LIMIT THEORY FOR EXPLOSIVELY COINTEGRATED SYSTEMS

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June 2007

COWLES FOUNDATION DISCUSSION PAPER NO. 1614

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Limit theory for explosively cointegrated systems[1]

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March 6, 2007

[1]Phillips thanks the NSF for research support under Grant No. SES-04-142254.

Abstract

A limit theory is developed for multivariate regression in an explosive cointegrated system. The asymptotic behavior of the least squares estimator of the cointegrating coefficients is found to depend upon the precise relationship between the explosive regressors. When the eigenvalues of the autoregressive matrix Θ are distinct, the centered least squares estimator has an exponential Θ^n rate of convergence and a mixed normal limit distribution. No central limit theory is applicable here and Gaussian innovations are assumed. On the other hand, when some regressors exhibit common explosive behavior, a different mixed normal limiting distribution is derived with rate of convergence reduced to $\sqrt{n}$. In the latter case, mixed normality applies without any distributional assumptions on the innovation errors by virtue of a Lindeberg type central limit theorem. Conventional statistical inference procedures are valid in this case, the stationary convergence rate dominating the behavior of the least squares estimator.

Keywords: Central limit theory, Exposive cointegration, Explosive process, Mixed normality.

AMS 1991 subject classification: 62M10; JEL classification: C22

1. Introduction

Autoregressions with an explosive root $|\theta| > 1$ came to prominence after the early work of White (1958) and Anderson (1959). Assuming Gaussian innovation errors, these authors derived a Cauchy limit theory for the centered least squares estimator with rate of convergence θ^n. The theory was generalized by Mijnheer (2002) to non Gaussian explosive processes generated by innovations satisfying a stability property. In each of these works, no central limit theory applies and the asymptotic distribution of the least squares estimator is characterised by the distributional assumptions imposed on the innovations.

In this paper, we consider an explosively cointegrated system

$$y_t = Ax_t + \varepsilon_t \tag{1}$$

$$x_t = \Theta x_{t-1} + u_t \tag{2}$$

$$\Theta = I_K + C, \quad C = \text{diag}(c_1, ..., c_K), \; c_i \in (-\infty, -2) \cup (0, \infty) \;\; \forall i,$$

where A is an $m \times K$ matrix of cointegrating coefficients, x_t is a $K-$vector of explosive autoregressions initialized at $x_0 = 0$, and $v_t = (\varepsilon_t', u_t')'$ is a sequence of independent, identically distributed $(0, \Sigma)$ random vectors with absolutely continuous density, where Σ is a positive definite matrix partitioned conformably with v_t as $\Sigma = \text{diag}(\Sigma_{\varepsilon\varepsilon}, \Sigma_{uu})$. We denote by $\theta_i = 1 + c_i$ the i-th diagonal element of Θ and by $\|\Theta\| = \max_{1 \le i \le K} |\theta_i|$ the spectral norm of Θ.

The asymptotic behavior of the least squares estimator

$$\hat{A}_n = \left(\sum_{t=1}^{n} y_t x_t' \right) \left(\sum_{t=1}^{n} x_t x_t' \right)^{-1}$$

is found to depend on the relationship between the regressors in (2), i.e. on the precise form of the matrix Θ. As Theorem 2.1 below shows, the rank of the limit matrix of the normalized sample second moments, and hence the order of magnitude of $\left(\sum_{t=1}^{n} x_t x_t'\right)^{-1}$, is determined exclusively by Θ. When Θ yields a nonsingular limit in Theorem 2.1, $\hat{A}_n - A$ is found to have a Θ^n rate of convergence and a mixed normal limiting distribution, under the assumption of Gaussian innovations (cf. Anderson, 1959). But when the limit moment matrix of Theorem 2.1 is singular, $\hat{A}_n - A$ has a degenerate mixed normal limiting distribution with convergence rate reduced to $n^{1/2}$. The asymptotics in the singular case are obtained by rotating the regression coordinates in a way that the singularity is eliminated and central limit theory applies. Consequently, the mixed normal limit theory in the singular case applies without any distributional assumptions on the innovation errors.

Explosive systems are useful in modeling periods of extreme behavior in economic and financial variables. Economic growth among the Asian dragons during the 1980s and recent growth in China provide examples of mildly explosive growth in macroeconomic variables. Hyperinflation in Germany in the 1920s and Yugoslavia in the 1990s are examples of some of the many historical instances of explosive behavior in prices. Financial bubbles in asset prices are another example, the recent rise and subsequent fall in price of internet stocks in the NASDAQ market creating and destroying some $8 trillion of shareholder wealth. To the extent that periods of explosive movement in such variables influence economic decisions or contaminate other variables, we may expect models of explosive cointegration such as (1) - (2) to be relevant in relating these variables. When there is a single source of the extreme movement, then such a system may also have explosively cointegrated regressors and the degeneracy described above may occur.

2. Results

We develop a limit theory for the centered least squares estimator

$$\hat{A}_n - A = \left(\sum_{t=1}^{n} \varepsilon_t x_t'\right)\left(\sum_{t=1}^{n} x_t x_t'\right)^{-1}.$$

It turns out that the asymptotic order of $\hat{A}_n - A$ depends on the rank of the limit of the normalized sample moment matrix $\sum_{t=1}^{n} x_t x_t'$. The latter can be derived, using a similar method to Anderson (1959), in terms of the random vector

$$X_\Theta = \sum_{j=1}^{\infty} \Theta^{-j} u_j, \tag{3}$$

where the series converges almost surely by virtue of the martingale convergence theorem.

2.1 Theorem. *The sample moment matrix of the explosive process (2) satisfies*

$$\Theta^{-n} \sum_{t=1}^{n} x_t x_t' \Theta^{-n} \to_{a.s.} \sum_{j=0}^{\infty} \Theta^{-j} X_\Theta X_\Theta' \Theta^{-j} \quad \text{as } n \to \infty, \tag{4}$$

where X_Θ is the random vector defined in (3).

Note that the almost sure limit $\sum_{j=0}^{\infty} \Theta^{-j} X_\Theta X_\Theta' \Theta^{-j}$ of the normalized sample moment matrix is not always non singular. Denote the i-th element of the random vector X_Θ by $X_\Theta^{(i)}$ and define the matrices

$$M_\Theta := \left[\frac{\theta_i \theta_j}{\theta_i \theta_j - 1} : i, j \in \{1, ..., K\}\right] \text{ and } \check{X}_\Theta := \operatorname{diag}\left(X_\Theta^{(1)}, ..., X_\Theta^{(K)}\right).$$

Since u_1 admits an absolutely continuous density, $X_{\Theta}^{(i)} \neq 0$ $a.s.$ for each i. Thus, the identity

$$\sum_{j=0}^{\infty} \Theta^{-j} X_{\Theta} X_{\Theta}' \Theta^{-j} = \check{X}_{\Theta} M_{\Theta} \check{X}_{\Theta}$$

implies that $\sum_{j=0}^{\infty} \Theta^{-j} X_{\Theta} X_{\Theta}' \Theta^{-j}$ is nonsingular whenever the matrix M_{Θ} is nonsingular, i.e. if and only if $c_i \neq c_j$ for all $i \neq j$ (cf. Lemma 4.3). On the other hand, when any two localising coefficients c_i, c_j are the same, the matrix M_{Θ} will have two identical columns and will, therefore, be singular.

We begin by discussing the non singular asymptotic moment matrix case.

2.2 Theorem. *For the explosive cointegrated system generated by (1) and (2) with $v_t =_d N(0, \Sigma)$ and $c_i \neq c_j$ for all $i \neq j$, the following limit theory applies as $n \to \infty$*

$$vec\left[\left(\hat{A}_n - A\right)\Theta^n\right] \Rightarrow MN\left(0, \left(\sum_{j=0}^{\infty} \Theta^{-j} X_{\Theta} X_{\Theta}' \Theta^{-j}\right)^{-1} \otimes \Sigma_{\varepsilon\varepsilon}\right).$$

2.3 Remarks.

(i) The assumption of Gaussian innovations is essential in order to obtain a mixed normal limiting distribution for the least squares estimator. This is because, despite being asymptotically equivalent to a martingale array (see (34)), the sample covariance does not satisfy the requirement of uniform asymptotic negligibility nor the Lindeberg condition (cf. section 3.2 of Hall and Heyde, 1980). As a result, no central limit theory applies in general and mixed normality requires Gaussian innovations, as in the AR(1) case of Anderson (1959).

(ii) When $v_t =_d N(0, \Sigma)$, $X_{\Theta} =_d N\left(0, \sum_{j=1}^{\infty} \Theta^{-j} \Sigma_{uu} \Theta^{-j}\right)$.

(iii) In the simplest case of a 2-equation system, $K = 1$, so x_t, $A = a$ and $\Theta = \theta$ are scalar. Letting Z be a $N(0, 1)$ variate, the previous remark yields

$$\begin{aligned} \left(\theta^2 - 1\right)^{1/2} X_{\Theta} &=_d N(0, \Sigma_{uu}) =_d \Sigma_{uu}^{1/2} Z, \\ \sum_{j=0}^{\infty} \Theta^{-j} X_{\Theta} X_{\Theta}' \Theta^{-j} &= \sum_{j=0}^{\infty} \theta^{-2j} X_{\Theta}^2 =_d \frac{\theta^2}{\left(\theta^2 - 1\right)^2} \Sigma_{uu} Z^2. \end{aligned}$$

Thus, Theorem 2.2 reduces to

$$\theta^n (\hat{a}_n - a) \Rightarrow MN\left(0, \frac{\Sigma_{\varepsilon\varepsilon}}{\Sigma_{uu} Z^2} \frac{\left(\theta^2 - 1\right)^2}{\theta^2}\right) =_d \left(\frac{\theta^2 - 1}{\theta}\right)\left(\frac{\Sigma_{\varepsilon\varepsilon}}{\Sigma_{uu}}\right)^{1/2} \frac{Y}{Z},$$

where Y and Z are independent $N(0,1)$ variates, or

$$\frac{\theta^{n+1}}{\theta^2-1}(\hat{a}-a) \Rightarrow \left(\frac{\Sigma_{\varepsilon\varepsilon}}{\Sigma_{uu}}\right)^{1/2} \mathcal{C},$$

where $\mathcal{C}$ is a standard Cauchy variate. In the general case, the exact form of the limiting distribution of Theorem 2.2 can be obtained by using a matrix quotient argument, as in Phillips (1985).

We now turn to the discussion of the limit theory in the case of two or more equal localising coefficients. We have seen that this case gives rise to a singular limit matrix for the sample variance, reflecting the fact that the regressors x_t are themselves explosively cointegrated. Since the mixing random matrix $\sum_{j=0}^{\infty} \Theta^{-j} X_\Theta X_\Theta' \Theta^{-j}$ is singular, the limit theory of Theorem 2.2 does not apply. The asymptotic behavior of the least squares estimator can be determined by a rotation of coordinates to isolate the explosive and non-explosive behavior, a method used by Park and Phillips (1988, 1989) in the setting of cointegrated processes. Here, however, the rotation is random and is determined by the limit vector X_Θ.

We start by grouping together the repeated diagonal elements of Θ. This can be done without loss of generality by premultiplying (2) by an appropriate permutation matrix (i.e. a square matrix consisting of zeros and ones that contains exactly one element 1 in each row and each column). If there are p groups of repeated diagonal elements of Θ the autoregressive matrix can be rearranged as

$$\begin{aligned} &\Phi = \operatorname{diag}(\Phi_1, \Phi_2) \\ &\Phi_1 = \operatorname{diag}\left(\theta_1 I_{r_1}, ..., \theta_p I_{r_p}\right) \\ &\Phi_2 = \operatorname{diag}\left(\varphi_1, ..., \varphi_{K-r}\right) \quad r = \textstyle\sum_{i=1}^{p} r_i \end{aligned} \tag{5}$$

where all φ_i and θ_i are diagonal elements of Θ with $\varphi_s \neq \theta_l$ for all s, l and $\varphi_i \neq \varphi_j$, $\theta_i \neq \theta_j$ for all $i \neq j$. This effectively rearranges the system of equations in (2) into a system of the form

$$\begin{bmatrix} x_{1t} \\ \cdots \\ x_{pt} \\ x_{p+1,t} \end{bmatrix} = \begin{bmatrix} \theta_1 x_{1t-1} \\ \cdots \\ \theta_p x_{pt-1} \\ \Phi_2 x_{p+1,t-1} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ \cdots \\ u_{pt} \\ u_{p+1,t} \end{bmatrix}, \tag{6}$$

where $x_{it} \in \mathbb{R}^{r_i}$ includes the regressors in (2) that contain the repeated root θ_i for each $i \in \{1, ..., p\}$ and $x_{p+1,t} \in \mathbb{R}^{K-r}$ includes the regressors that contain all distinct diagonal elements of Θ. Letting $\tilde{x}_t = \left(x_{1t}', ..., x_{pt}', x_{p+1,t}'\right)'$ and $\tilde{u}_t = \left(u_{1t}', ..., u_{pt}', u_{p+1,t}'\right)'$, (6) can be obtained from (2) as follows. Consider the $K \times K$ permutation matrix Π that transforms x_t into $\tilde{x}_t$: $\Pi x_t = \tilde{x}_t$. Then, using orthogonality of permutation matrices, (2) yields

$$\tilde{x}_t = \Pi\Theta x_{t-1} + \tilde{u}_t = \Pi\Theta\Pi'\Pi x_{t-1} + \tilde{u}_t = \Phi\tilde{x}_{t-1} + \tilde{u}_t, \tag{7}$$

where $\Phi = \Pi\Theta\Pi'$ has the explicit form given in (5) and, by orthogonality of Π, satisfies the useful identity

$$\Phi^{-j} = \Pi\Theta^{-j}\Pi' \quad \text{for all } j \in \mathbb{N}. \tag{8}$$

Similarly, we can write (1) in terms of $\tilde{x}_t$ as

$$y_t = A\Pi'\tilde{x}_t + \varepsilon_t = \Psi\tilde{x}_t + \varepsilon_t, \tag{9}$$

where $\Psi = A\Pi'$. Since $\hat{\Psi}_n - \Psi = \left(\hat{A}_n - A\right)\Pi'$, the asymptotic behavior of $\hat{A}_n$ is completely determined by that of $\hat{\Psi}_n$. In what follows, we show that only the first r rows of the permutation matrix Π will contribute to the limiting distribution of $\sqrt{n}\left(\hat{A}_n - A\right)$. It is therefore convenient to partition Π as

$$\Pi = \begin{bmatrix} \underset{r\times K}{\Pi_1} \\ \underset{(K-r)\times K}{\Pi_2} \end{bmatrix},$$

where, by the orthogonality of Π, Π_1 and Π_2 satisfy

$$\begin{array}{l} \Pi_1\Pi_1' = I_r,\ \Pi_2\Pi_2' = I_{K-r},\ \Pi_1\Pi_2' = 0 \\ \Pi_1'\Pi_1 + \Pi_2'\Pi_2 = I_K. \end{array} \tag{10}$$

In particular, the first line of (10) implies that $\text{rank}(\Pi_1) = r$ and $\text{rank}(\Pi_2) = K - r$. Conformably, we partition Φ^{-j} as $\Phi^{-j} = \text{diag}\left(\Phi_1^{-j}, \Phi_2^{-j}\right)$. The partitioned form of Π together with (8) then give rise to the identities

$$\Phi_1^{-j} = \Pi_1\Theta^{-j}\Pi_1',\ \Phi_2^{-j} = \Pi_2\Theta^{-j}\Pi_2',\ \Pi_1\Theta^{-j}\Pi_2' = 0 \tag{11}$$

$$\Theta^{-j} = \Pi_1'\Phi_1^{-j}\Pi_1 + \Pi_2'\Phi_2^{-j}\Pi_2 \tag{12}$$

for all $j \in \mathbb{N}$.

The limit theory for the cointegrated system (9) and (7) is derived by rotating the regression space in a direction orthogonal to

$$X_\Phi := \Pi_1 X_\Theta = \sum_{j=1}^{\infty}\left(\Pi_1\Theta^{-j}\Pi_1'\right)\Pi_1 u_j = \sum_{j=1}^{\infty}\Phi_1^{-j}\Pi_1 u_j,$$

where the last equality is obtained using (11). Corresponding to the partition of $\Pi_1 x_t$, define $X_\Phi = \left(X_{\Phi 1}', ..., X_{\Phi p}'\right)'$, $X_{\Phi i} \in \mathbb{R}^{r_i}$, and $H_{\Phi i} = X_{\Phi i}/\left(X_{\Phi i}'X_{\Phi i}\right)^{1/2}$ for each $i \in \{1, ..., p\}$. We consider an $r_i \times (r_i - 1)$ orthogonal complement $H_{\perp i}$ to each $H_{\Phi i}$

satisfying $H'_{\perp i} H_{\Phi i} = 0$ and $H'_{\perp i} H_{\perp i} = I_{r_i - 1}$ $a.s.$ for all $i \in \{1, ..., p\}$. Then

$$H = \begin{bmatrix} H'_{\perp 1} & 0 & ... & 0 & 0 \\ 0 & H'_{\perp 2} & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & H'_{\perp p} & 0 \\ H'_{\Phi 1} & 0 & ... & 0 & 0 \\ 0 & H'_{\Phi 2} & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & H'_{\Phi p} & 0 \\ 0 & 0 & ... & 0 & I_{K-r} \end{bmatrix} \tag{13}$$

is a $K \times K$ orthogonal matrix which can be partitioned as

$$H = \begin{bmatrix} U'_{\perp} \\ U'_{\Phi} \end{bmatrix}, \quad U_{\perp} = \begin{bmatrix} H_{\perp} \\ \underset{(K-r)\times(r-p)}{0} \end{bmatrix}, \quad U_{\Phi} = \begin{bmatrix} H_{\Phi} & \underset{r\times(K-r)}{0} \\ \underset{(K-r)\times p}{0} & I_{K-r} \end{bmatrix} \tag{14}$$
$$H_{\perp} = \operatorname{diag}(H_{\perp 1}, ..., H_{\perp p}), \quad H_{\Phi} = \operatorname{diag}(H_{\Phi 1}, ..., H_{\Phi p}).$$

By construction, the orthogonal complement matrix $H_{\perp}$ satisfies $H'_{\perp} X_{\Phi} = H'_{\perp} H_{\Phi} = 0$ and $H'_{\perp} H_{\perp} = I_{r-p}$ almost surely. Although $H_{\perp}$ is not unique, its outer product is uniquely defined by the relation

$$H_{\perp} H'_{\perp} = I_r - H_{\Phi} H'_{\Phi} \quad a.s. \tag{15}$$

(see e.g. 8.67 in Abadir and Magnus, 2005). Moreover, (14) implies a similar relationship between U_{Φ} and $U_{\perp}$, viz.,

$$U'_{\perp} U_{\Phi} = 0, \ \ U'_{\perp} U_{\perp} = I_{r-p} \ \text{ and } \ U_{\perp} U'_{\perp} = I_K - U_{\Phi} U'_{\Phi} \quad a.s. \tag{16}$$

Applying the orthogonal transformation H to the explosive regressor yields

$$\begin{aligned} z_t &= H\tilde{x}_t = [z'_{1t}, z'_{2t}]' \\ z_{1t} &= H'_{\perp} \Pi_1 x_t \in \mathbb{R}^{r-p} \quad z_{2t} = U'_{\Phi} \Pi x_t \in \mathbb{R}^{K-r+p} \end{aligned} \tag{17}$$

and

$$\begin{aligned} \hat{\Psi}_n - \Psi &= \left(\sum_{t=1}^{n} \varepsilon_t \tilde{x}'_t H' \right) \left(H \sum_{t=1}^{n} \tilde{x}_t \tilde{x}'_t H' \right)^{-1} H \\ &= \left(\sum_{t=1}^{n} \varepsilon_t z'_t \right) \left(\sum_{t=1}^{n} z_t z'_t \right)^{-1} H. \end{aligned} \tag{18}$$

With this rotation, the limit matrices of both $\sum_{t=1}^{n} z_t z'_t$ and $(\sum_{t=1}^{n} z_t z'_t)^{-1}$ are well defined after appropriate normalization. To see this, first observe that, in view of the identities

$$\left(H'_{\perp} \Phi_1^{-1} H_{\perp} \right)^j = H'_{\perp} \Phi_1^{-j} H_{\perp}, \ \ H'_{\perp} \Phi_1^{i} H_{\Phi} = 0 \quad \forall j \in \mathbb{N}, \, i \in \mathbb{Z} \tag{19}$$

z_{1t} satisfies the reverse autoregression

$$z_{1t} = \left(H_\perp' \Phi_1^{-1} H_\perp\right) z_{1t+1} - H_\perp' \Phi_1^{-1} \Pi_1 u_{t+1} \tag{20}$$

which, upon recursion, yields for each $t \leq n$

$$z_{1t} = \left(H_\perp' \Phi_1^{-1} H_\perp\right)^{n-t} z_{1n} - H_\perp' \sum_{j=1}^{n-t} \Phi_1^{-j} \Pi_1 u_{t+j}. \tag{21}$$

Proofs of (19) and (20) are given in Section 4. Using (10) and (12) the second term of (21) can be written as

$$\begin{aligned}
H_\perp' \sum_{j=1}^{n-t} \Phi_1^{-j} \Pi_1 u_{t+j} &= H_\perp' \Pi_1 \sum_{j=1}^{n-t} \left(\Pi_1' \Phi_1^{-j} \Pi_1\right) u_{t+j} \\
&= H_\perp' \Pi_1 \sum_{j=1}^{n-t} \left(\Theta^{-j} - \Pi_2' \Phi_2^{-j} \Pi_2\right) u_{t+j} \\
&= H_\perp' \Pi_1 \sum_{j=1}^{n-t} \Theta^{-j} u_{t+j} \\
&\to_{a.s.} H_\perp' \Pi_1 \sum_{j=1}^{\infty} \Theta^{-j} u_{t+j}
\end{aligned}$$

as $n \to \infty$ by the martingale convergence theorem. For the first term of (21), using (19) (15), (10) and (12) we obtain

$$\begin{aligned}
\left(H_\perp' \Phi_1^{-1} H_\perp\right)^{n-t} z_{1n} &= H_\perp' \Phi_1^{-(n-t)} H_\perp H_\perp' \Pi_1 x_n \\
&= H_\perp' \Phi_1^{-(n-t)} \left(I_r - H_\Phi H_\Phi'\right) \Pi_1 x_n \\
&= H_\perp' \Phi_1^{-(n-t)} \Pi_1 x_n \\
&= H_\perp' \Phi_1^{t} \Pi_1 \left(\Pi_1' \Phi_1^{-n} \Pi_1\right) x_n \\
&= H_\perp' \Phi_1^{t} \Pi_1 \left(\Theta^{-n} - \Pi_2' \Phi_2^{-n} \Pi_2\right) x_n \\
&= H_\perp' \Phi_1^{t} \Pi_1 \Theta^{-n} x_n \\
&= H_\perp' \Phi_1^{t} \left(H_\perp H_\perp' + H_\Phi H_\Phi'\right) \Pi_1 \Theta^{-n} x_n \\
&= \left(H_\perp' \Phi_1^{t} H_\perp\right) H_\perp' \Pi_1 \Theta^{-n} x_n \\
&\to_{a.s.} \left(H_\perp' \Phi_1^{t} H_\perp\right) H_\perp' \Pi_1 X_\Theta = 0
\end{aligned}$$

since $X_\Phi = \Pi_1 X_\Theta$. Thus, (21) implies that z_{1t} is an $\mathbb{R}^{r-p}$-valued stationary ergodic process with the following linear process representation:

$$z_{1t} = -H_\perp' \Pi_1 \zeta_{t+1} \;\; a.s. \quad \zeta_{t+1} = \sum_{j=1}^{\infty} \Theta^{-j} u_{t+j}. \tag{22}$$

The ergodic theorem then yields, as $n \to \infty$,

$$
\begin{aligned}
\frac{1}{n}\sum_{t=1}^{n} z_{1t}z_{1t}' &= H_\perp' \Pi_1 \left(\frac{1}{n}\sum_{t=1}^{n}\zeta_{t+1}\zeta_{t+1}'\right)\Pi_1' H_\perp \to_{a.s.} H_\perp'\Pi_1 E\left(\zeta_1\zeta_1'\right)\Pi_1' H_\perp \\
&= H_\perp'\Pi_1 \sum_{j=1}^{\infty}\Theta^{-j}\Sigma_{uu}\Theta^{-j}\Pi_1' H_\perp > 0, \qquad (23)
\end{aligned}
$$

where positive definiteness follows since $\Sigma_{uu} > 0$, Π_1 has full row rank equal to r and $H_\perp$ has full column rank equal to $r-p$. Thus, in the direction of $H_\perp$, the sample variance has the usual n^{-1} normalization that applies under stationarity. By standard inversion of a partitioned matrix (e.g. 5.18 in Abadir and Magnus, 2005) we obtain

$$
\begin{aligned}
\left(\sum_{t=1}^{n} z_t z_t'\right)^{-1} &= \begin{bmatrix} \sum_{t=1}^{n} z_{1t}z_{1t}' & \sum_{t=1}^{n} z_{1t}z_{2t}' \\ \sum_{t=1}^{n} z_{2t}z_{1t}' & \sum_{t=1}^{n} z_{2t}z_{2t}' \end{bmatrix}^{-1} = \begin{bmatrix} Z_1'Z_1 & Z_1'Z_2 \\ Z_2'Z_1 & Z_2'Z_2 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \left(Z_1'Q_2Z_1\right)^{-1} & -\left(Z_1'Q_2Z_1\right)^{-1}P_{2n}' \\ -P_{2n}\left(Z_1'Q_2Z_1\right)^{-1} & \left(Z_2'Z_2\right)^{-1} + P_{2n}\left(Z_1'Q_2Z_1\right)^{-1}P_{2n}' \end{bmatrix}, \qquad (24)
\end{aligned}
$$

where $Z_1 = [z_{11}', z_{12}', ..., z_{1n}']' \in \mathbb{R}^{n\times(r-p)}$, $Z_2 = [z_{21}', z_{22}', ..., z_{2n}']' \in \mathbb{R}^{n\times(K-r+p)}$,

$$
P_{2n} = \left(Z_2'Z_2\right)^{-1} Z_2'Z_1 \text{ and } Q_2 = I_n - Z_2\left(Z_2'Z_2\right)^{-1} Z_2'.
$$

Lemma 4.4 implies that $\|Z_2'Z_2\| = O_p\left(\|\Theta\|^{2n}\right)$, $\|P_{2n}\| = O_p\left(\|\Theta\|^{-n}\right)$ and

$$
\left(n^{-1}Z_1'Q_2Z_1\right)^{-1} = \left(n^{-1}Z_1'Z_1\right)^{-1} + O_p\left(n^{-1}\right).
$$

Thus, in view of (23), the large sample behavior of the sample moment matrix after rotation of the regression space is given by

$$
\begin{aligned}
\left(\frac{1}{n}\sum_{t=1}^{n} z_t z_t'\right)^{-1} &= \begin{bmatrix} \left(\frac{Z_1'Q_2Z_1}{n}\right)^{-1} & O_p\left(\|\Theta\|^{-n}\right) \\ O_p\left(\|\Theta\|^{-n}\right) & O_p\left(\|\Theta\|^{-2n}\right) \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{1}{n}\sum_{t=1}^{n} z_{1t}z_{1t}'\right)^{-1} + O_p\left(n^{-1}\right) & O_p\left(\|\Theta\|^{-n}\right) \\ O_p\left(\|\Theta\|^{-n}\right) & O_p\left(\|\Theta\|^{-2n}\right) \end{bmatrix} \\
&\to_p \begin{bmatrix} \left(H_\perp'\Pi_1\sum_{j=1}^{\infty}\Theta^{-j}\Sigma_{uu}\Theta^{-j}\Pi_1'H_\perp\right)^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \qquad (25)
\end{aligned}
$$

Having established a nonsingular limit for the sample moment matrix in the new regression coordinates the limit theory for the coefficient matrix Ψ in (9) is driven by the sample covariance $n^{-1/2}\sum_{t=1}^{n}\left(z_{1t}\otimes\varepsilon_t\right)$ which has a mixed normal asymptotic distribution

$$
\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left(z_{1t}\otimes\varepsilon_t\right) \Rightarrow MN\left(0, \left(H_\perp'\Pi_1\sum_{j=1}^{\infty}\Theta^{-j}\Sigma_{uu}\Theta^{-j}\Pi_1'H_\perp\right)\otimes\Sigma_{\varepsilon\varepsilon}\right) \qquad (26)
$$

by virtue of a martingale central limit theorem. The proof of (26) is given in Section 4. For the least squares estimator of Ψ, combining (18), (14) and (25) yields

$$\sqrt{n}\left(\hat{\Psi}_n - \Psi\right) = \left[\ \tfrac{1}{\sqrt{n}} \sum_{t=1}^{n} \varepsilon_t z_{1t}' \left(\tfrac{1}{n} \sum_{t=1}^{n} z_{1t} z_{1t}'\right)^{-1} H_{\perp}',\ \ 0\ \right] + o_p(1).$$

It is now straightforward to derive a limit theory for the original cointegrated system (1) and (2) by using the relationship $\hat{A}_n - A = \left(\hat{\Psi}_n - \Psi\right)\Pi$, so that

$$\begin{aligned}
\sqrt{n}\left(\hat{A}_n - A\right) &= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \varepsilon_t z_{1t}' \left(\frac{1}{n} \sum_{t=1}^{n} z_{1t} z_{1t}'\right)^{-1} H_{\perp}' \Pi_1 + o_p(1) \\
\sqrt{n} vec\left(\hat{A}_n - A\right) &= \left[\Pi_1' H_{\perp} \left(\frac{1}{n} \sum_{t=1}^{n} z_{1t} z_{1t}'\right)^{-1} \otimes I_m\right] \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left(z_{1t} \otimes \varepsilon_t\right) + o_p(1)
\end{aligned}$$

and the limit distribution of the least squares estimator follows as a consequence of (23) and (26).

2.4 Theorem. *For the explosive cointegrated system generated by (1) and (2) with $c_i = c_j$ for some $i \neq j$ the following limit theory applies as $n \to \infty$*

$$\sqrt{n} vec\left(\hat{A}_n - A\right) \Rightarrow MN\left(0, \Pi_1' H_{\perp} \left(H_{\perp}' \Pi_1 \sum_{j=1}^{\infty} \Theta^{-j} \Sigma_{uu} \Theta^{-j} \Pi_1' H_{\perp}\right)^{-1} H_{\perp}' \Pi_1 \otimes \Sigma_{\varepsilon\varepsilon}\right).$$

2.5 Remarks.

(i) The limit distribution of the least squares estimator is mixed Gaussian and singular, since $\text{rank}(H_{\perp}) = r - p$ and $\text{rank}(\Pi_1) = r$ implies that

$$\Pi_1' H_{\perp} \left(H_{\perp}' \Pi_1 \sum_{j=1}^{\infty} \Theta^{-j} \Sigma_{uu} \Theta^{-j} \Pi_1' H_{\perp}\right)^{-1} H_{\perp}' \Pi_1$$

is a singular matrix of rank $r - p$. Moreover, a $\sqrt{n}$ convergence rate applies, which is much slower than the usual Θ^n rate for explosive processes appearing in Theorem 2.2. This reduction in the convergence rate results from the fact that some regressors in certain directions are explosively cointegrated with a common explosive form, while the complementary set of regressors behave like stationary variates. These variates slow down the convergence rate and standard limit theory applies.

(ii) Unlike Theorem 2.2, Theorem 2.4 does not require any distributional assumptions on the innovations v_t. The limiting distribution of Theorem 2.4 is valid for non Gaussian innovations as a consequence of the central limit theorem applying for the sample covariance in (26).

(iii) In the polar case where all localising coefficients are equal, $\Theta = \theta I_K$, $r = K$, $p = 1$ and $\Pi_1 = I_K$, so Theorem 2.4 reduces to

$$\sqrt{\frac{n}{\theta^2 - 1}} vec\left(\hat{A}_n - A\right) \Rightarrow MN\left(0, H_\perp \left(H_\perp' \Sigma_{uu} H_\perp\right)^{-1} H_\perp' \otimes \Sigma_{\varepsilon\varepsilon}\right).$$

(iv) An interesting feature of the limit distribution of Theorem 2.4 is the relationship between the rank of the limiting covariance matrix and the order of cointegration between the explosive regressors. As noted in Remark 2.5 (i), the rank of the limiting covariance matrix is given by

$$(r - p)\, m = \left(\sum_{i=1}^{p} r_i - p\right) m,$$

where p is the number of repeated roots of Θ and r_i is the number of times that the repeated root θ_i appears in Θ. Hence, the limiting covariance matrix assumes its maximum rank, $(K-1)\, m$, when all diagonal elements of Θ are equal. On the other hand, the inequality $r \geq 2p$ implies that the minimum rank, m, occurs when $r = 2$ and $p = 1$, i.e. when Θ has exactly 2 equal diagonal elements. The rank of the limiting distribution of Theorem 2.4 reflects the fact that the orthogonal transformation H removes the singularity in (4) by cancelling out the effect of the regressors in (2) that are not cointegrated. The $\sqrt{n}$ limit theory of Theorem 2.4 is driven exclusively from the cointegrated part of x_t, i.e. the regressors in (2) that contain repeated explosive roots.

(v) In view of Theorem 2.4, the limit behavior of

$$\begin{aligned}\left(\frac{1}{n}\sum_{t=1}^{n} x_t x_t'\right)^{-1} &= \Pi' H' \left(\frac{1}{n}\sum_{t=1}^{n} z_t z_t'\right)^{-1} H \Pi \\ &\to_p \Pi_1' H_\perp \left(H_\perp' \Pi_1 \sum_{j=1}^{\infty} \Theta^{-j} \Sigma_{uu} \Theta^{-j} \Pi_1' H_\perp\right)^{-1} H_\perp' \Pi_1,\end{aligned}$$

and the fact that $\hat{\Sigma}_{\varepsilon\varepsilon} = n^{-1}\sum_{t=1}^{n} \hat{\varepsilon}_t \hat{\varepsilon}_t' \to_p \Sigma_{\varepsilon\varepsilon}$, where $\hat{\varepsilon}_t = y_t - \hat{A}_n x_t$, we obtain conventional asymptotic chi-squared distributions under the null hypothesis for regression Wald tests such as

$$W_n = g\left(\hat{A}_n\right)' \left[G_A \left\{\left(\sum_{t=1}^{n} x_t x_t'\right)^{-1} \otimes \hat{\Sigma}_{\varepsilon\varepsilon}\right\} G_A'\right]^{-1} g\left(\hat{A}_n\right), \quad G_A = \frac{\partial g}{\partial vec A'},$$

for some analytic restrictions of the form $H_0 : g(A) = 0$.

(vi) Note that the matrix

$$\Pi_1' H_\perp \left(H_\perp' \Pi_1 \sum_{j=1}^{\infty} \Theta^{-j} \Sigma_{uu} \Theta^{-j} \Pi_1' H_\perp \right)^{-1} H_\perp' \Pi_1$$

is invariant to the coordinate system defining $H_\perp$, so the limit theory of Theorem 2.4 is also invariant to the choice of coordinates.

We now provide a discussion of the asymptotic behavior of $\hat{A}_n - A$ in the direction of X_Θ. Recalling the partitioned form of Π and (14), the vector

$$H\Pi X_\Theta = \begin{bmatrix} U_\perp' \Pi X_\Theta \\ U_\Phi' \Pi X_\Theta \end{bmatrix} = \begin{bmatrix} H_\perp' \Pi_1 X_\Theta \\ U_\Phi' \Pi X_\Theta \end{bmatrix} = \begin{bmatrix} \underset{(r-p)\times 1}{0} \\ U_\Phi' \Pi X_\Theta \end{bmatrix}$$

cancels out the effect of $\left(\frac{Z_1' Z_1}{n}\right)^{-1}$ on the variance matrix in (24) and produces a typical explosive limit theory for $\hat{A}_n$. More specifically, letting $D := U_\Phi' \Phi U_\Phi$, (8), (18) and (24) yield

$$\begin{aligned}
\left(\hat{A}_n - A\right) \Theta^n X_\Theta &= \left(\hat{\Psi}_n - \Psi\right) \Phi^n \Pi X_\Theta \\
&= \left(\hat{\Psi}_n - \Psi\right) H' \left(H \Phi^n H'\right) H \Pi X_\Theta \\
&= \left(\hat{\Psi}_n - \Psi\right) H' \mathrm{diag}\left(U_\perp' \Phi^n U_\perp, U_\Phi' \Phi^n U_\Phi\right) H \Pi X_\Theta \\
&= \left(\sum_{t=1}^{n} \varepsilon_t z_t'\right) \left(\sum_{t=1}^{n} z_t z_t'\right)^{-1} \begin{bmatrix} \underset{(r-p)\times 1}{0} \\ D^n U_\Phi' \Pi X_\Theta \end{bmatrix} \\
&= \left(\sum_{t=1}^{n} \varepsilon_t z_{2t}'\right) \left[\left(Z_2' Z_2\right)^{-1} D^n + P_{2n} \left(Z_1' Q_2 Z_1\right)^{-1} P_{2n}' D^n\right] U_\Phi' \Pi X_\Theta \\
&\quad - \left(\sum_{t=1}^{n} \varepsilon_t z_{1t}'\right) \left(Z_1' Q_2 Z_1\right)^{-1} P_{2n}' D^n U_\Phi' \Pi X_\Theta. \qquad (27)
\end{aligned}$$

From the analysis preceding Theorem 2.4, we know that $\|P_{2n}\| = O_p\left(\|\Theta\|^{-n}\right)$, $\left(Z_1' Q_2 Z_1\right)^{-1} = O_p\left(n^{-1}\right)$ and $\sum_{t=1}^{n} \varepsilon_t z_{1t}' = O_p\left(n^{1/2}\right)$. Thus, since D is a diagonal matrix consisting of all distinct diagonal elements of Θ, $\|D\| = \|\Theta\|$ and the last term in (27) has asymptotic order $O_p\left(n^{-1/2}\right)$. On the other hand, using (16) and the fact that $U_\perp' \Phi^n U_\Phi = H_\perp' \Phi_1^n H_\Phi = 0$, we can write

$$\begin{aligned}
\sum_{t=1}^{n} \varepsilon_t z_{2t}' D^{-n} &= \sum_{t=1}^{n} \varepsilon_t x_t' \Pi' U_\Phi U_\Phi' \Phi^{-n} U_\Phi \\
&= \sum_{t=1}^{n} \varepsilon_t x_t' \Pi' \left(I_K - U_\perp U_\perp' \right) \Phi^{-n} U_\Phi \\
&= \sum_{t=1}^{n} \varepsilon_t x_t' \left(\Pi' \Phi^{-n} \Pi \right) \Pi' U_\Phi \\
&= \sum_{t=1}^{n} \varepsilon_t x_t' \Theta^{-n} \Pi' U_\Phi, \qquad (28)
\end{aligned}$$

so that $\sum_{t=1}^{n} \varepsilon_t z_{2t}' = O_p \left(\|\Theta\|^n \right)$, and the second term in (27) has asymptotic order $O_p \left(n^{-1} \right)$. Thus, Lemma 4.4 (a) and (28) yield

$$\begin{aligned}
\left(\hat{A}_n - A \right) \Theta^n X_\Theta &= \left(\sum_{t=1}^{n} \varepsilon_t z_{2t}' D^{-n} \right) \left(D^{-n} Z_2' Z_2 D^{-n} \right)^{-1} U_\Phi' \Pi X_\Theta + O_p \left(n^{-1/2} \right) \\
&= \sum_{t=1}^{n} \varepsilon_t x_t' \Theta^{-n} \Pi' U_\Phi \left(U_\Phi' \Pi \Theta^{-n} \sum_{t=1}^{n} x_t x_t' \Theta^{-n} \Pi' U_\Phi \right)^{-1} U_\Phi' \Pi X_\Theta.
\end{aligned}$$

The asymptotic behavior of $\hat{A}_n - A$ in the direction of X_Θ is determined by an argument identical to the non singular case of Theorem 2.2. For Gaussian innovations u_t, Lemma 4.2, (33) and (34) imply that $\Theta^{-n} \sum_{t=1}^{n} x_t x_t' \Theta^{-n}$ and $\sum_{t=1}^{n} \varepsilon_t x_t' \Theta^{-n}$ converge jointly in distribution, leading to a mixed normal limit, stated formally as follows.

2.6 Theorem. *For the explosive cointegrated system generated by (1) and (2) with $v_t =_d N(0, \Sigma)$ and $c_i = c_j$ for some $i \neq j$ the following limit theory applies on the direction of X_Θ*

$$\left(\hat{A}_n - A \right) \Theta^n X_\Theta \Rightarrow MN \left(0, X_\Theta' \Pi' U_\Phi \mathbb{V}^{-1} U_\Phi' \Pi X_\Theta \Sigma_{\varepsilon\varepsilon} \right)$$

where

$$\mathbb{V} = U_\Phi' \Pi \sum_{j=0}^{\infty} \Theta^{-j} X_\Theta X_\Theta' \Theta^{-j} \Pi' U_\Phi.$$

2.7 Remarks.

(i) The limit theory for the least squares estimator in the direction of X_Θ is mixed Gaussian with full rank covariance matrix of order m and the usual explosive rate of convergence. As in the non singular case, the assumption of Gaussian innovations is essential for the reasons explained in Remark 2.3 (i).

(ii) Rotation of the regression space in the direction of X_Θ determines the limit theory in the explosive direction resolving the singularity of the limiting moment matrix $\sum_{j=0}^{\infty} \Theta^{-j} X_\Theta X_\Theta' \Theta^{-j}$.

(iii) In the polar case of equal localising coefficients, we have $\Theta = \theta I_K$ with $\Pi = I_K$. Thus, Theorem 2.6 reduces to

$$\frac{\theta^{n+1}}{\sqrt{\theta^2 - 1}} \left(\hat{A}_n - A \right) X_\Theta \Rightarrow MN \left(0, \Sigma_{\varepsilon\varepsilon} \right).$$

3. Discussion

This paper provides a limit theory for explosively cointegrated systems. Both the normalisation and the limit distribution of the centred least squares estimate $\hat{A}_n - A$ are found to vary according to whether the regressors contain common explosive roots. When all the explosive roots are distinct, the Θ^n exponential rate of convergence and a full rank mixed normal limiting distribution apply under the assumption of Gaussian innovations. On the other hand, repeated explosive roots give rise to a degeneracy in the regression limit theory. This degeneracy is resolved analytically by an appropriate orthogonal rotation of the regression coordinates. The resulting limit theory is mixed normal and of reduced rank. The rank of the limit distribution depends on the number of repeated roots but is invariant to both the choice of coordinates and the distribution of the innovations. Thus, in the case where some explosive roots are common, an invariance principle holds.

The authors have shown that similar results to those given here hold for mildly explosive cointegrated systems with roots that approach unity at rates slower than n^{-1}. In particular, Magdalinos and Phillips (2006) consider models such as (1) and (2) with mildly explosive roots of the form

$$\Theta_n = I_K + \frac{C}{n^\alpha}, \ \alpha \in (0, 1), \quad C = \mathrm{diag}(c_1, ..., c_K) > 0.$$

For such systems, a mixed normal asymptotic distribution is derived for the least squares estimator with the mildly explosive rate of convergence, $n^\alpha \Theta_n^n$, when C has distinct diagonal elements and with the moderately stationary rate, $n^{(1+\alpha)/2}$, when C has repeated roots, corresponding to Theorems 2.2 and 2.4 respectively. An attractive feature of mildly explosive systems is that central limit theory applies in both cases and asymptotic mixed normality is valid without distributional assumptions on the innovations even when C has distinct diagonal elements. Such systems may also be more realistic for practical work.

4. Proofs

This section contains some technical lemmas as well as proofs of various statements and results in the paper. Throughout, we use the notation

$$\kappa_n := \lfloor n/2 \rfloor, \quad X_{\kappa_n} := \sum_{j=1}^{\kappa_n} \Theta^{-j} u_j, \tag{29}$$

$\mathcal{F}_t := \sigma(v_1, ..., v_t)$ for the natural filtration of the innovations, and let C be a bounding constant in $(0, \infty)$ that may assume different values. The above choice for κ_n is made for the sake of simplicity, and the results hold for any integer valued sequence κ_n satisfying

$$\sum_{n=1}^{\infty} n \|\Theta\|^{-2\kappa_n} < \infty \text{ and } \|\Theta\|^{-(n-\kappa_n)} \kappa_n^{1/2} \to 0 \text{ as } n \to \infty.$$

4.1 Lemma. *For κ_n and X_{κ_n} as defined in (29), we have*

$$\max_{\kappa_n+1 \le t \le n} \left\| \sum_{j=\kappa_n+1}^{t} \Theta^{-j} u_j \right\| = o_{a.s.}\left(\frac{1}{\sqrt{n}}\right) \quad \text{as } n \to \infty.$$

Proof. Using Doob's inequality for martingales we obtain, for each $\delta > 0$,

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left(\max_{\kappa_n+1 \le t \le n} \left\| \sum_{j=\kappa_n+1}^{t} \Theta^{-j} u_j \right\| > \frac{\delta}{\sqrt{n}}\right) &\le \frac{1}{\delta^2} \sum_{n=1}^{\infty} nE \left\| \sum_{j=\kappa_n+1}^{n} \Theta^{-j} u_j \right\|^2 \\
&= \frac{E\|u_1\|^2}{\delta^2} \sum_{n=1}^{\infty} n \sum_{j=\kappa_n+1}^{n} \|\Theta\|^{-2j} \\
&\le C \sum_{n=1}^{\infty} n \|\Theta\|^{-2\kappa_n} \\
&\le C \sum_{n=1}^{\infty} n \|\Theta\|^{-n} < \infty.
\end{aligned}$$

4.2 Lemma. *For κ_n and X_{κ_n} as defined in (29), we have, as $n \to \infty$,*

$$(\Theta^{-n} \otimes I_m) \sum_{t=1}^{n} (x_t \otimes \varepsilon_t) = \sum_{t=\kappa_n+1}^{n} (\Theta^{-n+t} \otimes I_m)(X_{\kappa_n} \otimes \varepsilon_t) + o_{a.s.}(1).$$

Proof. The lemma will follow by showing (30) and (31) below.

$$\left\| \sum_{t=1}^{\kappa_n} \left(\Theta^{-n+t} \otimes I_m\right) \left[\left(\Theta^{-t} x_t\right) \otimes \varepsilon_t\right] \right\| = o_{a.s.}(1) \tag{30}$$

$$\left\| \sum_{t=\kappa_n+1}^{n} \left(\Theta^{-n+t} \otimes I_m\right) \left(\sum_{j=\kappa_n+1}^{t} \Theta^{-j} u_j \otimes \varepsilon_t \right) \right\| = o_{a.s.}(1). \tag{31}$$

For (30), since $\|\Theta^{-t} x_t\| \leq \|\Theta^{-t} x_t - X_\Theta\| + \|X_\Theta\|$ and $\|\Theta^{-t} x_t - X_\Theta\| = o_{a.s.}(1)$, there exists a constant $C \in (0, \infty)$ such that

$$\left\|\Theta^{-t} x_t\right\| \leq C + \|X_\Theta\| \quad \forall t \geq 1 \ a.s. \tag{32}$$

with $\|X_\Theta\| < \infty$ $a.s.$ by the martingale convergence theorem. Thus, by ergodicity,

$$\begin{aligned}
&\left\| \sum_{t=1}^{\kappa_n} \left(\Theta^{-n+t} \otimes I_m\right) \left[\left(\Theta^{-t} x_t\right) \otimes \varepsilon_t\right] \right\| \\
\leq\ & \left(C + \|X_\Theta\|\right) \|\Theta\|^{-n} \sum_{t=1}^{\kappa_n} \|\Theta\|^{t} \|\varepsilon_t\| \\
\leq\ & \left(C + \|X_\Theta\|\right) \|\Theta\|^{-n} \left(\sum_{t=1}^{\kappa_n} \|\Theta\|^{2t} \right)^{1/2} \left(\sum_{t=1}^{\kappa_n} \|\varepsilon_t\|^2 \right)^{1/2} \\
=\ & O_{a.s.}\left(\|\Theta\|^{-(n-\kappa_n)} \kappa_n^{1/2} \right) = o_{a.s.}(1),
\end{aligned}$$

showing (30). For (31), Lemma 4.1 and the ergodic theorem yield

$$\begin{aligned}
&\left\| \sum_{t=\kappa_n+1}^{n} \left(\Theta^{-n+t} \otimes I_m\right) \left(\sum_{j=\kappa_n+1}^{t} \Theta^{-j} u_j \otimes \varepsilon_t \right) \right\| \\
\leq\ & \max_{\kappa_n+1 \leq t \leq n} \left\| \sum_{j=\kappa_n+1}^{t} \Theta^{-j} u_j \right\| \left(\sum_{t=\kappa_n+1}^{n} \|\Theta\|^{-n+t} \|\varepsilon_t\| \right) \\
\leq\ & \max_{\kappa_n+1 \leq t \leq n} \left\| \sum_{j=\kappa_n+1}^{t} \Theta^{-j} u_j \right\| \left(\sum_{t=\kappa_n+1}^{n} \|\Theta\|^{-2(n-t)} \right)^{1/2} \left(\sum_{t=\kappa_n+1}^{n} \|\varepsilon_t\| \right)^{1/2} \\
\leq\ & C\sqrt{n} \max_{\kappa_n+1 \leq t \leq n} \left\| \sum_{j=\kappa_n+1}^{t} \Theta^{-j} u_j \right\| \left(\frac{1}{n} \sum_{t=\kappa_n+1}^{n} \|\varepsilon_t\| \right)^{1/2} = o_{a.s.}(1).
\end{aligned}$$

Proof of Theorem 2.1. By (32) we obtain, almost surely,

$$\begin{aligned}
\left\| \sum_{t=1}^{\kappa_n} \Theta^{-(n-t)} \Theta^{-t} x_t x_t' \Theta^{-t} \Theta^{-(n-t)} \right\| &\leq \left(C + \|X_\Theta\|\right)^2 \sum_{t=1}^{\kappa_n} \|\Theta\|^{-2(n-t)} \\
&= O_{a.s.}\left(\|\Theta\|^{-2(n-\kappa_n)} \right).
\end{aligned}$$

Thus, Lemma 4.1 yields

$$\begin{aligned}\Theta^{-n}\sum_{t=1}^{n}x_t x_t'\Theta^{-n} &= \sum_{t=\kappa_n+1}^{n}\Theta^{-(n-t)}\Theta^{-t}x_t x_t'\Theta^{-t}\Theta^{-(n-t)} + o_{a.s.}(1) \\ &= \sum_{t=\kappa_n+1}^{n}\Theta^{-(n-t)}X_{\kappa_n}X_{\kappa_n}'\Theta^{-(n-t)} + o_{a.s.}(1) \\ &= \sum_{j=0}^{n-\kappa_n-1}\Theta^{-j}X_{\kappa_n}X_{\kappa_n}'\Theta^{-j} + o_{a.s.}(1), \qquad (33)\end{aligned}$$

and Theorem 2.1 follows immediately by the martingale convergence theorem.

Proof of Theorem 2.2. By (33) and Lemma 4.2,

$$\begin{aligned}&vec\left[\left(\hat{A}_n - A\right)\Theta^n\right] \\ = &\left[\left(\sum_{t=1}^{n}\Theta^{-n}x_t x_t'\Theta^{-n}\right)^{-1}\otimes I_m\right]\left(\Theta^{-n}\otimes I_m\right)\sum_{t=1}^{n}(x_t\otimes\varepsilon_t) \\ = &\left[\left(\sum_{j=0}^{n-\kappa_n-1}\Theta^{-j}X_{\kappa_n}X_{\kappa_n}'\Theta^{-j}\right)^{-1}\otimes I_m\right]\sum_{t=\kappa_n+1}^{n}\left(\Theta^{-n+t}X_{\kappa_n}\otimes\varepsilon_t\right) + o_{a.s.}(1).\end{aligned}$$

The assumption of Gaussian errors yields, conditional on $\mathcal{F}_{\kappa_n}$,

$$\sum_{t=\kappa_n+1}^{n}\left(\Theta^{-n+t}X_{\kappa_n}\otimes\varepsilon_t\right) =_d N\left(0, \left(\sum_{j=0}^{n-\kappa_n-1}\Theta^{-j}X_{\kappa_n}X_{\kappa_n}'\Theta^{-j}\right)\otimes\Sigma_{\varepsilon\varepsilon}\right) \qquad (34)$$

which leads to

$$\begin{aligned}vec\left[\left(\hat{A}_n - A\right)\Theta^n\right] &=_d \left[\left(\sum_{j=0}^{n-\kappa_n-1}\Theta^{-j}X_{\kappa_n}X_{\kappa_n}'\Theta^{-j}\right)^{-\frac{1}{2}}\otimes I_m\right]N\left(0, I_K\otimes\Sigma_{\varepsilon\varepsilon}\right) \\ &\Rightarrow MN\left(0, \left(\sum_{j=0}^{\infty}\Theta^{-j}X_{\Theta}X_{\Theta}'\Theta^{-j}\right)^{-1}\otimes\Sigma_{\varepsilon\varepsilon}\right),\end{aligned}$$

as $n\to\infty$, since $X_{\kappa_n}\to_{a.s.}X_\Theta$.

4.3 Lemma. *The determinant of the matrix*

$$W^{(s)} := \left[\frac{1}{w_i w_j - 1} : i, j \in \{1, ..., s\}\right]$$

is given by

$$\left|W^{(s)}\right| = \frac{1}{(w_1^2-1)\dots(w_s^2-1)}\prod_{j=1}^{s-1}\frac{(w_j-w_{j+1})^2(w_j-w_{j+2})^2\dots(w_j-w_s)^2}{(w_jw_{j+1}-1)^2(w_jw_{j+2}-1)^2\dots(w_jw_s-1)^2}.$$

Consequently, the matrix

$$M_W := \left[\frac{w_iw_j}{w_iw_j-1} : i,j\in\{1,\dots,s\}\right]$$

is nonsingular if and only if $w_i \neq w_j$ *for all* $i \neq j$.

Proof. We use induction. The result is immediate for $s = 2$. If we assume the result for $s-1$, and partition $W^{(s)}$ as

$$W^{(s)} = \begin{bmatrix} W^{(s-1)} & w \\ w' & \frac{1}{w_s^2-1}\end{bmatrix} \quad w := \left[\frac{1}{w_1w_s-1},\dots,\frac{1}{w_{s-1}w_s-1}\right]',$$

we have (e.g., 5.29 of Abadir and Magnus, 2005)

$$\left(w_s^2-1\right)\left|W^{(s)}\right| = \left|W^{(s-1)} - \left(w_s^2-1\right)ww'\right|.$$

Since the matrix on the right is equal to

$$\text{diag}\left(\frac{w_1-w_s}{w_1w_s-1},\dots,\frac{w_{s-1}-w_s}{w_{s-1}w_s-1}\right)W^{(s-1)}\text{diag}\left(\frac{w_1-w_s}{w_1w_s-1},\dots,\frac{w_{s-1}-w_s}{w_{s-1}w_s-1}\right)$$

and $\left|W^{(s-1)}\right|$ is known from the induction hypothesis, we obtain

$$\begin{aligned}\left|W^{(s)}\right| &= \frac{1}{w_s^2-1}\frac{(w_1-w_s)^2}{(w_1w_s-1)^2}\cdots\frac{(w_{s-1}-w_s)^2}{(w_{s-1}w_s-1)^2}\left|W^{(s-1)}\right| \\ &= \frac{1}{(w_1^2-1)\dots(w_s^2-1)}\prod_{i=1}^{s-1}\frac{(w_i-w_s)^2}{(w_iw_s-1)^2}\prod_{j=1}^{s-2}\frac{(w_j-w_{j+1})^2\dots(w_j-w_{s-1})^2}{(w_jw_{j+1}-1)^2\dots(w_jw_{s-1}-1)^2} \\ &= \frac{1}{(w_1^2-1)\dots(w_s^2-1)}\prod_{j=1}^{s-1}\frac{(w_j-w_{j+1})^2\dots(w_j-w_s)^2}{(w_jw_{j+1}-1)^2\dots(w_jw_s-1)^2}\end{aligned}$$

as required. Hence, $W^{(s)}$ is nonsingular if and only if $w_i \neq w_j$ for all $i \neq j$. The identity $M_W = \text{diag}(w_1,\dots,w_s)\,W^{(s)}\text{diag}(w_1,\dots,w_s)$ implies that nonsingularity of M_W is equivalent to nonsingularity of $W^{(s)}$.

4.4 Lemma. *Let* $D = U_\Phi'\Phi U_\Phi$. *The following hold as* $n\to\infty$:

(a) $D^{-n}Z_2'Z_2D^{-n} \to_{a.s.} U_\Phi'\Pi\left(\sum_{j=0}^{\infty}\Theta^{-j}X_\Theta X_\Theta'\Theta^{-j}\right)\Pi'U_\Phi > 0$ *a.s.*,

(b) $\|Z_2'Z_1\| = O_p\left(\|\Theta\|^n\right)$,

(c) $\left(\frac{Z_1'Q_2Z_1}{n}\right)^{-1} = \left(\frac{Z_1'Z_1}{n}\right)^{-1} + O_p\left(n^{-1}\right)$.

Proof. For part (a), first note that $U_\Phi'\Phi^{-j}U_\perp = H_\Phi'\Phi_1^{-j}H_\perp = 0$ for all $j \in \mathbb{N}$. Thus, using (17) and (16) we obtain

$$\begin{aligned} D^{-j}z_{2t} &= U_\Phi'\Phi^{-j}U_\Phi U_\Phi'\Pi x_t \\ &= U_\Phi'\Phi^{-j}\left(I_K - U_\perp U_\perp'\right)\Pi x_t \\ &= U_\Phi'\Phi^{-j}\Pi x_t - \left(U_\Phi'\Phi^{-j}U_\perp\right)U_\perp'\Pi x_t \\ &= U_\Phi'\Pi\left(\Pi'\Phi^{-j}\Pi\right)x_t \\ &= U_\Phi'\Pi\Theta^{-j}x_t, \end{aligned} \tag{35}$$

for all $j \in \mathbb{N}$, and similarly

$$D^{-j}U_\Phi'\Pi X_\Theta = U_\Phi'\Pi\Theta^{-j}X_\Theta. \tag{36}$$

The limit matrix of part (a) now follows immediately from (35) and Theorem 2.1:

$$\begin{aligned} D^{-n}Z_2'Z_2D^{-n} &= U_\Phi'\Phi^{-n}U_\Phi\sum_{t=1}^{n}z_{2t}z_{2t}'U_\Phi'\Phi^{-n}U_\Phi \\ &= U_\Phi'\Pi\Theta^{-n}\sum_{t=1}^{n}x_tx_t'\Theta^{-n}\Pi'U_\Phi \\ &\to_{a.s.} U_\Phi'\Pi\sum_{j=0}^{\infty}\Theta^{-j}X_\Theta X_\Theta'\Theta^{-j}\Pi'U_\Phi. \end{aligned}$$

In order to establish the nonsingularity of the limit matrix, note that, by (5),

$$D = U_\Phi'\Phi U_\Phi = \operatorname{diag}\left(\theta_1, ..., \theta_p, \varphi_1, ..., \varphi_{K-r}\right)$$

consists of all distinct diagonal elements of Θ. Then, denoting by d_i the i-th diagonal element of D, Lemma 4.3 implies that the matrix

$$M_D := \left[\frac{d_id_j}{d_id_j - 1} : i, j \in \{1, ..., K - r + p\}\right]$$

is nonsingular. Denoting by $S_\Phi^{(i)}$ the i-th element of the vector $S_\Phi = U_\Phi'\Pi X_\Theta$ and letting $\check{S}_\Phi :=\operatorname{diag}\left(S_\Phi^{(1)}, ..., S_\Phi^{(K-r+p)}\right)$, (36) gives

$$\begin{aligned} U_\Phi'\Pi\sum_{j=0}^{\infty}\Theta^{-j}X_\Theta X_\Theta'\Theta^{-j}\Pi'U_\Phi &= \sum_{j=0}^{\infty}D^{-j}S_\Phi S_\Phi'D^{-j} \\ &= \check{S}_\Phi M_D\check{S}_\Phi. \end{aligned}$$

The last matrix is nonsingular $a.s.$ since M_D is nonsingular and $S_\Phi^{(i)} \neq 0$ $a.s.$ for each i by absolute continuity of u_t.

For part (b), using a matrix Cauchy Schwarz inequality (e.g. 12.5 in Abadir and Magnus, 2005) we obtain

$$\begin{aligned}\|z_{1t}\|^2 &= \left\|H_\perp' \zeta_{t+1}\right\|^2 = tr\left(H_\perp H_\perp' \zeta_{t+1}\zeta_{t+1}'\right) \\ &\leq \left[tr\left(H_\perp H_\perp'\right)\right]^{1/2}\left[\left\|\zeta_{t+1}\right\|^2 tr\left(\zeta_{t+1}\zeta_{t+1}'\right)\right]^{1/2} \\ &= \left[tr\left(H_\perp' H_\perp\right)\right]^{1/2}\left\|\zeta_{t+1}\right\|^2 = (r-p)^{1/2}\left\|\zeta_{t+1}\right\|^2\end{aligned}$$

since $H_\perp' H_\perp = I_{r-p}$. Also, using (35) and a standard trace inequality we can write

$$\begin{aligned}E\|z_{2t}\|^2 &\leq E\left[tr\left(z_{2t}z_{2t}'\right)\right] \leq E\left[tr\left(U_\Phi U_\Phi'\right) tr\left(\tilde{x}_t\tilde{x}_t'\right)\right] \\ &= KE\|\tilde{x}_t\|^2 \leq \frac{KE\|\tilde{u}_1\|^2}{\|\Phi\|^2-1}\|\Phi\|^{2t}.\end{aligned}$$

Thus,

$$\begin{aligned}E\|vec\left(Z_2'Z_1\right)\| &\leq \sum_{t=1}^n E\left(\|z_{1t}\|\,\|z_{2t}\|\right) \leq \sum_{t=1}^n \left(E\|z_{1t}\|^2\right)^{1/2}\left(E\|z_{2t}\|^2\right)^{1/2} \\ &\leq (r-p)^{1/4}\left(E\|\zeta_1\|^2\right)^{1/2}\sum_{t=1}^n\left(E\|z_{2t}\|^2\right)^{1/2} \\ &\leq (r-p)^{1/4}\left(\frac{KE\|\tilde{u}_1\|^2 E\|\zeta_1\|^2}{\|\Phi\|^2-1}\right)^{1/2}\sum_{t=1}^n\|\Phi\|^t \\ &= O\left(\|\Phi\|^n\right)\end{aligned}$$

and the result follows since $\|\Phi\| = \|\Theta\|$. For part (c), note that part (a) implies that $\|Z_2'Z_2\| = O_p\left(\|\Theta\|^{2n}\right)$. Thus,

$$\frac{1}{n}\left\|Z_1'Z_2\left(Z_2'Z_2\right)^{-1}Z_2'Z_1\right\| \leq \frac{1}{n}\|Z_1'Z_2\|^2\|Z_2'Z_2\|^{-1} = O_p\left(\frac{1}{n}\right),$$

by part (b) and the result follows from the definition of Q_2.

Proof of (19). Since $\Phi_1^i = \text{diag}\left(\theta_1^i I_{r_1},...,\theta_p^i I_{r_p}\right)$, (14) gives, for all $i \in \mathbb{Z}$

$$H_\perp'\Phi_1^i H_\Phi = \text{diag}\left(\theta_1^i H_{\perp 1}'H_{\Phi 1},...,\theta_p^i H_{\perp p}'H_{\Phi p}\right) = 0.$$

On the other hand, (15) and the above identity for $i=-1$ yields

$$\begin{aligned}\left(H_\perp'\Phi_1^{-1}H_\perp\right)^2 &= H_\perp'\Phi_1^{-1}H_\perp H_\perp'\Phi_1^{-1}H_\perp \\ &= H_\perp'\Phi_1^{-1}\left(I_r - H_\Phi H_\Phi'\right)\Phi_1^{-1}H_\perp \\ &= H_\perp'\Phi_1^{-2}H_\perp - \left(H_\perp'\Phi_1^{-1}H_\Phi\right)H_\Phi'\Phi_1^{-1}H_\perp \\ &= H_\perp'\Phi_1^{-2}H_\perp,\end{aligned}$$

so we have proved the identity $\left(H_\perp'\Phi_1^{-1}H_\perp\right)^j = H_\perp'\Phi_1^{-j}H_\perp$ for $j=2$. The general case follows by straightforward induction.

Proof of (21). Using (10) and (11) the definition of z_{1t} yields

$$\begin{aligned} z_{1t} &= H_\perp' \Pi_1 x_t = H_\perp' \Pi_1 \left(\Theta^{-1} x_{t+1} - \Theta^{-1} u_{t+1}\right) \\ &= H_\perp' \Pi_1 \Theta^{-1} x_{t+1} - H_\perp' \Pi_1 \Theta^{-1} u_{t+1} \\ &= H_\perp' \Pi_1 \Theta^{-1} \left(\Pi_1' \Pi_1 + \Pi_2' \Pi_2\right) x_{t+1} - H_\perp' \Pi_1 \Theta^{-1} \left(\Pi_1' \Pi_1 + \Pi_2' \Pi_2\right) u_{t+1} \\ &= H_\perp' \left(\Pi_1 \Theta^{-1} \Pi_1'\right) \Pi_1 x_{t+1} - H_\perp' \left(\Pi_1 \Theta^{-1} \Pi_1'\right) \Pi_1 u_{t+1} \\ &= H_\perp' \Phi_1^{-1} \Pi_1 x_{t+1} - H_\perp' \Phi_1^{-1} \Pi_1 u_{t+1}. \end{aligned}$$

The second term has the form that appears in (21). For the first term, using the fact that $H_\perp' \Phi^{-1} H_\Phi = 0$, we can write

$$\begin{aligned} H_\perp' \Phi_1^{-1} \Pi_1 x_{t+1} &= H_\perp' \Phi_1^{-1} \left(H_\perp H_\perp' + H_\Phi H_\Phi'\right) \Pi_1 x_{t+1} \\ &= \left(H_\perp' \Phi_1^{-1} H_\perp\right) H_\perp' \Pi_1 x_{t+1} \\ &= \left(H_\perp' \Phi_1^{-1} H_\perp\right) z_{1t+1}, \end{aligned}$$

as required.

Proof of (26). Recalling the notation $\zeta_{t+1} = \sum_{j=1}^{\infty} \Theta^{-j} u_{t+j}$, (22) yields the following expression for the sample covariance:

$$\begin{aligned} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left(z_{1t} \otimes \varepsilon_t\right) &= -\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left(H_\perp' \Pi_1 \zeta_{t+1} \otimes \varepsilon_t\right) \\ &= -\left(H_\perp' \Pi_1 \otimes I_m\right) \sum_{t=1}^{n} \xi_{nt}, \end{aligned} \tag{37}$$

where $\xi_{nt} := n^{-1/2} \left(\zeta_{t+1} \otimes \varepsilon_t\right)$ is a martingale difference array with respect to $\mathcal{F}_{t+1}$, since ζ_{t+1} is $\sigma\left(v_{t+1}, v_{t+2}, ...\right)$ measurable. The conditional variance of $\sum_{t=1}^{n} \xi_{nt}$ is given by

$$\begin{aligned} \sum_{t=1}^{n} E_{\mathcal{F}_t}\left(\xi_{nt} \xi_{nt}'\right) &= \frac{1}{n} \sum_{t=1}^{n} \left[E_{\mathcal{F}_t}\left(\zeta_{t+1} \zeta_{t+1}'\right) \otimes \varepsilon_t \varepsilon_t'\right] \\ &= \frac{1}{n} \sum_{t=1}^{n} \left[E\left(\zeta_{t+1} \zeta_{t+1}'\right) \otimes \varepsilon_t \varepsilon_t'\right] \\ &= \left(\sum_{j=1}^{\infty} \Theta^{-j} \Sigma_{uu} \Theta^{-j}\right) \otimes \left(\frac{1}{n} \sum_{t=1}^{n} \varepsilon_t \varepsilon_t'\right) \\ &\to_{a.s.} \left(\sum_{j=1}^{\infty} \Theta^{-j} \Sigma_{uu} \Theta^{-j}\right) \otimes \Sigma_{\varepsilon\varepsilon}, \end{aligned}$$

by the ergodic theorem. Thus, provided that the Lindeberg condition

$$\sum_{t=1}^{n} E_{\mathcal{F}_t}\left(\left\|\xi_{nt}\right\|^2 \mathbf{1}\left\{\left\|\xi_{nt}\right\| > \delta\right\}\right) \to_p 0 \quad \delta > 0 \tag{38}$$

holds, Corollary 3.1 of Hall and Heyde (1980) and the Cramér Wold theorem imply that

$$\sum_{t=1}^{n} \xi_{nt} \Rightarrow N\left(0, \left(\sum_{j=1}^{\infty} \Theta^{-j} \Sigma_{uu} \Theta^{-j}\right) \otimes \Sigma_{\varepsilon\varepsilon}\right). \tag{39}$$

The proof of (38) is given below. The proof of (26) follows from (37) and (39).

Proof of (38). The Lindeberg condition (38) is equivalent to

$$\frac{1}{n}\sum_{t=1}^{n} \|\varepsilon_t\|^2 E_{\mathcal{F}_t}\left(\|\zeta_{t+1}\|^2 \mathbf{1}\left\{\|\zeta_{t+1}\| \|\varepsilon_t\| > \delta n^{1/2}\right\}\right) = o_p(1). \tag{40}$$

Applying the inequality

$$\mathbf{1}\left\{\|\zeta_{t+1}\| \|\varepsilon_t\| > \delta n^{1/2}\right\} \leq \mathbf{1}\left\{\|\zeta_{t+1}\| > \delta^{1/2} n^{1/4}\right\} + \mathbf{1}\left\{\|\varepsilon_t\| > \delta^{1/2} n^{1/4}\right\}$$

to (40), we deduce that (38) will follow if the following terms

$$\begin{aligned} S_{n1} &= \frac{1}{n}\sum_{t=1}^{n} \|\varepsilon_t\|^2 E\left(\|\zeta_{t+1}\|^2 \mathbf{1}\left\{\|\zeta_{t+1}\| > \delta^{1/2} n^{1/4}\right\}\right) \\ S_{n2} &= \frac{1}{n}\sum_{t=1}^{n} \|\varepsilon_t\|^2 \mathbf{1}\left\{\|\varepsilon_t\| > \delta^{1/2} n^{1/4}\right\} E\|\zeta_{t+1}\|^2 \end{aligned}$$

are $o_p(1)$. $S_{n1} \to 0$ in L_1 since, using the fact that ζ_{t+1} is a strictly stationary sequence with $E\|\zeta_1\|^2 < \infty$,

$$\begin{aligned} ES_{n1} &\leq \max_{1 \leq t \leq n} E\left(\|\zeta_{t+1}\|^2 \mathbf{1}\left\{\|\zeta_{t+1}\| > \delta^{1/2} n^{1/4}\right\}\right) E\|\varepsilon_1\|^2 \\ &= E\left(\|\zeta_1\|^2 \mathbf{1}\left\{\|\zeta_1\| > \delta^{1/2} n^{1/4}\right\}\right) E\|\varepsilon_1\|^2 \to 0. \end{aligned}$$

S_{n2} also tends to 0 in L_1 since

$$\begin{aligned} ES_{n2} &= E\|\zeta_1\|^2 \frac{1}{n}\sum_{t=1}^{n} E\left(\|\varepsilon_t\|^2 \mathbf{1}\left\{\|\varepsilon_t\| > \delta^{1/2} n^{1/4}\right\}\right) \\ &= E\|\zeta_1\|^2 E\left(\|\varepsilon_1\|^2 \mathbf{1}\left\{\|\varepsilon_1\| > \delta^{1/2} n^{1/4}\right\}\right) \to 0, \end{aligned}$$

by integrability of $\|\varepsilon_1\|^2$. Thus, (40) and (38) follow.

9. References

Abadir K.M. and J. R. Magnus (2005). *Matrix Algebra.* Econometric Exercises, vol.1. Cambridge University Press.

Anderson, T.W. (1959). "On asymptotic distributions of estimates of parameters of stochastic difference equations". *Annals of Mathematical Statistics*, 30, 676-687.

Hall, P. and C.C. Heyde (1980). *Martingale Limit Theory and its Application.* Academic Press.

Magdalinos, T and P. C. B. Phillips (2006). "Limit Theory for Cointegrated Systems with Moderately Integrated and Moderately Explosive Regressors". Working paper.

Mijnheer, J. (2002). "Asymptotic inference for AR(1) processes with (nonnormal) stable innovations. V. The explosive case". *Journal of Mathematical Sciences*, 111, 3854–3856.

Park, J. Y. and P.C.B. Phillips (1988). "Statistical Inference in Regressions with Integrated Processes: Part 1". *Econometric Theory* 4, 468-497.

Park, J. Y. and P.C.B. Phillips (1989). "Statistical Inference in Regressions with Integrated Processes: Part 2". *Econometric Theory* 5, 95-131.

Phillips, P. C. B. (1985). "The distribution of matrix quotients". *Journal of Multivariate Analysis* 16:1, 157–161.

White, J. S. (1958). "The limiting distribution of the serial correlation coefficient in the explosive case". *Annals of Mathematical Statistics* 29, 1188–1197.