Communication and Learning

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Abstract. We study the intergenerational accumulation of knowledge in an infinite-horizon model of communication. Each in a sequence of players receives an informative but imperfect signal of the once-and-for-all realization of an unobserved state. The state affects all players' preferences over present and future decisions. Each player observes his own signal but does not directly observe the realized signals or actions of his predecessors. Instead, he must rely on cheap-talk messages from the previous players to fathom the past. Each player is therefore both a receiver of information with respect to his decision, and a sender with respect to all future decisions. Senders' preferences are misaligned with those of future decision makers.

We ask whether there exist "full learning" equilibria — ones in which the players' posterior beliefs eventually place full weight on the true state. We show that, regardless of how small the misalignment in preferences is, such equilibria do not exist. This is so both in the case of private communication in which each player only hears the message of his immediate predecessor, and in the case of public communication, in which each player hears the message of all previous players. Surprisingly, in the latter case full learning may be impossible even in the limit as all players become infinitely patient. We also consider the case where all players have access to a mediator who can work across time periods arbitrarily far apart. In this case full learning equilibria exist.

1. Introduction

1.1. Motivation

Scientific progress routinely builds on knowledge accumulated over spans of many generations. Continued advances entail that each generation borrows from past discoverers and transmits its own discoveries to future generations. Isaac Newton famously compared this process to one of "standing on the shoulders of giants."[1] In many cases of course knowledge fails to progress, sometimes for highly prolonged periods. For instance, a thousand years passed from the time when "the Athenian philosopher Proclus made the last recorded astronomical observation in the ancient Greek world in 475 AD" to the writings of Copernicus who "set in hand the renewal of the scientific tradition." (Freeman, 2002, page xix).

This paper studies the accumulation of knowledge in an intergenerational model of learning and communication. In particular, we ask whether the participants in an ongoing society will eventually "fully learn" some unobserved state of the world: will they eventually place full weight on the true state?

To model intergenerational learning, we posit a dynastic sequence of representative "policy makers." Each policy maker occupies the decision making post for one period, but cares about the decisions of his successors even after he relinquishes his post. An individual's long run payoff is the discounted sum of stage payoffs, each of which depends on the policy chosen in that period and on an unobserved state.

The state can be interpreted as an unknown underlying fact that affects policy choices in society. For example, consider how the current body of knowledge on climate change affects (or should affect) environmental policy toward CO_2 emissions. The science in this case is clearly relevant to policy makers, and their responses are only slowly refined as more data on the true state of climate change accumulates. Historically, even basic knowledge has payoff-relevant consequences. For instance, innovations in the field of astronomy led to better navigation at sea.

In our model, aside from the difficulties of intergenerational information transmission, full learning is trivial. This is deliberately so since we want to focus on the effects of strategic communication. Our results on full learning are largely negative. In the presence of any bias in preferences that makes the intergenerational transmission of information strategically non-trivial, full learning will not occur in equilibrium.

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[1]In a 1676 letter to Robert Hooke. A popularized account of the context for this exceedingly well known quote and of the relationship between Newton and Hooke can be found in the introduction and the "Sir Isaac Newton" chapter of Hawking (2003).

1.2.   Overview

For simplicity, the state is assumed to be binary, taking values either 0 or 1 and each player/policy maker lives one period and chooses an action in the real line. Which action is more desirable depends on the value of the state. Since this is unknown, the best action depends on beliefs about the state. The optimal level of CO_2 abatement depends on the state of knowledge about how severe global warming really is.

At the end of the period each player receives an informative but imperfect signal of the state in $\{0, 1\}$. These signals are i.i.d. across time. Each player observes his own signal but does not directly observe the realized signals or actions of his predecessors. Instead, he must rely on cheap-talk messages about past signal realizations from the previous players to fathom the past. Each player is therefore both a *receiver* of information with respect to his own decision, and a *sender* with respect to all future decisions.

We consider two versions of this model. In one, communication is *public*. The t-th sender's message is directly available to all future policy makers. All messages are therefore publicly observed and common knowledge among all future decision makers. The public communication model is a natural benchmark if it is difficult or impossible for the t-th player to destroy the messages of prior generations.

A second version of the model assumes *private* communication. That is, the $t + 1$-th receiver is the sole recipient of the t-th sender's message. Under private communication, each decision maker is a "gate-keeper" who can choose precisely how much of his knowledge to convey to his successor. The private communication model is natural if past messages are easy to hide, manipulate, or falsify. The purely public and private communication models are intended as useful benchmarks, delimiting the intermediate cases that fall between these two prototypes.

Both the public and private communication models can be viewed as intergenerational versions of the classic information transmission model of Crawford and Sobel (1982) (henceforth CS). As in CS, nothing very interesting emerges without some misalignment of preferences between senders and receivers. As in their model, it is straightforward to show that in the absence of any misalignment, there is a truth-telling equilibrium in both the private and public communication models in which the sender each period truthfully reveals the value of the current signal and the reports on past signals received from his predecessors. In this truth-telling equilibrium, the Weak Law of Large Numbers implies that players' posterior beliefs eventually place full weight on the true state. Truth-telling allows society to fully learn the underlying state.

Our interest is in whether full learning occurs when there *is* a misalignment of preferences. As in CS, we examine an environment in which a decision maker's preferences over his current decision exhibit a bias that distorts his preferred action away from the preferred action of his predecessors. It is not hard to justify preference bias/misalignment in an intergenerational setting; decision makers often face costs and benefits that do not accrue to non-participants who vicariously benefit from the decision. Abatement of CO_2 levels is desirable but costly for the current decision makers. The current decision makers will also enjoy the benefits of future abatement levels, but without having to bear as much of the costs. This induces a bias in the players' preferences.

With any misalignment, regardless of its size, we show that in both the public and private communication models, truth-telling equilibria do not exist. In fact, our main results establish something considerably stronger. We show the impossibility of full learning in *any equilibrium*. Specifically, for any initial prior about the state, for any discount factor, for any precision of the signal (short of perfection), and in any Weak Perfect Bayesian Equilibrium, the probability that decision makers' posterior beliefs are within ε of learning the true state cannot converge to one through time. Indeed, in the public communication model we show that posterior beliefs about the state must eventually reach a threshold at which point learning ceases altogether. Only babbling can occur from that point on.

To get an intuitive sense for the difficulties involved in establishing that full learning cannot occur, it is useful to focus on the model of public communication. As we will do below, suppose that the players' preferences satisfy three simple properties. First of all, aside from the bias, the players' preferences are such that their most preferred action *equals* their belief that the state is equal to one. Suppose also that their payoff decreases monotonically as the distance between their most preferred action and the actual action taken goes up. Finally, the bias in preferences takes a simple form that induces players to bias the current action down by a fixed amount α relative to the bliss point we have just described.

To see that the bias in preferences may make the truthful revelation of information problematic just from one period to the next, one can reason in much the same way as in the CS world. Consider just a single player (sender) and his successor (receiver). Call the belief (that the state is equal to one) of the sender x. Suppose first that the sender's strategy is to reveal his signal $s \in \{0,1\}$. Since the signal is informative, the receiver will now have beliefs either x^0 (if the report is $s = 0$) or x^1 (if the report is $s = 1$) that satisfy $x^0 < x < x^1$. Because of the bias the receiver will take action $x^0 - \alpha$ in one case and $x^1 - \alpha$ in the other. Suppose on the other hand that the sender babbles and reveals no information. For instance,

he reports $s = 1$ regardless of what he really observes. Then the receiver's belief is the same as the sender's before he observes s, namely x. Hence the receiver takes action $x - \alpha$.

For the information to be truthfully transmitted between sender and receiver we need two incentive compatibility constraints to be satisfied. The sender must *both* (weakly) prefer action $x^0 - \alpha$ to action $x^1 - \alpha$ when his belief is x^0 *and* he must (weakly) prefer action $x^1 - \alpha$ to action $x^0 - \alpha$ when his belief is x^1. Since the sender prefers actions that are closer to his belief and $x^1 > x^0$, it is clear that he prefers action $x^1 - \alpha$ to action $x^0 - \alpha$ when his belief is x^1. One of the two incentive compatibility constraints does not bind.
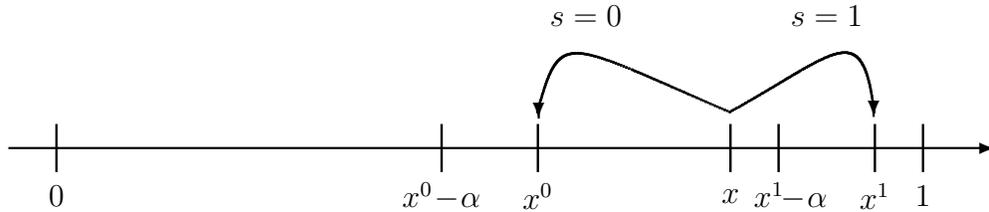
Figure 1: One Period Incentive Compatibility

The second constraint on the other hand might be impossible to satisfy. This constraint is satisfied if $x^0 - \alpha$ is no further from x^0 than is $x^1 - \alpha$, as is for instance the case in Figure 1. Since the constraint can be written as $2\alpha \leq x^1 - x^0$, it is clear that if $x^1 - x^0$ is small relative to α then the second incentive compatibility constraint cannot be satisfied. If this is the case, after observing $s = 0$, the sender prefers the action that the receiver would take if he believed the sender's message and was told that $s = 1$. Hence there is no equilibrium in which the sender's messages fully reveal the signal s.

It is critical to notice that the possibility of truthful revelation just one period ahead depends on the the *relative* size of $x^1 - x^0$ and α. Given this, it is immediate that one period ahead truth-telling will be possible when x takes "intermediate" values (provided that α is not too large), while it will become impossible as x approaches either zero or one. This is simply because Bayes' rule tells us that the updating effect will be larger for intermediate values of x and will shrink all the way to zero as x goes to the extremes of $[0, 1]$.

Armed with these observations, it is now possible to delve further into the difficulties of establishing that full learning cannot take place in our dynamic model.

To begin with, suppose that we had an equilibrium with perpetual truth-telling. This is a particularly simple way to achieve full learning of course. Suppose for the sake of argument that the true state is one. Then, in equilibrium, the beginning-of-period belief of player t — denoted x_t throughout the paper — converges to one (with probability one). This of

course means that the difference between x_t updated with a signal of one (denoted x_t^1) and x_t updated with a signal of zero (denoted x_t^0) must eventually become arbitrarily small. So eventually in this putative truthful equilibrium we must be in a situation in which $2\alpha > x_t^1 - x_t^0$ and hence looking ahead only one period truthful revelation is no longer possible. This, however is not sufficient to dismiss perpetual truth-telling as a possible equilibrium. The reason is that we are in a dynamic model in which players look indefinitely ahead at discounted payoffs.

A player t who finds that $2\alpha > x_t^1 - x_t^0$ before deciding to deviate or not from the putative truth-telling equilibrium, will have to reason through the following. Suppose that he observes a time t signal — denoted s_t throughout the paper — equal to zero. Since $2\alpha > x_t^1 - x_t^0$, as far as his period $t + 1$ payoff, player t clearly prefers to deviate and report $s_t = 1$. However, think now of his payoff in period $t + \tau$, with τ possibly large. Taking the truthful strategies of the others as given, the beliefs between $t + 1$ and $t + \tau$ will go up and down through time, following the realizations of the signals each period. Deviating in period t and reporting $s_t = 1$ when in fact $s_t = 0$ will simply take the entire sequence of beliefs two "notches" up relative to where it would be if t did not deviate (in the same sense that x_t^0 is two notches down from x_t^1 — it takes two realizations of the signal equal to one for Bayes' rule to take beliefs from x_t^0 to x_t^1). But then consider a sequence of signals that takes the beliefs gradually towards the center of the interval $[0, 1]$. As we noted before, (provided that α is not too large) the one-period-ahead calculation in this case tells the player not to deviate from truthful revelation. So, with some probability, deviation at t has the same effect (in period $t + \tau$) than deviating when the beliefs are in an intermediate position and hence there is no gain from deviation at $t + \tau$. In fact deviations could entail a sizeable loss when the beliefs are in the intermediate range of $[0, 1]$, especially if the signal is reasonably precise. Of course, the outcome of calculation for the entire path following a deviation will depend crucially on the signal precision (how fast the beliefs move through time), and on how much the players discount the future. As it turns out, the computation is not trivial. One of the main reasons for this is that while the loss from deviating when beliefs are intermediate is multiplied by a probability that approaches zero, the gain from deviating shrinks to zero as well when x_t goes to either one or zero, because in these cases $x_t^1 - x_t^0$ must shrink to zero too.

In order to rule out *any* equilibrium with full learning, a further layer of difficulties arises. These stem from the fact that full learning can in principle obtain in equilibria that are very different from the simple truth-telling equilibrium hypothesized above. In particular,

full learning could obtain in an equilibrium that fails to be Markov in the sense that its continuation equilibria could be different after different histories leading to the same belief. This structure of equilibrium can in principle be used to "punish" deviations from truth-telling behavior. The evaluation of continuation payoffs we have sketched above becomes more subtle as a result.

To see the extra difficulties involved in ruling out full learning in general, consider again a player t who finds that $2\alpha > x_t^1 - x_t^0$, and observes a signal $s_t = 0$. Just as before, in a full learning equilibrium this must happen at some point with probability one. Looking one period ahead, player t prefers to report $s_t = 1$ as before. However, we can no longer be sure that the following players use truth-telling strategies. The continuation path of play could be very different after a deviation at t. As before, it is possible that a sequence of signals eventually takes the beliefs back towards the middle of $[0, 1]$, when the deviation at t entails a loss. Because the equilibrium need not be Markov in the first place, the continuation equilibrium could for instance be such that the beliefs get "trapped" in this range for long periods (because of prolonged spells of babbling) *only* following a report of $s_t = 1$ at t. So, by deviating player t may be "punished" in that the future path of beliefs may yield the losses associated with beliefs in the intermediate range of $[0, 1]$ with much higher probability than in the case of a truth-telling strategy profile. Our results establishing that full learning cannot take place overcome the difficulties we have described mostly by bounding continuation payoffs in a variety of ways.

Once the impossibility of full learning is established, it is reasonable to turn to the question of whether a weaker result might be attainable in the limit as the players become infinitely patient. We refer to possible sequences of equilibria in which full learning occurs as discounting approaches zero as the case of *limit learning*. As it turns out, the possibility of limit learning equilibria depends on which specification of the model, public or private communication, is considered. Surprisingly, a more positive result is obtained in the private communication model in which a richer set of deviations is available to the players. We show that limit learning is possible in the private communication case when the bias is not too large relative to the precision of the signal. By contrast, we show that limit learning is *not* possible in the public communication model if the signal precision is not too large. The pattern that seems to emerge from these results — parameter configurations in which limit learning is possible with private communication and impossible with public communication — is that "more" communication is not necessarily a good thing.

The natural next question is whether a more positive picture for learning emerges when

mediated communication is allowed. By mediated communication, we have in mind a centralized gatekeeper who receives the message of each sender but is careful to release a highly filtered version of the information to each policy maker. In fact, we show that full learning equilibria exist in the mediated communication model when the bias is not large relative to the signal's precision, regardless of how heavily the players discount the future. The idea, roughly, is that the gatekeeper releases a summary statistic after sufficiently many signal realizations. The bias remains small relative to the difference in future beliefs induced by one's message. The incentives to reveal (to the mediator) truthfully one's information are therefore preserved.

If anything, we view the possibility of full learning with a mediator as confirming the difficulties of learning the state in our set-up. The availability of a mediator who can work across time periods arbitrarily far apart from each other is surely to be considered highly contrived and unlikely in any application.

1.3. Related Literature

This paper sits at the intersection of two literatures: the one on strategic information transmission and the one on social learning.

As we mentioned already, the social learning side of our model stripped of the strategic information transmission is deliberately kept trivial. The social learning literature has focused on reasons why full learning might not occur other than strategic information transmission, for instance costly information acquisition or herding behavior. In our case previous actions are not observed, and i.i.d. informative signals about the true state are observed regardless of any decisions the players take. The Weak Law of Large Numbers guarantees that, provided the information about signals is properly transmitted from one player to the next, the true state will eventually be learned. The literature on social learning is vast and it would be foolish to attempt to summarize it here. We simply refer to the surveys by Gale (1996) and Sobel (2000), and further references in Moscarini, Ottaviani, and Smith (1998).

The literature on strategic information transmission is also way too large for us to attempt an overview here. Farrell and Rabin (1996), Sobel (2007) and Krishna and Morgan (2008) survey this literature, while Krishna and Morgan (2001) and Ottaviani and Sørensen (2006) have extensive bibliographic references. Within the strategic information transmission literature, we specifically comment on a few contributions that we feel are more closely related to ours.

Our model is probably closest to that of Spector (2000) who examines a repeated game

of communication in a large population with a multi-dimensional policy space. Everyone in the population shares the same underlying policy preferences. The population is partitioned into two groups characterized by different priors about an unobserved state. Each individual receives an informative signal of the state. The two groups alternate in the role of senders and receivers with a new sender and receiver randomly drawn from the appropriate group each period.

Unlike ours, Spector's players are myopic. Hence he does not look at the dynamic component of strategic information transmission. Rather, his focus is on the effect of conflict on a multi-dimensional policy space which, surprisingly, in a steady-state reduces to at most a single-dimensional disagreement.

A dynamic version of the basic sender-receiver game is found in Taub (1997). He models a dynamic game with a single sender and receiver. Both sender and receiver have ideal policy positions that vary across time according to serially correlated processes that are uncorrelated with one another. There are obvious similarities between our paper and his — both involve learning via accumulated messages. In Taub's model, however, the learning (though not full learning) is possible due to the serial correlation in the sender's bias. In our case whatever learning occurs takes place automatically, provided information is properly transmitted through time.

Li (2007) considers sequential arrival of information in a sender-receiver set up. Her focus is on what the sequence of reports by the sender reveals about his *ability* (the quality of his signals). Intriguingly, she finds that a sender who "changes his mind" may be a signal of high ability. In our set up all signals have the same precision since they are i.i.d. and hence there is nothing to infer about the sender's ability in the sense of Li (2007).

Golosov, Skreta, Tsyvinski, and Wilson (2008) also consider a dynamic sender receiver game. Their set up is close to the original model in CS. A state is realized once and for all and observed by the sender. Then, in each of a finite number of periods, the sender sends a message to the same receiver who takes an action each time. They find that there are equilibria that do not have the simple partitional structure found in CS. In the model they analyze the receiver does not learn the state, but more information is transmitted than in the static set up.

Finally, this paper builds on our previous work on dynastic communication in Anderlini, Gerardi, and Lagunoff (2007, 2008), Anderlini and Lagunoff (2005), and Lagunoff (2006).[2]

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[2]Related models are also found in Kobayashi (2007) and Lagunoff and Matsui (2004).

These papers consider dynastic games in which messages are about the past history of play. Because there are no "objective" types (i.e., no exogenous states or payoff types), there is no learning in the traditional sense of the word. These contributions focus on whether social memory, embodied in the beliefs of the new entrants, is accurate, and what pitfalls may befall a society if it is not.

1.4.   Outline

The rest of the paper is organized as follows. In Section 2 we specify the model and equilibrium concept, and we define formally what it means that full learning occurs. Sections 3 and 4 contain our main results, asserting that full learning is impossible in our model. Section 5 is concerned with the limit case in which the players become infinitely patient and shows that in some cases full learning becomes possible in the limit but in some other cases it does not. Section 6 considers the case of mediated communication and Section 7 briefly concludes.

For ease of exposition, all proofs have been placed in an Appendix. In the numbering of equations and all other items, a prefix of "A" points to the Appendix. For the sake of brevity, some proofs have been omitted from the Appendix as well. These can be found in a supplement to the paper available at http://www.anderlini.net/learning-omitted-proofs.pdf

2.   The Set Up

2.1.   Model

Time is discrete, and indexed by $t = 0, 1, 2, \ldots$ In period 0 Nature selects the state ω which can be either 1 or 0. State $\omega = 1$ is chosen with probability $r \in (0, 1)$. Nothing else takes place in period 0, and no-one in the model observes Nature's choice of ω, which is determined once and for all.

There is a period-by-period flow of imperfect information about the state ω. We take this to have the simplest form that will allow for an interesting model: a sequence of imperfect but informative conditionally i.i.d. signals. The realized signal in period t is denoted s_t, and its symmetric structure is parameterized by a single number $p \in (1/2, 1)$ as

$$\Pr\left\{ s_t = 1 \;\middle|\; \omega = 1 \right\} = \Pr\left\{ s_t = 0 \;\middle|\; \omega = 0 \right\} = p \tag{1}$$

With obvious terminology, we refer to p as the signal's quality.[3]

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[3]While the assumption of conditionally i.i.d. signals is important for our arguments, the symmetry of the signal structure could easily be dispensed with.

There is a countable infinity of players indexed by $t = 1, 2, \ldots$ Player t chooses an action $a_t \in \mathbb{R}$ at time t, and derives state-dependent utility from his own choice and from the choices $a_{t+\tau}$ with $\tau = 1, 2, \ldots$ of all his successors.

Let $u(a_t, \omega)$ be a concave function reaching a maximum at $a_t = \omega$ for $\omega = 0, 1$. Let $v(a_t)$ be a weakly convex increasing function. Fix a sequence of actions $a_1, \ldots, a_t, \ldots$ and a state ω. Then the utility of player t is written as

$$(1 - \delta) \left[\sum_{\tau=0}^{\infty} \delta^\tau u(a_{t+\tau}, \omega) - v(a_t) \right] \tag{2}$$

with δ a common discount factor in $(0, 1)$. In (2), the term $-v(a_t)$ embodies the bias in the players' preferences . Each player t bears an extra cost for higher levels of the current action a_t. Aside from this, for given beliefs, the players' preferences are aligned. Of course, what matters to player t is the expected value of (2), given his beliefs about ω and the sequence of all future actions $\{a_{t+\tau}\}_{\tau=1}^{\infty}$.

We work with a specific form of (2) that captures our desiderata and at the same time ensures tractability. We take u to be a quadratic loss function, and v to be linear so that (2) is written as

$$-(1 - \delta) \left[\sum_{\tau=0}^{\infty} \delta^\tau (\omega - a_{t+\tau})^2 + 2\alpha a_t \right] \tag{3}$$

In (3), the bias term is conveniently parameterized by α. In fact it is immediate to check that if player t believes that $\omega = 1$ with probability x_t, then his choice of a_t to maximize (3) is simply

$$a_t = x_t - \alpha \tag{4}$$

Because of (4), we refer to α simply as *the* bias. Throughout, we assume that $\alpha \in (0, 1/2)$.[4]

Player t's information comes from his predecessor(s) in the form of payoff irrelevant messages. He does not observe directly any part of the past history of play or realized signals.

In the case of *private communication* player t observes a message m_{t-1} sent by player $t-1$, which is observed by no-one else. In the case of *public communication* player t observes all messages $m_{t-\ell}$ (with $\ell = 1, \ldots, t-1$) sent by all preceding players. It is useful to establish a

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[4]The restriction $\alpha < 1/2$ simplifies some of our computations, and only disposes of uninteresting cases.

piece of notation for what player t observes by way of messages that encompasses both the private and public communication cases. We let $m^{t-1} = m_{t-1}$ in the private communication case, while $m^{t-1} = (m_1, \ldots, m_{t-1})$ in the public communication case. (Set $m_0 = m^0 = \emptyset$.)

The choice of message spaces is completely irrelevant to our main results. In fact, they hold for *any* choice of message spaces. This will be reflected in the formal statement of each of our propositions below. For the time being we simply establish that the message space available to each player t from which to pick m_t is denoted by $M_t \neq \emptyset$. (Set $M_0 = \emptyset$.) We also let M represent the entire collection $\{M_t\}_{t=0}^{\infty}$ of message spaces.

After observing m^{t-1}, player t picks an action a_t. After choosing a_t he observes the informative but imperfect signal s_t, and after observing s_t he decides which message m_t to send. Note that, crucially, player t does not learn the state before sending m_t. This can be interpreted as his period t payoff simply accruing after he sends m_t, or as his experiencing a payoff that depends on his action a_t and the signal s_t in an appropriate way.[5]

Given this time line, player t's strategy has two components. One that returns an action a_t as a function of m^{t-1}, and one that returns a message m_t as a function of m^{t-1} and s_t. We denote the first by λ_t and the second by σ_t. Formally

$$\lambda_t : m^{t-1} \mapsto a_t \qquad \text{and} \qquad \sigma_t : (m^{t-1}, s_t) \mapsto m_t$$

Notice that there is a sense in which the λ_t part of t's strategy is trivial. Given his beginning-of-period belief $x_t \in [0, 1]$ that $\omega = 1$, using (3) and (4) we know that in any equilibrium it will be the case that $a_t = x_t - \alpha$. But since the only mechanism for transmitting information are the messages described above, it is clear that x_t is entirely determined by the message-sending behavior of t's predecessors $(\sigma_1, \ldots, \sigma_{t-1})$ together with the observed m^{t-1}. Thus, all strategic behavior (and hence any equilibrium) is entirely pinned down by a profile $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_t, \ldots)$.

To avoid unnecessary use of notation, and since this does not cause ambiguity, from now on we will refer to $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_t, \ldots)$ as a strategy profile, leaving it as understood that the associated $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_t, \ldots)$, determining actions via (3) and (4), is to be considered as well. Of course, actions and messages sent in equilibrium will be determined by (3), (4) and the players' on- and off-path beliefs. Luckily off-path beliefs need not be specified with any piece of notation. The reason is simply that off-path beliefs will *never* play any role in our

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[5]The latter would be the analogue of what is routinely done in the large literature on repeated games with imperfect private monitoring. See Mailath and Samuelson (2006) for an overview and many references.

results. Since the on-path beliefs are entirely pinned down by $\boldsymbol{\sigma}$ via Bayes' rule, referring to this profile will always suffice.

As imaginable from the preceding discussion of off-path beliefs, throughout the paper by Equilibrium we mean a Weak Perfect-Bayesian Equilibrium (henceforth WPBE) of the game at hand (see for instance Fudenberg and Tirole, 1991). This is the weakest equilibrium concept that embodies sequential rationality that applies to our set up. Our main focus is on ruling out certain outcomes as possible in any equilibrium. Therefore, a weaker equilibrium concept strengthens our main results. It is in fact also the case that whenever we construct an equilibrium in any of our arguments below (Propositions 3 and 5 below), then the equilibrium will also satisfy the more stringent requirements of a Sequential Equilibrium in the sense of Kreps and Wilson (1982).

Throughout the paper, we only consider pure strategies. We strongly conjecture that our main results hold when mixed strategies are allowed. Note that, since $u(\cdot, \omega) - v(\cdot)$ is strictly concave in a_t, mixing in actions is clearly impossible in any equilibrium. Moreover, recall that our main results concern the absence of equilibria in which the state is learned through time. Mixed message strategies seem very unlikely to help in this respect, even if one could construct strategy profiles that guarantee the indifferences needed in any mixed strategy equilibrium.

2.2. Full Learning

Our main question (mostly answered in the negative) is whether the accumulated memory will eventually uncover, or fully learn, the true state. Therefore, we need to be precise about what we mean by full learning in our model.

Fix a strategy profile $\boldsymbol{\sigma}$. As we noted above, the belief of player t at the beginning of period t that $\omega = 1$ is entirely determined by $\boldsymbol{\sigma}$ and the m^{t-1} he observes. Denote it by $x_t(\boldsymbol{\sigma}, m^{t-1})$ with the dependence on $\boldsymbol{\sigma}$ or m^{t-1}, or both, sometimes suppressed as convenient.

Now fix a realization $\mathbf{s}$ of the entire stochastic process $\{s_t\}_{t=1}^{\infty}$. Via the strategy profile $\boldsymbol{\sigma}$, the realization $\mathbf{s}$ clearly determines m^{t-1} for every $t = 1, 2, \ldots$ This makes it clear that a given profile $\boldsymbol{\sigma}$ (together with a realization of the state ω and the signal structure (1) parameterized by p) defines a stochastic process $\{x_t\}_{t=1}^{\infty}$ governing the (on-path) evolution of the players' posterior. The probabilities (conditional when needed) of events in this process will be indicated by $\Pr_{\boldsymbol{\sigma}}$.[6]

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[6]It is clear that $\boldsymbol{\sigma}$ also induces a stochastic process $\{m_t\}_{t=0}^{\infty}$ governing the (on-path) evolution of messages

We are now ready to define formally what it means that full learning obtains given a profile $\boldsymbol{\sigma}$.

Definition 1. _Full Learning: We say that a profile $\boldsymbol{\sigma}$ induces full learning of the state if and only if the associated stochastic process $\{x_t\}_{t=1}^{\infty}$ converges in probability to the true state ω._

In symbols, we say that full learning of the state obtains given $\boldsymbol{\sigma}$ if and only if for every $\varepsilon > 0$ there exists a $\bar{t} \geq 1$ such that $t > \bar{t}$ implies

$$\Pr_{\boldsymbol{\sigma}} \left\{ x_t > 1 - \varepsilon \ \Big| \ \omega = 1 \right\} > 1 - \varepsilon \qquad \text{and} \qquad \Pr_{\boldsymbol{\sigma}} \left\{ x_t < \varepsilon \ \Big| \ \omega = 0 \right\} > 1 - \varepsilon$$

We can now proceed with the first of our main results.

3. No Full Learning with Private Communication

With private communication full learning of the state cannot occur in equilibrium. This is so regardless of the discount factor, signal precision, bias or prior.

Proposition 1. _No Full Learning with Private Communication: Fix any M, δ, p, α and r. Then there is no equilibrium $\boldsymbol{\sigma}^*$ of the model with private communication that induces full learning of the state._

By the Proposition, posterior beliefs do not converge in probability to the true state. Concretely, this means that for a set of signal realizations arising with positive probability, beliefs are not close to the truth at infinitely many dates.

What follows is an informal sketch of the argument we use in the Appendix to prove Proposition 1. It is convenient to divide it into three parts.

The first part of the argument shows that off-path behavior can be safely ignored in the second and third parts. To see that this is the case consider an equilibrium $\boldsymbol{\sigma}^*$, and then proceed to delete from the players' strategy sets all messages that are sent with ex-ante probability zero. The strategy profile $\hat{\boldsymbol{\sigma}}^*$ that we obtain in this way clearly induces the same stochastic process $\{x_t\}_{t=1}^{\infty}$ on the players' posterior given any realization of the state ω. Moreover, the profile $\hat{\boldsymbol{\sigma}}^*$ must also be an equilibrium in the game with the reduced message

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sent by each player. With a slight abuse of notation the probabilities (conditional when needed) of events in this process will be indicated in the same way by $\Pr_{\boldsymbol{\sigma}}(\cdot)$. The argument of the operator will ensure that this does not cause any ambiguity. Finally, because $\boldsymbol{\sigma}$ pins down the on-path evolution of both $\{x_t\}_{t=1}^{\infty}$ and $\{m_t\}_{t=0}^{\infty}$, it also induces _updated_ probabilities that ω is one or zero, conditional on any (positive probability) event concerning $\{x_t\}_{t=1}^{\infty}$ or $\{m_t\}_{t=0}^{\infty}$, or both. These will also be identified using the notation $\Pr_{\boldsymbol{\sigma}}(\cdot)$, with the arguments used to avoid any ambiguity.

spaces. This is simply because by deleting messages from the players' strategy sets we only reduce the possible deviations to be considered to verify that a given strategy profile is in fact an equilibrium. Hence, there is no loss of generality in considering only message spaces and equilibria like $\hat{\sigma}^*$, in which all messages are sent with positive probability.

The second part of the argument concerns continuation payoffs as a function of a player's beliefs, when the beliefs are sufficiently close to one. In particular, consider a player's continuation payoff computed after he has chosen an action, but before he observes the signal, as a function of the message he receives, conditional on the true state ω. We show that if m and m' are two messages that are associated with beliefs x and x' that are both arbitrarily close to one, then the continuation payoffs that they induce are arbitrarily close to each other, both conditional on the true state being one and on the true state being zero. This is not hard to show when we are conditioning on $\omega = 1$. Using the incentive constraints of the player, it must be that the continuation payoffs are close to each other since the conditioning event ($\omega = 1$) has probability close to one in the player's beliefs. If the payoffs were not close to each other one of the two "types" (either the one receiving m or the one receiving m') would necessarily have an incentive to behave like the other. To show that the continuation payoffs are close to each other when conditioning on $\omega = 0$ involves an extra difficulty since the conditioning event has probability close to zero in the player's beliefs and hence in this case the conditional continuation payoffs have a vanishingly small effect on the player's incentives. We circumvent this difficulty by ruling out some possible equilibrium continuation paths, which in turn can be done using the incentives of subsequent players.

The third part of the argument proceeds by contradiction. Suppose then that we have an equilibrium with full learning. Conditional on $\omega = 1$, the players' beliefs must converge to one; learning cannot stop. So some player $t - 1$ must have a beginning-of-period belief arbitrarily close to one. For the purposes of this illustration we focus on the case in which this player fully transmits his end-of-period belief to player t. In other words, after observing s_{t-1} equal to zero or to one he passes on to t his end-of-period belief, x' or x, using messages m' or m respectively.[7] Since $t - 1$'s beginning-of-period belief is arbitrarily close to one, x and x', and m and m' are as in the second part of the argument above.

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[7]Notice that clearly it must be that $x > x'$. Notice also that since we are in the private communication case, the picture may be different from the case we are concentrating on here. In particular, $t - 1$ may send different messages after observing s_{t-1} equal to zero and to one, but may at the same time fail to transmit his end-of-period belief to player t. This is because other "types" of player $t - 1$ may be using the same messages on the equilibrium path. Of course, in the Appendix we treat an exhaustive set of possible cases.

Now consider the continuation payoff of player $t-1$, at the end of the period, as a function of the message he sends. This can be divided into the payoff that accrues to him at t, and the stream of payoffs that accrue to him from $t+1$ onward — the continuation payoffs of player t about which we know from the second part of the argument. Assume that player $t-1$ has observed $s_{t-1} = 0$ and hence that he is supposed to send message m' to player t, thus inducing him to have belief x'.

As far as period t is concerned, player $t-1$ now has an incentive to deviate and send message m. This is for exactly the same reason that we pointed out in Subsection 1.2 above, with the help of Figure 1. It follows directly from the fact that $x-x'$ is small. Notice that the gain from deviation that accrues to player $t-1$ at time t shrinks to zero as x and x' approach one and hence $x-x'$ approaches zero. Nevertheless, this gain can be bounded below by a function linear in $x-x'$. In fact it is bounded below by $(x-x')\alpha$.

Next, consider the change in continuation payoffs from $t+1$ onwards that is generated by a deviation from $t-1$'s part to send m instead of m'. This is hard to put a sign on. The difficulty is similar to the one we outlined in Subsection 1.2 above. A single deviation at $t-1$ could make a large difference a long way down the line. However, we do know that this payoff difference has a lower bound of the form $(x-x')A$, with A a term that depends on t's continuation payoffs. This is because $t-1$'s continuation payoffs from $t+1$ onwards are averages of player t's continuation payoffs from the second part of the argument. Using the fact that t's continuation payoffs are all close together as in the second part of the argument, we can then show that as $x-x'$ shrinks to zero, the term A in the lower bound also shrinks to zero. Hence, as x and x' approach one and hence $x-x'$ approaches zero, the gain to $t-1$ from deviating that accrues to him at time t must be larger than all possible losses from the deviation in periods after t. So, player $t-1$ will want to deviate and send m after observing $s_{t-1} = 0$. This destroys the putative equilibrium in which full learning occurs.

We can now move on to our second main result.

4.   No Full Learning with Public Communication

Just as in the case of private communication, with public communication full learning of the state cannot occur in equilibrium. In this case too, this is so regardless of the discount factor, signal precision, bias or prior.

Proposition 2. *No Full Learning with Public Communication: Fix any M, δ, p, α and r. Then there is no equilibrium $\boldsymbol{\sigma}^*$ of the model with public communication that induces full learning of the state.*

The conclusion of Proposition 2 is the same as for Proposition 1: posterior beliefs do not converge in probability to the true state. In this case, this is true despite the fact that decision makers can access the entire history of messages that precedes them.

As with Proposition 1, a formal proof of the claim is in the Appendix. Here we give an informal sketch of the argument, which for convenience is divided into five parts.

The first part of the argument consists of the observation that in the public communication case, since we are only considering pure strategies each player's behavior at the message stage can only be of two varieties. He can either babble and reveal nothing, or he can fully reveal his signal to all subsequent players. Hence, there is no loss of generality in setting the message space of all players equal to $\{0, 1\}$. There is also no loss of generality in considering (as we do both here and in the Appendix) only the case in which all players who babble send message one, and all players who reveal their signal send message one when the signal is one, and message zero when the signal is zero.

The second part of the argument considers the possible continuation payoffs of a player, say t, who truthfully reveals his signal to his successor $t + 1$. These are the possible continuation payoffs at the message stage of a player t whose end-of-period belief is the same as the beginning-of-period belief of $t + 1$. As the strategy profile of all players from $t + 1$ on (the "continuation profile") varies, so will the continuation payoff of player t. It can be shown that as we consider all possible continuation profiles (regardless of whether they are part of an equilibrium or not), the continuation payoff of t achieves a maximum when the continuation profile is truthful revelation from all players from $t + 1$ on, and achieves a minimum when the continuation profile is babbling for all players from $t + 1$ on.

The third part of the argument concerns a different thought experiment concerning t's continuation payoffs at the end of period t. In particular, we consider the hypothetical situation in which t's end-of-period belief is zero (so that he is certain that $\omega = 0$), but $t + 1$ and all subsequent players think that t's end-of-period belief is $x_{t+1} > 0$.[8] Somewhat abusing terminology we will refer to x_{t+1} as a "fresh prior" for $t + 1$ and all subsequent players. When x_{t+1} is close to one we are able to show that, as we consider all possible continuation profiles (regardless of whether they are part of an equilibrium or not), t's continuation payoff achieves a minimum when the continuation profile is babbling for all players from $t + 1$ on.

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[8]This is obviously not something that can take place given any strategy profile when communication is public. Nevertheless, it is a useful case to consider as an "anchor" to look at t's continuation payoffs after possible deviations from a putative equilibrium with full learning. This is exactly how this part of the argument will feed into the next one.

In this case, unlike in the second part of the argument above, considering directly all possible continuation profiles in comparison with babbling turns out not to be a practical line of argument. We proceed by simplifying the problem with an observation. It obviously suffices to show that the babbling continuation profile is a solution to the problem of *minimizing t*'s continuation payoff, given t's belief of zero and x_{t+1} close to one. However, once we consider this minimization problem, it is not hard to see that only continuation profiles that are Markov — so that behavior only depends on beliefs and not on which history leads to those beliefs — need to be considered. This simplifies the problem considerably, and allows us to reach the stated conclusion. It is interesting to notice that restricting attention to Markov strategies for the purpose of this characterization is a way to circumvent precisely the difficulty of "trapped" beliefs that we outlined in Subsection 1.2 above.

The fourth part of the argument simply puts together what we know from the second and third parts. Notice that the continuation payoff of player t for a given continuation strategy profile and a given fresh prior x_{t+1} is actually linear in t's end-of-period belief. This, using what we know about how t's continuation payoff changes as we change the continuation profile, has the following implication. Whenever the fresh prior x_{t+1} is near one and t's end-of-period belief is anywhere below x_{t+1}, the continuation payoff to t is minimized when the continuation profile is babbling for all players from $t + 1$ on.

The fifth part closes the argument by contradiction, assuming that there is a putative equilibrium in which full learning occurs. Clearly, this implies that we must be able to find a player t with a beginning-of-period belief arbitrarily close to one who reveals his signal to player $t + 1$. Let x_{t+1}^0 and x_{t+1}^1 be the beginning-of-period belief of $t + 1$ after he receives message zero and one respectively. In this putative equilibrium these are also the end-of-period beliefs of player t after observing a signal of zero and one respectively. Moreover, both x_{t+1}^0 and x_{t+1}^1 are arbitrarily close to one and they obviously satisfy $x_{t+1}^0 < x_{t+1}^1$.

Now consider player t, after observing $s_t = 0$. His end-of-period belief is x_{t+1}^0. His putative equilibrium strategy tells him to send message zero. Upon receiving this message $t+1$'s belief will be x_{t+1}^0. Hence, using what we know from the second part of the argument, an upper bound on t's continuation payoff at this point can be obtained by computing his payoff when the continuation profile is truthful. Suppose instead that t deviates and sends message one. Since the putative equilibrium prescribes that he is truthful, his message is believed by $t + 1$ and all subsequent players. After the deviation the end-of-period belief of t is x_{t+1}^0, but $t + 1$ and all subsequent players believe it to be x_{t+1}^1. So, x_{t+1}^1 is now just like the fresh prior above for $t + 1$ and all subsequent players. Therefore from part four of the argument we know

that in this case a lower bound on t's continuation payoff can be obtained assuming that the continuation profile is babbling from $t + 1$ on.

Since the two continuation profiles involved in the bounds are perpetual truth-telling in one case and perpetual babbling in the another, the comparison is relatively simple to carry out. This computation tells us that a player t with end-of-period belief x_{t+1}^0 prefers the continuation payoff he gets under a babbling continuation profile with $t+1$ and all subsequent players having a fresh prior of x_{t+1}^1 to the continuation payoff he gets under a truth-telling continuation profile with player $t + 1$ having a belief of x_{t+1}^0. Hence, after observing $s_t = 0$ player t wants to deviate from the putative equilibrium and send $t + 1$ a message of one, instead of a message of zero as the equilibrium prescribes.

Finally, as we noted in Subsection 1.2 above, it should be pointed out that our argument yields a stronger result than the formal statement of Proposition 2. In particular, the bounds on continuation payoffs that hold for any continuation profile (equilibrium or not) and their comparison as in the fifth part of the argument above show that there are thresholds (upper and lower in fact) to the possible equilibrium beliefs. For every given δ, p α and r, these correspond to points at which learning ceases altogether. Only babbling can occur after they are reached.

5. Infinite Patience

5.1. Limit Learning

Intuitively, as the discount factor δ increases the players' preferences become closer to complete alignment. The bias term only affects the current period and the weight of the current period in each player's preferences obviously decreases as δ grows.

Since the main source of the difficulties in learning the state is the mis-alignment in the players' preferences, intuitively we would expect learning to become easier as the players' patience grows. We say that *limit learning* is possible if there are equilibria with full learning in the limit as δ approaches 1.

Our next task is to investigate whether limit learning is possible in our model. The answer is that, among other things, it depends on the type of communication allowed. In the model with public communication we are able to show that when p is not too large limit learning is impossible. On the other hand in the model with private communication we can show that, if α is not too large relative to p, then limit learning can occur.

Before proceeding, we need to make the notion of limit learning precise.

Definition 2. _Limit Learning: We say that limit learning of the state is possible if and only if as δ approaches 1 we can find a sequence of equilibria along which the true state is learned with a precision and a probability both approaching 1._

Fix p, α and r. In symbols, we say that limit learning is possible if and only if for every ε > 0 we can find a $\bar{t} \geq 1$ and a $\bar{\delta} < 1$ such that for every $\delta > \bar{\delta}$ there is a collection of message spaces M and an equilibrium $\boldsymbol{\sigma^*}(\delta)$ such that for every $t > \bar{t}$

$$\text{Pr}_{\boldsymbol{\sigma^*}(\delta)} \left\{ x_t > 1 - \varepsilon \ \Big| \ \omega = 1 \right\} > 1 - \varepsilon \quad \text{and} \quad \text{Pr}_{\boldsymbol{\sigma^*}(\delta)} \left\{ x_t < \varepsilon \ \Big| \ \omega = 0 \right\} > 1 - \varepsilon$$

We are now ready for our first result on limit learning.

5.2. Private Communication

Provided the bias is not too large relative to the signal's quality, limit learning is possible when communication is private.

Proposition 3. _Limit Learning with Private Communication: Fix any r and suppose that $\alpha < p - 1/2$. Then limit learning is possible in the model with private communication._[9]

At this point it is worth mentioning again that, as we alluded to in Subsection 2.1, the limit learning of the statement of Proposition 3 can be sustained with Sequential Equilibria rather than just with WPBE.

A formal proof of Proposition 3 is in the Appendix, where we construct an appropriate sequence of Sequential Equilibria. As with our previous two results we only give a sketch here. For the sake of clarity, this is divided into four parts. Throughout this sketch, for simplicity we imagine that the prior r is equal to one half. If this were not the case the review phase we sketch below would not be symmetric making the outline a lot more cumbersome to follow. The role of the condition $\alpha < p - 1/2$ would also be lengthier to explain.

The argument begins with two observations. The condition $\alpha < p - 1/2$ in the statement of the proposition has an immediate interpretation. This condition tells us that any player with a belief of $1 - p$ or less will strictly prefer action $-\alpha$ to action $1 - \alpha$ to be taken in the future. Moreover, these preferences will be reversed for any player with a belief of p or more. Since $r = 1/2$ it is clear that the belief of any player who knows (or believes) the history of

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[9]In the Appendix we prove a slightly stronger statement. The one given here is phrased to allow direct comparability with the statement of Proposition 4 below. The proof of Proposition 3 in the Appendix yields a collection M of message spaces and an equilibrium $\boldsymbol{\sigma^*}(\delta)$ satisfying the requirements of Definition 2 which do not in fact depend on δ but only on the value of ε (or equivalently $\bar{\delta}$) in Definition 2.

signals to contain more zeroes than ones will have a belief of $1-p$ or less, while any player who knows (or believes) the history of signals to contain more ones than zeroes will have a belief of p or more. So, the majority of signals being zero or one will determine a player's strict preferences between future actions $-\alpha$ and $1-\alpha$ as above.

The second part of the argument outlines the mechanics of the review phase that characterizes the equilibria we construct to support limit learning. In the first T periods of play, with T a large odd number, equilibrium consists of a review phase. The purpose of the review phase is to establish whether the majority of signals in the first T periods is zero or one. The review phase is said to have outcome zero or one according to which one of these events occur. Of course if the proportion of zeroes or of ones is very large in the initial part of the review phase, its outcome may be determined early.

During the first T periods, unless the outcome has already been determined, the players collectively count the number of signals equal to one that have been realized so far. Each player adds one to his predecessor's message if he observes a signal of one, and leaves the count unchanged if he observes a signal of zero. He then sends the new tally as a message to the next player. If at any point the outcome of the review phase is determined, the current player sends a message that indicates that the outcome of the review phase has been determined and whether it is zero or one. From that point on, up to T and beyond, only that message is sent forward through time each period and thus reaches all subsequent players. No further information is ever added.

The third part of the argument consists of noticing that, using the Weak Law of Large Numbers, making the review phase longer and longer, we can ensure that the beliefs of all players from $T+1$ onwards are arbitrarily close to the true state (within ε of it), with arbitrarily large probability (at least $1-\varepsilon$) as required for limit learning to occur. Intuitively, this is clear from the fact that the signal is informative and, in equilibrium, all players from $T+1$ onwards only know whether the majority of a large number T of realizations of the signal is zero or one.

The fourth part of the argument consists of the verification that no player wants to deviate from the prescribed behavior. It is useful to consider two cases. Begin with any player $t \geq T+1$, after the review phase is necessarily over. As we noted in the third part of the argument, this player's belief is either arbitrarily close to zero or arbitrarily close to one, depending on the message he receives about the outcome of the review phase. If he deviates and reports an outcome of the review phase different from the one he received, all subsequent actions will be arbitrarily close to $-\alpha$ and his belief arbitrarily close to one in one case, and will be

arbitrarily close to $1 - \alpha$ with his belief arbitrarily close to zero in the other case. Therefore, because of what we know from the first part of the argument, no player $t \geq T + 1$ stands to gain from deviating from the proposed equilibrium.

Next, consider any player $t \leq T$ when the outcome of the review phase may or may not have been determined. Consider his continuation payoff accruing from time $T + 1$ onward. Note that, whatever t tells his successor, the actions from $T + 1$ onward will all be arbitrarily close to either $-\alpha$ or $1 - \alpha$, according to whether the outcome of the review phase will be declared to be zero or one.

Now consider what the belief of player t would be at the beginning of period $T + 1$, if he could observe the signal realizations in all periods from t to T. Assume that all players, including t, obey the equilibrium strategies we have described. Using Bayes' rule, this belief would clearly be p or more for those realizations of signals that entail an outcome of the review phase equal to one, and would be $1 - p$ or less for those realizations of signals that entail an outcome of the review phase equal to zero. If he plays according to his equilibrium strategy, from $T + 1$ onwards player t's payoff will be the one corresponding to an action close to $1 - \alpha$ in the former case and the one corresponding to an action close to $-\alpha$ in the latter case. It then follows from the first part of the argument that, if t plays according to his equilibrium strategy, the continuation actions from $T + 1$ onwards will be matched to what he prefers, contingent on every possible realizations of signals.

If on the other hand t deviates from his equilibrium strategy, it is not hard to see that the deviation will make his beliefs and preferred continuation actions from $T + 1$ onwards mismatched with positive probability. Since δ is appropriately high, gains accruing before $T + 1$ can be ignored. Hence, no player $t \leq T$ wants to deviate from the proposed equilibrium.

Our next concern is limit learning in the public communication case.

5.3.   Public Communication

Provided that the signal quality is not too high, limit learning cannot occur when communication is public. Propositions 3 and 4 taken together imply that, for some configurations of parameters, limit learning is possible under private communication but is impossible when communication is public.

Proposition 4. *No Limit Learning with Public Communication: Fix any α and r and suppose that $p < \sqrt{2}/(1 + \sqrt{2})$. Then limit learning is impossible in the model with public communication.*

Proposition 4 is proved in the Appendix. As previously, a brief sketch of the argument follows.

Fix a discount factor δ and an equilibrium. From the proof of Proposition 2 we know that there is a greatest lower bound to the values that the belief of any player can take in this equilibrium. Denote this bound by $\underline{x}$, and denote $\underline{x}^0$ and $\underline{x}^1$ the beliefs obtained by updating $\underline{x}$ on the basis of a signal of zero and one respectively. In the terminology of Subsection 1.2, the beliefs $\underline{x}^0$ and $\underline{x}^1$ are one notch down and one notch up from $\underline{x}$ respectively.

The argument proceeds by contradiction. Suppose that the proposition is false, then as δ approaches one we must be able to find a sequence of equilibria with $\underline{x}$ approaching zero, so that limit learning can occur. (Limit learning of course requires more, but this is clearly a necessary condition for it to be possible at all.)

Since $\underline{x}$ is a greatest lower bound on beliefs, we must be able to find a player, say t, who has a beginning-of-period belief $\underline{x}^1$ and reveals his signal truthfully to subsequent players. So, after observing $s_t = 0$, if t follows his equilibrium strategy, the beginning-of-period belief of all players from $t + 1$ onwards is $\underline{x}$ and the action $\underline{x} - \alpha$ is chosen in every period.

Now focus on t entertaining a deviation to reporting a signal realization of one after observing $s_t = 0$. It is convenient to look at t's continuation payoff component by component. Fix a given period ahead, say $t + \tau$, and a sequence of realized signals between t and $t + \tau$. Two cases are possible. Given the sequence of signals, either at $t + \tau$ the lower bound has been achieved and hence action $\underline{x} - \alpha$ is chosen, or the bound has not been achieved and an action $\hat{x} - \alpha > \underline{x} - \alpha$ is chosen instead. In the former case, clearly player t is indifferent between following his equilibrium strategy and the hypothesized deviation.

Consider then the case in which action $\hat{x} - \alpha > \underline{x} - \alpha$ is chosen at $t + \tau$. Notice that since player $t + \tau$ does not know that t deviated, his beginning-of-period belief is two notches up from what it would have been if he had known about t's deviation. Clearly, the latter is how t evaluates this particular component of his continuation payoff after the hypothesized deviation. Moreover, t's updated belief must be at least $\underline{x}^0$ since the lower bound has not been reached and hence $t + \tau$'s beginning-of-period belief must be at least $\underline{x}^1$.

The condition that $p < \sqrt{2}/(1 + \sqrt{2})$ of the statement of the proposition now allows us to close the argument. In fact when p is low in this sense, so that each notch of updating makes an appropriately small difference to beliefs, straightforward algebra is sufficient to show the following. Assume that $\underline{x}$ is sufficiently low, and consider any player with a belief x at least as large as $\underline{x}^0$. Denote by x^{11} the belief two notches up from x. Then the player with belief x strictly prefers action $x^{11} - \alpha$ to action $\underline{x} - \alpha$. However, this implies that for $\underline{x}$ sufficiently

low, player t finds the deviation hypothesized above strictly profitable. This destroys the putative equilibrium with $\underline{x}$ arbitrarily low corresponding to δ arbitrarily close to one, and hence closes the argument for Proposition 4.

Our next and last concern is the case of communication that takes place via a mediator.

6.   Mediated Communication

The fact that communication can take place via a mediator (Forges, 1986, Myerson, 1982) seems a particularly contrived case in our set up. On top of the standard requirements for mediated communication, in our case the mediator would have to work across time periods. In fact, our possibility result below involves a mediator who works across a number of time periods that grows without bound.

In spite of this, we now turn to this model. The reason is that in this case our main results are overturned: full learning can occur. Precisely because mediated communication seems so unnatural in this context, in our view this strengthens our original negative results. But of course this is a matter of interpretation.

The structure of the equilibria that yield this possibility also seems of independent interest. They are a natural variation on those that yielded limit learning in Proposition 3 above.

In the mediated communication model, we take it to be the case that each player t reports to a mediator the signal s_t that he observes, so that we can simply choose $M_t = \{0,1\}$ for every t. The mediator then recommends an action to each player t on the basis of the reports of all players from 1 to $t-1$.

This structure can be shown to be without loss of generality (Forges, 1986, Myerson, 1982), but this is besides the point in our case. Since we are establishing a *possibility* result, doing so for one particular class of mediated communication protocols is all that is required.

A mediated communication protocol $\mathbf{M}$ is a sequence of maps $\{\mu_t\}_{t=0}^{\infty}$. Each μ_t is from $\{0,1\}^{t-1}$ into $\mathbb{R}$. The interpretation is that $\mu_{t-1}(m_1, \ldots, m_{t-1})$ is the action that the mediator recommends to player t if the reports of the previous players are $(m_1, \ldots, m_{t-1})$. Consistently with the notation we established in Subsection 2.1, we set $m^{t-1} \equiv \mu_{t-1}(m_1, \ldots, m_{t-1})$.

We call $\mathbf{M}^*$ a mediated communication equilibrium if it satisfies the standard conditions that, given $\mathbf{M}^*$, all players are willing to truthfully reveal the signal they observe, and all players are willing to take whatever action $a_t = m^{t-1}$ prescribed by $\mathbf{M}^*$. Notice that any given $\mathbf{M}^*$, mapping sequences of signals into actions, induces a stochastic process $\{a_t\}_{t=1}^{\infty}$.

Recall that, using (3) and (4), player t is willing to take action a_t if and only if his

belief is equal to $x_t = a_t + \alpha$. Therefore, in line with Definition 1, we say that a mediated communication equilibrium $\mathbf{M}^*$ induces full learning of the state if and only if the induced stochastic process $\{a_t + \alpha\}_{t=1}^{\infty}$ converges in probability to the true state.

Our next result asserts that full learning can obtain in the mediated communication case.

Proposition 5. *Full Learning With a Mediator: Fix δ, p and r. Suppose that $\alpha < p - 1/2$. Then there is a mediated communication equilibrium $\mathbf{M}^*$ that induces full learning of the state.*

The proof of Proposition 5 is omitted for reasons of space, given that it is largely a notational and reinterpretation exercise once one has the details of the proof of Proposition 3. As with our other results, we provide a sketch of the argument here.

The communication equilibrium $\mathbf{M}^*$ that we propose induces full learning of the state using an infinite sequence of review phases like the one in the proof of Proposition 3, increasing in length through time. For ease of exposition we will refer to the mediator recommending an action to a player and to the belief induced by the mediator's message interchangeably. The former equals the latter minus α so that no ambiguity will ensue.

Fix a small number ε. Number the phases as 1 through to n and beyond, and let T_n be the length of each phase. Recall that δ, p and r are given, and that, by assumption, the condition $\alpha < p - 1/2$ of Proposition 3 is also satisfied here.

All players in each review phase, say n, receive a message from the mediator recommending an action based on the outcome of review phase $n - 1$ (the empty message in the case of review phase 1), and report truthfully to the mediator the signal they observe.

Each review phase n is sufficiently long (but may exceed the minimum length required for this as will be clear below) to ensure that, as in the proof of Proposition 3, via the Weak Law of Large Numbers, the belief of players in review phase $n + 1$ are within ε^n of the true state with probability at least $1 - \varepsilon^n$. This can be done by picking T_n large enough, just as in the proof of Proposition 3. For ease of reference below, call this the precision requirement for T_n. Notice that of course as n grows large and hence ε^n shrinks to zero this is sufficient to achieve full learning of the state as required.

We then need to check that no player wants to deviate from any of the prescriptions of $\mathbf{M}^*$ in any of the review phases. In one respect, this is simpler than in the case of Proposition 3. This is so because no player can affect the actions of other players in the same review phase. The mediator does not reveal to them any information pertaining to the current review phase. So, the incentives to reveal truthfully his signal for a player in review phase

$n-1$ only depend on the actions of players in review phases n and beyond. It then follows that by picking T_n sufficiently large we can ensure that the incentives for players in review phase $n-1$ are determined by their (strict) preferences over actions that will be taken during review phase n, regardless of the fixed value of the discount factor δ. Call this the incentive requirement for T_n.

We then pick each T_n to be large enough so that both the precision requirement and the incentive requirement are met, and, aside from one caveat, the argument is complete.

The caveat is that in our sketch of the proof of Proposition 3 we dealt with a single symmetric review phase that for simplicity was assumed to begin with a prior r equal to one half. Clearly, with multiple review phases, as n increases the staring point of each review phase will be further and further away from the symmetry guaranteed by $r = 1/2$. This point can be dealt in the same way as in the formal proof of Proposition 3 presented in the Appendix. The outcome of each review phase will not be determined just by simple majority, and the length requirements of T_n mentioned above will have to take these asymmetries into account.

7.   Conclusions

This paper examines a model of learning through intergenerational strategic communication. We study the polar cases of private and public communication. The main results show that in either model, full learning is impossible. In this sense, "standing on the shoulders of giants" has its limits.

Demonstrating the impossibility of full learning involves a number of difficulties in both cases. Even under a set of simplifying assumptions, dealing with the continuation payoffs of the players who are both senders and receivers of information in an open-ended model is a non-trivial task. In most cases, a direct evaluation of continuation payoffs proves an intractable route to follow in our model. As a consequence our main results are proved using a variety of techniques to bound these payoffs appropriately.

We also ask whether limit learning can be sustained in the sense of a sequence of equilibria that yields full learning in the limit, as discounting shrinks to zero. The two cases of private and public communication yield different results in this case. For some configurations of parameters, limit learning is possible with private communication but it is impossible in the public communication case.

We also consider the case of communication via a mediator who can work across time periods that are arbitrarily far apart. In this case, full learning can be sustained in equilib-

rium. Because of the implausibility of mediated communication in our set up, we view this last result as reinforcing the message of our two main results, asserting that full learning is impossible in our model.

Our results generalize to the case in which the state can take any value in a finite set, and to correspondingly richer signal structures, provided that the imperfect signals are still informative and i.i.d.

We do not know how far our results generalize to models in which the signal quality can change through time. These changes could take place either exogenously because of, say, technological advances, or endogenously, because of investment choices on the part of the players. Both cases seem worth exploring, but are obviously well beyond the scope of the present paper.

Appendix

A.1. The Proof of Proposition 1

Definition A.1: *Fix any δ, p, α and r. Consider M and $\hat{M}$. Let $\boldsymbol{\sigma}^*$ and $\hat{\boldsymbol{\sigma}}^*$ be equilibrium strategy profiles when the message spaces are M and $\hat{M}$ respectively.*

We say that $\boldsymbol{\sigma}^*$ and $\hat{\boldsymbol{\sigma}}^*$ are outcome equivalent if and only if they induce the same stochastic process $\{x_t\}_{t=1}^{\infty}$ on the players' posteriors for every realization of the state ω.

Lemma A.1: *Fix any M, δ, p, α, r and an equilibrium strategy profile $\boldsymbol{\sigma}^*$. Then there exist an $\hat{M} = \{\hat{M}_t\}_{t=0}^{\infty}$ with $\hat{M}_t \subseteq M_t$ for every t and a $\hat{\boldsymbol{\sigma}}^*$ which is an equilibrium given δ, p, α, r and $\hat{M}$ as follows.*

Given $\hat{\boldsymbol{\sigma}}^*$ and $\hat{M}$, all messages are sent with (ex-ante) positive probability, and $\hat{\boldsymbol{\sigma}}^*$ is outcome equivalent to $\boldsymbol{\sigma}^*$ according to Definition A.1.

Proof: Construct $\hat{M} = \{\hat{M}_t\}_{t=0}^{\infty}$ from $M = \{M_t\}_{t=0}^{\infty}$ by deleting all messages that are sent with ex-ante probability zero given $\boldsymbol{\sigma}^*$. Construct $\hat{\boldsymbol{\sigma}}^*$ by letting $\hat{\sigma}_t^*(m_{t-1}, s_t) = \sigma_t^*(m_{t-1}, s_t)$ for every t, every $m_{t-1} \in \hat{M}_{t-1}$ and every $s_t = \{0, 1\}$.

Clearly, $\hat{\boldsymbol{\sigma}}^*$ is an equilibrium when the message spaces are $\hat{M}$ since we have simply reduced the set of possible deviations. Also, by construction, the strategy profile $\hat{\boldsymbol{\sigma}}^*$ is outcome equivalent to the original strategy profile $\boldsymbol{\sigma}^*$. ∎

Remark A.1: *In view of Lemma A.1, we only consider specifications of M and equilibria $\boldsymbol{\sigma}^*$ in which all messages are sent with ex-ante positive probability throughout the rest of the proof of Proposition 1.*

Before proceeding further, some extra notation is needed. The entire batch is defined taking as given an M and an equilibrium strategy profile $\boldsymbol{\sigma}^*$ as in Remark A.1.

To begin with, define

$$X_t^B = \bigcup_{m_{t-1} \in M_{t-1}} x_t(m_{t-1}) \tag{A.1}$$

so that X_t^B is the set of all possible beginning-of-period beliefs of player t that $\omega = 1$. Because of Remark A.1 these can all be computed using Bayes' rule.

We let $x_t^E(m_{t-1}, s_t)$ denote the end-of-period belief of player t that $\omega = 1$ when he receives message m_{t-1} and observes signal $s_t = 0, 1$ during period t. Notice that using Bayes rule

$$x_t^E(m_{t-1}, s_t) = \frac{x_t(m_{t-1})\Pr(s_t|\omega = 1)}{x_t(m_{t-1})\Pr(s_t|\omega = 1) + [1 - x_t(m_{t-1})]\Pr(s_t|\omega = 0)} \tag{A.2}$$

Consider player t and any message $m_{t-1} \in M_{t-1}$ that player t can receive. We let $U_t(m_{t-1}, \omega)$ be the discounted continuation payoff of player t if the state is ω and he behaves as if he has received message m_{t-1} (in this case, we say player t *behaves according to* m_{t-1}). The payoff is computed after player t has chosen his action and before he observes his signal s_t. The continuation payoff is normalized so that weight $(1 - \delta)$ is given to the expected payoff in period $t + 1$ and weight $(1 - \delta)\delta^\tau$ is given to the expected payoff in each period $t + \tau + 1$.

Formally, for any $\tau \geq 1$, let $a_{t+\tau}(m_{t-1}, s^{t, t+\tau-1})$ denote the action that player $t + \tau$ will choose if player t behaves according to m_{t-1} and the sequence of signals between period t and period $t + \tau - 1$ is $s^{t, t+\tau-1} = (s_t, s_{t+1}, \ldots, s_{t+\tau-1})$.[10] Then $U_t(m_{t-1}, \omega)$ is given by

$$U_t(m_{t-1}, \omega) = -(1 - \delta) \sum_{\tau=1}^{\infty} \delta^{\tau-1} \sum_{s^{t, t+\tau-1} \in \{0,1\}^\tau} \Pr(s^{t, t+\tau-1}|\omega)[\omega - a_{t+\tau}(m_{t-1}, s^{t, t+\tau-1})]^2 \tag{A.3}$$

Lastly, notice that $U_t(m_{t-1}, \omega)$ as defined in (A.3) satisfies the recursive relationship

$$\begin{aligned} U_t(m_{t-1}, \omega) = {}&\Pr(s_t = 0|\omega) \left\{ -(1 - \delta)[\omega - a_{t+1}(m_{t-1}, s^{t,t} = 0)]^2 + \delta U_{t+1}[\sigma_t(m_{t-1}, s_t = 0), \omega] \right\} + \\ &\Pr(s_t = 1|\omega) \left\{ -(1 - \delta)[\omega - a_{t+1}(m_{t-1}, s^{t,t} = 1)]^2 + \delta U_{t+1}[\sigma_t(m_{t-1}, s_t = 1), \omega] \right\} \end{aligned} \tag{A.4}$$

Lemma A.2: *Fix an M and $\boldsymbol{\sigma}^*$ as in Remark A.1. For any $\eta > 0$ there exists an $\varepsilon > 0$ such that the following is true for every t. Suppose that m_{t-1} and m'_{t-1} are two messages in M_{t-1} with $\min\{x_t(m_{t-1}), x_t(m'_{t-1})\} > 1 - \varepsilon$ and $x_t(m_{t-1}) \neq x_t(m'_{t-1})$. Then*

$$|U_t(m_{t-1}, 1) - U_t(m'_{t-1}, 1) - U_t(m_{t-1}, 0) + U_t(m'_{t-1}, 0)| < \eta \tag{A.5}$$

Proof: Omitted. Available at http://www.anderlini.net/learning-omitted-proofs.pdf

We are now ready to proceed with the main part of the argument for Proposition 1.

Proof of Proposition 1: Fix any δ, p, α and r. Let also an M and $\boldsymbol{\sigma}^*$ as in Remark A.1. Suppose by way of contradiction that the equilibrium $\boldsymbol{\sigma}^*$ induces full learning as in Definition 1. Distinguish between the following two possibilities.

Case (a). For every $\varepsilon > 0$, we can find a $\bar{t}$ and a pair of messages $m_{\bar{t}-1}$ and $m'_{\bar{t}-1}$ with $\min\{x_{\bar{t}}(m_{\bar{t}-1}), x_{\bar{t}}(m'_{\bar{t}-1})\} > 1 - \varepsilon$ and $x_{\bar{t}}(m_{\bar{t}-1}) \neq x_{\bar{t}}(m'_{\bar{t}-1})$.

and

Case (b). There exists $\tilde{\varepsilon}$ such that for every t there is at most one element of X_t^B (the set of all possible beginning-of-period equilibrium beliefs of player t as in (A.1)) strictly greater than $1 - \tilde{\varepsilon}$.

Clearly one (and only one) of Cases *(a)* and *(b)* must hold.

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[10]Throughout the rest of the paper, for any $t' \geq t$, the array $(s^t, \ldots, s^{t'})$ will be denoted by $s^{t,t'}$.

We begin with Case *(a)*. Let $\eta = (1 - \delta)\alpha/2\delta$. Using Lemma A.2, we can pick an $\varepsilon \in (0, \alpha/2)$, a $\bar{t}$ and a pair of messages $m_{\bar{t}-1}$ and $m'_{\bar{t}-1}$ as specified in Case *(a)* such that

$$|U_{\bar{t}}(m_{\bar{t}-1}, 1) - U_{\bar{t}}(m'_{\bar{t}-1}, 1) - U_{\bar{t}}(m_{\bar{t}-1}, 0) + U_{\bar{t}}(m'_{\bar{t}-1}, 0)| < \eta = \frac{(1 - \delta)\alpha}{2\delta} \tag{A.6}$$

To keep notation usage down, during this part of the argument we let $m_{\bar{t}-1} = m$, $m'_{\bar{t}-1} = m'$, $x_{\bar{t}}(m_{\bar{t}-1}) = x$ and $x_{\bar{t}}(m'_{\bar{t}-1}) = x'$. Without loss of generality, assume $x > x'$.

Clearly, there exists a type $(\hat{m}_{\bar{t}-2}, \hat{s}_{\bar{t}-1})$ of player $\bar{t}-1$ who has the end-of-period belief $x_t^E(\hat{m}_{\bar{t}-2}, \hat{s}_{\bar{t}-1}) \geq x'$ and sends message m'. We now show that type $(\hat{m}_{\bar{t}-2}, \hat{s}_{\bar{t}-1})$ has an incentive to deviate and send message m. To keep notation usage down, during this part of the argument we let $\hat{x} = x_t^E(\hat{m}_{\bar{t}-2}, \hat{s}_{\bar{t}-1})$.

Let $\varphi(\hat{x})$ denote the difference between the continuation payoff of type $(\hat{m}_{\bar{t}-2}, \hat{s}_{\bar{t}-1})$ after he sends message m and the continuation payoff after he sends the equilibrium message m'. Using (3) and (A.4), the quantity $\varphi(\hat{x})$ can be written as

$$\varphi(\hat{x}) = (1 - \delta)\bar{\varphi}(\hat{x}) + \delta(A\hat{x} + B) \tag{A.7}$$

where

$$\bar{\varphi}(\hat{x}) = -\hat{x}(1 - x + \alpha)^2 - (1 - \hat{x})(x - \alpha)^2 + \hat{x}(1 - x' + \alpha)^2 + (1 - \hat{x})(x' - \alpha)^2$$

and

$$A = U_{\bar{t}}(m, 1) - U_{\bar{t}}(m', 1) + U_{\bar{t}}(m', 0) - U_{\bar{t}}(m, 0) \quad \text{and} \quad B = -U_{\bar{t}}(m', 0) + U_{\bar{t}}(m, 0) \tag{A.8}$$

Notice that it follows from the incentive constraints of player $\bar{t}$ (to whom the second term of the payoff difference (A.7) applies) that

$$Ax + B \geq 0 \geq Ax' + B \tag{A.9}$$

which, since $x > x'$, implies that $A \geq 0$.

We now show that $\varphi(\hat{x})$ is strictly positive and, therefore that type $(\hat{m}_{\bar{t}-2}, \hat{s}_{\bar{t}-1})$ of player $\bar{t} - 1$ has an incentive to deviate from his equilibrium strategy. We need to distinguish between two cases depending on whether $\hat{x}$ is smaller or greater than x. We start with the case $\hat{x} \in [x', x)$.

A lower bound for the expression $\bar{\varphi}(\hat{x})$ can be computed in the following way

$$\bar{\varphi}(\hat{x}) = (x - x')(2\alpha + 2\hat{x} - x - x') > (x - x')2(\alpha - \varepsilon) > (x - \hat{x})\alpha \tag{A.10}$$

where the first inequality follows from $\hat{x} > 1 - \varepsilon$ (and $x < 1$ and $x' < 1$), while the second inequality follows from $\varepsilon < \alpha/2$.

Consider now again the entire payoff difference $\varphi(\hat{x})$. We have

$$\varphi(\hat{x}) > (1 - \delta)(x - \hat{x})\alpha + \delta(A\hat{x} + B) =$$
$$(1 - \delta)(x - \hat{x})\alpha - \delta(x - \hat{x})A + \delta(Ax + B) \geq$$
$$(x - \hat{x})[(1 - \delta)\alpha - \delta A] > (x - \hat{x})(1 - \delta)\frac{\alpha}{2} > 0$$

where the first inequality comes from (A.7) and (A.10), the second follows from (A.9), the third follows from

(A.6) and the definition of A in (A.8), and the last one from the fact that we are considering the case in which $\hat{x} \in [x', x)$.

Consider now the case $\hat{x} \geq x$. We have that (recall that $A \geq 0$)

$$\varphi(\hat{x}) = (1-\delta)(x-x')(2\alpha + 2\hat{x} - x - x') + \delta(A\hat{x} + B) \geq$$
$$(1-\delta)(x-x')(2\alpha + x - x') + \delta(Ax + B) > 0$$

This proves that type $(\hat{m}_{\bar{t}-2}, \hat{s}_{\bar{t}-1})$ of player $\bar{t}-1$ has an incentive to deviate from the equilibrium prescription of sending message m', and hence concludes the argument for Case *(a)*.

We now turn to Case *(b)*. By our contradiction hypothesis, $\boldsymbol{\sigma^*}$ induces full learning of the state. Therefore, since we are in Case *(b)* as identified above, we can find $\tilde{t}$ such that for every $t > \tilde{t}$ there exists a unique beginning-of-period belief for player t above $1 - \tilde{\varepsilon}$. This belief must be the largest element in X_t^B and we denote it by $x_t^{\max}$.

Next, fix ε' such that

$$\frac{(1-\tilde{\varepsilon})p}{(1-\tilde{\varepsilon})p + \tilde{\varepsilon}(1-p)} < 1 - \varepsilon' \tag{A.11}$$

Given ε' we can find a period $t(\varepsilon') > \tilde{t}$ such that for every $t > t(\varepsilon')$

$$\frac{x_t^{\max}(1-p)}{x_t^{\max}(1-p) + (1-x_t^{\max})p} > 1 - \varepsilon' \tag{A.12}$$

For every t let $M_{t-1}^{\max} \subset M_{t-1}$ be the set of messages that induce belief $x_t^{\max}$, so that

$$M_{t-1}^{\max} = \{m_{t-1} \in M_{t-1} \mid x_t(m_{t-1}) = x_t^{\max}\}$$

Because $\boldsymbol{\sigma^*}$ induces full learning, the sequence $\{x_t^{\max}\}_{t=1}^{\infty}$ must converge to one. Using (A.11) and (A.12) it is then easy to show that there must exist a period $\bar{t}-1 > t(\varepsilon')$ and two types $(m_{\bar{t}-2}, s_{\bar{t}-1})$ and $(m'_{\bar{t}-2}, s'_{\bar{t}-1})$ in $M_{\bar{t}-2}^{\max} \times \{0,1\}$ which send two distinct messages, say m and m' respectively, such that

$$x_{\bar{t}}(m) > 1 - \varepsilon' \qquad \text{and} \qquad x_{\bar{t}}(m') < 1 - \tilde{\varepsilon}$$

It is also easy to check that when ε' is sufficiently small type $(m'_{\bar{t}-2}, s'_{\bar{t}-1})$ has an incentive to deviate and send message m instead of m' as the equilibrium prescribes. The analysis is very similar to the analysis of Case *(a)* and we omit the details. The argument in Case *(b)* is therefore concluded. Hence, the proof of the proposition is now complete. ∎

A.2.   The Proof of Proposition 2

We begin with some observations that do not require proof.

Remark A.2*: Given that we restrict attention to pure strategies, there is no loss of generality in assuming that the set of messages M_t available to player $t = 1, 2, \ldots$ is $\{0, 1\}$. This is what we do in the remainder of the proof.*

Hence, let $M^0 = \emptyset$ and define, for each $t \geq 1$, $M^t = \{0,1\}^t$. M^t is the set of messages profiles of length t. Denote an arbitrary element by $m^t = (m_1, \ldots, m_t) \in M^t$. Note that M^t contains the message profiles observed by player $t+1$.

Remark A.3: *Given the restriction to pure strategies, it is without loss of generality to restrict attention to strategy profiles $\boldsymbol{\sigma}$ such that:*

$$\sigma_t(m^{t-1}, s_t = 1) = 1 \tag{A.13}$$

for every $t = 1, 2, \ldots$, and every $m^{t-1} \in M^{t-1}$. Thus, if the signal s_t is revealed, then this done by matching messages and signals. Moreover if the the signal is not revealed, then the message "1" is sent regardless of the signal observed.

We say that player t adopts a truthful *strategy if*

$$\sigma_t(m^{t-1}, s_t = 0) = 0$$

for every $m^{t-1} \in M^{t-1}$. We say that a strategy profile is truthful if each player adopts a truthful strategy. We denote the truthful strategy profile by $\boldsymbol{\sigma}^{TR}$.

We also say that player t adopts a babbling *strategy if*

$$\sigma_t(m^{t-1}, s_t = 0) = 1$$

for every $m^{t-1} \in M^{t-1}$, so that, using (A.13) he sends message "1" regardless of s_t. We say that a strategy profile is babbling if each player adopts a babbling strategy. We denote the babbling strategy profile by $\boldsymbol{\sigma}^{B}$.

Remark A.4: *Fix a strategy profile $\boldsymbol{\sigma}$. For every $t \geq 1$ and every $m^t \in M^t$, recall that $\mathrm{Pr}_{\boldsymbol{\sigma}}(m^t|\omega)$ is the probability induced by strategy profile $\boldsymbol{\sigma}$ that player $t+1$ observes the message profile m^t when the state is ω. We also let $m^t(s^t, \boldsymbol{\sigma}) \in M^t$ denote the message profile generated by $\boldsymbol{\sigma}$ when the signal profile is s^t.*

Let

$$\mathcal{M}_{\boldsymbol{\sigma}}^t = \bigcup_{s^t \in \{0,1\}^t} m^t(s^t, \boldsymbol{\sigma}) \tag{A.14}$$

Notice that $\mathcal{M}_{\boldsymbol{\sigma}}^t$ is the set of messages that are observed by $t+1$ with positive probability under $\boldsymbol{\sigma}$, and that this set is independent of ω. This is simply because the quality of the signals p is less than one and therefore every sequence of signals s^t has positive probability, regardless of the state.

Before proceeding further, some extra notation is needed. In the remainder of the proof, we will often use the notation $x \in [0, 1]$, and variations of it like x' to denote a player's *prior*, in other words, the probability that this player attaches to the *original draw by nature* of ω being equal to 1. While in the main body of the paper this is just defined as $r \in (0, 1)$, we find it convenient to proceed in this way here because of the way this prior will be made to change at various points in the proof. This will turn out to be a convenient way to take into account the players' possible beliefs.

Fix a profile $\boldsymbol{\sigma}$, and a prior for player t equal to $x \in [0, 1]$. By Bayes' rule, after observing any $m^{t-1} \in \mathcal{M}_{\boldsymbol{\sigma}}^{t-1}$, player t's beginning-of-period belief that $\omega = 1$ must be

$$\mathrm{Pr}_{\boldsymbol{\sigma}}(\omega = 1|m^{t-1}, x) = \frac{x \, \mathrm{Pr}_{\boldsymbol{\sigma}}(m^{t-1} \,|\, \omega = 1)}{x \, \mathrm{Pr}_{\boldsymbol{\sigma}}(m^{t-1} \,|\, \omega = 1) + (1 - x)\mathrm{Pr}_{\boldsymbol{\sigma}}(m^{t-1} \,|\, \omega = 0)} \tag{A.15}$$

Recall that by (4), if his belief is as in (A.15), player t then chooses action $a_t = \mathrm{Pr}_{\boldsymbol{\sigma}}(\omega = 1|m^{t-1}, x) - \alpha$. To streamline the notation further, we let

$$\beta_t(s^{t-1}, \boldsymbol{\sigma}, x) = \mathrm{Pr}_{\boldsymbol{\sigma}}(\omega = 1 \,|\, m^{t-1}(s^{t-1}, \boldsymbol{\sigma}), x) \tag{A.16}$$

This is the beginning-of-period belief of player t if he has prior x, the players use strategy profile $\boldsymbol{\sigma}$, and the sequence of realized signals is s^{t-1}.

Some of our steps below are more easily conceptualized thinking by of a fictitious "player 0" added to the game. Player 0 does not make any choice, his payoff depends on the actions of future players and on ω in the same way as all other players. Given a sequence of actions $\{a_1, \ldots, a_t, \ldots\}$ and a state ω, player 0's payoff is

$$-(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} (\omega - a_t)^2$$

Our fictitious player 0 is also allowed to have a *different prior* from all other players $1, 2, \ldots$ This will be particularly helpful below when possible deviations are entertained.

Fix a prior x for player 0 and let the prior of all other players be x'. Fix also a profile $\boldsymbol{\sigma}$. Given these we know what (contingent) actions future players will take. We then let $V_x(\boldsymbol{\sigma}, x', t)$ be the expected one-period payoff that player 0 will receive in period t. Using (A.16) and (4), this can be computed as

$$V_x(\boldsymbol{\sigma}, x', t) = - \sum_{s^{t-1} \in \{0,1\}^{t-1}} \left[x \Pr(s^{t-1} | \omega = 1)(1 - \beta_t(s^{t-1}, \boldsymbol{\sigma}, x') + \alpha)^2 \right.$$
$$\left. + \ (1 - x) \Pr(s^{t-1} | \omega = 0)(\beta_t(s^{t-1}, \boldsymbol{\sigma}, x') - \alpha)^2 \right] \tag{A.17}$$

When $t = 1$, (A.17) reduces to

$$V_x(\boldsymbol{\sigma}, x', 1) \ = \ -x(1 - x' + \alpha)^2 - (1 - x)(x' - \alpha)^2 \tag{A.18}$$

Fix x, x' and $\boldsymbol{\sigma}$ as above. Given any $T \geq 1$ we let $V_x^T(\boldsymbol{\sigma}, x')$ denote the discounted expected payoff to player zero in the first T periods. Hence, using (A.17)

$$V_x^T(\boldsymbol{\sigma}, x') = (1 - \delta) \sum_{t=1}^{T} \delta^{t-1} V_x(\boldsymbol{\sigma}, x', t) \tag{A.19}$$

Still fixing x, x' and $\boldsymbol{\sigma}$ as above, we let $V_x(\boldsymbol{\sigma}, x')$ denote the overall discounted expected payoff of player zero. Hence, using (A.19)

$$V_x(\boldsymbol{\sigma}, x') = \lim_{T \to \infty} V_x^T(\boldsymbol{\sigma}, x') \tag{A.20}$$

Finally, observe that, since this is true period-by-period, all three quantities in (A.17), (A.19) and (A.20) are linear in x.

Lemma A.3: *Fix any p, α and δ. For any $x \in [0, 1]$ and any strategy profile $\boldsymbol{\sigma}$,*

$$V_x(\boldsymbol{\sigma}^{TR}, x) \ \geq \ V_x(\boldsymbol{\sigma}, x).$$

Proof: It suffices to show that for every $t \geq 1$

$$V_x(\boldsymbol{\sigma}^{TR}, x, t) \geq V_x(\boldsymbol{\sigma}, x, t). \tag{A.21}$$

Consider first $t = 1$. It is apparent from (A.18) that $V_x(\boldsymbol{\sigma}, x, 1)$ does not vary with $\boldsymbol{\sigma}$. The profile $\boldsymbol{\sigma}$ only enters payoffs from date $t = 2$ onward. Hence, (A.21) is trivially satisfied for $t = 1$.

Next, consider any date $t+1$ with $t \geq 1$. For any profile $\boldsymbol{\sigma}$, (A.16) and (A.17) imply

$$V_x(\boldsymbol{\sigma}, x, t+1) = -\sum_{m^t \in \mathcal{M}_{\boldsymbol{\sigma}}^t} \left[x\text{Pr}_{\boldsymbol{\sigma}}(m^t \mid \omega = 1)(1 - \text{Pr}_{\boldsymbol{\sigma}}(\omega = 1 \mid m^t, x) + \alpha)^2 + \right.$$
$$\left. (1-x)\text{Pr}_{\boldsymbol{\sigma}}(m^t \mid \omega = 0)(\text{Pr}_{\boldsymbol{\sigma}}(\omega = 1 \mid m^t, x) - \alpha)^2 \right] \tag{A.22}$$

Using (A.15) and some simple algebra we get

$$x \sum_{m^t \in \mathcal{M}_{\boldsymbol{\sigma}}^t} \text{Pr}_{\boldsymbol{\sigma}}(m^t \mid \omega = 1)(1 - \text{Pr}_{\boldsymbol{\sigma}}(\omega = 1 \mid m^t, x)) =$$
$$(1-x) \sum_{m^t \in \mathcal{M}_{\boldsymbol{\sigma}}^t} \text{Pr}_{\boldsymbol{\sigma}}(m^t \mid \omega = 0)\text{Pr}_{\boldsymbol{\sigma}}(\omega = 1 \mid m^t, x) \tag{A.23}$$

Expanding the square terms of the right-hand side of (A.22) and using (A.23), we can further rewrite $V_x(\boldsymbol{\sigma}, x, t+1)$ as

$$V_x(\boldsymbol{\sigma}, x, t+1) = -\sum_{m^t \in \mathcal{M}_{\boldsymbol{\sigma}}^t} \left[x\text{Pr}_{\boldsymbol{\sigma}}(m^t | \omega = 1)(1 - \text{Pr}_{\boldsymbol{\sigma}}(\omega = 1 | m^t, x))^2 \right.$$
$$\left. + (1-x)\text{Pr}_{\boldsymbol{\sigma}}(m^t | \omega = 0)(\text{Pr}_{\boldsymbol{\sigma}}(\omega = 1 | m^t, x))^2 \right] - \alpha^2$$

And finally, using (A.16), we get

$$V_x(\boldsymbol{\sigma}, x, t+1) =$$
$$-\sum_{s^t \in \{0,1\}^t} [x\text{Pr}(s^t | \omega = 1)(1 - \beta_{t+1}(s^t, \boldsymbol{\sigma}, x))^2 + (1-x)\text{Pr}(s^t | \omega = 0)(\beta_{t+1}(s^t, \boldsymbol{\sigma}, x))^2] - \alpha^2 \tag{A.24}$$

Equation (A.24) is particulary useful to evaluate the difference between time-$t+1$ payoffs induced $\boldsymbol{\sigma}^{TR}$ and $\boldsymbol{\sigma}$. Under the truthful strategy profile $\boldsymbol{\sigma}^{TR}$, equality (A.16) reduces to

$$\beta_{t+1}(s^t, \boldsymbol{\sigma}^{TR}, x) = \frac{x\text{Pr}(s^t | \omega = 1)}{x\text{Pr}(s^t | \omega = 1) + (1-x)\text{Pr}(s^t | \omega = 0)} = \text{Pr}(\omega = 1 | s^t, x) \tag{A.25}$$

Then using (A.24) and (A.25) we can write,

$$V_x(\boldsymbol{\sigma}^{TR}, x, t+1) - V_x(\boldsymbol{\sigma}, x, t+1) =$$
$$\sum_{s^t \in \{0,1\}^t} \left\{ - \left[x\text{Pr}(s^t | \omega = 1)(1 - \text{Pr}(\omega = 1 | s^t, x))^2 + (1-x)\text{Pr}(s^t | \omega = 0)(\text{Pr}(\omega = 1 | s^t, x))^2 \right] \right. \tag{A.26}$$
$$\left. + \left[x\text{Pr}(s^t | \omega = 1)(1 - \beta_{t+1}(s^t, \boldsymbol{\sigma}, x))^2 + (1-x)\text{Pr}(s^t | \omega = 0)(\beta_{t+1}(s^t, \boldsymbol{\sigma}, x))^2 \right] \right\}$$

Now fix any $s^t \in \{0,1\}^t$ and consider the corresponding element in the summation sign in (A.26). Dividing though by $x\text{Pr}(s^t | \omega = 1) + (1-x)\text{Pr}(s^t | \omega = 0)$ we get

$$- \left[\text{Pr}(\omega = 1 | s^t, x)(1 - \text{Pr}(\omega = 1 | s^t, x))^2 + \text{Pr}(\omega = 0 | s^t, x)(\text{Pr}(\omega = 1 | s^t, x))^2 \right]$$
$$+ \left[\text{Pr}(\omega = 1 | s^t, x)(1 - \beta_{t+1}(s^t, \boldsymbol{\sigma}, x))^2 + \text{Pr}(\omega = 0 | s^t, x)(\beta_{t+1}(s^t, \boldsymbol{\sigma}, x))^2 \right] \tag{A.27}$$

To check that this difference is nonnegative, observe that the quantity

$$\text{Pr}(\omega = 1 | s^t, x)(1 - z)^2 + \text{Pr}(\omega = 0 | s^t, x)z^2$$

as a function of z achieves a minimum at $z = \Pr(\omega = 1|s^t, x)$. Consequently, $V_x(\boldsymbol{\sigma}^{TR}, x, t+1) \geq V_x(\boldsymbol{\sigma}, x, t+1)$ for $t \geq 1$, and hence the proof of Lemma A.3 is now complete. ∎

Lemma A.4: *Fix any p, α and δ. For any $x \in [0, 1]$ and any strategy profile $\boldsymbol{\sigma}$,*

$$V_x(\boldsymbol{\sigma}, x) \ \geq \ V_x(\boldsymbol{\sigma}^B, x).$$

Proof: The comparison between $V_x(\boldsymbol{\sigma}, x)$ and $V_x(\boldsymbol{\sigma}^B, x)$ can be carried out manipulating the former as in the proof of Lemma A.3, and evaluating the latter observing that the players' beliefs do not change through time under the babbling profile $\boldsymbol{\sigma}^B$. The details are omitted for the sake of brevity. ∎

Remark A.5: *Fix a prior r and a signal quality p. Suppose that someone could observe a signal profile of arbitrary length, comprising k_1 realizations of value 1 and k_0 realizations of value 0. To state the next result we need to introduce two further pieces of notation. Let $k = k_1 - k_0$. Then, using Bayes' rule and simple algebra, the belief of this observer that $\omega = 1$ would be given by*

$$\gamma(r, p, k) \ = \ \cfrac{1}{1 + \cfrac{1-r}{r} \left(\cfrac{1-p}{p}\right)^k}$$

Hence,

$$\Gamma(r, p) \ = \ \{x \in (0, 1) \mid x \ = \ \gamma(r, p, k) \quad \text{for some} \quad k = 0, \pm 1, \pm 2, \dots \} \tag{A.28}$$

denotes the set of beliefs that can be generated by observing arbitrary signal profiles of any length given r and p.

Lemma A.5: *Fix any p, α and δ. Then there exists an $\bar{x} < 1$ such that for every $x \in [\bar{x}, 1] \cap \Gamma(r, p)$ and every strategy profile $\boldsymbol{\sigma}$,*

$$V_0(\boldsymbol{\sigma}, x) \ \geq \ V_0(\boldsymbol{\sigma}^B, x).$$

Proof: Omitted. Available at http://www.anderlini.net/learning-omitted-proofs.pdf

Lemma A.6: *Fix p, α, and δ. There exists $\bar{x} < 1$ such that for every $x' \in [\bar{x}, 1] \cap \Gamma(r, p)$, every $x \leq x'$ and every strategy profile $\boldsymbol{\sigma}$,*

$$V_x(\boldsymbol{\sigma}, x') \geq V_x(\boldsymbol{\sigma}^B, x')$$

Proof: Recall that for any pair $(\boldsymbol{\sigma}, x')$ the payoff $V_x(\boldsymbol{\sigma}, x')$ is linear in x. Therefore, the claim is an immediate consequence of Lemmas A.4 and A.5. ∎

Proof of Proposition 2: Consider a player t, who observes the sequence of messages m^{t-1}, the signal $s_t = 0$ and holds the associated end-of-period belief x. Using Bayes' rule, if instead of observing $s_t = 0$ he were to observe $s_t = 1$, his posterior would be equal to

$$h(x) = \frac{xp^2}{xp^2 + (1-x)(1-p)^2} \tag{A.29}$$

Clearly $h(x) > x$ for every $x \in (0, 1)$.

Now suppose by way of contradiction that there is an equilibrium $\boldsymbol{\sigma}^*$ which induces full learning of the state as in Definition 1. Then, for any $\varepsilon > 0$ we can find a t and a sequence of messages $m^{t-1} \in \mathcal{M}_{\boldsymbol{\sigma}^*}^{t-1}$ such that

$$\Pr\nolimits_{\boldsymbol{\sigma}^*}(\omega = 1 | m^{t-1}, r) > 1 - \varepsilon \tag{A.30}$$

and

$$\boldsymbol{\sigma}_t^*(m^{t-1}, s_t = 0) = 0 \tag{A.31}$$

Let x denote the end-of-period belief of player t if he observes m^{t-1} and the signal $s_t = 0$. Clearly this can be computed applying Bayes' Rule to the left-hand side of (A.30). We take ε sufficiently small so that $x > \bar{x}$, where $\bar{x}$ is as in Lemma A.6.

Suppose that type $(m^{t-1}, s_t = 0)$ follows the equilibrium strategy and reveals his signal as required by (A.31). It then follows from Lemma A.3 that the continuation payoff (at the message stage) of type $(m^{t-1}, s_t = 0)$ is bounded above by $V_x(\boldsymbol{\sigma}^{TR}, x)$.

Suppose now that instead type $(m^{t-1}, s_t = 0)$ deviates and sends message one. Using $h(x) > x > \bar{x}$, in this case it follows from Lemma A.6 that his continuation payoff (at the message stage) is bounded below by $V_x(\boldsymbol{\sigma}^B, h(x))$.

It follows that $\psi(x) = V_x(\boldsymbol{\sigma}^B, h(x)) - V_x(\boldsymbol{\sigma}^{TR}, x)$ is a lower bound on the gain from deviating from equilibrium at the message stage for type $(m^{t-1}, s_t = 0)$. Using simple algebra we can write $\psi(x)$ as

$$-x(1 - h(x) + \alpha)^2 - (1-x)(h(x) - \alpha)^2 + (1-\delta)[x(1 - x + \alpha)^2 + (1-x)(x - \alpha)^2]$$
$$+ (1-\delta)\sum_{t=1}^{\infty}\delta^t \sum_{s^t \in \{0,1\}^t} \left\{ x \Pr(s^t|\omega = 1)\left[1 - \frac{x \Pr(s^t|\omega = 1)}{x \Pr(s^t|\omega = 1) + (1-x)\Pr(s^t|\omega = 0)} + \alpha\right]^2 \right.$$
$$\left. + (1-x)\Pr(s^t|\omega = 0)\left[\frac{x \Pr(s^t|\omega = 1)}{x\Pr(s^t|\omega = 1) + (1-x)\Pr(s^t|\omega = 0)} - \alpha\right]^2 \right\}$$

Notice next that $\psi(1) = 0$ and $h'(1) = [(1-p)/p]^2$. Therefore $\psi'(1)$ can be computed as

$$\psi'(1) = -\alpha^2 + 2\alpha\left(\frac{1-p}{p}\right)^2 + (1-\alpha)^2 + (1-\delta)[\alpha^2 - 2\alpha - (1-\alpha)^2] +$$
$$(1-\delta)\sum_{t=1}^{\infty}\delta^t \sum_{s^t \in \{0,1\}^t}\left[\Pr(s^t|\omega = 1)\alpha^2 - 2\alpha\Pr(s^t|\omega = 1)\frac{\Pr(s^t|\omega = 0)}{\Pr(s^t|\omega = 1)} - \Pr(s^t|\omega = 0)(1-\alpha)^2\right]$$

which after simplification yields

$$\psi'(1) = 2\alpha\left[\left(\frac{1-p}{p}\right)^2 - 1\right] < 0$$

The latter clearly implies that for x sufficiently close to one we must have that $\psi(x) > 0$. Hence we have shown that, for x sufficiently close to one, type $(m^{t-1}, s_t = 0)$ has an incentive to deviate from the putative equilibrium $\boldsymbol{\sigma}^*$ at the message stage. Therefore the proof of the proposition is now complete. ∎

A.3.   Proof of Proposition 3

Fix r and p, and recall from (A.28) that $\Gamma(r,p)$ is the set of beliefs that can be generated by observing arbitrary signal profiles of any length.

Let z denote the largest element of $\Gamma(r,p)$ not exceeding $1/2$. In other words, let $z = \max\{x \in \Gamma(r,p) \mid x \in (0,1/2]\}$. Consider a player with a beginning-of-period belief equal to z. After observing signals zero and one his beliefs are denoted z_0 and z_1 respectively. Using Bayes' rule, these can be trivially computed as

$$z_0 = \frac{z(1-p)}{z(1-p)+(1-z)p} \qquad \text{and} \qquad z_1 = \frac{zp}{zp+(1-z)(1-p)} \tag{A.32}$$

From (A.32) and the way we have defined z it is evident that $0 < z_0 \leq 1-p$ and $1/2 < z_1 \leq p$. Since by assumption $\alpha < p - 1/2$ it then follows easily that for some $\eta > 0$ it must be that for every $a' \in [-\alpha, \eta - \alpha]$ and every $a'' \in [1-\alpha-\eta, 1-\alpha]$ we have that both of the following inequalities hold

$$-z_0(1-a')^2 - (1-z_0)a'^2 > -z_0(1-a'')^2 - (1-z_0)a''^2$$

$$\tag{A.33}$$

$$-z_1(1-a')^2 - (1-z_1)a'^2 < -z_1(1-a'')^2 - (1-z_1)a''^2$$

In other words, any player who believes that $\omega = 1$ with probability z_0 prefers any $a' \in [-\alpha, \eta - \alpha]$ to any $a'' \in [1-\alpha-\eta, 1-\alpha]$. Moreover these preferences are reversed for any player who believes that $\omega = 1$ with probability z_1.

It useful to observe at this point that since $z \in \Gamma(r,p)$, there must exist an integer K (positive or negative) that satisfies

$$z = \frac{rp^K}{rp^K + (1-r)(1-p)^K} \tag{A.34}$$

Intuitively, K has the following property. Suppose that, beginning with a prior r, a player can observe a sequence of signals with the property that the number of signals equal to one minus the number of signals equal to zero is K. Then the player's belief that $\omega = 1$ is precisely z. For the rest of the proof we let T be an integer such that $T - 1 + K$ is an even positive integer.

Our construction relies on strategies that embody a "review phase" of length T, with T large. During this review phase the strategies keep track of the difference between the number of signals equal to one minus the number of signals equal to zero. If this difference is more than K, then the "outcome" of the review phase is "one," while if the difference is less than or equal to K then the "outcome" of the review is "zero."

It turns out that it is convenient for the argument that follows to write the basics of the review phase in terms of T and the number of signals equal to one, rather than in terms of the difference we have just mentioned. This is why we define the review phase "outcome function" $R_T : \{0,1\}^T \to \{0,1\}$ as follows. Consider an arbitrary sequence of signals $s^T = (s_1, \ldots, s_T)$ of length T, and let

$$R_T(s^T) = \begin{cases} 1 & \text{if} \quad \sum_{t=1}^{T} s_t > \dfrac{T-1+K}{2} \\[2mm] 0 & \text{if} \quad \sum_{t=1}^{T} s_t \leq \dfrac{T-1+K}{2} \end{cases}$$

The Weak Law of Large Numbers implies that for every $\varepsilon > 0$, there exists a $\bar{T}$ such that for every $T > \bar{T}$

$$\Pr[R_T(s^T) = i \mid \omega = i] > 1 - \varepsilon \qquad \forall\, i \in \{0, 1\} \tag{A.35}$$

and

$$\Pr[\omega = i \mid R_T(s^T) = i] > 1 - \varepsilon \qquad \forall\, i \in \{0, 1\} \tag{A.36}$$

We are now ready to specify the message spaces and strategies that induce limit learning of the state as required. Note that these will not depend on δ even though this is allowed in principle by the statement of Proposition 3 (see footnote 9).

Fix an arbitrary $\varepsilon > 0$ and an $\eta > 0$ as in (A.33). Then using (A.36) find an integer T that satisfies (A.35) and such that

$$\bar{x}_1 = \Pr[\omega = 1 \mid R_T(s^T) = 1] > \max\{1 - \varepsilon, 1 - \eta\}$$

$$\bar{x}_0 = \Pr[\omega = 1 \mid R_T(s^T) = 0] < \min\{\varepsilon, \eta\}$$

and

$$\frac{\bar{x}_1(1 - p)}{\bar{x}_1(1 - p) + (1 - \bar{x}_1)p} > z_1 \qquad \frac{\bar{x}_0 p}{\bar{x}_0 p + (1 - \bar{x}_0)(1 - p)} < z_0 \tag{A.37}$$

Recall that our claim is that limit learning can obtain in a Sequential Equilibrium in the sense of Kreps and Wilson (1982) (see Subsection 2.1), rather than just in a Perfect Bayesian Equilibrium. We accomplish this by constructing the message spaces in a particularly careful way, so that once the strategies are specified *all* messages will be sent in equilibrium with positive probability. In this way there will be nothing further to prove since off-path beliefs will not even need to be defined. With this in mind, we are now ready to proceed with the construction of message spaces.

In addition to the standard messages corresponding to integers, consider two additional ones labeled 0^* and 1^*. The integer messages are interpreted as counting the number of signals equal to one, while the 0^* and 1^* are used to "declare" the outcome of the review phase. The five possibilities for M_t so that no messages are ever off-path are then as follows

(i) If $t < T$, $t \le \dfrac{T - 1 + K}{2}$ and $t < \dfrac{T + 1 - K}{2}$ then $M_t = \{0, 1, \ldots, t\}$

(ii) If $t < T$, $t > \dfrac{T - 1 + K}{2}$ and $t < \dfrac{T + 1 - K}{2}$ then $M_t = \{0, 1, \ldots, \dfrac{T - 1 + K}{2}, 1^*\}$

(iii) If $t < T$, $t \le \dfrac{T - 1 + K}{2}$ and $t \ge \dfrac{T + 1 - K}{2}$ then $M_t = \{t - \dfrac{T - 1 - K}{2}, \ldots, t, 0^*\}$

(iv) If $t < T$, $t > \dfrac{T - 1 + K}{2}$ and $t \ge \dfrac{T + 1 - K}{2}$ then $M_t = \{t - \dfrac{T - 1 - K}{2}, \ldots, \dfrac{T - 1 + K}{2}, 0^*, 1^*\}$

(v) If $t \ge T$ then $M_t = \{0^*, 1^*\}$

We are now ready to construct the strategy profile $\boldsymbol{\sigma}^*$ that induces limit learning of the state as required. Intuitively, all players $t \le T$ truthfully participate in the review phase. They truthfully count the tally of signals equal to one they are given by their predecessors on the basis of the actual signal they observe. (Player 1 has no predecessor, so he just counts to one or to zero according to the signal he observes.) Once the review

phase is declared either by 0^* or 1^* all players babble, indefinitely into the future.

Formally, $\sigma_1^*(m_0, s_1) = s_1$ for all $s_1 \in \{0, 1\}$. For all $t \geq 2$, the strategy σ_t^* is then defined by

$$\sigma_t^*(m_{t-1}, s_t) = \begin{cases} m_{t-1} & \text{if } m_{t-1} \in \{0^*, 1^*\}, \text{ regardless of } s_t \\[2mm] m_{t-1} + 1 & \text{if } m_{t-1} + 1 \leqslant \dfrac{T-1+K}{2} \text{ and } s_t = 1 \\[3mm] 1^* & \text{if } m_{t-1} = \dfrac{T-1+K}{2} \text{ and } s_t = 1 \\[3mm] m_{t-1} & \text{if } m_{t-1} > t + \dfrac{K-T-1}{2} \text{ and } s_t = 0 \\[3mm] 0^* & \text{if } m_{t-1} \leq t + \dfrac{K-T-1}{2} \text{ and } s_t = 0 \end{cases} \qquad (A.38)$$

Observe that if the players use the strategy profile $\boldsymbol{\sigma}^*$ defined in (A.38), then, using (A.35) and (A.36), in any period $t > T$ the beginning-of-period belief x_t will be greater than $1 - \varepsilon$ (smaller than ε) with probability at least $1 - \varepsilon$ when the state is $\omega = 1$ ($\omega = 0$). Hence limit learning obtains under $\boldsymbol{\sigma}^*$, as required.

It remains to check that $\boldsymbol{\sigma}^*$ is in fact an equilibrium for δ sufficiently large.

Notice that in any period $t \geq T$ only two messages will ever be sent, and only the two actions $\bar{x}_0 - \alpha$ and $\bar{x}_1 - \alpha$ will ever be chosen. It then follows easily from (A.33) and (A.37) that no player wants to deviate from the proposed equilibrium after T.

Finally, consider any player $t \leq T$. If he follows the proposed equilibrium strategy defined in (A.38), from $T+1$ onwards the following will take place. The action $\bar{x}_1 - \alpha$ will be played if the prior r updated on the basis of the first T signal realizations yields a belief at least as large as z_1. Conversely, $\bar{x}_0 - \alpha$ will be played if the prior r updated on the basis of the first T signal realizations yields a belief at most equal to z_0. These two actions are the only ones that will possibly be taken in period $T+1$ and all subsequent periods. Hence it follows from (A.33) that a player t who plays according to σ_t^* will induce his preferred action among these two with probability one in every period greater than T. Deviating from σ_t^* will necessarily reduce this probability to be strictly less than one. While following the equilibrium strategy might be costly for player t in periods before T, this is clearly sufficient to show that he will not want to deviate from the proposed equilibrium if δ is close enough to one. ■

A.4. Proof of Proposition 4

In view of Remarks A.2 and A.3 we restrict attention to message spaces $M_t = \{0, 1\}$ for every t and to strategy profiles that satisfy (A.13) since this is without loss of generality.

We begin by observing that, as we remarked already, inspection of the proof of Proposition 2 reveals that if we fix any equilibrium $\boldsymbol{\sigma}^*$ then there exists an $\bar{x} < 1$ such that following any sequence of signals, the beginning-of-period belief x_t of any player t must satisfy $x_t \leq \bar{x}$, with equality holding for some t.

Using a similar line of argument it is possible to show that if we fix any equilibrium $\boldsymbol{\sigma}^*$ then there exists an $\underline{x} > 0$ such that following any sequence of signals, the beginning-of-period belief x_t of any player t must satisfy $x_t \geq \underline{x}$, with equality holding for some t. We omit the details for the sake of brevity.

Recall that in (A.29) we defined $h : [0, 1] \to [0, 1]$ as the belief x updated after the observation of two signals of one. Using the fact that by assumption $p < \sqrt{2}/(1 + \sqrt{2})$ and Bayes' rule, simple algebra is then sufficient to show that there exists an $\eta > 0$ such that for any $x' \in (0, \eta)$ and any $x \geq x'(1-p)/[x'(1-p) + (1-x')p]$ it must be that

$$-x[1 - h(x) + \alpha]^2 - (1-x)[h(x) - \alpha]^2 \quad > \quad -x[1 - x' + \alpha]^2 - (1-x)[x' - \alpha]^2 \qquad (A.39)$$

Now suppose by way of contradiction that limit learning of the state is possible under the hypotheses of the proposition. Let $\boldsymbol{\sigma}^*(\delta)$ be the sequence of equilibria needed for limit learning to be possible as in Definition 2 as ε shrinks to zero and δ approaches one. For each such equilibrium, let $\underline{x}_\delta$ be the lower bound on beginning-of-period beliefs as we just defined above.

Since η as needed for (A.39) to hold is in fact fixed, it follows from our contradiction hypothesis that choosing δ sufficiently large we can ensure that $\underline{x}_\delta < \eta$.

Hence there is at least one player, say $t + 1$, who with positive probability has beginning-period-belief $\underline{x}_\delta < \eta$. This implies that with positive probability player t has a beginning-of-period belief of $\underline{x}_\delta p/[\underline{x}_\delta p + (1 - \underline{x}_\delta)(1 - p)]$ and that his strategy must be to truthfully reveal his signal in this case. This is because in this way player t's belief after observing a signal of zero is precisely $\underline{x}_\delta$, and hence the lower bound on beginning-of-period beliefs is achieved in period $t + 1$.

Now consider the incentives of player t after observing $s_t = 0$. If he reports his signal truthfully as prescribed by the equilibrium then the action $\underline{x}_\delta - \alpha$ will be chosen in every subsequent period. This is because the lower bound on beliefs $\underline{x}_\delta$ will have been achieved, and this implies that all players will babble in the continuation equilibrium prescribed by $\boldsymbol{\sigma}^*(\delta)$.

Suppose now instead that player t with beginning-of-period belief $\underline{x}_\delta p/[\underline{x}_\delta p + (1 - \underline{x}_\delta)(1 - p)]$ deviates and reports a signal of one instead. This report is believed because the putative equilibrium $\boldsymbol{\sigma}^*(\delta)$ prescribes that t should tell the truth in this case.

Consider now an arbitrary period $t + \tau$ with $\tau \geq 1$ and an arbitrary sequence of signals $s^{t,t+\tau-1} = (s_t, \ldots, s_{t+\tau-1})$ with $s_t = 0$. Two cases are possible. The first case is that in period $t + \tau$ the action $\underline{x}_\delta - \alpha$ is chosen because the belief has reached its lower bound $\underline{x}_\delta$. In this case player t is clearly indifferent between telling the truth as prescribed by the equilibrium or deviating and reporting $m_t = 1$.

The second case is that $s^{t,t+\tau-1}$ is such that by period $t + \tau$ the lower bound for beliefs has not been reached and perpetual babbling has not necessarily begun. Let x be the belief that player $t+\tau$ would have at this point if he knew that player t had deviated. Since $t + \tau$ obviously does not know that t has deviated, his belief is simply $h(x)$. So, since the lower bound has not been reached we must have $h(x) > \underline{x}_\delta$. This implies that $x \geq \underline{x}_\delta(1 - p)/[\underline{x}_\delta(1 - p) + (1 - \underline{x}_\delta)p]$.

To sum up, in the second case in period $t+\tau$ an action $h(x) - \alpha$ is taken, where x is some belief satisfying $x \geq \underline{x}_\delta(1 - p)/[\underline{x}_\delta(1 - p) + (1 - \underline{x}_\delta)p]$. How does player t evaluate this situation? His updated belief must be in fact x. This is because in period t he observed a signal of zero, but in fact reported a signal of one, and a belief of x updated on the basis of two signals of one is precisely $h(x)$. Given this belief, it follows from inequality (A.39) that player t strictly prefers action $h(x) - \alpha$ to action $\underline{x}_\delta - \alpha$.

We conclude that the deviation is strictly profitable for t in every period $t+1, t+2, \ldots$ provided that the lower bound in beliefs has not been reached. Since the probability that the bound has not been reached in any period is obviously strictly positive, we can then conclude that the proposed deviation is strictly profitable. ∎

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