibility consistent with efficiency? Obviously, this is trivially the case if the optimal allocation

[12]Note also that, as is clear from the left panel, the restriction on achievable payoffs imposed by the lowest seller's type reservation payoff is not equivalent to the restriction that the seller obtains the *ex ante* payoff $\mathbb{E}[(v(0)-c(t))^+] = 2/3$. Consider the vertex that minimizes the seller's payoff, subject to the buyer's payoff being zero. The requirement that the seller's lowest type gets at least $v(0) - c(0)$ drives the seller's *ex ante* payoff up to $17/18 > 2/3$. In this example, driving the seller's *ex ante* payoff down to $\mathbb{E}[(v(0) - c(t))^+]$ is only possible in some equilibrium for high enough values of the buyer's payoff.

under full commitment is such that no seller's type trades with interior probability. If some seller's types do trade with interior probability, sufficient conditions can be given in terms of the buyer's gain function Y (see (7)). Because $Y(0) = 0$ and $Y'(0) > 0$, yet $Y(1) < 0$, Y admits a smallest local maximizer $\underline{t}$. Note that $\underline{t}$ solves $v(\underline{t}) - c(\underline{t}) = \underline{t}c'(\underline{t})$ (assuming differentiability at this point for the sake of this discussion). Let also $\bar{t}$ denote the smallest strictly positive root of Y. We show in appendix C that efficiency is attainable in bargaining if

$$\forall t \geq \bar{t}, \int_{\underline{t}}^{t} (v(s) - c(t))ds \geq 0. \tag{9}$$

This condition is satisfied in the examples typically given in the literature (for instance, Samuelson's two-step example), but obviously, as our example above shows (left panel), it is not always true that efficiency can be achieved. Note that the condition becomes easier to satisfy as gains from trade $(v(t) - c(t))$ increase, and information rents $(tc'(t))$ decrease (both $\underline{t}$ and $\bar{t}$ then increase). We summarize this discussion as follows.

> *Constrained efficiency can be achieved by bargaining as $\delta \to 1$ (even when the first-best outcome cannot) if gains from trade are high, or information rents low enough.*

Because bargaining can achieve the same degree of efficiency as any (incentive compatible, individually rational) mechanism that satisfies veto-incentive compatibility, this implies that market institutions may only improve upon bargaining if they constrain the buyer somehow, in a way that weakens the veto-incentive compatibility constraint. This seems rather demanding, but not impossible. For instance, screening contracts by the uninformed party (here, the buyer), as in Rothschild and Stiglitz (1976), dispense with the requirement of veto-incentive compatibility: the uninformed party offers (and commits) to a menu of price and quantity pairs, and the informed party chooses from them. This is not quite as demanding in terms of commitment as full commitment, although the difference is small (see Mylovanov, 2008). In any event, there is little to gain from less constraining trading institutions. Note, for instance, that communication will not expand the set of equilibrium outcomes. (Formally, the set of allocations that are achieved by communication equilibria is the same as those achieved by perfect Bayesian equilibria in the bargaining game, as $\delta \to 1$). Fortunately, as discussed, circumstances in which veto-incentive compatibility does not reduce efficiency are quite common, and in those circumstances, as little commitment as bargaining suffices.

How do equilibrium outcomes in bargaining compare with the unique equilibrium outcome derived by Deneckere and Liang, when the buyer makes all the offers? In our two examples, the seller does worse in the latter equilibrium outcome than in any equilibrium outcome of our game. However, it is easy to construct examples in which this is not the case. In fact, the following can be shown (details available upon request).

Lemma 1

i. The allocation from the unique limit equilibrium outcome of the game in which the buyer makes all the offers is an equilibrium allocation in the game in which the seller makes all the offers if and only if it gives the lowest seller's type his reservation payoff (i.e., $v(0) - c(0)$), provided that the discount factor is sufficiently close to one.

ii. For δ close enough to one, the game in which the seller makes all the offers admits an equilibrium outcome that is strictly more efficient than the limit equilibrium outcome of the game in which the buyer makes all the offers.

The first statement should come as no surprise given that the allocation that results from the bargaining game in which the buyer makes all the offers must be veto-incentive compatible. This follows from the "skimming" property in bargaining: because, from any history onward, the remaining seller's types are all types above some threshold z_n, and because the buyer's continuation payoff must be non-negative, it must be that $B(z_n) \geq 0$.[13]

The second statement is immediately implied by the first, given that the buyer secures a strictly positive payoff when he makes the offers, yet within the set of veto-incentive compatible allocations, efficiency is maximized when the buyer gets zero profits.

Of course, this lemma compares the best equilibrium outcome in one game with the unique equilibrium outcome in the other. There might be equilibria in the game in which the seller makes all the offers that are more inefficient that the equilibrium outcome when the buyer makes offers. Rather surprisingly, our example illustrates that this need not be true, however. As is obvious from the right panel of Figure 1, efficiency might be necessarily higher when the seller makes all the offers. This makes apparent that having the seller make all the offers does not simply "expand" the set of equilibria.

4 Main Proofs

4.1 Proof of Proposition 1 and Le 2

The proof of Theorem 2 will be divided in several steps. First, we establish Proposition 1, which immediately implies Theorem 2.2, given Theorem 1. We will then show how this, along with some other observations, can be used to establish Theorem 2.1.

The proof of Proposition 1 is itself divided into three parts. First, we show that, given an allocation (x, p), the condition that $B(t)$ be non-negative for all t is necessary for the allocation to be implementable in the veto-incentive compatible program. Second, we turn to sufficiency.

[13]If z_n and z_{n+1} denote consecutive threshold types, the inequality $B(t) \geq 0$ for $t \in (z_n, z_{n+1})$ follows from the fact that the types in $[z_n, t]$ are the most unprofitable ones (for the buyer) above z_n.

We first show that the conditions are sufficient if the functions c and v are step functions. Then we show how, by appropriate limiting arguments, the result follows for any functions c and v satisfying the assumptions of the model.

4.1.1 Proof of Proposition 1

As mentioned, the argument is divided into three steps. First comes necessity.

Lemma 2 *If (x,p) is an allocation that is implementable in the veto-incentive compatible program, then, for all $t \in T$,*

$$B(t) = \int_t^1 (x(s)(v(s) - p(s))ds \geq 0.$$

Proof. Fix an allocation (x,p) that is implementable in the veto-incentive compatible program, and let μ denote the corresponding mechanism. Observe that, for all $t \in T$,

$$\begin{aligned}
\int_t^1 x(s)p(s)ds &= \int_t^1 \int_{\mathbb{R}_+} p\mu(s)[1,dp]ds \\
&\leq \int_t^1 \int_{\mathbb{R}_+} \left(\int_T v(u)d\nu(u|p) \right) \mu(s)[1,dp]ds \\
&\leq \int_t^1 \int_{\mathbb{R}_+} \left(\int_{u \geq t} v(u)d\nu(u|p) \right) \mu(s)[1,dp]ds \\
&= \int_t^1 x(s)v(s)ds.
\end{aligned}$$

The first equality follows from the definition of the function p (see (2)). The first inequality is implied by veto-incentive compatibility; the second follows from the monotonicity of v; the last equality, from the law of iterated expectations. This establishes the claim. ■

We now show sufficiency in the special case in which c and v are step functions.

Lemma 3 *If c and v are step functions, and (x, p) is an allocation that is implementable in the full commitment program, and such that, for all $t \in T$,*

$$B(t) = \int_t^1 (x(s)(v(s) - p(s))ds \geq 0,$$

then (x, p) is also implementable in the veto-incentive compatible program.

Proof. Since c and v are step functions, we may equivalently describe the environment as finite: there are N types, with cost and values

$$c_1 \leq c_2 \leq \cdots \leq c_N, \text{ and } v_1 \leq v_2 \leq \cdots \leq v_N.$$

To avoid some trivial but distracting complications, we shall assume that the inequalities involving costs are strict: $\forall i < n$, $c_i < c_{i+1}$. The probability of each type (i.e., the length of each step) is denoted q_i.[14] An allocation, then, reduces to a pair of vectors $x = (x_1, \ldots, x_N)$, $p = (p_1, \ldots, p_N)$.

The hypothesis that $B(t) \geq 0$ for all $t \in T$ implies that, for all $J = 1, \ldots, N$,

$$\sum_{i=J}^{N} x_i q_i v_i \geq \sum_{i=J}^{N} x_i q_i p_i. \tag{10}$$

We shall show that any incentive-compatible, individually rational allocation satisfying this condition can be implemented in the veto-incentive compatible program, using N prices. The proof is by induction on the number of types (uniformly over all cost, values and probabilities).

[14] More precisely, the number of types N is the number of types $t_i \in T$ for which either c or v (or both) has a discontinuity. The length of the interval refers to the intervals defined by the corresponding partition of T.

Note that this is true for $N = 1$. In that case, the buyer's individual rationality constraint implies $p_1 \le v_1$ (which trivially implies our hypothesis), while the seller's individual rationality constraint implies $p_1 \ge c_1$. Note then that any such allocation (x_1, p_1) with $p_1 \in [c_1, v_1]$ satisfies the veto-incentive compatibility constraint: conditional on p_1, the buyer assigns probability one to the (unique) type 1, and since $v_1 \ge p_1$, his payoff conditional on this event is positive.

Assume then that, whenever there are N types, and for any collection of costs, values and probabilities $\{(c_1, v_1, q_1), \ldots, (c_N, v_N, q_N)\}$ (with $0 \le c_i < v_i$, here and in what follows), any incentive compatible, individually rational allocation $\{(x_1, p_1), \ldots, (x_N, p_N)\}$ that satisfies (10) can be implemented in the veto-incentive compatible program with N (not necessarily distinct) prices. Consider the case of $N + 1$ types, with cost, values and probabilities $\{c_i, v_i, q_i\}_{i=1}^{N+1}$. Fix some incentive compatible, individually rational allocation

$$\{(x_1, p_1), \ldots, (x_{N+1}, p_{N+1})\},$$

satisfying (10). The argument is divided into three steps.

Step 1. Note that, by (10) with $J = N + 1$, $p_{N+1} \le v_{N+1}$. Also, incentive compatibility implies that $p_N \le p_{N+1}$. [15] It follows that there exists $z \in [0, x_{N+1}/x_N]$ such that

$$z x_N p_N + (x_{N+1} - z x_N)\, v_{N+1} = x_{N+1} p_{N+1}. \tag{11}$$

To see this, note that, for $z = 0$, the left-hand side reduces to $x_{N+1} v_{N+1}$, which is at least as large as the right-hand side, while for $z = x_{N+1}/x_N$, the left-hand side reduces to $x_{N+1} p_N$, which is at most as large as the right-hand side. Fix some z satisfying (15). Note that $z \le 1$, because

[15] The argument is standard: considering the two incentive compatibility conditions involving types N and $N + 1$ only, it follows that $x_N \ge x_{N+1}$ and $p_N \le p_{N+1}$.

$x_N \geq x_{N+1}$.

Step 2. Consider the game in which there are N types, with costs and values $\{\hat{c}_i, \hat{v}_i, \hat{q}_i\}_{i=1}^{N}$, defined as follows. Costs are unchanged: $\hat{c}_i := c_i$, all $i = 1, \ldots, N$. Values are given by

$$\hat{v}_i := v_i \text{ for } i < N, \text{ and } \hat{v}_N := \frac{q_N v_N + q_{N+1} z v_{N+1}}{q_N + q_{N+1} z},$$

(note that $\hat{v}_N \geq v_N \geq \hat{c}_N$), while probabilities are

$$\hat{q}_i := \frac{q_i}{\sum_{i \leq N} q_i + q_{N+1} z} \text{for } i < N, \text{ and } \hat{q}_N := \frac{q_N + q_{N+1} z}{\sum_{i \leq N} q_i + q_{N+1} z}.$$

We claim that the allocation $\{(x_i, p_i)\}_{i=1}^{N}$ (derived from $\{(x_i, p_i)\}_{i=1}^{N+1}$) is implementable, in this new environment, in the veto-incentive compatible program.

First, because costs are the same in this environment as in the original environment, individual rationality and incentive compatibility for all seller's types is implied by the fact that these were satisfied by the allocation $\{(x_i, p_i)\}_{i=1}^{N+1}$ in the original environment.

Therefore, to show that this allocation is implementable in the veto-incentive compatible program, given the induction hypothesis, it suffices to show that, for all $J \leq N$,

$$\sum_{i=J}^{N} x_i \hat{q}_i \hat{v}_i \geq \sum_{i=J}^{N} x_i \hat{q}_i p_i.$$

(Note that individual rationality for the buyer is the special case $J = 1$.) Simplifying,

$$\sum_{i=J}^{N} x_i \hat{q}_i \left(\hat{v}_i - p_i\right) = \frac{1}{q_N + q_{N+1} z} \left[\sum_{i=J}^{N-1} x_i q_i \left(v_i - p_i\right) + q_N x_N \left(v_N - p_N\right) + q_{N+1} x_N z \left(v_{N+1} - p_N\right) \right].$$

Adding and subtracting $(x_{N+1} - x_N z)\, v_{N+1}$ to the expression inside the square brackets yield

$$\sum_{i=J}^{N} x_i \hat{q}_i (\hat{v}_i - p_i) = \frac{1}{q_N + q_{n+1} z} \left[\begin{array}{c} \sum_{i=J}^{N-1} x_i q_i (v_i - p_i) + q_N x_N (v_N - p_N) + \\ q_{N+1} (x_{N+1} v_{N+1} - x_N z p_N - (x_{N+1} - x_N z)\, v_{N+1}) \end{array} \right].$$

Using the definition of z, we finally obtain

$$\sum_{i=J}^{N} x_i \hat{q}_i (\hat{v}_i - p_i) = \frac{1}{q_N + q_{N+1} z} \left[\sum_{i=J}^{N+1} x_i q_i (v_i - p_i) \right] \geq 0,$$

where the last inequality uses that, by assumption, the allocation satisfies (10).

Therefore, by the induction hypothesis, the allocation $\{(x_i, p_i)\}_{i=1}^{N}$ is implementable in the veto-incentive compatible program, in this new environment, with N prices. Let $\{\hat{r}_1, \ldots, \hat{r}_N\}$ be the prices that implement this allocation in the veto-incentive compatible program, and $\{\hat{x}_1(r), \ldots, \hat{x}_N(r)\}_{r \in \{\hat{r}_1, \ldots, \hat{r}_N\}}$ be the probabilities assigned to these prices.

Step 3. We now construct a set of prices $\{r_1, \ldots, r_{N+1}\}$ and probabilities $\{x_1(r), \ldots, x_{N+1}(r)\}$, $r \in \{r_1, \ldots, r_{N+1}\}$, that implement $\{(x_1, p_1), \ldots, (x_{N+1}, p_{N+1})\}$ in the veto-incentive compatible program, in the original environment.

The prices are given by

$$\{r_1, \ldots, r_{N+1}\} = \{\hat{r}_1, \ldots, \hat{r}_N\} \cup \{v_{N+1}\}.$$

The probabilities are given by, for $i < N + 1$,

$$x_i(r) = \hat{x}_i(r), \forall r \in \{\hat{r}_1, \ldots, \hat{r}_N\}, \text{ and } x_i(v_{N+1}) = 0,$$

and

$$x_{N+1}(r) = z\hat{x}_N(r)\, \forall r \in \{\hat{r}_1, \ldots, \hat{r}_N\}, \text{ and } x_{N+1}(v_{N+1}) = x_{N+1} - zx_N.$$

It is immediate to see that, conditional on any given $r \in \{\hat{r}_1, \ldots, \hat{r}_N\}$, the conditional value is the same as in the modified environment, so that the buyer's veto-incentive compatibility constraint holds. This is also true if $r = v_{N+1}$, because the only seller's type trading at this price is type $N+1$. Furthermore, by construction, buyer i trades with probability x_i and receives an average price p_i. This completes the proof. ■

Finally, we can show sufficiency for arbitrary cost and value functions.

Lemma 4 *If (x, p) is an individually rational and incentive compatible allocation such that, for all $t \in T$,*

$$B(t) = \int_t^1 x(s)[v(s) - p(s)]ds \geq 0,$$

then (x, p) is implementable in the veto-incentive compatible program.

Proof. Fix an allocation (x, p) that satisfies the assumptions of the lemma. Consider a sequence of partitions $\mathcal{P}^n = \{t_1^n, \ldots, t_n^n\}$, with $t_1^n = 0, t_n^n = 1$, $\max_i |t_i^n - t_{i+1}^n| < K/n$ for some constant K independent of n, and such that $D \subseteq \mathcal{P}^n$, where D is the set of discontinuities of either v or c (without loss of generality, assume that n is large enough to include this finite set).

We now define a sequence of functions $c^n, v^n : T \to \mathbb{R}_+$ as follows: for all $t < 1$, set $c^n(t) := c(t_j^n)$ for $t \in \left[t_j^n, t_{j+1}^n\right)$, $c^n(1) := c(t_{n-1}^n)$, as well as, for all $t < 1$, $v^n(t) := v(t_{j+1}^n)$ for $t \in \left[t_j^n, t_{j+1}^n\right)$, $v^n(1) := v(t_n^n)$.

Further, define the sequence of allocations x^n, p^n as follows: for all $t \in T$, set $x^n(t) := x(t_j^n)$, and $p^n(t) := p(t_j^n)$ for $t \in \left[t_j^n, t_{j+1}^n\right)$, $j < n-1$ ($t \in \left[t_j^n, t_{j+1}^n\right]$ if $j = n-1$.)[16]

[16]Note that the functions v^n, c^n as well as the allocations x^n, p^n are right-continuous.

Note that the allocation (x^n, p^n) is incentive compatible and individually rational for the seller given the functions (c^n, v^n) (because the choices of the types in the set $\mathcal{P}^n$ are incentive compatible and individually rational given the original allocation (x, p).) Define

$$B_j^n := \int_{t_j^n}^1 x^n(s)[v^n(s) - p^n(s)]ds.$$

Because $x(t_{j+1}^n) \leq x(t) \leq x(t_j^n)$ and $p(t_{j+1}^n) \leq p(t) \leq p(t_j^n)$ (by incentive compatibility) for $t \in [t_j^n, t_j^n + 1)$, $j < i-1$, we can pick these sequences such that, because $B(t_j^n) \geq 0$ (the lemma's hypothesis), it is also the case that also $B_j^n \geq 0$ for all j (clearly, $B_n^n = 0$). Therefore, the allocation (x^n, p^n) is individually rational for the buyer given (c^n, v^n) and further, given Lemma 4, this allocation is veto-incentive compatible in the game with cost and value functions (c^n, v^n). Let μ^n denote the corresponding mechanism. The mechanism μ^n defines a function x^n specifying the probability of trade given some message t, and a joint distribution $\tilde{\mu}^n$ on $T \times \mathbb{R}_+$ in case that there is a trade for each type.[17] Let $\hat{\mu}^n$ denote the product distribution whose marginals coincide with those of $\tilde{\mu}^n$. Note that incentive compatibility and veto-incentive compatibility are restrictions on the marginal distributions only, so that any mechanism inducing the pair x^n and $\hat{\mu}^n$ also implements (x^n, p^n). Note that, by construction, (x^n, p^n) converge (pointwise) to (x, p), and similarly, (c^n, v^n) converge pointwise to (c, v). Also, since we can replace the set of prices $\mathbb{R}_+$ by the compact interval $[0, v(1)]$ (because $v(1)$ is an upper bound on the price that can be in the support of any mechanism that is veto-incentive compatible), a subsequence of the sequence $\{\hat{\mu}^n\}$ (without loss of generality the sequence itself) must converge weakly to some distribution $\hat{\mu}$. It follows from Theorem 3.2. of Billinsgley (1968) that $\hat{\mu}$ must itself be a product distribution, and that the marginals of $\hat{\mu}^n$ converge weakly to the marginals of $\hat{\mu}$. Therefore, for all prices p,

[17] More precisely, $x = \mu(\cdot)[1, \mathbb{R}_+]$, as defined in Section 2, and the distribution $\tilde{\mu}$ is the joint distribution $\nu((1, \cdot), \cdot)$, where ν is the conditional distribution defined in Section 2 as well.

the marginal distribution $\hat{\mu}^n(\cdot|p)$ converges weakly to $\hat{\mu}(\cdot|p)$, and so it follows that, for all p,

$$\int_T \hat{\mu}(t|p)(t-p)dt \geq 0,$$

which is precisely the requirement of veto-incentive compatibility. Therefore, along with x, $\hat{\mu}$ defines a veto-incentive compatible mechanism. (Incentive compatibility and individual rationality are satisfied by hypothesis, given the limiting allocation (x,p)). ■

Note that Lemma 2 and 4 immediately imply Proposition 1.

4.1.2 Proof of Theorem 2.1

We shall now show that the payoff set of the veto-incentive compatible program can be obtained by assuming that:

1. the highest type of the seller that trades with positive probability has a zero payoff;

2. all local incentive compatibility constraints are binding.

Let us refer to this payoff set as Π^V. Note that this set is compact and convex. Both claims will be established by considering the boundary of Π^V. Because both properties are preserved under convex combinations, the result follows for the entire set. Also, given (x,p), let $\bar{t} := \sup\{t \in T : x(t) > 0\}$.

Clearly, $(0,0)$ is an extreme point of this set, and because it is achieved by the allocation $(x,p) = (0,0)$, the claims are trivially valid for this point. We further divide this boundary into $\Pi^V_- := \{(\pi^S, \pi^B) \in \mathbb{R}^2 : \pi^B = \max_{(\pi_1,\pi_2)\in\Pi^V} \pi_2 \text{ s.t. } \pi_1 \leq \pi^S\}$ and $\Pi^V_+ := \{(\pi^S, \pi^B) \in \mathbb{R}^2 : \pi^B = \max_{(\pi_1,\pi_2)\in\Pi^V} \pi_2 \text{ s.t. } \pi_1 \geq \pi^S\}$. As will be clear, Π^V_+ intersects the axis $\{(\pi^S, 0) : \pi^S \in \mathbb{R}\}$, so that $\Pi^V = \text{co } \{(0,0)\} \cup \Pi^V_+ \cup \Pi^V_-$, where, given any set A, co A denotes the convex hull of A.

Let us now establish three claims for $\Pi_+^V \cup \Pi_-^V$ simultaneously. If (x,p) achieves $\pi \in \Pi_+^V \cup \Pi_-^V$, then

1. $\lim_{s\downarrow t} \pi^S(s|t) = \pi^S(t)$ for all t. Suppose that this is not the case. First, consider the case in which the payoff is in Π_+^V. Take the supremum over $\hat{t}$ such that $\pi\left(\hat{t}\right) > \lim_{s\downarrow t} \pi\left(s \mid \hat{t}\right)$. Clearly, $\hat{t}$ is a point of discontinuity of $c\,(t)$ and $x\,(t)$. Consider then the following alternative allocation (x', p'), defined by

$$
\begin{aligned}
x'(t) = x(t) + \varepsilon \text{ if } t \in [\hat{t}, \hat{t}+\varepsilon),\ x'(t) = x(t) \quad & \text{otherwise;} \\
\bar{p}'(t) = \bar{p}(t) + \varepsilon c(t+\varepsilon) \text{ if } t \in [\hat{t}, \hat{t}+\varepsilon),\ \bar{p}'(t) = \bar{p}(t) \quad & \text{otherwise.}
\end{aligned}
$$

   It is straightforward to see that, for small enough $\varepsilon > 0$, this is incentive-compatible, satisfies $B\,(t) \geq 0$ for all t and strictly improves the buyer's payoff, while weakly improving the seller's payoff. Consider next the case in which the payoff of (x,p) belongs to Π_-^V. Take the supremum over $\hat{t}$ such that $\pi\left(\hat{t}\right) > \lim_{s\downarrow t} \pi\left(s \mid \hat{t}\right)$. Clearly, $\hat{t}$ is a point of discontinuity of $c\,(t)$. Thus consider the alternative allocation (x', p'), defined by

$$
\begin{aligned}
x'(t) = x(t) \quad & \text{for all } t \in [0,1], \\
\bar{p}'(t) = \bar{p}(t) - \varepsilon \text{ if } t \in [0, \hat{t});\ \bar{p}'(t) = \bar{p}(t) \quad & \text{otherwise.}
\end{aligned}
$$

   It is straightforward to check that for small $\varepsilon > 0$ this allocation is implementable. Moreover, it decreases the seller's payoff and increases the buyer's payoff, which contradicts the assumption that the payoff is in Π_-^V.

2. $\pi^S(\bar{t}) = 0$, where $\bar{t} := \sup\{t \leq 1 : x(t) > 0\}$ is the highest seller's type that trades with positive probability. Again, consider first the case in which the payoff is in Π_+^V. Modify the allocation by increasing $p(t)$ (for all t such that $x(t) > 0$) by some (small)

$\varepsilon > 0$, contradicting the hypothesis that $\pi \in \Pi_{+}^{V}$. Suppose next that $\pi \in \Pi_{-}^{V}$. Suppose towards a contradiction that this is not the case. Fix some small $\eta > 0$ and let $t^{*} = \sup\{t : x(t) - x(\bar{t}_{-}) > \eta\}$. Since the allocation is right-continuous we must have $x(t^{*}) \leq x(\bar{t}) + \eta$. Thus, define $\hat{p}$ such that $p\left(t_{-}^{*}\right) - x\left(t_{-}^{*}\right) c\left(t_{-}^{*}\right) = \hat{p} - [x(\bar{t}) + \eta] c\left(t_{-}^{*}\right)$, and consider the alternative allocation

$$
\begin{aligned}
\hat{x}(t) = x(\bar{t}) + \eta \text{ if } t \in [t^{*}, \bar{t}),\ \hat{x}(t) = x(t) \quad & \text{otherwise;} \\
\hat{p}(t) = \hat{p} \text{ if if } t \in [t^{*}, \bar{t}),\ \hat{p}(t) = p(t) \qquad & \text{otherwise.}
\end{aligned}
$$

The payoff of each seller's type weakly decreases in this alternative allocation, while the payoff of the buyer strictly increases. If the payoff of the seller remains constant, we are done. Suppose that the seller's payoff decreases by $\alpha > 0$. There exists $\varepsilon > 0$ such that $\int_{0}^{\bar{t}} \varepsilon dt = \alpha$. Thus, increase all prices by ε, so that the seller's overall payoff does not change. This is incentive compatible and increases the buyer's payoff. Thus, since the increase in surplus goes to the buyer, it is enough to show that $B(t) \geq 0$, all t. Note that the variation in the buyer's *ex ante* payoff is

$$
\begin{aligned}
\Delta B(0) &= \int_{0}^{\bar{t}} (\Delta x(t)(v(t) - c(t)))\, dt - \int_{0}^{\bar{t}} (\Delta \bar{p}(t) - \Delta x(t) c(t))\, dt \\
&= \int_{0}^{\bar{t}} (\Delta x(t)(v(t) - c(t)))\, dt > 0,
\end{aligned}
$$

where $\Delta x(t) := x'(t) - x(\bar{t})$ and $\Delta \bar{p}(t) := \bar{p}'(t) - \bar{p}(t)$. Furthermore,

$$
\Delta x(t)(v(t) - c(t)) + (\Delta \bar{p}(t) - \Delta x(t) c(t)) < 0,
$$

if and only if $t < t^{**} \in [t^{*}, \bar{t})$. Thus $\Delta B(t) \geq 0$ for all t, which completes the argument.

3. $x(0) = 1$. Suppose towards a contradiction that $x(0) < 1$. Since the cost function is piecewise right-continuous and piecewise differentiable we take an interval $[0, \eta]$ such that the allocation is differentiable on that interval. Fix $n' \in \mathbb{N}$ such that $1/n' < \eta$, and consider the following alternative allocation $(x_n, \bar{p}_n)$ defined as

$$x_n(t) = x(t) + (1 - x(0)) \text{ if } t \in [0, \tfrac{1}{n}),\ x_n(t) = x(t) \qquad \text{otherwise;}$$
$$\bar{p}_n(t) = \bar{p}(t) + c\left(\tfrac{1}{n}\right)(1 - x(0)) \text{ if } t \in [0, \eta),\ \bar{p}(t) = \bar{p}(t) \quad \text{otherwise.}$$

Notice that there exists $m > n'$ such that this allocation is implementable (and is also a Pareto improvement for all $n > m$). If $\pi \in \Pi_+^V$, this is an immediate contradiction. If instead $\pi \in \Pi_-^V$, let $k > 0$ be the maximal subgradient of the payoff set at π. Now notice that for each n the payoff of the buyer increases by $(1 - x(0)) \int_0^{\frac{1}{n}} (v(s) - c(1/n))\, ds$, while the payoff of the seller increases by $(1 - x(0)) \int_0^{\frac{1}{n}} (c(s) - c(1/n))\, ds$. Thus the ratio of the increase in the payoff of the buyer and the seller is arbitrarily large as $n \to \infty$, and for n large enough, both payoffs can be increased at a rate greater than k, a contradiction.

Note that we have now established Theorem 2, because the representation of the expected payments $\bar{p}$ given there follows immediately from the first two claims.

4.2 Proof of Theorem 3

This theorem is established by dividing the set of extreme points of the relevant payoff set into three different cases, according to whether this extreme point lies to the "north-east," "north-west," or "south-west" of the payoff set (i.e., according to the signs of the weights on the seller's and buyer's payoff whose linear combination this extreme point maximizes.) Arguments for one case require minor modifications to be valid in the other cases.[18] For brevity, we only provide the complete proof for the case of positive weights, that is, we consider extreme points that lie on the Pareto-frontier.

The proof is divided into two steps. First, it is shown that allocations for which x is a step function satisfying some properties can be implemented as equilibria of the game. Second, we show that every vertex of the equilibrium payoff set is the limit of a sequence of such allocations.

We first define a certain class of allocations (x, p).

4.2.1 Regular Allocations

Recall that, for all $0 \leq t_1 < t_2 \leq 1$, $v_{t_1}^{t_2} = \mathbb{E}[v(t) | t \in [t_1, t_2)]$.

Definition 1 *The allocation (x, p) is regular if there exists $0 = t_0 < t_1 < \cdots < t_K \leq 1$, for some finite K, such that*

1.
$$x(t) = \begin{cases} x_k & \text{if } t \in [t_{k-1}, t_k), \ k = 1, \ldots, K, \\ 0 & \text{if } t \geq t_K, \end{cases}$$
with $1 = x_1 > \cdots > x_K > 0$;

2.
$$p(t) = \begin{cases} p_k & \text{if } t \in [t_{k-1}, t_k), \ k = 1, \ldots, K, \\ 0 & \text{if } t \geq t_K, \end{cases}$$
with $v(0) < p_1 < \cdots < p_K$;

[18] For instance, in the "south-west" region, the local incentive constraints are binding "downward," and the definition of regular allocations must be modified accordingly.

3. for each $k = 1, \ldots, K-1$,

$$x_k \left(p_k - c\left(t_{k,-}\right)\right) = x_{k+1}\left(p_{k+1} - c\left(t_{k,-}\right)\right),$$

where $t_{k,-} = \lim_{t \uparrow t_k} t$ *(recall that* c *is right-continuous) and*

$$x_K \left(p_K - c\left(t_{K,-}\right)\right) \geq 0;$$

4. We have

$$B(0) \geq 0, B(t_1) = \cdots = B(t_{K-2}) = 0, \ \textit{and } B(t_{K-1}) > 0;$$

5. Furthermore,

$$v_{t_{K-2}}^{t_{K-1}} > c(t_{K-1,-});$$

6. Finally, $\pi^S(0) \geq v(0) - c(0)$.

That is, a regular allocation is a step allocation such that local incentive compatibility constraints hold at each jump, the contribution to the buyer's payoff of each interval of types $[t_k, 1]$ is zero except for $k = 0, K-1$, and positive for $t = 0, K-1$ (strictly so for $t = K-1$). Furthermore, the expected valuation of the buyer over the penultimate interval of types exceeds the cost of the seller's highest type in the previous interval, and the seller's lowest type must guarantee his security payoff.

A regular allocation need not be an equilibrium allocation in the discrete-time game, because of the indivisibilities that discrete periods introduce. This indivisibility becomes less and less problematic as $\delta \to 1$, and we show that we can choose (x^δ, p^δ) such that

$$\|(x^\delta, p^\delta)\| \to \|(x, p)\|,$$

uniformly in t, as $\delta \to 1$. The following lemma will be established in the next two subsections.

Lemma 5 *Fix a regular allocation (x, p). There exists a sequence of equilibria $\sigma^\delta \in E(\delta)$ such that the corresponding sequence of allocations (x^δ, p^δ) converges to (x, p) as $\delta \to 1$, uniformly in $t \in T$.*

We first turn to the definition of this allocation (x^δ, p^δ).

4.2.2 The Allocation (x^δ, p^δ)

Fix some regular allocation (x, p). In what follows, we assume that $K > 2$.[19] Fix δ and $\varepsilon < \min\{p_1 - v(0), v_{t_{K-2}}^{t_{K-1}} - c(t_{K-1,-})\}$. Further, pick $p_1^\delta, \ldots, p_{K-2}^\delta, \hat{p}^\delta, \tilde{p}^\delta \in \mathbb{R}_+$ such that, for all $k = 1, \ldots, K-2$, $\varepsilon/4 < p_k - p_k^\delta < 3\varepsilon/4$, as well as $\varepsilon/4 < v_{t_{K-2}}^{t_K} - \hat{p}^\delta < 3\varepsilon/4$ and $\varepsilon/4 < v_{t_{K-1}}^{t_K} - \tilde{p}^\delta < 3\varepsilon/4$. Set $T_1^\delta = 0$, and consider the following system in $T_2^\delta < \cdots < T_K^\delta, \beta^\delta$:

$$\delta^{T_k^\delta}\left(p_k^\delta - c(t_{k,-})\right) = \delta^{T_{k+1}^\delta}\left(p_{k+1}^\delta - c(t_{k,-})\right), \quad k = 2, \ldots, K-3,$$

$$\delta^{T_{K-2}^\delta}\left(p_{K-2}^\delta - c(t_{K-2,-})\right) = \delta^{T_{K-1}^\delta}\left(\beta^\delta\left(\hat{p}^\delta - c(t_{K-2,-})\right) + \left(1 - \beta^\delta\right)\delta\left(v_{t_{K-2}}^{t_{K-1}} - c(t_{K-2,-})\right)\right),$$

$$\delta^{T_{K-1}^\delta + 1}\left(v_{t_{K-2}}^{t_{K-1}} - c(t_{K-1,-})\right) = \delta^{T_K^\delta}\left(\tilde{p}^\delta - c(t_{K-1,-})\right),$$

$$\delta^{T_{K-1}^\delta}\beta^\delta\left(\hat{p}^\delta - c(t_{K,-})\right) + \delta^{T_K^\delta}\left(1 - \beta^\delta\right)\left(\tilde{p}^\delta - c(t_{K,-})\right) = x_K\left(p_K - c(t_{K,-})\right),$$

$$\delta^{T_{K-1}^\delta}\int_{t_{K-2}}^{t_K}\left(v(s) - \hat{p}^\delta\right) ds = \delta^{T_K^\delta}\int_{t_{K-1}}^{t_K}\left(v(s) - \tilde{p}^\delta\right) ds. \tag{12}$$

[19] The proof for $K = 2$ is very similar, but requires slightly different notations.

We may assume that this system in $\delta^{T_2^\delta}, \ldots, \delta^{T_K^\delta}, \beta^\delta$ has full rank (slightly change the values of the p^δ variables otherwise). By perturbing the values of the p^δ variables, it thus follows from the implicit function theorem that there exists $\delta_\varepsilon < 1$ such that for any $\delta > \delta_\varepsilon$, the values of the p^δ variables can be chosen so that $0 < \beta^\delta < 1$, $T_1^\delta, \ldots, T_K^\delta$ are integers, and such that

$$\max\{\max_{k \leq K-2} |\delta^{T_k^\delta} - x_k|, |\beta^\delta \delta^{T_{K-1}^\delta} + \left(1 - \beta^\delta\right) - x_{K-1}|, |\beta^\delta \delta^{T_{K-1}^\delta} + \left(1 - \beta^\delta\right) \delta^{T_K^\delta} - x_K|\} \leq \varepsilon.$$

Also, given equalities (13) and (14), the sequence β^δ converges to β, defined as the solution to the affine equation

$$x_{K-1} \left[\beta \left(v_{t_{K-2}}^{t_K} - c\left(t_{K,-}\right)\right) + (1-\beta) \left(\frac{v_{t_{K-2}}^{t_{K-1}} - c\left(t_{K-1,-}\right)}{v_{t_{K-1}}^{t_K} - c\left(t_{K-1,-}\right)}\right) \left(v_{t_{K-1}}^{t_K} - c\left(t_{K,-}\right)\right)\right] = x_K \left(p_K - c\left(t_{K,-}\right)\right). \tag{13}$$

Because $B\left(t_{K-2}\right) = 0$, $B\left(t_{K-1}\right) > 0$ and $v_{t_{K-2}}^{t_{K-1}} > c\left(t_{K-1,-}\right)$, β must lie in $(0,1)$, and satisfy

$$x_{K-1} \left[\beta + (1-\beta) \left(\frac{v_{t_{K-2}}^{t_{K-1}} - c\left(t_{K-1,-}\right)}{v_{t_{K-1}}^{t_K} - c\left(t_{K-1,-}\right)}\right)\right] = x_K. \tag{14}$$

The values β^δ will play the role of a probability in our equilibrium construction.

We now finally define the allocation $\left(x^\delta, p^\delta\right)$ as

$$x^\delta\left(t\right) := \begin{cases} \delta^{T_k^\delta} & \text{if } t \in \left[t_{k-1}, t_k\right),\ k = 1, \ldots, K = 2, \\ \beta^\delta \delta^{T_{K-1}^\delta} + \left(1 - \beta^\delta\right) & \text{if } t \in \left[t_{K-2}, t_{K-1}\right), \\ \beta^\delta \delta^{T_{K-1}^\delta} + \left(1 - \beta^\delta\right) \delta^{T_K^\delta} & \text{if } t \in \left[t_{K-1}, t_K\right), \\ 0 & \text{if } t \geq t_K, \end{cases}$$

and

$$p^{\delta}(t) := \begin{cases} p_k^{\delta} & \text{if } t \in [t_{k-1}, t_k), \ k = 1, \ldots, K = 2, \\ \frac{\delta^{T_{K-1}^{\delta}}\left(\beta^{\delta} p_{K-1}^{\delta} + \delta\left(1-\beta^{\delta}\right) v_{t_{K-2}}^{t_{K-1}}\right)}{x^{\delta}(t)} & \text{if } t \in [t_{K-2}, t_{K-1}), \\ \frac{\delta^{T_{K-1}^{\delta}} \beta^{\delta} p_{K-1}^{\delta} + \delta^{T_K^{\delta}}\left(1-\beta^{\delta}\right) v_{t_{K-1}}^{t_K}}{x^{\delta}(t)} & \text{if } t \in [t_{K-1}, t_K), \\ 0 & \text{if } t \geq t_K. \end{cases}$$

Note that for every $k = 1, \ldots, K-2$, we have

$$p_k^{\delta} < p_k \leq v_{t_{k-1}}^{t_k},$$

where the second inequality follows from the fact that $B(0) \geq 0$, $B(t_1) = \cdots = B(t_{K-2}) = 0$.

As the last step in the proof of Lemma 5, we show that the allocation $\left(x^{\delta}, p^{\delta}\right)$ can be implemented in the bargaining game when the discount factor is δ.

4.2.3 The equilibrium σ^{δ} of the bargaining game

First, we describe the players' on-path behavior. Then we turn to the off-path behavior.

In the first period of the game, the seller's types in $[t_0, t_1)$ make the offer p_1^{δ} and the buyer accepts it.

Consider now the types in the interval $[t_{k-1}, t_k)$, $k = 2, \ldots, K-2$. In period $n = 1, \ldots, T_k^{\delta} - 1$, they make a losing offer equal to $v(1)$. In period T_k^{δ}, the types in $[t_{k-1}, t_k)$ make the offer p_k^{δ} and the buyer accepts it.

Next, consider the types in $[t_{K-2}, t_{K-1}) \cup [t_{K-1}, t_K) = [t_{K-2}, t_K)$. In period $n = 1, \ldots, T_{K-1}^{\delta} - 1$, they make the losing offer $v(1)$. In period T_{K-1}^{δ} the types in $[t_{K-2}, t_K)$ offer $\hat{p}^{\delta}$. The buyer accepts the offer $\hat{p}^{\delta}$ with probability β^{δ}.

Suppose that the buyer rejects $\hat{p}^{\delta}$. In the following period, period $T_{K-1}^{\delta} + 1$, the types in

$[t_{K-2}, t_{K-1})$ offer $v_{t_{K-2}}^{t_{K-1}}$ and the buyer accepts the offer. How about the types in $[t_{K-1}, t_K)$? In period $n = T_{K-1}^{\delta} + 1, \ldots, T_K^{\delta} - 1$, they make the losing offer $v(1)$. In period T_K^{δ} they offer $\tilde{p}^{\delta}$ and the buyer accepts the offer.

Finally, each type $t \geq t_K$ makes the losing offer $v(1)$ in every period.

To see that this behavior is part of an equilibrium, consider all possible deviations in turn. Suppose that in a certain period n, a type t makes an offer that the buyer is supposed to accept with probability one. Suppose that the buyer deviates and rejects the offer. Then the seller of type t keeps making the same offer until the buyer accepts it. On the other hand, the buyer accepts the serious offer in the first period in which it is made.

If at any point the seller makes an offer greater than $v(0)$ and different from the serious offers described above, the buyer rejects it. Of course, the buyer accepts any offer smaller than $v(0)$.

It is simple to verify that the strategy profile just described constitutes an equilibrium (or rather, that there exists a belief system along which this strategy profile is an equilibrium). By construction, each type $t \in [t_{k-1}, t_k)$, $k = 1, \ldots, K-1$, is indifferent between his own strategy and the strategy of type $t' \in [t_k, t_{k+1})$. Thus, any type $t \in [0, 1]$ does not any incentive to mimic the equilibrium behavior of another type t'. Also, type t does not have any incentive to make offers that are not used in equilibrium since the buyer will reject them.

Conditional on receiving an offer that has to be accepted with probability one, the buyer's expected payoff is weakly positive. Thus, he has an incentive to accept the offer.

Finally, consider the buyer in period T_{K-1}^{δ}. If he rejects the offer $\hat{p}^{\delta}$, then in the following period the types in $[t_{K-2}, t_{K-1})$ will offer $v_{t_{K-2}}^{t_{K-1}}$. By definition, if the buyer accepts the offer $v_{t_{K-2}}^{t_{K-1}}$, his expected payoff is equal to zero. This and equality (12) imply that in period T_{K-1}^{δ} the buyer is indifferent between accepting and rejecting the offer $\hat{p}^{\delta}$.

The off-path behavior can be easily made sequentially rational by assuming that following any deviation the buyer assigns probability one to the event that the seller's type is $t = 0$. (This might be seen as an extreme belief revision, but it is convenient, and other possibilities would do just as well.)

4.2.4 Proof of Theorem 3, Conclusion

The previous subsections have shown that any regular allocation can be achieved as an equilibrium allocation in the bargaining game as $\delta \to 1$. Note that the set of equilibrium payoffs that can be achieved in the bargaining game is a subset of the set of payoffs spanned by the allocations described in Theorem 3, because the constraint $\pi^S(0) \geq v(0) - c(0)$ must hold, as explained before the theorem. Also, equilibrium allocations must satisfy veto-incentive compatibility. Therefore, one direction of the Theorem 3 is obvious. The other direction will be established if we can show that every extreme point of the set of veto-incentive compatible payoffs giving the seller his security payoff can be approximated arbitrarily closely by regular allocations. This is the content of Lemma 6. Recall that, for brevity, we restrict ourselves here to the case of extreme points of the payoff set that lie on the Pareto-frontier.

Lemma 6 *For every extreme point* (π^S, π^B) *(on the north-east boundary) of the payoff set that can be achieved by veto-incentive compatible allocations for which* $\pi^S(0) \geq v(0) - c(0)$*, and every* $\varepsilon > 0$*, there exists a regular allocation whose payoff is within distance* ε *of* (π^S, π^B)*.*

Proof. See Appendix A. ■

This concludes the proof of Theorem 3.

5 Extensions

5.1 Non-monotonic Values

We have maintained throughout the assumption that both the seller's cost, and the buyer's value are non-decreasing. Of course, there is no loss of generality in assuming that one of these functions is non-decreasing. So let us assume that types are ordered so that only the cost function is non-decreasing, and maintain all other assumptions (besides monotonicity). In particular, gains from trade are bounded away from zero for all t, and, to avoid trivialities, the seller's highest cost exceeds the buyer's average value. Does there exist a similarly tractable characterization of the veto-incentive compatible program when the value function is not necessarily increasing? In that case, it is easy to see that $B(t) \geq 0$ for all t is no longer a necessary condition, although it remains a sufficient condition for implementability. This suggests that positive correlation singles out the collection of intervals $\{[t, 1] : t < 1\}$ as the relevant one for the domains of the integral constraints $B(t)$. We view it as an important next step to identify what the "right" collection of intervals is, if any, over which the expected buyer's payoff must be positive, when values are not positively correlated, before turning to more general environments with limited commitment and private information.

In the absence of such a characterization, we might still ask the question: under which conditions is the *ex ante* efficient (i.e., surplus-maximizing) allocation of the commitment program also implementable in the veto-incentive program, or even in the bargaining game as frictions disappear? The answer to this question is surprisingly simple. Recall that the *ex ante* efficient mechanism under full commitment takes a very simple form, with (at most) two thresholds t_1 and t_2, with $0 < t_1 \leq t_2 \leq 1$. If $t_1 = t_2$, it is trivial to implement the allocation in the game, and, *a fortiori*, in the veto-incentive compatible program, so let us assume that $t_2 > t_1$. We have the following necessary and sufficient condition, which generalizes Proposition 1, at the cost of being stated in terms of endogenous variables (t_1, t_2).

Proposition 2 *If $t_2 > t_1$, the ex ante efficient allocation of the commitment program is implementable in the bargaining game as $\delta \to 1$ if and only if*

$$c(t_2) \leq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v\left(t\right).$$

Proof. Sufficiency follows closely the construction in 4.2.3 and is omitted. Necessity is established in Appendix B. ■

In fact, the proof of necessity makes clear that it is equally necessary for veto-incentive compatibility, so that this condition is also necessary and sufficient for implementability in the veto-incentive compatible program.

5.2 Payoffs vs. Allocations

Our characterizations of veto-incentive compatibility, as well as limiting equilibrium outcomes in the bargaining game, were cast in terms of the agents' expected payoffs, not in terms of the allocations themselves. This is no coincidence. *Not every incentive-compatible allocation whose payoffs satisfy the conditions of the characterization need be implementable.* What we have characterized is the projection of the implementable allocations onto the expected payoffs. We have no direct characterization in terms of allocation only. For instance, not every allocation that gives the seller's lowest type a profit $\pi^S(0) \geq v(0) - c(0)$ need be implementable. Indeed, suppose that there are three equiprobable types of seller (and buyer), and we consider parameters such that the highest cost, c_3, is strictly lower than the expected value of the lower two values, $(v_1 + v_2)/2$. Further, consider an incentive compatible allocation in which the buyer's expected payoff is zero, the highest seller's type does not trade, but the second highest does; this seller's intermediate type gets a strictly positive profit, and the seller's lowest type gets a payoff exceeding $v(0) - c(0)$, so that, by our results, the resulting expected payoffs are equilibrium payoffs in the bargaining game when frictions are sufficiently small.[20]

Yet this specific allocation, which requires the seller's high type not to trade, cannot be implemented in the bargaining game. To see this, note that the buyer will never accept an offer that gives him a strictly negative payoff, and therefore, because the buyer's expected payoff is zero, it must be that his expected payoff is also zero, conditional on any offer that is submitted with positive probability, after any history. By the martingale property of beliefs, there is a sequence of equilibrium offers along which the buyer's expected value, conditional on these offers, is non-decreasing, and therefore, at least as large as $(v_1 + v_2)/2 > c_3$. This sequence of offers must involve offers accepted with positive probability, for otherwise the seller's intermediate type would not be willing to follow it. By mimicking this sequence of offers, the seller's highest type guarantees a strictly positive profit, a contradiction.

5.3 Limited Commitment on the Seller's Side

Veto-incentive compatibility weakens the commitment assumption made in the full commitment program on the buyer's side. As discussed, this is a relaxation that is relevant for many actual market institutions. Furthermore, our characterization of the equilibrium payoffs in the bargaining game suggests that this is the "right" relaxation, namely, the absence of commitment on either side, as captured by the bargaining game, appears to impose no further constraints on

[20] Such an example is easy to find with a mathematical software: for instance, it occurs for the parameters $c_1 = 1, c_2 = 5970/2142, c_3 = 175/51$, and $v_1 = 134/65, v_2 = 2458/509, v_3 = 5$. The allocation is $x_1 = 1, x_2 = 1309475796/1359864155, x_3 = 0, p_1 = 926734382/271972831, p_2 = 898659860/271972831, p_3 = 0$.

achievable payoffs, aside from the security payoff that the seller must secure.[21]

It is then natural to ask whether one could derive results that mirror those of Section 3.2 in which the seller's commitment, instead of, or in addition to, the buyer's commitment is relaxed. While we shall not attempt to obtain a characterization for each possible case, we discuss here the relationship between the different sets of allocations and payoffs. As we shall see, limited commitment on the seller's side is arguably less of a problem than on the buyer's side.

Unlike the buyer, the seller gets an opportunity to influence the terms at which the trade would take place. Therefore, there are two possible ways of modeling the absence of commitment on the seller's side. A mechanism is *ex post* individually rational for the seller if the price p that is offered to the buyer is always higher than the cost of the seller's reported type t:

$$\forall t \in T : \int_{[0,c(t))} \mu(t)[1, dp] = 0.$$

This guarantees that the seller never loses from the mechanism, but it does not give him the authority to actually prevent the trade. Alternatively, we might endow the seller with the ability to block the trade given the realized price. This notion, in line with Forges' original definition of veto-incentive compatibility, is more demanding than *ex post* individual rationality: the ability to block the trade affects the seller's incentives to report his type truthfully, as the payoff from making a given report must include the option value from blocking the trade if the realized price happens to be below the seller's actual cost. To be more formal, we re-define the payoff of the

[21] Of course, in bargaining, the seller is not formally allowed to withdraw an offer that he makes, but why would he? Acceptance by the buyer reveals no information, so a seller that anticipates withdrawing an offer might as well not submit it.

type t seller that reports s, from a given mechanism μ, as

$$\hat{\pi}^S(s|t) = \int_{\mathbb{R}_+} \mathbf{1}_{\{p \geq c(t)\}}(p - c(t))\mu(s)[1, dp].$$

A mechanism is *seller veto-incentive compatible* if it is incentive compatible given the payoff $\hat{\pi}$, and the allocation (x, p) is implementable in the seller's veto-incentive compatible program if there is a mechanism that is seller veto-incentive compatible and induces the allocation (x, p), according to eqns. (1)–(2), taking into account that trade does not take place for prices below $c(t)$. To distinguish this notion from veto-incentive compatibility as defined in Section 2, the latter will now be referred to as buyer veto-incentive compatibility.

Does seller veto-incentive compatibility, or even *ex post* individual rationality restrict the set of implementable allocations, or the set of achievable payoff vectors? In a nutshell, the answer is no, as far as payoffs are concerned, and sometimes, as far as allocations are concerned, but only if it comes in addition to buyer veto-incentive compatibility. More precisely, we have the following proposition.

Proposition 3

i. The set of implementable allocations (and thus, of achievable payoff vectors) in the full commitment program remains unchanged if seller veto-incentive compatibility is imposed.

ii. The set of implementable allocations (and thus, of achievable payoff vectors) in the buyer veto-incentive compatible program remains unchanged if seller ex post individual rationality is imposed.

iii. The set of achievable payoff vectors in the buyer veto-incentive compatible program remains unchanged if seller veto-incentive compatibility is imposed.

Because seller veto-incentive compatibility implies seller *ex post* individual rationality, we have omitted some relationships that follow from the proposition. For instance, from (i), it follows that seller *ex post* individual rationality does not restrict the set of implementable allocations in the full commitment program. Furthermore, all remaining inclusions are strict: that is, for some parameters, the set of implementable allocations in the buyer veto-incentive compatible program is strictly reduced if seller veto-incentive compatibility is imposed, and, as we know, the set of implementable allocations in the veto-incentive compatible program is strictly contained in the set of allocations of the full commitment program, for some parameters.

The proofs of the claims in Proposition 3, some of which follow arguments that are similar to the other proofs in the paper, are sketched in Appendix D.[22] Additional details, as well as examples establishing the strict inequalities, are available from the authors.

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[22]More precisely, we show in Appendix D that seller *ex post* individual rationality does not restrict the set of allocations that can be achieved in the (buyer) veto-incentive compatible program, and that, as far as payoffs are concerned, we can also impose seller veto-incentive compatibility. In both cases, attention is restricted to finite types, and the result in the general case follows by standard limiting arguments.

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6 Omitted Proofs

Appendix A: Proof of Lemma 6

As mentioned, we restrict ourselves to the case of extreme points that lie on the Pareto-frontier here. Considering points on the "north-west" and "south-west" of the relevant payoff set require relatively straightforward modifications.

We proceed with a series of claims.

Claim 5 *Consider an interval $I = [t_1, t_2)$ such that $B(t) = 0$ for all $t \in I$ and (c, v) are C^1 on I. Assume that $(x(t_2), p(t_2)) = (x_2, p_2) \gg 0$ and $x(t)$ satisfies the following differential equation* *on I:*

$$-x(t)\,v'(t) = x'(t)\,(v(t) - c(t)).$$

For every $\varepsilon > 0$ there exists a regular allocation (x^n, p^n) such that $\sup_{s\in I}\|(x^n, p^n) - (x,p)\| < \varepsilon$ and

$$x^n(t_{2,-})\,(p^n(t_{2,-}) - c(t_{2,-})) = \bar{p}(t_2) - x(t_2)\,c(t_{2,-}).$$

Proof. For each $n \in \mathbb{N}$, we consider the uniform mesh of $[t_1, t_2]$: $I_1^n := [t_1, t_1 + \frac{t_2 - t_1}{n}), \ldots, I_n^n := [t_2 - \left(\frac{t_2-t_1}{n}\right), t_2)$. For each I, we write

$$v(I) := \mathbb{E}\left[v(t) \mid t \in I\right].$$

Next, we define the regular allocation for each n. In this allocation, $x_n(t)$ is constant over each of interval I_i^n. Thus, we define $\{x_1^n, \ldots, x_n^n\}$ such that $x^n(t) := x_i^n$ if $t \in I_i^n$. We define the value x_n^n by

$$x_n^n\,(v(I_1^n) - c(t_{2,-})) = \bar{p}(t_2) - x(t_2)\,c(t_{2,-}),$$

and, recursively, x_{k-1}^n as the solution to

$$x_{k-1}^n\left(v\left(I_{k-1}^n\right) - c(t_{k-1})\right) = x_{k-1}^n\,(v(I_k^n) - c(t_{k-1})).$$

Notice that we can write

$$
\begin{aligned}
x_{n-k}^{n} &= \left(\frac{v\left(I_{n-k+1}^{n}\right)-c\left(t_{n-k}\right)}{v\left(I_{n-k}^{n}\right)-c\left(t_{n-k}\right)}\right)\times\left(\frac{v\left(I_{n-k+2}^{n}\right)-c\left(t_{n-k+1}\right)}{v\left(I_{n-k+1}^{n}\right)-c\left(t_{n-k+1}\right)}\right)\times \\
&\quad \cdots\times\left(\frac{v\left(I_{n}^{n}\right)-c\left(t_{n-1}\right)}{v\left(I_{n-1}^{n}\right)-c\left(t_{n-1}\right)}\right) \\
&= \left(1+\left(\frac{v\left(I_{n-k+1}^{n}\right)-v\left(I_{n-k}^{n}\right)}{v\left(I_{n-k}^{n}\right)-c\left(t_{n-k}\right)}\right)\right)\times\left(1+\left(\frac{v\left(I_{n-k+2}^{n}\right)-v\left(I_{n-k+1}^{n}\right)}{v\left(I_{n-k+1}^{n}\right)-c\left(t_{n-k+1}\right)}\right)\right)\cdot \\
&\quad \times\left(\frac{v\left(I_{n}^{n}\right)-c\left(t_{n-1}\right)}{v\left(I_{n-1}^{n}\right)-c\left(t_{n-1}\right)}\right).
\end{aligned}
\tag{15}
$$

Thus, using (15) we have

$$
\begin{aligned}
\log\left(x_{n-k}^{n}\right) &= \log\left(1+\left(\frac{v\left(I_{n-k+1}^{n}\right)-v\left(I_{n-k}^{n}\right)}{v\left(I_{n-k}^{n}\right)-c\left(t_{n-k}\right)}\right)\right)+\log\left(1+\left(\frac{v\left(I_{n-k+2}^{n}\right)-v\left(I_{n-k+1}^{n}\right)}{v\left(I_{n-k+1}^{n}\right)-c\left(t_{n-k+1}\right)}\right)\right)+\cdots \\
&= \frac{1}{n}\left[\frac{\log\left(1+\left(\frac{v\left(I_{n-k+1}^{n}\right)-v\left(I_{n-k}^{n}\right)}{v\left(I_{n-k}^{n}\right)-c\left(t_{n-k}\right)}\right)\right)}{\left(\frac{1}{n}\right)}\right]+\frac{1}{n}\log\left(\frac{1+\left(\frac{v\left(I_{n-k+2}^{n}\right)-v\left(I_{n-k+1}^{n}\right)}{v\left(I_{n-k+1}^{n}\right)-c\left(t_{n-k+1}\right)}\right)}{\left(\frac{1}{n}\right)}\right)+\cdots \\
&= \left(\frac{1}{n}\right)\left[\frac{v'\left(t_{n-k}\right)}{v\left(t_{n-k}\right)-c\left(t_{n-k}\right)}\right]+\left(\frac{1}{n}\right)\left[\frac{v'\left(t_{n-k+1}\right)}{v\left(t_{n-k+1}\right)-c\left(t_{n-k+1}\right)}\right]+\cdots+R_{n} \\
&= \int_{t_{n-k}}^{t_{2}}\left(\frac{v'\left(t_{n}\left(s\right)\right)}{v\left(t_{n}\left(s\right)\right)-c\left(t_{n}\left(s\right)\right)}\right)ds+R_{n},
\end{aligned}
$$

where $t_{n}\left(s\right)\in\left(s-\frac{1}{n},s+\frac{1}{n}\right)$ for all s and $\limsup_{n}R_{n}\left(t_{n-k}\right)=0$. Furthermore, notice that $\left|\left(\frac{v'(t_{n}(s))}{v(t_{n}(s))-c(t_{n}(s))}\right)\right|\leq\sup_{s\in I}\left|\left(\frac{v'(s)}{v(s)-c(s)}\right)\right|=\gamma$. Thus, by dominated convergence, for every $z\in I$, we have

$$
\begin{aligned}
\lim_{n}\left|\log x\left(z\right)-\log x^{n}\left(z\right)\right| &\leq \lim_{n}\int_{t_{1}}^{t_{2}}\left|\left(\frac{v'\left(t_{n}\left(s\right)\right)}{v\left(t_{n}\left(s\right)\right)-c\left(t_{n}\left(s\right)\right)}\right)-\left(\frac{v'\left(s\right)}{v\left(s\right)-c\left(s\right)}\right)\right|ds \\
+\limsup_{n}\left|\max_{s}R_{n}\left(s\right)\right| &= 0.
\end{aligned}
$$

This establishes uniform convergence of $\log x^n(z)$. Because $x^n(t) \geq x(t_2)$ for every t and for every n this establishes convergence of x^n. ■

Claim 6 *Consider an implementable allocation (x, p) with support $[0, \bar{t}]$. There are two countable collection of open sets:*

i)

$$\mathcal{A} = \left\{A_i : i \in \mathbb{N},\ A_i = (t_1^i, t_2^i) : B\left(t_1^i\right) = B\left(t_2^i\right) = 0 \textit{ and } B(t) > 0 \textit{ if } t \in A_i\right\},$$

ii)

$$\mathcal{B} = \left\{B_i : i \in \mathbb{N},\ B_i = (t_1^i, t_2^i) :\ B(t) = 0 \textit{ if } t \in B_i\right\},$$

such that $\mu(\cup_i A_i) + \mu(\cup_i B_i) = \bar{t}$, where μ is the Lebesgue measure.

Proof. Because B is continuous, the set $\{t \in [0, 1] : B(t) > 0\}$ is open. This establishes i).

Notice that $\mathcal{A}$ is a countable collection of sets. Therefore, $\mu(\cup_i A_i) = \mu(\cup_i \mathrm{cl}\ A_i)$, where cl F denotes the closure of set F. Therefore, the set $[0, \bar{t}]/\cup_i \mathrm{cl}\ A_i$ is open and has measure $\bar{t} - \mu(\cup_i A_i)$. This establishes ii). ■

Claim 7 *Consider a finite collection of sets $\{I_j\}_{j=1}^n, I_j = [t_1^j, t_2^j)$, and an allocation (x, p) such that:*

i) $B\left(t_1^j\right) = B\left(t_2^j\right) = 0$ for every j;

ii)

$$\inf_{s \in I_j} \left[\bar{p}(s) - x(s)c(s)\right] > 0.$$

Then there exists $\varepsilon^* > 0$ and $K > 0$ such that if $\varepsilon \leq \varepsilon^*$ and if $\max\{dv(\mathcal{I}_j), dc(\mathcal{I}_j)\} \leq \varepsilon$

then $dx(\mathcal{I}_j) \leq K\varepsilon^*$ *for every* $j < J$*, where*

$$dg(\mathcal{I}_j) := \left| \sup_{t\in\mathcal{I}_j} g(t) - \inf_{t\in\mathcal{I}_j} g(t) \right|.$$

Proof. Let $\lambda := \min_t [v(t) - c(t)] > 0$. Thus $\inf_{t\in\mathcal{I}_j} v(t) - \sup_{t\in\mathcal{I}_j} c(t) \geq \frac{\lambda}{2} > 0$. Take $\varepsilon^* = \left(\frac{\lambda}{4}\right)$. Notice that

$$x(t_{j,-})(p(t_{j,-}) - c(t_{j,-})) \geq x(t_{j-1})(p(t_{j-1}) - c(t_{j,-})). \tag{16}$$

We have

$$x(t_{j,-})(p(t_{j,-}) - c(t_{j,-})) \leq x(t_{j,-})(v(t_{j-1}) - c(t_{j-})) + x(t_{j,-})\varepsilon, \tag{17}$$

and

$$x(t_{j-1})(p(t_{j-1}) - c(t_{j,-})) \geq x(t_{j-1})(v(t_{j-1}) - c(t_{j,-})) - \varepsilon x(t_{j-1}). \tag{18}$$

Thus

$$x(t_{j,-})(v(t_{j-1}) - c(t_{j,-})) + x(t_{j,-})\varepsilon \geq x(t_{j-1})(v(t_{j-1}) - c(t_{j,-})) - \varepsilon x(t_{j-1}), \tag{19}$$

which implies

$$\varepsilon(x(t_{j,-}) + x(t_{j-1})) \geq (x(t_{j-1}) - x(t_{j,-}))(v(t_{j-1}) - c(t_{j,-})).$$

Since $\varepsilon^* = \left(\frac{\lambda}{4}\right)$, we have $(v(t_{j-1}) - c(t_{j,-})) > \left(\frac{\lambda}{2}\right)$ thus:

$$(x(t_{j-1}) - x(t_{j,-})) \leq \left(\frac{\varepsilon(x(t_{j,-}) + x(t_{j-1}))}{(v(t_{j-1}) - c(t_{j,-}))}\right) \leq \left(\frac{2\varepsilon}{\left(\frac{\lambda}{2}\right)}\right) = \left(\frac{\varepsilon}{\lambda}\right),$$

establishing the claim. ■

Corollary 8 *If* $\{I_j\}_{j=1}^{n}$ *is a sequence of sets satisfying the assumptions of the Claim above such that* $\sum_j \max\{dv(\mathcal{I}_j), dc(\mathcal{I}_j)\} \leq \varepsilon$ *then* $\sum_j dx(\mathcal{I}_j) \leq K\varepsilon$.

We may finally prove the following strengthening of Lemma 6.

Lemma 7 *For every extreme point* (π^S, π^B) *of the payoff set that can be achieved by veto-incentive compatible allocations for which* $\pi^S(0) \geq v(0) - c(0)$*, and every* $\varepsilon > 0$*, there exists a regular allocation* (x_n, p_n) *such that* $\|(x,p) - (x_n, p_n)\| < \varepsilon$.

Proof. We assume that there exists $t_2 < \hat{t}$ such that $B(t_2) = 0$ and $B(t) > 0$ for all $t \in (t_2, \hat{t})$.[23] Our goal is to construct a sequence of regular allocations (x_n^*, p_n^*) arbitrarily close to (x,p) such that $\lim_n (x_n^*, p_n^*) = (x,p)$. Fix $\varepsilon > 0$. First, we construct a sequence of (non-regular) allocations $\{(x_n, p_n)\}_{n=1}^{\infty}$.

- Step 1: For every $n \in \mathbb{N}$, we set $(x_n(t), p_n(t)) := (x(t), p(t))$ if $t > t_2$.

- Step 2: For every $n \in \mathbb{N}$ we consider a mesh of $[0, t_2)$, $\{\mathcal{I}_j^n\}_{j=1}^{M_n} := \{[t_{1,1}^n, t_{2,1}^n), \ldots, [t_{1,M_n}^n, t_{2,M_n}^n)\}$ containing 3 classes of intervals:

  i) Type A intervals $\mathcal{A}_n$: intervals such that $B(t) > 0$ for every $t \in \text{int } \mathcal{I}_j^n$ and $B(t) = 0$ if $t \in \text{cl } \mathcal{I}_j^n$;

  ii) Type B intervals $\mathcal{B}_n$: intervals such that $B(t) = 0$ for every $t \in \text{int } \mathcal{I}_j^n$;

  iii) Type C intervals $\mathcal{C}_n := \{\mathcal{I}_j^n\}_{j=1}^{M_n} / (\mathcal{A}_n \cup \mathcal{B}_n)$.

  For this mesh we insist that:

  iv) $\sum_{\mathcal{I}_j^n \in \mathcal{C}_n} \max\{|dc(\mathcal{I}_j^n)|, |dv(\mathcal{I}_j^n)|\} < \left(\frac{1}{n}\right)$.

[23] The alternative case requires minor adjustments in the argument.

Without loss we take A_n, B_n and C_n such that all discontinuity points belong to the boundary of these sets.

- Step 3: We define an allocation (x_n, p_n) as follows:

$$p_n(t) := \begin{cases} p(t) \text{ if } t \in \mathcal{I}_j^n \text{ and } \mathcal{I}_j^n \in \mathcal{A}_n \cup \mathcal{B}_n, \\ v\left(\mathcal{I}_j^n\right) \text{ otherwise, and:} \end{cases} \tag{20}$$

$$\log x_n(t_-) + \log\left(p_n(t_-) - c(t_-)\right) = \log x_n(t) + \log\left(p_n(t) - c(t_-)\right). \tag{21}$$

Assume that $\log x_n(t) = \log x(t)$ for all $t \geq t'$ and assume that $\mathcal{I}_j^n = [t_{j,1}^n, t_{j,2}^n)$ with $\mathcal{I}_j^n \in \mathcal{C}_n$ and $t' = t_{j,2}^n$. Assume that $t_{j,1-}^n \in \mathcal{A}_n \cup \mathcal{B}_n$. We have

$$\begin{aligned} \log x_n\left(t_{j,1-}^n\right) &= \log\left(p_n\left(t_{j,1}^n\right) - c\left(t_{j,1-}^n\right)\right) - \log\left(p_n\left(t_{j,1-}^n\right) - c\left(t_{j,1-}^n\right)\right) + \log x\left(t_{j,2}^n\right) \\ &\quad + \int_{t_{j,1}^n}^{t_{j,2}^n} d\log\left(p_n(s) - c(s)\right) ds \\ &= \log\left(p\left(t_{j,1-}^n\right) - c\left(t_{j,1-}^n\right)\right) - \log\left(p_n\left(t_{j,1-}^n\right) - c\left(t_{j,1-}^n\right)\right) \\ &\quad + \log x\left(t_{j,2}^n\right) + \int_{t_{j,1}^n}^{t_{j,2}^n} d\log\left(p(s) - c(s)\right) ds \\ &\quad + \log\left(p_n\left(t_{j,1}^n\right) - c\left(t_{j,1-}^n\right)\right) - \log\left(p\left(t_{j,1}^n\right) - c\left(t_{j,1-}^n\right)\right) \\ &\quad + \left[\int_{t_{j,1}^n}^{t_{j,2}^n} d\log\left(p_n(s) - c(s)\right) ds - \int_{t_{j,1}^n}^{t_{j,2}^n} d\log\left(p(s) - c(s)\right) ds\right]. \end{aligned}$$

Since $t^n_{j,1-} \in \mathcal{A}_n \cup \mathcal{B}_n$, we have $p\left(t^n_{j,1-}\right) = p_n\left(t^n_{j,1-}\right)$. Hence

$$\begin{aligned}
\log x_n\left(t^n_{j,1-}\right) &= \log x\left(t^n_{j,1-}\right) \\
&\quad + \left[\log\left(p_n\left(t^n_{j,1}\right) - c\left(t^n_{j,1-}\right)\right) - \log\left(p\left(t^n_{j,1}\right) - c\left(t^n_{j,1-}\right)\right)\right] \\
&\quad + \left[\int_{t^n_{j,1}}^{t^n_{j,2}} d\log\left(p_n\left(s\right) - c\left(s\right)\right) ds - \int_{t^n_{j,1}}^{t^n_{j,2}} d\log\left(p\left(s\right) - c\left(s\right)\right) ds\right].
\end{aligned}$$

Thus, for every t, we have

$$\begin{aligned}
\left|\log x_n\left(t\right) - \log x\left(t\right)\right| &\leq \sum_{\mathcal{I}^n_j \in \mathcal{C}_n} \left|\log\left(p_n\left(t^n_{j,1}\right) - c\left(t^n_{j,1-}\right)\right) - \log\left(p\left(t^n_{j,1}\right) - c\left(t^n_{j,1-}\right)\right)\right| \qquad (22) \\
&\quad + \sum_{\mathcal{I}^n_j \in \mathcal{C}_n} \left|\int_{t^n_{j,1}}^{t^n_{j,2}} d\log\left(p_n\left(s\right) - c\left(s\right)\right) ds - \int_{t^n_{j,1}}^{t^n_{j,2}} d\log\left(p\left(s\right) - c\left(s\right)\right) ds\right|.
\end{aligned}$$

Thus, for every t, we have

$$\begin{aligned}
\left|\log x_n\left(t\right) - \log x\left(t\right)\right| &\leq \sum_{\mathcal{I}^n_j \in \mathcal{C}_n} \left|\log\left(p_n\left(t^n_{j,1}\right) - c\left(t^n_{j,1-}\right)\right) - \log\left(p\left(t^n_{j,1}\right) - c\left(t^n_{j,1-}\right)\right)\right| \qquad (23) \\
&\quad + \sum_{\mathcal{I}^n_j \in \mathcal{C}_n} \left|\int_{t^n_{j,1}}^{t^n_{j,2}} d\log\left(p_n\left(s\right) - c\left(s\right)\right) ds - \int_{t^n_{j,1}}^{t^n_{j,2}} d\log\left(p\left(s\right) - c\left(s\right)\right) ds\right|.
\end{aligned}$$

Furthermore, since there exists $\gamma > 0$ and $n' \in \mathbb{N}$ such that if $n > n'$ and $t < t_2$, we have $\log\left(p_n\left(s\right) - c\left(s\right)\right) \geq \gamma$ and $\log\left(p\left(s\right) - c\left(s\right)\right) \geq \gamma$ for all $s < t_2$ and $n > n'$ then (iv) implies that $\log x_n\left(t\right) \rightarrow \log x\left(t\right)$ uniformly.

- Step 4: Consider $n \in \mathbb{N}$ such that $\left\|\left(x_n, p_n\right) - \left(x, p\right)\right\| < \frac{\varepsilon}{2}$.

Now consider the allocation $\left(x_n, p_n\right)$. We will construct a regular allocation $\left(x^*_m, p^*_m\right)$ such that $\left\|\left(x_n, p_n\right) - \left(x^*_m, p^*_m\right)\right\| < \frac{\varepsilon}{2}$. We consider the class $\mathcal{B}_n$ (remember that n is kept fixed).

We will construct a sequence of regular allocations $\{(x_m^*, p_m^*)\}_{m=1}^{\infty}$ such that

$$\lim_m \|(x_n, p_n) - (x_m^*, p_m^*)\| = 0.$$

For each m, we define a partition $\left\{\mathcal{I}_j^{M_m}\right\}$. This partition consists of all intervals $\mathcal{I}_j^n \in \mathcal{A}_n \cup \mathcal{C}_n$ and of a mesh of the intervals $\mathcal{I}_j^n \in \mathcal{B}_n$ into m subintervals of the same length. This generates a class of intervals $\mathcal{B}_m$ such that if $t \in \mathcal{I}_j^m, \mathcal{I}_j^m \in \mathcal{B}_m$, then $B(t) = 0$ for all $t \in \text{int } \mathcal{I}_j^m$. We define p_m^* by

$$p_m^*(t) = \begin{cases} v\left(\mathcal{I}_j^M\right) & \text{if } \mathcal{I}_j^m \in \mathcal{B}_m, \\ p_n(t) & \text{otherwise.} \end{cases} \tag{24}$$

We use (21) to define $x_m^*(t)$. Next, it follows from Claim 5 that for every $\vartheta > 0$, there exists $m' \in \mathbb{N}$ such that if $m > m'$ then for each interval $\mathcal{I}_j^n \in \mathcal{B}_m$, and for each $t \in \text{int } \mathcal{I}_j^n$, we have

$$\left|x_m^*(t) - \xi_m^j(t)\right| < \vartheta, \tag{25}$$

where $\xi_m^n(t)$ satisfies: i) $\xi_m^j\left(t_{2,j-}^n\right)\left(p_n\left(t_{2,j-}^n\right) - c\left(t_{2,j-}^n\right)\right) = x_m^*\left(t_{2,j}^n\right)\left(p_m^*\left(t_{2,j}^n\right) - c\left(t_{2,j-}^n\right)\right)$ and ii) $-\xi_m^j(t)\, v'(t) = \xi_m^j(t)'\left(v(t) - c(t)\right)$ for every $t \in \text{int } \mathcal{I}_j^n$. This allow us to conclude that, for every $\mathcal{I}_j^n \in \mathcal{B}_n$, $\left|\log x_m^*(t) - \log \xi_m^j(t)\right|$ converge uniformly to zero. Assume that there are J_1 of these terms.

Now, consider an interval $\mathcal{I}_j^k \in \left\{\mathcal{I}_j^{M_m}\right\}$, $\mathcal{I}_j^k = [t_{1,k}^m, t_{2,k}^m)$ with $t_{2,k}^m \in \mathcal{I}_j^n$ for some $\mathcal{I}_j^n \in \mathcal{B}_n$. Notice that the number of these intervals is the same for each m. Furthermore, by construction, since we have assumed that (c, v) are continuous on each one of these intervals,

$$\begin{aligned}
& \log x_m^*\left(t_{2,k-}^m\right)+\log\left(p_m^*\left(t_{2,k-}^n\right)-c\left(t_{2,k-}^n\right)\right) \\
= & \log x_m^*\left(t_{2,k}^m\right)+\log\left(p_m^*\left(t_{2,k}^n\right)-c\left(t_{2,k-}^n\right)\right) \\
= & \log x_m^*\left(t_{2,k}^m\right)+\log\left(p_n^*\left(t_{2,k}^n\right)-c\left(t_{2,k-}^n\right)\right) \\
& +\left[\log\left(p_m\left(t_{2,k}^n\right)-c\left(t_{2,k-}^n\right)\right)-\log\left(p_n\left(t_{2,k}^n\right)-c\left(t_{2,k-}^n\right)\right)\right].
\end{aligned}$$

Now, notice that each of the J_2 terms

$$\left|\log\left(p_m^*\left(t_{2,k}^n\right)-c\left(t_{2,k-}^n\right)\right)-\log\left(p_n\left(t_{2,k}^n\right)-c\left(t_{2,k-}^n\right)\right)\right|$$

converges to zero.

Thus, take $\eta>0$. Consider m^* such that if $m>m^*$ then for every $\mathcal{I}_j^n \in \mathcal{B}_n$ and for every $t \in \mathcal{I}_j^n$, $\left|\log x_m^*(t)-\log \xi_m^j(t)\right|<\frac{\eta}{2J_1}$, and

$$\left|\log\left(p_m^*\left(t_{2,k}^n\right)-c\left(t_{2,k-}^n\right)\right)-\log\left(p_n\left(t_{2,k}^n\right)-c\left(t_{2,k-}^n\right)\right)\right|<\frac{\eta}{2J_2},$$

for all of the terms identified in the paragraph above. Thus, we conclude that for every $t \in [0,1]$ $\left|\log x_m^*(t)-\log x_n(t)\right|<\eta$. Since $p_m(t) \rightarrow p_n(t)$ pointwise, by dominated convergence we conclude that $\lim_m \left\|(x_n, p_n)-(x_m^*, p_m^*)\right\|=0$. Finally, divide every allocation (p_m^*, x_m^*) by $x_m^*(0)$ to guarantee that $x_m^*(t) \in [0,1]$ for every t. This concludes the proof.

■

Appendix B: Proof of Proposition 2

Recall that, in the *ex ante* efficient allocation, the seller's expected transfers $\bar{p}(t)$ are given by

$$\bar{p}(t) = \begin{cases} (1-x)\,c(t_1) + xc(t_2) & t \in [0, t_1)\,, \\ xc(t_2) & t \in [t_1, t_2]\,, \\ 0 & t > t_2. \end{cases}$$

Define the set $\hat{T}$ as

$$\hat{T} := \{t \in [0, t_2] : v(t') \leq v(t) \text{ for every } t' \in [0, t_2]\}\,.$$

Throughout we assume that the set $\hat{T}$ is nonempty (this is not guaranteed by our assumptions, and minor adjustments are necessary otherwise). To ease notation, we let $\hat{v}$ denote the value of the function v over the set $\hat{T}$.

Suppose that $c(t_2) > v_{t_1}^{t_2}$. We want to show that it is impossible to construct a collection of distributions $(\mu(\cdot|t))_{t\in[0,t_2]}$ over the interval $[0, \hat{v}]$ which satisfy the following three conditions:

i) for every $t \in [0, t_2]$,

$$\int_0^{\hat{v}} d\mu(p|t) = x(t)\,, \tag{26}$$

ii) for every $t \in [0, t_2]$,

$$\int_0^{\hat{v}} p d\mu(p|t) = \bar{p}(t)\,, \tag{27}$$

iii) for all $p \in [0, \hat{v}]$,

$$\int_0^{t_2} (v(t) - p)\, d\mu(p|t) = 0.$$

(Recall that under the *ex ante* efficient mechanism the buyer's expected payoff is equal to zero).

We approximate the function v by a sequence of step functions v^n, $n \in \mathbb{N}$. In particular, each

v^n satisfies

i) for every $t \in [0, t_2]$,

$$v(t) \leq v^n(t) \leq \hat{v},$$

ii) for every $t \in [0, 1]$,

$$0 \leq v^n(t) - v(t) \leq \frac{1}{n},$$

iii) if t and t' belong to the same step of v^n, then $x(t) = x(t')$.

Finally, for each $n \in \mathbb{N}$, we let $I^n \subset [0, t_2]$ denote the union of the intervals over which the function v^n takes the value $\hat{v}$.

Fix $n \in \mathbb{N}$. For each $p < \hat{v}$ we have

$$\int_0^{t_2} (v^n(t) - p)\, d\mu(p|t) = \varepsilon^n(p),$$

for some $\varepsilon^n(p) \geq 0$. After dividing both sides by $\hat{v} - p$ and rearranging terms, we have

$$\int_{t \in I^n} d\mu(p|t) + \int_{t \in [0,t_2] \setminus I^n} \left(1 - \frac{\hat{v} - v^n(t)}{\hat{v} - p}\right) d\mu(p|t) = \frac{\varepsilon^n(p)}{\hat{v} - p} \geq 0.$$

We integrate the two sides of the equality over p, and get

$$z^n := \int_{t \in I^n} \int_0^{\hat{v}} d\mu(p|t)\, dt + \int_{t \in [0,t_2] \setminus I^n} \int_0^{\hat{v}} \left(1 - \frac{\hat{v} - v^n(t)}{\hat{v} - p}\right) d\mu(p|t)\, dt \geq 0.$$

For each $t \in [0, t_2] \setminus I^n$, let $\bar{\mu}(\cdot|t)$ denote the distribution that assigns probability $x(t)$ to the offer $\bar{p}(t)/x(t)$ (with probability $1 - x(t)$ no offer is made). Notice that the function $\frac{1}{p - \hat{v}}$ is concave in p. This, together with conditions (26) and (27), implies that, for each $n \in \mathbb{N}$,

$$\bar{z}^n := \int_{t \in I^n} \int_0^{\hat{v}} d\bar{\mu}\left(p|t\right) dt + \int_{t \in [0,1] \setminus I^n} \int_0^{\hat{v}} \left(1 - \frac{\hat{v} - v^n\left(t\right)}{\hat{v} - p}\right) d\bar{\mu}\left(p|t\right) dt \geq tz^n \geq 0. \tag{28}$$

We take the limit of $\bar{z}^n$ as n goes to infinity, so that

$$\bar{z} := \lim_{n \to \infty} \bar{z}^n = t_1 + \left(t_2 - t_1\right) x - \frac{\int_0^{t_1} \left(\hat{v} - v(t)\right) dt}{\hat{v} - (1-x)c(t_1) - xc(t_2)} - x \frac{\int_{t_1}^{t_2} \left(\hat{v} - v(t)\right) dt}{\hat{v} - c(t_2)} =$$
$$\frac{t_1\left(v_0^{t_1} - (1-x)c(t_1) - xc(t_2)\right)}{\hat{v} - (1-x)c(t_1) - xc(t_2)} - \frac{x(t_2 - t_1)\left(c(t_2) - v_{t_1}^{t_2}\right)}{\hat{v} - c(t_2)} < \frac{t_1\left(v_0^{t_1} - (1-x)c(t_1) - xc(t_2)\right) - x(t_2 - t_1)\left(c(t_2) - v_{t_1}^{t_2}\right)}{\hat{v} - (1-x)c(t_1) - xc(t_2)} = 0,$$

where the inequality follows from the fact that $c\left(t_2\right) > v_{t_1}^{t_2}$, and the last equality follows from the definition of x in equation (6). However, $\bar{z}$ being strictly negative contradicts the fact that it is the limit of a sequence of nonnegative numbers (see condition (28)).

Appendix C: A Sufficient Condition for the Efficient Mechanism to be Implemented in the Bargaining Game

Recall that $Y : [0,1] \to \mathbb{R}$ is defined as

$$Y\left(t\right) := \int_0^t \left(v\left(s\right) - c\left(t\right)\right) ds = \int_0^t \left(v\left(s\right) - c\left(s\right) - sc'\left(s\right)\right) ds.$$

Our assumptions imply that, as mentioned, $Y\left(0\right) = 0$, $Y'\left(0\right) > 0$ and $Y\left(1\right) < 0$. Let $\underline{t}$ denote the smallest local maximizer of the function Y. Also, let $\bar{t}$ denote the smallest strictly positive root of Y. For any t let $\mu\left(t\right)$ denote the mechanism under which the types below t trade with probability one at the price $c\left(t\right)$ and the types above t do not trade. Notice that if $Y\left(t\right) \geq 0$, then the mechanism $\mu\left(t\right)$ is incentive compatible and individually rational.

Consider the efficient mechanism under full commitment. We know that there exist $0 < t_1 \leq$

$t_2 \leq 1$ such that the seller's types in $[0, t_1)$ trade with probability 1, while the types in $[t_1, t_2]$ trade with probability $x(t_1, t_2) \in [0, 1)$ (all other types of the seller do not trade). Recall that the buyer's individual rationality constraint holds with equality. Thus, we have

$$0 = \int_0^{t_1} (v(s) - c(t_1))\, ds + x \left(t_1 c(t_1) + \int_{t_1}^{t_2} v(s)\, ds - t_2 c(t_2) \right) =$$
$$Y(t_1) + x \int_{t_1}^{t_2} (v(s) - c(s) - sc'(s))\, ds = Y(t_1) + x (Y(t_2) - Y(t_1)).$$

Therefore, we can express $x(t_1, t_2)$ as

$$x(t_1, t_2) = \frac{Y(t_1)}{Y(t_1) - Y(t_2)}.$$

Consider the case in which $t_2 > t_1$, i.e., there is a set of types who trade with a probability larger than zero but smaller than one. First, we must have $Y(t_2) - Y(t_1) < 0$, otherwise we may increase x and improve efficiency. This immediately implies $Y(t_1) > 0$. Second, under the optimal mechanism $Y(t_2) < 0$. In fact, if $Y(t_2) \geq 0$, it is possible to implement the mechanism $\mu(t_2)$, which is more efficient than the original one. In particular, this implies that $t_2 > \bar{t}$.

Finally, we must have $t_1 \geq \underline{t}$. Suppose that $t_1 < \underline{t}$. Fix t_2 of the original mechanism and choose $t_1' \in (t_1, \underline{t}]$. Consider the mechanism under which the types in $[0, t_1')$ trade with probability 1 while the types in $[t_1', t_2]$ trade with probability

$$x(t_1', t_2) = \frac{Y(t_1')}{Y(t_1') - Y(t_2)} > \frac{Y(t_1)}{Y(t_1) - Y(t_2)} = x(t_1, t_2),$$

where the inequality follows from $Y(t_1') > Y(t_1)$ and $Y(t_2) < 0$. Of course, the new mechanism is more efficient than the original one since the types in $[t_1, t_2]$ trade with a larger probability while the types outside this interval trade with the same probability as under the original mechanism.

We summarize our results:

Fact 9 *Let t_1 and t_2 denote the endpoints of the first two steps of the optimal mechanism. Then $t_1 \geq \underline{t}$, and $t_2 \geq \bar{t}$.*

We are now ready to provide a sufficient condition to implement the efficient mechanism in the bargaining game (when the players are sufficiently patient).

Condition 10 *For any $t \geq \bar{t}$*

$$\int_{\underline{t}}^{t} \left(v\left(s\right) - c\left(t\right)\right) ds \geq 0.$$

We now explain why the above condition is sufficient. Fix $0 < \tilde{t} \leq 1$, and consider the function $\varphi : \left[0, \tilde{t}\right] \to \mathbb{R}$ given by

$$\varphi\left(t\right) := \int_{t}^{\tilde{t}} \left(v\left(s\right) - c\left(\tilde{t}\right)\right) ds.$$

Under our assumptions, if $\varphi\left(t'\right) \geq 0$ for some t', then $\varphi\left(t\right) > 0$ for every $t \in \left(t', \tilde{t}\right)$. Recall that the function v is increasing. Let t'' denote the value in $\left[0, \tilde{t}\right]$ such that $v\left(t''\right) = c\left(\tilde{t}\right)$ (let $t'' = \tilde{t}$ if $v\left(\tilde{t}\right) < c\left(\tilde{t}\right)$). The function φ is increasing $\left[0, t''\right]$. By definition, φ is positive above t''.

Therefore, fix $t_2 \geq \bar{t}$. Our condition guarantees that for each $t_1 \in \left[\underline{t}, t_2\right]$,

$$\int_{t_1}^{t_2} \left(v\left(s\right) - c\left(t_2\right)\right) ds \geq 0,$$

which implies the result, by Proposition 1.

Appendix D: Proof of Proposition 3 (Sketch)

This appendix sketches the proofs of the two harder statements in Proposition 3. We first show that the set of allocations in the buyer veto-incentive compatible program is the same whether or not one imposes *ex post* seller individual rationality. We then show that, as far as payments are concerned, the latter requirement can even be strengthened to seller veto-incentive compatibility. In both cases, for simplicity, we restrict attention to finite types. The extension to our set-up with a continuum of types follows by standard limiting arguments.

Lemma 8 *Assume that c and v are step functions with n steps such that $c_1 < c_2 < \cdots < c_N$, and (x, p) is an allocation that is implementable in the veto-incentive compatible program. Then there exists a measure μ which induce this allocation such that, for all $t \in T$, we have*

$$\int_{[0,c(t))} \mu(t)[1, dp] = 0.$$

Proof. Since (c, v) are step functions we can consider the model with N types in which the probability of each type is q_i. We write $\{\mu_i\}_{i=1}^{N}$ for the distribution of offers faced by type i.

Step 1: We divide the type space into 3 subsets:

$$\begin{aligned} T_1 &:= \{i \in \{1, \ldots, N\} : p_i > v_i\}, \\ T_2 &:= \{i \in \{1, \ldots, N\} : p_i < v_i\}, \\ T_3 &:= \{i \in \{1, \ldots, N\} : p_i = v_i\}. \end{aligned}$$

Step 2: For $k \leq j$, define

$$L_k^j := \sum_{i=k}^{j} q_i (x_i (v_i - p_i)).$$

Step 3: Notice that $L_0^N = B(0) \geq 0$, and let J^* be the lowest type i such that $L_0^i \geq 0$. Here we show how to construct an allocation satisfying the properties above for the special case that $J^* = N > 1$. The general proof considers a partition of the type space $\{1, \ldots, i_1\}, \{i_1 + 1, \ldots, i_2\}, \ldots, \{i_K + 1, \ldots, N\}$ and applies this procedure to each set separately.

Step 4: We will present an algorithm which delivers the desired result.

Step 4.1: Let k_1 be the smallest element in T_2.

There are 2 cases to consider:

Case 1:

$$q_1 x_1 \left(v_1 - p_1\right) + q_{k_1} x_{k_1} \left(v_{k_1} - p_{k_1}\right) < 0.$$

Case 2:

$$q_1 x_1 \left(v_1 - p_1\right) + q_{k_1} x_{k_1} \left(v_{k_1} - p_{k_1}\right) \geq 0.$$

Case 1: Notice that since $k_1 > 1$, we have $p_{k_1} \geq p_1$. From type k_1's individual rationality constraint, we have $p_{k_1} \geq c_{k_1}$. Also, there exists $\lambda \in (0,1)$ such that

$$\lambda q_1 x_1 \left(v_1 - p_1\right) + q_{k_1} x_{k_1} \left(v_{k_1} - p_{k_1}\right) = 0. \tag{29}$$

Next, notice that

$$p_1 = \alpha p_{k_1} + (1 - \alpha) v_1, \tag{30}$$

for some $\alpha \in (0, 1]$. Thus, applying (30) into (29) we have

$$0 = \lambda q_1 x_1 (1 - \alpha) \left(v_1 - v_1\right) + \lambda q_1 x_1 \alpha \left(v_1 - p_{k_1}\right) + q_{k_1} x_{k_1} \left(v_{k_1} - p_{k_1}\right). \tag{31}$$

Next, we use (31) to show that $x = x^1 + \hat{x}^1$, where

$$x_i^1 := \begin{cases} \lambda x_1 \text{ if } i = 1, \\ x_{k_1} \text{ if } i = k_1, \\ 0 \text{ otherwise,} \end{cases}$$

and $\hat{x}^1 := x - x^1 \geq 0$. For the allocation (x^1, p), we construct a measure $\{\mu_i^1\}_{i=1}^N$ such that:

a. $\left(\int d\mu^1, \int p d\mu^1\right) = (x^1, p)$;

b. If $x_i^1 > 0$ then $\mu_i^1[0, c_i) = 0$.

For that, we define $\mu_i^1 := 0$ if $i \notin \{1, k_1\}$ and

$$\mu_1^1(\tilde{p}) := \begin{cases} \lambda x_1 \alpha \text{ if } \tilde{p} = p_{k_1} \\ \lambda x_1 (1-\alpha) \text{ if } \tilde{p} = v_1 \\ 0 \text{ otherwise} \end{cases} \quad \mu_{k_1}^1(\tilde{p}) := \begin{cases} x_{k_1} \text{ if } \tilde{p} = p_{k_1}, \\ 0 \text{ otherwise.} \end{cases}$$

Case 2: There exists $(\zeta, \gamma) \in (0,1] \times (0,1]$ such that

$$\begin{aligned} p_1 &= \zeta p_{k_1} + (1-\zeta) v_1, \\ 0 &= q_1 x_1 (1-\zeta)(v_1 - v_1) + q_1 x_1 \zeta (v_1 - p_{k_1}) + \gamma q_{k_1} x_{k_1} (v_{k_1} - p_{k_1}). \end{aligned}$$

Thus, we define

$$x_i^1 := \begin{cases} x_1 \text{ if } i = 1, \\ \gamma x_{k_1} \text{ if } i = k_1, \\ 0 \text{ otherwise,} \end{cases}$$

and $\hat{x}^1 := x - x^1 \geq 0$. For the allocation (x^1, p), we construct measures $\{\mu_i^1\}_{i=1}^N$ by setting $\mu_i^1 := 0$ if $i \notin \{1, k_1\}$ and

$$\mu_1^1(\tilde{p}) := \begin{cases} x_1 \zeta \text{ if } \tilde{p} = p_{k_1} \\ x_1 (1-\zeta) \text{ if } p = v_1 \\ 0 \text{ otherwise} \end{cases} \quad \mu_{k_1}^1(\tilde{p}) = \begin{cases} \gamma x_{k_1} \text{if } \tilde{p} = p_{k_1}, \\ 0 \text{ otherwise.} \end{cases}$$

Step 4.2: Assume that $x = \sum_{i=1}^{M} x^i + \hat{x}^M$. There are two possibilities:

Case i. $\left\{i \in \{1, \ldots, N\} : \hat{x}_i^M > 0\right\} \cap T_1 \neq \varnothing$.

Case ii. $\left\{i \in \{1, \ldots, N\} : \hat{x}_i^M > 0\right\} \subseteq T_2 \cup T_3$.

Assume that $\hat{x}_i^{M-1}$ is such that $\sum_{i=1}^{N} q_i \left(\hat{x}_i^{M-1}(v_i - p_i)\right) \geq 0$ and $\sum_{i=1}^{J} q_i \left(\hat{x}_i^{M-1}(v_i - p_i)\right) < 0$ if $J < N$. We claim:

Claim 11 *If **Step 4.1** is applied to $\hat{x}_i^{M-1}$, $\hat{x}_i^{M-1} = x_i^M + \hat{x}_i^M$ with $\left\{i \in \{1, \ldots, N\} : \hat{x}_i^M > 0\right\} \cap T_1 \neq \varnothing$, then $\sum_{i=1}^{N} q_i \left(\hat{x}_i^M (v_i - p_i)\right) \geq 0$ and $\sum_{i=1}^{J} q_i \left(\hat{x}_i^M (v_i - p_i)\right) < 0$ if $J < N$.*

Proof: The first conclusion follows since $\sum_{i=1}^{N} q_i \left(\hat{x}_i^{M-1}(v_i - p_i)\right) = \sum_{i=1}^{N} q_i \left(\hat{x}_i^M (v_i - p_i)\right)$. For the second, let k_{M-1} be the largest element of $\left\{i \in \{1, \ldots, N\} : \hat{x}_i^{M-1} > 0\right\} \cap T_2$. There are two possibilities:

a. $J < k_{M-1} \leq N$. In this case,

$$\begin{aligned} & \sum_{i \leq J} q_i \left(\hat{x}_i^M (v_i - p_i)\right) \\ = & \sum_{i \leq J} q_i \left(\hat{x}_i^{M-1}(v_i - p_i)\right) + \sum_{i \leq J, i \in T_1} q_i \left(\left(\hat{x}_i^{M-1} - \hat{x}_i^M\right)(v_i - p_i)\right) < 0, \end{aligned}$$

where we used the fact that $\sum_{i \leq J} q_i \left(\hat{x}_i^{M-1}(v_i - p_i)\right) < 0$ by assumption, $\left(\hat{x}_i^{M-1} - \hat{x}_i^M\right) \geq 0$ and $\mathbf{1}_{\{i \in T_1\}}(v_i - p_i) < 0$.

b. $k_{M-1} \leq J < N$. In this case,

$$\begin{aligned} 0 &> \sum_{i \leq J} q_i \left(\hat{x}_i^{M-1}(v_i - p_i)\right) \\ &= \sum_{i \leq J} q_i \left(\hat{x}_i^{M-1}(v_i - p_i)\right) + \sum_{i \leq N} q_i \left(\left(\hat{x}_i^M - \hat{x}_i^{M-1}\right)(v_i - p_i)\right) \\ &= \sum_{i \leq J} q_i \left(\hat{x}_i^M (v_i - p_i)\right), \end{aligned}$$

where we used the fact that $k_{M-1} \leq J$ implies

$$0 = \sum_{i \leq N} q_i \left(\left(\hat{x}_i^M - \hat{x}_i^{M-1} \right) (v_i - p_i) \right) = \sum_{i \leq J} q_i \left(\left(\hat{x}_i^M - \hat{x}_i^{M-1} \right) (v_i - p_i) \right).$$

■

From Claim 11, we can apply **Step 4.1** into $\hat{x}_i^M$ to obtain x^{M+1} and $\hat{x}^{M+1}$ and $\left\{\mu_i^{M+1}\right\}_{i=1}^N$ such that:

a'. $\left(\int d\mu^{M+1}, \int p d\mu^{M+1}\right) = (x^{M+1}, p)$;

b'. If $x_i^{M+1} > 0$ then $\mu_i^{M+1}[0, c_i) = 0$.

Notice that this procedure can take (at most) $N - 1$ rounds. In order to complete the Lemma we move to **Case ii.**

Case ii: In this case, define $\left\{\mu_i^{M+1}\right\}_{i=1}^N$ by:

$$\mu_i^{M+1}(\tilde{p}) := \begin{cases} \hat{x}_i^M \text{ if } \tilde{p} = p_i \\ 0 \text{ otherwise.} \end{cases}$$

Step 5: Assume the algorithm described in **Step 4.1** and **Step 4.2** was applied to the allocation x such that $x = \sum_{j=1}^{K} x^j + \hat{x}^k$. Thus it is straightforward to verify that the measure $\{\mu_i\}_{i=1}^N$ defined by $\mu_i(\tilde{p}) := \sum_{j=1}^{K+1} \mu_i^j(\tilde{p})$ is such that $(x, p) = \left(\int d\mu, \int p d\mu\right)$ and $\mu_i[0, c_i) = 0$. This completes the proof. ■

We now turn to the other nontrivial claim: seller veto-incentive compatibility does not restrict the set of payoffs that can be achieved in the buyer veto-incentive compatible program. Here as well, attention is restricted to finite types.

Lemma 9 *Assume that the type space is finite and let* (π^B, π^S) *be a vertex of the payoff frontier achieved in the (buyer) veto-incentive compatible program. There exists a seller veto-incentive compatible measure* $\mu = \{\mu_i\}_{i=1}^N$ *that achieves this payoff.*

Proof. Assume that there are N types.[24] It can be shown that if (π^B, π^S) is a vertex of the payoff frontier then it achieved by an allocation (x, p) for which there exists a partition of the type space: $\{\mathcal{P}_j\}_{j=1}^K$ with $\mathcal{P}_1 = \{1, \ldots, i_1\}$ and $\mathcal{P}_j = \{i_{j-1}+1, \ldots, i_j\}$, with $i_K \geq 1$ such that:[25]

i. If $j < K$, then if $i, i' \in \mathcal{P}_j$ we have $p_i = p_{i'} = \mathbb{E}[v \mid \mathcal{P}_j]$.

ii. If $j = K$, then we have either a. or b. below:

a. $(p_i, x_i) = (p_N, x_N)$ for all $i \in \mathcal{P}_K$;

b. $\mathcal{P}_K = I_1 \cup I_2$ where $I_1 = \{i_{k-1}+1, \ldots, i_l\}$ and $I_2 = \{i_l + 1, \ldots, N\}$ with $i_{k-1} \leq i_l < N$ is such that $(p_i, x_i) = (p', x')$ if $i \in I_1$ and $(p_i, x_i) = (p'', x'')$ if $i \in I_2$ with $c_{i_l} \leq \mathbb{E}[v \mid i \in I_1]$ and $p' < p''$.

Here, we prove the more challenging case b.

Step 1: Defining μ_i for $i \notin \mathcal{P}_K$ by:

$$\mu_i(\tilde{p}) := \begin{cases} x_i \text{ if } \tilde{p} = p_i, \\ 0 \text{ otherwise.} \end{cases}$$

Step 2: To define μ_i for $i \in \mathcal{P}_K$, there are two cases to consider:

Case 1: $p' \leq \mathbb{E}[v \mid i \in I_1]$.

[24] For simplicity of exposition we assume that all types trade with positive probability.
[25] A proof is available upon request.

In this case we let

$$\mu_i(\tilde{p}) := \begin{cases} x' \text{ if } \tilde{p} = p' \text{ and } i \in I_1 \\ 0 \text{ if } \tilde{p} \neq p' \text{ and } i \in I_1 \end{cases} \quad \mu_i(\tilde{p}) = \begin{cases} x'' \text{ if } \tilde{p} = p'' \text{ and } i \in I_2, \\ 0 \text{ if } \tilde{p} \neq p'' \text{ and } i \in I_2. \end{cases}$$

It is straightforward to check that μ is veto-incentive compatible for the seller.

Case 2: $p' > \mathbb{E}[v \mid i \in I_1]$.

In this case, notice that since the allocation is incentive compatible we must have

$$B_{i_{k-1}+1} = \sum_{i \geq i_{k-1}+1} q_i x_i (v_i - p_i) \geq 0, \tag{32}$$

Furthermore, because $p' \in (\mathbb{E}[v \mid i \in I_1], p'')$, there exists $\alpha \in (0,1)$ such that

$$p' = \alpha \mathbb{E}[v \mid i \in I_1] + (1-\alpha) p''. \tag{33}$$

Thus, notice that from (32) and (33),

$$\begin{aligned} 0 &\leq \sum_{i \in I_1} q_i x_i (v_i - p_i) + \sum_{i \in I_2} q_i x_i (v_i - p'') \\ &= \sum_{i \in I_1} \alpha q_i x_i (v_i - \mathbb{E}[v \mid i \in I_1]) \\ &\quad + \sum_{i \in I_1} (1-\alpha) q_i x_i (v_i - p'') + \sum_{i \in I_2} q_i x_i (v_i - p''). \end{aligned}$$

Thus, $\sum_{i \in I_1} (1-\alpha) q_i x_i (v_i - p'') + \sum_{i \in I_2} q_i x_i (v_i - p'') = B_{i_{k-1}+1} \geq 0$.

Therefore, we define μ_i by

$$\mu_i\left(\tilde{p}\right) := \begin{cases} \alpha x' \text{ if } \tilde{p} = \mathbb{E}\left[v \mid i \in I_1\right] \text{ and } i \in I_1 \\ \left(1-\alpha\right) x' \text{ if } \tilde{p} = p'' \text{ and } i \in I_1 \\ 0 \text{ if } \tilde{p} \notin \left\{p', p''\right\} \text{ and } i \in I_1 \end{cases} \qquad \mu_i\left(\tilde{p}\right) := \begin{cases} x'' \text{ if } \tilde{p} = p'' \text{ and } i \in I_2, \\ 0 \text{ if } \tilde{p} \neq p'' \text{ and } i \in I_2. \end{cases}$$

It is straightforward to verify that the allocation constructed is veto-incentive compatible for the seller. This completes the proof. ■