PRICING WITHOUT PRIORS

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Abstract

We consider the problem of pricing a single object when the seller has only minimal information about the true valuation of the buyer. Specifically, the seller only knows the support of the possible valuations and has no further distributional information.

The seller is solving this choice problem under uncertainty by minimizing her regret. The pricing policy hedges against uncertainty by randomizing over a range of prices. The support of the pricing policy is bounded away from zero. Buyers with low valuations cannot generate substantial regret and are priced out of the market. We generalize the pricing policy without priors to encompass many buyers and many qualities.

Keywords: Monopoly, Optimal Pricing, Regret, Multiple Priors, Distribution Free.

Jel Classification: C79, D82

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1 Introduction

We consider the problem of seller who has to price a given product with minimal information about the willingness to pay of the buyer. We offer a solution to the pricing problem of the seller by analyzing the pricing policy under regret minimization.

"There always is a first time." With growing and globalizing markets the number of situations in which market participants have little information about their environment appears to be increasing. Market surveys can be costly and time consuming. Unless stakes are high, with market places evolving and trading partners changing, it is useful to know how to set a price without the need to gather additional information. The traditional decision theory determines the optimal price according to the prior belief. Yet there is little guidance as how to form these initial beliefs.

We formally model the problem of optimal pricing with minimal information and build on the axiomatic literature on decision making under uncertainty. The objective function of the seller is to minimize the regret from a given pricing policy. The regret of the seller is the difference between the profit under *complete information* and the realized profit under *incomplete information*. The regret of the seller can be positive for two reasons: (*i*) the buyer has a low valuation relative to the price and hence does not purchase the object, or (*ii*) he has a high valuation relative to the price and hence the seller could have obtained a higher revenue. The notion of regret contains a benchmark against which the realized profit is measured and offers a trade-off which determines the optimal policy.[1]

The idea of a minimax regret rule was first suggested by Savage (1951) in his reading of Wald (1950). A decision theoretic axiomatization of regret was provided by Milnor (1954) and more recently by Stoye (2007). It is noteworthy that the axiomatic foundations for the *minimax regret criterion* do not refer to regret at all, rather they relax the axiom of independence of irrelevant alternatives. Namely, the irrelevance of the alternative is only maintained if it would not change the choice outcome under complete information. In this way, the minimax regret criterion captures the idea of a decision maker who is concerned about foregone opportunities, and hence the term "regret". We wish to emphasize that the concern for regret arises from the axioms and not from any emotional or behavioral considerations. In particular, there is no

[1] The notion of regret shares features with the notion of competitiveness which is central in optimal design problems analyzed in computer science (see the recent survey to online design problems by Borodin and El-Yaniv (1998)). The competitiveness of a policy is the ratio (rather than the difference) of realized profit against maximal profit under complete information. Neeman (2003) analyzes the competitiveness of the second price auction and Bergemann and Valimaki (2006) survey robust models in mechanism design.

need for the decision maker to learn the true state of the world after making her decision.

How should the seller price under the minimax regret criterion? The pricing policy has to resolve the conflict between the regret which arises with low prices against the regret associated with high prices. If the seller offers a low price, regret can arise through the arrival of a high valuation buyer. On the other hand, if the seller offers a high price, regret can be caused by a valuation just below the offered price. It then becomes evident that a single price will always expose the seller to substantial regret. Consequently, the seller can decrease her exposure by offering many prices in the form of a random pricing policy. With a random pricing policy, the seller diminishes the likelihood of large regret.

The intuition regarding the regret minimizing policy is easy to establish in comparison to the optimal revenue maximizing policy for a given distribution. An optimal policy for a given distribution of valuations is always to offer the object at a deterministic price. In contrast a regret minimizing policy will offer many prices (with varying probability). With a single price, the risk of missing a trade at a valuation just below the given price is substantial. On the other hand, if the seller were simply to lower the price, she would miss the chance of extracting revenue from higher valuation customers. She resolves this conflict by offering low prices to the low valuation customers with positive probability.

We shall contrast the policy under the minimax regret criterion with the *maximin utility criterion* which seeks to maximize the worst case outcome. In the setting here, the worst outcome arises when the buyer has a valuation below the offered price. The maximin utility criterion forces the seller to set the price equal to the lowest possible valuation (provided it generates positive profits). This conservative point view fails to provide a trade-off in terms of foregone opportunities by focussing exclusively on the worst case scenario from the perspective of profits.

The current analysis complements our earlier work on robust monopoly pricing in Bergemann and Schlag (2007). There we considered a robust version of the classic problem of optimal monopoly pricing with incomplete information. In the robust version of the problem the seller only knows that demand will be in a *small* neighborhood of a *given* model distribution. We characterized the optimal pricing policy under two distinct, but related, decision criteria with multiple priors: (i) maximin expected utility and (ii) minimax expected regret. The resulting optimal pricing policy under either criterion depends on the model distribution and the size of the neighborhood. In the current contribution we do not allow for any prior information about the valuation of the buyer nor do we allow for variation in the uncertainty faced by the seller. In particular, we cannot say how the seller would be responding to an increase in uncertainty. The absence of prior information then allows us to focus on the trade-offs inherent to an environment without information.

A recent paper by Eren and Ryzin (2006) considers a product differentiation problem without prior information and under regret minimization. They consider a market with differentiated products (either horizontal or vertical) and determine the optimal product positioning without market information. Perakis and Roels (2006) consider the inventory problem of the newsvendor model with partial information under regret minimization.

2 Model

Consider a seller of a good who faces a single potential buyer. The seller sets a price p for a unit of the good. The buyer wishes to buy at most one unit of the good and has a value v, his willingness to pay, belonging to a closed interval such that $v \in [\underline{v}, 1]$ where $\underline{v} \geq 0$.[2] The net utility of the buyer of purchasing the product at price p is given by

$$u(v, p) = v - p.$$

The marginal cost of the seller is constant and equal to $c \in [0, 1)$, and the cost c is incurred only if the good is sold. The profit of the seller equals

$$\pi(p, v) \triangleq (p - c) \mathbb{I}_{\{v \geq p\}}, \tag{1}$$

where $\mathbb{I}_{\{v \geq p\}}$ is the indicator function specifying:

$$\mathbb{I}_{\{v \geq p\}} = \begin{cases} 0, & \text{if} \quad v < p, \\ 1, & \text{if} \quad v \geq p. \end{cases}$$

The value v of the good is private information to the buyer and unknown to the seller. The only information the seller has is that $v \in [\underline{v}, 1]$. Clearly, the buyer purchases the good if $v \geq p$ and does not purchase if $v < p$.

We solve the problem in which the seller seeks to minimize the maximal expected regret. The *regret* of the seller charging price p is determined as the difference between the maximal profit the seller could make if she knew the value v and the profit she makes by setting p. The maximal profit when knowing v is given by

$$\max_p \pi(p, v) = \max\{v - c, 0\},$$

and we obtain the following formula for regret:

$$r(p, v) \triangleq \max\{v - c, 0\} - (p - c) \mathbb{I}_{\{v \geq p\}}. \tag{2}$$

[2]The normalization to 1 is without loss of generality and the value v can interpreted as the relative value in relation to the maximum possible value.

The regret is equal to the foregone profits of the seller due to not knowing the true value of the buyer. The regret is non-negative and can only vanish if $p = v$ or if $v \leq c$. The seller experiences strictly positive regret in two different cases: (i) the good is sold but the buyer would have been willing to pay more, so $p < v$ and $r(p, v) = v - p$ or (ii) the good was not sold but the willingness to pay of the buyer exceeded the cost or $p > v > c$ and $r(p, v) = v - c$. An upper bound on the valuation of the buyer is needed to ensure that the regret is finite.

The pricing policy with regret can be determined as an equilibrium strategy of a zero-sum game between the seller and adversarial nature. In the zero-sum game, the payoff to the seller is equal to $-r(p, v)$, to nature it is equal to $r(p, v)$ for a given realization of price p and valuation v. (The equilibrium behavior of the buyer is incorporated in the definition of regret given in (2)). The seller may use a mixed pricing strategy $\Phi \in \Delta\mathbb{R}$ and nature may choose a distribution over valuations, denoted by $F \in \Delta[\underline{v}, 1]$. The regret of the seller choosing a mixed pricing policy $\Phi \in \Delta\mathbb{R}$ given a valuation v is defined by the expected regret, so

$$r(\Phi, v) = \int r(p, v)\, d\Phi(p),$$

and by extension the expected regret given Φ and F is given by:

$$r(\Phi, F) = \int \int r(p, v)\, d\Phi(p)\, dF(v).$$

A pair of strategies (Φ^*, F^*) is a Nash equilibrium of the zero game if it forms a saddle point:

$$r(\Phi^*, F) \leq r(\Phi^*, F^*) \leq r(\Phi, F^*), \quad \forall \Phi, \forall F. \tag{3}$$

The pricing strategy Φ^* is said to *attain minimax regret* and the equilibrium strategy of nature F^* is called a *least favorable demand*. The value $r^* \triangleq r(\Phi^*, F^*)$ is referred to as the value of the minimax regret.

The behavior in the minimax regret problem has a well-known relationship to Bayesian decision making. The pricing policy Φ^* that attains minimax regret also maximizes the expected profits of a Bayesian decision maker who is endowed with a least favorable demand F^* as prior. In this sense it is as if the minimax regret approach selects a specific prior.

3 Pricing without Priors

The regret of the seller arises from two, qualitatively different kind of exposures. If the valuation of the buyer is very high, then the regret may arise from having offered a price too low relative to the valuation. We refer to this as the *upward exposure*. On the other hand, by having offered a price too high, the buyer risks to have a valuation below the price and the regret of the seller arises from not selling at all. Correspondingly, we refer to this as the *downward exposure*. At every given price p, the seller faces both a downward and an upward exposure. In this context, a deterministic price policy will always leave the seller exposed to substantial regret and the regret can be significantly reduced by offering a probabilistic pricing policy. We observe that a buyer with a low valuation cannot generate substantial regret and hence we may expect that the seller will never offer a price to sell to a customer with a low valuation. Consequently, the lower bound on the valuations given by $\underline{v}$ will only play a role in the determination of the equilibrium if it is not too low. A critical value for the lower bound $\underline{v}$ is given by $c + (1-c)/e$ and we define:

$$\kappa \triangleq \max\left\{\underline{v}, c + (1-c)/e\right\}.$$

The seller may "hedge" against regret and resolve the dilemma of facing both downward and upward exposure by "trying her luck" in a well calibrated manner. If the seller is to be indifferent in her pricing policy against the least favorable demand, then the marginal profit must be zero over the range of prices which the seller offers. In the language of optimal monopoly pricing this means that the virtual utility of different prices has to be constant and equal to zero:

$$p - c - \frac{1 - F^*(p)}{f^*(p)} = 0. \tag{4}$$

In turn for nature to be indifferent between different valuations, it must be that the regret:

$$r\left(v, \Phi^*(p)\right) = v - c - \int_{p \le v} (p - c)\, d\Phi^*(p),$$

is constant for those valuations (which satisfy $v \ge c$). By differentiating with respect to v we obtain:

$$1 - (p - c)\,\phi^*(p) = 0,$$

or

$$\phi^*(p) = \frac{1}{p - c}. \tag{5}$$

It is now reasonable to guess that the distributions of seller and of nature share the same support over some interval $[a, b] \subseteq [\max\{\underline{v}, c\}, 1]$. We observe that the upper bound of the interval has to be $b = 1$ as an increase in the valuation from $v = b$ to $v = 1$ could otherwise strictly increase the regret of the seller. On the other hand, given the interval $[a, 1]$, nature may always choose a valuation just below a. This choice of valuation would yield a regret arbitrarily close to $a-c$ as the seller would fail to sell the good with prices $p \geq a$. In consequence the regret will be equal to $a-c$. The value of a is lowest if the distribution Φ^* of prices does not display a mass point and is obtained at $a = c \cdot 1 + (1-c) \cdot (1/e)$ as we have:

$$\int_{c+(1-c)/e}^{1} \frac{1}{p-c} dp = 1.$$

The equilibrium strategies are then identified by the lowest possible a subject to (i) the indifference conditions (4) and (5), (ii) the requirement that Φ^* and F^* are well-defined distributions and that $a \geq \underline{v}$. The later conditions will imply that the least favorable demand F^* has a mass point at the upper end of the interval and that the pricing policy will have a mass point at the lower end of the interval if $\underline{v} > c + (1-c)/e$.

Proposition 1 (Pricing without Priors)
The unique minimax regret strategy is given by Φ^**:*

$$\Phi^*(p) = \begin{cases} 0 & \text{if } 0 \leq p < \kappa, \\ 1 + \ln \frac{p-c}{1-c} & \text{if } \kappa \leq p \leq 1, \end{cases} \tag{6}$$

and Φ^* *has a point mass at* $p = \underline{v}$ *if and only if* $\underline{v} > c + (1-c)/e$.

Proof. A least favorable demand is given by F^* with:

$$F^*(v) = \begin{cases} 0 & \text{if } 0 \leq v < \kappa, \\ 1 - \frac{\kappa-c}{v-c} & \text{if } \kappa \leq v < 1, \\ 1 & \text{if } v = 1. \end{cases} \tag{7}$$

Given the pair (Φ^*, F^*) we need to verify the saddlepoint condition (3). The expected regret for a given price p is

$$r(p, F^*) = \kappa - c + \int_\kappa^1 \frac{\kappa-c}{v-c} dv - (\kappa - c) = (\kappa - c) \ln \frac{1-c}{\kappa-c}, \quad \text{for } p \in [\kappa, 1],$$

and

$$r(p, F^*) = \kappa - c + \int_\kappa^1 \frac{\kappa-c}{v-c} dv - (p-c) > (\kappa-c) \ln \frac{1-c}{\kappa-c} \text{ for } 0 \leq p < \kappa.$$

Similarly, the expected regret from given valuation v is

$$\begin{aligned} r(\Phi^*, v) &= v - c - \int_\kappa^v dp - (\kappa - c)\left(1 + \ln \frac{\kappa-c}{1-c}\right) \\ &= -(\kappa-c)\ln \frac{\kappa-c}{1-c}, \quad \text{for } v \in [\kappa, 1], \end{aligned} \tag{8}$$

and

$$r\left(\Phi^{*}, v\right)=\max \{v-c, 0\}<-(\kappa-c) \ln \frac{\kappa-c}{1-c} \text { for } \underline{v} \leq v<\kappa .$$

We have thus verified that (Φ^*, F^*) satisfies (3). The uniqueness of Φ^* follows as nature has to be indifferent over all $v \in (\kappa, 1]$. ■

The solution Φ^* of the regret minimization problem *simultaneously* determines a least favorable demand F^* given by (7) *and* a performance guarantee for the seller in terms of the maximal regret given by (8).

We observe that if the seller were restricted to choose a deterministic price policy, then the regret minimizing price would have to balance the upside exposure $1-p$ and the downside exposure $p-c$ in a single price p.

Corollary 1 (Deterministic Pricing)
If the seller is constrained to pure strategies, then

$$p^{*}=\begin{cases}\frac{1}{2}(1+c) & \text{if } \underline{v}<\frac{1}{2}(1+c), \\ \underline{v} & \text{if } \underline{v} \geq \frac{1}{2}(1+c).\end{cases}$$

The associated regret r^* for the seller is naturally higher under the restriction to pure strategies. At this point, it may be instructive to briefly consider a possible alternative objective in the presence of large uncertainty, namely to choose a price that maximizes the minimum profit. Here the seller chooses a price (distribution) Φ^* such that:

$$\Phi^{*} \in \arg \min_{\Phi} \sup_{v} \pi(\Phi, v).$$

With the maximin criterion, the seller chooses a price policy Φ^* that puts all the mass on $p = \underline{v}$ if $\underline{v} > c$ and is indifferent over all prices in $[c, 1]$ if $\underline{v} \leq c$. Under the minimax criterion, the seller is exclusively concerned with missing sales at valuations above marginal cost and hence she sets the price equal to the lowest possible valuation provided $\underline{v} > c$. If however $\underline{v} \leq c$ then all prices achieve the same minimal profit equal to 0 and every price above c is a solution to the maximin problem.

4 Discussion

Robustness In this paper we considered the optimal pricing of a single object with minimal information about the nature of the demand. Specifically, the information of the seller consisted of the interval of possible valuations without any additional distributional information. As the seller minimized her regret, randomization over prices played an important role. It is used to protect the seller against suffering from foregone opportunities. We argued that the optimal price policy under minimax regret can be understood in the classic expected utility (profit) framework as an optimal pricing rule under a specific prior. Yet the randomization over many prices would never emerge as the unique optimal pricing policy in the expected utility setting as there is always an optimal price which is deterministic.

In Bergemann and Schlag (2007) we consider the problem of optimal pricing when the seller has some prior information given by a model distribution and by a specified neighborhood around the model distribution in which the true demand distribution is known to be. The resulting model can be interpreted as a robust version of the classic problem of optimal monopoly pricing.

This paper and Bergemann and Schlag (2007) make distinct extreme assumptions about multiple priors. Here, the set of multiple priors is the set of all demand distributions, there it is a small neighborhood around a model distribution. Many intermediate scenarios are interesting for future research. In particular, it seems natural to analyze a dynamic version of the robust pricing problem in which the uncertainty decreases over time due to the sampling of information.

Many Buyers We defined the pricing problem of the seller as offering a single product for a single buyer with an unknown valuation. The model and the results allow a further interpretation, namely as offering the same product simultaneously to a finite number or a continuum of buyers. The notion of regret is subadditive with equality holding when all buyers have the same valuation and hence the problem of minimizing (average) regret when facing many small buyers or a single large buyer leads to the same solution as outlined in proposition 1.

Product Differentiation In the current model, the buyer has a binary choice between accepting or rejecting a single product. A natural generalization of the model would allow for many different qualities of the same product class as in Mussa and Rosen (1978). There, the marginal willingness to pay for quality is constant and given by v and the cost of providing quality q is given by a convex cost function $c(q)$. Without prior information, the seller would now like to offer a menu of qualities to as to minimize her regret. The optimal menu $(q^*(v), p^*(v))$ would offer a combination of qualities $q^*(v)$ and prices $p^*(v)$ such that the buyers would self-select and such that the regret is minimized. With complete information, the seller would choose for every value v, the first best quantity $q_{FB}(v)$ which maximizes the social surplus $v \cdot q - c(q)$. The regret of the seller is the difference between the maximal net revenue and the realized net revenue. The regret minimization again requires that the regret is constant across all types which receive offers from the seller, i.e. for all payoff types v with $q^*(v) > 0$ and the solution of the regret minimization problem is given by the following differential equation in v :

$$v \cdot q^{*\prime}(v) - c'(q^*(v)) = 0,$$

which can be solved after imposing the relevant boundary conditions.

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