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By

Barry O'Neill and Bezalel Peleg

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COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/

Lexicographic Composition of Simple Games

Barry O'Neill
University of California, Los Angeles

Bezalel Peleg
Hebrew Univesity of Jerusalem

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Abstract: A two-house legislature can often be modelled as a proper simple game whose outcome depends on whether a coalition wins, blocks or loses in two smaller proper simple games. It is shown that there are exactly five ways to combine the smaller games into a larger one. This paper focuses on one of the rules, *lexicographic composition,* where a coalition wins in $G_1 \Rightarrow G_2$ when it either wins in G_1, or blocks in G_1 and wins in G_2. It is the most decisive of the five. A *lexicographically decomposable* game is one that can be represented in this way using components whose player sets partition the whole set. Games with veto players are not decomposable, and anonymous games are decomposable if and only if they are decisive and have two or more players. If a player's benefit is assessed by any semi-value, then for two isomorphic games a player is better off from having a role in the first game than having the same role in the second. Lexicographic decomposability is sometimes compatible with equality of roles. A relaxation of it is suggested for its practical benefits.

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Section 1. Introduction

Group decision rules have been modeled by simple games, defined as those in which a coalition either wins or not, with no outcomes in between (von Neumann and Morgenstern 1944, Shapley and Shubik 1954, Peleg 1984, Taylor and Zwicker 1999, Peleg and Sudholter 2003). One stream of research developed an algebra in which a coalition wins in the product (sum) of two smaller games whenever it wins in both (either) of them (Shapley 1964, Owen 1964, Billera 1980, and others). Shapley (1967) defined the more general idea of a committee, a group of players who can be treated as a decision unit within the larger game.

This paper considers a class of composition rules, then focuses on one of them. It shows that there are five ways to combine two proper games into a third one, where success in the combined game depends only on whether a coalition wins, blocks or loses without blocking in the components. Among the five is "lexicographic composition," where a first group of players has the right to decide, but if there is a deadlock the power passes to a separate group. The name ”lexicographic” is suggested by the ordering of words in the dictionary, which puts ”azure” before ”babble” – that is, considers the word’s first letter, then its second, etc. The analogy for games is that a coalition’s power in the second game, however great, is irrelevant if its rivals win the first game. The lexicographic rule is especially interesting because it has been overlooked theoretically, has practical advantages and is occasionally used. One attractive feature is associativity, which allows simple games to be strung together without consideration to their grouping, and another is that among the five rules it is least prone to stalemate.

Section 2 gives definitions and example and Section 3 gives some basic properties of the lexicographic rule, showing that there are only five ways of combining two proper games to make a proper game. The lexicographic rule is the most decisive of them. Section 4 relates power and the order of play: if the two components are identical a player would prefer to be in the first game, when interests are measured by any semi-value. The next two sections treat decomposability. Even if a game is not played as a physical sequence it may be representable in this way since the criterion is the game’s winning coalitions, not its realization in the world. We ask when a certain game, even though played as a unit, is equivalent to a decomposed game, and investigate the case where the component games partition the player set of the larger one. Section 5 lists the smallest decomposable games, and Section 6 gives some conditions guaranteeing or precluding decomposability. Section 7 considers whether lexicographic composition is compatible with equality, interpreted here as anonymity, i.e., all players having the same abstract role. The section presents some ways to generate LD games that are anonymous, and are both anonymous overall with anonymous components. Section 8 discusses a weakening of the lexicographic rule that is in actual use.

Section 2. Definitions and examples

Let $\mathcal{P}$ be an infinite set called the *players,* whose subsets are called *coalitions.* A *simple game* G is a set of coalitions such that:

(1) $S \in G$ and $S \subseteq S'$ imply $S' \in G$ (monotonicity);

(2) the set of minimal coalitions of G is non-empty and finite;

(3) every minimal coalition is non-empty and finite.

A simple game will be called just a *game* and its minimal coalitions are its *minimal winning coalitions,* $M(G)$. Thanks to monotonicity the latter define the game. Thus $\{12, 13, 23\}^+$ specifies a game in which the minimal winning coalitions are 12, 13 and 23, where "12" is an abbreviation for the coalition $\{1, 2\}$, etc., and where $\{12, 13, 23\}^+$ is the set comprising the listed coalitions plus all their supersets with respect to $\mathcal{P}$. Four examples of non-games are: the empty set of no coalitions, which is excluded by condition (2); the set of all coalitions, which is excluded by condition (3) since its minimal coalition is the empty coalition; the set $\{12, 13, 14, ...\}^+$, which is excluded by condition (2); and the set $\{S : \overline{S}$ is finite$\}$ (where $\overline{S}$ designates $\mathcal{P}\backslash S$.) Although winning coalitions of infinite size are quite valid, the last example has no minimal winning coalitions and so violates condition (2).

The coalitions in the set $B(G) = \{S : S,\ \overline{S} \notin G\}$ are called the *blocking coalitions* of G. Thus a coalition is blocking when neither it nor its complement win. A coalition *loses* if it does not win or block. Those players who appear in some minimal winning coalition are designated $P(G)$ and called the *players of* G. For an arbitrary game $P(G_i)$ will be designated P_i, and $\#P_i$ will be n_i. It is useful to have notation for sets of coalitions that contain only players of G, so we let $\widehat{G} = \{S : S \subseteq P(G)\ \&\ S \in G\}$ and

$\widehat{B}(G) = \{S : S \subseteq P(G)\ \&\ S \in B(G)\}$. In contrast to G and $B(G)$, these are finite. For an arbitrary game G_i, the sets $B(G_i)$ and $\widehat{B}(G_i)$ will be abbreviated B_i and $\widehat{B}_i$.

A *veto player* is one who is in all of G's coalitions. A game is *proper* if $S,\ S' \in G$ implies $S \cap S' \neq \phi$, that is, if two disjoint coalitions cannot both win. The set of proper simple games is denoted $\mathcal{G}_{Pr}$. (Although improper games are useful, the games discussed here will all be proper.) A *decisive* game is a proper game with $B(G) = \phi$. The *product* of two games is the game $G_1 \cap G_2$ and the *sum* is $G_1 \cup G_2$.

An *automorphism* of a game G is a permutation of $P(G)$ that leaves G invariant. A game G is *anonymous* if for every $i,\ j \in P(G)$ it has an automorphism mapping i into j. Anonymity is weaker than *player symmetry*, which requires that every permutation of $P(G)$ be an automorphism. For example, the game whose players are the vertices of a pentagon and whose minimal winning coalitions are three players in a sequence is anonymous but not symmetrical since only some player triples can win. Games G_1 and G_2 are *isomorphic* if there is a bijection $f : P_1 \to P_2$ such that $\widehat{G}_2 = f(\widehat{G}_1)$, and such a bijection is called an *isomorphism* of the games.

Given proper games G_1 and G_2, their *lexicographic composition* $G_1 \Rightarrow G_2$ (read "G_1 then G_2") is defined as $G_1 \cup (B_1 \cap G_2)$, so that a coalition wins in it if it either wins in G_1, or blocks in G_1 and wins in G_2. (The next section will show that $G_1 \Rightarrow G_2$ is itself a proper game.) A game G is said to be *lexicographically decomposable* (LD) if there exist games $G_1,\ G_2$ with $P_1 \cup P_2 = P$ and $P_1 \cap P_2 = \phi$ such that $G = (G_1 \Rightarrow G_2)$.

Any decisive game G can be written $G \Rightarrow G_2$, but that does not constitute decomposability because the player sets do not partition P. A valid example is a two-party negotiation where a disagreement sends the issue to an arbitrator. The first game is $\{12\}^+$, two negotiators following the unanimity rule, and the second is the single-player unanimity game $\{3\}^+$. Their composition is the 2-of-3 majority game, and thus $\{12, 13, 23\}^+ = (\{12\}^+ \Rightarrow \{3\}^+)$. A more complex example is the American presidential electoral system where a deadlock in the Electoral College sends the choice to the House of Representatives with one vote per state. If the House also deadlocks, what happens next in choosing a president is murky but some experts have construed the rules as implying a longer lexicographic string.

A non-decisive symmetrical majority game, e.g., one requiring 5 votes out of 7, is not LD. This can be seen by supposing the game has such a representation and choosing S as a minimal winning coalition in G_1. Construct S' by replacing one member of S by a player from G_2. Since S wins in G, S and S' are of equal size and the game is symmetrical, then S' wins in G. However, since S' does not win in G_1 it must block in G_1, and its new player must win alone in G_2. However this implies that G is decisive, contrary to the premise. Non-decomposability holds not just for non-decisive symmetrical games but for all non-decisive anonymous games, as shown in Theorem 6.

Another non-LD rule is the United Nations Security Council whose minimal winning coalitions have all 5 permanent members along with exactly 4 out of the 10 non-permanent ones. According to Theorem 4, the existence of a veto player precludes a decomposition. The reason, roughly put, is that if the second game becomes relevant, the overall outcome will depend on whether the first game was blocked by a veto player or by a group of non-veto players, but in lexicographic compositions the details of how the first game was blocked are irrelevant when the decision passes to the second.

Section 3. Basic properties and the five rules of composition

Proposition 1 (Closure). For $G_1, G_2 \in \mathcal{G}_{Pr}$, $(G_1 \Rightarrow G_2) \in \mathcal{G}_{Pr}$.

Proof. If G_1 is decisive then $G = G_1 \cup (B_1 \cap G_2)$ is identical to G_1 and the claim follows. If G_1 is non-decisive, then G satisfies the non-emptiness condition for a simple game since it is a superset of G_1. The monotonicity condition, that for $S \in G$ and $S \subseteq S'$, $S' \in G$, holds since every superset of a blocking set in G_1 is either blocking or winning in G_1, and every superset of a winning set in G_2 is winning in G_2. Properness requires that every pair $S,\ S' \in G$ intersect. If both coalitions are in G_1 or both in G_2 this is true since these games are proper, and if $S \in G_1$ and $S' \in G_2$, then since S' wins in G it blocks in G_1 and intersects all of that game's winning coalitions. $\boxdot$

Proposition 2. For G_1, $G_2 \in \mathcal{G}_{Pr}$, $B(G_1 \Rightarrow G_2) = B(G_1) \cap B(G_2)$.

This follows directly from the definition of composition. Note that the blocking coalitions are the same as for the sum game $G_1 \cup G_2$.

Theorem 1. The following two propositions are equivalent
(1) The binary operator $\circ$, from $\mathcal{G}_{Pr} \times \mathcal{G}_{Pr}$ to $\mathcal{G}_{Pr}$, is such that whether S wins in $G_1 \circ G_2$ depends only on whether S wins, blocks, or loses in G_1 and G_2.
(2) $G_1 \circ G_2$ is one of the following:

(i) the *lexicographic* rule $G_1 \Rightarrow G_2$, (or its reversal $G_2 \Rightarrow G_1$);

(ii) the *degenerate* rule G_1, (or its reversal G_2);

(iii) the *tally* rule $(G_1 \cap G_2) \cup (G_1 \cap B_2) \cup (B_1 \cap G_2)$ in which a coalition wins by winning in more component games than its complement;

(iv) the *veto* rule $G_1 \cap (G_2 \cup B_2)$ in which a coalition wins by winning in the first component game and not losing in the second, (or its reversal $G_2 \cap (G_1 \cup B_1)$);

(v) the *product* rule $G_1 \cap G_2$.

Proof: To show that (2) is sufficient for (1) note that the rules listed depend are functions of membership in B_i and G_i and so depend only on whether a coalition wins, blocks or loses in each component. Proposition 1 stated that the lexicographic rule yields a proper game and the same can be shown in a analogous way for the other rules.

To show that (1) implies (2) note that any such operator can be depicted as in Figure 1, left, whose cells show the outcomes for $G_1 \circ G_2$ for all possible outcomes in the components. The possible outcomes, a coalition winning, blocking or losing are designated $W,\ B$, and L (which here stand for events rather than sets of coalitions.) The diagonal cells must have the values shown: the $W - W$ cell must be W by the first and second conditions of the definition of a simple game; the $L - L$ cell must be L because of the first and third conditions of the definition; and the $B - B$ cell must be B since otherwise the substitution of S for $\overline{S}$ would show that the composition game is improper. The off-diagonal cells are subject to the following restrictions, which are generated by switching S and $\overline{S}$: if a (respectively $b,\ c$) is W then f (respectively $e,\ d$) is L; if a $(b,\ c)$ is B then f $(e,\ d)$ is B; if a $(b,\ c)$ is L then f $(e,\ d)$ is W.

These considerations imply that an operator is fully specified by the values of $a,\ b$, and c. The next step is to show that the monotonicity of the game $G_1 \circ G_2$ implies that those values are ordered as in Figure 1 (left), where an arrow $a \rightarrow b$ indicates that a is equal to or higher than b and where it is taken that a W is higher than a B is higher than an L.

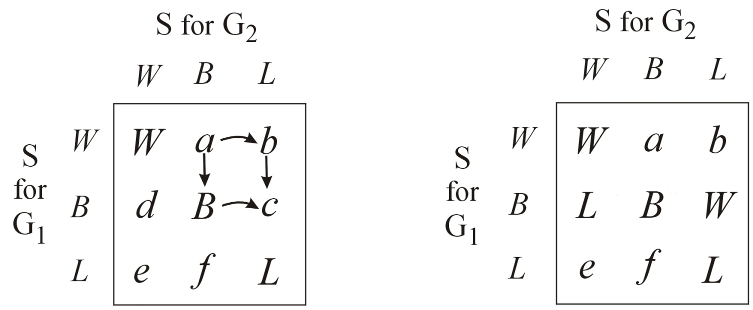


Figure 1: The ordering conditions for an operator, and an example that violates them.

The arrow indicating that "B" be equal to or higher than "c" will be derived as an example. An example of an operator that violates it has $c = W$ and therefore $d = L$ as in Figure 1 (right). Choose G_1 and G_2 as non-decisive games with disjoint player sets and choose S as a coalition that blocks in both. By the table S blocks in $G_1 \circ G_2$. The coalition $S \cup P(G_2)$ blocks in G_1 and wins in G_2. It must block or win in $G_1 \circ G_2$ since it is a superset of S and $G_1 \circ G_2$, as a game, is monotonic, but the table assigns L to the $B - W$ cell. The configuration of Figure 1 (right) is thus impossible. Similar arguments can be made for the other three arrows.

There are exactly eight orders consistent with the partial order in Figure 1. This can be established by constructing a tree where the possiblilities for a branch to the possibilities for b, etc. The result is the five rules and their three reversals. $\boxdot$

Their tables are shown below, where the tables of the reversals of the lexicographic, degenerate and veto rules are understood as the transposes.

	W	B	L
W	W	W	W
B	W	B	L
L	L	L	L

Lexicographic

W	W	W
B	B	B
L	L	L

Degenerate

W	W	B
W	B	L
B	L	L

Tally

W	W	B
B	B	B
B	L	L

Veto

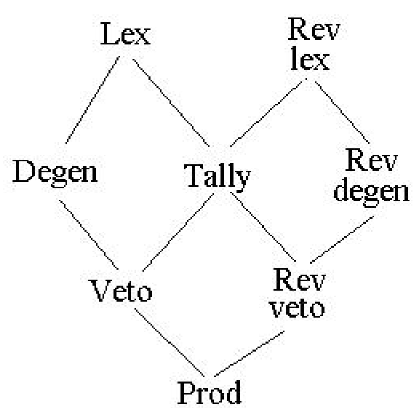
W	B	B
B	B	B
B	B	L

Product

Tables 1-5.

One composition rule is said to be *more decisive than* another if the set of coalitions that win under the former are a superset of those that win under the latter. For example, the lexicographic rule is more decisive than the veto rule, since winning in G_1 is sufficient in the former but is only one of two requirements in the latter.

Proposition 3. The following partial order gives the relative decisiveness of the eight rules:



Proof: Relative decisiveness is determined by the inclusion relations among the sets of W cells of the above tables. $\boxdot$

Shapley (1967) defined a *committee* of a game G as a simple game G' with $P(G') \subseteq P(G)$ such that for any $S \subseteq P(G)$, if $S \backslash P(G') \notin G$ and $S \cup P(G') \in G$, then $S \in G$ if and only if $S \cap P(G') \in G'$. That is, whenever players of G' are crucial to a coalition's success in G, it is because they win in G'. In the lexicographic case, for example, G_1 is not a committee because some of its members can produce a win in G by blocking in G_1 even though they do not win in G_1.

Proposition 4. In the degenerate and product rules both games are committees, in the veto rule only the first game is a committee, in the lexicographic rule only the second game is a committee, and in the tally rule neither game is a committee.

Proof: The lexicographic case will be sufficient to show the form of the argument. To show that G_1 is not a committee: since G is LD, G_1 possesses a blocking coalition, say K. Letting $S = K \cup P_2$, we have $S \backslash P_1 \notin G$, $S \cup P_1 \in G$, and $S \in G$, but $S \cap P_1 \notin G_1$. Next, to show that G_2 is a committee: if $S \backslash P_2 \notin G$ and $S \cup P_2 \in G$, then $S \cap P_1$ blocks in G_1, so that S wins in G if and only if $S \cap P_2$ wins in G_2. ⊡

Proposition 5. The lexicographic and product rules are associative, but the degenerate, tally and veto rules are not.

Proof: In the lexicographic case it is required to prove that for $G_1, G_2, G_3 \in \mathcal{G}_{Pr}$, $[(G_1 \Rightarrow G_2) \Rightarrow G_3] = [G_1 \Rightarrow (G_2 \Rightarrow G_3)]$. The left side is $G_1 \cup (B_1 \cap G_2) \cup [B(G_1 \Rightarrow G_2) \cap G_3]$ and the right side is $G_1 \cup [B_1 \cap (G_2 \cup (B_2 \cap G_3))]$. Applying Proposition 2 to the former and expanding the right side of the latter as a union shows that both are identical to $G_1 \cup (B_1 \cap G_2) \cup (B_1 \cap B_2 \cap G_3)$. The associativity of the the product follows directly from its definition. The tally rule (W, B, L) is not associative since $(W \circ W) \circ L = W \circ L = B$ (using the obvious notation), whereas $W \circ (W \circ L) = W \circ B = W$. The veto rule (W, B, B) is not associative because $(W \circ W) \circ L = W \circ L = B$ whereas $W \circ (W \circ L) = W \circ B = W$, and similarly for the reversals. ⊡

Proposition 6. The tally and product rules are commutative, but the lexicographic, degenerate and veto rules are not.

Proof: Commutativity of an operator is equivalent to symmetry of its table under an exchange of rows and columns. ⊡

Proposition 6 states that $G_1 \Rightarrow G_2$ and $G_2 \Rightarrow G_1$ can be different, but Proposition 5 implies that $G_1 \Rightarrow G_2 \Rightarrow ... \Rightarrow G_n$ is well-defined. The latter is an attractive property since it implies that committee decisions can be sequenced without the complexity of subgroups. The product rule, which is prevalent in actual use, is even simpler because it is both associative and commutative, but it is at the bottom on decisiveness while the lexicographic rule is at the top. These are practical reasons for considering the latter in the appropriate situations.

Section 4. The advantage of being in the first component

One criterion for choosing a procedure is how it allocates power. A certain group of players may deserve more power either due to their expertise or their ethical right to make the decision. Other things equal, does the lexicographic procedure grant more power to those in the first game or in the second? Having the last word sounds attractive, but the next theorem states that when the two component games are isomorphic and when a player's benefit is evaluated by any of a broad class of measures, the advantage goes to the first group.

The class is the semi-values and defining it requires the concept of a game *in coalitional function form*, in which a function v assigns a real number to each subset of players. (The relation of such games to simple games is straightforward: a simple game's *coalitional function* assigns 1 to a coalition if it is in G and 0 otherwise; the resulting coalitional function game v *corresponds to* the simple game G.) A semi-value ψ associates with each v an *additive game* ψv (one satisfying $v(S \cup S') = v(S) + v(S')$ for $S \cap S' = \phi$.) This is equivalent to assigning to each player i the real number $\psi v(\{i\})$, which can be interpreted as i's benefit from playing v. A *semi-value* is a ψ with these four properties:

Linearity: If $v = u + w$, then $\psi v = \psi u + \psi w$.

S*ymmetry*: For any game v and permutation π of players, $\psi(\pi v) = \pi(\psi v)$. (The game πv is defined as the one assigning to coalition S the value $v(\pi^{-1}S)$.)

Monotonicity: If v is monotonic, then ψv is monotonic.(Monotonicity here requires that for all $S \subseteq S', v(S) \leq v(S')$.)

Dummy: If $v(S \cup \{i\}) = v(S) + v(\{i\})$ for $i \notin S$, then $\psi v(\{i\}) = v(\{i\})$.

Theorem 2: Let G_1 and G_2 be isomorphic games with $P_1 \cap P_2 = \phi$, let f be a bijection from P_1 to P_2 such that $\widehat{G}_2 = f(\widehat{G}_1)$, and let v be the coalitional function game corresponding to $G = (G_1 \Rightarrow G_2)$. For any semi-value ψ and $i \in P_1$, $\psi v(\{i\}) \geq \psi v(\{f(i)\})$.

Proof: Construct the bijection $g : P(G) \to P(G)$ such that $g(i) = f(i)$ for $i \in P_1$, and $g(j) = f^{-1}(j)$ for $j \in P_2$. Then for $j \in P_2$,
$\psi v(\{j\}) = \sum_{S \subseteq P(G)\backslash\{j\}} p(\#S)\ [v(S + \{j\}) - v(S)]$, where $p(\#S)$ is a probability vector (Dubey, Neyman and Weber 1981). Let $i = g(j)$. Then
$\psi v(\{i\}) = \sum_{S \subseteq P(G)\backslash\{j\}} p(\#S)\ [v(g(S) + \{i\}) - v(g(S))]$. The claim is that for $S \subseteq P(G)\backslash\{j\}$, $v(S + \{j\}) - v(S) = 1$ implies $v(g(S) + \{i\}) - v(g(S)) = 1$. The first equality implies $S + \{j\} \in B_1 \cap G_2$ whereas $S \notin G_2$. Therefore $g(S) + \{i\} \in G_1 \cap B_2$ whereas $g(S) \notin G_1$. The definition of $G_1 \Rightarrow G_2$ completes the proof. ⊡

The *Banzhaf score* of player i in a game G is defined as
$Bsc(i, G) = \#\{S :\ S \in \widehat{G}\ \&\ S\backslash i \notin \widehat{G}\}$, that is, the number of coalitions of players of G that contain i and need i to order to win. The *Banzhaf measure* is i's Banzhaf score normalized by the number of coalitions containing i, $Banz(i, G) = Bsc(i, G)/2^{n-1}$. It is meant to assess the player's power and its calculation for decomposable games shows why being in the second group makes one weaker.

Theorem 3:
$$\begin{aligned} Banz(i, G_1 \Rightarrow G_2) &= Banz(i, G_1) && \text{for } i \in P_1, \\ &= Banz(i, G_2)\ \#\widehat{B}(G_1)/2^{n_1} && \text{for } i \in P_2. \end{aligned}$$

Proof: For a game H and $i \in P(H)$, define $W(i, H) = \{S : i \in S \in \widehat{H}\}$ and $B(i, H) = \{S : i \in S \in \widehat{B}(H)\}$. Then $Bsc(i, H) = 2\ \#W(i, H) - \#\widehat{H}$ (see Dubey and

Shapley, 1979). For $G = (G_1 \Rightarrow G_2)$ and $i \in P_1$,
$Bsc(i,G) = 2\ [2^{n_2}\ \#W(i,G_1) + \#B(i,G_1)\ \#\widehat{G}_2] - 2^{n_2}\ \#\widehat{G}_1 - \#\widehat{B}(G_1)\ \#\widehat{G}_2$. Since $\#\widehat{B}(G_1) = 2\ \#B(i,G_1)$, then $Bsc(i,G) = 2^{n_2}[2\ \#W(i,G_1) - \#\widehat{G}_1]$. Dividing both sides by $2^{n_1+n_2-1}$ yields $Banz(i,G) = Banz(i,G_1)$, the first formula of the theorem. For $i \in P_2$,
$Bsc(i,G) = 2\ [\#\widehat{B}(G_1)\ \#W(i,G_2) + 2^{n_2-1}\ \#\widehat{G}_1] - 2^{n_2}\ \#\widehat{G}_1 - \#\widehat{B}(G_1)\ \#\widehat{G}_2$. Hence
$Bsc(i,G) = \#\widehat{B}(G_1)\ [2\ \#W(i,G_2) - \#\widehat{G}_2]$. Dividing by $2^{n_1+n_2-1}$ yields the second formula.
⊡

The value $\#\widehat{B}(G_1)$ can be interpreted as G_1's lack of decisiveness since it is twice the minimum number of changes required in the game's coalitional function to make it decisive. Then $\#\widehat{B}(G_1)/2^{n_1}$ can be understood as indecisivness normalized according to the size of the game. It is the probability that a blocking coalition forms, if each player is in or out of the coalition independently with probability 1/2. The theorem thus indicates that the Banzhaf measure discounts a player's power for going second and this discounting is greater the more decisive that G_1 is.

Section 5. The smallest LD games

Simple games of a fixed player set are finite in number, so they can be listed (e.g., von Neumann and Morgenstern 1944, Shapley 1962), and this section gives some of this "descriptive" theory. The counts of small LD games are in Table 6, whose entries refer to isomorphism classes, i.e., games unique up to permutations of the players. In line with the present definition of the player set, games with "dummies" are not included. The first column was generated by a computer program available from the first-listed author, which produced all possible games with players distinguished and then eliminated isomorphic duplicates in a fairly efficient way. The second column is calculated from Table IV of Loeb and Conway (2000). The LD games were generated by combining all pairs of smaller component games and eliminating duplicates. For completeness the Table should have a final column for games that are LD and anonymous, but in view of Theorem 6 below, it would have the same entries as the decisive and anonymous counts, for $n > 1$.

n	all	anonymous	decisive	dec & anon	LD
1	1	1	1	1	0
2	1	1	0	0	0
3	3	2	1	1	1
4	9	2	1	0	2
5	69	4	4	1	8
6	3441	13	23	1	42

Table 6. Numbers of isomorphism classes of games.

The forms of decomposable games for n up to 6 are listed in Table 7, which shows a weighted quota representation if the game has one, or otherwise gives its minimal winning coalitions, and then gives some selected decompositions. To save space, it lists only those six-player decisive games whose second game involves two or more players. There are 20 further games that have only representations $G_1 \Rightarrow U_1$, and these can be generated from a consulting a listing of the six-player games (e.g., Muroga *et al.*, 1962) and separating out an arbitrary player, as in Theorem 4 below.

The table uses these further definitions:

$M_{n,k}$: the *k-of-n majority game*.

U_n : the *unanimity game* of n players, $M_{n,n}$.

C_n : the *chief player game* of n players (von Neumann and Morgenstern, 1944), in which a distinguished player plus one other wins (non-decisive.)

A_n : the *apex game* of n players, in which a distinguished "apex" player plus one other wins, and the coalition of all non-apex players wins (decisive.)

$k|\overline{w}$ or $k|w_1w_2...w_n$: the *weighted majority game* of n players with quota k and weight vector $\overline{w}$ in which a coalition wins if its total weight is at least the quota.

<u>3-player, decisive (1)</u>

$3|111\ (M_{3,2}) : U_2 \Rightarrow U_1$

<u>4-player, non-decisive (1)</u>

$4|2211 : U_2 \Rightarrow U_2$

<u>4-player, decisive (1)</u>

$3|2111\ (A_4) : U_3 \Rightarrow U_1,\ C_3 \Rightarrow U_1$

<u>5-player, non-decisive (4)</u>

$6|33111 : U_2 \Rightarrow U_3$

$\{123, 145, 245, 345\}^+ : U_3 \Rightarrow U_2$

$6|33211 : U_2 \Rightarrow C_3$

$6|42211 : C_3 \Rightarrow U_2$

<u>5-player, decisive (4)</u>

$3|11111\ (M_{5,3}) : M_{4,3} \Rightarrow U_1$

$4|31111\ (C_5) : U_4 \Rightarrow U_1,\ C_4 \Rightarrow U_1$

$4|22111 : 4|2111 \Rightarrow U_1,\ U_2 \Rightarrow U_2 \Rightarrow U_1$

$5|32211 : 5|2211 \Rightarrow U_1,\ 5|3211 \Rightarrow U_1,\ 5|3221 \Rightarrow U_1$

<u>6-player, non-decisive (19)</u>

$\{1234, 156, 256, 356, 456\}^+ : U_4 \Rightarrow U_2$

$6|222211 : M_{4,3} \Rightarrow U_2$

$8|622211 : C_4 \Rightarrow U_2$

$8|442211 : U_2 \Rightarrow U_2 \Rightarrow U_2, U_2 \Rightarrow 4|2211$

$\{12, 134, 156, 23456\}^+ : 5|3211 \Rightarrow U_2$

$\{123, 124, 156, 256, 3456\}^+ : 5|2211 \Rightarrow U_2$

$10|644211 : 5|3221 \Rightarrow U_2$

$8|422211 : 4|2111 \Rightarrow U_2$

$9|633211 : C_3 \Rightarrow C_3$

$\{123, 145, 146, 245, 246, 345, 346\}^+ : U_3 \Rightarrow C_3$

$9|633111 : C_3 \Rightarrow U_3$

$\{123, 1456, 2456, 3456\}^+ : U_3 \Rightarrow U_3$

$8|441111 : U_2 \Rightarrow U_4$

$6|331111 : U_2 \Rightarrow M_{4,3}$

$8|443111 : U_2 \Rightarrow C_4$

$10|553211 : U_2 \Rightarrow 5|3211$

$10|552211 : U_2 \Rightarrow 5|2211$

$10|553221 : U_2 \Rightarrow 5|3221$

$8|442111 : U_2 \Rightarrow 4|2111$

<u>6-player, decisive, with $\#P_2 \geq 2$ (3)</u>

$6|332111 : U_2 \Rightarrow U_3 \Rightarrow U_1$

$6|422111 : C_3 \Rightarrow M_{3,2}$

$\{123, 145, 245, 345, 146, 156, 246, 256, 346, 356\} : U_3 \Rightarrow U_2 \Rightarrow U_1$

Table 7. LD games for small n.

Some games have several decompositions. In some cases one can split a component further, as in $8|442211$, and in others one can divide a non-homogeneous player set in different ways, as in $3|2111$. The latter case shows that, unlike game sums and products, there is no unique lexicographic factorization into "prime" games.

Section 6. Conditions for decomposability

Theorem 4. A proper game with a veto player is not LD.

Proof: Assume that G has a veto player v and a decomposition $G_1 \Rightarrow G_2$. Then $v \in P_1$, otherwise the coalition P_1 could win in G without v. The coalition $P_1 - \{v\}$ blocks in G_1, since otherwise v would be a dictator, violating the non-triviality requirement $P = P_1 \cup P_2$. Since $(P_1 - \{v\}) \cup P_2$ blocks in G_1 and wins in G_2, it wins in G, contradicting the premise that v has a veto. $\boxdot$

The next theorem shows that any decisive game of more than one player is LD and that weighted majority games have an especially easy representation.

Theorem 5. Let G be a decisive game with $n \geq 2$, let $i \in P(G)$, and let G_{-i} be the $(n-1)$-person game whose minimal winning coalitions are those of G omitting any containing player i. Then $G = (G_{-i} \Rightarrow \{i\}^+)$. Let $G = k|\overline{w}$ be a decisive weighted majority game and let $\overline{w}_{-i}$ be the weight vector w with w_i omitted. Then $G = (k|\overline{w}_{-i} \Rightarrow \{i\}^+)$.

Proof. Since G_{-i} and $k|\overline{w}_{-i}$ are non-empty, monotonic and proper, the compositions are well-defined. Regarding the first claim, G being decisive and a coalition blocking in G_{-i} imply that it is the player left out of G_{-i} who renders the coalition non-blocking in G. Similarly in the second claim, failing to attain the quota k in $k|\overline{w}_{-i}$ means that the dropped player holds the weight that would put the coalition over the quota. $\boxdot$

Theorem 6: Given that a proper simple game G is anonymous, it is LD if and only if it is decisive and $n \geq 2$.

Proof: Given anonymity, it is clear that decisiveness and two or more players imply LD, since G can be expressed $G_1 \Rightarrow U_1$, with G_1 constructed as in Theorem 5. To show the other direction of implication, assume that $G = (G_1 \Rightarrow G_2)$ is anonymous with $n \geq 2$ but not decisive, to generate a contradiction. This implies that G_2 is also non-decisive. For $i = 1, 2$ define $w_i = min\{\#S : S \in G_i\}$ and $b_i = min\{\#S : S \in B_i\}$. We proceed in several steps.

1) Let $T_1 \in \widehat{B}_1$ and π be an automorphism of G. If $\pi(T_1) \subseteq P_1$ then $\pi(T_1) \in \widehat{B}_1$. Indeed, choose $T_2 \in \widehat{B}_2$ such that $\#T_2 = b_2$. Then $T_1 \cup T_2 \in \widehat{B}$. Hence $\pi(T_1) \cup \pi(T_2) \in \widehat{B}$. As $\pi(T_1) \subseteq P_1$ and $\#(\pi(T_2) \cap P_2) \geq b_2$, we may conclude that $\pi(T_2) \subseteq P_2$. Hence $\pi(T_1) \in \widehat{B}_1$.

2) We now show that $w_1 \geq b_1 + w_2$. Let $S_1 \in \widehat{G}_1$ such that $\#S_1 = w_1$, let $i \in S_1$ and $j \in P_2$.There exists an automorphism π of G such that $\pi(i) = j$. Since $\pi(S_1) \in \widehat{G}$, it follows that $w_1 \geq b_1 + w_2$.

3) We now claim that for every $k \in P_2$ there exists $S_2 \in \widehat{G}_2$ such that $k \in S_2$ and $\#S_2 = w_2$. Let $\widetilde{S}_2 \in \widehat{G}_2$ with $\#\widetilde{S}_2 = w_2$, $\widetilde{k} \in \widetilde{S}_2$ and $T_1 \in \widehat{B}_1$ such that $\#T_1 = b_1$, $T_1 \cup \widetilde{S}_2 \in \widehat{G}$. Let π be an automorphism of G such that $\pi(\widetilde{k}) = k$. By 2), $S_2 = \pi(T_1 \cup \widetilde{S}_2) \cap P_2$ has all the desired properties.

4) Let, again, $S_1 \in \widehat{G}_1$ such that $\#S_1 = w_1$, let $i \in S_1$ and $j \in P_2$. There exists an automorphism π of G such that $\pi(i) = j$, $\pi(S_1) \cap P_1 \in \widehat{B}_1$. Let $T_1 = \pi^{-1}(\pi(S_1) \cap P_1)$. By 1), $T_1 \in \widehat{B}_1$. We distinguish now the following cases:

4a) For some $k \in P_2$, $\pi(k) \in P_2$. By 3) we may choose $S_2 \in \widehat{G}_2$ satisfying $\#S_2 = w_2$ and $k \in S_2$. $T_1 \cup S_2 \in \widehat{G}$, hence $\pi(S_2) \in \widehat{G}_2$. But this implies that $\pi(S_1)$ and $\pi(S_2)$ are disjoint winning coalitions of G_2, a contradiction.

4b) $\pi(P_2) \subseteq P_1$. Choose $T_2 \in \widehat{B}_2$. Then $T_1 \cup T_2 \in \widehat{B}$, but $\pi(T_1) \cup \pi(T_2) \subseteq P_1$, which is impossible. ⊡

Section 7. Decomposability and equality

A central issue of democratic theory is how citizens can participate in their governance in a way that is both equitable and efficient. On considerations of pure democracy every issue should be debated by everyone, but limitations of time mean that most decisions are assigned to elected leaders or bureaucrats. Lexicographic systems can save time since they often bypass some of the players, and this section discusses whether they can be designed to be equitable as well. The mathematical interpretation of equitability will be anonymity, that all players have the same role in the game. This is very strict criterion and weaker ones like assigning equal power according to some measure would be easier to satisfy. Note that "equitable" is generally different than "fair", since certain parties sometimes have a right to a special role, so this analysis would be relevant only when parties were considered a priori equal.

The problem is to find games that are both LD and anonymous. One method starts with two anonymous games, one of them decisive and the other LD, and forms a single LD anonymous game. Composition by the quotient method is defined as follows. Let G be a game with $P(G) = \{1, ...n\}$ and let H_i, $i = 1, ...n$, be games with disjoint player sets. The *composition* game using *quotient* G and *components* H_i is $G[H_1, ...H_n]$ where $S \in G[H_1, ...H_n]$ iff there exists $T \in \widehat{G}$ such that $S \in \cap_{i \in T} H_i$ (Shapley, 1967). A coalition wins in the large game if it wins in a winning set of component games, as if the latter were individual players.

Proposition 7: Let G be LD and representable as $(G_1 \Rightarrow G_2)$, with $P(G) = \{1, ...n\}$, $P(G_1) = \{1, ...k\}$ and $P(G_2) = \{k+1, ...n\}$. If H_i, $i \in P(G)$, are decisive games with disjoint player sets then $G[H_1, ... H_n] = (G_1[H_1, ...H_k] \Rightarrow G_2[H_{k+1}, ...H_n])$. If H_i are isomorphic to each other and G and H_1 are anonymous then $G[H_1, ...H_n]$ is anonymous.

Proof: To prove the first claim let $S \in G[H_1, ...H_n]$. Then there exists $T \in \widehat{G}$ such that $S \in \cap_{i \in T} \ H_i$. If $T \in \widehat{G}_1$, then $S \in G_1[H_1, ...H_k]$. Otherwise $T = B_1 \cup T_2$, where $B_1 \in \widehat{B}(G_1)$ and $T_2 \in \widehat{G}_2$. As $S \in \cap_{i \in B_1} H_i$, it is blocking in $G_1[H_1, ...H_k]$. As $S \in \cap_{i \in T_2} H_i$, it is winning in $G_2[H_{k+1}, ...H_n]$. Hence S is winning in $G_1[H_1, ...H_k] \Rightarrow G_2[H_{k+1}, ...H_n]$.

Conversely, let $S \in G_1[H_1, ...H_k] \Rightarrow G_2[H_{k+1}, ...H_n]$. If $S \in G_1[H_1, ...H_k]$ then $S \in G[H_1, ...H_n]$. If $S = \widetilde{B}_1 \cup S_2$ where $\widetilde{B}_1$ is blocking in $G_1[H_1, ...H_k]$ and $S_2 \in G_2[H_{k+1}, ...H_n]$, then there exist $B_1 \in \widehat{B}(G_1)$ and $T_2 \in \widehat{G}_2$ such that $S \in (\cap_{i \in B_1} H_i) \cap (\cap_{i \in T_2} H_i)$. (Here we use the assumption that $H_1, ...H_n$ are decisive.) As $B_1 \cup T_2 \in \widehat{G}$, then $S \in G[H_1, ... H_n]$.

For the second claim, which was noted by Loeb and Conway (2000, 401), let f_{ij} be an isomorphism between H_i and H_j. For every π_1, π_2, automorphisms of G and H_1

respectively, we define $\pi = \pi(\pi_1, \pi_2) : \cup_{i=1}^n P(H_i) \to \cup_{i=1}^n P(H_i)$ by the following rule: let $j \in P(H_i)$; then $\pi(j) = f_{1,\pi_1(i)} \circ \pi_2 \circ f_{1i}^{-1}(j)$. As π is an automorphism of $G[H_1, ...H_n]$ that game is transitive. ⊡

A simple application of this proposition takes the quotient as $M_{3,2}$, which has the decomposition $M_{3,2} = (U_2 \Rightarrow U_1)$, and also uses $M_{3,2}$ as component games. This produces the identity $M_{3,2}[M_{3,2},\ M_{3,2},\ M_{3,2}] = (U_2[M_{3,2},\ M_{3,2}] \Rightarrow M_{3,2})$, which can be seen either as three subcommittees voting simultaneously or as two voting and the third committee resolving ties. By construction not only is the composition game anonymous but so are the two lexicographic components.

Another way to find decompositions of anonymous games is to start with any decisive anonymous game and decompose it. (Theorem 6 states that decisiveness is necessary and sufficient in this context.) There are many such starting games and for the special interest of their simplicity we look at only those that are not only anonymous themselves but yield anonymous components. The simplest examples are decisive majority games, so that $M_{7,4} = (M_{6,4} \Rightarrow U_1)$ for example. Another instance involves the 7-player Fano game of Figure 2 (von Neumann and Morgenstern, 1944). Deleting the point in the center of the triangle produces a structure equivalent to the six-person Octahedral Game Oct_6, defined as the game with players at the vertices and with minimal winning coalitions as the four faces shaded in the figure. The latter game is proper since every pair of shaded faces intersect, and it is clearly anonymous. Thus $Fano_7 = (Oct_6 \Rightarrow U_1)$. The operation can also be understood from the opposite direction, as a composition starting with Oct_6 and adding a seventh player in the middle of the octahedron, who wins by combining with any two diagonally opposite players. Since the diagonally opposite pairs are the only minimal blocking coalitions of Oct_6 the new game is decisive, and itcan be verified to be the Fano game.

A third such example starts with the game whose players lie at the vertices of a pentagon and whose minimal winning coalitions are any three in a row. It can be called $Pent_5$. It is not decisive, since two non-adjacent players can block. Adding a sixth player who wins in combination with any of the blocking pairs generates $\{123,\ 234,\ 345,\ 145,\ 125,\ 136,\ 146,\ 246,\ 256,\ 356\}^+$. This is an interesting game first

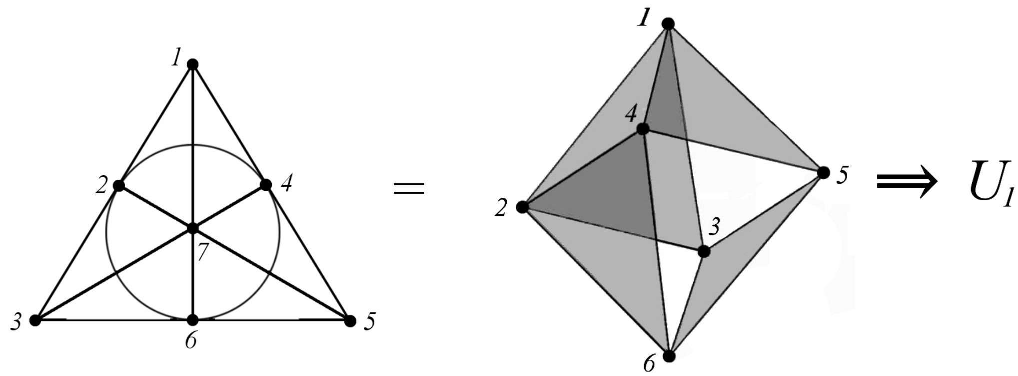


Figure 2: Fano_7 decomposes to the Octahedral game then the one-player game.

identified by Gurk and Isbell (1959) as the unique smallest game that is anonymous and decisive with an even number of players. A representation that brings out its anonymous character was offered by Zvonkin (reported by Loeb and Conway, 2000) who put the 6 players at the 12 vertices of a icosahedron with diagonally opposite vertices assigned to the same player and defined the minimal winning coalitions as any three players on the same (triangular) face. Accordingly it can be called the six-player icosahedral game $Icos_6$, and we have $Icos_6 = (Pent_5 \Rightarrow U_1)$.

A larger example involves the game based on the 11-point Steiner system $St(4, 5, 11)$, which is unique for its parameters. It has 11 players, 66 minimal winning coalitions of 5 players each with the property that any 4-tuple or players appears exactly once. It decomposes as $St(4, 5, 11) = (G_1 \Rightarrow U_1)$ where G_1 has minimal winning coalitions based on a 10-player design: 36 sets of 5 players each in which each triple of players appears exactly three times.

Anonymous decisive games like these, that decompose as $G_1 \Rightarrow U_1$ with G_1 anonymous, are relatively few because their automorphism group must satisfy a restrictive property. A game is said to be *k-transitive* if for any two ordered k-tuples of players there exists an automorphism carrying the first k-tuple into the second. Clearly, for $k \geq 2$

k-transitivity implies $(k-1)$-transitivity, and 1-transitivity is just anonymity. According to the proposition below the operation described here requires at least 2-transitivity, since separating a player from a game that is not 2-transitive, like $Pent_5$, distinguishes some of the players left behind and the first component will not be anonymous.

Proposition 8. Let G be an anonymous proper game that is LD as $G_1 \Rightarrow U_1$. Then G_1 is anonymous if and only if G is 2-transitive.

Proof: Let G's automorphism group be $(\Pi, \{1, ...n\})$, where Π are permutations that act on the player set $\{1, ...n\}$. This has a subgroup $(\{\pi \in \ \Pi : \pi(n) = n\}, \{1, ...n\})$, and the group $(\Pi_1, \{1, ...n-1\})$, defined as the latter's restriction to $\{1, ...n-1\}$ will be G_1's automorphism group. The theorem is true if the following holds (where transitivity has the obvious meaning applied to groups rather than games): if Π is 1–transitive then Π_1 is 1-transitive if and only if Π is also 2-transitive. The forward direction of the implication was shown by Ledermann (1973). To show the backward direction note that the 2-transitivity of G implies that any pair (i, n) and (j, n) both lie in some orbit, which implies G_1's anonymity. $\boxdot$

An enumeration of small anonymous decisive games (Loeb and Conway, 2000) shows how constraining 2-transitivity is. For $n = 1$ to 12 there are 57,317 types of such games up to isomorphism but only 11 of them are 2-transitive. They are the six simple majority games with odd numbers of players, as well as the three games just discussed - the Icosahedral, Fano and Steiner quadruple system games - and finally two 10-player games, each of which has two non-isomorphic kinds of winning coalitions and so is more complicated than the previous ones. Thus the search yields a few interesting types but stipulating that the components be anonymouse is quite restrictive.

Section 8. Lexicographic composition with deference.

A lexicographic system can give more relative power to some parties, which is justifiable when they are experts or they have a normative claim to be making the decision. For example, expertise and legitimacy are both significant in the design of arbitration procedures, since the principal parties should know better what is good for them and they have a special right to make a decision about their lives. In the US presidential case ties in the Electoral College are broken by the House of Representatives. While the power indices for the combination of the two bodies do not seem to have appeared in the literature, it seems likely in light of Section 4 that the former body has far more power from possessing the first move and having a decisive rule. The arrangement plausibly reflects a greater right of the Electoral College to choose a president because it was selected by the citizenry just for that purpose.

The way to confer more power is to put a group in the first position and make that game relatively decisive. There is an irony in this, however: suppose the first game requires a 2/3 majority, and that a coalition achieves only 60% of the first group and blocks in the second. Even though the first group has expressed a preference this fact is discarded and the result is left undecided. If the first group has a special claim to decide, then the rules should give them some degree of deference after a block in the second group. A system that includes such deference was used at the 1984 Pan-American Games (Shapley, n.d.). The *New York Times* reported that an American boxer was "grim and baffled" about why he had lost the match (Litsky 1984). The rules used were: a main panel of five expert judges sat at ringside and voted for one fighter or the other. A 5-0 or 4-1 vote settled it but a 3-2 split sent the matter to another five-person panel where a majority of 5-0 or 4-1 settled it. However a 3-2 split in either direction in the back up panel gave the match to whichever fighter had won 3-2 in the first panel.

This kind of rule can be defined using the lexicographic operator. A game of *lexicographic composition with deference* is defined as a proper simple game expressible as $G_1 \Rightarrow G_2 \Rightarrow G_1'$ where $G_1,\ G_2,\ G_1',\ \in \mathcal{G}_{Pr},\ P(G_1) \cap P(G_2) = \varnothing$ and $G_1 \subseteq G_1'$. The boxing rules are then $M^1_{5,4} \Rightarrow M^2_{5,4} \Rightarrow M^1_{5,3}$ where the superscripts specify the player sets, $P_1 = \{1, 2, 3, 4, 5\}$ being the first set of judges and $P_2 = \{6, 7, 8, 9, 10\}$ the backup.

The following proposition indicates another way of expressing such games. Its proof is straightforward.

Proposition 9: If G is a game of lexicographic composition with deference, representable as $G_1 \Rightarrow G_2 \Rightarrow G_1'$, then $G = G_1 \cup (G_2 \Rightarrow G_1')$.

The deference construction was meant to generate anonymity. In the boxing rules the first group has 68% of the power, according to the Shapley-Shubik index. The method was designed for contexts where fairness requires inequality, but we can still ask whether such rules can be anonymous. Indeed one example are:$U^1_3 \Rightarrow U^2_2 \Rightarrow M^1_{3,2}$, which is equivalent to $M_{5,3}$, and $M^1_{5,4} \Rightarrow U^2_2 \Rightarrow M^1_{5,3}$, which is $M_{7,4}$. (The player sets are $\{1, 2, 3\}$ and $\{4, 5\}$ in the former, and $\{1, 2, 3, 4, 5\}$ and $\{6, 7\}$ in the latter.) These two representations are instances of a general rule stated in the next proposition. A *decisive majority game* M_n is defined as one with an odd number n of players and winning coalitions of size $(n + 1)/2$. Such a game can be decomposed as lexicographic sequences with deference, but the particular way noted involves separating only two players.

Proposition 10. For $n \geq 5$ and odd, $M_n = (M^1_{n-2,(n+1)/2} \Rightarrow U^2_2 \Rightarrow M^1_{n-2})$, where $P_1 = \{1, 2, ...\ n - 2\}$ and $P_2 = \{n - 1,\ n\}$.

Proof. The right hand side has three types of minimal winning coalitions: ${}_{n-2}C_{(n+1)/2}$ coalitions of $(n + 1)/2$ players of P_1, who win in the first component; ${}_{n-2}C_{(n-1)/2}$ coalitions of $(n - 1)/2$ players of P_1 who block in the first component and 2 players from P_2 who win in the second; 2 ${}_{n}C_{(n-1)/2}$ coalitions of $(n - 1)/2$ players of P_1 who block in the first component and win in the third, and 1 player in P_2 who blocks in the second. The three groups are disjoint and their numbers sum to ${}_{n}C_{(n+1)/2}$, which is the number of distinct coalitions of size $(n + 1)/2$,which define the simple majority game. ⊡

Many real decision systems are vaguely like this in that they are "almost" lexicographic. Even when the first group is blocked its views are not completely discarded. Under final offer arbitration a negotiation failure sends the decision to an arbitrator who must choose between only two outcomes, those nominated by each negotiator. In the US presidential system, the House of Representatives must choose from the three candidates who received the highest votes in the Electoral College. Often higher courts have to take the factual judgements of lower courts as given and can reverse them only for legal error. These systems favor the first group by letting it limit the range of choice, but since this goes beyond the dichotomy of outcomes of simple games, they are more readily modeled in the extensive form (e.g., Moulin 1983).

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