EXACT DISTRIBUTION THEORY IN STRUCTURAL ESTIMATION WITH AN IDENTITY

By

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Exact Distribution Theory in Structural Estimation with an Identity*

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Abstract

Some exact distribution theory is developed for structural equation models with and without identities. The theory includes LIML, IV and OLS. We relate the new results to earlier studies in the literature, including the pioneering work of Bergstrom (1962). General IV exact distribution formulae for a structural equation model without an identity are shown to apply also to models with an identity by specializing along a certain asymptotic parameter sequence. Some of the new exact results are obtained by means of a uniform asymptotic expansion. An interesting consequence of the new theory is that the uniform asymptotic approximation provides the exact distribution of the OLS estimator in the model considered by Bergstrom (1962). This example appears to be the first instance in the statistical literature of a uniform approximation delivering an exact expression for a probability density.

Keywords: Exact distribution, Identity, IV estimation, LIML, Structural equation, Uniform asymptotic expansion.

JEL classification: C30

Dedication

In memory of Rex Bergstrom whose pioneering paper in *Econometrica* 1962 opened up a new understanding of the comparative properties of simultaneous equations estimators by deriving their exact finite sample distributions.

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1 Introduction

Bergstrom (1962) and Basmann (1961) are recognized as the two pioneering articles on the exact distribution of simultaneous equations estimators. Both papers dealt with special models, Bergstrom's dealing with a just identified two equation income determination model with one exogenous variable and Basmann's with an overidentified structural equation model with two endogenous variables and four exogenous variables. Bergstrom's study, like that of Basmann, was motivated by the desire to learn more about the finite sample distributional characteristics of (consistent) simultaneous equations estimators like limited information maximum likelihood (LIML) and instrumental variables (IV), especially in comparison to those of (inconsistent) least squares (OLS). Accordingly, he derived exact mathematical forms for the density functions of the LIML and OLS estimators (under Gaussian innovations), graphed the densities and computed probabilities of concentration about the true value of the parameter for the two estimators. The results provided clear support for the use of simultaneous equations estimators like LIML even in very small samples.

Later research has generally reinforced this conclusion about the superiority of the LIML procedure in small samples, at least for correctly specified equations. The conclusion is particularly interesting because it applies in spite of the fact that the distribution of LIML is known to have heavy tails (Phillips, 1984, 1985) and can be bimodal (Phillips and Hajivassiliou, 1987; Nelson and Startz, 1990; Woglom, 2001; Phillips, 2006). While the bimodality does not show up in the sketch of the frequency functions shown in Bergstrom (1962), it does become apparent under different parameter configurations, particularly those that reflect poor instrumentation, and when a wider support is considered. The topic is of some ongoing interest and relates in important ways to the recently studied and practically important phenomenon of weak instrumentation in structural estimation - see Forchini (2006), Hillier (2006) and Phillips (2006) for further discussion, analysis and references in the context of the intervening literature.

The formulae derived by Bergstrom and Basmann are quite different in form and seem difficult to relate. Moreover, Bergstrom's results bear no obvious relation to general formulae for exact distributions that were obtained in the subsequent literature – specifically, Phillips (1980) in the case of the IV estimator, and Phillips (1984, 1985) in the case of LIML. Bergstrom's results have also been somewhat neglected in the ensuing literature, even by Nelson and Startz (1990) who use precisely the same exact distribution as Bergstrom's density for LIML in their study of bimodality.

The stochastic income determination model studied by Bergstrom is a case of strong endogeneity, where there is a structural behavioral equation and a structural identity. The role of the identity is important in the distribution theory because it provides a magnet for an alternative centering, pulling consistent estimators like IV and LIML away from the relevant parameter in the behavioral relation and thereby naturally inducing a bimodality, as discussed in Phillips (2006).

Identities are common in structural systems and it is therefore of some interest to relate Bergstrom's particular results to a more general theory. The present paper contributes by providing alternative derivations of Bergstrom's main findings, both for the LIML and OLS estimators. In particular, we provide an alternative reduction of the (more complex) distributional result for OLS. Next, we relate the LIML and OLS results for models with a structural identity to the general exact distribution theory for models without identities, showing how the results for models with an identity arise from a specialization along a certain asymptotic parameter sequence. Part of the development involves the derivation of some new exact distribution theory for the IV estimator which facilitates the required asymptotic expansion.

The asymptotics used here are of some independent interest and utilize the theory of uniform asymptotic expansions. En route, we are able to generalize earlier results given in Holly and Phillips (1979) for saddlepoint (SP) approximations, in which the approximations had restricted domains - one for each tail of the distribution. The uniform approximation does not have this restriction. Finally, it is shown that the leading term in the uniform asymptotic expansion provides the exact distribution for a structural model estimator when the model has a structural identity. To the author's knowledge, this is the first instance in the statistical or econometric literature where a uniform approximation delivers an exact distribution.[1]

2 Bergstrom's model and results

The model considered in Bergstrom (1962) is

$$y_t = \alpha + \beta x_t + u_t \tag{1}$$

$$x_t = y_t + \gamma z_t \tag{2}$$

where the (spending propensity) parameter β is assumed to satisfy $\beta < 1$.The condition $\beta \neq 1$ is needed for the existence of a reduced form and a non trivial data generating mechanism. Equation (1) has two measured endogeneous variables y_t, x_t, and a stochastic disturbance u_t that is assumed to be $iid\ N(0, \sigma^2)$ for the development of an exact finite sample theory. Equation (2) is a structural identity involving an observed instrumental variable z_t that is assumed to be strictly exogenous and fixed, so that the workings are effectively conditioned on the sample $\{z_t : t = 1, ..., n\}$.

The parameter γ did not specifically appear in Bergstrom (1962) and is there set to unity, but it is useful to control the relevance of the instrument z_t in the

[1] We might compare this result with the three special cases (normal, inverse normal, and chi-squared), considered by Daniels (1980), where the SP approximation to the distribution of the sample mean of an iid sequence is know to be exact, up to normalization. In the present case, of course, we are dealing with statistics that are substantially more complex than sample means in form, and where the exact distribution does not have a simple analytic form, but is instead given by an infinite series. We also note that Marsh (1998) found that the SP approximation to the density of a ratio of independent central chi-squared variates (an F ratio) is exact upon normalization.

system (c.f. Phillips, 2006) and is convenient to use as a known scale coefficient here, although it could readily be absorbed into z_t and its effects measured in terms of the signal from the instrument. When $\gamma \to 0$, the instrument z_t becomes irrelevant to the determination of y_t and x_t, and we end up with the identity $x_t = y_t$ in place of (2). On the other hand, when $\gamma \to \infty$, the system is dominated by the signal from z_t. In view of the identity (2) and the exogeneity of z_t, the degree of endogeneity as measured by the correlation coefficient of x_t and u_t is unity, so that there is strong endogeneity in the system.

Bergstrom (1962) allowed for the presence of an intercept in (1). However, setting $\alpha = 0$ is inconsequential and only affects the degrees of freedom and noncentrality measure, so we proceed without an intercept in what follows, making adjustments as needed to relate results to those of Bergstrom. When $\alpha = 0$, the reduced form is

$$y_t = \pi_y z_t + v_t, \quad \pi_y = \frac{\beta\gamma}{1-\beta}, \; v_t = \frac{1}{1-\beta} u_t \tag{3}$$

$$x_t = \pi_x z_t + v_t, \quad \pi_x = \frac{\gamma}{1-\beta}, \tag{4}$$

and β is just identified by the relation

$$\beta = \pi_y / \pi_x = 1 - \gamma / \pi_x. \tag{5}$$

Correspondingly, the LIML estimator of β is $\hat{\beta} = \hat{\pi}_y / \hat{\pi}_x = 1 - \gamma / \hat{\pi}_x$, where $\hat{\pi}_y$ and $\hat{\pi}_x$ are the reduced form least squares estimates, obtained from (3) - (4) with no intercept. The estimate $\hat{\beta}$ is the same as the IV estimator with z_t as instrument. Since $\hat{\pi}_y$ and $\hat{\pi}_x$ are Gaussian, Bergstrom's expression for the exact density of $\hat{\beta}$ is immediate from simple variate transformation and has the form

$$\mathrm{pdf}(b) = \frac{\lambda_n^{1/2}}{\sqrt{2\pi}} \frac{1-\beta}{\sigma} \frac{1}{(1-b)^2} \exp\left\{ -\frac{\lambda_n}{2\sigma^2} \left(\frac{b-\beta}{1-b} \right)^2 \right\}, \tag{6}$$

where $\lambda_n = \gamma^2 \sum_{t=1}^n z_t^2$ is the noncentrality parameter (Bergstrom normalized this parameter to the sample size so that $\lambda_n = n$). The density (6) is the same as that studied in Nelson and Startz (1990) because the model studied in that paper is observationally equivalent to a structural equation with a parameterized identity (for details, see Phillips, 2006).

Bergstrom found the following form of the exact density of the OLS estimator, which holds for even values of $n \geq 4$

$$\begin{aligned} \mathrm{pdf}(b) &= (-1)^{\frac{n-4}{2}} \frac{4\sqrt{n}(1-\beta) e^{-\frac{\lambda_n}{2\sigma^2}}}{2^{\frac{n-1}{2}} \sqrt{\pi}\sigma^2 (1-b)^2} \frac{e^q}{z^2} \sum_{j=0}^{\frac{n-4}{2}} \frac{\Gamma(n-2-j)}{j! \Gamma\left(\frac{n-2-2j}{2}\right)} \left(\frac{2}{z}\right)^{n-4-2j} \\ &\quad \left(\frac{4\sqrt{n}}{\sigma z} + 2 \right)^j (-1)^{n-4-j} \left\{ e^{-q} - \sum_{k=0}^{n-3-j} \frac{(-q)^k}{k!} \right\}, \end{aligned} \tag{7}$$

for

$$1 \neq b \neq \frac{1+\beta}{2}, \tag{8}$$

and where

$$z = \frac{\sqrt{n}\,(2b-\beta-1)}{\sigma\,(1-\beta)}, \quad q = \frac{n\,(1+\beta-2b)\,(1-\beta)}{2\sigma^2\,(1-b)^2}.$$

Notably, (7) is a finite series and involves no special transcendental functions. Bergstrom found (7) by a sequence of variate transformations and integrations that reduce dimension, the final steps involving the reduction of an integral involving a complicated factor in the integrand which Bergstrom expanded in a finite series using the binomial expansion, which was valid for even values of $n \geq 4$ and for values of the argument satisfying (8).

In particular, from equations (16) and (21) of Bergstrom (1962) and after notational translation ($T \longmapsto n$ and $\lambda_n^2 \longmapsto n$) we have

$$\mathrm{pdf}\,(b) = \frac{\sqrt{n}(1-\beta)}{\sigma\,(1-b)^2}\,g\left(\frac{\sqrt{n}\,(2b-\beta-1)}{\sigma\,(1-b)}\right), \quad b \neq 1, \tag{9}$$

where

$$g\,(z) = k\int_0^{-\left(\frac{z^2}{2}+\frac{\sqrt{n}z}{\sigma}\right)} \frac{4q}{z^2}\left\{-\frac{4q^2}{z^2}-\left(\frac{4\sqrt{n}}{\sigma z}+2\right)q\right\}^{\frac{n-4}{2}} e^q dq, \quad \text{for } z \neq 0, \tag{10}$$

and $k = \left\{2^{(n-1)/2}\sqrt{\pi}\Gamma\left(\frac{n-2}{2}\right)e^{\frac{n}{2\sigma^2}}\right\}^{-1}$.

Using this formulation we can obtain a simpler and more general expression for the density by expressing the integral (10) in terms of a confluent hypergeometric function whose series form (e.g. Lebedev, 1972) is ${}_1F_1\,(a,b;x) = \sum_{j=0}^{\infty}\frac{(a)_j}{j!(b)_j}x^j$, where $(a)_j$ is the forward factorial $(a)_j = a\,(a+1)\,...(a+j-1)$. The result is stated as follows and proved in the Appendix.

Proposition 1 *The exact density of the OLS estimator of β in the Gaussian model (1)-(2) with $\gamma = 1$ and $\lambda_n = n$ has the form*

$$pdf(b) = \frac{n^{\frac{n-1}{2}}\Gamma\left(\frac{n}{2}\right)e^{-\frac{n}{2\sigma^2}\left\{\frac{(b-\beta)^2}{(1-b)^2}\right\}}(1-\beta)^{n-1}}{2^{(n-1)/2}\sigma^{n-1}\sqrt{\pi}\Gamma\,(n-1)\,|1-b|^n}\;{}_1F_1\left(\frac{n}{2}-1, n-1; -\frac{n\,(1-\beta)\,(1+\beta-2b)}{2\sigma^2\,(1-b)^2}\right). \tag{11}$$

This expression for the density holds for all $n \geq 2$ and for all b. As will be shown below, the density (11) is, in fact, continuous as the argument b passes through unity. The density is also positive at $b = 1$, unlike the density of the LIML estimator (6), which is easily seen to have a zero at $b = 1$ (c.f., Phillips and Wickens, 1978; Nelson and Startz, 1990).

3 The LIML Density as a Specialization

We now proceed to show how the exact LIML density (6), and later the exact OLS density (11), can be derived as a special case of exact results for a model without an identity. This specialization is accomplished by taking an appropriate limit of the densities in the general case to correspond with the singular covariance matrix structure that characterizes the model with an identity. The process and the results are of some independent interest, not least because past exact results for the general case have been derived explicitly under a nonsingular (Wishart) distribution assumption and normalizing transformations that set the covariance structure of the endogenous variables to an identity matrix. The process described below is given for the two endogenous variable case to keep the length and notation of the present contribution manageable and so that the Bergstrom results are appropriately highlighted as the specialization of primary interest. Results for the general case of $m+1$ endogenous variables raise some additional complications and will be reported elsewhere.

We write the general two endogenous variable model (ignoring included exogenous variables) as

$$y_t = \beta x_t + u_t \tag{12}$$

$$x_t = \gamma_x z_t + u_{xt}. \tag{13}$$

When $u_{xt} = \frac{u_t}{1-\beta}$ and $\gamma_x = \frac{\gamma}{1-\beta}$, (13) is the reduced form equation (4), viz.,

$$x_t = \frac{\gamma}{1-\beta} z_t + \frac{u_t}{1-\beta}, \tag{14}$$

and then

$$y_t = \beta x_t + u_t = \frac{\beta\gamma}{1-\beta} z_t + \frac{\beta u_t}{1-\beta} + u_t = \frac{\beta\gamma}{1-\beta} z_t + \frac{u_t}{1-\beta},$$

corresponding to the reduced form equation (3). Moreover, equation (13) is

$$x_t = \frac{\gamma}{1-\beta} z_t + y_t - \frac{\beta\gamma}{1-\beta} z_t = \gamma z_t + y_t,$$

giving the identity (2). Hence, model (12) - (14) is equivalent to the structural model with identity (1) - (2) when $u_{xt} = \frac{u_t}{1-\beta}$ and $\gamma_x = \frac{\gamma}{1-\beta}$.

We proceed with a finite sample analysis of model (12) - (13). First assume that the covariance structure $\Sigma > 0$ of (u_t, u_{xt}) is nondegenerate. The covariance matrix of the reduced form is

$$\Omega = B^{-1}\Sigma B'^{-1} = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \Sigma \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix} = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \Omega_{22} \end{bmatrix}, \quad \text{say,}$$

where $B = \begin{bmatrix} 1 & -\beta \\ 0 & 1 \end{bmatrix}$, and which we can write in alternate form as

$$\Omega = \begin{bmatrix} \omega_{11} & \rho\omega_{11}^{1/2}\Omega_{22}^{1/2} \\ \rho\omega_{11}^{1/2}\Omega_{22}^{1/2} & \Omega_{22} \end{bmatrix}, \quad \text{with } \rho = \Omega_{22}^{-1/2}\omega_{21}/\omega_{11}^{1/2}.$$

We transform the model to standardized form as shown in Phillips (1982, 1983), leading to the new system

$$\begin{aligned} y_t &= \beta^* x_t + u_t^* && (15) \\ x_t &= \gamma_x^* z_t + u_{xt}^*, && (16) \end{aligned}$$

where

$$\beta^* = \frac{\Omega_{22}^{1/2}}{\omega_{11}^{1/2}\left(1-\rho^2\right)^{1/2}}\left(\beta - \Omega_{22}^{-1}\omega_{21}\right), \quad \gamma_x^* = \Omega_{22}^{-1/2}\gamma_x,$$

(see thereom 3.3.1 and equation (3.54) of Phillips, 1982) Then, using the notation $\delta = \omega_{11}^{1/2}\Omega_{22}^{-1/2}$, we have

$$\beta^* = \frac{\Omega_{22}^{1/2}}{\omega_{11}^{1/2}\left(1-\rho^2\right)^{1/2}}\left(\beta - \Omega_{22}^{-1}\omega_{21}\right) = \frac{1}{\delta\left(1-\rho^2\right)^{1/2}}\left(\beta - \rho\delta\right), \tag{17}$$

and the corresponding transformation for the estimator is

$$r^* = \frac{1}{\delta\left(1-\rho^2\right)^{1/2}}\left(r - \rho\delta\right), \tag{18}$$

where r^* and r denote realizations of the corresponding estimates of β^* and β. We observe that for the Bergstrom model, $\omega_{11} = \Omega_{22} = \frac{\sigma^2}{1-\beta}$, so that $\delta = 1$, and the correlation coefficient is $\rho = 1$. We may therefore expect a correspondence between exact results for this model and the Bergstrom model upon passing $\rho^2 \to 1$. Note that passing $\rho^2 \to 1$ corresponds to $\beta^* \to \pm\infty$ in the standardized model.

For this structural model without an identity, the exact distribution of the LIML/IV estimator has the known following form in the just identified case

$$\text{pdf}\left(r^*\right) = \frac{e^{-\frac{\mu^2}{2}\left(1+\beta^{*2}\right)}}{\pi\left(1+r^{*2}\right)}\,{}_1F_1\left(1, \frac{1}{2}; \frac{\mu^2}{2}\frac{\left(1+\beta^* r^*\right)^2}{1+r^{*2}}\right),$$

where the noncentrality parameter is

$$\mu^2 = n\frac{\gamma_x^2}{\Omega_{22}} = n\frac{\gamma^2/\left(1-\beta\right)^2}{\sigma^2/\left(1-\beta\right)^2} = \frac{n\gamma^2}{\sigma^2} = \frac{\lambda_n}{\sigma^2} \tag{19}$$

(see Phillips, 1980, 1983). Hence, for the untransformed system we have the exact density

$$\operatorname{pdf}(r) = \frac{1}{\delta\left(1-\rho^2\right)^{1/2}} \frac{e^{-\frac{\mu^2}{2}\left(1+\beta^{*2}\right)}}{\pi\left(1+r^{*2}\right)} \, {}_1F_1\left(1, \frac{1}{2}; \frac{\mu^2}{2} \frac{\left(1+\beta^* r^*\right)^2}{1+r^{*2}}\right), \tag{20}$$

upon substituting for β^* and r^* using (17) and (18).

We now examine expression (20) and seek to take limits as $\rho^2 \to 1$, leading to the degenerate case. The result is stated formally as follows.

Proposition 2 *The limiting form of the exact density of the LIML estimator in the just identified case has the form*

$$\lim_{\rho^2 \to 1} pdf(r) = \frac{\left(\mu^2\right)^{1/2}}{\sqrt{2\pi}} \frac{|\beta-\delta|}{(r-\delta)^2} \exp\left\{-\frac{\mu^2}{2} \frac{(r-\beta)^2}{(r-\delta)^2}\right\}, \tag{21}$$

in the case of a model with a structural identity.

When $\delta = 1$ and noting that $\mu^2 = \lambda_n/\sigma^2$ from (19), the limiting density (21) is

$$\operatorname{pdf}(r) = \frac{\lambda_n^{1/2}}{\sqrt{2\pi}\sigma} \frac{|\beta-1|}{(r-1)^2} \exp\left\{-\frac{\lambda_n}{2\sigma^2} \frac{(r-\beta)^2}{(r-1)^2}\right\}, \tag{22}$$

corresponding to the Bergstrom result (6) for LIML in this just identified case. Thus, Bergstrom's LIML density is just a specialization of the density that holds for a structural model without an identity.

A similar specialization holds for the OLS estimator, as would now be anticipated. However, the limiting argument to prove this correspondence involves greater technical complexity because the exact density of the OLS estimator presently in the literature is not sufficiently general to permit a uniform limiting approximation along the required path. In fact, use of existing formulae for the exact density produces limiting approximations that are only valid on a restricted domain as shown in the following section, corresponding to earlier work by Holly and Phillips (1979) on saddlepoint approximations and Phillips (1980) on Laplace approximations.

The first step in a more general approach is therefore to provide a suitable reformulation of that density. We accomplish this reformulation by finding a new expression for the exact density of the general IV estimator in a structural model without an identity. This new expression turns out to be of independent interest and have other uses which are not pursued here.

4 The OLS Density as a Specialization

4.1 Restricted approximants

We start with the exact OLS density, which may be obtained from the results in Phillips (1980). For the standardized model and in the two endogenous variable case that is relevant here, the exact OLS density has the following form

$$\begin{aligned}\mathrm{pdf}\left(r^*\right) &= \frac{e^{-\frac{\mu^2}{2}\left(1+\beta^{*2}\right)}\Gamma\left(\frac{n+1}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{n}{2}\right)\left(1+r^{*2}\right)^{(n+1)/2}} \sum_{j=0}^{\infty} \frac{\left(\frac{n-1}{2}\right)_j}{j!\left(\frac{n}{2}\right)_j} \left(\frac{\mu^2}{2}\beta^{*2}\right)^j \\ &\quad \times {}_1F_1\left(\frac{n+1}{2}, \frac{n}{2}+j; \frac{\mu^2}{2}\frac{\left(1+\beta^* r^*\right)^2}{1+r^{*2}}\right). \qquad (23)\end{aligned}$$

which is found by setting $N = n-1$ in equation (13) of Phillips (1980)[2]. In this expression, as earlier, the notations β^* and r^* refer to the standardized model and are given in (17) and (18). In unstandardized form, we have $\mathrm{pdf}\left(r\right) = \delta^{-1}\left(1-\rho^2\right)^{-1/2} \mathrm{pdf}\left(r^*\right)$ with the corresponding substitutions

$$\beta^{*2} = \frac{1}{\delta^2\left(1-\rho^2\right)}\left(\beta-\rho\delta\right)^2, \quad 1+r^{*2} = \frac{\delta^2+r^2-2r\rho\delta}{\delta^2\left(1-\rho^2\right)}, \qquad (24)$$

and

$$\frac{\left(1+\beta^* r^*\right)^2}{1+r^{*2}} = \frac{\left[\delta^2+\beta r-\rho\delta\left(r+\beta\right)\right]^2}{\delta^2\left(1-\rho^2\right)\left\{\delta^2+r^2-2r\rho\delta\right\}}. \qquad (25)$$

Each expression in (24) and (25) diverges as $\rho^2 \to 1$, enabling an asymptotic expansion. In particular, both factors of the following expression, which appears in the density formula (23),

$$\left(\frac{\mu^2}{2}\beta^{*2}\right)^j \; {}_1F_1\left(\frac{n+1}{2}, \frac{n}{2}+j; \frac{\mu^2}{2}\frac{\left(1+\beta^* r^*\right)^2}{1+r^{*2}}\right) \qquad (26)$$

diverge as $\rho^2 \to 1$. When these expansions, which are given in the Appendix, are employed in (23), we obtain the following approximation to the OLS density in a model with a structural identity.

[2] The setting $N = n-1$ does not take into account an intercept term in the structural equation, which leads to a further loss of one degree of freedom, so that $N = n-2$ in that event, a fact that will be used later.

Proposition 3 *Limiting approximations to the exact density of the OLS estimator in a structural model with an identity are given by*

$$\lim_{\rho\to 1} \text{pdf}\,(r) = \left(\frac{\mu^2}{2\pi}\right)^{1/2} \frac{\exp\left\{-\frac{\mu^2}{2}\frac{(r-\beta)^2}{(r-\delta)^2}\right\}}{(r-\delta)^2} \frac{|\beta-\delta|^{(n+1)/2}}{|2r-\beta-\delta|^{(n-1)/2}}, \tag{27}$$

$$\lim_{\rho\to -1} \text{pdf}\,(r) = \left(\frac{\mu^2}{2\pi}\right)^{1/2} \frac{\exp\left\{-\frac{\mu^2}{2}\frac{(r-\beta)^2}{(r+\delta)^2}\right\}}{(r+\delta)^2} \frac{|\beta+\delta|^{(n+1)/2}}{|2r-\beta+\delta|^{(n-1)/2}}, \tag{28}$$

as $\rho \to \pm 1$.

When $\delta = 1$, $\mu^2 = \lambda_n/\sigma^2$, and n is replaced by $n-1$ to account for the presence of an intercept, as in the Bergstrom case, formulae (27) and (28) become

$$\lim_{\rho\to 1} \text{pdf}\,(r) = \frac{\lambda_n^{1/2}\,|\beta-1|^{n/2}\exp\left\{-\frac{\lambda_n}{2\sigma^2}\frac{(r-\beta)^2}{(r-1)^2}\right\}}{\sqrt{2\pi}\sigma\,(r-1)^2\,|2r-\beta-1|^{(n-2)/2}}, \tag{29}$$

$$\lim_{\rho\to -1} \text{pdf}\,(r) = \frac{\lambda_n^{1/2}\exp\left\{-\frac{\lambda_n}{2\sigma^2}\frac{(r-\beta)^2}{(r+1)^2}\right\}|\beta+1|^{n/2}}{\sqrt{2\pi}\sigma\,(r+1)^2\,|2r-\beta+1|^{(n-2)/2}}, \tag{30}$$

with (29) corresponding to the Bergstrom case where $\rho = 1$.

Both formulae (27) and (28) represent limiting approximations, rather than the exact OLS density. The reason is that upon using the expansion of (26) in (23), the resulting infinite series is summable as a binomial series only over a restricted range for the density and this restricted domain is violated when $\rho = \pm 1$. The details are given in the Appendix. Interestingly, this violation does not prevent the limits (29) and (30) from existing and, as discussed later in the paper, these limits can be validated as large concentration parameter (λ_n) approximants. A similar limitation on the domain of application applies to the saddlepoint approximation of Holly and Phillips (1979) and the Laplace approximation given in Phillips (1980).

Interestingly, the approximations (29) and (30) have zeros in their support: (29) at $r = 1$, and (30) at $r = -1$. Thus, the approximation (29) corresponding to the Bergstrom case where $\rho = 1$ has a zero at $r = 1$, like the LIML density (22), but also has a singularity at $r = (1+\beta)/2$, which is not present in the exact density.

We now proceed to develop an alternative form of the exact density which is useful in deriving a valid limit as $\rho^2 \to 1$. It will be convenient to perform the development for the case of the exact density of a IV estimator, as in Phillips (1980), so that the final result has wider applicability. We continue to confine attention to the two endogenous variable case and let L be a parameter representing the degree of overidentification, using the same notation as Phillips (1980). The Bergstrom OLS case is covered in this notation when $L = n - 2$, because Bergstrom (1962) has an intercept in the structural equation and the constant counts as an additional variable and further adjusts the degrees of freedom.

4.2 A new form for the IV exact density

We start by considering the following two representations of the IV exact density. The first is analogous to (23) above and is given in Phillips (1980), the second is derived by a simple rearrangement of the multiple series

$$
\begin{aligned}
\operatorname{pdf}(r^*) &= \frac{e^{-\frac{\mu^2}{2}\left(1+\beta^{*2}\right)}\Gamma\left(\frac{L+2}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{L+1}{2}\right)\left(1+r^{*2}\right)^{(L+2)/2}} \sum_{j=0}^{\infty} \frac{\left(\frac{L}{2}\right)_j}{j!\left(\frac{L+1}{2}\right)_j}\left(\frac{\mu^2}{2}\beta^{*2}\right)^j \\
&\quad {}_1F_1\left(\frac{L+2}{2}, \frac{L+1}{2}+j; \frac{\mu^2}{2}\frac{\left(1+\beta^* r^*\right)^2}{1+r^{*2}}\right) \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{e^{-\frac{\mu^2}{2}\left(1+\beta^{*2}\right)}}{B\left(\frac{1}{2}, \frac{L+1}{2}\right)\left(1+r^{*2}\right)^{(L+2)/2}} \sum_{j=0}^{\infty} \frac{\left(\frac{L+2}{2}\right)_j}{j!\left(\frac{L+1}{2}\right)_j}\left(\frac{\mu^2}{2}\frac{\left(1+\beta^* r^*\right)^2}{1+r^{*2}}\right)^j \\
&\quad {}_1F_1\left(\frac{L}{2}, \frac{L+1}{2}+j; \frac{\mu^2}{2}\beta^{*2}\right). \qquad (32)
\end{aligned}
$$

We observe that in both these representations the series involve terms with confluent hypergeometric functions, viz.,

$$
{}_1F_1\left(\frac{L+2}{2}, \frac{L+1}{2}+j; \frac{\mu^2}{2}\frac{\left(1+\beta^* r^*\right)^2}{1+r^{*2}}\right) \text{ and } {}_1F_1\left(\frac{L}{2}, \frac{L+1}{2}+j; \frac{\mu^2}{2}\beta^{*2}\right),
$$

in which both the parameter j and the argument (the third parameter of the function) may be large. This means that the validity of the conventional (large third parameter) asymptotic expansion of the ${}_1F_1$ function (see (35) below) is essentially restricted to cases where j is finite. Use of this type of asymptotic approximation, as we have seen above, leads to series in j that are summable only over a restricted domain of values of the argument r^*. In consequence, the resulting approximants lack uniformity and therefore limits that are taken as $\rho^2 \to 1$ do not produce an exact density for a model with a structural identity.

In order to achieve an approximation that is valid over the entire domain of r^* we need to develop a uniform asymptotic expansion. This can be accomplished by the following technique. First, we find an alternate representation of the density which involves a ${}_1F_1$ function that does not have a large or variable parameter. To do so, we will make use of the following integral representation, Kummer relation and conventional (large parameter) asymptotic expansion of the ${}_1F_1$ function (e.g., Lebedev, 1972):

$$
\begin{aligned}
{}_1F_1(a, b; x) &= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{st}t^{a-1}(1-t)^{b-a-1}\,dt, && (33)\\
{}_1F_1(a, b; x) &= e^x\,{}_1F_1(b-a, b; -x), && (34)\\
{}_1F_1(a, b; x) &= \frac{\Gamma(b)}{\Gamma(a)}e^x x^{a-b}\left\{1+O\left(x^{-1}\right)\right\}. && (35)
\end{aligned}
$$

Using (34) and (33) in (32) we have

$$
\begin{aligned}
\mathrm{pdf}\left(r^{*}\right) &= \frac{e^{-\frac{\mu^2}{2}}}{B\left(\frac{1}{2},\frac{L+1}{2}\right)\left(1+r^{*2}\right)^{(L+2)/2}} \sum_{j=0}^{\infty} \frac{\left(\frac{L+2}{2}\right)_j}{j!\left(\frac{L+1}{2}\right)_j} \left(\frac{\mu^2}{2}\frac{\left(1+\beta^{*}r^{*}\right)^2}{1+r^{*2}}\right)^j \\
&\quad \times {}_1F_1\left(j+\frac{1}{2},\frac{L+1}{2}+j;-\frac{\mu^2}{2}\beta^{*2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{e^{-\frac{\mu^2}{2}}}{B\left(\frac{1}{2},\frac{L+1}{2}\right)\left(1+r^{*2}\right)^{(L+2)/2}} \sum_{j=0}^{\infty} \frac{\left(\frac{L+2}{2}\right)_j}{j!\left(\frac{L+1}{2}\right)_j} \left(\frac{\mu^2}{2}\frac{\left(1+\beta^{*}r^{*}\right)^2}{1+r^{*2}}\right)^j \\
&\quad \times \frac{\Gamma\left(\frac{L+1}{2}+j\right)}{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(\frac{L}{2}\right)} \int_0^1 e^{-\frac{\mu^2}{2}\beta^{*2}t} t^{j-\frac{1}{2}} \left(1-t\right)^{\frac{L}{2}-1} dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{e^{-\frac{\mu^2}{2}}}{B\left(\frac{1}{2},\frac{L+1}{2}\right)\left(1+r^{*2}\right)^{(L+2)/2}} \\
&\quad \times \int_0^1 e^{-\frac{\mu^2}{2}\beta^{*2}t} \left\{ \sum_{j=0}^{\infty} \frac{\left(\frac{L+2}{2}\right)_j}{j!\left(\frac{L+1}{2}\right)_j} \left(t\frac{\mu^2}{2}\frac{\left(1+\beta^{*}r^{*}\right)^2}{1+r^{*2}}\right)^j \frac{\Gamma\left(\frac{L+1}{2}+j\right)}{\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(\frac{L}{2}\right)} \right\} t^{-\frac{1}{2}} \left(1-t\right)^{\frac{L}{2}-1} dt \\
&= \frac{e^{-\frac{\mu^2}{2}}\Gamma\left(\frac{L+1}{2}\right)}{B\left(\frac{1}{2},\frac{L+1}{2}\right)\Gamma\left(\frac{L}{2}\right)\Gamma\left(\frac{1}{2}\right)\left(1+r^{*2}\right)^{(L+2)/2}} \\
&\quad \times \int_0^1 e^{-\frac{\mu^2}{2}\beta^{*2}t} \left\{ \sum_{j=0}^{\infty} \frac{\left(\frac{L+2}{2}\right)_j}{j!\left(\frac{1}{2}\right)_j} \left(t\frac{\mu^2}{2}\frac{\left(1+\beta^{*}r^{*}\right)^2}{1+r^{*2}}\right)^j \right\} t^{-\frac{1}{2}} \left(1-t\right)^{\frac{L}{2}-1} dt \\
&= \frac{e^{-\frac{\mu^2}{2}}\Gamma\left(\frac{L+2}{2}\right)}{\Gamma\left(\frac{L}{2}\right)\pi\left(1+r^{*2}\right)^{(L+2)/2}} \\
&\quad \times \int_0^1 e^{-\frac{\mu^2}{2}\beta^{*2}t} {}_1F_1\left(\frac{L+2}{2},\frac{1}{2};\left(\frac{\mu^2 t}{2}\frac{\left(1+\beta^{*}r^{*}\right)^2}{1+r^{*2}}\right)\right) t^{-\frac{1}{2}} \left(1-t\right)^{\frac{L}{2}-1} dt, \qquad (36)
\end{aligned}
$$

which is a new form for the exact density of the IV estimator. The hypergeometric function in (36) has the two parameters $\left(\frac{L+2}{2},\frac{1}{2}\right)$ which are fixed for any given L, which facilitates a uniform asymptotic development as $\rho^2 \to 1$.

Note that when $1+\beta^{*}r^{*}=0$ (or $r^{*}=-1/\beta^{*}$), expression (36) for the exact density reduces immediately to

$$
\begin{aligned}
\operatorname{pdf}(r^*) &= \frac{e^{-\frac{\mu^2}{2}}\Gamma\left(\frac{L+2}{2}\right)}{\Gamma\left(\frac{L}{2}\right)\pi\left(1+r^{*2}\right)^{(L+2)/2}}\int_0^1 e^{-\frac{\mu^2}{2}\beta^{*2}t}t^{-\frac{1}{2}}\left(1-t\right)^{\frac{L}{2}-1}dt \\
&= \frac{e^{-\frac{\mu^2}{2}}\Gamma\left(\frac{L+2}{2}\right)}{\Gamma\left(\frac{L}{2}\right)\pi\left(1+r^{*2}\right)^{(L+2)/2}}\frac{\sqrt{\pi}\Gamma\left(\frac{L}{2}\right)}{\Gamma\left(\frac{L+1}{2}\right)}\,{}_1F_1\left(\frac{1}{2},\frac{L+1}{2};-\frac{\mu^2}{2}\beta^{*2}\right) \\
&= \frac{e^{-\frac{\mu^2}{2}\left(1+\beta^{*2}\right)}\,{}_1F_1\left(\frac{L}{2},\frac{L+1}{2};\frac{\mu^2}{2}\beta^{*2}\right)}{B\left(\frac{1}{2},\frac{L+1}{2}\right)\left(1+r^{*2}\right)^{(L+2)/2}}, \quad r^*=-\frac{1}{\beta^*}, \qquad (37)
\end{aligned}
$$

upon application of (34). Observe that (37) corresponds to formula (31) for the exact density in this case.

4.3 A uniform asymptotic expansion of the IV density

Transforming to original coordinates, we find that for $\mu^2>0$ and $1+\beta^*r^*\neq 0$

$$
\frac{\mu^2 t}{2}\frac{\left(1+\beta^*r^*\right)^2}{1+r^{*2}} = \frac{\mu^2 t}{2}\frac{\left[\delta^2+\beta r-\rho\delta\left(r+\beta\right)\right]^2}{\delta^2\left(1-\rho^2\right)\left\{\delta^2+r^2-2r\rho\delta\right\}} = O\left(\frac{1}{1-\rho^2}\right) \quad \text{for all } t\neq 0, \qquad (38)
$$

as $\rho^2\to 1$. Hence, we may employ the usual asymptotic expansion (35) of the ${}_1F_1$ function in (36), giving

$$
\begin{aligned}
&{}_1F_1\left(\frac{L+2}{2},\frac{1}{2};\left(\frac{\mu^2 t}{2}\frac{\left(1+\beta^*r^*\right)^2}{1+r^{*2}}\right)\right) \\
&= \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{L}{2}+1\right)}e^{\frac{\mu^2 t}{2}\frac{\left(1+\beta^*r^*\right)^2}{1+r^{*2}}}\left(\frac{\mu^2 t}{2}\frac{\left(1+\beta^*r^*\right)^2}{1+r^{*2}}\right)^{\frac{L+1}{2}}\left\{1+O\left(1-\rho^2\right)\right\}, \qquad (39)
\end{aligned}
$$

in this case.

Using (39) in (36) and the representation (33), we have

$$
\begin{aligned}
\operatorname{pdf}(r^*) &= \frac{e^{-\frac{\mu^2}{2}}\left(\frac{\mu^2}{2}\frac{\left(1+\beta^*r^*\right)^2}{1+r^{*2}}\right)^{\frac{L+1}{2}}}{\Gamma\left(\frac{L}{2}\right)\pi^{1/2}\left(1+r^{*2}\right)^{(L+2)/2}} \\
&\quad\times\int_0^1 e^{-\frac{\mu^2}{2}\beta^{*2}t}e^{\frac{\mu^2 t}{2}\frac{\left(1+\beta^*r^*\right)^2}{1+r^{*2}}}t^{\frac{L}{2}}\left(1-t\right)^{\frac{L}{2}-1}dt\left\{1+O\left(1-\rho^2\right)\right\} \\
&= \frac{e^{-\frac{\mu^2}{2}}\left(\frac{\mu^2}{2}\frac{\left(1+\beta^*r^*\right)^2}{1+r^{*2}}\right)^{\frac{L+1}{2}}}{\Gamma\left(\frac{L}{2}\right)\pi^{1/2}\left(1+r^{*2}\right)^{(L+2)/2}}\frac{\Gamma\left(\frac{L+2}{2}\right)\Gamma\left(\frac{L}{2}\right)}{\Gamma\left(L+1\right)} \\
&\quad\times{}_1F_1\left(\frac{L+2}{2},L+1;\frac{\mu^2}{2}\left\{\frac{\left(1+\beta^*r^*\right)^2}{1+r^{*2}}-\beta^{*2}\right\}\right)\left\{1+O\left(1-\rho^2\right)\right\}
\end{aligned}
$$

$$= \frac{\Gamma\left(\frac{L+2}{2}\right)\left(\frac{\mu^2}{2}\right)^{\frac{L+1}{2}} e^{-\frac{\mu^2}{2}} \left|1+\beta^* r^*\right|^{L+1}}{\Gamma(L+1)\,\pi^{1/2}\left(1+r^{*2}\right)^{L+3/2}}$$

$$ {}_1F_1\left(\frac{L}{2}+1, L+1; \frac{\mu^2}{2}\frac{1-\beta^{*2}+2\beta^* r^*}{1+r^{*2}}\right)\left\{1+O\left(1-\rho^2\right)\right\}, \qquad (40)$$

which is valid irrespective of the sign of $1-\beta^{*2}+2\beta^* r^*$ as ${}_1F_1$ is an entire function.

We state the result formally as follows.

Proposition 4 *A uniform asymptotic approximation to the exact IV density is given by*

$$\text{pdf}\left(r^*\right) \sim \frac{\Gamma\left(\frac{L+2}{2}\right)\left(\frac{\mu^2}{2}\right)^{\frac{L+1}{2}} e^{-\frac{\mu^2}{2}} \left|1+\beta^* r^*\right|^{L+1}}{\Gamma(L+1)\,\pi^{1/2}\left(1+r^{*2}\right)^{L+3/2}}\ {}_1F_1\left(\frac{L}{2}+1, L+1; \frac{\mu^2}{2}\frac{1-\beta^{*2}+2\beta^* r^*}{1+r^{*2}}\right). \qquad (41)$$

This uniform approximation is valid as $\rho^2 \to 1$ provided $\mu^2 > 0$. The asymptotic approximation also holds as $\mu^2 \to \infty$ provided $1+\beta^* r^* \neq 0$, in which case the exact density has the form (37).

Note that the approximation (41) is valid as $\mu^2 \to \infty$ provided $1+\beta^* r^* \neq 0$, so that (41) delivers an alternative approximant for conventional large sample size, large concentration parameter asymptotics. This approximant is more general than the approximation given in Phillips (1980) and it holds over the support of the density except when $r^* = -1/\beta^*$, unlike the approximants obtained in Phillips (1980) and Holly and Phillips (1979), which hold only in the tails. When $r^* = -1/\beta^*$, the exact density has the simple form given by (37).

Note also that the approximate density (41) has finite moments to integer order L, reproducing precisely the moment existence property of the exact density (31).

4.4 The IV and OLS Densities as Exact Approximations

We start by transforming the density (40) to original coordinates. Using (52), (53), and (51) in the Appendix, we have

$$\begin{aligned}
1+2\beta^* r^* - \beta^{*2} &= \frac{\delta^2 + 2\beta r - 2r\rho\delta - \beta^2}{\delta^2\left(1-\rho^2\right)}, \\
1+r^{*2} &= \frac{\delta^2 + r^2 - 2r\rho\delta}{\delta^2\left(1-\rho^2\right)}, \\
1+\beta^* r^* &= \frac{\delta^2 + \beta r - \rho\delta\left(r+\beta\right)}{\delta^2\left(1-\rho^2\right)},
\end{aligned}$$

so that as $\rho \to \pm 1$

$$\begin{aligned} \frac{1+2\beta^* r^* - \beta^{*2}}{1+r^{*2}} &= \frac{\delta^2 + 2\beta r - 2r\rho\delta - \beta^2}{\delta^2 + r^2 - 2r\rho\delta} \\ &\to \begin{cases} \frac{(\beta-\delta)(2r-\beta-\delta)}{(r-\delta)^2} & \rho \to 1 \\ \frac{(\beta+\delta)(2r-\beta+\delta)}{(r+\delta)^2} & \rho \to -1 \end{cases} . \end{aligned}$$

Then, from (17) and (40), we have

$$\begin{aligned} \mathrm{pdf}(r) &= \frac{1}{\delta(1-\rho^2)^{1/2}} \mathrm{pdf}(r^*) \\ &= \frac{1}{\delta(1-\rho^2)^{1/2}} \frac{\Gamma\left(\frac{L+2}{2}\right)\left(\frac{\mu^2}{2}\right)^{\frac{L+1}{2}} e^{-\frac{\mu^2}{2}} \left|\frac{\delta^2+\beta r-\rho\delta(r+\beta)}{\delta^2(1-\rho^2)}\right|^{L+1}}{\Gamma(L+1)\pi^{1/2}\left\{\frac{\delta^2+r^2-2r\rho\delta}{\delta^2(1-\rho^2)}\right\}^{L+3/2}} \\ &\quad \times {}_1F_1\left(\frac{L}{2}+1, L+1; \frac{\mu^2}{2}\frac{\delta^2+2\beta r-2r\rho\delta-\beta^2}{\delta^2+r^2-2r\rho\delta}\right)\left\{1+O\left(1-\rho^2\right)\right\} \end{aligned}$$

$$\begin{aligned} &= \frac{\Gamma\left(\frac{L+2}{2}\right)\left(\frac{\mu^2}{2}\right)^{\frac{L+1}{2}} e^{-\frac{\mu^2}{2}} \left|\delta^2+\beta r-\rho\delta(r+\beta)\right|^{L+1}}{\Gamma(L+1)\pi^{1/2}\left\{\delta^2+r^2-2r\rho\delta\right\}^{L+3/2}} \\ &\quad \times {}_1F_1\left(\frac{L}{2}+1, L+1; \frac{\mu^2}{2}\frac{\delta^2+2\beta r-2r\rho\delta-\beta^2}{\delta^2+r^2-2r\rho\delta}\right)\left\{1+O\left(1-\rho^2\right)\right\} \end{aligned}$$

Hence, as $\rho \to 1$ we have

$$\begin{aligned} \lim_{\rho\to 1} \mathrm{pdf}(r) &= \frac{\Gamma\left(\frac{L+2}{2}\right)\left(\frac{\mu^2}{2}\right)^{\frac{L+1}{2}} e^{-\frac{\mu^2}{2}} \left|\delta^2+\beta r-\delta(r+\beta)\right|^{L+1}}{\Gamma(L+1)\pi^{1/2}(r-\delta)^{2L+3}} \\ &\quad \times {}_1F_1\left(\frac{L}{2}+1, L+1; \frac{\mu^2}{2}\frac{(\beta-\delta)(2r-\beta-\delta)}{(r-\delta)^2}\right) \\ &= \frac{\Gamma\left(\frac{L+2}{2}\right)\left(\frac{\mu^2}{2}\right)^{\frac{L+1}{2}} e^{-\frac{\mu^2}{2}} \left|(\beta-\delta)(r-\delta)\right|^{L+1}}{\Gamma(L+1)\pi^{1/2}(r-\delta)^{2L+3}} \\ &\quad \times {}_1F_1\left(\frac{L}{2}+1, L+1; \frac{\mu^2}{2}\frac{(\beta-\delta)(2r-\beta-\delta)}{(r-\delta)^2}\right) \end{aligned}$$

$$= \frac{\Gamma\left(\frac{L+2}{2}\right)\left(\frac{\mu^2}{2}\right)^{\frac{L+1}{2}} e^{-\frac{\mu^2}{2}} \left|\beta-\delta\right|^{L+1}}{\Gamma(L+1)\pi^{1/2}\left|r-\delta\right|^{L+2}} \, {}_1F_1\left(\frac{L}{2}+1, L+1; \frac{\mu^2}{2}\frac{(\beta-\delta)(2r-\beta-\delta)}{(r-\delta)^2}\right). \tag{42}$$

Using (34), this limit may be written in the alternate form

$$\frac{\Gamma\left(\frac{L+2}{2}\right)\left(\frac{\mu^2}{2}\right)^{\frac{L+1}{2}}|\beta-\delta|^{L+1}e^{-\frac{\mu^2}{2}\frac{(r-\beta)^2}{(r-\delta)^2}}}{\Gamma(L+1)\,\pi^{1/2}|r-\delta|^{L+2}}\,{}_1F_1\left(\frac{L}{2},L+1;-\frac{\mu^2}{2}\frac{(\beta-\delta)(2r-\beta-\delta)}{(r-\delta)^2}\right).$$

Similarly, when $\rho \to -1$ we find

$$\begin{aligned}\lim_{\rho\to 1}\mathrm{pdf}(r) &= \frac{\Gamma\left(\frac{L+2}{2}\right)\left(\frac{\mu^2}{2}\right)^{\frac{L+1}{2}}e^{-\frac{\mu^2}{2}}|(\beta+\delta)|^{L+1}}{\Gamma(L+1)\,\pi^{1/2}|r+\delta|^{L+2}} \\ &\quad\times{}_1F_1\left(\frac{L}{2}+1,L+1;\frac{\mu^2}{2}\frac{(\beta+\delta)(2r-\beta+\delta)}{(r+\delta)^2}\right) \\ &= \frac{\Gamma\left(\frac{L+2}{2}\right)\left(\frac{\mu^2}{2}\right)^{\frac{L+1}{2}}|\beta+\delta|^{L+1}e^{-\frac{\mu^2}{2}\frac{(r-\beta)^2}{(r+\delta)^2}}}{\Gamma(L+1)\,\pi^{1/2}|r+\delta|^{L+2}} \\ &\quad\times{}_1F_1\left(\frac{L}{2},L+1;-\frac{\mu^2}{2}\frac{(\beta+\delta)(2r-\beta+\delta)}{(r+\delta)^2}\right). \qquad (43)\end{aligned}$$

These results for the two cases $\rho \to \pm 1$ may be formalized as follows.

Proposition 5 *The exact density of the IV estimator with L degrees of overidentification in a model with two endogenous variables and a structural identity is*

$$pdf(r)=\frac{\Gamma\left(\frac{L+2}{2}\right)\left(\frac{\mu^2}{2}\right)^{\frac{L+1}{2}}|\beta\mp\delta|^{L+1}e^{-\frac{\mu^2}{2}\frac{(r-\beta)^2}{(r-\delta)^2}}}{\Gamma(L+1)\,\pi^{1/2}|r\mp\delta|^{L+2}}\,{}_1F_1\left(\frac{L}{2},L+1;-\frac{\mu^2}{2}\frac{(\beta\mp\delta)(2r-\beta\mp\delta)}{(r\mp\delta)^2}\right), \qquad (44)$$

according as the correlation between the endogenous variables $\rho = \pm 1$.

Setting $\mu^2 = n/\sigma^2$, $\delta = 1$, $\rho = 1$, and $L = n-2$, expression (44) reduces to Bergstrom's exact OLS density as given in the form (11) of Proposition 2. Moreover, setting $\mu^2 = n/\sigma^2 = \lambda_n/\sigma^2$, $\delta = 1$, $\rho = 1$, and $L = 0$ in (44), and noting that the ${}_1F_1$ function is unity because the series terminates at the first term in this case, we have

$$\mathrm{pdf}(r)=\frac{\lambda_n^{1/2}|\beta-1|\,e^{-\frac{\lambda_n}{2\sigma^2}\frac{(r-\beta)^2}{(r-1)^2}}}{(2\pi)^{1/2}\sigma(r-1)^2},$$

which is Bergstrom's exact LIML density (6) for the just identified case. Thus, (44) is a general formula that includes exact results for OLS, IV and LIML for the case of a structural identity in the system.

As remarked in the Introduction, this set of results seems to be the first of its kind where a uniform asymptotic approximation produces exact finite sample densities.

4.5 Some Properties of the IV and OLS Exact Densities

(a) It is apparent from the form of the IV density (44) that integer moments of the distribution are finite up to the degree of overidentification L, just as in the case of models without a structural identity.

(b) When $L = 0$, the ${}_1F_1$ function in (44) is unity, so the exact density has a zero at $r = \delta$, just as in the case of Bergstrom's LIML estimator. But when $L > 0$, the density does not have a zero at $r = \delta$, as shown below.

(c) Taking a large μ^2 asymptotic approximation to (44) using (35), we obtain, noting the equivalence of (44) to

$$\begin{aligned}\mathrm{pdf}\,(r) &= \frac{\Gamma\left(\frac{L+2}{2}\right)\left(\frac{\mu^2}{2}\right)^{\frac{L+1}{2}} e^{-\frac{\mu^2}{2}} \left|\beta \mp \delta\right|^{L+1}}{\Gamma\left(L+1\right)\pi^{1/2}\left|r \mp \delta\right|^{L+2}} \\ &\quad \times {}_1F_1\left(\frac{L}{2}+1, L+1; \frac{\mu^2}{2}\frac{\left(\beta \mp \delta\right)\left(2r - \beta \mp \delta\right)}{\left(r \mp \delta\right)^2}\right) \qquad (45)\end{aligned}$$

$$\begin{aligned} &= \frac{\Gamma\left(\frac{L+2}{2}\right)\left(\frac{\mu^2}{2}\right)^{\frac{L+1}{2}} e^{-\frac{\mu^2}{2}} \left|\beta \mp \delta\right|^{L+1}}{\Gamma\left(L+1\right)\pi^{1/2}\left|r \mp \delta\right|^{L+2}} \\ &\quad \times \frac{\Gamma\left(L+1\right)}{\Gamma\left(\frac{L}{2}+1\right)} \left\{\frac{\mu^2}{2}\frac{\left(\beta \mp \delta\right)\left(2r - \beta \mp \delta\right)}{\left(r \mp \delta\right)^2}\right\}^{-\frac{L}{2}} e^{\frac{\mu^2}{2}\frac{\left(\beta \mp \delta\right)\left(2r - \beta \mp \delta\right)}{\left(r \mp \delta\right)^2}} \left\{1 + O\left(\mu^{-2}\right)\right\} \\ &= \frac{\left(\frac{\mu^2}{2}\right)^{\frac{L+1}{2}} \left|\beta \mp \delta\right|^{L+1}}{\pi^{1/2}\left|r \mp \delta\right|^{L+2}} \left\{\frac{\mu^2}{2}\frac{\left(\beta \mp \delta\right)\left(2r - \beta \mp \delta\right)}{\left(r \mp \delta\right)^2}\right\}^{-\frac{L}{2}} e^{-\frac{\mu^2}{2}\left\{1 - \frac{\left(\beta \mp \delta\right)\left(2r - \beta \mp \delta\right)}{\left(r \mp \delta\right)^2}\right\}} \left\{1 + O\left(\mu^{-2}\right)\right\} \\ &= \frac{\left(\frac{\mu^2}{2\pi}\right)^{\frac{1}{2}} \left|\beta \mp \delta\right|^{\frac{L}{2}+1} e^{-\frac{\mu^2}{2}\frac{\left(r-\beta\right)^2}{\left(r \mp \delta\right)^2}}}{\left(r \mp \delta\right)^2 \left|2r - \beta \mp \delta\right|^{\frac{L}{2}}} \left\{1 + O\left(\mu^{-2}\right)\right\},\end{aligned}$$

which corresponds to (27) and (28) when $L = n-2$, thereby validating the latter formulae as large μ^2 or concentration parameter approximants to the exact OLS distribution.

(d) We take limits of the density as $r \to \pm\delta$ when $L > 0$. Using (35), note that

$$\begin{aligned} &{}_1F_1\left(\frac{L}{2}, L+1; -\frac{\mu^2}{2}\frac{\left(\beta \mp \delta\right)\left(2r - \beta \mp \delta\right)}{\left(r \mp \delta\right)^2}\right) \\ &= \frac{\Gamma\left(L\right)}{\Gamma\left(\frac{L}{2}\right)} e^{-\frac{\mu^2}{2}\frac{\left(\beta \mp \delta\right)\left(2r - \beta \mp \delta\right)}{\left(r \mp \delta\right)^2}} \left\{-\frac{\mu^2}{2}\frac{\left(\beta \mp \delta\right)\left(2r - \beta \mp \delta\right)}{\left(r \mp \delta\right)^2}\right\}^{-\frac{L}{2}-1} \left\{1 + O\left(\left(r \mp \delta\right)^2\right)\right\} \\ &= \frac{\Gamma\left(L\right)}{\Gamma\left(\frac{L}{2}\right)} e^{\frac{\mu^2}{2}\frac{\left(\beta - r\right)^2 - \left(r \mp \delta\right)^2}{\left(r \mp \delta\right)^2}} \left\{\frac{\left(r \mp \delta\right)^2}{\frac{\mu^2}{2}\left(\delta \mp \beta\right)\left(2r - \beta \mp \delta\right)}\right\}^{\frac{L}{2}+1} \left\{1 + O\left(\left(r \mp \delta\right)^2\right)\right\},\end{aligned}$$

so that as $r \to \pm\delta$, we have

$$\begin{aligned}
& \lim_{r\to\pm\delta} \mathrm{pdf}\,(r) \\
= & \lim_{r\to\pm\delta} \left\{ \frac{\Gamma\left(\frac{L+2}{2}\right)\left(\frac{\mu^2}{2}\right)^{\frac{L+1}{2}} |\beta \mp \delta|^{L+1} e^{-\frac{\mu^2}{2}\frac{(r-\beta)^2}{(r-\delta)^2}}}{\Gamma(L+1)\,\pi^{1/2}\,|r\mp\delta|^{L+2}} \right. \\
& \left. \times\ {}_1F_1\left(\frac{L}{2}, L+1; -\frac{\mu^2}{2}\frac{(\beta\mp\delta)(2r-\beta\mp\delta)}{(r\mp\delta)^2}\right)\right\}
\end{aligned}$$

$$\begin{aligned}
= & \lim_{r\to\pm\delta} \left\{ \frac{\Gamma\left(\frac{L+2}{2}\right)\left(\frac{\mu^2}{2}\right)^{\frac{L+1}{2}} |\beta \mp \delta|^{L+1}}{\Gamma(L+1)\,\pi^{1/2}\,|r\mp\delta|^{L+2}} \frac{\Gamma(L)}{\Gamma\left(\frac{L}{2}\right)} e^{-\frac{\mu^2}{2}} \left\{ \frac{(r\mp\delta)^2}{-\frac{\mu^2}{2}(\beta\mp\delta)(2r-\beta\mp\delta)} \right\} \right\}^{\frac{L}{2}+1} \\
= & \frac{|\beta\mp\delta|^{L+1} e^{-\frac{\mu^2}{2}}}{\mu(2\pi)^{1/2}\left\{-(\beta\mp\delta)(\pm\delta-\beta)\right\}^{\frac{L}{2}+1}} \\
= & \frac{|\beta\mp\delta|^{L+1} e^{-\frac{\mu^2}{2}}}{\mu(2\pi)^{1/2}\left\{(\beta\mp\delta)^2\right\}^{\frac{L}{2}+1}} = \frac{|\beta\mp\delta|^{L+1} e^{-\frac{\mu^2}{2}}}{\mu(2\pi)^{1/2}|\beta\mp\delta|^{L+2}} = \frac{e^{-\frac{\mu^2}{2}}}{\mu(2\pi)^{1/2}|\beta\mp\delta|},
\end{aligned}$$

so that the limit is well defined in both cases.

5 Conclusion

The results given here integrate Bergstrom's (1962) findings with later research over the succeeding decades on exact distribution theory in structural models. The results apply to a structural equation with two endogenous variables. This model is important in practice and corrresponds with the simple income determination model studied by Bergstrom. We expect similar findings to hold in the general case with a structural equation containing $m+1$ endogenous variables for $m \geq 2$. Those results will be reported elsewhere.

The exact distributional results have been derived under Gaussian distributional assumptions. But the same results apply as limit distributions in the context of structural models with weak instruments, as may be shown by central limit arguments under general martingale difference sequence errors as in Phillips (1989) and Staiger and Stock (1997).

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7 Appendix

Proof of Proposition 1

First observe that for $z = \frac{\sqrt{n}(2b-\beta-1)}{\sigma(1-b)}$ the upper limit of the integral in (10) is

$$\begin{aligned} A &= -\left(\frac{z^2}{2} + \frac{\sqrt{n}z}{\sigma}\right) = -\frac{n\left(2b-\beta-1\right)^2}{2\sigma^2\left(1-b\right)^2} - \frac{n\left(2b-\beta-1\right)}{\sigma^2\left(1-b\right)} \\ &= -\frac{n\left(2b-\beta-1\right)\left\{\left(2b-\beta-1\right)+2\left(1-b\right)\right\}}{2\sigma^2\left(1-b\right)^2} \\ &= -\frac{n\left(2b-\beta-1\right)\left(1-\beta\right)}{2\sigma^2\left(1-b\right)^2} = \frac{n\left(1-\beta\right)\left(1+\beta-2b\right)}{2\sigma^2\left(1-b\right)^2}, \end{aligned}$$

and then

$$\begin{aligned} g\left(z\right) &= k\int_0^A \frac{4q}{z^2}\left\{-\frac{4q^2}{z^2} - \left(\frac{4\sqrt{n}}{\sigma z}+2\right)q\right\}^{\frac{n-4}{2}} e^q dq \\ &= \frac{2^{n-2}k}{\left(z^2\right)^{(n-2)/2}}\int_0^A q^{\frac{n-2}{2}}\left\{A-q\right\}^{\frac{n-4}{2}} e^q dq \\ &= \frac{2^{n-2}k}{\left(z^2\right)^{(n-2)/2}} B\left(\frac{n}{2}-1,\frac{n}{2}\right) A^{n-2}\ {}_1F_1\left(\frac{n}{2}, n-1; A\right) \qquad (46) \\ &= \frac{2^{n-2}ke^A}{\left(z^2\right)^{(n-2)/2}} B\left(\frac{n}{2}-1,\frac{n}{2}\right) A^{n-2}\ {}_1F_1\left(\frac{n}{2}-1, n-1; -A\right), \qquad (47) \end{aligned}$$

using an integral representation of the confluent hypergeometric function ${}_1F_1$ (e.g. Gradshteyn and Ryhzik, 2000, formula 3.383.1; but we note the misprint in their formula 3.383.1 in which their $u^{u+\nu-1}$ should read $u^{\mu+\nu-1}$) for (46) and the Kummer relation ${}_1F_1\left(\frac{n}{2}, n-1; A\right) = e^A\ {}_1F_1\left(\frac{n}{2}, n-1; -A\right)$ for (47). Proceeding, we note that

$$\begin{aligned} ke^A &= \frac{e^{\frac{n(1-\beta)(1+\beta-2b)}{2\sigma^2(1-b)^2}}}{2^{(n-1)/2}\sqrt{\pi}\Gamma\left(\frac{n-2}{2}\right)e^{\frac{n}{2\sigma^2}}} = \frac{e^{-\frac{n}{2\sigma^2}\left\{1-\frac{(1-\beta)(1+\beta-2b)}{(1-b)^2}\right\}}}{2^{(n-1)/2}\sqrt{\pi}\Gamma\left(\frac{n-2}{2}\right)} \\ &= \frac{e^{-\frac{n}{2\sigma^2}\left\{\frac{(1-b)^2-(1-\beta)(1+\beta-2b)}{(1-b)^2}\right\}}}{2^{(n-1)/2}\sqrt{\pi}\Gamma\left(\frac{n-2}{2}\right)} = \frac{e^{-\frac{n}{2\sigma^2}\left\{\frac{(b-\beta)^2}{(1-b)^2}\right\}}}{2^{(n-1)/2}\sqrt{\pi}\Gamma\left(\frac{n-2}{2}\right)}, \end{aligned}$$

since $\left(1-b\right)^2 - \left(1-\beta\right)\left(1+\beta-2b\right) = \left(b-\beta\right)^2$, and $z = \frac{\sqrt{n}(2b-\beta-1)}{\sigma(1-b)}$

$$\begin{aligned} \frac{A^{n-2}}{\left(z^2\right)^{\frac{n-2}{2}}} &= \frac{n^{n-2}\left(1-\beta\right)^{n-2}\left(1+\beta-2b\right)^{n-2}}{2^{n-2}\sigma^{2n-4}\left(1-b\right)^{2n-4}}\frac{\sigma^{n-2}\left|1-b\right|^{n-2}}{n^{(n-2)/2}\left|2b-\beta-1\right|^{n-2}} \\ &= \frac{n^{(n-2)/2}\left(1-\beta\right)^{n-2}}{2^{n-2}\sigma^{n-2}\left|1-b\right|^{n-2}}. \end{aligned}$$

Hence,

$$g(z) = \frac{2^{n-2}ke^{A}}{(z^2)^{\frac{n-2}{2}}} B\left(\frac{n}{2}-1,\frac{n}{2}\right) A^{n-2}\ {}_1F_1\left(\frac{n}{2}-1,n-1;-A\right) \quad (48)$$

$$= \frac{2^{n-2}e^{-\frac{n}{2\sigma^2}\left\{\frac{(b-\beta)^2}{(1-b)^2}\right\}}}{2^{(n-1)/2}\sqrt{\pi}\Gamma\left(\frac{n-2}{2}\right)} \frac{n^{(n-2)/2}(1-\beta)^{n-2}}{2^{n-2}\sigma^{n-2}|1-b|^{n-2}} B\left(\frac{n}{2}-1,\frac{n}{2}\right)\ {}_1F_1\left(\frac{n}{2}-1,n-1;-A\right)$$

$$= \frac{n^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)e^{-\frac{n}{2\sigma^2}\left\{\frac{(b-\beta)^2}{(1-b)^2}\right\}}(1-\beta)^{n-2}}{2^{(n-1)/2}\sigma^{n-2}\sqrt{\pi}\Gamma(n-1)|1-b|^{n-2}}\ {}_1F_1\left(\frac{n}{2}-1,n-1;-A\right). \quad (49)$$

Next, using (49) in (9) we obtain the following alternate form for the OLS density

$$\operatorname{pdf}(b) = \frac{\sqrt{n}(1-\beta)}{\sigma(1-b)^2} g\left(\frac{\sqrt{n}(2b-\beta-1)}{\sigma(1-b)}\right)$$

$$= \frac{n^{\frac{n-1}{2}}\Gamma\left(\frac{n}{2}\right)e^{-\frac{n}{2\sigma^2}\left\{\frac{(b-\beta)^2}{(1-b)^2}\right\}}(1-\beta)^{n-1}}{2^{(n-1)/2}\sigma^{n-1}\sqrt{\pi}\Gamma(n-1)|1-b|^{n}}\ {}_1F_1\left(\frac{n}{2}-1,n-1;-A\right). \quad (50)$$

giving the stated result.

Proof of Proposition 2 First, we establish some simple algebraic correspondences. Calculations reveal that

$$1+\beta^{*2} = 1+\frac{(\beta-\rho\delta)^2}{\delta^2(1-\rho^2)} = \frac{\delta^2+\beta^2-2\beta\rho\delta}{\delta^2(1-\rho^2)},$$

$$1+\beta^* r^* = 1+\frac{1}{\delta(1-\rho^2)^{1/2}}(\beta-\rho\delta)\frac{1}{\delta(1-\rho^2)^{1/2}}(r-\rho\delta)$$

$$= 1+\frac{(\beta-\rho\delta)(r-\rho\delta)}{\delta^2(1-\rho^2)} = \frac{\delta^2+\beta r-\rho\delta(r+\beta)}{\delta^2(1-\rho^2)}, \quad (51)$$

and then

$$1+2\beta^* r^*-\beta^{*2} = 1+\frac{2(\beta-\rho\delta)(r-\rho\delta)}{\delta^2(1-\rho^2)}-\frac{(\beta-\rho\delta)^2}{\delta^2(1-\rho^2)}$$

$$= \frac{\delta^2+2\beta r-2r\rho\delta-\beta^2}{\delta^2(1-\rho^2)}, \quad (52)$$

and

$$1+r^{*2} = 1+\frac{(r-\rho\delta)^2}{\delta^2(1-\rho^2)} = \frac{\delta^2+r^2-2r\rho\delta}{\delta^2(1-\rho^2)}. \quad (53)$$

It follows that

$$\frac{\left(1+\beta^{*}r^{*}\right)^{2}}{1+r^{*2}}=\frac{\left\{\frac{\delta^{2}+\beta r-\rho\delta(r+\beta)}{\delta^{2}(1-\rho^{2})}\right\}^{2}}{\frac{\delta^{2}+r^{2}-2r\rho\delta}{\delta^{2}(1-\rho^{2})}}=\frac{\left[\delta^{2}+\beta r-\rho\delta\left(r+\beta\right)\right]^{2}}{\delta^{2}\left(1-\rho^{2}\right)\left\{\delta^{2}+r^{2}-2r\rho\delta\right\}}, \tag{54}$$

which diverges as $\rho^2 \to 1$. Hence, to find the limiting density for the degenerate case where there is an identity in the model, we may employ the first term in the (large argument) asymptotic expansion of the confluent hypergeometric function in (20) above, viz.

$$\begin{aligned}
& {}_1F_1\left(1,\frac{1}{2};\frac{\mu^{2}}{2}\frac{\left(1+\beta^{*}r^{*}\right)^{2}}{1+r^{*2}}\right)\\
= & {}_1F_1\left(1,\frac{1}{2};\frac{\mu^{2}}{2}\frac{\left[\delta^{2}+\beta r-\rho\delta\left(r+\beta\right)\right]^{2}}{\delta^{2}\left(1-\rho^{2}\right)\left\{\delta^{2}+r^{2}-2r\rho\delta\right\}}\right)\\
= & \sqrt{\pi}\left(\frac{\mu^{2}}{2}\right)^{1/2}\exp\left\{\frac{\mu^{2}}{2}\frac{\left[\delta^{2}+\beta r-\rho\delta\left(r+\beta\right)\right]^{2}}{\delta^{2}\left(1-\rho^{2}\right)\left\{\delta^{2}+r^{2}-2r\rho\delta\right\}}\right\}\\
& \times\left\{\frac{\left[\delta^{2}+\beta r-\rho\delta\left(r+\beta\right)\right]^{2}}{\delta^{2}\left(1-\rho^{2}\right)\left\{\delta^{2}+r^{2}-2r\rho\delta\right\}}\right\}^{1/2}\left\{1+O\left(1-\rho^{2}\right)\right\},
\end{aligned} \tag{55}$$

(e.g., see Lebedev, 1972). We now have

$$\begin{aligned}
\operatorname{pdf}(r) = & \frac{1}{\delta\left(1-\rho^{2}\right)^{1/2}}\frac{e^{-\frac{n}{2}\mu^{2}\left(1+\beta^{*2}\right)}}{\pi\left(1+r^{*2}\right)}\,{}_1F_1\left(1,\frac{1}{2};\frac{n}{2}\mu^{2}\frac{\left(1+\beta^{*}r^{*}\right)^{2}}{1+r^{*2}}\right)\\
= & \frac{\sqrt{\pi}\left(\frac{\mu^{2}}{2}\right)^{1/2}}{\delta\left(1-\rho^{2}\right)^{1/2}}\frac{\exp\left\{-\frac{n}{2}\mu^{2}\frac{\delta^{2}+\beta^{2}-2\beta\rho\delta}{\delta^{2}(1-\rho^{2})}+\frac{n}{2}\mu^{2}\frac{\left[\delta^{2}+\beta r-\rho\delta(r+\beta)\right]^{2}}{\delta^{2}(1-\rho^{2})\left\{\delta^{2}+r^{2}-2r\rho\delta\right\}}\right\}}{\pi\frac{\delta^{2}+r^{2}-2r\rho\delta}{\delta^{2}(1-\rho^{2})}}\\
& \times\left\{\frac{\left[\delta^{2}+\beta r-\rho\delta\left(r+\beta\right)\right]^{2}}{\delta^{2}\left(1-\rho^{2}\right)\left\{\delta^{2}+r^{2}-2r\rho\delta\right\}}\right\}^{1/2}\left\{1+O\left(1-\rho^{2}\right)\right\}.
\end{aligned} \tag{56}$$

Observe that

$$\begin{aligned}
& -\frac{\mu^{2}}{2}\frac{\delta^{2}+\beta^{2}-2\beta\rho\delta}{\delta^{2}\left(1-\rho^{2}\right)}+\frac{\mu^{2}}{2}\frac{\left[\delta^{2}+\beta r-\rho\delta\left(r+\beta\right)\right]^{2}}{\delta^{2}\left(1-\rho^{2}\right)\left\{\delta^{2}+r^{2}-2r\rho\delta\right\}}\\
= & -\frac{\mu^{2}}{2}\frac{1}{\delta^{2}\left(1-\rho^{2}\right)}\left\{\frac{\left(\delta^{2}+\beta^{2}-2\beta\rho\delta\right)\left\{\delta^{2}+r^{2}-2r\rho\delta\right\}-\left[\delta^{2}+\beta r-\rho\delta\left(r+\beta\right)\right]^{2}}{\left\{\delta^{2}+r^{2}-2r\rho\delta\right\}}\right\}\\
= & -\frac{\mu^{2}}{2}\frac{1}{\delta^{2}\left(1-\rho^{2}\right)}\left\{\frac{\delta^{2}\left(r-\beta\right)^{2}\left(1-\rho^{2}\right)}{\left\{\delta^{2}+r^{2}-2r\rho\delta\right\}}\right\}=-\frac{\mu^{2}}{2}\frac{\left(r-\beta\right)^{2}}{\left\{\delta^{2}+r^{2}-2r\rho\delta\right\}},
\end{aligned}$$

so that (56) becomes

$$
\begin{aligned}
\operatorname{pdf}(r) &= \frac{\sqrt{\pi}\left(\frac{\mu^2}{2}\right)^{1/2}}{\delta\left(1-\rho^2\right)^{1/2}} \frac{\exp\left\{-\frac{\mu^2}{2}\frac{(r-\beta)^2}{\left\{\delta^2+r^2-2r\rho\delta\right\}}\right\}}{\pi\frac{\delta^2+r^2-2r\rho\delta}{\delta^2(1-\rho^2)}} \\
&\quad \times \left\{\frac{\left[\delta^2+\beta r-\rho\delta\left(r+\beta\right)\right]^2}{\delta^2\left(1-\rho^2\right)\left\{\delta^2+r^2-2r\rho\delta\right\}}\right\}^{1/2} \\
&= \left(\frac{\mu^2}{2\pi}\right)^{1/2} \frac{\exp\left\{-\frac{\mu^2}{2}\frac{(r-\beta)^2}{\left\{\delta^2+r^2-2r\rho\delta\right\}}\right\}}{\left(\delta^2+r^2-2r\rho\delta\right)} \left\{\frac{\left[\delta^2+\beta r-\rho\delta\left(r+\beta\right)\right]^2}{\left\{\delta^2+r^2-2r\rho\delta\right\}}\right\}^{1/2} \left\{1+O\left(1-\rho^2\right)\right\}.
\end{aligned}
$$

Then, as $\rho \to 1$ we have

$$
\begin{aligned}
\operatorname{pdf}(r) &\to \left(\frac{\mu^2}{2\pi}\right)^{1/2} \frac{\exp\left\{-\frac{\mu^2}{2}\frac{(r-\beta)^2}{\left\{\delta^2+r^2-2r\rho\delta\right\}}\right\}}{\left(\delta^2+r^2-2r\rho\delta\right)} \left\{\frac{\left[\delta^2+\beta r-\rho\delta\left(r+\beta\right)\right]^2}{\left\{\delta^2+r^2-2r\rho\delta\right\}}\right\}^{1/2} \\
&= \left(\frac{\mu^2}{2\pi}\right)^{1/2} \frac{\exp\left\{-\frac{\mu^2}{2}\frac{(r-\beta)^2}{(r-\delta)^2}\right\}}{(r-\delta)^2} \left\{\frac{(r-\delta)^2(\beta-\delta)^2}{(r-\delta)^2}\right\}^{1/2} \\
&= \frac{\left(\mu^2\right)^{1/2}}{\sqrt{2\pi}} \frac{|\beta-\delta|}{(r-\delta)^2} \exp\left\{-\frac{\mu^2}{2}\frac{(r-\beta)^2}{(r-\delta)^2}\right\}.
\end{aligned}
$$

Noting that $\mu^2 = n\gamma^2/\sigma^2 = \lambda_n/\sigma^2$ from (19), we have

$$
\begin{aligned}
\operatorname{pdf}(r) &= \frac{\lambda_n^{1/2}}{\sqrt{2\pi}\sigma} \frac{|\beta-\delta|}{(r-\delta)^2} \exp\left\{-\frac{\lambda_n}{2\sigma^2}\frac{(r-\beta)^2}{(r-\delta)^2}\right\} \\
&= \frac{\lambda_n^{1/2}}{\sqrt{2\pi}\sigma} \frac{|1-\beta|}{(1-b)^2} \exp\left\{-\frac{\lambda_n}{2\sigma^2}\left(\frac{b-\beta}{1-b}\right)^2\right\} \quad \text{for } \delta = 1,
\end{aligned}
$$

giving the stated correspondence with Bergstrom's expression.

Proof of Proposition 3 Expanding the ${}_1F_1$ function as $\rho^2 \to 1$ and $\frac{\mu^2}{2}\frac{(1+\beta^* r^*)^2}{1+r^{*2}} \to \infty$, we have (e.g. Lebedev, 1972, p. 271)

$$
\begin{aligned}
& {}_1F_1\left(\frac{n+1}{2}, \frac{n}{2}+j; \frac{\mu^2}{2}\frac{(1+\beta^* r^*)^2}{1+r^{*2}}\right) && (57)\\
&= \frac{\Gamma\left(\frac{n}{2}+j\right)}{\Gamma\left(\frac{n+1}{2}\right)} e^{\frac{\mu^2}{2}\frac{(1+\beta^* r^*)^2}{1+r^{*2}}} \left\{\frac{\mu^2}{2}\frac{(1+\beta^* r^*)^2}{1+r^{*2}}\right\}^{1/2-j} \left\{1+O\left(1-\rho^2\right)\right\} \\
&= \frac{\Gamma\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)_j}{\Gamma\left(\frac{n+1}{2}\right)} e^{\frac{\mu^2}{2}\frac{(1+\beta^* r^*)^2}{1+r^{*2}}} \left\{\frac{\mu^2}{2}\frac{(1+\beta^* r^*)^2}{1+r^{*2}}\right\}^{1/2-j} \left\{1+O\left(1-\rho^2\right)\right\},
\end{aligned}
$$

we find that as $\rho^2 \to 1$

$$
\begin{aligned}
& \left(\frac{\mu^2}{2}\beta^{*2}\right)^j {}_1F_1\left(\frac{n+1}{2}, \frac{n}{2}+j; \frac{\mu^2}{2}\frac{(1+\beta^* r^*)^2}{1+r^{*2}}\right) \\
&= \frac{\left(\frac{n}{2}\right)_j \Gamma\left(\frac{n}{2}\right) e^{\frac{\mu^2}{2}\frac{(1+\beta^* r^*)^2}{1+r^{*2}}}}{\Gamma\left(\frac{n+1}{2}\right)} \left\{\frac{\mu^2}{2}\frac{(1+\beta^* r^*)^2}{1+r^{*2}}\right\}^{1/2} \\
&\quad \times \left(\frac{\mu^2}{2}\frac{\beta^{*2}\left(1+r^{*2}\right)}{(1+\beta^* r^*)^2}\right)^j \left\{1+O\left(1-\rho^2\right)\right\}. && (58)
\end{aligned}
$$

Substituting the expansion (58) in (23) we obtain

$$
\begin{aligned}
\operatorname{pdf}(r^*) &= \frac{e^{-\frac{\mu^2}{2}\left\{\left(1+\beta^{*2}\right)-\frac{(1+\beta^* r^*)^2}{1+r^{*2}}\right\}}}{\pi^{1/2}\left(1+r^{*2}\right)^{(n+1)/2}} \left\{\frac{\mu^2}{2}\frac{(1+\beta^* r^*)^2}{1+r^{*2}}\right\}^{1/2} \\
&\quad \times \sum_{j=0}^{\infty} \frac{\left(\frac{n-1}{2}\right)_j}{j!} \left\{\frac{\mu^2}{2}\frac{\beta^{*2}\left(1+r^{*2}\right)}{(1+\beta^* r^*)^2}\right\}^j \left\{1+O\left(1-\rho^2\right)\right\} \\
&= \frac{\mu e^{-\frac{\mu^2}{2}\frac{(r^*-\beta^*)^2}{1+r^{*2}}} |1+\beta^* r^*|^{1/2}}{(2\pi)^{1/2}\left(1+r^{*2}\right)^{(n+2)/2}} \left[1-\frac{\beta^{*2}\left(1+r^{*2}\right)}{(1+\beta^* r^*)^2}\right]^{-\frac{n-1}{2}} \left\{1+O\left(1-\rho^2\right)\right\} && (59)\\
&= \left(\frac{\mu^2}{2\pi}\right)^{1/2} \frac{e^{-\frac{\mu^2}{2}\frac{(r^*-\beta^*)^2}{1+r^{*2}}} |1+\beta^* r^*|^{1/2}}{\left(1+r^{*2}\right)^{(n+2)/2}} \left\{\frac{1+2\beta^* r^*-\beta^{*2}}{(1+\beta^* r^*)^2}\right\}^{-\frac{n-1}{2}} \left\{1+O\left(1-\rho^2\right)\right\} \\
&= \left(\frac{\mu^2}{2\pi}\right)^{1/2} \frac{e^{-\frac{\mu^2}{2}\frac{(r^*-\beta^*)^2}{1+r^{*2}}} |1+\beta^* r^*|^{n}}{\left(1+r^{*2}\right)^{(n+2)/2}\left(1+2\beta^* r^*-\beta^{*2}\right)^{(n-1)/2}} \left\{1+O\left(1-\rho^2\right)\right\}, && (60)
\end{aligned}
$$

since

$$
\left(1+\beta^{*2}\right)-\frac{(1+\beta^* r^*)^2}{1+r^{*2}} = \frac{(r^*-\beta^*)^2}{1+r^{*2}},
$$

$$1-\frac{\beta^{*2}\left(1+r^{*2}\right)}{\left(1+\beta^{*}r^{*}\right)^{2}}=\frac{1+2\beta^{*}r^{*}-\beta^{*2}}{\left(1+\beta^{*}r^{*}\right)^{2}}$$

and where the binomial expansion is used in (59). The validity of the representation (59) therefore depends on the condition

$$\frac{\beta^{*2}\left(1+r^{*2}\right)}{\left(1+\beta^{*}r^{*}\right)^{2}}<1,\ \text{ or } 1+2\beta^{*}r^{*}-\beta^{*2}>0, \tag{61}$$

The leading term in the expansion (60) corresponds to the asymptotic (large sample size) approximation given in Phillips (1980, equation (15)) and the saddlepoint approximation in Holly and Phillips (1979, equation (25)) under the same condition (61).

Translating (60) to unstandardized coordinates, we have

$$\operatorname{pdf}(r)=\frac{1}{\delta\left(1-\rho^{2}\right)^{1/2}}\operatorname{pdf}\left(r^{*}\right)$$

$$=\frac{\left(\frac{\mu^{2}}{2\pi}\right)^{1/2}\exp\left\{-\frac{\mu^{2}}{2}\frac{(r-\beta)^{2}}{\delta^{2}+r^{2}-2r\rho\delta}\right\}\left|\delta^{2}+\beta r-\rho\delta\left(r+\beta\right)\right|^{n}}{\left\{\delta^{2}+r^{2}-2r\rho\delta\right\}^{(n+2)/2}\left\{\delta^{2}+2\beta r-2r\rho\delta-\beta^{2}\right\}^{(n-1)/2}}\left\{1+O\left(1-\rho^{2}\right)\right\}. \tag{62}$$

We next consider limits of this expression as $\rho^2 \to 1$. When $\rho \to 1$, the component factors behave as follows

$$\begin{aligned}
\delta^{2}+r^{2}-2r\rho\delta &\rightarrow (r-\delta)^{2}, \\
\delta^{2}+2\beta r-2r\rho\delta-\beta^{2} &\rightarrow (\beta-\delta)(2r-\beta-\delta), \\
\delta^{2}+\beta r-\rho\delta(r+\beta) &\rightarrow (r-\delta)(\beta-\delta),
\end{aligned}$$

whereas, when $\rho \to -1$, we have

$$\begin{aligned}
\delta^{2}+r^{2}-2r\rho\delta &\rightarrow (r+\delta)^{2}, \\
\delta^{2}+2\beta r-2r\rho\delta-\beta^{2} &\rightarrow (\beta+\delta)(2r-\beta+\delta), \\
\delta^{2}+\beta r-\rho\delta(r+\beta) &\rightarrow (r+\delta)(\beta+\delta).
\end{aligned}$$

Thus, as $\rho \to 1$, (62) becomes

$$\begin{aligned}
\lim_{\rho\to 1}\operatorname{pdf}(r) &= \left(\frac{\mu^{2}}{2\pi}\right)^{1/2}\frac{\exp\left\{-\frac{\mu^{2}}{2}\frac{(r-\beta)^{2}}{(r-\delta)^{2}}\right\}}{|r-\delta|^{n+2}}\frac{|(r-\delta)(\beta-\delta)|^{n}}{|(\beta-\delta)(2r-\beta-\delta)|^{(n-1)/2}} \\
&= \left(\frac{\mu^{2}}{2\pi}\right)^{1/2}\frac{\exp\left\{-\frac{\mu^{2}}{2}\frac{(r-\beta)^{2}}{(r-\delta)^{2}}\right\}}{|r-\delta|^{n+2}}\frac{|(r-\delta)|^{n}\,|\beta-\delta|^{(n+1)/2}}{|2r-\beta-\delta|^{(n-1)/2}} \\
&= \left(\frac{\mu^{2}}{2\pi}\right)^{1/2}\frac{\exp\left\{-\frac{\mu^{2}}{2}\frac{(r-\beta)^{2}}{(r-\delta)^{2}}\right\}}{(r-\delta)^{2}}\frac{|\beta-\delta|^{(n+1)/2}}{|2r-\beta-\delta|^{(n-1)/2}},
\end{aligned}$$

which, for $\delta = 1$ and $\mu^2 = \lambda_n/\sigma^2$, becomes

$$\lim_{\rho \to 1} \text{pdf}(r) = \frac{\lambda_n^{1/2} |\beta - 1|^{(n+1)/2} \exp\left\{-\frac{n\gamma^2}{2\sigma^2}\frac{(r-\beta)^2}{(r-1)^2}\right\}}{\sqrt{2\pi}\sigma (r-1)^2 |2r - \beta - 1|^{(n-1)/2}}. \tag{63}$$

Similarly, when $\rho \to -1$ we obtain

$$\lim_{\rho \to -1} \text{pdf}(r) = \left(\frac{\mu^2}{2\pi}\right)^{1/2} \frac{\exp\left\{-\frac{\mu^2}{2}\frac{(r-\beta)^2}{(r+\delta)^2}\right\}}{(r+\delta)^2} \frac{|\beta + \delta|^{(n+1)/2}}{|2r - \beta - \delta|^{(n-1)/2}}, \tag{64}$$

which for $\delta = 1$ and $\mu^2 = \lambda_n/\sigma^2$, becomes

$$\lim_{\rho \to -1} \text{pdf}(r) = \frac{\lambda_n^{1/2} |\beta + 1|^{(n+1)/2} \exp\left\{-\frac{n\gamma^2}{2\sigma^2}\frac{(r-\beta)^2}{(r+1)^2}\right\}}{\sqrt{2\pi}\sigma (r+1)^2 |2r - \beta - 1|^{(n-1)/2}}. \tag{65}$$

Both formulae (64) and (65) represent limiting approximations, rather than the exact OLS density. The reason is that upon using the expansion (58) in (23), the resulting infinite series is summable as a binomial series as in (59) only over a restricted range for the density, represented by condition (61). As remarked above, this same limitation applies to the saddlepoint approximation of Holly and Phillips (1979) and the Laplace approximation given in Phillips (1980). In effect, the asymptotic expansion (58) does not take account of the fact that the parameter j may be large in the second argument of the ${}_1F_1$ function (57).

Next consider what happens when we translate the condition (61) into unstandardized coordinates when $\rho = 1$. Observe that in original coordinates, using (24) and (25), we have

$$\frac{\beta^{*2}(1 + r^{*2})}{(1 + \beta^* r^*)^2} = \frac{(\beta - \rho\delta)^2 \left\{\delta^2 + r^2 - 2r\rho\delta\right\}}{\left[\delta^2 + \beta r - \rho\delta(r + \beta)\right]^2}$$

and for $\rho^2 = 1$

$$\frac{(\beta - \rho\delta)^2 \left\{\delta^2 + r^2 - 2r\rho\delta\right\}}{\left[\delta^2 + \beta r - \rho\delta(r + \beta)\right]^2} = \begin{cases} \frac{(\beta-\delta)^2(r-\delta)^2}{[(r-\delta)(\beta-\delta)]^2} = 1 & \rho = 1 \\ \frac{(\beta+\delta)^2(r+\delta)^2}{[(r+\delta)(\beta+\delta)]^2} = 1 & \rho = -1 \end{cases}.$$

Hence, the validity condition for the summation of (59) is violated when $\rho^2 = 1$. However, the approximate density (62) exists and is valid for all $\rho^2 < 1$ and, rather interestingly, this approximate density has well defined limits as $\rho \to \pm 1$ given by (64) and (65). As discussed in the paper, these limits can themselves be validated in terms of large concentration parameter approximants to the exact density.