MANAGING STRATEGIC BUYERS

By

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Managing Strategic Buyers*

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Abstract

We consider the problem of a monopolist who must sell her inventory before some deadline, facing n buyers with independent private values. The monopolist posts prices but has no commitment power. The seller faces a basic trade-off between imperfect price discrimination and maintaining an effective reserve price. When there is only one unit and only a few buyers, the seller essentially posts unacceptable prices up to the very end, at which point prices collapse in a series of jumps to a "reserve price" that exceeds marginal cost. When there are many buyers, the seller abandons this reserve price in order to more effectively screen buyers. Her optimal policy then replicates a Dutch auction, with prices decreasing continuously over time.

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1 Introduction

1.1 Revenue Management

The revenue management literature addresses the pricing of goods sharing three essential characteristics: (i) there is a fixed quantity of resource for sale, (ii) the resource is perishable (i.e., there is a time after which it is valueless), and (iii) consumers have heterogeneous valuations. Revenue management is practiced in a variety of industries, including airlines, apparel, entertainment, freight, hotels, pipelines and rental cars.

In the standard revenue management model, each buyer must be served immediately upon arrival or forever lost, and the only relevant price from a buyer's point of view is the current one. In contrast, this paper examines revenue management with buyers who strategically choose their time of purchase, continually deciding whether to buy today or wait in hope of buying at a lower price tomorrow.

We consider a monopolist facing a fixed, known number of strategic buyers with unit demand but privately-known valuations. There is a fixed terminal date at which the good is consumed, and after which any remaining unsold unit has no value. In the intervening period, the seller can set a price in each of a finite number of instants, which buyers may either accept and hence end the game, or reject and push the game to the next price revision. We focus attention on the limiting case in which the delay between successive offers is arbitrarily small. The seller cannot make commitments, in the sense that the prices she posts must be sequentially rational.

Our main results are the following. First, the seller's lack of commitment restricts her ability to implement her favorite mechanism. More precisely, we show that the seller faces a basic trade-off between a reserve price and price discrimination. If she insists on a positive reserve price, she must give up on her ability to finely price discriminate among buyers.

It should come as no surprise to the reader familiar with the Coase conjecture that lack of commitment can be very costly to the seller. However, we show that the deadline endows the seller with considerable commitment power. While she cannot get both a reserve price and finely price discriminate, she can achieve the revenue from either strategy.

First, the seller can be assured of a payoff at least as high as the static monopoly profit. In particular, a seller with two opportunities to set prices can secure a payoff at least as large as a seller with only one opportunity, simply by charging an initial price so high that every buyer rejects, and then duplicating the behavior that secures the lowest equilibrium payoff of a seller who can set only one price. Iterating this argument gives the result. This makes it clear that, despite some similarities, the revenue management problem differs fundamentally from that of a durable-goods monopolist.[1]

Second, we show that the seller can also secure the revenue from a Dutch auction without a reserve price (which might be greater or lower than the monopoly profit), under a mild regularity condition on the distribution of buyers' types.[2] This initially sounds obvious, since the seller can seemingly replicate a Dutch auction by simply decreasing the price in sufficiently fine increments. However, the seller setting her current price cannot commit to (and indeed often will not) run a Dutch auction in subsequent periods, and some of the possible future alternatives may be better not only for the seller, but also for high valuation buyers (i.e., the incentives of the seller and of a buyer with a given valuation need not be opposed). But then high-valuation buyers will not be willing to pay as much today as they would in a Dutch auction, potentially scuttling the seller's ability to secure the Dutch auction revenue. To obtain the result, we must accordingly show that *regardless* of what equilibrium continuation behavior is expected, the seller's current price attracts at least all those buyers' valuations who would accept this price in a Dutch auction.

This analysis suggests that the seller uses one of two pricing strategies. First, the seller can achieve a sequence of prices culminating in a positive "reserve" price, but then can only imprecisely discriminate among buyers. In the extreme, this strategy effectively leads to the static monopoly price. Alternatively, the seller can achieve sharper price discrimination, but only at the cost of subsequent price reductions that erode the reserve price. In the extreme, this strategy leads to the payoff of a Dutch auction with no reserve price. The relative performance of these two strategies is clear in the two limiting cases of a single buyer and a large number of buyers. In the case of one buyer, the best the seller can do is to charge the static monopoly price in the last period. With many buyers, running a Dutch auction with no reserve price outperforms charging the static monopoly price.

Further results for general distributions are difficult to obtain. First, the game exhibits a double nonstationarity—equilibrium continuation play and continuation payoffs depend on both the specification of residual demand and on the time remaining. In addition, for a fixed sequence of prices, the buyers' acceptance decisions emerge as an equilibrium in a game played by the buyers. This game exhibits strategic complementarities—the more likely other buyers are to wait, the more tempting it is to wait—and hence, in some cases, there

[1] Talluri and van Ryzin [28, p. 365], for example, argue that customers of seemingly perishable revenue-managed goods are unlikely to buy more than one unit during the life cycle of the product, making the product effectively infinitely lived. In addition, customers for such goods are exhausted over time, and customers are aware that the monopolist finds it difficult to commit to her price, giving a revenue-management problem many durable-goods features. A good example of revenue-management price dynamics reminiscent of durable-goods problems is provided by the cruise-line industry (see Talluri and van Ryzin [28, pp. 560–561] or Coleman, Meyer and Scheffman [16]), in which significant, last-minute discounts are common and customers often wait in order to purchase at deep discounts. Similarly, last-minute deals are sold, often through a variety of intermediaries, by theaters and firms in the travel industry.

[2] We assume the multiplicative inverse of the distribution of valuations is convex, a weaker assumption than the commonly-invoked condition of log-concavity.

are multiple possibilities for equilibrium buyer behavior. As we show, for some valuation distributions, there are then multiple equilibria in the overall game.

To make further progress, we then specialize to the case of uniformly-distributed buyer valuations. We prove that the equilibrium is unique and characterize equilibrium strategies.

First, this allows us to delineate the seller's revenue more precisely. When the number of buyers exceeds a critical threshold, the seller relinquishes altogether her ability to keep prices high and lets the price drop to zero. Our second bound is then tight: her revenue is equal to the Dutch auction. Recall that our first bound is tight with one buyer, in which case the seller effectively bunches together all buyers' valuations above the static monopoly price. For an intermediate number of buyers, she is able to secure a revenue that exceeds both bounds, by optimally trading off a larger number of bunches with a lower, but positive reserve price that strictly decreases with the number of buyers.

More importantly, specializing the distribution to the uniform case allows us to describe the actual price path that the seller employs. When there are few buyers, so that the seller opts for a positive reserve price, the price drops in the twinkling of an eye, as in the Coase conjecture, but not to the lowest valuation, and only very near the deadline, instead of the very beginning. Because she takes into account her inability to commit, the monopolist ends up charging prices that are higher, until near the very end, than those she would set if she could commit. The seller essentially sits on the choke price until then, and sales occur in the last instants, at which point the price drops in a rapid sequence of discrete price cuts.

When there are enough buyers, so that her revenue is equal to the Dutch auction revenue, the price drops to the lowest valuation, but not in the twinkling of an eye. Indeed, the price then decreases continuously over time, at a level that is always below what it would be if the seller could commit.

1.2 The Literature

There are four related bodies of work. First, as we have noted, a large revenue management literature has examined the case of a seller who faces sequentially-arriving buyers. (See Talluri and van Ryzin [28] for an introduction, as well as (for example) Bitran and Mondschein [8] and Gallego and van Ryzin [19], and see Gershkov and Moldovanu [20] for an extension to heterogeneous objects.) The standard assumption in this literature is that the buyers are myopic, i.e., they base their decision on a comparison of the prevailing price with their valuation. The seller then faces an option pricing problem, as she must continually compare the value of selling to the current buyer against the option of waiting for a future buyer. However, no intertemporal links appear in the buyers' calculations. In contrast, our buyers remain until the good is sold and are fully strategic, constantly trading off buying the good today or waiting for a chance to buy later at a lower price. As Besanko and Winston [7] argue, mistakenly treating forward-looking customers as myopic may have an important impact on revenue. Comparatively few papers discuss the case of a monopolist with scarce supply and no commitment power selling to forward-looking customers. Given the technical difficulties, Aviv and Pazgal [4] and Jerath, Netessine, and Veeraraghavan [22] do so in a model with two periods.

Second, our paper is most closely related to Chen [13] and Bulow and Klemperer [11]. Chen [13] considers a model that is equivalent to ours, though hers is cast in terms of a single buyer bidding to purchase from multiple sellers (rather than a single seller bidding to sell to multiple buyers). This allows us to rely on Chen for an existence-for-equilibrium result.[3] Chen's primary characterization result is that an equilibrium must be one of two types: either the price remains bounded away from the lowest valuation, in which case there is imperfect price discrimination, or prices fall to the lowest valuation, in which case prices finely discriminate among customers. This mirrors our primary general result. We proceed to sharpen this result by specializing to the uniform distribution, while Chen considers an extension of the model in which the agents discount at different rates and there are limits on the frequency with which prices can be posted. Technically, the primary difference is that Chen does not consider the possibility of multiple equilibria. We postpone until Section 5 a discussion of how our work is related to Bulow and Klemperer [11].

Third, much of the richness and the difficulty in the problem facing our seller arises out of her inability to commit to future prices. McAfee and Vincent [24] and Skreta [27] examine a seller who conducts a sequence of auctions and a sequence of optimal mechanisms, respectively, without the ability to make commitments The most important difference between our analysis and that of McAfee and Vincent [24] or Skreta [27] is that the latter papers allow their sellers to commit to a mechanism within each period. In the limit as the discount factor gets large, the sequential-mechanisms problem becomes trivial in our setting—the seller should simply wait until the last period and implement an optimal mechanism. In addition, direct mechanisms make it difficult to tell just what commitment power is allowed the seller. We typically interpret direct mechanisms not as literal descriptions of the interaction between seller and buyers, but as a way of analyzing an underlying indirect mechanism. Depending on the nature of the latter, allowing the seller to commit to a direct mechanism in each period may invest her with enormous commitment powers. As a result, we consider it important to take an indirect-mechanism approach that is specific about the actions available to the seller in each period.

Finally, we have noted that the revenue management problem with strategic buyers shares some features of durable-goods monopoly problems (e.g., Ausubel and Deneckere [3] and Gul, Sonnenschein and Wilson [21]). The durable-good setting differs from ours in its infinite horizon and in the fact that there are as many goods as buyers. The scarcity of the good in our setting changes the issues surrounding price discrimination, with the impetus for buying early at a high price now arising out of the fear that another agent will snatch

[3]Notice that Chen's existence result does not require her assumption of increasing virtual valuations, which we do not impose.

the good in the meantime, rather than discounting.[4] Like a durable-goods monopolist, our seller would still be better off if able to commit, though cases can arise in which our seller's difficulty is that she would prefer that future prices be *lower* (and hence future demand brisk, heightening the urgency or purchasing now), but cannot commit to lowering them.

2 The Model

2.1 The Environment

We consider a dynamic game between a single seller, with one unit for sale, and n buyers. The good is purchased and consumed at a fixed future date that we normalize to 1, and is valueless thereafter. The seller has the interval $[0, 1]$ of time in which to make an agreement with a buyer on the purchase of the good. The buyer and seller discount at the same rate r, and there is then no loss of generality in normalizing this interest rate to be zero.[5]

The seller can post a price at each time $\{\Delta, 2\Delta, \ldots, 1\}$ (restricting attention throughout to values of Δ that divide 1 without remainder). We can thus think of the seller as facing a finite horizon of length $T_\Delta = \frac{1}{\Delta}$. Since our arguments will typically involve reasoning backwards from the final period and the number of periods will vary with Δ, we find it convenient to let $t = 1, \ldots, T_\Delta$ index the number of *remaining* periods, so that T_Δ is the first and 1 the last period. At each period t, the seller posts a price $p_t \in \mathbb{R}$. After observing the price, buyers simultaneously and independently accept or reject. If the price is accepted by at least one buyer, the game ends with a transaction at this posted price between the seller and a buyer randomly selected from among the accepting buyers. If the offer is rejected, the game moves on to the next period.[6]

[4] Kahn [23] introduces an element of scarcity within a period by examining a durable-goods monopolist with increasing costs, showing that this allows the seller to escape the zero-profit conclusion of the Coase conjecture. Similarly, a sufficiently small capacity constraint (a stylized form of increased costs) introduces scarcity within a period and allows positive profits. McAfee and Wiseman [25] show that capacity constraints have this effect even if the seller can choose to increase the capacity constraint in any period at a nominal cost. Bagnoli, Salant and Swierzbinski [6] and von der Fehr and Kühn [17] clarify the circumstances under which a Coasian firm can effectively commit, while Cho [14] examines an alternative source of commitment, arising out of the assumption that the good deteriorates while held by the seller.

[5] Since the buyer purchases and consumes the good at time 1, it is only a normalization to take all prices to be time-1 prices and eliminate discounting from the model. We might alternatively consider a model in which the buyer consumes at time 1 but pays immediately upon concluding an agreement. The commonality of the discount factor ensures that the equilibrium is unchanged, with a time-τ price p equivalent for both agents to a time-1 price of $pe^{-r(1-\tau)}$. It then simplifies the notation to present the analysis in terms of time-1 prices, while recognizing that a (nonlinear) transformation of time is required to convert the price paths of Section 4.3 to paths in "real time."

[6] Formally, an outcome of the game is a vector $(\mathbf{v}, t, p_t, i)$, $i = 1, \ldots, n$, or $(\mathbf{v}, 0, \varnothing)$; with the interpretation that the realized profile of valuations is $\mathbf{v} = (v_1, \ldots, v_n)$ and the price p_t is accepted in period t by buyer i if the outcome is $(\mathbf{v}, t, p_t, i)$, and that no buyer ever accepts in case $(\mathbf{v}, 0, \varnothing)$.

Each buyer has a private valuation (or type) v, that is independently and identically drawn from some distribution F with support $[0,1]$, once and for all. A buyer of valuation v who receives the object at price p garners payoff $v-p$. The seller has a zero reservation value, with her payoff (or revenue) being the price at which she sells the good.[7]

A *nontrivial* history $h^t \in H^t$ is a history after which the game is not effectively over, i.e. a sequence $(p_{T_\Delta}, \ldots, p_{t+1})$ of prices that were posted by the seller and rejected by all buyers (we set $H^{T_\Delta} = \varnothing$). A behavior strategy for the seller is a finite sequence $\{\sigma_S^t\}_{t=1}^{T_\Delta}$, where σ_S^t is a probability transition from H^t into $\mathbb{R}$, mapping the history of prices h^t into a probability distribution over prices. A behavior strategy for buyer i is a finite sequence $\{\sigma_i^t\}_{t=1}^{T_\Delta}$, where σ_i^t is a probability transition from $[0,1] \times H^t \times \mathbb{R}$ into $\{0,1\}$, mapping buyer i's type, the history of prices, and the current price into a probability of acceptance.[8]

The seller has no commitment power—each price must be sequentially rational, given the history of previous play and anticipations of optimal continuation play. "Real world" sellers may often work hard at making price commitments, perhaps by offering a guarantee that they will refund the difference should a consumer subsequently discover a lower price. Such devices appear most likely to allow commitments to constant price paths. Such commitments are worthless in the current setting, in the sense that a seller without commitment can always (if there are at least two buyers) do better than committing to constant prices. The full-commitment solution, which we calculate as a benchmark, involves a descending sequence of prices that may sometimes lie above and sometimes below the corresponding no-commitment sequence. Commitments of this complexity may be sufficiently demanding in some circumstances as to make the no-commitment solution interesting.

We will focus our attention on perfect Bayesian equilibria in which buyers use symmetric strategies: $\sigma_i^t = \sigma_j^t$, all i, j, t.[9] That is, a buyer's strategy depends on his valuation, not on his identity. Existence of such an equilibrium follows from standard arguments. See, in particular, Chen [13, Proposition 1]. Clearly, the specification of prices that are rejected by all buyers in an equilibrium is to a large extent arbitrary. Therefore, statements about uniqueness are understood to be made up to the specification of such unacceptable prices, and (similarly) up to the specification of behavior on the part of measure-zero sets of indifferent buyers.

[7]Formally, the seller's von Neumann-Morgenstern utility function takes the value p_t if the outcome is $(\mathbf{v}, t, p_t, i)$, and zero otherwise. Buyer i's utility is $v_i - p_t$ if the outcome is $(\mathbf{v}, t, p_t, i)$ and zero otherwise. We define the seller's and buyers' expected utilities over lotteries of outcomes in the standard fashion.

[8]That is, for each $h^t \in H^t$, $\sigma_S^t(h^t)$ is a probability distribution over $\mathbb{R}$, and the probability $\sigma_S^t(\cdot)[A]$ assigned to any Borel set $A \subset \mathbb{R}$ is a measurable function of h^t, and similarly for σ_i^t.

[9]Because the only off-path histories are triggered by the uninformed party, generalizing Fudenberg and Tirole's [18, Definition 8.2] definition to our infinite game raises no particular difficulty.

2.2 The Buyers' Game

Because a buyer with a zero value rejects all positive prices, nontrivial histories are on the equilibrium path. Therefore, each individual buyer faces a "non-strategic" stopping problem, choosing the point (if any) along a sequence of prices at which to accept and end the game, taking as given the behavior of other buyers. Higher-valuation buyers are more anxious to purchase than lower-valuation buyers. As a result, a buyer's optimal stopping time is non-increasing in his valuation (as long as there are at least two buyers). Buyers with higher valuations accept earlier, and the buyers who accept in a given period t are those (if any) whose valuations exceed a critical threshold v_t. Section A.1 uses a familiar single-crossing argument to prove:

Lemma 1. *Let $n \geq 2$. Fix an equilibrium, and suppose period t has been reached without a price having been accepted. Then the seller's posterior belief is that the buyers' valuations are identically and independently drawn from the distribution $F(v)/F(v_{t+1})$, with support $[0, v_{t+1}]$, for some $v_{t+1} \in (0, 1]$.*

Let us fix a (for expositional reasons, pure) strategy for the seller, and consider the resulting game between buyers. Given the sequence of equilibrium prices, each period t is characterized by a critical (or marginal) buyer's valuation v_t, with higher-valuation buyers accepting and lower-valuation buyers rejecting. Relative to these thresholds, a buyer with valuation v accepts the price p_t that maximizes

$$\begin{aligned} & F(v_{t+1})^{n-1} \sum_{j=0}^{n-1} \frac{1}{j+1} \binom{n-1}{j} \left(1 - \frac{F(v_t)}{F(v_{t+1})}\right)^j \left(\frac{F(v_t)}{F(v_{t+1})}\right)^{n-1-j} (v - p_t) \\ = \;& F(v_{t+1})^{n-1} \frac{1 - [F(v_t)/F(v_{t+1})]^n}{1 - F(v_t)/F(v_{t+1})} \frac{v - p_t}{n}. \end{aligned} \tag{1}$$

The term $F(v_{t+1})^{n-1}$ is the probability that there is no higher-valuation buyer who will consign buyer v to a zero payoff by accepting a higher price. The term $1/(j+1)$ in the summation is the probability that the buyer is awarded the good if j other buyers accept the posted price. The binomial expression is the probability that j such buyers accept this price, and $v - p_t$ is the resulting payoff. Higher prices are relatively more valuable to higher-valuation buyers (the single-crossing condition), and it is then a relatively straightforward calculation that if buyer v_t prefers price p_t to all subsequent prices, then so do all higher-valuation buyers, giving the result.

If the critical thresholds v_t are interior (meaning that some buyers do accept in each period), then buyer v_t must be indifferent between accepting p_t and waiting until the following period (in a continuation game with $t - 1$ periods to go and buyers' valuations known to be no larger than v_t). If buyer v_t accepts, his payoff (conditional on period t having been reached) is

$$\sum_{j=0}^{n-1} \frac{1}{j+1}\binom{n-1}{j}\left(1-\frac{F(v_t)}{F(v_{t+1})}\right)^{j}\left(\frac{F(v_t)}{F(v_{t+1})}\right)^{n-1-j}(v_t-p_t) \;=\; \frac{1-[F(v_t)/F(v_{t+1})]^n}{1-F(v_t)/F(v_{t+1})}\frac{v_t-p_t}{n}. \tag{2}$$

This matches the payoff given in (1), except that we are now conditioning on having reached period t. By waiting one more period instead, buyer v_t gets

$$\left(\frac{F(v_t)}{F(v_{t+1})}\right)^{n-1}\frac{1-[F(v_{t-1})/F(v_t)]^n}{1-F(v_{t-1})/F(v_t)}\frac{v_t-p_{t-1}}{n}. \tag{3}$$

The interpretation of this expression is analogous to that of (2), with prices and thresholds being indexed by period $t-1$ rather than t. The new term $[F(v_t)/F(v_{t+1})]^{n-1}$ is the probability that no other buyer accepts the good at price p_t, and captures the fact that the payoff from waiting until period $t-1$ is zero if some other buyer accepts the seller's offer in period t.

In a symmetric equilibrium, if prices p_t and p_{t-1} are such that buyer v_t is indifferent between purchasing with t or $t-1$ periods to go, the expressions in (2) and (3) must be equal, that is,

$$\frac{F(v_{t+1})^n-F(v_t)^n}{F(v_{t+1})-F(v_t)}(v_t-p_t)=\frac{F(v_t)^n-F(v_{t-1})^n}{F(v_t)-F(v_{t-1})}(v_t-p_{t-1}). \tag{4}$$

We now see the first indication that there might be multiple solution to the game between buyers. When there is more than one buyer,[10] equation (4) provides a second-order difference equation for the sequence $\{v_t\}$, whose boundary conditions are that $v_{T_\Delta}=1$, and $v_1=p_1$: in the first period, the seller has not yet learned anything about the buyers' values, and in the last period, a buyer accepts if and only his value exceeds the outstanding price. Boundary value problems for difference equations often have multiple solutions, and indeed may have no solution (see, for example, Agarwal [1]).

More generally, for a given pure strategy of a seller, the outcome of the game between buyers can be summarized by a set of thresholds v_t with the property that the period t that maximizes (1) for buyer v must be such that $v\in[v_t,v_{t+1})$. This is an equilibrium problem, which not only may have multiple *interior* solutions in which (4) is satisfied in each period, but also admit corner solutions featuring some periods in which all buyers optimally reject.

This possibility of multiple solutions for a given sequence of prices is no surprise given the strategic complementarity in the buyers' problem. A buyer's incentive to wait increases in the probability that he assigns to other buyers waiting as well. His incentive to accept in a given period is highest if he expects all other buyers to accept immediately.

[10] It is clear that, with a single buyer, the indifference condition can only hold if $p_t=p_{t-1}$. Unless prices are constant, a single buyer will purchase in the period in which the price is lowest.

2.3 The Seller's Problem

The seller chooses a sequence of prices, leading to payoff

$$\pi_{T_\Delta} = \sum_{t=1}^{T_\Delta} (F(v_{t+1})^n - F(v_t)^n) p_t, \tag{5}$$

where $F(v_{t+1})^n - F(v_t)^n$ is the probability that the highest valuation among the buyers lies in the interval $[v_t, v_{t+1})$, and hence that the object is sold at price p_t.

The obvious first step in analyzing the seller's problem is to write her period-t payoff recursively as

$$\pi_t(v_{t+1}) = \left(1 - \left(\frac{F(v_t)}{F(v_{t+1})}\right)^n\right) p_t + \left(\frac{F(v_t)}{F(v_{t+1})}\right)^n \pi_{t-1}(v_t), \tag{6}$$

where we set $\pi_0 = 0$. The payoff $\pi_t(v_{t+1})$ is the payoff of the seller, given that t periods remain and conditional on the buyers' values being drawn from $[0, v_{t+1}]$. This payoff is given by the current price p_t multiplied by the probability of acceptance, plus the continuation payoff $\pi_{t-1}(v_t)$ multiplied by the probability of rejection. This recursive formulation has the advantage of focussing attention on the seller's sequential rationality constraints, which appear in the requirement that the seller's strategy maximize $\pi_t(v_{t+1})$, conditional on reaching period t with valuations on $[0, v_{t+1}]$, for every possible t and v_{t+1} consistent with some history.

Suppose that, for every valuation v_t, there is a unique equilibrium in the continuation game featuring $t-1$ remaining periods and buyer valuations drawn from $[0, v_t]$, and that this equilibrium is interior, in the sense that the threshold buyer v_t in every period is just indifferent between accepting the current-period price and waiting. Then the seller's period-t problem is to choose p_t to maximize (6), subject to the incentive constraint (4) (which fixes v_t), with $\pi_{t-1}(v_t)$, v_{t-1} and p_{t-1} (the latter two appearing in (4)) fixed by the equilibrium of the continuation game. Moreover, in this case, we can characterize equilibrium behavior by solving backwards, first finding the equilibrium of the game with one period to go (as a function of v_2), then using this to maximize (6) subject to (4) for $t = 2$, and so on.

In general, because multiple sequences of thresholds $\{v_t\}$ might be consistent with the sequence of equilibrium prices, we cannot be sure that there is a unique equilibrium in the continuation game, given $t-1$ periods to go and valuations drawn from $[0, v_t]$. The seller's period t problem is then to choose p_t to maximize (6), subject either to (4) (or an inequality version of (4) in the case of a corner solution where no buyers accept the current price), with $\pi_{t-1}(v_t)$, v_{t-1} and p_{t-1} that are consistent with one of the (possibly many) equilibria of the continuation game. Moreover, which equilibrium appears in the continuation game depends arbitrarily upon the price the seller sets in period t, and indeed upon the entire history of prices set by the seller up to then.

Among the buyers, the only possible out-of-equilibrium move is an acceptance, which ends the game, and the possibility that multiple thresholds may be consistent with a given price sequence arise out of payoff complementarities. For the seller, most prices will be out-of-equilibrium prices. The ability to attach different continuation equilibria to various out-of-equilibrium prices expands the prospects for multiplicity.

A simple example shows that multiple equilibria, yielding different revenues, are a real phenomenon.

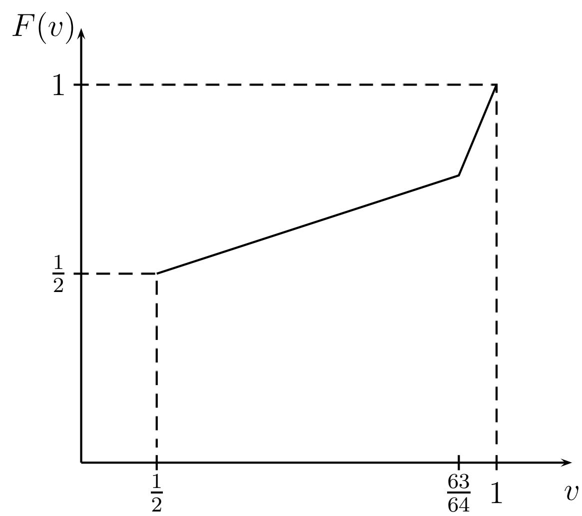

Figure 1. Distribution of valuations in Example 1.

Example 1. There are two buyers and two periods. Buyers' valuations are independently drawn from the distribution F, given by:

$$F(v) = \begin{cases} 0 & \text{if} \quad v < \frac{1}{2}, \\ \frac{1}{2} & \text{if} \quad v = \frac{1}{2}, \\ \frac{1}{2} + \frac{8}{15}\left(v - \frac{1}{2}\right) & \text{if} \quad v \in \left(\frac{1}{2}, \frac{63}{64}\right), \\ \frac{91}{120} + \frac{232}{15}\left(v - \frac{63}{64}\right) & \text{if} \quad v \in \left(\frac{63}{64}, 1\right). \end{cases}$$

See Figure 1. The monopoly price for this specification of demand is $\frac{1}{2}$, and the final price p_1 will also be $\frac{1}{2}$, no matter what price is chosen in the initial period.

Consider the price $\overline{p} = \frac{3}{4}$. We have $1 - \overline{p} = \frac{1}{2}(1 - p_1)$. This ensures that if the initial price is $\overline{p}$ and all buyers reject, then a type-1 buyer is just indifferent between accepting (for payoff $1 - \overline{p}$) and rejecting (for a final payoff of $\frac{1}{2}(1 - p_1)$). As a result, for any initial price $\frac{3}{4}$ or larger, there is a corner-solution equilibrium for the buyers in which all buyers reject this initial price.

Now consider an arbitrary initial price p_2. Condition (4), identifying the interior buyer v_2 who is just indifferent between accepting and rejecting, assuming that all types $v \geq v_2$ accept, is now

$$\frac{1 - F(v_2)^2}{1 - F(v_2)}(v_2 - p_2) = F(v_2)(v_2 - p_1).$$

Notice that $v_1 = p_1$, i.e., a buyer accepts the final price if and only if his valuation is at least as large as the price. The seller's payoff, given by (6), is $(1 - F(v_2))^2 p_2 + F(v_2)^2 p_1$. Maximizing this payoff subject to this constraint, we find that the seller will choose[11]

$$p_2 = \frac{43}{56} > \frac{3}{4} = \overline{p}.$$

This establishes the existence of multiple equilibria. In one of these equilibria, the seller sets price $\frac{43}{56}$ in the initial period, with buyers accepting if and only if their valuations are at least $\frac{31}{32}$. Should the initial price be rejected, the seller sets price $\frac{1}{2}$ in the final period, accepted by a buyer if and only if his valuation is at least $\frac{1}{2}$. In the alternative equilibrium, any initial price exceeding $\frac{3}{4}$ is rejected by every buyer. The seller sets the price $\frac{3}{4}$ in the initial period, accepted by buyers if and only if their valuations are at least $\frac{97}{104}$ $(< \frac{31}{32})$, followed by price $\frac{1}{2}$ in the final period. The former equilibrium gives the seller a higher revenue than the latter.

In Section 3, we work with bounds on equilibrium payoffs and behavior that must hold in any equilibrium. Section 4 specializes to the case of a uniform distribution, for which we can show that the continuation equilibrium is invariably unique, allowing an explicit calculation of the equilibrium.

3 Main Results

3.1 The Seller's Trade-Off

What would be the seller's favorite outcome? In the commonly considered case of increasing virtual valuations, the revenue-maximizing mechanism is straightforward, consisting of a Dutch auction with a positive reserve price.[12] This mechanism combines perfect separation

[11]One can rearrange the buyers' indifference condition to obtain $p_2 = \frac{v_2 + F(v_2)/2}{1 + F(v_2)}$, and then express the seller's payoff as

$$(1 - F(v_2))^2 p_2 + F(v_2)^2 p_1 \;=\; (1 - F(v_2))^2 \frac{v_2 + F(v_2)/2}{1 + F(v_2)} + \frac{F(v_2)^2}{2}.$$

Differentiating, we find that the seller maximizes her revenue by choosing a value of p_2 that makes indifferent the buyer whose type v_2 satisfies $v_2 - F(v_2)v_2 + F(v_2)/2 = 0$, which we can solve for $v_2 = \frac{31}{32}$. We can then calculate that $p_2 = 43/56$.

[12]Recall that the *virtual valuation* of a buyer with value v is defined as $v - (1 - F(v))/f(v)$.

among those buyers to whom the good is sold with a positive terminal price that excludes low-valuation buyers.

Unfortunately for the seller, this allocation is not an equilibrium outcome of the game. If the terminal price is to be positive, then in any previous period k the seller must be charging the same price to all buyers in a non-negligible interval of valuations. The idea is that the final critical buyer's valuation is determined by setting the monopoly price given the posterior distribution, and hence can be nonzero only if it is smaller than the upper bound on this distribution, and so smaller than the previous-period critical buyer's valuation. A similar argument allows us to work backward through the chain of critical buyers, at each step finding a non-negligible interval of buyers charged the same price. Therefore, if the seller insists on a positive reserve price, she must imprecisely discriminate in the prices she induces the buyers to accept.

Formally, let $v_{\Delta k}$ be the valuation of the buyer who is just indifferent between accepting and rejecting the period-k price (cf. Lemma 1), given period length Δ. Section A.2 proves:

Proposition 1. *Suppose that F has no atoms. If $\lim_{\Delta \to 0} v_{\Delta 1} > 0$, then for all k,*

$$\lim_{\Delta \to 0} v_{\Delta k+1} > \lim_{\Delta \to 0} v_{\Delta k}.$$

Therefore, the buyer cannot both finely price discriminate, and exclude low-valuation buyers. Nevertheless, as we show next, the deadline endows the seller with considerable commitment power. She can obtain the revenue from her favorite reserve price (without further price discrimination), or alternatively the revenue from perfect price discrimination (without a reserve price). In the process, we see both that commitments to a constant price sequence would be of no value, and that our seller confronts a quite different situation than that facing a durable-goods monopolist.

3.2 Static Monopoly

We first prove that the seller can guarantee herself the static monopoly profit. Let $\underline{\pi}_\Delta(n)$ and $\overline{\pi}_\Delta(n)$ denote the seller's lowest and highest equilibrium payoff of the game with n buyers and period length Δ, respectively. Notice that $\overline{\pi}_1(n) = \underline{\pi}_1(n) = \pi_1(n)$ is the static monopoly payoff with n buyers, being uniquely defined by

$$\pi_1(n) = \max_{p \in [0,1]} p(1 - F(p)^n). \tag{7}$$

Proposition 2. *The seller can guarantee the static monopoly payoff. More generally, the opportunity to revise prices more quickly increases both the lower and upper bound on the seller's equilibrium payoff: If $\Delta < \Delta'$ and hence $T_\Delta > T_{\Delta'}$, then*

$$\underline{\pi}_\Delta(n) \geq \underline{\pi}_{\Delta'}(n), \ \textit{and}\ \overline{\pi}_\Delta(n) \geq \overline{\pi}_{\Delta'}(n),$$

and so

$$\underline{\pi}_\Delta(n) \geq \overline{\pi}_1(n).$$

Note that this result requires no distributional assumption. The basic idea behind it is rather straightforward. For any equilibrium in a game with $k-1$ remaining periods, there is an equilibrium in the k-period game in which the seller sets an unacceptably high initial price, and then play duplicates that of the $k-1$-period equilibrium. Additional periods can thus only increase the largest and smallest equilibrium payoff. Notice that since $\pi_1(n)$ is the best that can be achieved by a seller committed to an optimal constant price scheme, such commitments are not valuable.[13]

3.3 Dutch Auction

Next, we turn to fine price discrimination. Let $\pi^D(n)$ denote the expected revenue from a Dutch auction with n bidders and a zero reserve price. This is the seller's revenue in the equilibrium considered by Bulow and Klemperer [11] in an infinite-horizon, continuous-time game in which the seller has no commitment power.

Proposition 3. *Suppose that $1/F$ is convex. As prices get revised frequently enough, the seller can guarantee the revenue of an optimal auction with zero reserve price:*

$$\liminf_{\Delta \to 0} \underline{\pi}_\Delta(n) \geq \pi^D(n). \tag{8}$$

The assumption that $1/F$ is convex is implied by log-concavity of F, an assumption that is very often made in information economics, and satisfied by most familiar distributions. See Bagnoli and Bergstrom [5]. It is a strictly weaker assumption: $1/F$ is convex for the Cauchy distribution, one of the few familiar distributions that fails long-concavity.[14] The convexity of $1/F$ implies that $1/F$ is almost-everywhere differentiable, and hence that F admits a density f. Furthermore, this density admits only downward jumps, and must be bounded away from 0 except possibly at the upper end of the support ($f(x) = 0$ for some $x < 1$ implies $f = 0$ on $[x, 1]$).

Proposition 3 sounds so plausible that it might come as a surprise that its proof requires any assumption at all. After all, why can't the seller just move her prices down the demand curve? Indeed, it is straightforward to show that if the seller could *commit* to working

[13] In the case of the uniform distribution, examined in the next section, we can strengthen the weak inequalities in Propositions 2 to strict ones, and establish (9) below for all Δ, rather than simply in the limit as $\Delta \to 0$.

[14] Log-concavity of F means that $\ln F$ is concave, or alternatively, $\ln 1/F$ is convex, which is obviously more stringent a requirement than $1/F$ being convex. Bagnoli and Bergstrom [5, Table 3] mention that the Cauchy distribution fails log-concavity, and it is straightforward to verify that the second derivative of its multiplicative inverse is positive.

through a succession of sufficiently close and equally spaced prices, then any optimal buyer response would give her a payoff approaching (as Δ gets small and the price gaps shrink to zero) that of a Dutch auction.[15] To see where our intuition goes wrong from there, consider the case of finite types $v_0 < v_1 < \cdots < v_k < \cdots < v_K$, and suppose we knew that, as long as only buyers with valuations belonging to $\{v_1, \ldots, v_k\}$ (or a subset thereof) remained, the seller could guarantee herself nearly the Dutch auction revenue. Could we extend this conclusion to the case in which type v_{k+1} is the highest type in the support of the seller's beliefs?

Consider the highest price p_{k+1} that the seller can post, yet attract the buyer with valuation v_{k+1}. This price is determined by the utility that such a buyer would secure if he were to deviate and reject it. The difficulty is that we do not know that the seller will actually use the Dutch auction once this price is rejected (and she rationally assigns probability 0 to any buyer having the valuation v_{k+1} or higher). We have assumed that she *can* secure the Dutch auction revenue in the continuation, but this does not imply that she *will* implement the Dutch auction. For example, she might prefer a price sequence that leads to some positive reserve price (as is the case with few buyers and a uniform distribution, see Section 4). Such an alternative strategy might yield a higher utility than the Dutch auction to the buyer whose valuation is v_{k+1}. In particular, it is easy to construct examples of distributions for which there is an allocation that is preferred both by the seller, and the buyer with valuation v_{k+1}, to the Dutch auction. In this case, the price the seller must post to ensure that the buyer whose valuation is v_{k+1} is willing to accept might be strictly lower than the corresponding price in the Dutch auction, and as a result, her revenue is strictly smaller than in the Dutch auction.

The key step in proving Proposition 3 is then to find conditions under which the seller and currently highest-valuation buyer have opposing interests, in the sense that mechanisms the seller might consider as an alternative to the Dutch auction are worse for the highest-valuation buyer, hence allowing the seller to trade with the buyer at a price at least as high as that of the Dutch auction. This requires some restriction on F.

Let us say that a feasible, incentive compatible allocation can be obtained via a price sequence if, for any interval of buyer types receiving the good with equal probability (except for a bottom interval of types who may receive the good with probability zero), the probability that the good is allocated to a buyer in this interval equals the probability that the highest buyer valuation lies in that interval.[16] The structure of our problem restricts the

[15]This follows from Athey [2, Theorem 6, proof]: incentive compatibility implies that equilibrium strategies are increasing in types, so that any sequence of such equilibrium strategies, indexed by the mesh of the price grid, must have a convergent subsequence, and its limit must be an equilibrium of the standard Dutch auction. But the latter admits a unique equilibrium.

[16]If there is a (maximal) interval of types $[v, v')$, $v < v'$, which obtain the good with probability q, then it must be that $nq(F(v') - F(v)) = F(v')^n - F(v)^n$. Not every feasible, incentive-compatible allocation can be obtained via a price sequence. As a simple example, let there be two buyers with valuations uniformly

seller to such mechanisms. Section A.4.3 proves:[17]

Lemma 2. *Suppose $1/F$ is convex. Then any feasible, incentive-compatible allocation that can be obtained via a price sequence and that provides the seller a revenue at least as large as that of the Dutch auction gives the highest-valuation buyer a strictly lower payoff than does the Dutch auction.*

Of course, with a continuum of types, it is not possible to define the continuation as simply as with finitely many types, and the finite (though arbitrarily large) number of periods available to the seller also creates some difficulties. As a result, the proof of Proposition 3 is neither easy nor short. The interested reader is referred to Section A.4.

The lower bound that Proposition 2 derives has an important corollary: in the case of increasing virtual valuations, the seller's revenue must increase with the number of buyers. Bulow and Klemperer [12] show that, if the virtual valuation is increasing in v, a standard assumption in mechanism design, the payoff from a zero-reserve-price English auction with $n+1$ bidders exceeds the payoff from an optimal mechanism with n bidders. Because the former is a lower bound on the lowest equilibrium payoff $\underline{\pi}_\Delta(n+1)$ for Δ small enough if $1/F$ is convex, and the latter by definition an upper bound on the (highest) equilibrium payoff $\overline{\pi}_\Delta(n)$, the following corollary obtains.

Corollary 1. *Suppose that $1/F$ is convex, and that the virtual valuation is increasing. Then more buyers are better for the seller, i. e., for every n,*

$$\liminf_{\Delta\to 0} \underline{\pi}_\Delta(n+1) \geq \limsup_{\Delta\to 0} \overline{\pi}_\Delta(n). \tag{9}$$

3.4 Comparing the Two Bounds

Propositions 2 and 3 provide two lower bounds on the seller's payoffs. Which of these bounds is the sharper one depends on the distribution and the number of buyers. Can the seller actually do better than either in the dynamic game? As we shall see in Section 4, sometimes, although as Proposition 1 indicates, the seller cannot have the best of both worlds. Either the seller insists on a reserve price, or she insists on finely discriminating among different valuations. In the first case, she must give up such discrimination, treating buyers with widely different valuations identically, while in the second case she must give up on the reserve price, letting prices drift all the way down to zero. How this trade-off is best resolved depends on the number of buyers. Consider the following two extreme cases.

distributed on $[0,1]$. Let $q(v)$ be the probability that a buyer receives the good, conditional on being of valuation v. Then there exists a feasible and incentive compatible allocation with $q(v) = v$ for $v \leq \frac{1}{2}$ and $q(v) = \frac{1}{2}$ otherwise, but no price sequence achieves such an allocation.

[17] If F is concave (a special case of $1/F$ convex), then we can prove a stronger and much more convenient result, namely that the Dutch auction is the favorite allocation of the buyer whose valuation is highest, among all feasible and incentive-compatible allocations, but concavity of F is an uncomfortably strong assumption.

One Buyer: With one buyer, it follows from Samuelson [26] that among all mechanisms, the optimal ones are equivalent to having the seller make a take-it-or-leave-it offer to the buyer. As the seller can always do so by posting a price of 1 in every period but the last, every equilibrium must then yield this maximal payoff to the seller. The seller then obtains the static monopoly payoff (which exceeds the zero revenue of a Dutch auction). In every equilibrium, all prices but that posted in the last period must be unacceptable to all buyers, and the price in the last period must be the monopoly price.

A Large Number of Buyers: When the number of buyers is very large, the payoff from a Dutch auction without reserve price exceeds the monopoly payoff. To see this, we invoke some arguments based on extreme value theory (Blumrosen and Holenstein [9, Theorems 4 and 5]). The extent to which the Dutch auction falls short of complete surplus extraction is given by the difference between the first and the second order statistic of the buyers' value. This difference is roughly $2/[nf(1)]$, and hence the Dutch auction revenue is

$$1 - \frac{2}{n}\frac{1}{f(1)} + O\left(\frac{1}{n^2}\right), \tag{10}$$

where $|O(1/n^2)| \leq M/n^2$ for some $M > 0$. Under monopoly pricing, the seller must make sure that her price approaches 1 (as the number of buyers grows) sufficiently slowly so as to ensure that with high probability, at least one buyer has a valuation exceeding the price. This probability increases exponentially fast as the price is lowered, and hence a given probability can be achieved with a price that is within a distance of only the order $\ln n/n$ from 1. This in turn allows us to write the monopoly payoff as

$$1 - \frac{\ln n}{n}\frac{1}{f(1)} + O\left(\frac{1}{n^2}\right). \tag{11}$$

Comparing (10) and (11), it is immediate that the Dutch auction revenue is larger than the monopoly revenue when there are sufficiently many buyers.

Many questions remain. As mentioned above, we have not shown whether these lower bounds on revenue can be tight. What prices would the seller use, and what would the resulting allocation be? While our informal discussion suggests that the seller might achieve these bounds by the obvious corresponding pricing strategy, this has not been established so far. While the proofs of the earlier propositions involve specific price sequences, these were exhibited to show what the seller could do, not what she actually does. Even if the seller were to obtain precisely the revenue of a Dutch auction without reserve, this would not still imply that she does so by duplicating the prices of such an auction, as there are many allocations that achieve such a revenue.[18] Even less is known about the price *path*, that

[18] However, because we know that she secures the Dutch auction revenue, we know that she must replicate the Dutch auction *if* she lets the price drop to zero, at least when virtual valuations are increasing.

is, about when different prices are charged. Even if the seller were to replicate the Dutch auction revenue by using a very fine grid of prices, she might wait until the last instants to do so, or instead go through all these prices at the very beginning of the game.

The following section provides answers to these questions in the special case of the uniform distribution.

4 Uniformly Distributed Buyer Valuations

This section assumes throughout that the distribution is uniform: $F \sim \mathcal{U}[0,1]$. In this case, we show that the equilibrium is unique and exhibits convenient homogeneity properties. Subsection 4.1 provides an explicit, if technical, description of this equilibrium. Subsection 4.2 sharpens our earlier results on revenue and describes the equilibrium price range, while Subsection 4.3 solves for the equilibrium price path.

4.1 Equilibrium Strategies

We hereafter assume that there are at least two buyers. With the uniform distribution, the buyers' indifference condition given by (4) simplifies considerably, as it becomes

$$\begin{aligned}\frac{v_{t+1}^n - v_t^n}{v_{t+1} - v_t}(v_t - p_t) &= \frac{v_t^n - v_{t-1}^n}{v_t - v_{t-1}}(v_t - p_{t-1}) \qquad (12)\\ &= \frac{v_t^n - v_{t-1}^n}{v_t - v_{t-1}}(v_t - v_{t-1}) + \frac{v_t^n - v_{t-1}^n}{v_t - v_{t-1}}(v_{t-1} - p_{t-1})\\ &= (v_t^n - v_{t-1}^n) + \cdots + (v_2^n - v_1^n) = v_t^n - v_1^n, \qquad (13)\end{aligned}$$

where the third equality obtains by repeated substitution. Note that this can be re-written as

$$\frac{p_t}{v_{t+1}} = \frac{v_t}{v_{t+1}} - \frac{1 - v_t/v_{t+1}}{1 - (v_t/v_{t+1})^n}\left(\left(\frac{v_t}{v_{t+1}}\right)^n - \left(\frac{v_1}{v_{t+1}}\right)^n\right), \qquad (14)$$

giving us the basic incentive constraint for the seller's problem. This in turn suggests some characteristics of the equilibrium, namely that the ratios $v_1/v_2, \ldots, v_t/v_{t+1}$ are fixed by optimality considerations and the price p_t is a linear function of v_{t+1}.

The seller's payoff with t periods to go, given by (6), simplifies to

$$\pi_t(v_{t+1}) = \frac{v_{t+1}^n - v_t^n}{v_{t+1}^n}p_t + \frac{v_t^n}{v_{t+1}^n}\pi_{t-1}(v_t). \qquad (15)$$

Using (14) to incorporate the incentive constraint, the seller's problem is to maximize

$$\frac{\pi_t(v_{t+1})}{v_{t+1}} = \frac{v_t}{v_{t+1}}\left(1 - \left(\frac{v_t}{v_{t+1}}\right)^{n-1}\right) + \left(\frac{v_1}{v_{t+1}}\right)^n\left(1 - \frac{v_t}{v_{t+1}}\right) + \left(\frac{v_t}{v_{t+1}}\right)^{n+1}\frac{\pi_{t-1}(v_t)}{v_t}. \qquad (16)$$

This again suggests an equilibrium solution in which the payoff $\pi_t(v_{t+1})$ is linear in v_{t+1}.

Section A.5 shows that there exists a unique equilibrium. In this equilibrium, the seller's price and revenue in each period t are a linear function of the period $t+1$ threshold v_{t+1}. The argument proceeds by induction. It is straightforward, given the uniform distribution, to calculate that this is the case in the last period. We then work backward, showing that in each period, the buyers' indifference condition given by (14) has a unique solution (for a given price set by the seller, and given the results obtained for subsequent periods), and that the seller's problem given by (16) has only interior solutions, in the sense that some buyers accept the price in each period. The remaining task is to show that there is a unique interior solution, at which point the argument becomes somewhat technical. We let $q_t = v_{t+1}/v_1$, and then note that (after some manipulation) the sequence of values q_t, describing the ratios of the values of indifferent buyers, must be an increasing solution to the second-order difference equation

$$q_{t+1}^n - \frac{nq_t^n - 1}{q_t} q_{t+1} + n(q_t^{n-1} q_{t-1} - 1) - q_{t-1}^n = 0, \tag{17}$$

with the boundary conditions

$$q_0 = 1, \qquad q_1 = (n+1)^{\frac{1}{n}}.$$

Section A.5 shows this sequence exists and is unique. This gives:

Proposition 4. *Suppose the seller's period-t posterior belief has support $[0, v_{t+1}]$ after some nontrivial history. (If $t = T_\Delta$, take $v_{T_\Delta+1} = 1$.) Then there is a unique perfect Bayesian equilibrium in the resulting continuation game. In particular, the seller sets the period-t price*

$$p_t = \left[1 - \frac{q_t - q_{t-1}}{q_t^n - q_{t-1}^n}\left(q_{t-1}^{n-1} - q_{t-1}^{-1}\right)\right] \frac{q_t}{q_{t-1}} v_{t+1}, \tag{18}$$

and given an arbitrary *price $\hat{p}_t$, each buyer i with valuation $v_i \geq v(\hat{p}_t, v_{t+1}, t)$ accepts the price and each buyer i with $v_i < v(\hat{p}_t, v_{t+1}, t)$ rejects it, where $v(\hat{p}_t, v_{t+1}, t)$ is the unique value v solving*

$$1 - \frac{\hat{p}_t}{v} = \frac{v_{t+1} - v}{v_{t+1}^n - v^n} v_{t+1}^{n-1} \left(1 - q_{t+1}^{-n}\right). \tag{19}$$

The path of equilibrium behavior is straightforward to trace, even if the statement of the strategies is somewhat formidable. A sufficient statistic for continuation play in each period t is the upper bound v_{t+1} on the buyers' valuations. Given this bound, the seller can calculate the optimal price p_t according to (18), and this price will be accepted by buyers in the interval $[v_t, v_{t+1})$, where the cutoffs v_t evolve (in equilibrium) according to

$$v_t = \frac{q_t}{q_{t-1}} v_{t+1}. \tag{20}$$

Should a deviation to an out-of-equilibrium price on the part of the seller push us off the equilibrium path in period t, we will depart from this progression of marginal buyer valuations, with next period's value of v_t now defined by (19) rather than (20). Once we have obtained this new value, however, we face a continuation game with a unique perfect Bayesian equilibrium, defined by (18)–(19). Buyer deviations have no effect on continuation strategies.

4.2 Equilibrium Prices

We now describe the range of equilibrium prices used by the seller, in the limit in which prices can be revised arbitrarily quickly. Let $p_{\Delta t}$ denote the equilibrium price set by the seller when there are t periods to go (including the current one), given period length Δ. Given $p_{\Delta t}$, let $v_{\Delta t} \in [0, 1]$ denote the valuation of the buyer who is indifferent between accepting and rejecting in period t, characterized in Proposition 3. (Set $v_{\Delta t} = 1$ if every buyer rejects, and $v_{\Delta t} = 0$ if every buyer accepts.)

Proposition 5. *Fix a period length Δ and a number of buyers $n \geq 2$. The sequences $\{p_{\Delta t}, v_{\Delta t}\}_{t=1}^{T_\Delta}$ of equilibrium prices and types take values in $(0, 1)$ and are strictly increasing in t (i.e., decreasing over time). Further:*

(5.1) For $n < 6$, $\lim_{\Delta \to 0} v_{\Delta 1} > 0$ *and* $\lim_{\Delta \to 0} \pi_\Delta(n) > \pi^D(n)$.

(5.2) For $n \geq 6$, $\lim_{\Delta \to 0} v_{\Delta 1} = 0$ *and* $\lim_{\Delta \to 0} \pi_\Delta(n) = \pi^D(n)$.

(5.3) $\lim_{\Delta \to 0} v_{\Delta 1}$ *is decreasing in n.*

Sequential rationality forces the seller to set a series of prices that decline over time, in each period skimming off an upper interval of high-valuation buyers. As Δ shrinks and price-revision opportunities come more frequently, the seller sets a higher and higher initial price $p_{\Delta T_\Delta}$, using her frequent price revisions to skim off smaller intervals in each period and hence more effectively price discriminate among the buyers. If $p_{\Delta T_\Delta}$ increases sufficiently rapidly as Δ shrinks, the higher starting price and smaller skimming intervals will counteract the more frequent price revisions and the terminal price $p_{\Delta 1}$ will never fall to zero—the seller commits to a reserve price. If $p_{\Delta T_\Delta}$ increases more slowly as Δ shrinks, the more frequent price revisions will more than make up for the higher initial price and smaller intervals, and $p_{\Delta 1}$ will approach zero—no commitment.

The larger is the number of buyers, the lower does the seller allow the ultimate price to drop (Proposition 5.3). Because the final and only serious price that she posts with one buyer matches the optimal reserve price under commitment, and the latter is independent of the number of buyers, it follows that the seller always lets the price eventually drop below what it would be if she could commit. However, one can construct examples of distributions in which this is not the case (that is, in which the seller would like to commit to a *lower* price; see Appendix B.4 for such an example).

With five or fewer buyers (the specific number is obviously an artefact of the uniform distribution of buyer values), the finite horizon allows the seller to commit to a positive reserve price, no matter how long the horizon. As long as there are multiple buyers, the resulting revenue is larger than either of our bounds. It then follows from Proposition 1 that only a very few prices have a nonnegligible probability of being accepted. Indeed, the set of prices that she charges has only one accumulation point, the choke price.

Instead, if there are more than six buyers, then the seller's final price approaches zero (Proposition 5.2). As the proof in Section A.6 makes clear, the closure of the set of prices that are charged at some point or another includes all prices from zero to the choke price, so that the outcome of the game mimics the outcome of the Dutch auction. In this case, the seller's lack of commitment power poses no difficulties in discriminating between buyers, but she abandons all hope of maintaining a reserve price. Incidentally, this implies that the revenue lower bound provided by the Dutch auction is actually achieved.

4.3 Pricing Dynamics

The previous subsection has described which prices the seller sets. We are interested here in characterizing *when* the seller charges these prices. That is, we study the limiting path of prices and indifferent types as the period length Δ goes to zero. We maintain the assumption that F is uniform throughout.

4.3.1 Noncommitment

We begin with our standard assumption that the seller cannot commit to future prices. Recall that $v_{\Delta t}$ denotes the indifferent buyer's valuation in the unique equilibrium of the game, with t instants to go. Given any period length Δ and given the sequence of indifferent types $\{v_{\Delta T_\Delta}, \ldots, v_{\Delta 1}\}$ maximizing the payoff of a seller with commitment, define the step function

$$v_\Delta(x) = v_{\Delta t} \text{ for all } x \in \left[\frac{t-1}{T_\Delta}, \frac{t}{T_\Delta}\right), \quad v_\Delta(1) = 1,$$

where, in keeping with our use of t to identify the number of remaining pricing opportunities, we think of x as the time remaining before hitting the terminal horizon. Our next proposition establishes that the (continuous extension of the) limit

$$v(x) = \lim_{\Delta \to 0} v_\Delta(x) \tag{21}$$

exists, and identifies this limit. This gives us the path of valuations of the indifferent buyers, as a function of time. Similarly, the proposition identifies the corresponding limit $p(x)$ of the analogously defined price function $p_\Delta(x)$, with $p(x)$ identifying the path of equilibrium prices as a function of time. Clearly, with only one buyer, only the last posted price matters, and we accordingly assume $n \geq 2$.

Proposition 6.

[6.1] For $2 \leq n < 6$, the functions v and p both converge to $v(x) = 1$ and $p(x) = 1$ on $[0, 1)$.

[6.2] For $n \geq 6$, the limiting function v describing the path of indifferent buyers induced by a seller who cannot commit to prices is well-defined, and equal to

$$v(x) = x^{\frac{3}{n+1}}, \tag{22}$$

while the corresponding price function is given by

$$p(x) = \frac{n-1}{n} x^{\frac{3}{n+1}}. \tag{23}$$

The seller's trade-off between reserve price and price discrimination thus reappears in the seller's use of time. When $n < 6$ (and as Δ gets very small), all of the pricing action occurs in the very last instants of the horizon. The seller sets prices very close to the choke price, with very little chance of an acceptance, until the last instants, at which point the price p and the marginal buyer v cascade in chunks to nonzero terminal values. For any $\varepsilon > 0$, the probability that a trade occurs before time $1 - \varepsilon$ goes to zero as Δ gets small.

In contrast, for larger values of n, marginal valuations and prices both decline continuously as time passes (v and p both increase in x). While the seller achieves the Dutch-auction revenue in this case, the seller does not adopt the obvious approximation of the Dutch auction, namely a sequence of equally-spaced price intervals. Instead, $v(x)$ and $p(x)$ are both concave, indicating that the seller reduces price slowly and moves though buyer valuations in the early stages, setting ever-larger price reductions and slicing off ever-larger chunks of buyers as the endpoint approaches.

Combining (22)–(23), we find that when $n \geq 6$, a buyer of valuation v purchases the object (if a competitor does not snatch it first) at price[19]

$$p(v) = \frac{n-1}{n} v.$$

[19] As a check on this result, we can then calculate that the seller's revenue when facing n buyers is

$$\int_0^1 p(v) n v^{n-1} dv = \int_0^1 (n-1) v^n dv = \frac{n-1}{n+1} = \pi^D(n),$$

in keeping with Proposition 5.2. In this calculation, $p(v)$ is the price paid by a buyer of type v and nv^{n-1} is the density of the highest bidder's valuation, obtained by noting there are n candidates for the highest bidder and for each valuation v the probability that it is higher than the other valuations is v^{n-1}.

The price is thus a linear function of the buyer's valuation, with a slope that is increasing in the number of buyers. Hence, as the number of buyers increases, the seller gains from the fact that the likelihood of a high-valuation buyer increases, and also from the fact that increased competition among buyers pushes each buyer to pay a price closer to his valuation.

4.3.2 Commitment

We can gain some insight into the seller's pricing incentives by examining the case in which the seller can commit to her entire sequence of prices. We define the limiting functions $v(x)$ and $p(x)$ exactly as before, but with $\{v_{\Delta T_\Delta}, \ldots, v_{\Delta 1}\}$ now being the sequence of indifferent buyers in the (unique) equilibrium of the game with commitment.

At first glance, the commitment case is trivial.[20] Because virtual valuations are increasing in the case of a uniform distribution, a Dutch auction with reserve price $\frac{1}{2}$ is optimal, and commitment gives the seller the ability to come arbitrarily close (as $\Delta \to 0$) to this outcome. But how does the seller do this? What sequence of prices does the seller set in order to come close to the optimal-auction payoff? As it turns out, a seller who can commit does not spread prices equally apart, and the optimal distribution of prices that results asymptotically translates into a specific price path. The counterpart of Proposition 6 is

Proposition 7. *Let $n \geq 2$. The limiting function v (cf. (21)) describing the path of indifferent buyers induced by a seller who can commit to prices is well-defined, and equal to*

$$v(x) = \frac{1}{2}\left(\left(2^{\frac{n+1}{3}} - 1\right)x + 1\right)^{\frac{3}{n+1}}, \tag{24}$$

while the corresponding price function is given by

$$p(x) = \frac{(n-1)v(x)^n + 2^{-n}}{nv(x)^{n-1}}. \tag{25}$$

The seller fully takes advantage of the entire time horizon, and decreases prices $p(x)$ (and thus, the indifferent type $v(x)$) continuously over time as the terminal point approaches (x decreases). As expected, $v(1) = 1$ and $v(0) = 1/2$, so that the seller begins (at $x = 1$) slicing off the highest-type buyers, moving downward to a valuation of $1/2$ (at $x = 0$). The function v is concave in x (it is affine in x for $n = 2$), so that the seller runs through buyers more rapidly as time goes on, and is increasing in n. Prices are also increasing in n, and of course increasing in x—prices decline over time—but they are not concave in x. Rather, they are convex for x low enough, and concave for high enough values of x (this higher interval being empty if and only if $n \leq 3$). Prices decrease relatively rapidly at the beginning and end of the interval, progressing somewhat more slowly in the middle.

[20] One might worry about equilibrium multiplicity of the game between buyers, but with the uniform distribution, the equilibrium is again unique.

Figure 2 illustrates these results. While Proposition 6 calculates the type of the marginal buyer and the price as a function of the time remaining, we make Figure 2 more intuitive by translating these into functions that give marginal valuations and prices as a function of the *time that has elapsed*. Notice that the function v initially picks out marginal buyers whose types are arbitrarily close to 1, while the prices that make these buyers indifferent are quite a bit lower.

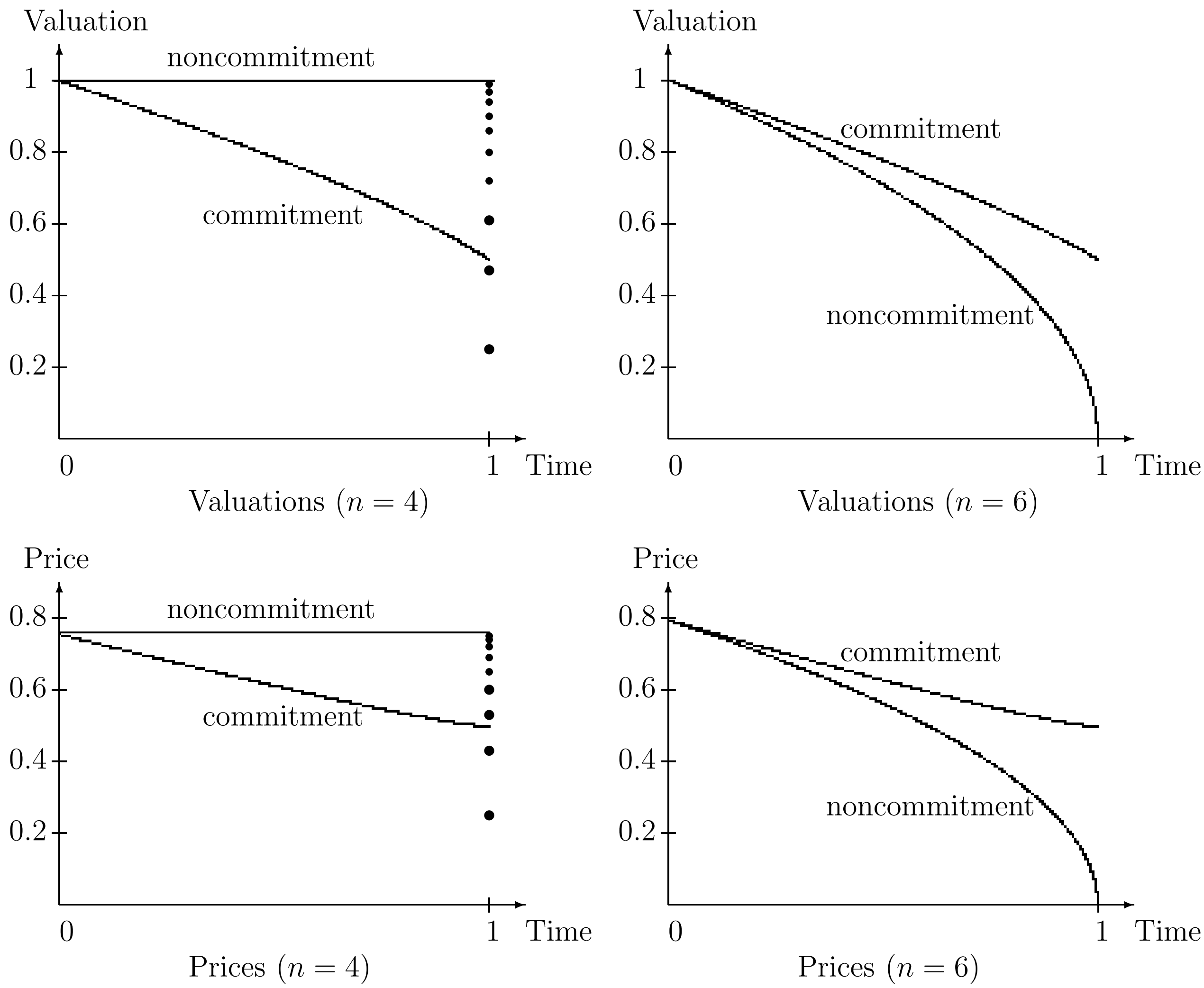

Figure 2. Limiting (as $\Delta \to 0$) critical valuations v and prices p, as a function of the time that has elapsed.

There is some arbitrariness in the price path under noncommitment and few buyers. The price over the interval $[0, 1)$ must be high enough that there are no sales, and many price paths will have this effect. We have also taken the liberty to associate a collection of indifferent valuations and terminal prices with time 1, even though this is impossible in a continuous-time model, as a reminder that as Δ gets small, the seller packs an arbitrarily large number of nontrivial prices and indifferent buyer valuations into a vanishingly small final instant of time.

When there are few buyers, the seller would like to preserve a positive reserve price. In the absence of commitment, the great obstacle to doing so is the future temptation to slash prices. The seller attenuates these future price reductions by initially reducing prices more slowly than would be the case without commitment. When there are many buyers, the seller abandons the reserve price. The continual pressure to reduce prices thus leads to price reductions that are more rapid than would be the case under commitment.

5 Discussion

Unknown number of buyers. We have assumed that our seller knows how many buyers she faces. What if this is not the case? The obvious alternative is to consider a model in which the number of sellers is determined by a Poisson process. If values are drawn uniformly from the unit interval, the resulting model is quite tractable.[21] In this case, the seller's unique equilibrium strategy always entails a positive terminal price. As the price falls without a purchase in our model, the seller draws the inference that all of the buyers happen to have low valuations, while remaining convinced of the number of buyers. The importance of price discrimination remains unaltered, and (when there are sufficiently many buyers) the seller's decision to sacrifice the reserve price in the interests of price discrimination remains unaltered.

As the price falls without a purchase in a model with a Poisson-distributed number of buyers, the seller draws the inference not only that the buyers have low valuations, but also that there are simply not many buyers there. Eventually, the seller becomes very pessimistic about the number of buyers, and a reasoning analogous to the one applying to the case of a low, but known number of buyers implies here as well that the equilibrium continuation path of play entails a positive terminal price.

Multiple Units. Suppose the seller has more than one unit for sale, with buyers' valuations drawn from the unit interval according to the uniform distribution. Letting k be the number of units, we assume $n \geq k+5$, which suffices to ensure that the seller's price eventually declines to zero. Let $p_{kn}(v)$ be the price paid by a buyer of valuation v, in the limiting case of arbitrarily short time periods, when there are k objects and n buyers. Let π_{kn} be the seller's payoff when selling k objects to n buyers. Arguments analogous to those of the case

[21]The Poisson process allows especially convenient calculations, as the problem takes on a recursive structure much like that induced by the uniform distribution of valuations in our fixed-number-of-buyers model.

$k = 1$ give[22]

$$p_{kn}(v) \;=\; \frac{n-k}{n}v, \tag{26}$$

$$\pi_{kn} \;=\; k\frac{n-k}{n+1}. \tag{27}$$

A buyer of valuation v thus pays more for the object when facing more competitors, but pays less when there are more objects for sale. The seller's revenue is increasing in the number of buyers, and is increasing in the number of objects as long as there are at least twice as many buyers as objects. If the seller has too many objects for sale, she would be better off destroying some of them before offering the remainder for sale to the buyers.

Suppose now that the seller begins with k objects and n buyers, and consider the limiting case of vanishingly small period lengths Δ. The price drops until some buyer of type v buys the first object at price $\frac{n-k}{n}v$. At this point, the price jumps upward to $\frac{(n-1)-(k-1)}{n-1}v = \frac{n-k}{n-1}v$, as the seller now continues with the equilibrium strategy given one less object and one less buyer, with the remaining buyers' valuations distributed on $[0, v]$. The price continues to fall until another buyer of type v' purchases at $\frac{n-k}{n-1}v'$, at which point the price jumps to $\frac{n-k}{n-2}v'$. This continues until a single object is left, to be eventually sold to a buyer of type v'' at price $\frac{n-k}{n-(k-1)}v''$.

Figure 3 illustrates these dynamics. The seller begins with two objects and lets the price fall, decreasing the indifferent buyer type, until the first purchase occurs. The price now jumps upward as the seller switches to the appropriate single-object price path, while the identity of the indifferent buyer continues to decline from the valuation of the buyer who purchased.

[22] The derivations are much more complicated with multiple objects and do not yield closed-form solutions for the functions $v(x)$ and $p(x)$. Section B.3 sketches the arguments. To provide some insight into these functions, one can verify both that π_{kn} is the expected value of a $k+1$-st price auction with n bidders, and that π_{kn} and p_{kn} satisfy the recursion

$$\pi_{kn} = \int_0^1 nv^{n-1}[p_{kn}(v) + v\pi_{k-1,n-1}]dv,$$

where $v\pi_{k-1,n-1}$ is the continuation value of selling $k-1$ objects to $n-1$ buyers with valuations uniformly distributed on $[0, v]$.

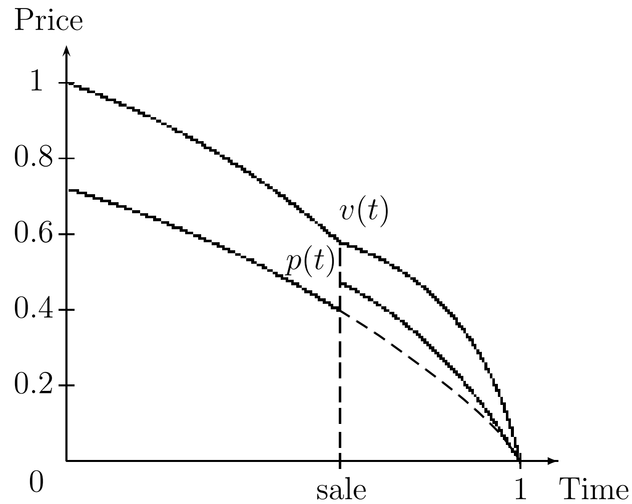

Figure 3: Prices and marginal valuations for $n = 7$ (with a sale at time $t = .6$)

The price jumps in this progression are reminiscent of the frenzies in Bulow and Klemperer [11]. Each sale in their model raises the possibility of a frenzy, in which additional buyers purchase at the price of the most recent sale, or even a price increase, in the event that more buyers than there are remaining objects attempt to purchase at the most recent sale price. The revenue earned by our seller (for sufficiently large n) matches that of Bulow and Klemperer's. Bulow and Klemperer work directly in continuous time and impose conditions on the path of prices set by the seller, including that price must decline continuously to zero in the absence of a sale, that a sale must be followed by repeated opportunities for additional buyers to purchase at the sale price, and that the price must jump upward if these opportunities for additional purchases lead to excess demand for the good. The result is one of the many continuous-time price paths that maximize the seller's revenue. Our analysis begins in discrete time and places no restrictions beyond sequential rationality on the seller's prices, in the process selecting one of the optimal continuous-time price paths as the limit of the optimal pricing scheme with very short, discrete pricing periods.

A Appendix: Proofs

A.1 Proof of Lemma 1

Fix a candidate equilibrium. Suppose that a buyer with valuation v finds it optimal to accept price p in some period t. Then it must be the case that

$$\sum_{k=0}^{n-1} q(k)\frac{1}{1+k}(v-p) \geq q(0)\sum_{p_\ell \in P} \rho(p_\ell)(v-p_\ell),$$

where $q(k)$ is the probability that k other buyers accept the price p, P is the finite set of prices the seller will set in the remaining $t-1$ periods under the candidate equilibrium, and $\rho(p_\ell)$ is the probability the buyer will purchase the good at such a price. Notice that $\rho(p_\ell)$ combines the buyer's subsequent decisions of when to say yes to a price, as well as the probability that other buyers will accept either p_ℓ or an earlier price. We have $\sum_{p_\ell \in P} \rho(p_\ell) \leq 1$, and hence the derivative in v of the left side of this inequality is at least as large as that of the right. If $q(0) < 1$ or $\rho(p_\ell)$ attaches nonunitary probability to prices equal to p, the derivative of the left side is strictly larger than that of the right, and hence all buyers with valuations $v' > v$ find it strictly optimal to also accept the offer, giving the result. If $q(0) = 1$ and $\rho(p_\ell)$ attaches probability 1 to prices equal to p, then all buyers are indifferent between accepting and rejecting the current price. In this case, there is no loss of generality in taking the accepting set of buyers to be an upper interval of buyer types. ■

A.2 Proof of Proposition 1

Suppose $v_{\Delta 1} > 0$. Then $v_{\Delta 2}$ must be such that

$$v_{\Delta 1} = \arg\max v(1-(F_2(v))^n),$$

where F_2 is the posterior cumulative distribution over the buyers' types, given that these types are contained in the interval $v_{\Delta 2}$. If F_2 has a strictly positive density, then this ensures that $v_{\Delta 2} > v_{\Delta 1}$ and, along with $\lim_{\Delta\to 0} v_{\Delta 1} > 0$, that $\lim_{\Delta\to 0} v_{\Delta 2} > \lim_{\Delta\to 0} v_{\Delta 1}$. An induction argument then establishes the corresponding inequality for every k. ■

A.3 Proof of Proposition 2

If $\Delta = 1$, the seller literally gets only one chance to set a price, and is a static monopoly, ensuring that she earns at least $\pi_1(n)$. We now proceed by induction. Fix a period length Δ' and corresponding $T_{\Delta'}$, and let period-length Δ permit one additional pricing period. One possibility open to the seller under period length Δ is to set an unacceptable offer in the first period, in which case play will continue with some equilibrium of the resulting continuation game, which must also be an equilibrium in the game with period length Δ'. Setting an unacceptable price in the first period thus ensures a payoff of at least $\underline{\pi}_{\Delta'}(n)$, and hence any equilibrium with period length Δ must bring at least this payoff (or $\underline{\pi}_{\Delta}(n) \geq \underline{\pi}_{\Delta'}(n)$). Similarly, considering strategies in which the seller makes a unacceptable offer in the first period followed by an equilibrium giving payoff $\overline{\pi}_{\Delta'}(n)$ gives $\overline{\pi}_{\Delta}(n) \geq \overline{\pi}_{\Delta'}(n)$. ■

A.4 Proof of Proposition 3

The proof of Proposition 3 is broken into several steps. Our first objective is to show that, if $1/F$ is convex, then the seller can achieve no allocation that the highest type would prefer to the Dutch auction, and that yields a higher revenue to the seller.[23] Proving this will take the next four subsections. Section A.4.1 defines a sequence of maximization programs, indexed by the revenue that the seller can guarantee. Starting with the assumption that the seller can guarantee nothing, we derive what utility the buyer would achieve from his favorite allocation, and therefore, what the highest price would be that he would not be able to resist accepting. These prices provide a basis for a new lower bound on the seller's revenue, as she can choose to "move down" the demand curve by using a fine grid of those prices (note that these prices would be accepted by the buyer *independently* of his expectations about the continuation). If it is common knowledge among buyers that, with enough periods to go, the seller can guarantee a revenue close to this lower bound, we can use this program again, to re-compute the highest utility that a given buyer's type can hope for, across all allocations raising that much revenue. We then again derive the price that he cannot resist accepting, etc. Section A.4.2 shows that the sequence of programs converges to a fixed point. To study this fixed point, Section A.4.3 discretizes the problem and shows that the revenue from the fixed point in the discrete problem, under an assumption that corresponds to $1/F$ convex in the discrete set-up, is at least as large as the Dutch auction revenue. Section A.4.4 uses limiting arguments to establish the same result for the continuous program that we have started with.

Finally, we apply this result to the dynamic game in Section A.4.5.

A.4.1 The Program

Let us say that a function $q_v : [0,1] \to [0, F(v)^{n-1}]$ is *admissible* if it lies in the closure of the set of functions with the properties that

$$q_v \text{ is non} - \text{decreasing and right} - \text{continuous}, \tag{28}$$

[23] We do not know whether this is true for all feasible, incentive-compatible allocations, or only for those satisfying our admissibility requirement (see Section A.4.1 below).

and there exists a strictly increasing sequence of valuations $0 = v_0, v_1, \ldots, v_M = 1$ such that for every $i \in \{1, \ldots, M-1\}$, $q_v - F(v)^{n-1}$ is non-increasing on (v_i, v_{i+1}) and

$$\int_{v_i}^{v_{i+1}} (q_v(s) - F^{n-1}(s))dF(s) = 0, \tag{29}$$

with the last two properties also holding on $[0, v_1)$ if q_v is not identically zero on this interval.

We will think of q_v as an allocation contemplated by buyer v. Condition (28) is an obvious feasibility condition. Condition (29) is a strengthening of Border's [10, Proposition 3.1] condition guaranteeing the implementability of a reduced-form auction. If, for every interval $[v_i, v_{i+1})$ with $i \geq 1$, we required either q_v to be constant or $q_v - F(v)^{n-1}$ to be identically zero on this interval (with the same conditions holding on $[0, v_1)$ if q is not zero on this interval), then we would have an obvious necessary condition for q_v to be obtained via a sequence of prices. More precisely, (29) would then correspond to a deterministic sequence of prices, which implies that, except for some lower interval of types whose probability of obtaining the unit is zero, it must be that the probability that the good is allocated to a player with a type in a higher interval is equal to the probability that the highest bidder's valuation lies in this interval.[24] In particular, if there is a (maximal) interval of types $[s, s')$, $s < s'$, which obtain the good with probability q, then it must be that (recall that $1/F$ is assumed to be convex, and hence F has no upward jumps)

$$nq(F(s') - F(s)) = F(s')^n - F(s)^n.$$

We find it convenient to require that q_v is admissible, a condition that is weaker than being implementable by a price sequence but still stronger than Border's condition, because it simplifies a subsequent continuity argument (Claim 4 below). Given the sequence $v_0, v_1, \ldots, v_M$, we refer to an interval $[v_i, v_{i+1})$ as a *bunch*. If $q_v = 0$ on $[0, v_1)$, we say that the buyers in this interval are *rationed*.

Consider the following maximization program, parameterized by $v \in [0, 1]$ and $H : [0, 1] \to \mathbb{R}$:

$$U_v(v|H) = \max_{q_v} \int_0^v q_v(t)dt$$

over functions $q_v : [0, 1] \to [0, F(v)^{n-1}]$, subject to the constraints that

$$q_v \text{ admissible} \tag{30}$$

[24]Note that, in the dynamic game, the seller might randomize over sequence of prices. However, she is only willing to do so if all raise the same revenue, so that each sequence in the support of her mixed strategy satisfies (28)–(31), and so the maximum $U(v|H)$ must be achieved by such a deterministic sequence.

and

$$\begin{aligned} J(v) &:= \int_0^v \left(s q_v(s) - \int_0^s q_v(t)dt \right) dF(s) \\ &= \int_0^v \left(s q_v(s) - U_v(s) \right) dF(s), \\ &\geq H(v), \end{aligned} \tag{31}$$

where we write $U_v(s)$ for the payoff of type $s \leq v$, i.e.

$$U_v(s) = \int_0^s q_v(t)dt.$$

Condition (31) states that the chosen allocation must raise an (unconditional) revenue at least equal to H. The objective $U(v|H)$ is then the (unconditional) payoff of a bidder with type v. The interpretation of this program is clear: this is the highest payoff a buyer of type v can conceivably expect from a price sequence that raises a revenue (per buyer) of at least $H(v)$. Let also H^D, U^D, q^D denote the Dutch auction revenue, utility, and allocation.

Fix $H^0 \leq H^D$, and define inductively $U^k = U(\cdot|H^k)$, and H^{k+1} via, for all v,

$$H^{k+1}(v|H^k) := \int_0^v (sF(s)^{n-1} - \min\{U_s^k(s), U^D(s)\})dF(s).$$

The interpretation of H^{k+1} is that it is the (unconditional) revenue (per buyer) from a descending price auction accruing from the types in $[0, c]$, in which type s accepts as soon as the price reaches the level $s - \min\{U_s^k(s), U^D(s)\}$. The minimum ensures that the Dutch revenue H^D is a fixed point of the map defined by the recursion. Given H^k, we shall refer to any maximizing function q_v as q_v^k (note that it may not be unique).

A.4.2 Existence of a Fixed Point

Note that, for $H \leq H^D$, the feasible set of the program is non-empty, because q^D is feasible. We proceed with a few obvious claims.

Claim 1: If $\tilde{H}^k \geq H^k$ (pointwise), then $\tilde{H}^{k+1} \geq H^{k+1}$. This follows from the fact that $\tilde{H}^k \geq H^k \Rightarrow \forall v : U(v|\tilde{H}^k) \leq U(v|H^k)$.

Claim 2: H^{k+1} is bounded above (for instance, $H^{k+1}(v) \leq v$), for all k.

Claim 3: Given $H^k \leq H^D$, H^{k+1} is non-decreasing in v. This follows from the definition of H^{k+1} and the positivity of its integrand.

Note that Claims 1 and 2 imply that, starting from H^0, the sequence $\{H^k\}$ must converge. The limit H^∞ must be a fixed point of the map from H^k to H^{k+1}, and so must be a differentiable function of v, given that U^{∞} is continuous in v. By Dini's theorem, given Claim 1 and the continuity of H^{∞}, the convergence of $\{H^k\}$ to H^{∞} must be uniform. Because $H^0 \leq H^D$ and H^D is a fixed point of the map, $H^{\infty} \leq H^D$.

Claim 4: The objective function $U(v|\cdot)$ is continuous on $[0, H^D(v))$, for all $v \in [0,1]$. Clearly, it is non-increasing (because tightening the constraint in (31) can only lower the objective). To see that it is continuous, one need only note that for any admissible q_v other than the Dutch auction, a convex combination of q_v and the Dutch auction yields an admissible allocation that smoothly increases revenue and smoothly affects the payoff to the buyer as the weight on the Dutch auction increases. Hence, the constraint (31) can be satisfied strictly at an arbitrarily cost in terms of the objective. In particular, for every $\varepsilon^k \in (0,1)$, there exists ε^{k+1} such that $H^{k+1}(\cdot|(1-\varepsilon^k)H^k) \geq (1-\varepsilon^{k+1})H^{k+1}(\cdot|H^k)$, with $\lim_{\varepsilon^k \to 0} \varepsilon^{k+1} = 0$.

Claim 5: If $\lim_{l\to\infty} |F_l - F| = 0$ (in the L_1 norm), then for all $k \geq 1$, $\left|U_l^k - U^k\right| \to 0$ and $\left|H_l^k - H^k\right| \to 0$, where U_l^k and H_l^k are the sequences of utilities and revenues obtained from a sequence of continuous distributions F_l. To see this, fix k and suppose that $\left|H_l^k - H^k\right| \to 0$. Given $q_v(\cdot)$, define $q_{v,\ell}(t)$ as (i) $q_{v,\ell}(t) = F_l^{n-1}(t)$ if $t \in \{s \in [0,1] : q_v(s) = F^{n-1}(t)\}$: (ii) $q_{v,\ell}(t) = (F_l^n(t'') - F_l^n(t'))/[n(F_l(t'') - F_l(t'))]$ if $[t', t''] = \mathrm{cl}\, \{s \in [0,1] : q_v(s) = F^{n-1}(t)\}$ and $t' < t''$. Note that $q_{v,l}$ satisfies the admissibility constraints because q_v does. Given Claim 4, because $H_l^k(v) \to H^k(v) < H^D(v)$, we can define a sequence $\{q_{v,l}^m : m \in \mathbb{N}\}$ with $\lim_m q_{v,l}^m = q_{v,l}$ for all l, v, and such that $q_{v,l}^m$ satisfies (31) for all m, l large enough. Because $\lim_{m,l} q_{v,l}^m = q_v$, it follows that $\left|U_l^k - U^k\right| \to 0$. It follows then from the definitions of H^{k+1}, H_l^{k+1} that $\left|H_l^{k+1} - H^{k+1}\right| \to 0$, concluding the induction step.

Claim 6: It follows from uniform convergence that

$$\lim_{l\to\infty} |F_l - F| = 0 \Rightarrow \lim_{l\to\infty} |H_l^{\infty} - H^{\infty}| = 0.$$

A.4.3 Discrete Approximation

Suppose we have a discrete distribution of types. There are M different valuations, indexed by $1, 2, \ldots, M$, with valuations $\frac{1}{M}, \frac{2}{M}, \frac{3}{M}, \ldots, 1$, and with probabilities $f_1, \ldots, f_M$.

Suppose $q_m(j)$ is the probability that buyer j receives the object, and $p_m(j)$ the buyer's expected payment. Then j's utility, using the incentive constraint for the second line, is

$$\begin{aligned} U_m(j) &= \Delta j q_m(j) - p_m(j) \\ &= \Delta j q_m(j-1) + p_m(j-1) \\ &= \Delta q_m(j) + U_m(j-1). \end{aligned}$$

Iterating, we have

$$U_m(j) = \Delta \sum_{i=1}^{j-1} q_m(i). \tag{32}$$

The expected price is given by

$$p_m(j) = \Delta j q_m(j) - U_m(j) = \Delta j q_m(j) - \Delta \sum_{i=1}^{j-1} q_m(i).$$

Expected revenue per buyer from types $1, \ldots, m$ is given by (rearranging the double summation to get the second equality, and letting $f + i := f(i)$ and $F_i := F(i)$)

$$\begin{aligned} \sum_{i=1}^{m} p_m(i) f_i &= \sum_{i=1}^{m} \left(\frac{1}{M} i q_m(i) - \frac{1}{M} \sum_{k=1}^{i-1} q_m(k) \right) f_i \\ &= \Delta \sum_{i=1}^{m} \left(i - \frac{F_M - F_i}{f_i} \right) f_i q_m(i). \end{aligned}$$

Under the Dutch auction, we have

$$q_m^D(j) = \frac{1}{n} \frac{F_j^n - F_{j-1}^n}{F_j - F_{j-1}}.$$

The admissibility requirement is then that q_m be increasing and that we can partition the buyers into subintervals, so that if buyer $m's$ solution bunches together the buyers from j to k, then

$$\begin{aligned} n \sum_{i=j}^{k} q_m(i) f_i &= n \sum_{i=j}^{k} \frac{1}{n} \frac{F_i^n - F_{i-1}^n}{F_i - F_{i-1}} f_i \\ &= n \sum_{i=j}^{k} \frac{1}{n} \left(F_i^n - F_{i-1}^n \right) \\ &= F_k^n - F_{j-1}^n, \end{aligned}$$

with $q_m(i) - F_i^{n-1}$ decreasing.

The maximization problem we consider is now given by

$$\begin{aligned} U_m^k(m) &= \max_{q_m(i), i=1,\ldots,m} \Delta \sum_{i=1}^{m} q_m(i) \\ s.t. \quad & q_m \text{ admissible} \\ & \sum_{i=1}^{m} \left(\Delta i q_m(i) - U_m^k(i) \right) f_i \geq R_m^k, \end{aligned}$$

where

$$R_m^{k+1} = \sum_{i=1}^{m} \left(\Delta i q_m(i) - U_i^k(i)\right) f_i.$$

We now construct an induction argument. Notice first that for buyer 1, there is a unique solution to the maximization problem, which duplicates the outcome of the Dutch auction. This gives the point of departure for the induction.

Now suppose that every buyer $1, \ldots, m$ solves the maximization problem with the Dutch auction. We show that buyer $m+1$ also does so. The steps in the argument are:

- Buyer $m+1$ could choose $q_{m+1}(j)$ to be the Dutch auction. This would satisfy the constraints of the maximization problem and give $U_{m+1} = U_{m+1}^D$. If this is indeed the solution to the maximization problem, this step of the induction is finished.
- Suppose the solution gives some $U_{m+1} > U_{m+1}^D$. Then R_{m+1} falls short of R_{m+1}^D by $f_{m+1}(U_{m+1} - U_{m+1}^D)$.
- It must then be that buyer $m+1$'s solution to the maximization problem holds fixed $R_{m+1} + f_{m+1}U_{m+1}$. But since we have a fixed point, we must then also hold fixed $J_{m+1} + f_{m+1}U_{m+1} = R_{m+1}^D$.

We now show that any departure of q_{m+1} from the Dutch auction can only decrease the total $J_{m+1} + f_{m+1}U_{m+1} = R_{m+1}^D$. Notice that this proves Lemma 2 for this discrete version of the problem, with the lemma itself then following from limiting arguments (below).

Suppose first that there is a nontrivial minimal bunch, consisting of valuations in $\{\underline{v}, \ldots, \overline{v}\}$. Admissibility ensures that we can divide the types in the bunch into a lower interval of types, for whom $q > q_m^D(v)$, and an upper interval for whom $q < q_m^D(v)$. We now note that we can move from the Dutch auction on $\{\underline{v}, \ldots, \overline{v}\}$ to the bunch in a finite sequence of steps, at each step choosing a type j in the lower interval whose current probability $q(j)$ falls short of $q_m(j)$ and a type $k > j$ in the upper interval whose current probability $q(k)$ exceeds $q_m(k)$, and then increasing q_j and decreasing q_k, while preserving the value $f_j q_j + f_k q_k$, until either $q_j = q$ or $q_k = q$. We then fix the resulting values and proceed to the next step. In a finite sequence of steps, this leads from the Dutch auction to the bunch.

We argue that each step reduces $J_{m+1} + f_{m+1}U_{m+1}$. Letting

$$\frac{dq_k}{dq_j} = -\frac{f_j}{f_k},$$

we need to show (where Φ denotes virtual valuation),

$$\begin{aligned}
\frac{d}{dq_j}\left(J_{m+1} + f_{m+1}U_{m+1}\right) &= \frac{d}{dq_j}\left(f_{m+1}(q_j + q_k) + [q_j f_j \Phi_j + q_k f_k \Phi_k]\right) \\
&= f_{m+1}\left(1 - \frac{f_j}{f_k}\right) + f_j\left(j - \frac{F_{m+1} - F_j}{f_j}\right) - f_j\left(k - \frac{F_{m+1} - F_k}{f_k}\right) < 0.
\end{aligned}$$

This can be simplified to obtain

$$\left(1-\frac{f_j}{f_k}\right)(-F_m+F_j) < f_j(k-j)+\frac{f_j}{f_k}(F_k-F_j).$$

Since $k > j$ and $F_m > F_j$, this inequality is obviously satisfied if $f_j \leq f_k$. If $f_j > f_k$, we can note that a sufficient condition for the inequality is

$$f_j - f_k < f_j(k-j) + f_j,$$

which again obviously holds.

There remains the possibility that buyer $m+1$ might want to ration buyers at the bottom. Rationing at the bottom decreases the buyer's utility. Hence, this can improve on the Dutch auction outcome for the buyer only if combined with a higher bunch, where the creation of the bunch increases utility (at the cost of decreased revenue), with the rationing at the bottom then increasing revenue enough (relative to the attendant utility decline) to restore the value of $f_{m+1}U_{m+1} + J_{m+1}$. We show this cannot occur.

In the course of creating a bunch involving types j and $k > j$, utility increases (in q_j) at rate

$$1-\frac{f_j}{f_k}$$

(notice we must have $f_j < f_k$ if the buyer is to have any hope of increasing payoff via such an adjustment) while revenue decreases at rate

$$f_j\Phi_j + f_k\Phi_k\left(-\frac{f_j}{f_k}\right) \quad = \quad \left(1-\frac{f_j}{f_k}\right)(-F_{m+1}) + F_j - \frac{f_j}{f_k}F_k - f_j(k-j).$$

In the course of rationing by decreasing q_ℓ, utility decreases at rate -1, while revenue increases at rate

$$-f_\ell\Phi_\ell = -f_\ell + F_{m+1} - F_\ell$$

for some type ℓ that is rationed. We need the net effect on revenue of comparable changes in utility to be negative. Hence we need

$$(-f_\ell + F_{m+1} - F_\ell)\left(1-\frac{f_j}{f_k}\right) + \left(1-\frac{f_j}{f_k}\right)(-F_{m+1}) + F_j - \frac{f_j}{f_k}F_k - f_j(k-j) < 0.$$

This is

$$-(f_\ell + F_\ell)\left(1-\frac{f_j}{f_k}\right) + F_j - \frac{f_j}{f_k}F_k - f_j(k-j) < 0.$$

The most difficult version of this inequality occurs when $k = j+1$, giving

$$(F_j - f_\ell - F_\ell)\left(1-\frac{f_j}{f_{j+1}}\right) < 2f_j.$$

Now noting that $F_j - f_\ell - F_\ell < F_j$, we can rewrite this as

$$F_j(f_{j+1} - f_j) < 2f_j f_{j+1}.$$

As M gets small, this condition is implied by the requirement that $1/F$ is convex.

A.4.4 Characterization of the Fixed Point

We now construct a limiting argument, showing that $H^\infty \to H^D$. Suppose that $1/F$ is convex. By Claim 5, we might as well assume that $1/F$ is strictly convex (otherwise, consider $F_l = ((1-\lambda_l)F^{-1} + \lambda_l G^{-1})^{-1}$, where $\lambda_l \to 0$, $\lambda_l > 0$ and G^{-1} is a strictly convex distribution.). Then define the sequence of discrete distributions F_j with

$$f_j^m = \int_{j/m}^{(j+1)/m} f(s)ds,$$

where $m \in \mathbb{N}$ and $j \in \{0, \ldots, m-1\}$. Note that, because $1/F$ is strictly convex, for all m large enough, $F_j(f_{j+1} - f_j) < 2f_j f_{j+1}$ and the conclusions from the previous subsection apply. Define now the (continuous) distribution F^m as, for all $t \in [0, 1]$,

$$\tilde{F}_m(t) = \int_0^t \sum_{j=0}^{m-1} f_j^m 1_{s \in [j/m, (j+1)/m)} ds.$$

Because $t \to j/m$ for all $t \in [j/m, (j+1)/m)$, it follows that, inductively for all k, $\tilde{U}_m^k(t) \to U_m^k(j)$ for all $t \in [j/m, (j+1)/m)$, all j, where $\tilde{U}_m(t)$ is the utility of type t at the k-th step in the continuous program, with distribution $\tilde{F}_m$, and $U_m^k(j)$ is the utility of type j in the discrete approximation, as well as $\tilde{H}_m^k(t) \to H_m^k(j)$, where $H_m^k(j)$ is the (per-buyer, unconditional) revenue in the discrete approximation, which itself satisfies $\lim_m H_m^k(j) \to H^D(t)$, where $t = \lim j/m$. Therefore, $\tilde{H}_m^k \to H^D$, and because $\tilde{F}_m \to F$, it follows from Claim 5 that $H^\infty = H^D$ for the distribution F.

A.4.5 Proof of Proposition 3

We now turn our attention to the dynamic game. Note that, because H^k is non-decreasing, incentive compatibility implies that

$$U(v|H^k) - U(v - 1/m|H^k) < \frac{1}{m}.$$

Given the sequence $\{H^k\}$ introduced in the previous subsection (with $H^0 := \lim_{\varepsilon \to 0} H_\varepsilon^0$), we define the price function $p(\cdot|H^k)$, or p^k as $p(v|H^k) = v - U(v|H^k)$ for all v. It follows from the inequality above that $p(\cdot|H^k)$ is increasing in v.

The key observation is the following. Suppose that the buyers' valuations have support $[0, v]$, and that it is common knowledge that the buyer can secure the continuation payoff $(1-\varepsilon)H^k$ from next period onward. (This means that he can secure $(1-\varepsilon)H^k(s)$ if the support is $[0, s]$, for all s.) If the seller posts the price $p(v - 1/m|(1-\varepsilon)H^k)$ in the current period for some integer m, then we claim that all buyers of type $s \geq v - 1/m$ must accept. All types cannot reject: if so, the best that buyer of type v can hope for in the continuation is $U_v(v|(1-\varepsilon)H^k)$, yet by accepting his payoff is $v - p(v-1/m|(1-\varepsilon)H^k) > v - p(v|(1-\varepsilon)H^k) = U_v(v|(1-\varepsilon)H^k)$. By the same reasoning, the highest type that rejects must be no larger than $v - 1/m$ (though he may be strictly lower).

We claim that for each $k \geq 0$, and each $\varepsilon^k > 0$, there exists T^k such that it is common knowledge that the seller can secure at least the continuation payoff $(1-\varepsilon^k)H^k$ in the game with T^k periods to go or more. This is obviously true for $k = 0$ (and $H^0 = 0$). Let us assume it is true for some integer k, and we shall show that it holds for $k+1$.

Fix $\varepsilon^k > 0$ and $m \in \mathbb{N}$. Define $H^m_{\varepsilon^k}$ as the revenue function (per buyer) computed as follows. Given that buyers have valuation with support $[0, v]$, the seller chooses the sequence of prices

$$p((m-1)v/m|(1-\varepsilon^k)H^k), p((m-2)v/m|(1-\varepsilon^k)H^k), \ldots, p(0|(1-\varepsilon^k)H^k),$$

in descending order, and the buyer of type s accepts the price $p(iv/m|(1-\varepsilon^k)H^k)$ if $s \in [iv/m, (i+1)v/m)$. Fix m. We argue that with $T^k + m$ periods to go, the seller can guarantee $H^m_{\varepsilon^k}$. Fix v, and let $\{p^i_v : i = 0, \ldots, m-1\}$ denote the price sequence above (lower indices for lower prices). We claim, by induction on $j = 0, \ldots, m-1$ that with $T^k + j$ periods to go, the seller does at least as well as if she were to use the sequence $p^j_v, \ldots, p^0_v$. This is obvious for $j = 0$, because $p^0_v = 0$. Assume that it is true for some $j < m-1$. If the seller sets the price p^{j+1}_v in the initial period, by our earlier observation, all buyers, if any, with values above $(j+1)v/m$ must accept (if types with value below $(j+1)v/m$ accept, all the better, because prices are decreasing), and in the continuation, we have assumed that the seller does at least as well as by continuing to use the price sequence $\{p^i_v : i = 0, \ldots, j\}$. [Of course, the seller's actual continuation strategy might be very different, because, in particular, she must secure a continuation payoff of $(1-\varepsilon^k)H^k$, but whatever the strategy it follows, the strategy above provides a lower bound.]

Note now that, for all v, $\lim_{m\to\infty} H^m_{\varepsilon^k}(v) = H^{k+1}(v|(1-\varepsilon^k)H^k)$ uniformly in v. Yet as we have remarked in the previous section, for every $\varepsilon^k \in (0,1)$, there exists ε^{k+1} such that $H^{k+1}(\cdot|(1-\varepsilon^k)H^k) \geq (1-\varepsilon^{k+1})H^{k+1}(\cdot|H^k)$, with $\lim_{\varepsilon^k\to 0}\varepsilon^{k+1} = 0$. Therefore, for every $\varepsilon^{k+1} > 0$, we can find $T^{k+1} \in \mathbb{N}$ such that the seller can guarantee (per buyer) revenue of at least $(1-\varepsilon^{k+1})H^{k+1}$ in the game in which at least T^{k+1} periods remain.

We conclude that for every integer k, and every $\varepsilon > 0$, there exists $T \in \mathbb{N}$ such that the seller can guarantee revenue of at least $(1-\varepsilon)H^k$ in the game with T periods or more. Because H^k converges to the Dutch auction revenue H^D, as established in the previous subsection, and because $\varepsilon > 0$ is arbitrary, the proposition follows. ■

A.5 Proof of Proposition 4

We fix Δ (and hence the number of periods T_Δ) and use an induction argument on the number of remaining periods to show that, with t periods to go and beliefs about buyers' types that are uniform over a set $[0, v_{t+1}]$,

(i) the perfect Bayesian equilibrium in the continuation game is unique,

(ii) the seller's payoff equals $\mu_t v_{t+1}$ for some μ_t that is independent of v_{t+1}, and

(iii) the period-t price is such that buyers accept if and only if their valuation exceeds that of an indifferent type v_t given by some $\gamma_t v_{t+1}$, where $\gamma_t \in (0,1)$ is independent of v_{t+1}.

To show this, we use the seller's first-order conditions to determine a recursion (and initial values) that characterize the sequences γ_t and μ_t. We show that these define a unique sequence, with the property that $\gamma_t < 1$ for all t. We then show that these values achieve a maximum of the seller's objective function.

A.5.1 The Last Period

Consider the last period ($t = 1$) and let the seller's posterior belief be that the buyers' valuations are uniformly distributed on $[0, v_2]$. Then buyer v_i accepts the price p_1 if and only if $p_1 \leq v_i$, and the seller chooses $p_1 = v_1$ to maximize

$$\left(1 - \left(\frac{v_1}{v_2}\right)^n\right) v_1 = \left(\left(1 - \left(\frac{v_1}{v_2}\right)^n\right) \frac{v_1}{v_2}\right) v_2,$$

so indeed $v_1 = \gamma_1 v_2$ is linear in v_2, where γ_1 maximizes

$$(1 - \gamma_1^n)\, \gamma_1, \qquad \text{and hence} \qquad \gamma_1 = (n+1)^{-1/n}.$$

The value of the problem, $\pi_1(v_2)$, is then

$$\pi_1(v_2) = \mu_1 v_2, \quad \text{where } \mu_1 = \frac{n}{n+1}\gamma_1,$$

and so v_1 is indeed linear in v_2 as well. This solution is obviously unique.

A.5.2 The Induction Step

Now fix t and assume that for any $\tau < t$ periods to go, and for every uniform distribution of buyer valuations on $[0, v_{\tau+1}]$, the equilibrium is unique and characterized by values μ_τ and $\gamma_\tau < 1$ such that the seller sets a price accepted by all buyers with types above $\gamma_\tau v_{\tau+1}$, for an expected continuation revenue of $\mu_\tau v_{\tau+1}$. Consider the game with t periods to go, and beliefs that are uniform over $[0, v_{t+1}]$.

The buyer's indifference condition. From (12), and letting $\gamma_t := v_t/v_{t+1}$, the buyer's indifference condition can be written as

$$\frac{1-\gamma_t^n}{1-\gamma_t}\left(v_t - p_t\right) = \gamma_t^{n-1}\frac{1-\gamma_{t-1}^n}{1-\gamma_{t-1}}\left(v_t - p_{t-1}\right). \tag{33}$$

This equation identifies the critical buyer, given an arbitrary price p_t. Higher-valuation buyers will accept the price and lower-valuation buyers reject it. As in (13), this is a telescoping sum, so that

$$\frac{1-\gamma_t^n}{1-\gamma_t}\left(1-\frac{p_t}{v_t}\right) = \gamma_t^{n-1}\left(1-\prod_{\tau=1}^{t-1}\gamma_\tau^n\right). \tag{34}$$

Using $\gamma_t = \frac{v_t}{v_{t+1}}$ and our subsequently introduced convention $q_t := \frac{v_{t+1}}{v_1}$, we obtain the characterization of buyer behavior given by (19). Note that, given $\gamma_1, \ldots, \gamma_{t-1}$, equation (34) pins down γ_t uniquely as a function of p_t/v_{t+1}. (Write $p_t/v_t = (p_t/v_{t+1})/\gamma_t$, divide both sides by γ_t^{n-1} and note that the left-hand side is monotonic in γ_t, while the right-hand side is independent of it.) This implies that, with $t+1$ periods to go, given the posted price by the seller, there can be at most one critical type that is indifferent between accepting or not. To conclude from this that the buyer's optimal behavior is uniquely determined given the posted price, and the equilibrium strategies in the continuation, we must also argue that, for a given price, either there exists an interior, indifferent type, or all buyers' types find it optimal to reject, but not both. Suppose, for the sake of contradiction, that this were the case. This means, in particular, that given the price p_t, it is optimal for the highest buyer's type to reject it, if all buyers do so. In that case, rejecting yields as payoff

$$\mathcal{U}_{t-1} = \frac{1-\gamma_{t-1}^n}{n(1-\gamma_{t-1})}(1-p_{t-1}),$$

where we normalize this highest type to 1, without loss of generality. (We use here an argument by induction, so that it is optimal for the highest type to accept the price p_{t-1} in the following period, and the ratio over successive types will be then given by the equilibrium value of γ_{t-1}, which is independent of the actual value of the highest type.) By deviating and accepting instead the current price, while all other buyers are waiting, the buyer's highest type gets

$$1-p_t.$$

We wish to show that this is inconsistent with another type being indifferent between accepting and rejecting. If there is such a type, (33) yields that

$$\frac{1-\gamma_t^n}{1-\gamma_t}\left(\gamma_t - p_t\right) = \gamma_t^{n-1}n(\gamma_t\mathcal{U}_{t-1}) = n\gamma_t^n\mathcal{U}_{t-1},$$

where the factor γ_t reflects the fact that, with $t-1$ periods, the payoff of the highest type, if he is v_t, is a fraction γ_t of what it would be if he were type $v_{t+1} = 1$. Because we wish to show that $1 - p_t \geq \mathcal{U}_{t-1}$, we must argue that

$$n\gamma_t^n (1-p_t) \geq \frac{1-\gamma_t^n}{1-\gamma_t}(\gamma_t - p_t) = \gamma_t^n \left(1 - \prod_{\tau=1}^{t-1} \gamma_\tau^n\right),$$

i.e., that

$$n(1-p_t) \geq 1 - \prod_{\tau=1}^{t-1} \gamma_\tau^n.$$

From (34), we have that

$$1 - p_t = (1-\gamma_t)\left(1 + \frac{\gamma_t^n}{1-\gamma_t^n}\left(1 - \prod_{\tau=1}^{t-1} \gamma_\tau^n\right)\right),$$

and so, rearranging, we must show that

$$n(1-\gamma_t) + \left(\frac{n\gamma_t^n(1-\gamma_t)}{1-\gamma_t^n} - 1\right)\left(1 - \prod_{\tau=1}^{t-1} \gamma_\tau^n\right) \geq 0.$$

If the second term is positive, we are done, and if not, a sufficient condition is that

$$n(1-\gamma_t) + \left(\frac{n\gamma_t^n(1-\gamma_t)}{1-\gamma_t^n} - 1\right) = n\frac{1-\gamma_t}{1-\gamma_t^n} - 1 \geq 0,$$

which is obviously true, as $(1-\gamma_t^n) \leq n(1-\gamma_t)$.

The seller's profit. As in (6), the seller's profit π_t in period t is given by

$$\pi_t(v_{t+1}) = \left(1 - \left(\frac{v_t}{v_{t+1}}\right)^n\right) p_t + \left(\frac{v_t}{v_{t+1}}\right)^n \pi_{t-1}(v_t),$$

given the posted price p_t. Assuming that this price is such that there is an indifferent buyer's type, we can proceed as in (14), using (34) to eliminate the price and rewrite this as

$$\begin{aligned}
\pi_t(v_{t+1}) &= (1-\gamma_t^n)(p_t - v_t) + (1-\gamma_t^n) v_t + \gamma_t^n \pi_{t-1}(v_t) \\
&= -(1-\gamma_t)\sum_{\tau=1}^{t-1}(1-\gamma_\tau^n)\left(\prod_{l=\tau+1}^{t} \gamma_l^{n-1}\right) v_{\tau+1} + (1-\gamma_t^n) v_t + \gamma_t^n \pi_{t-1}(v_t).
\end{aligned}$$

Dividing by v_{t+1}, and letting $\mu_{t-1} := \frac{\pi_{t-1}(v_t)}{v_t}$, we have

$$\begin{aligned}
\frac{\pi_t(v_{t+1})}{v_{t+1}} &= -(1-\gamma_t)\sum_{\tau=1}^{t-1}(1-\gamma_\tau^n)\left(\prod_{l=\tau+1}^{t}\gamma_l^{n-1}\right)\frac{v_{\tau+1}}{v_{t+1}} + (1-\gamma_t^n)\frac{v_t}{v_{t+1}} + \gamma_t^n\mu_{t-1}\frac{v_t}{v_{t+1}} \\
&= -(1-\gamma_t)\sum_{\tau=1}^{t-1}(1-\gamma_\tau^n)\left(\prod_{l=\tau+1}^{t}\gamma_l^{n}\right) + \gamma_t(1-\gamma_t^n) + \gamma_t^{n+1}\mu_{t-1} \\
&= (1-\gamma_t)\prod_{\tau=1}^{t}\gamma_\tau^n + \gamma_t\left(1-\gamma_t^{n-1}\right) + \gamma_t^{n+1}\mu_{t-1}, \qquad (35)
\end{aligned}$$

which is an expression that is independent of v_{t+1}, and we may thus define $\mu_t := \frac{\pi_t(v_{t+1})}{v_{t+1}}$.

The seller's maximization. Because the behavior of the buyers is uniquely pinned down by the prices, we can maximize the seller's payoff with respect to γ_t (and we shall see that, indeed, the optimum is interior). The first and second derivatives of the seller's objective (35) are

$$(n\gamma_t^{n-1} - (n+1)\gamma_t^n)\prod_{\tau=1}^{t-1}\gamma_\tau^n + 1 - n\gamma_t^{n-1} + (n+1)\gamma_t^n\mu_{t-1}, \qquad (36)$$

and

$$((n-1)n\gamma_t^{n-2} - n(n+1)\gamma_t^{n-1})\prod_{\tau=1}^{t-1}\gamma_\tau^n - (n-1)n\gamma_t^{n-2} + n(n+1)\gamma_t^{n-1}\mu_{t-1},$$

respectively. Together with (34), this allows us to obtain the characterization of prices given in (18). The second derivative can be rewritten as

$$\frac{n}{\gamma_t}\left((n\gamma_t^{n-1} - (n+1)\gamma_t^n)\prod_{\tau=1}^{t-1}\gamma_\tau^n - n\gamma_t^{n-1} + (n+1)\gamma_t^n\mu_{t-1}\right) - n\gamma_t^{n-2}\prod_{\tau=1}^{t-1}\gamma_\tau^n + n\gamma_t^{n-2}.$$

When the first derivative equals zero, the terms in parenthesis in this second derivative equal negative one, giving a second derivative of

$$n(-\gamma_t^{-1} - \gamma_t^{n-2}\prod_{\tau=1}^{t-1}\gamma_\tau^n + \gamma_t^{n-2}),$$

which is negative if $\gamma_t \in (0,1]$. Hence, whenever the first derivative has an interior solution, the second (evaluated at that solution) is negative. This in turn ensures that if the first-order condition induced by (36) has an interior solution, that solution is unique and is a global maximizer.

Uniqueness. We must now show that the first-order condition induced by (36) has a unique, interior solution. Hence, we must show that (36) determines a sequence $\{\gamma_t\}$ with each $\gamma_t \in (0,1)$. Let $q_t = (\prod_{\tau=1}^{t} \gamma_\tau)^{-1}$, so $\gamma_t = q_{t-1}/q_t$. We can then rewrite the first-order condition (36) as

$$(n+1)\left(q_{t-1}^{-n} - \mu_{t-1}\right)\left(\frac{q_{t-1}}{q_t}\right)^n + n\left(1 - q_{t-1}^{-n}\right)\left(\frac{q_{t-1}}{q_t}\right)^{n-1} = 1, \tag{37}$$

and the seller's maximization problem given by (35) to get

$$\mu_t = \left(1 - \frac{q_{t-1}}{q_t}\right) q_t^{-n} + \frac{q_{t-1}}{q_t}\left(1 - \left(\frac{q_{t-1}}{q_t}\right)^{n-1}\right) + (\frac{q_{t-1}}{q_t})^{n+1}\mu_{t-1},$$

that is,

$$\mu_t q_t^{n+1} - q_t = q_t q_{t-1}\left(q_t^{n-1} - q_{t-1}^{n-1}\right) + \mu_{t-1} q_{t-1}^{n+1} - q_{t-1}. \tag{38}$$

Now let $\xi_t := \mu_t q_t^{n+1} - q_t$. Notice that the first definition give (20). Then we can rewrite (37) and (38) as

$$(n+1)\,\xi_{t-1} = n\left(q_{t-1}^n - 1\right) q_t - q_t^n q_{t-1}, \tag{39}$$

and

$$\xi_t = \xi_{t-1} + q_t q_{t-1}\left(q_t^{n-1} - q_{t-1}^{n-1}\right). \tag{40}$$

We then combine (39) and (40) to get

$$n\left(q_t^n - 1\right) q_{t+1} - q_{t+1}^n q_t = n\left(q_{t-1}^n - 1\right) q_t - q_t^n q_{t-1} + (n+1)\, q_t q_{t-1}\left(q_t^{n-1} - q_{t-1}^{n-1}\right),$$

or rearranging,

$$q_{t+1}^n - n\frac{q_t^n - 1}{q_t} q_{t+1} + n\left(q_t^{n-1} q_{t-1} - 1\right) - q_{t-1}^n = 0, \tag{41}$$

which holds for $t \geq 1$ provided we adopt the convention $q_0 = 1$ and recall that $q_1 = (n+1)^{1/n}$. This gives us the difference equation given by (17).

Observe now that the sequence $\{\gamma_t\}$ is in $(0,1)$ if and only if the sequence $\{q_t\}$ is strictly increasing. The following lemma establishes that this is the case:

Lemma 3. *Consider the polynomial P defined by*

$$P(x) := x^n - n\frac{q_t^n - 1}{q_t} x + n\left(q_t^{n-1} q_{t-1} - 1\right) - q_{t-1}^n. \tag{42}$$

For each $q_{t-1} < q_t$ with $q_t > 1$, P admits a unique real root strictly larger than q_t.

Proof. Assume throughout that $q_{t-1} < q_t$. The polynomial P has two real roots if n is even, and three if n is odd. To see this, observe that for n even, it is a convex function that is negative for $x = q_t$, since

$$\begin{aligned} P(q_t) &= q_t^n - n\frac{q_t^n - 1}{q_t}q_t + n\left(q_t^{n-1}q_{t-1} - 1\right) - q_{t-1}^n \quad \leq \quad 0 \\ &\Leftrightarrow \quad q_t^n - q_{t-1}^n \leq nq_t^{n-1}\left(q_t - q_{t-1}\right), \end{aligned}$$

which is the case since the function $x \mapsto x^n$ is convex for $n \geq 2$. Observe that this also establishes that P admits a real root larger than q_t. If n is odd, then P is concave on $\mathbb{R}_-$ and convex on $\mathbb{R}_+$. Further, $P(0) = n\left(q_t^{n-1}q_{t-1} - 1\right) - q_{t-1}^n \geq 0$, and (as noted) $P(q_t) \leq 0$. So, in all cases, P uniquely admits a real root x that is strictly larger than q_t. $\square$

This ensures that each γ_t is contained in the interval $(0, 10$, which in turn ensures there is a unique equilibrium. $\blacksquare$

A.6 Proof of Proposition 5

A.6.1 Characterizing v_t

We investigate the sequence of critical valuations $\{v_t\}$, leading to the demonstration of Proposition 5.1 and 5.2. The heart of the argument is contained in the following three lemmas. Let $x(q_t, q_{t-1})$ denote the unique root larger than q_t solving (42).

Lemma 4.

(4.1) The root $x(q_t, q_{t-1})$ is contained in $(q_t, q_t + (q_t - q_{t-1}))$.

(4.2) For $q_{t-1} < q_t$, $x(q_t, q_{t-1})$ is strictly decreasing in q_{t-1}, and holding q_t/q_{t-1} fixed, the ratio $x(q_t, q_{t-1})/q_t$ is an increasing function of q_t.

Recall that $q_t = \frac{v_{t+1}}{v_1}$. Hence, Lemma 4.1 indicates that as the seller moves up the interval of possible buyer valuations (i.e., moves earlier in the sequence of periods $(T_\Delta, T_\Delta - 1, \ldots, 1)$), she slices off smaller and smaller intervals of buyer valuations to which to sell: $v_t - v_{t-1}$ is decreasing in t. Intuitively, the seller discriminates more finely among higher-valuation buyers. Lemma 4.2 assembles some technical results to be used in proving Lemma 5.

Proof. For (4.1), let $q_{t-1} = q(1 - \alpha)$, for some $\alpha \in (0, 1)$ and $q \geq 1$, $q_t = q$ and consider $P(q(1 + \alpha))$. Now,

$$P(q(1 + \alpha)) = (1 + \alpha)^n q^n - (1 - \alpha)nq^n + n\alpha(1 - 2q^n) > 0,$$

because

$$(1 + \alpha)^n - (1 - \alpha)^n > 2n\alpha,$$

as the left-hand side is convex in α with derivative equal to $2n$ at $\alpha = 0$. Therefore, it must be that $q(1+\alpha) > x$ and so $x - q_t \leq q_t - q_{t-1}$.

The first part of (4.2) is immediate, since $dP/dq_{t-1} > 0$. As for the second part, observe that we can rewrite (41) as

$$r_t^n - r_{t-1}^{-n} - n(r_t - r_{t-1}^{-1}) - \frac{n}{q_t^n}(1 - r_t) = 0,$$

where $r_t := q_{t+1}/q_t$ for all t. Fixing r_{t-1}, it follows that r_t is increasing in q_t, since the left-hand side is increasing in r_t (note that $r_t > 1$) and decreasing in q_t. □

Lemma 5. *Consider a sequence u_t with $q_0 = u_0$, $q_1 \geq u_1$, and for every $t \geq 2$, $u_{t+1} \leq x(u_t, u_{t-1})$. Then $q_t \geq u_t$ for all t.*

Proof. The proof is by induction on t. Observe that, for $t = 1$, by construction both $q_1 \geq u_1$ and $q_1/q_0 \geq u_1/u_0$. Assume now that, for some $t \geq 1$, both $q_\tau \geq u_\tau$ and $q_\tau/q_{\tau-1} \geq u_\tau/u_{\tau-1}$ for all $\tau \leq t$. It follows that

$$\frac{q_{t+1}}{q_t} = \frac{x(q_t, q_{t-1})}{q_t} \geq \frac{x(u_t, \frac{u_t}{q_t} q_{t-1})}{u_t} \geq \frac{x(u_t, u_{t-1})}{u_t} \geq \frac{u_{t+1}}{u_t}.$$

The first inequality follows from the second part of Lemma 4.2, given that $u_t \leq q_t$. The second inequality follows from the facts that $\frac{u_t}{q_t} q_{t-1} \leq u_{t-1}$ (by the induction hypothesis) and $x(q_t, q_{t-1})$ is decreasing in its second argument (the first part of Lemma 4.2). The final inequality follows the fact that $x(u_t, u_{t-1})$ is an upper bound on u_{t+1}. Since $q_0 = u_0$, the conclusion that $q_{t+1} \geq u_{t+1}$ follows from this inequality and the induction hypothesis. □

Lemma 6. *Consider the sequence $\{u_t\}_{t=0}^{\infty}$ defined by $u_t = (1 + n(t-1)t/2)^{1/n}$, for all $t \geq 0$. The sequence u_t diverges and, for all $t \geq 1$ and all $n \geq 6$, $u_t \leq q_t$.*

Proof. Divergence is immediate from the definition of u_t. We can calculate that $u_0 = 1 = q_0$ and $u_1 = 1 < (n+1)^{\frac{1}{n}} = q_1$. The result then follows from Lemma 5 and the fact that, for every $t \geq 2$, $u_{t+1} \leq x(u_t, u_{t-1})$. This last inequality is established via a tedious calculation. Details are presented in Section B.1. □

Establishing statements (5.1) and (5.2) of Proposition 5 is now straightforward. Recall that, in an optimal auction with zero reserve price, the expected revenue is given by

$$\pi^D(n) = \frac{n-1}{n+1}.$$

This value is therefore an upper bound on the expected revenue that the seller can hope for in the dynamic game as $\Delta \to 0$, *if* $\lim_{\Delta \to 0} v_{\Delta 1} = 0$, or equivalently $\lim_{t \to \infty} q_t = \infty$. For $n \geq 6$, it follows from Lemma 5 that $\lim_{t \to \infty} q_t = \infty$ and hence $\lim_{\Delta \to 0} v_{\Delta 1} = 0$. The best the seller can hope for, as $\Delta \to 0$, is therefore $\pi^D(n)$. Because $q_t - q_{t-1}$ is decreasing in t (Lemma 4.1), it is bounded, and therefore $\lim_{\Delta\to 0} \max_{t \le T_\Delta} v_{\Delta t} - v_{\Delta,t-1} = 0$, and so also $\lim_{\Delta\to 0} \max_{t\le T_\Delta} p_{\Delta t} - p_{\Delta,t-1} = 0$, where $p_{\Delta t}$ is the price charged with t periods to go in the game with period Δ and hence T_Δ stages. It then follows from Proposition 1 in Chwe [15] that the expected revenue converges to $\pi^D(n)$. This gives the second conclusion of Proposition 5.

What if $n < 6$? We can explicitly compute the first terms of μ_t for $n \in \{2, \ldots, 5\}$, and observe that $\mu_t > \pi^D(n)$ for $t = 1$ if $n = 2, 3$, $t = 4$ if $n = 4$, and $t = 36$ if $n = 5$. Since one feasible strategy for the seller is to set $p_\tau = 1$ until period $t = 1$ (if $n = 2, 3$), $t = 4$ (if $n = 4$) or $t = 36$ (if $n = 5$) and then obtain value μ_t, the seller's optimal strategy must give a payoff exceeding $\pi^D(n)$, and hence $\lim_{\Delta\to 0} \mu_{T_\Delta} > \pi^D(n)$. The preceding argument establishes that a necessary condition for such a limiting payoff is that $\lim_{\Delta\to 0} v_{\Delta 1} > 0$. This establishes the first part of Proposition 5.

A.6.2 Declining Terminal Prices

We prove here that $\lim_{\Delta\to 0} v_{\Delta 1}$ is decreasing in n, giving Proposition 5.3. In particular, $\lim_{\Delta\to 0} v_{\Delta 1}$ is then lower than the last price quoted in an optimal auction with commitment, as the reserve price (which is the limit of the lowest price in the dynamic game with commitment) equals 1/2, which is $\lim_{\Delta\to 0} v_{\Delta 1}$ when $n = 1$.

The result is proved in several steps. First, recall that $v_1 = \gamma_1 v_2$, where $\gamma_1 = (n+1)^{-n}$. Now consider the following auction, parameterized by v. First, the auctioneer continuously lowers the price until the indifferent type is v. At this stage, if the unit is still not accepted, she offers the price $w = \gamma_1 v$, i.e. the monopoly price on the residual demand. If it is also rejected, the auction is over. We may compute the revenue from such an auction by first computing the probability $q(x)$ that a buyer of type x wins the object. This equals 0 if $x < w$, $(v^n - w^n)/(n(v-w))$ if $x \in [w, v)$, and x^{n-1} for $x \ge v$. The price that type x accepts is as usual $p(x) = q(x) - \int_0^x q(t)dt/q(x)$, and expected revenue $R_n(w)$, which equals $\int_0^1 p(x)dF^n(x)$, is then

$$R_n(w) = \frac{n-1}{n+1} - (n((n+1)^{1/n} - 1)w - 1)w^n,$$

which is a function of w that is increasing up to $((n+1)^{1+1/n} - n - 1)^{-1}$, and then decreasing.

Consider $n = 2, \ldots, 5$. We first claim that, given $w = \lim_{\Delta\to 0} v_{\Delta 1}$, the revenue $R_n(w)$ exceeds the limiting revenue from the equilibrium of the dynamic game (as $\Delta \to 0$). Indeed, consider the two allocations corresponding to each mechanism, the auction described above, and the allocation from the limit. In both cases, buyers' types below w do not get the unit; types in $[w, v)$ get it only if there is no type above v, with the same probability in both cases ($v = \lim_{\Delta\to 0} v_{\Delta 2}$, since the price in the last period is the monopoly price on the residual demand). So the difference originates from types above v. However, for such types, the auction described above achieves an efficient allocation, while this is not necessarily true in the other case. Since with a uniform distribution, the virtual valuation is strictly increasing in types, it follows that $R_n(w)$ exceeds the revenue from the limit of the dynamic game, and hence from the dynamic game, independently of the length of the horizon (since the seller's payoff increases with T).

By considering the first terms of the sequences μ_t (recall that it is a non-decreasing sequence) we obtain that $\lim_{\Delta\to 0} \mu_{T_\Delta} > 4/10$ for $n = 2$, $\lim_{\Delta\to 0} \mu_{T_\Delta} > .515$ for $n = 3$, and $\lim_{\Delta\to 0} \mu_{T_\Delta} > .6019$ for $n = 4$. Yet $R_n(w)$ exceeds those values only if $w > 4/10$ (for $n = 2$), $w > .32$ (for $n = 3$) and $w > .24$ (for $n = 4$). Since the sequence $1/q_t$ is decreasing, with $\lim_{t\to\infty} 1/q_t = \lim_{\Delta\to 0} v_{\Delta 1}$, it is now easy to verify that, after computing the first few terms, $\lim_{t\to 0} 1/q_t$ is less than .4 for $n = 3$, less than .32 for $n = 4$ and less than .2 for $n = 5$. It follows that $\lim_{\Delta\to 0} v_{\Delta 1}$ is decreasing in n for $n = 3, 4, 5$. Since this limit is 0 for $n \geq 6$, the same holds for all $n > 2$, and clearly the conclusion also holds for $n = 1$ and $n = 2$ (in the former case, the only price accepted with positive probability is 1/2, while in the latter case, by computing the first few terms, it is verified that $\lim_{\Delta\to 0} v_{\Delta 1} < 1/2$.) ■

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B Appendix: Not for Publication

B.1 Details, Proof of Lemma 6

Our purpose is to prove that, for all $t \geq 2$ and $n \geq 6$,

$$(1 + nt(t+1)/2)^{\frac{1}{n}} \leq x((1 + n(t-1)t/2)^{\frac{1}{n}}, (1 + n(t-2)(t-1)/2)^{\frac{1}{n}}).$$

or, equivalently, for all $t \geq 1$ and $n \geq 6$,

$$(1 + n(t+1)(t+2)/2)^{\frac{1}{n}} \leq x((1 + nt(t+1)/2)^{\frac{1}{n}}, (1 + n(t-1)(t-2)/2)^{\frac{1}{n}}).$$

(At this point, letting $x = 1/t$ and $y = 1/n$, one can rewrite this inequality as a function on the unit square and then gain some confidence in its veracity by using a program such as *Mathematica* to plot it.) Upon manipulation, this is equivalent to showing that, for all $t \geq 1$, $n \geq 6$,

$$4t\left(2 + nt\left(t+1\right)\right)^{1/n} + \left(2 + nt\left(t+1\right)\right)\left(2 + n\left(t-1\right)t\right)^{1/n} - nt\left(t+1\right)\left(2 + n\left(t+1\right)\left(t+2\right)\right)^{1/n} \leq 0,$$

or

$$\left(1 + \frac{n\left(t+1\right)}{1 + \frac{nt(t+1)}{2}}\right)^{1/n} - \left(\frac{2}{nt\left(t+1\right)} + 1\right)\left(1 - \frac{nt}{1 + \frac{nt(t+1)}{2}}\right)^{1/n} - \frac{4}{n\left(t+1\right)} \geq 0. \qquad (43)$$

This will be done in two steps.

B.1.1 The Case $t = 1$

In that case, we must show that

$$g_L\left(n\right) := n\left(\left(1 + 3n\right)^{1/n} - 1\right) \geq 2\left(1 + n\right)^{1/n} + 1 =: g_R\left(n\right).$$

Observe that, for $x > 0$,

$$\frac{d}{dx}\left(x \ln\left(1 + x^{-1}\right)\right) = \ln\left(1 + \frac{1}{x}\right) - \frac{1}{1+x} \geq 0,$$

where the last step follows from the standard inequality $\ln x \geq (1+x)^{-1}$ applied to $1/x$. It follows that g_R is decreasing in n.

Consider now the function g_L. Its second derivative with respect to n is

$$\frac{\left(1 + 3n\right)^{\frac{1}{n} - 2}}{n^3}\lambda\left(n\right),$$

where

$$\lambda(n) = (1+3n)\ln(1+3n)\left((1+3n)\ln(1+3n) - 6n\right) - 9(n-1)n^2.$$

We claim that λ is negative $\forall n \geq 1$. To see this, observe first that

$$\frac{d^3\lambda}{dn^3} = \frac{-54}{(3n+1)^2}\left(1+3n+9n^2 - 2(1+3n)\ln(1+3n)\right) < 0,$$

because

$$1+3n+9n^2 \geq 2(1+3n)\ln(1+3n),$$

which is because, from the standard inequality $\ln\left(1+\frac{1}{x}\right) \leq \frac{1}{\sqrt{x^2+x}}$, it follows that $\ln(1+3n) \leq 3n/\sqrt{1+3n}$. Taking squares in the resulting inequality and collecting terms yield the desired result.

Therefore,

$$\frac{d^2\lambda}{dn^2} = 18\left(\frac{1}{1+3n} + \ln(1+3n) + \ln^2(1+3n) - (1+3n)\right)$$

is decreasing, and it is negative for $n=1$, so it is negative for all $n \geq 1$.

In turn, this implies that

$$\frac{d\lambda}{dn} = 3\left(2(1+3n)\ln^2(1+3n) - 9n^2 - 6n\ln(1+3n)\right)$$

is decreasing, and it is negative for $n=1$, so it is negative for all $n \geq 1$. Repeating once more the argument, this establishes that λ is decreasing, and again it is negative for $n=1$, and therefore for all $n \geq 1$.

We have now established that $d^2g_L/dn^2 \leq 0$ for all $n \geq 1$. Thus, dg^L/dn is decreasing in n. However, $\lim_{n\to\infty} dg^L/dn = 0$, and so $dg^L/dn \geq 0$. This proves that g^L is an increasing function.

This part of the proof is concluded by observing that $g^L(6) > g^R(6)$. Since g^L is increasing, while g^R is decreasing, the inequality follows for all $n \geq 6$.

B.1.2 The General Case, $t > 1$

B.1.2a A Sufficient Inequality. Recall that, from Taylor's theorem,

$$(1+x)^{1/n} \geq 1 + \frac{x}{n} - \frac{n-1}{2n^2}x^2 + \frac{(n-1)(2n-1)}{6n^3}x^3 - \frac{(n-1)(2n-1)(3n-1)}{24n^4}x^4$$
$$+\frac{(n-1)(2n-1)(3n-1)(4n-1)}{120n^5}x^5 - \frac{(n-1)(2n-1)(3n-1)(4n-1)(5n-1)}{720n^6}x^6,$$

and similarly,

$$(1-x)^{1/n} \leq 1 - \frac{x}{n} - \frac{n-1}{2n^2}x^2 - \frac{(n-1)(2n-1)}{6n^3}x^3 - \frac{(n-1)(2n-1)(3n-1)}{24n^4}x^4$$
$$-\frac{(n-1)(2n-1)(3n-1)(4n-1)}{120n^5}x^5 - \frac{(n-1)(2n-1)(3n-1)(4n-1)(5n-1)}{720n^6}x^6.$$

We now apply these two bounds to the left side of (43), inserting $x = \frac{n(t+1)}{1+\frac{nt(t+1)}{2}}$ and $x = nt/\left(1 + \frac{nt(t+1)}{2}\right)$ respectively. We obtain a rational function whose denominator is positive (being a square) and whose numerator is twice the following polynomial in n of degree 6:

$$a_6 n^6 + a_5 n^5 + a_4 n^4 + a_3 n^3 + a_2 n^2 + a_1 n + a_0,$$

with

$$\begin{aligned}
a_0 &= -4t^6 + 24t^5 - 120t^4 + 480t^3 - 1440t^2 - 2880t - 2880,\\
a_1 &= 48t^7 + 78t^6 + 1330t^5 + 670t^4 - 7818t^3 - 9454t^2 - 6166t,\\
a_2 &= 12t^9 + 24t^8 + 852t^7 + 890t^6 - 9240t^5 - 23184t^4 - 21588t^3 - 13104t^2 - 1950t,\\
a_3 &= 360t^9 + 990t^8 - 3030t^7 - 12645t^6 - 15635t^5 - 4805t^4 + 3285t^3 + 5990t^2 + 1730t,\\
a_4 &= 60t^{11} + 240t^{10} - 930t^9 - 5370t^8 - 11580t^7 - 15376t^6\\
&\quad -16824t^5 - 19620t^4 - 14730t^3 - 6960t^2 - 1410t,\\
a_5 &= -180t^{11} - 945t^{10} - 2115t^9 - 2610t^8 - 168t^7 + 5322t^6\\
&\quad +11830t^5 + 16105t^4 + 12093t^3 + 4904t^2 + 836t,\\
a_6 &= 3t(1+t)(1+2t)(-80 + t(1+t)(-272 + t(1+t)(-126 + t(1+t)(8 + 5t(1+t))))).
\end{aligned}$$

We must show that this polynomial is positive.

B.1.2b Preliminary Observations. Observe first that a_6 is positive for $t \geq 2$. Indeed, the last factor is a polynomial of degree 4 in $x = t(1+t)$, namely

$$-80 - 272x - 126x^2 + 8x^3 + 5x^4.$$

Since the coefficients change signs only once, Descartes' rule implies that there is at most one strictly positive root. Since this polynomial is negative when evaluated at $x = 0$, and positive when evaluated at $x = 6$ (i.e. $t = 2$), the root must be in $(0, 2)$, and so the polynomial is positive for all $t \geq 2$.

Observe that, by Descartes' rule, a_1 can have at most one strictly positive root. The coefficient a_1 is negative for $t = 2$ and positive for $t = 3$, so that the unique root is in $(2, 3)$, and so a_1 is negative for $t \geq 2$. Similarly, a_2 can have at most one strictly positive root.

The coefficient a_2 is negative for $t = 3$ and positive for $t = 4$, so the unique root is in $(3, 4)$, and so $a_2 > 0$ for $t \geq 4$. Similarly, a_3 can have at most two strictly positive roots. Further, the signs of a_3 at $t = 1/2$, $t = 1$ and $t = 4$ alternate, so that here again, there is no root for $t \geq 4$, and so $a_3 > 0$ for $t \geq 4$. By the same method, a_4 can have at most one strictly positive root, and a_4 is negative for $t = 4$ and positive for $t = 5$, so $a_4 > 0$ for $t \geq 5$. Finally, a_5 can have at most one strictly positive root, and it is positive at $t = 1$ and negative at $t = 2$, so it is strictly negative for $t \geq 2$.

We need two further facts. First, $-a_5 > -a_0$ for $t \geq 2$. To see this, let us compute the difference

$$\begin{aligned} a_5 - a_0 \quad = \quad & -180t^{11} - 945t^{10} - 2115t^9 - 1610t^8 - 168t^7 + 5326t^6 + 11806t^5 \\ & +16225t^4 + 11613t^3 + 6344t^2 + 3716t + 2880, \end{aligned}$$

so, again by Descartes rule, there can be at most one positive root of the difference, and the difference is positive for $t = 1$ and negative for $t = 2$, and so this difference is negative for $t \geq 2$.

Second, we claim that $-a_5/a_6$ is increasing for $t \geq 4$. To see this, observe that the derivative of the ratio a_5/a_6 is equal to the ratio of the following numerator, over a denominator which is positive since it is a square,

$$\begin{aligned} & -150t^{17} - 2820t^{16} - 19500t^{15} - 363160t^{14} + 129880t^{13} + 933852t^{12} + 2769050t^{11} \\ & +53161174t^{10} + 6507696t^9 + 5494474t^8 + 3239750t^7 + 2186194t^6 + 2877454t^6 \\ & +3504246t^5 + 2839892t^4 + 1532112t^3 + 550712t^2 + 120816t + 11904, \end{aligned}$$

so it has at most one strictly positive root, and it is positive for $t = 3$ and negative for $t = 4$. So the ratio $-a_5/a_6$ is increasing for $t \geq 4$ and so always less than its limit, which equals

$$\lim_{t\to\infty} -\frac{a_5}{a_6} = 6.$$

B.1.2c The Result For $n > 6$. We are now ready to get our result, at least in the case $n > 6$ for now. We use the Lagrange-McLaurin theorem.[25] Given some polynomial of degree n, with real coefficients $\{a_i\}$, let $m = \sup\{i | a_i < 0\}$, and $B = \sup\{-a_i | a_i < 0\}$. Then any real root r of the polynomial satisfies

$$r < 1 + \left(\frac{B}{a_n}\right)^{\frac{1}{n-m}}.$$

[25] Riccardo Benedetti and Jean-Jacques Risler, *Real algebraic and semi-algebraic sets* (Hermann, Paris, 1990), Theorem 1.2.2.).

Given our previous analysis, it follows that, applying the theorem to the polynomial in n for $t \geq 5$, any real root is less than

$$1 - \frac{a_5}{a_6} < 7.$$

This establishes the inequality (*) for the case $n > 6$ and $t \geq 5$. For $n > 6$ but for each $t = 2, 3, 4$, we can compute

$$1 - \max\left\{-\frac{a_0}{a_6}, -\frac{a_1}{a_6}, -\frac{a_2}{a_6}, -\frac{a_3}{a_6}, -\frac{a_4}{a_6}, -\frac{a_5}{a_6}\right\},$$

which of course is independent of n. It is still less than 7 for both $t = 3, 4$. In both cases, the maximum is achieved by $-a_5/a_6$. In the case $t = 2$, the maximum is achieved by $-a_4/a_6$, and in that case the bound on the root is only $n < 30$. However, we can directly verify that for $t = 2$ and each value $n = 7, \ldots, 30$, the polynomial is positive.

B.1.2d The result for $n = 6$. We are left with proving the result for the case $n = 6$. Plugging into the polynomial in n, we obtain the following polynomial in t,

$$\begin{gathered}155520t^{11} + 1321920t^{10} + 4324320t^9 + 8065440t^8 - 40593600t^7 - 168237440t^6 \\ -321927240t^5 - 358960440t^4 - 234969960t^3 - 83572680t^2 - 12520920t - 5760.\end{gathered}$$

Once more, by Descartes' rule, there can be at most one strictly positive root, and since this polynomial is negative for $t = 2$, and positive for $t = 3$, we are done —except for the case $n = 6$ and $t = 2$. Evaluating the original inequality for that one case concludes the proof.

B.2 Proof of Propositions 6 and 7

The proofs of Propositions 7 and 6 have much in common. The commitment argument is simpler, and it is accordingly easiest to first establish Proposition 7, and then extend the argument to the noncommitment case of Proposition 6.

B.2.1 Proof of Proposition 7

B.2.1a The Seller's Payoff with Commitment We first express the seller's payoff in terms of the indifferent buyers' valuations. Fix a period length Δ and hence number of periods T_Δ, and then suppress Δ in the notation. The seller's payoff with commitment can be written as

$$\Pi = (1 - v_T^n)\, p_T + \left(v_T^n - v_{T-1}^n\right) p_{T-1} + \cdots + (v_2^n - v_1^n)\, p_1, \tag{44}$$

where

$$\begin{aligned}\frac{v_{t+1}^n - v_t^n}{v_{t+1} - v_t}(v_t - p_t) &= \frac{v_t^n - v_{t-1}^n}{v_t - v_{t-1}}(v_t - p_{t-1}) = v_t^n - v_{t-1}^n + \frac{v_t^n - v_{t-1}^n}{v_t - v_{t-1}}(v_{t-1} - p_{t-1}) \\ &= v_t^n - v_{t-2}^n + \frac{v_{t-1}^n - v_{t-2}^n}{v_{t-1} - v_{t-2}}(v_{t-2} - p_{t-2}) = \cdots = v_t^n - v_1^n,\end{aligned}$$

so that

$$\left(v_{t+1}^n - v_t^n\right) p_t = \left(v_{t+1}^n - v_t^n\right) v_t - (v_{t+1} - v_t)\left(v_t^n - v_1^n\right). \tag{45}$$

Substituting (45) into (44), we have

$$\begin{aligned}\Pi &= (1 - v_T^n)\, v_T + \left(v_T^n - v_{T-1}^n\right) v_{T-1} + \cdots + (v_2^n - v_1^n)\, v_1 \\ &\quad - (1 - v_T)\left(v_T^n - v_1^n\right) - (v_{T-1} - v_T)\left(v_{T-1}^n - v_1^n\right) - \cdots - (v_3 - v_2)\left(v_2^n - v_1^n\right) \\ &= v_T - (1 - v_{T-1})\, v_T^n - (v_T - v_{T-2})\, v_{T-1}^n - \cdots - (v_3 - v_1)\, v_2^n + (1 - v_2 - v_1)\, v_1^n.\end{aligned}$$

We can think of the seller as choosing the identities of the indifferent buyers in order to maximize this payoff. Taking derivatives with respect to these valuations (and setting $v_{T+1} = 1$), we obtain the first-order conditions

$$n v_t^{n-1} = \frac{v_{t+1}^n - v_{t-1}^n}{v_{t+1} - v_{t-1}} \quad (t = 2, \ldots, T), \tag{46}$$

$$n v_1^{n-1}(1 - v_2 - v_1) = v_2^n - v_1^n. \tag{47}$$

The first formula can be re-written as

$$\sigma_t^n - n\sigma_t = \sigma_{t-1}^{-n} - n\sigma_{t-1}^{-1},$$

where $\sigma_t := v_{t+1}/v_t$.

B.2.1b Two Preliminary Inequalities This section collects two useful technical results.

Lemma 7. *Let $h(x) := x^n - nx$. Then, for $n \geq 2$,*

$$h(2 - x) \geq h(x) \quad (x \in [0, 1]), \text{ and } \lim_{x \uparrow 1} \frac{h^{-1} \circ h(x) - 1}{x - 1} = -1, \tag{48}$$

where h^{-1} is the inverse of $h : [0, \infty) \to \mathbb{R}$.

Proof. Because the function $y \mapsto y^n$ is convex,

$$(1 + y)^n - (1 - y)^n \geq 2ny,$$

for $y \in [0,1]$, so that, for $x = 1 - y$,

$$(2-x)^n - n(2-x) \geq x^n - nx,$$

i.e. $h(2-x) \geq h(x)$. Now, observe that the limit is simply the derivative of $h^{-1} \circ h(x)$ at 1. Because $h'(1) = 0$,

$$h(1-\varepsilon) - h(1) = \frac{h''(1)}{2}\varepsilon^2 + o\left(\varepsilon^3\right), h(1+\delta) - h(1) = \frac{h''(1)}{2}\delta^2 + o\left(\delta^3\right),$$

and so, if $h(1-\varepsilon) = h(1+\delta) \to h(1)$, it follows that $\varepsilon/\delta \to 1$, so that $\left(h^{-1} \circ h\right)'(1) = 1$. □

Lemma 8. *For all $n \geq 2$, there exists $K > 0$ such that, for all $t \geq 1$,*

$$h\left(\left(1 + \frac{1}{t+K}\right)^{-\frac{3}{n+1}}\right) \geq h\left(\left(1 + \frac{1}{t+1+K}\right)^{\frac{3}{n+1}}\right). \tag{49}$$

Proof. For $n = 2$, it is easy to verify that the two sides are equal, independently of the value of K. Consider $n > 2$. Taking a Taylor expansion, we have that

$$h\left((1+y)^{-\frac{3}{n+1}}\right) - h\left(\left(1 + \frac{y}{1+y}\right)^{\frac{3}{n+1}}\right) = \frac{3n(n-1)(n-2)(2n-1)}{5}y^5 + o\left(y^6\right),$$

so that there exists $\bar{y}$ such that, for all $y \in [0, \bar{y}]$,

$$h\left((1+y)^{-\frac{3}{n+1}}\right) \geq h\left(\left(1 + \frac{y}{1+y}\right)^{\frac{3}{n+1}}\right).$$

Letting $K := \bar{y}^{-1} - 1$, the result follows. □

B.2.1c Properties of the Commitment Solution We now use these inequalities to characterize the sequence $\{v_t\}_{t=1}^{\infty}$ of critical buyer types.[26] Fix $v_1 \in (0,1)$ and $\sigma_1 > 1$ and consider the sequence $\{v_t\}_{t=1}^{\infty}$ defined by v_1, σ_1 and

$$\sigma_t^n - n\sigma_t = \sigma_{t-1}^{-n} - n\sigma_{t-1}^{-1}, \text{ i.e. } h(\sigma_t) = h\left(\sigma_{t-1}^{-1}\right),$$

for $h(x) = x^n - nx$. Observe that, since h is decreasing on $[0,1]$, and increasing on $[1,\infty)$, $\sigma_t \geq 1$ for all t. Further, because $h(x) \geq h\left(x^{-1}\right)$ for all $x \geq 1$, it is strictly decreasing in t, with limit given by 1.

[26] For a fixed Δ, only the first T_Δ terms in the infinite sequence we study will be relevant, but the entire infinite sequence will come into play as $\Delta \to 0$.

Lemma 9. *1. For all n, the sequence $\{v_t\}$ is concave, with*

$$\lim_{t\to\infty} \frac{v_{t+1}-v_t}{v_t-v_{t-1}} = 1.$$

2. For all n, there exists $K>0$ such that

$$\sigma_t \geq \left(1+\frac{1}{t+K}\right)^{\frac{3}{n+1}}.$$

3. For all n, and $m \in \mathbb{N}$,

$$\underline{\lim}_{t\to\infty} \frac{v_{mt}}{v_t} \geq m^{\frac{3}{n+1}}.$$

We use Lemma 9.2 in the proof of Lemma 9.3, and use Lemmas 9.1 and Lemma 9.3 in Section A.4.3.

Proof. First, observe that

$$v_{t+1} - v_t \leq v_t - v_{t-1} \Leftrightarrow \sigma_t \leq 2 - \sigma_{t-1}^{-1},$$

for $\sigma_t = v_{t+1}/v_t$. Now

$$h(\sigma_t) = h\left(\sigma_{t-1}^{-1}\right) \leq h\left(2-\sigma_{t-1}^{-1}\right),$$

where the last inequality follows from (48), given that $\sigma_{t-1}^{-1} \leq 1$. Since h is increasing for $x \geq 1$, and both $\sigma_t \geq 1$ and $2-\sigma_{t-1}^{-1} \geq 1$, it follows that indeed $\sigma_t \leq 2-\sigma_{t-1}^{-1}$, so that the sequence v_t is concave. Further, since

$$\frac{v_{t+1}-v_t}{v_t-v_{t-1}} = \frac{\sigma_t - 1}{1-\sigma_{t-1}^{-1}} = \frac{h^{-1}\circ h\left(\sigma_{t-1}^{-1}\right)-1}{1-\sigma_{t-1}^{-1}},$$

and $\lim_t \sigma_t = 1$, it follows from $\lim_{x\uparrow 1}\left(h^{-1}\circ h(x) - 1\right)/(1-x) = 1$ that

$$\lim_t \left(v_{t+1}-v_t\right)/\left(v_t - v_{t-1}\right) = 1.$$

Given σ_1, fix K such that both $\sigma_1 \geq \left(1+\frac{1}{1+K}\right)^{\frac{3}{n+1}}$ and (49) is satisfied. Let

$$\nu_t := \left(1+\frac{1}{t+K}\right)^{\frac{3}{n+1}}.$$

By induction, we show that $\sigma_t \geq \nu_t$. By definition of K, $\sigma_1 \geq \nu_1$. Suppose now that $\sigma_{t-1} \geq \nu_{t-1}$. Since h is decreasing on $[0,1]$, and given (49),

$$h(\sigma_t) = h\left(\sigma_{t-1}^{-1}\right) \geq h\left(\nu_{t-1}^{-1}\right) \geq h(\nu_t),$$

and since h is increasing on $[1, \infty)$,

$$\sigma_t \geq \nu_t.$$

Observe that

$$\frac{v_{mt}}{v_t} = \Pi_{\tau=t}^{mt-1} \sigma_\tau \geq \Pi_{\tau=t}^{mt-1} v_\tau = \left(\frac{mt+K}{t+K} \right)^{\frac{3}{n+1}},$$

so that

$$\underline{\lim}_t \frac{v_{mt}}{v_t} \geq m^{\frac{3}{n+1}}.$$

$\square$

Lemma 9 tells us about the sequence $\{v_t\}_{t=1}^{\infty}$ given a value v_1. We must next identify the appropriate value v_1. One strategy available to the seller is to set a price with t periods to go equal to $\frac{1+t/T}{2}$, causing v_1 to converge to $\frac{1}{2}$ as Δ gets small (and hence T_Δ large). It follows from standard results (Athey [2]) that her revenue then converges to the revenue of the optimal auction. Conversely, her revenue converges to the revenue of the optimal auction only if $p_1 = v_1$ converges to $1/2$ as Δ gets small, allowing us to take $v_1 = \frac{1}{2}$. It follows from the first-order conditions (46)–(47) that v_2 then converges to v_1, so that asymptotically the entire sequence $\{v_t\}_{t=1}^{\infty}$ is contained in $[0, 1]$.

B.2.1d The Limit $\Delta \to 0$ We now consider the limit $\Delta \to 0$. Consider the sequence of functions $v_\Delta(x)$ on $[0, 1]$ defined as follows. For any period length Δ, define the step function

$$v_\Delta(x) = v_{\Delta t} \text{ for all } x \in \left[\frac{t-1}{T_\Delta}, \frac{t}{T_\Delta} \right), \; v_\Delta(1) = 1.$$

Pick a subsequence of functions $\{v_\Delta(x)\}$ that converges on the rationals, to some limit function. Because each sequence is non-decreasing, so must be the limit, and let $x \mapsto v(x)$ denote the right-continuous extension of this limit. Since the sequence $\{v_t\}$ is concave (Lemma 9.1), the function v must be concave, and it is therefore continuous on $(0, 1)$, and admits left- and right-derivatives everywhere on $(0, 1)$.

Because the sequence σ_t defined by a value of σ_1 and the recursion $h(\sigma_t) = h\left(\sigma_{t-1}^{-1}\right)$ is increasing in σ_1, and given that $v_{T_\Delta} = 1$, it follows that the value of σ_1 solving the commitment problem for fixed v_1 is decreasing in v_1. Since $\lim_{\Delta \to 0} v_1 = 1/2$, σ_1 is bounded above in Δ, so that, since for a fixed σ_1,

$$\lim_{t \to \infty} \frac{v_{t+1} - v_t}{v_t - v_{t-1}} = 1,$$

it follows also that, for all values $k > 0$ such that $kT \in \mathbb{N}$,

$$\lim_{T \to \infty} \frac{v_{kT+1} - v_{kT}}{v_{kT} - v_{kT-1}} = 1.$$

It follows that the left- and right derivatives of v agree everywhere, so that v is differentiable on $(0,1)$. Therefore, considering the equation

$$nv\left(x\right)^{n-1}\left(v\left(x+\delta\right)-v\left(x-\delta\right)\right)=\left(v\left(x+\delta\right)^{n}-v\left(x-\delta\right)^{n}\right),$$

we might use a Taylor expansion to the third degree as $\delta\to 0$, to obtain

$$n\left(n-1\right)v\left(x\right)^{n-3}v'\left(x\right)\left[v\left(x\right)v''\left(x\right)+\frac{\left(n-2\right)}{3}v'\left(x\right)^{2}\right]\delta^{3}+o\left(\delta^{4}\right).$$

Because $\underline{\lim}_{t\to\infty}\frac{v_{mt}}{v_t}\geq m^{\frac{3}{n+1}}$ for all m (Lemma 9.3), $v'\left(x\right)>0$. Hence it must be that

$$v\left(x\right)v''\left(x\right)+\frac{\left(n-2\right)}{3}v'\left(x\right)^{2}=0.$$

This differential equation has as general solution

$$v\left(x\right)=K_1\left(x+K_2\right)^{\frac{3}{n+1}},$$

for constants K_1, K_2, and our boundary conditions $v\left(1\right)=1/2$, $v\left(1\right)=1$ allow us to identify these constants, giving (24):

$$v\left(x\right)=\frac{1}{2}\left(\left(2^{\frac{n+1}{3}}-1\right)x+1\right)^{\frac{3}{n+1}}.$$

Since

$$\frac{v_{t+1}^{n}-v_{t}^{n}}{v_{t+1}-v_{t}}\left(v_t-p_t\right)=v_t^n-v_1^n,$$

and $\lim_{\varepsilon\to 0}\frac{v(x+\varepsilon)^n-v(x)^n}{v(x+\varepsilon)-v(x)}=nv(x)^{n-1}$, it follows that $nv(x)^{n-1}(v(x)-p(x))=v(x)^n-v(0)^n$, and the expression (25) for $p(x)$ follows. ■

B.2.2 Proof of Proposition 6

Our characterization of the non-commitment solution builds on our proof of Proposition 7. We first derive an asymptotic estimate of the sequence q_t/q_{t+1} (from (17). The polynomial (41) that defines q_t can be rewritten as

$$q_{t+1}^{n}-q_{t-1}^{n}=\frac{n}{q_t}(q_t^n(q_{t+1}-q_{t-1})-(q_{t+1}-q_t)).$$

Since the sequence q_t diverges, we may ignore the second term from the right-hand side, and so, defining $s_t=q_t/q_{t+1}$ (i.e., in terms of the notation of Section A.3.2, $s_t=r_t^{-1}$), we have, for large t,

$$s_t^{-n}-s_{t-1}^{n}-n(s_t^{-1}-s_{t-1})\approx 0.$$

As we also know that $s_t \to 1$, we let $s_t = 1 - \varepsilon_t$, and, so using Taylor expansions to the third order,

$$3\varepsilon_t^2 + (n+4)\varepsilon_t^3 - 3\varepsilon_{t-1}^2 + (n-2)\varepsilon_{t-1}^3 \approx 0.$$

Since $\varepsilon_t \to 0$, this implies that $\lambda_t := \varepsilon_t/\varepsilon_{t-1} \to 1$. Rewriting this equation, we have

$$3(\varepsilon_t - \varepsilon_{t-1})(1+\lambda_t)\varepsilon_{t-1} + ((n+4)\lambda_t^2 + (n-2)\lambda_t^{-1})\varepsilon_t\varepsilon_{t-1}^2 = 0,$$

so, approximately,

$$\varepsilon_t - \varepsilon_{t-1} + \frac{n+1}{3}\varepsilon_{t-1}\varepsilon_t = 0.$$

If we let $\mu_t = (n+1)\varepsilon_t/3$, this gives

$$\mu_{t-1} - \mu_t = \mu_t\mu_{t-1},$$

or

$$1/\mu_t - 1/\mu_{t-1} = 1,$$

so we obtain that $\mu_t = (t+C)^{-1}$, for a constant C (possibly infinite). That is, for large t, either $\varepsilon_t = 0$ or $\varepsilon_t = \frac{3}{n+1}t^{-1}$. However, recall that we already know (cf. Lemma 5) that

$$\frac{q_t}{q_{t+1}} \leq \frac{u_t}{u_{t+1}} = \left(1 - \frac{nt}{1 + \frac{nt(t+1)}{2}}\right)^{1/n} < 1 - \frac{n}{t},$$

and so the possibility that $\varepsilon_t = 0$ could be ruled out. We conclude that $s_t = 1 - \frac{3}{(n+1)t}$ asymptotically.[27]

It also follows that

$$\lim_t \frac{q_{t+1} - q_t}{q_t - q_{t-1}} = \lim_t \frac{s_t^{-1} - 1}{1 - s_{t-1}} = \lim \frac{t}{t-1} = 1.$$

Therefore, if we define, as in the case with commitment, the sequence of functions $v_\Delta(x)$ on $[0,1]$ as the step function

$$v_\Delta(x) = v_t \text{ for all } x \in \left[\frac{t-1}{T_\Delta}, \frac{t}{T_\Delta}\right),\ v_\Delta(1)(1) = 1,$$

[27]Observe that we made approximations sequentially in the process of deriving this solution. If we plug in our solution into the recursion involving s_t and s_{t-1}, we find that the second approximation is of the order $o(t^{-3})$, and the term that was ignored in the initial polynomial is of the same order, so the order of approximations is irrelevant. Also, observe that, since the term that is being ignored is of order $o(t^{-3})$, yet the slope of the function $s_t \mapsto s_t^{-n} - s_{t-1}^n - n(s_t^{-1} - s_{t-1})$ at 1 is equal to $o(t^{-1})$, the impact of the approximation is of the order $o(t^{-2})$, so that even the cumulative impact of the approximations is negligible, justifying the approximation.

and, following what has been done with commitment, we pick a subsequence of functions $\left\{v^T(x)\right\}$ that converges on the rationals, to some limit function (which, because each sequence is non-decreasing, is non-decreasing as well, as well as concave since the sequence q_t is), and we let $x \mapsto v(x)$ denote the right-continuous extension of this limit, it follows that the left- and right-derivatives coincide everywhere on $(0,1)$. Now,

$$v'(x) = \lim_{\Delta \to 0} \frac{v_{t+1} - v_t}{\Delta} = \lim_{\Delta \to 0} T \frac{q_t - q_{t-1}}{q_T} = \lim_{\Delta \to 0} \frac{T}{t} \frac{q_t}{q_T} t \frac{q_t - q_{t-1}}{q_t} = \frac{3}{n+1} \frac{v(x)}{x},$$

with boundary condition $v(1) = 1$. This gives $v(x) = x^{\frac{3}{n+1}}$, or (22). Since

$$\frac{v_{t+1}^n - v_t^n}{v_{t+1} - v_t} (v_t - p_t) = v_t^n - v_0^n,$$

the solution (23) for $p(x)$ follows.

B.3 Multiple Units

This section derives the price function (26) and payoff function (26). We provide the preliminary analysis for any number of units, and then specialize to the case of two units.

B.3.1 The buyer's indifference condition

As in the single-object argument, we begin by identifying indifferent buyers. Suppose that there are k units left. Define

$$\phi_t = k \sum_{j=0}^{n-1} \frac{\binom{n-1}{j}}{j+1} \gamma_t^{n-1-j} (1-\gamma_t)^j + \sum_{j=0}^{k-2} \binom{n-1}{j} \left(1 - \frac{k}{j+1}\right) \gamma_t^{n-1-j} (1-\gamma_t)^j,$$

where, as usual, $\gamma_t = v_t / v_{t+1}$. By accepting now, the buyer with valuation v_t gets

$$\phi_t (v_t - p_t).$$

By waiting one period instead, he gets

$$\gamma_t^{n-1} \phi_{t-1} (v_t - p_{t-1}) + \sum_{j=1}^{k-1} \binom{n-1}{j} (1-\gamma_t)^j \gamma_t^{n-1-j} W_{k-j,t} v_t,$$

where $W_{k-j,t}$ is the normalized expected payoff when only $k-j$ units are left (and the number of bidders has gone down to $n-j$) and t periods to go. Indifference requires the two to be equal. Observe that, defining

$$\phi_t (v_t - p_t) = M_t v_{t+1}, \tag{50}$$

the buyer's indifference condition becomes

$$M_t v_{t+1} = \gamma_t^{n-1}\phi_{t-1}(v_t - v_{t-1}) + \sum_{j=1}^{k-1}\binom{n-1}{j}(1-\gamma_t)^j \gamma_t^{n-1-j} W_{k-j,t} v_t + \gamma_t^{n-1} M_{t-1} v_t. \quad (51)$$

B.2.2 The sellers's maximization problem

The seller's payoff is

$$\begin{aligned} S_{t+1}v_{t+1} &= \max\left\{\gamma_t^n S_t v_t + \sum_{j=1}^{k-1}\binom{n}{j}(1-\gamma_t)^j\gamma_t^{n-j}(jp_t + Y_{k-j,t}v_t) + k\sum_{j=k}^{n}\binom{n}{j}(1-\gamma_t)^j\gamma_t^{n-j}p_t\right\} \\ &= \max\left\{\begin{array}{c} \gamma_t^n S_t v_t - \left[\sum_{j=1}^{k-1} j\binom{n}{j}(1-\gamma_t)^j\gamma_t^{n-j} + k\sum_{j=k}^{n}\binom{n}{j}(1-\gamma_t)^j\gamma_t^{n-j}\right](v_t - p_t) \\ +\sum_{j=1}^{k-1}\binom{n}{j}(1-\gamma_t)^j\gamma_t^{n-j}(Y_{k-j,t} - j)v_t - k\sum_{j=k}^{n}\binom{n}{j}(1-\gamma_t)^j\gamma_t^{n-j}v_t \end{array}\right\}, \end{aligned}$$

where $Y_{k-j,t}$ is the seller's normalized continuation payoff when only $k-j$ units are left, with t periods to go. Observe now that

$$\sum_{j=1}^{k-1} j\binom{n}{j}(1-\gamma_t)^j\gamma_t^{n-j} + k\sum_{j=k}^{n}\binom{n}{j}(1-\gamma_t)^j\gamma_t^{n-j} = n(1-\gamma_t)\phi_t,$$

so we may re-write the seller's payoff as

$$S_{t+1} = \max\left\{\begin{array}{c} \gamma_t^{n+1}S_t - n(1-\gamma_t)\left[\gamma_t^n\phi_{t-1}(1-\gamma_{t-1}) + \sum_{j=1}^{k-1}\binom{n-1}{j}(1-\gamma_t)^j\gamma_t^{n-j}W_{k-j,t} + \gamma_t^n M_{t-1}\right] \\ +\sum_{j=1}^{k-1}\binom{n}{j}(1-\gamma_t)^j\gamma_t^{n+1-j}(Y_{k-j,t} - j) - k\sum_{j=k}^{n}\binom{n}{j}(1-\gamma_t)^j\gamma_t^{n+1-j} \end{array}\right\},$$

which is a function to be maximized over γ_t. This can be written more compactly as

$$S_{t+1} = \max\left\{\gamma_t^{n+1}S_t + h(\gamma_t)\right\}. \quad (52)$$

B.2.3 The seller's maximization

Taking derivatives of (52) with respect to the γ_t, the seller's first-order conditions are

$$S_t = -h'(\gamma_t)/((n+1)\gamma_t^n), \quad (53)$$

and therefore, using (53 in (52),

$$h'(\gamma_{t+1}) = \gamma_{t+1}^n(\gamma_t h'(\gamma_t) - (n+1)h(\gamma_t)). \quad (54)$$

Writing h as

$$h(\gamma_t) = g(\gamma_t) - n(1-\gamma_t)\gamma_t^n M_{t-1} \quad (55)$$

and using this expression to substitute for h in (54) gives

$$g'(\gamma_{t+1}) - n(n-(n+1)\gamma_{t+1})\gamma_{t+1}^{n-1}M_t = \gamma_{t+1}^{n}(\gamma_t g'(\gamma_t) - (n+1)g(\gamma_t) + n\gamma_t^n M_{t-1}). \quad (56)$$

We further have, from the price recursion (51)

$$M_t = A_t + \gamma_t^n M_{t-1}, \quad (57)$$

with

$$A_t = \gamma_t^n \phi_{t-1}(1-\gamma_{t-1}) + \sum_{j=1}^{k-1}\binom{n-1}{j}(1-\gamma_t)^j \gamma_t^{n-j} W_{k-j,t}. \quad (58)$$

Using (58) in (57) to eliminate M_{t-1}, we solve for

$$M_t = \frac{g'(\gamma_{t+1}) - \gamma_{t+1}^n(\gamma_t g'(\gamma_t) - (n+1)g(\gamma_t) - nA_t)}{n^2\gamma_{t+1}^{n-1}(1-\gamma_{t+1})}. \quad (59)$$

Therefore, inserting in (56),

$$\frac{g'(\gamma_{t+1}) - \gamma_{t+1}^n(\gamma_t g'(\gamma_t) - (n+1)g(\gamma_t) - nA_t)}{\gamma_{t+1}^{n-1}(1-\gamma_{t+1})} - n^2 A_t =$$
$$\gamma_t \frac{g'(\gamma_t) - \gamma_t^n(\gamma_{t-1}g'(\gamma_{t-1}) - (n+1)g(\gamma_{t-1}) - nA_{t-1})}{(1-\gamma_t)}.$$

The expression $\gamma_t g'(\gamma_t) - (n+1)g(\gamma_t) - nA_t$ can be further simplified. Indeed,

$$\gamma_t g'(\gamma_t) - (n+1)g(\gamma_t) - nA_t = \sum_{j=1}^{k-1} j(1-\gamma_t)^{j-1}\gamma_t^{n-j}\left(\binom{n-1}{j}n(1-\gamma_t)W_{k-j,t} - \binom{n}{j}\gamma_t Y_{k-j,t}\right)$$
$$+\sum_{j=1}^{k-1}\binom{n}{j}j^2(1-\gamma_t)^{j-1}\gamma_t^{n+1-j} + k\sum_{j=k}^{n}\binom{n}{j}j(1-\gamma_t)^{j-1}\gamma_t^{n+1-j}.$$

B.2.4 The function $v(x)$

We now let $k=2$ and seek the function $v(x)$, giving the identity of the indifferent buyer given that there are two units for sale and the length of time to the deadline is x. Given $k=2$, we have

$$Y_{1,t} \approx \frac{n\frac{q_{t-1}}{q_t} - \left(\frac{q_{t-1}}{q_t}\right)^n}{n+1}, \text{ and } W_{1,t} \approx \frac{1}{n}\left(\frac{v_t}{v_{t+1}}\right)^{n-1}.$$

Observe that

$$\frac{1}{\gamma(x)} - 1 \approx \frac{v'(x)}{v(x)}.$$

If we let $t+1 = x+\varepsilon$, $t = x$ and $t-1 = x-\varepsilon$, we can approximate $Y_{1,t}$ by

$$\frac{1}{n}\left((n-1)\left(1+\frac{3\varepsilon}{nx}\right)^{-1} - \left(1+\frac{3\varepsilon}{nx}\right)^{1-n}\right),$$

(recall that there is one fewer buyer) and $W_{1,t}$ by

$$\frac{1}{n-1}\left(1+\frac{3\varepsilon}{nx}\right)^{1-(n-1)}.$$

Finally, we can approximate γ_t as follows:

$$\gamma_{t+1} = \left(1+\frac{v'(x)}{v(x)}\varepsilon + \left(\frac{v''(x)}{v(x)} - \left(\frac{v'(x)}{v(x)}\right)^2\right)\varepsilon^2\right)^{-1},$$

$$\gamma_t = \left(1+\frac{v'(x)}{v(x)}\varepsilon\right)^{-1},$$

$$\gamma_{t-1} = \left(1+\frac{v'(x)}{v(x)}\varepsilon - \left(\frac{v''(x)}{v(x)} - \left(\frac{v'(x)}{v(x)}\right)^2\right)\varepsilon^2\right)^{-1},$$

and do an asymptotic expansion in ε around 0, obtaining

$$(n^2(n+1)w(x)^4 - 2nw'(x)^2 + w(x)^2(3+n(3n+1)w'(x)))\varepsilon^3 + o(\varepsilon^4) = 0,$$

where $w(x) := v'(x)/v(x)$. We also know that $v(0)=0, v(1)=1$. Calculating the valuations $v(x)$ is thus a matter of solving the ordinary differential equation.

$$n^2(n+1)w(x)^4 - 2nw'(x)^2 + w(x)^2(3+n(3n+1)w'(x)) = 0. \tag{60}$$

B.2.5 The price function $p(x)$ and payoff π

Turning now to the price $p(x)$, from $\phi_t(v_t - p_t) = M_t v_{t+1}$ (cf. (50)), it follows that

$$p_t = v_t - \frac{M_t}{\phi_t}v_{t+1} = v_{t+1}\left(\gamma_t - \frac{M_t}{\phi_t}\right).$$

We have expression (59) for M_t, and thus attention turns to computing

$$\gamma_t - \frac{M_t}{\phi_t}.$$

Using our approximations W, X and γ, it is straightforward to verify that, in the case $k=2$,

$$\lim_{\varepsilon\to 0}\gamma_t - \frac{M_t}{\phi_t} = \frac{n-2}{n}.$$

This in turn gives the price function

$$p(x) = \frac{n-2}{n} v(x).$$

It is then straightforward that the seller's payoff is given by $2\frac{n-2}{n-1}$.

B.4 Committing to Lower Prices

This section provides an example in which the seller cannot commit to charging a low enough price in the second stage, and an example in which the seller cannot commit to charging a high enough price.

B.3.1 The Model

Assume that there are two buyers and two periods. Buyers have one of three possible valuations, v_1, v_2, or $v_3 = 1$, with $v_1 < v_2 < 1$. A buyer has valuation v_i with probability ρ_i, where

$$\rho_1 = \frac{1}{8}, \qquad \rho_2 = \frac{1}{4}, \qquad \rho_3 = \frac{5}{8}.$$

Conditional on all buyers being of type v_1 or v_2 in the last period, the seller's choice is obviously between charging v_1 or v_2. She chooses the latter, higher price if and only if

$$\Delta = \left(1 - \left(\frac{\rho_1}{\rho_1 + \rho_2}\right)^n\right) v_2 - v_1 > 0.$$

There are four obvious pure strategies in the two-period game: selling to type v_3 first, and then to type v_2; selling to type v_3, and then to v_1; selling to types v_3 and v_2 first, and then to v_1; and finally, selling to no one first, and then to type v_3. The seller could also wait and sell to some larger subset of types in the second period, but it is clear that this is worse than some strategy in which type v_3 accepts in the first period. (Of course, the latter strategy may not satisfy sequential rationality). We consider these strategies are in turn.

B.3.1a Selling to type v_3, and then to type v_2. Denote the price charged in the first period by p_{32} (the second price is v_2), and the expected payoff by V_{32}. The price p_{32} must satisfy

$$\frac{1-(\rho_1+\rho_2)^n}{1-(\rho_1+\rho_2)}(v_3 - p_{32}) = (\rho_1+\rho_2)^{n-1}\frac{1-(\rho_1/(\rho_1+\rho_2))^n}{1-\rho_1/(\rho_1+\rho_2)}(v_3 - v_2),$$

and the payoff V_{32} must satisfy

$$V_{32} = (1-(\rho_1+\rho_2)^n)p_{32} + (\rho_1+\rho_2)^n\left(1 - \left(\frac{\rho_1}{\rho_1+\rho_2}\right)^n\right) v_2.$$

Solving, we find that

$$V_{32} = (1-\rho_1^n)v_3 - \frac{1-\rho_1}{\rho_2}((\rho_1+\rho_2)^n - \rho_1^n)(v_3 - v_2).$$

B.3.1b Selling to type v_3, and then to type v_1. Denote the price charged in the first period by p_{31} (the second price is v_1), and the expected payoff by V_{31}. The price p_{31} must satisfy

$$\frac{1-(\rho_1+\rho_2)^n}{1-(\rho_1+\rho_2)}(v_3 - p_{31}) = (\rho_1+\rho_2)^{n-1}(v_3 - v_1),$$

and the payoff V_{31} must satisfy

$$V_{31} = (1-(\rho_1+\rho_2)^n)p_{32} + (\rho_1+\rho_2)^n v_1.$$

Solving, we find that

$$V_{31} = v_3 - (\rho_1+\rho_2)^{n-1}(v_3 - v_1).$$

B.3.1c Selling to type v_2, and then to type v_1. Denote the price charged in the first period by p_{21} (the second price is v_1), and the expected payoff by V_{21}. The price p_{21} must satisfy

$$\frac{1-\rho_1^n}{1-\rho_1}(v_2 - p_{21}) = \rho_1^{n-1}(v_2 - v_1),$$

and

$$V_{21} = (1-\rho_1^n)p_{21} + \rho_1^n v_1.$$

Solving, we find that

$$V_{21} = v_2 - \rho_1^{n-1}(v_2 - v_1).$$

B.3.1d Selling to type v_3 in the second period. Clearly, this yields a payoff of $V_3 = (1-(\rho_1+\rho_2)^n)v_3$.

B.1.2 Case 1: $(v_1, v_2) = (1/8, 1/4)$. The seller cannot commit to a low price

It is easy to check that $\Delta = 7/72 > 0$—conditional on the buyers not being of type v_3, it is optimal to set the price to v_2 in the one-stage game. However, we have that

$$\frac{43}{64} = V_{31} > \begin{cases} V_{32} = 21/32 \\ V_{21} = 15/64 \\ V_3 = 5/8 \end{cases}.$$

That is, the optimal two-stage strategy is to sell to high types first, and then to all types.

This also dominates all schemes involving mixing (since if type v_2 or type v_3 is supposed to randomize in the first period, this lowers the probability of acceptance, relative to the same type accepting with probability one in the first period, as well as the price paid in the first period, and it does not affect the price in the second).

But since $\Delta > 0$, the seller cannot achieve this payoff, because in the second stage, she cannot help but charge a high price. Therefore, this is an example in which the seller cannot commit to charge low enough a price in the second stage.

B.3.3 Case 2: $(v_1, v_2) = (4/5, 8/9)$. The seller cannot commit to a high price

It is easy to check that $\Delta = -4/405 > 0$. Conditional on the buyers not being of type v_3, it is optimal to set the price to v_1 in the one-stage game. However, we have

$$\frac{539}{576} = V_{32} > \begin{cases} V_{31} = 37/40 \\ V_{21} = 79/90 \\ V_3 = 5/8 \end{cases}.$$

This also dominates all schemes involving mixing (for the same reasons as before).

But since $\Delta < 0$, the seller cannot achieve this payoff, since in the second stage, she cannot help but charge a low price. This is therefore an example in which the seller cannot commit to charge high enough a price in the second stage.