ASSORTATIVE MATCHING AND REPUTATION

By

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Assortative Matching and Reputation[*]

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Abstract

Consider Becker's (1973) classic matching model, with unobserved fixed types and stochastic publicly observed output. If types are complementary, then matching is assortative in the known Bayesian posteriors (the 'reputations').

We discover a robust failure of Becker's result in the simplest *dynamic* two type version of this world. Assortative matching is generally neither efficient nor an equilibrium for high discount factors. In a novel labor theoretic rationale, we show that assortative matching fails around the highest (lowest) reputation agents for 'low-skill (high-skill) concealing' technologies. We then find that as the number of production outcomes grows, almost all technologies are of either form.

Our theory implies the dynamic result that high-skill matches eventually break up. It also reveals that the induced information rents create discontinuities in the wage profile. This in turn produces life-cycle effects: young workers are paid less than their static marginal product, and old workers more.

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1 Introduction

This paper began as a joint project by a tenured faculty member and a graduate student. It is reasonable to assume that the reputation of the faculty member was substantially more established than the graduate student's. The Beatles broke up after it was clear that their members were highly talented, and each member went on to match with individuals having unestablished reputations. While in the first example an individual with an established reputation matched with another whose reputation was not established, in the second example a partnership of likes dissolved once reputations were established. In this paper, we argue that beyond the obvious personal match interplay, there are real economic forces at play in these match break-ups. These examples motivate two questions: What static matchings can we expect to see when reputations matter? How will an individual's partners change over time?

Becker's (1973) considered a static Walrasian complete information matching model, where output depended solely on exogenous types. He showed that there is *positive assortative matching* (PAM) — here, everyone matches with a same reputation partner — iff types are productive complements. Extending his work to a static world of uncertainty, complementarity again delivers the same assorting result (our Proposition 0).

What we propose here instead is a twist in the spirit of Becker (1973) — namely, with fixed types where matching decisions are the only choice variables. But we venture that matches quite often yield not only current output, but also information about types. Specifically, this matching concern arises with *both* incomplete information about types *and* time dynamics. Lest the model degenerate, we further assume that output is random. And finally, we posit that output is publicly observable; this both agrees with the economics we have in mind, and ensures that the Welfare Theorems hold, again closely hewing to Becker's model. Thus, it suffices to explore whether PAM constitutes an efficient matching allocation. We later illustrate how publicly observed stochastic output and unobserved skills/talents are quite typically paired features of the world we live in. *Publicly* observed output also induces a reputational concern. For simplicity, we assume just two underlying types, 'good' or 'bad'. Everyone is then summarized by *the public posterior chance that he is 'good'* — his *reputation*.

In this setting, we show that PAM fails despite the globally complementary production, Becker (1973). While we offer some applications below, we believe that our paper in and of itself is an key foundational result for matching papers with incomplete information.[1] A model with public information is the natural first step, and we argue that our result is far subtler than one might think at first. Indeed, a naive view of this failure runs as follows: Informational and productive concerns run at cross purposes, and thus for sufficiently patient agents, the former effect dominates. *This logic fails,* and it is instructive to see why. We show that PAM is informationally *superior* for some ranges of reputations. Tradeoffs are the stuff of interesting economics, and informational and productive efficiency only sometimes oppose each other. For with PAM, output variation owes equally to the two parties, and so both likewise benefit from Bayes updating; however, PAM can *never* be informationally efficient for *all* reputations. We show that if individuals are patient enough, there is some *range* of reputations where information efficiency dominates productive efficiency, and PAM fails.

Determining *why and where* PAM fails is critical for our finding, and fleshing out this subtlety is a key economic and technical contribution of the paper. Indeed, we show that these failures occur for high or low reputation agents, or both. We then characterize when either case obtains depending on the stochastic productive interaction. Finally, we flesh out the implications for matching dynamics: Partnerships of like agents will dissolve as they prove themselves as either highly skilled — or not so. Matches of unlike agents, like tenured faculty with student coauthors, then form.

While our PAM failure is a robust finding, the tradeoff between information and productive efficiency becomes knife-edged as the patience rises. For in that case, the average present value of both current output and newly acquired information falls. The resulting horse race between information and productive efficiency is almost too close to call. It requires a *very* precise characterization of the extremal behavior of the value function. This paper turns on this new property of optimal learning.

PAM Failure in a Two Period Model. Our analysis of the PAM failure begins with a two period model. Becker's result yields PAM in the final period. This, in turn, yields a *fixed* convex continuation value function. We then deduce that *expected continuation values* are boundedly strictly convex in the reputation of *one's partner.* We show that this induces strict gains from rematching any assortatively mated in-

[1] Indeed, several recent important papers have ventured into this arena: See, eg., Bar-Isaac (2004). Niederle and Roth (2004) specifically matches two key features of our model: complementarity, uncertain types, and publicly revealed signals. Our welfare theorems, in particular, prove that matching after information revelation is not only more efficient but also obtains in a Walrasian equilibrium.

terior agents with 0 *or* 1 (i.e. surely bad or surely good individuals), *or* both. This implication of future value convexity works against the force of current production complementarity. For since the static production losses from non-assortative matching in the first period are bounded, PAM cannot be optimal with enough weight on the future (Proposition 1). This either-or argument is also novel to the learning literature, and might well prove useful for analyzing other problems.

PAM Failure in an Infinite Horizon Model. Economists care about two period models because they are parables of the richer dynamic world we inhabit — where the 'today' and 'tomorrow' more generally mean 'current' and 'future'. Yet this is not always so: Finitely repeated games, for instance, can have remarkably different equilibrium sets than the same game repeated infinitely often. In the same vein, is our two-period analysis representative of the general setting? The simple answer is no. But while our findings *a priori* hang in the balance (future values being endogenous), we do rescue a slightly less dramatic failure of PAM. It too turns on a trade-off between value convexity due to learning and static input complementarity.

To see where our earlier logic goes wrong, we observe that the two period analysis critically relies on fixed continuation values. Consequently, with enough patience, the informational gains to non-assortative matching dominate the (bounded) static losses. But with an infinite horizon, the continuation value is endogenous to the discount factor, and in a troubling fashion: As is well-known, it 'flattens out' with rising patience. This captures the fact that no piece of information matters much today since one can just as well acquire it tomorrow. So as the discount factor rises to 1, current production and information acquired in a match *both* become vanishingly important. Thus, the payoff losses may forever outweigh the informational gains for any level of patience.

The behavior of the value function is more subtle than suggested by the preceding paragraph. While it is true that the value function becomes less convex for any fixed reputation, it actually becomes *more* convex in a neighborhood of the extremes 0 and 1; thus, we are led once again to check whether PAM fails near these extremes. But now we must turn to a labor economics story: We call the technology *low-skill concealing* if matches with one or two 'bad' agents are statistically similar. Consider pairwise music production. If we assume that a great song requires two good co-creators (like Lennon and McCartney), then this technology is low-skill concealing. We show in Proposition 2 that PAM fails for high reputations when production is sufficiently low-skill concealing, and for low reputations with sufficiently *high-skill concealing* technologies. In this case, matching middling types with those near 0 is certainly unrewarding. Less obviously so, we prove that matching middling types with those near 1 *is* profitable.

Not all technologies are high- or low-skill concealing. There are examples in which the information effect *reinforces* the static output effect near 0 and 1, yielding PAM for any level of patience. In what sense then is the PAM failure eventually generally true? We show in Proposition 3 that for randomly chosen production technologies, the chance of a high- or low-skill concealing converges to one, as the number of production outcomes grows. We then provide simulation evidence that suggests that these conditions are *extremely* likely to obtain in practice, even with few production outcomes.

Wage Profile Effects. In addition to our findings about the structure of matches, we offer some insights into the distribution of wages. Suppose that a high reputation individual matches with a lower one. In this case, the information rent he provides about his partner must more than compensate for the static efficiency loss; in return, he receives an equilibrium wage premium. This premium strictly exceeds the benefit conferred by his greater productivity. As a result, the relationship between expected ability and wages is neither monotonic nor even continuous when PAM fails — even though expected output is monotonic and continuous in reputation.

Now consider a situation with a dual PAM failure, where high reputations match down and low reputations match up. Proposition 3 suggests this is generally true. In this case, both high and low reputation agents are paid more than their current productivity, and middling reputation agents less than their static productivity. This has life-cycle implications for wages: Younger workers, often with unfocused reputations, will sacrifice wages to pay information rents; conversely, older workers will be paid in excess of their productivity, given their established high *or* low reputations.

We acknowledge our informational effect to be secondary to Becker's productive force for PAM. But a clear testable implication of our findings is that assortative matching failures should be observed more often when (*i*) there is uncertainty about types, (*ii*) separate contributions to production are unobservable, and (*iii*) productive interactions are frequent (a proxy for patience).

Match Break-Ups. We conclude the paper by shifting our focus from the failure of PAM to the behavior of the individual agents in the market. Unlike other matching models with fixed types, ours affords an economically interesting built-in micro-story as well. For while the market is in steady-state, individuals proceed through their life-cycle. Their reputations randomly change over time, converging towards the underlying true types. So with enough patience, if two genuinely good types are paired, then we should expect their reputations to rise as time passes. Eventually, these reputations will enter the region where PAM fails, and the partnership will efficiently dissolve.

Related Work. Kremer and Maskin (1996) is the one twist (of which we are aware) on Becker's original static complete information matching model where PAM fails. However, it starts by assuming a production function without complementarity.

Our paper enters a very small literature of matching in equilibrium models with incomplete information. Jovanovic (1979) considers a model where the slow revelation of information about worker types causes turnover. Chade (2004) is clearly the paper most closely related work to ours. He extends Becker's work to uncertain types, but assumes private information, re-inforcing PAM, by way of a new 'acceptance curse'. A number of extensions of Becker's work assume search frictions (see Burdett and Coles (1997), Smith (1997), and Shimer and Smith (2000)), but the thrust of the results is to develop additional conditions under which PAM still obtains. Unlike these search papers, we show that Becker's finding robustly unravels given an informational friction.

Paper Outline. Section 2 develops Becker's static model with uncertain types. In Section 3, we build intuition in a stylized two period setting, in which we deduce our key insight about convexity versus complementarity. In Section 4, we set up our dynamic model, define a Pareto optimum and competitive equilibrium, establish the Welfare Theorems, and deduce existence. Our theory will thereby apply both to the efficient and equilibrium analyses; however, our serious interest in the planner's problem is for the information it provides us about individual agents; for the planner's multipliers are precisely the agents' private present values of wages. We explore the infinite horizon model in Section 5, where we first develop our skill-concealing conditions. We appendicize all technical proofs, and try to provide in-text intuitive arguments.

2 Becker's Static Model, but with Uncertain Types

Uncertain Types and Stochastic Output. We consider the simplest world with uncertain types, where each agent can either be 'good' (G) or 'bad' (B). Only nature knows the types. Agents match in pairs to produce output, and enjoy a symmetric production role. Everyone is risk neutral. There are $N > 1$ possible distinct nonnegative *output levels* q_i. For each pair of matched types, there is an implied distribution over output levels. Output q_i is realized by pairs $\{G, G\}$, $\{G, B\}$, and $\{B, B\}$ with respective chances h_i, m_i, and ℓ_i. Being probabilities, we have $\sum_i h_i = \sum_i m_i = \sum_i \ell_i = 1$. The respective *expected outputs* is $H = \sum h_i q_i, M = \sum m_i q_i$, and $L = \sum \ell_i q_i$, while we define vectors $h = (h_i)$, $m = (m_i)$, and $\ell = (\ell_i)$. Stochastic output is essential to our thrust, as we seek a model in which uncertainty about types realistically persists over time; we do not want true types revealed after the first period. Figure 1 summarizes:

Chance of Output q_i

	B	G
B	ℓ_i	m_i
G	m_i	h_i

Expected Output

	B	G
B	L	M
G	M	H

Figure 1: **Match Output.**

Our main assumptions are thus that individual skill levels are not fully known, and that separate contributions to joint production are not observable. We can think of many examples for which these assumptions are apropos: academic co-authoring, movie production, the advertising industry, the legal profession, consulting — and even pro sports (is the quarterback's coach, the offensive line coach, or the head coach more responsible for an American football team's success?). Indeed, a large fraction of output (recommendations, reports, sales) within most organizations is produced by *teams*, and outside of the pure marriage setting, one's reputation is often at least as important as the static output from the current partnership. For example, the graduate student co-author may rightly care much more about the paper's effect on the market's perception of his ability than about the paper's actual quality. As Kremer (1993) has underscored, stochastic joint production is even a major feature of high-tech industrial production. The motivational O-Ring space example he provides is low-skill concealing. Production ranging from automobile prototypes to drug development also fits this mold.

Bayesian Reputations. Each of the continuum of individuals has a publicly observed chance $x \in [0,1]$ that his type is G. Call x his *reputation.* So a match between agents with reputations x and y yields output $q_i \geq 0$ $(i = 1, \ldots, N)$ with probability

$$p_i(x, y) = xyh_i + [x(1-y) + y(1-x)]m_i + (1-x)(1-y)\ell_i.$$

The expected output of this match is

$$Q(x, y) = xyH + [x(1-y) + y(1-x)]M + (1-x)(1-y)L$$

Since $q_i > 0$ for some i, $Q(x, y) > 0$ when $0 < x, y < 1$, and matching is always optimal.

A twice differentiable function Q is strictly *supermodular iff* $Q_{12} > 0$, or types are productive complements. We use this common expression for complementarity, denoting it SPM; the reverse case of substitutes, or *submodular* production, is denoted SBM. Clearly, $Q_{12}(x, y)$ is constant for a bilinear function like Q.

Assumption 1 (Supermodularity) $\pi \equiv Q_{12}(x, y) = H + L - 2M > 0.$

Thus, $\pi > 0$ is the *premium* to assortatively pairing (G, G) and (B, B) over cross matching two pairs (B, G). We restrict attention to the case of $\pi > 0$, i.e. the most favourable case for PAM. But we only require that we avoid the knife-edge case $\pi = 0$.

First we consider the social planner's problem. Given a density g and cdf G over agents' reputations $x \in [0, 1]$, the planner chooses a matching to maximize the expected value of output. Let $F(\cdot|y)$ be the conditional cdf of match partners for y.[2] The planner cannot match more of a particular type than available. Thus, he solves:

$$\mathcal{V}(g) = \max_F \int_{[0,1]^2} Q(x, y) dF(x|y) dy \tag{1}$$

$$\text{s.t.} \quad \text{Feasibility:} \quad F(1|y) \leq g(y) \ \forall y \tag{2}$$

We say that x and y are matched if (x, y) lies in the *matching set*, namely the joint support of $F(x|y)$. In this setting, *positive assortative matching* (PAM) obtains iff

[2] Some looseness will persist until the appendix, as we postpone using measures. Our intention is to be intuitive. For example, given $g(y) = y^2$ and PAM, $F(x|y) = 0$ for all $x < y$, and $F(x|y) = y^2$ for all $x \geq y$. Note that there may not exist a density corresponding to the optimal F. For example, if there is a single optimal partner x^* for y then $F(x|y)$ jumps up in a single atom at $x = x^*$.

like-reputation agents match. So in a graph, the PAM matching set coincides with the 45 degree line. *Negative assortative matching* (NAM) obtains when every reputation x matches only with the (opposite) reputation $y(x)$ solving $G(y(x)) = 1 - G(x)$. Then:[3]

Proposition 0 (Becker (1973)) *Given SPM, PAM solves the static social planner's maximization problem.*[4] *Given SBM, the efficient matching entails NAM.*

Assumption 1 shall henceforth remain in force. We thereby maintain the most favorable SPM case for PAM, as we ultimately show that assortative matching fails.

In a competitive equilibrium, each worker x chooses the partner y that maximizes his (expected) wage $w(x|y)$, achieving his *value* $v(x)$. Also, wages of matched workers exhaust output,[5] and market clear. Altogether, a *competitive equilibrium* (CE) is a triple (F, v, w) where F obeys the feasibility constraint (2), while v and w satisfy:

- Output Shares: $$w(x|y) + w(y|x) = Q(x, y) \tag{3}$$
- Worker Value Maximization: $$v(x) = \max_y w(x|y)$$

Becker proved the welfare theorems, which Theorem 2 revisits in a dynamic setting.

Theorem 0 (Becker (1973)) *The First and Second Welfare Theorems obtain, and the decentralizing wage is $w(x \mid y) = Q(x, y) - v(y)$ for any matched pair.*

Observe how PAM implies that $v(x) = Q(x, x)/2$, which is strictly convex by SPM:

$$Q(x, x) = x^2 H + 2x(1 - x)M + (1 - x)^2 L = \pi x^2 + 2(M - L)x + L$$

This convexity is exploited in the last period of the two period model in §3.

3 Matching in a Metaphorical Two Period World

To build essential intuition for our infinite horizon results, we consider a stylized two period model with payoffs in periods $t = 0$ and $t = 1$ weighted by $1 - \delta \in [0, 1)$ and δ.

[3] Our propositions are descriptive matching results, and theorems technical equilibrium results.

[4] Becker proved this for the discrete case. For our purposes, Lorentz (1953) is more appropriate as he proved the formal result in the continuum case (albeit unaware of any economic context).

[5] There is much flexibility in defining wages for matches that are never realized in equilibrium. We find interpreting offered wages as output shares intuitively appealing. Note that with this specification, equilibrium pins down wages for both realized and unrealized matches.

This is in the spirit of standard Bellman equations, such as (8). Obviously, $\delta < 1/2$ in any two period model with strict time preference. But investigating what happens in the two period model as $\delta \to 1$ is often suggestive of infinite horizon models as $\delta \to 1$. As it turns out, in the two period model we can exploit the *fixed strict convexity* of the second period value function, whereas the value function is *endogenous* to the discount factor in the infinite horizon model. This dodges a hard (but necessary) complication, allowing us to prove a strong impossibility result in the two period model.

Our two period results are of independent interest, as they apply, for instance, to any overlapping generations setting where individuals enjoy finite productive lives — *as long as each agent knows when he will exit the labor pool.* If exit rates are stochastic, then the correct model is one with an infinite horizon.

3.1 Bayesian Updating and Continuation Values

In period one, agents are matched, and then produce publicly observed output after which their reputations are updated. Assume that a match with y yields output q_i. By Bayes rule, agent x's *posterior reputation* equals

$$z_i(x, y) \equiv z(q_i, x, y) \equiv p_i(1, y)x/p_i(x, y) \tag{4}$$

Also, Assumption 1 precludes $h = m = \ell$, and thus the dynamic economy is not a trivial repetition of the static one. For if $h \neq m$ or $\ell \neq m$, then all reputations but 0 and 1 shift with positive chance after each match: $z_i(x, y) \neq x$ for some i, if $x \neq 0, 1$.

The *expected continuation value of the match* (x, y) plays a central role in this paper:

$$\Psi^v(x, y) \equiv \psi^v(x|y) + \psi^v(y|x), \quad \text{where} \quad \psi^v(x|y) \equiv E[v(z(\tilde{q}, x, y))] \tag{5}$$

That is, $\psi^v(x|y)$ is the expected continuation value of agent x when matched with y at the start of the first period. That $v(x)$ is strictly convex not surprisingly yields the same true of $\psi^v(x|y)$. But this delivers a critical reciprocal form of convexity. In the appendix, we show that $\psi^v(x|y)$ is strictly convex in *one's partner's reputation* y:

$$\psi^v_{yy}(x|y) = \pi \sum_i p_i(x, y)[z_{iy}(x, y)]^2 > 0 \tag{6}$$

3.2 Value Convexity versus Production Supermodularity

We now deduce an unqualified failure of PAM that is unique to this setting, but which cleanly demonstrates the opposition between the value convexity we have found and production supermodularity. To better flesh out the contrast, observe that if individuals only cared about future output, then $\psi^v(x|\cdot)$ would be type x's match payoff function. In that event, PAM would require that Ψ^v be SPM on the matching set. We now show that this supermodularity requirement cannot possibly be met.

Proposition 1 *Fix $x \in (0,1)$. Given matches $(0,0)$, (x,x), and $(1,1)$, the total expected continuation value is strictly raised by rematching x with either 0 or 1.*

Quick Proof: Since $\psi^v(x|y)$ is strictly convex in y, it is also quasi-convex. Thus, either $\psi^v(x|0) > \psi^v(x|x)$ or $\psi^v(x|1) > \psi^v(x|x)$. Since agents 0 and 1 have the same posterior reputation regardless of partner, this implies that either:

$$\psi^v(x|0) + \psi^v(0|x) > \psi^v(x|x) + \psi^v(0|0) \quad \text{or} \quad \psi^v(x|1) + \psi^v(1|x) > \psi^v(x|x) + \psi^v(1|1) \ \square$$

Assume PAM in period zero, and pick any reputation $x \in (0,1)$. Re-match as many of the x's with 0 or 1 as possible (the choice governed by Proposition 1). The period one informational gains from the above rematching are strictly positive; the period zero production losses are finite, and so are swamped by the gains for δ high enough.

Corollary 1 *In the two period model, PAM is not a CE for large enough $\delta < 1$.*

Ours is an *either-or* PAM failure with enough patience. It is not simply that agents with extreme reputations 0 and 1 are both informationally valuable, helping those with interior reputations reveal their types — since all match output variance owes to the uncertain type of x. We next illustrate how the informational effect may reinforce the static output effect near one extreme and conflict near the other; thus, PAM fails near one extreme with enough patience, but obtains for any level of patience near the other.

3.3 Illustrative Example of Assortative Matching Failure

Let $(q_1, q_2) = (0, 2)$, $h = (1/2, 1/2)$, and $m = \ell = (1, 0)$. This yields SPM output, since $\pi = H+L-2M = 1$. For our parametrization, a matched pair (x, y) produces output q_2

with chance $p_2(x,y) = xy/2$. Reputation x updates to $z_1(x,y) = (2-y)x/(2-xy)$ after the low output, and to $z_2(x,y) = 1$ after the high output.

Given PAM in period two, the value of reputation x at the start of the second period is: $v(x) \equiv Q(x,x)/2 = x^2$. Now, agent x's expected continuation value is

$$\psi^v(x|y) \equiv p_1(x,y)v(z_1(x,y)) + p_2(x,y)v(z_2(x,y)) = \left(1 - \frac{xy}{2}\right)\left(\frac{(2-y)x}{2-xy}\right)^2 + \frac{xy}{2}$$

The present value of the match (x,y) is $V(x,y) \equiv (1-\delta)xy + \delta\Psi^v(x,y)$, where

$$\Psi^v(x,y) \equiv \psi^v(x|y) + \psi^v(y|x) = \frac{x^2(1-y) + y^2(1-x) + xy}{1 - xy/2}$$

To illustrate the either-or nature of our rematching result, Proposition 1, let $x = 0.5$. The period one informational gains from rematching 0.5 are:

$$\psi^v(0.5|1) - \psi^v(0.5|0.5) \approx 0.043 \quad \text{and} \quad \psi^v(0.5|0) - \psi^v(0.5|0.5) \approx -0.013 \qquad (7)$$

Thus, reputation 1/2 has a higher continuation value matching with a reputation 1, but a *lower* continuation value matching with a reputation 0. However, the average informational gain from matching half the 0.5s with 0 and half with 1 equals $0.03 > 0$. The period zero static production loss from this rematching equals 1/4. Thus, the rematching increases the total expected value if $(1-\delta)/4 < 0.03\delta$, or $\delta > 0.89$.

Of course, this does not imply that PAM is efficient for all $\delta < 0.89$, because we have only considered one alternative rematching. The efficient matching maximizes $\int_{[0,1]^2} V(x,y)dF(x|y)dy$ subject to $F(1|y) \leq g(y)\ \forall\ y$. If $\Psi^v_{12} > 0$ then $V(x,y)$ would be SPM, and the efficient and competitive equilibrium matching would be PAM. Evaluating the cross partial Ψ^v_{12} along the diagonal $y = x$ yields:

$$\Psi^v_{12}(x,x) = \frac{4(2 - 8x + 7x^2 - x^4)}{(2-x^2)^3} \gtrless 0 \qquad \text{for} \quad x \lessgtr 0.36$$

Here, the learning effect reinforces the productive SPM effect for low reputations x, but opposes it for high reputations, consistent with (7). Figure 2 illustrates the solution for $\delta = 0.99$, given an initial uniform density over reputations and no entry.

For some insight into this pattern, notice that $m = \ell$. Thus, the output distribution

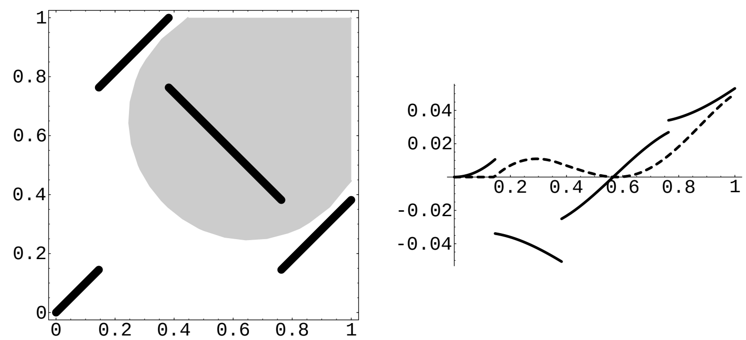

Figure 2: **Two Period Example.** On the left, we depict the shaded SBM total value region (where $V_{xy} < 0$), and the resulting discontinuous optimal matching graph $\mathscr{G} = \{(x, y(x)), 0 \le x \le 1\}$ (solid line). On the right, we plot the equilibrium wage function $w(x) \equiv w(x|y(x))$ (solid line). Given the high discount rate $\delta = 0.99$, the wage $w(x|y(x))$ is almost entirely an information rent $\psi^v(y(x)|x) - v(y(x))$ — whose discontinuity forces a jump in the wage profile. We superimpose the surplus in optimal values over assortative matching values. The diagram is the result of a numerical Walrasian tâtonnement process.

is the same for both good and bad partners of a reputation 0. Thus, we learn nothing about an interior reputation agent $x \in (0,1)$ when he matches with 0. But if such an x assortatively matches, this *is* informationally valuable — it may well be a $\{G, G\}$ match. Thus, matching with 0 is informationally inferior to assortative matching.

Here is an intuition for the shape of the matching set $\mathscr{G}$ in the left panel. First, by local optimality considerations, $\mathscr{G}$ is decreasing whenever the match value $V(x,y)$ is SBM (shaded region). Second, it cannot exit the SPM region on a downward slope. Third, by the uniform density on reputations, $\mathscr{G}$ has slope ± 1 whenever continuous.[6]

Observe how agents with high reputations are willing to match 'down' (the solid line in right panel of Figure 2), because they are paid a positive wage for doing so. Note how *the wage profile must jump each time the matching shifts from PAM to NAM.* This occurs mostly for productive reasons, though the information rent in (11) also jumps.

Underscoring its importance, the allocational and wage profile effects of reputational matching in this example are vast — over 80% of all agents non-assortatively match yielding an increasing informational rent payment for two thirds of agents. The income distribution is right-skew if production is supermodular, stochastic, and high skill.[7]

[6]See Kremer and Maskin (1996) for formal characterizations of solutions in a one-shot matching model where match values are not SPM, and their wage profile thus jumps for this static reason.

[7]That PAM increases the income share of the talented is known (see Kremer and Maskin (1996)).

4 Dynamic Matching with Reputational Concerns

4.1 Pareto Optimum and Competitive Equilibrium

Extend the two period model to periods $t = 0, 1, 2, \ldots$, and introduce deaths: agents live to the next period with *survival chance* $\sigma < 1$. We must strike a delicate balance between identifying agents with reputations and identifying agents as individuals. If individuals were completely anonymous, then we could not apply our results to real world partnerships. But if reputation is not a sufficient statistic for past output realizations, then we both violate the general equilibrium spirit of Becker's work, and present ourselves with unsurmountable and unnecessary technical obstacles. Instead of either extreme, we adopt the following compromise:

Assumption: *The entire output history of currently matched individuals is observable; however, once a partnership dissolves, only the reputation of each individual is recorded.*

With this assumption, reputation is a sufficient statistic for the information from all previous matches, and yet we may still speak of partnerships in a meaningful sense.

Let γ be the planner's discount factor, and $\delta \equiv \sigma\gamma$ the agents' discount factor. Given an initial density g over reputations, the planner chooses the matching set in each period to maximize the (normalized) present value of output, respecting the feasibility constraint and the law of motion for the reputation density. Thus, the planner solves:[8]

$$\mathcal{V}(g) = \max_F \left[(1-\delta) \int_{[0,1]^2} Q(x,y) dF(x|y) dy + \gamma \mathcal{V}(\bar{g} + \sigma B(F)) \right] \tag{8}$$

subject to feasibility (2). Here $\bar{g}(x)$ is the inflow density of reputation x agents, and $B(F)$ is the Bayes-updated density over reputations given the matching cdf F. (Its exact formula is in the appendix.) Thus, $\bar{g} + \sigma B(F)$ is next period's reputation density.

In solving this problem, the planner trades off higher expected output today for a more informative density over reputations tomorrow. This trade-off is at the heart of our paper. A *steady state Pareto optimum* (PO) is a triple (F, v, g) such that, given g, (F, v) solve the planner's problem, and $g = \bar{g} + \sigma B(F)$. Just as in the analysis of the modified golden rule in growth models, the social planner does not maximize across steady states. Instead, she chooses an optimal matching in each period and *then* we

[8] I.e. a unit payoff per period is worth $\mathcal{V}^T = 1 + \gamma\mathcal{V}^T$; setting $\mathcal{V} \equiv (1-\delta)\mathcal{V}^T$ yields $\mathcal{V} = (1-\delta) + \gamma\mathcal{V}$.

impose steady state. Our results actually obtain both in *and* out of steady state; however, we focus on the steady state for simplicity. In the appendix, we prove:

Theorem 1 *A steady state Pareto optimum exists.*

The FOC for this problem are:

$$\begin{array}{rcl} (x,y) \in supp(F) \Rightarrow [v(x)+v(y)-(1-\delta)Q(x,y)-\delta\Psi^v(x,y)] & = & 0 \\ v(x)+v(y)-(1-\delta)Q(x,y)-\delta\Psi^v(x,y) & \geq & 0 \end{array} \tag{9}$$

where $v(x)$ is the multiplier on the constraint (2), the shadow value of an agent x.

The FOCs (9) are economically intuitive. For any matched pair, the sum of the shadow values of the two agents this period equals the total value to the planner of matching them. Further, for *any* pair, the sum of the shadow values today weakly exceeds the value the planner could achieve by matching them.

Next, there are many ways to conceptualize a *competitive equilibrium* (CE) in a matching environment. We assume that agents can either (*i*) hire a match partner, offer him a sure wage, and take the output residual as profit; (*ii*) hire himself out to another agent; or (*iii*) stay unemployed, earning nothing. To this end, let $w(x|y)$ be the wage that agent x earns if matched with y. Anticipating a welfare theorem to come, we overuse notation, letting $v(x)$ denote the maximum discounted sum of wages that x can earn. A *steady state CE* is a 4-tuple (F, v, w, g), where $g = \overline{g} + \sigma B(F)$, F obeys constraint (2), wages are output shares (3), and a dynamic maximization obtains:

- Worker Maximization: $$v(x) = \max_y[(1-\delta)w(x|y) + \delta\psi^v(x|y)] \tag{10}$$

We now turn to the Welfare Theorems. We believe that Mortensen (1982) is the first paper to show that they are not to be taken for granted in a matching setting. But his impediment was search frictions, while ours is information. Observe that while agents' true types are unknown, all information is public; thus, no externality arises, as wages should fully incorporate all informational rents. Moreover, the production technology is linear in measures of matched agents, and thus meets the standard convex technology requirement.[9] One nonstandard feature of our welfare theorems is that the allocation

[9]This is not surprising, given the similarity between ours and a static assignment problem. Gretsky, Ostroy and Zame (1992) prove that the PO, CE, and the Core coincide in a continuum assignment problem. The dynamic nature of our problem prevented us from applying their results.

F is embellished with the values v, as are also claiming coincidence of the planner's shadow values and the agents' 'private values'. This innovation is an essential ulterior purpose in our paper of the welfare theorems — we can therefore refer to both shadow values and private values simply as *values*, and the map $x \mapsto v(x)$ as the *value function.*

Theorem 2 (Welfare Theorems) *If (F, v, w, g) is a steady state CE, then (F, v, g) is a steady state PO. Conversely, if (F, v, g) is a steady state PO, then (F, v, w, g) is a steady state CE, where for all matched pairs (x, y), the wage $w(x|y)$ of x satisfies:*

$$w(x|y) = \overbrace{Q(x,y) - v(y)}^{\textit{Static Wage}} + \overbrace{\frac{\delta}{1-\delta}[\psi^v(y|x) - v(y)]}^{\textit{Information Rent}} \tag{11}$$

To properly internalize all externalities, the wage that reputation x receives is the sum of two parts. First is the difference between total match output and the outside option of his partner y (i.e. Becker's static wage function). Second, there is the *information rent*, or the discounted difference between y's continuation value and his outside option. (The information rent that x pays y is embedded in $w(y|x)$.) That informational benefits are publicly observed sustains the welfare theorems, since agents may extract the benefit they confer. The informativeness of output for agent y depends on the reputation of his partner x. Naturally, some partners are more informative for y. The more informative x is for y, the higher is his equilibrium wage when matched to y.

To gain some insight into why this wage decentralizes the Pareto optimum, consider a pair (x, y) matched in equilibrium. Worker Maximization (10) requires

$$v(x) = (1-\delta)w(x|y) + \delta\psi^v(x|y) = (1-\delta)\left[Q(x,y) - v(y) + \frac{\delta}{1-\delta}[\psi^v(y|x) - v(y)]\right] + \delta\psi^v(x|y)$$

using our computed wage (11). With some simplification, we get:

$$v(x) + v(y) = (1-\delta)Q(x,y) + \delta\left[\psi^v(x|y) + \psi^v(y|x)\right]$$

which holds if (x, y) are matched in the Pareto optimum by the planner's FOC (9).

Finally, we consider existence. Theorem 1 proved that a steady state PO exists; also, any steady state PO can be decentralized as an CE, by Theorem 2. Altogether:

Corollary 2 *There exists a steady state competitive equilibrium.*

4.2 Value Functions

To better understand why $\psi^v(y|x) - v(y)$ is an "information rent", we must argue that values are convex in reputations. Thus, agents are willing to pay to have the output signal revealed. The convexity of the value function in beliefs for a single agent learning problem is well-known (see Easley and Kiefer (1988)). We are aware of no other paper which asserts the convexity of the *multipliers* in a planner's problem. We hereby exploit the equivalence between the CE and PO, and use the planner's problem to give insights into the agents' private values $v(x)$.

Lemma 1 *The value function $x \mapsto v(x)$ is strictly convex.*

Although the result is non-standard, it admits the standard intuition that information is valuable — only here, its value is to the planner. Why? Suppose that a signal is revealed about the true type of an agent with reputation x. This variance in the reputation has mean x (i.e. is a 'fair' gamble), and cannot harm the planner — for she could choose to ignore it. By standard decision theory, an agent is risk averse (i.e. hates all fair income gambles) iff his utility function is concave. Inverting this logic, the value function must be convex, and strictly so as the information is strictly beneficial; for the planner is not indifferent across all matches when $\pi = H + L - 2M \neq 0$. For then, there is a productive reason to prefer either assortative or non-assortative matches.

Not only is information about one's own type valuable, but so too is information about one's partner. This reciprocal convexity is a critical ingredient in our paper.

Lemma 2 *The expected continuation value $\psi^v(x|y)$ is strictly convex in x and y.*

The intuition is one step upstream from that of value convexity. If we have better information about x's partner, then we have better information about the new reputation that x will acquire after output from his match is observed.

5 Infinite Horizon Matching Results

5.1 A Delicate Tradeoff

Our two period result obtains because the continuation value function is *fixed*, given PAM in the final period (by Becker); further, it is boundedly and *strictly convex.* Thus, PAM fails with sufficient patience, given our either-or inequality in Proposition 1.

But with no last period, the continuation value function depends on the discount factor — and in precisely a fashion that entirely undermines the two period logic of Proposition 1: For as the discount factor δ rises, the value function[10] v_δ flattens out, losing its convexity in the limit. Not only do the static losses from assortative-matching vanish, but so too do the informational gains. (In the same vein, the risk premium vanishes as a risk averter becomes 'less concave'.) As such, we may have been misled, and an infinite horizon model is called for to resolve this race to infinite patience.

A matching is *productively efficient* if no other matching leads to higher output in the current period. A matching is *informationally efficient* if no other matching leads to higher expected discounted continuation output. A necessary condition for either efficiency notion is that they are satisfied locally, so that a marginal matching change cannot increase output today or in the future. Now, consider the following rough intuition for a marginal change in an assortative match. Compare the sum of match values of (x, x) and $(x+\Delta, x+\Delta)$ to the sum of match values of $(x, x+\Delta)$ and $(x+\Delta, x)$. By a second order Taylor Series, the static losses from such a change are:

$$Q(x,x) + Q(x+\Delta, x+\Delta) - 2Q(x, x+\Delta) \approx Q_{12}(x,x)\Delta^2$$

The dynamic change in continuation value from such a rematching is likewise:

$$\Psi^\delta(x,x) + \Psi^\delta(x+\Delta, x+\Delta) - 2\Psi^\delta(x, x+\Delta) \approx \Psi^\delta_{12}(x,x)\Delta^2$$

We thus have the following net marginal change from matching assortatively:

$$\overbrace{(1-\delta)Q_{12}}^{\textit{Static Production Gains}} \quad + \quad \overbrace{\delta\Psi^\delta_{12}}^{\textit{Dynamic Informational Losses}} \tag{12}$$

For PAM to be efficient this weighted sum cannot be negative. In the infinite horizon model, the static losses from not matching assortatively vanish as $\delta \to 1$ (as they did in the finite horizon model). However, we have a knife edged trade-off in the limit, as the informational gains also vanish. Indeed, for any fixed x the continuation value v_δ is a function of δ, *and* flattens out as $\delta \to 1$. Indeed, reputations converge

[10]We add a δ subscript to highlight the dependence of v_δ on discounting. We also switch to the more compact notation ψ^δ and Ψ^δ rather than ψ^{v_δ} and Ψ^{v_δ}.

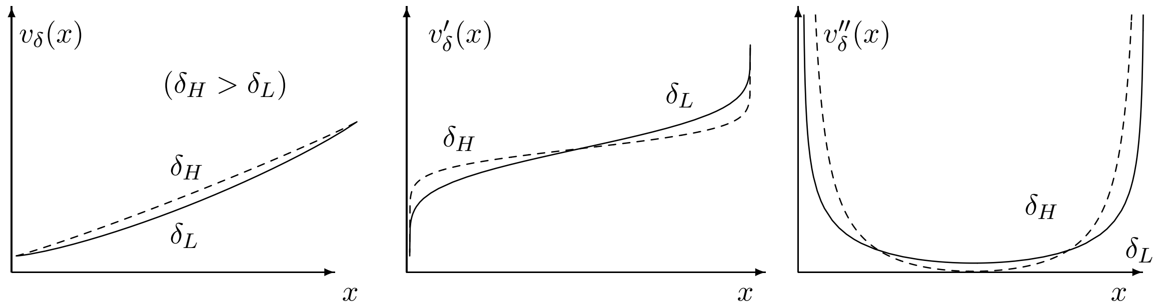

Figure 3: **Value Function v_δ and Derivatives v'_δ and v''_δ.** This graph depicts the value function flattening, and the convexity explosion near $0, 1$: $\lim_{x\to 0} v''_\delta(x) = \lim_{x\to 1} v''_\delta(x) = \infty$.

to 0 or 1 (see Proposition 4), and the chance that an agent with initial reputation x converges to 1 is x. Thus, as $\delta \to 1$, the sequences of value functions converges $v_\delta(x) \to xv_\delta(1) + (1-x)v_\delta(0)$, which implies that Ψ_{12}^δ converges to 0 as $\delta \to 1$.

Actually the asymptotic behavior of the value function is more complex than the prior 'flattening' logic suggests. While the convexity at any fixed interior x converges to zero, the integral $\int_0^1 v''_\delta(x)dx$ is a constant in δ. This apparent contradiction is resolved by noting that as δ converges to 1, all of the convexity accumulates at the extremes 0 and 1. See the right panel of Figure 3. Since convexity was sufficient to establish the failure of PAM in the two period model, this suggests that we should try to establish our failure result near the extremes.

Proposition 1 established that such a tradeoff exists between informational and production efficiency for some reputations. In §5.2, we establish that such a tradeoff always exists near $x = 0$ or $x = 1$. But we are still not done, for then we must show that informational efficiency dominates as $\delta \to 1$. This need not occur because the marginal value of information *for given probability shifts* increases in the convexity of the value function near the extremes. But by the same token, Bayes' rule updates very slowly near the extremes: i.e. *the belief variance falls.* On balance, the informational value of marginal changes in matching near extremes can still vanish. Thus, whether the informational effect dominates the productive effect still hangs in doubt.

It turns out that we need much more than explosive extremal convexity to resolve these difficulties. We need in fact a very precise asymptotic approximation of the value function. In general determining an asymptotic functional form for the *optimal value* function is an intractable problem, and at any rate has never been done, to our knowledge. Instead we proceed by contradiction: We study the assortative matching policy, and derive its asymptotic behavior. Our analysis relies on a new finding in Anderson and Smith (2004) that if a *fixed policy* in learning problems generates a convex static payoff, then the second derivative of the value function explodes at a *geometric rate* near extremes 0 and 1 — conveniently where our noted PAM failures occur. Again, this precise characterization of the value function is required: Had the second derivative a slightly different functional form near the extremes, Becker's static effect could well dominate for all large discount factors.

Shorn of the details, there are three logically separate steps that we must take to establish our main result.

1 Sign the information effect near the reputational extremes: When there is a conflict between informational and productive efficiency?

2 Characterize when informational efficiency dominates productive efficiency.

3 Show that informational efficiency wins this tradeoff for large δ as $N \to \infty$.

5.2 Informational vs. Productive Efficiency

A. The Sign of the Information Effect. Ultimately we will use our asymptotic approximation for v_δ to precisely characterize when there exists a tradeoff between informational and productive efficiency near the extremes. However, one might wonder why this is necessary at all. In fact, consider the following seductive loose intuition: Since type $x = 0$ (alternatively, $x = 1$) has a known reputation, production uncertainty in an $(x, 0)$ match only owes to the stochastic production and to the uncertain type of the first party; for an (x, x) match, production uncertainty is increased by the uncertain type of the second party. Thus, the $(x, 0)$ match must provide more information about the true type of x than the (x, x) match. However, in our two period example of §3.3, there was a tradeoff for large x, while the information effect actually reinforced the static output effect for x below a threshold. Since there is *not always* a conflict between informational and productive efficiency, this simple intuition is lacking.

To understand why this simple intuition falls short, let us first clarify the meaning of informational efficiency. Two matches $(x, 0)$ are *more informative* than the matches $(0, 0)$ and (x, x) if they yield a weakly higher expected continuation value for x and 0. By Blackwell's Theorem, this holds *iff* the conditional distribution of signals generated by an $(x, 0)$ match *dominates* (is sufficient for) the (x, x) match. Only this can allow one *a priori* to declare non-assortative matching informationally inefficient. Indeed, the intuition in the last paragraph suggests that $(x, 0)$ and $(x, 1)$ might *both* dominate (x, x). As it happens, this may be impossible. For some parameters, either $(x, 0)$ does not dominate (x, x), or $(x, 1)$ does not dominate (x, x), for any x! (See Appendix B.)

Although $(x, 0)$ and $(x, 1)$ do not both dominate (x, x), our proof of Proposition 1 establishes that — for *any* convex value function v — the expected continuation value of any $0 < x < 1$ was strictly increased by rematching with either 0 or 1. Hence, either $(x, 0)$ or $(x, 1)$ dominates (x, x). But when the value function v depends on the discount factor, as in the infinite horizon model, this simple argument fails.

With a continuum of stochastically heterogeneous agents, a precise characterization of the optimal value function in our model is surely an intractable exercise. Fortunately, we can thankfully sidestep this with a direct counterfactual approach: We first assume PAM and derive the resulting value function; we then show that PAM is not globally optimal for that value function. How? Given PAM, we prove (appendix Claim 7(i)) that $v''_\delta(x)$ geometrically explodes near $x = 0$ and $x = 1$. Specifically, we show that

$$v''_\delta(x) \propto x^{-\alpha_\delta}, \quad \text{where } \alpha_\delta \text{ solves } 1 \equiv \delta \textstyle\sum_i \ell_i (m_i/\ell_i)^{2-\alpha_\delta} \tag{13}$$

when $\delta < 1$ large enough. Why does this matter? Given this functional form, we show (in the appendix) that informational and productive efficiency generally conflict near $x = 0$ or $x = 1$. This result requires the full strength of (13) — not only the explosive geometric form, but also the specific formula for the rate α_δ.

We now introduce an asymmetric correlation function for three vectors a, b, c. Let:

$$S(a, b, c) \equiv \sum_i \left[(b_i - a_i) \log \left(\frac{b_i}{c_i} \right) + \frac{b_i^2 - a_i c_i}{c_i} \right]$$

We know of no precedent for this expression. Generally, $S(a, b, c) \geq 0$ when the vectors a, b, c are close. For instance, $S(a, b, b) = 0$, $S(a, a, c) > 0$, and $S(a, b, a) > 0$.[11]

Lemma 3 *There exists $\delta^* < 1$, such that for all $\delta > \delta^*$, PAM is not informationally efficient near $x = 0$ if $S(h, m, \ell) > 0$ and near $x = 1$ if $S(\ell, m, h) > 0$.*

[11]Indeed, $S(a, a, c) \equiv \sum_i (a_i - c_i) a_i / c_i > \sum_i (a_i - c_i) = 0$, while $S(a, b, a) \equiv \sum_i [(b_i - a_i) \log (b_i/a_i) + (b_i^2/a_i) - a_i] = \sum_i [(b_i - a_i) \log (b_i/a_i) + (b_i^2/a_i) - b_i] = \sum_i [\text{non-negative terms}] + S(b, b, a) > 0$.

With our v'' approximation (13), the Appendix shows that near $x = 0$, the cross partial of the expected continuation value function obeys $\Psi^{\delta}_{12}(x,x) \propto -C(\delta)x^{1-\alpha_\delta} S(h,m,\ell)$. Thus, $\Psi^{\delta}_{12}(x,x)$ shares the same sign as $-S(h,m,\ell)$ near $x = 0$, and $-S(\ell,m,h)$ near $x = 1$. So a conflict between dynamic informational and static production concerns — namely, $\Psi^{\delta}_{12}(x,x) < 0 < Q_{12}(x,x)$ — arises near $x = 0$ or $x = 1$ iff $S(\cdot) > 0$.

For more intuition, consider the extreme case $m = \ell$ (and so $m \neq h$, by our Assumption 1). Then the superior (G,G) matches can be statistically distinguished from the lesser matches (B,G) and (B,B), but the latter two cannot be so nuanced.[12] Now consider the signal from an x paired with 0 or 1 in this case. Type 0 provides no information, as (B,G) and (B,B) yield the same output distribution. Thus, we should not expect PAM failures near $(x,x) = (0,0)$. On the other hand, $S(m,m,h) > 0$, so that PAM is not informationally efficient near $(x,x) = (1,1)$.

Case	Near $x = 0$	Near $x = 1$
$m = \ell$	PAM	NAM
$m = h$	NAM	PAM
$h = \ell$	NAM	NAM

Table 1: **Informational Efficiency at the Extremes.** These are the three knife-edge technologies and the local matching required for informational efficiency.

Note that in the extreme case, $m = \ell$, it takes two 'good' agents for stochastically better production; when at least one agent is low quality, the quality of the other agent is irrelevant. With this observation, let's call this a *perfectly high skill* technology. By the same token, $h = m$ yields a *perfectly low skill* technology. (Alternatively, such technologies may be more accurately called *low/high skill concealing*, respectively.)

Note that $S(h,m,\ell) > 0$ in the perfectly high skill case, while $S(\ell,m,h) > 0$ in the perfectly low skill case. Inspired by these two extreme cases, we call a technology *high skill* iff $S(h,m,\ell) > 0$, and *low skill* iff $S(\ell,m,h) > 0$. While supermodularity rules out technologies that are simultaneously perfectly high and perfectly low skill (i.e. $h = m = \ell$), a technology can be both high and low skill. We soon argue in §5.3 that such technologies are 'common'.

[12] The story in Kremer (1993) of production success requiring no mistakes by all parties is an excellent example of such a technology. More positively, one can imagine that creative work really identifies whether both parties were talented.

B. The Race to Perfect Patience. We have argued that $S(h, m, \ell) > 0$ forces a tradeoff between informational and productive efficiency near $x = 0$. We now wish to show that informational efficiency dominates for high discount factors. Recalling (12), PAM will not obtain at the discount factor $\delta < 1$ if the following obtains for any x:

$$(1 - \delta)Q_{12}(x, x) + \delta\Psi^{\delta}_{12}(x, x) < 0. \tag{14}$$

The first positive term vanishes as $\delta \to 1$, but it could still be that $\Psi^{\delta}_{12}(x, x) < 0$ vanishes more quickly than does $(1 - \delta)$. This is all to say that the outcome of our claimed race to infinite patience is not at all clear *a priori*. However, it turns out that having $S(\cdot) > 0$ is sufficient to predict a clear winner:

Proposition 2 *There exists $\delta^* < 1$ such that $\forall\ \delta > \delta^*$, in the infinite horizon model, PAM fails in a neighborhood of $(0, 0)$ or respectively $(1, 1)$ iff:*

$$S(h, m, \ell) > 0 \quad \textit{or respectively} \quad S(\ell, m, h) > 0 \tag{15}$$

Specifically, PAM fails for low reputation (respectively, high reputation) agents with high skill technology (respectively, low skill technology) if δ is high enough.

For an idea of how large the deviations from PAM may be, we now extend the two-period example in §3.3 to an infinite horizon. In the left panel of Figure 4, we have graphed the value function in the infinite horizon model for three discount factors $\delta \in \{0, 0.9, 0.99\}$ as solid lines. At $\delta = 0$, the value functions of the models coincide. Observe how the value function flattens out as $\delta \uparrow 1$. For comparison purposes, we depict the final Becker value function v^0 for $\delta = 1$ as a dashed line. By contrast, this pure patience value function is strictly convex.

The right panel of Figure 4 shades in the agents for whom PAM fails. Reflecting the diminished convexity with longer horizon models, the PAM failure set is strictly smaller at all discount factors δ in the infinite horizon. Note that PAM obtains in the infinite horizon model for any $\delta \leq 0.9$. By contrast, PAM failed for any $\delta > 0.79$ in the two period model. No $x \geq 0.14$ is matched assortatively at $\delta = 0.99$.

The bottom line is that the PAM failure asserted for two period metaphorical models obtains with no last period provided one of the inequalities in (15) holds. We now argue in the next section that the key condition (15) holds quite generally.

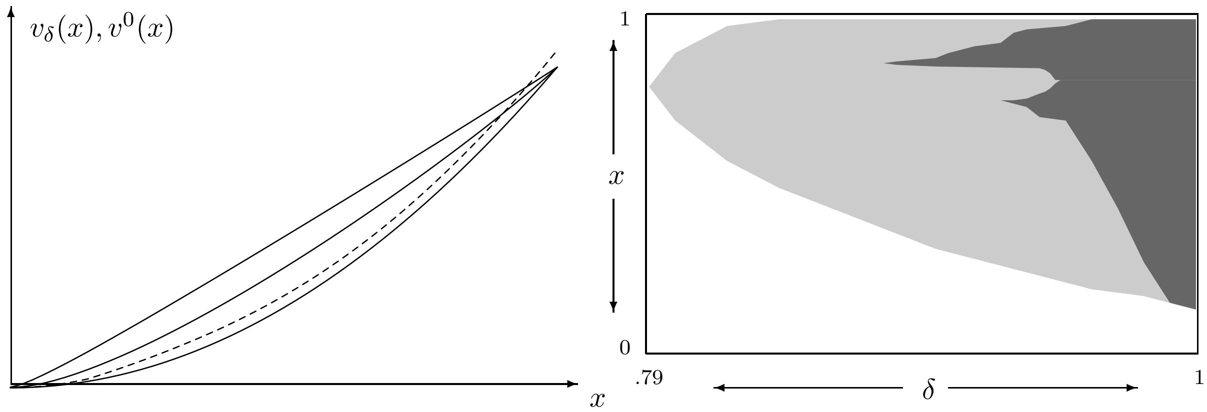

Figure 4: **Folk Theorem Depiction: Infinite Horizon vs. Two-Period Models.** In the left panel, the value function v_δ in the infinite horizon model for $\delta \in \{0, 0.9, 0.99\}$ is depicted as a solid line, while the two period model v^0 for $\delta = 1$ is a dashed line. In the right panel, the dark shaded region is the correspondence between δ and the set of reputations x (vertical axis) that do not match assortatively in the infinite horizon model. The lightly shaded region is the same correspondence for the two-period model.

5.3 PAM Fails for 'Almost All' Production Technologies

We have given sufficient conditions for non-assortative matching at the extremes, and yet we claimed an almost general failure of PAM. How can we justify our assertion? For indeed, with fixed N, the inequalities (15) are *not* always satisfied, and therefore PAM may well obtain for *both* high *and* low reputations. Here is the simplest parameterized example of this phenomenon: Let $h = (3\varepsilon^2, 1/2 - 3\varepsilon^2, 1/2)$, $m = (\varepsilon, 1 - 2\varepsilon, \varepsilon)$, and $\ell = (1/2, 1/2 - 3\varepsilon^2, 3\varepsilon^2)$, where $q = (q_1, q_2, q_3) = (0, 0.1, 1)$. For this technology, we have $S(h, m, \ell) = S(\ell, m, h) \approx$ (constant) $+ \log(\varepsilon)/2 < 0$ for small ε. Observe that $h = (0, 1/2, 1/2)$, $m = (0, 1, 0)$, and $\ell = (1/2, 1/2, 0)$ in the $\varepsilon \to 0$ limit, violating our full support assumption. Thus, PAM reveals the types of those matched, and in fact both partners quickly update to 0 or 1: If q_2 is observed (half chance), then nothing is learned, and we start anew. Otherwise, if q_1 is observed, then both types are surely good; if q_3 is observed, then both types are surely bad. For this example, *PAM is optimal even for high δ in the infinite horizon model.*

We now assert that this example, while robust for fixed $N = 3$, is vanishingly rare when the number N of production outcomes grows:

Proposition 3 *With technologies (h, m, ℓ, q) drawn from an atomless distribution, PAM fails (i.e. inequalities* (15) *hold) near* 0 and 1 *with chance tending to* 1 *as* $N \to \infty$.

Despite the nature of this result, simulations suggest that the PAM failures are extremely likely even for low N. For example, with parameters (h, m, ℓ, q) uniformly generated on the unit simplex, we found simultaneous violations of (15) only 43, 18, 5, and 1 out of *one billion* times for the $N = 3, 4, 5, 6$ cases respectively.

Yet again, the *exact* asymptotic form of the value v is critical for Proposition 3. To see this, suppose instead that $v(x) = x \log x - x + \textit{constant}$. Then $v'(x) = \log x$ and $v''(x) = x^{-1}$. This value function is convex and further v'' is unbounded near $x = 0$; however, v'' lies just outside the geometric family (13), and in fact, Proposition 3 fails.

5.4 Long Run Match Dynamics

Our focus until now has been on a failure of PAM *in the large* — that is, on the distribution of matches in an economy. Despite having a dynamic model, we have been asking an essentially static question: when does an agent with reputation x self-match?

In fact, our model also admits a rich unfolding *micro* story: agents are born and then form and dissolve partnerships as their reputations evolve over time. We finally focus our lens on this intriguing subplot with turnover, and the breakup of seemingly successful partnerships. Let's turn to the limit behavior of an individual's reputation.

Proposition 4 *Fix an agent with reputation x^t at time-$t = 0, 1, 2, \ldots$ (with x^0 given).*
(a) If he is not eventually matched forever with the same partner, then $x^t \to 0$ or 1 with time-0 chances $1 - x^0$ and x^0, for generic technologies.
(b) If he is eventually matched with the same partner (initially y^0), then generically, $x^t \to 0$, $1/2$, 1 with ex ante chances $(1-x^0)(1-y^0)$, $(1-x^0)y^0 + x^0(1-y^0)$, and $x^0 y^0$.

Indeed, reputations are beliefs about underlying types, and thus are martingales: Namely, one's current reputation is the expected future reputation. (This is even true with endogenous matching, by Easley and Kiefer (1988).) It is then well-known that they converge to stationary reputations. Assume first case (a). Intuitively, as long as someone does not have the same-reputation partner each period, any reputation $x \in (0, 1)$ is subject to change as information is revealed. (This is generically true, as the appendix argues.) Thus, his type is eventually revealed: G or B, or $x = 0$ or $x = 1$.

In case (b), the market's ability to learn an agent's type is frustrated by its lack of knowledge of his partner's type. The applicable 'match state space' here is really

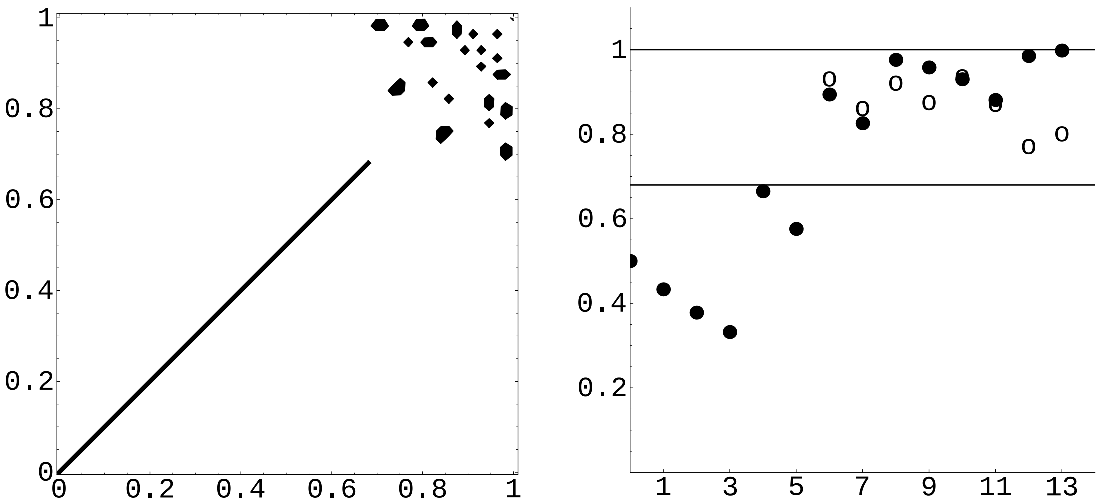

Figure 5: **The Matching Set and a Possible Time Path for x_t and $y(x_t)$.** In the left hand panel we display the steady state matching set when the mass of entrants is uniform, and $\delta = 0.95$, $q = (0, 1)$, $h = (1/2, 1/2)$, and $m = \ell = (0.95, 0.05)$. In the right hand panel will graph a sample time path $t = \{0, 1, \ldots, 13\}$ for x_t and his partner y_t.

$\{(G,G)$, (B,B), (G,B), and $(B,G)\}$. Since production is symmetric, state (G,B) is indistinguishable from (B,G). Thus, if matches are permanent we may end up with $x^\infty = y^\infty = 1/2$ — that is, it may be clear from observing the (infinite) past output realizations that one person in the match is good and one bad, but given that only joint output is observable, there is no way to tell who is good.

Now that we know how individual agents reputations behave in the long run we can discuss the micro structure of match dynamics.

Corollary 3 *Assume δ high enough and production sufficiently low-skill (resp. high-skill) concealing. Then any match of two good (resp. bad) agents eventually breaks up.*

Convergence to $(1/2, 1/2)$ obtains for a long-lived match $\{G, B\}$. The limit is $(1, 1)$ for $\{G, G\}$ and $(0, 0)$ for $\{B, B\}$, by Proposition 4. If, in addition, the technology is both high and low skill, then Proposition 2 implies that as we approach (G, G) or (B, B), like reputation agents cannot stay matched, and so they must break up. So genuinely good type matches, such as the Beatles, will form when they enjoy middle reputations, and sever when their stars rise high enough, with a high skill technology.

In Figure 5 we present a sample path for an individual agent who enters the market with initial reputation $x_0 = 1/2$. In the left hand panel, we have shaded the matching set. In the right hand panel, we have plotted a sample time path for an agent's reputation (x_t) and the reputation of his partner (y_t), starting with $x_0 = 1/2$. Assortative matching obtains below $x = 0.68$. In the example, the agent enters the non assortative region fairly early in his career (i.e. it is not the case that agent's are assortatively matched 'most' of the time) — from period $t = 6$ onward.

These dynamic results also have implications for the life cycle behavior of wages vs. productivity. In each period, a new cohort $\bar{g}$ arrives, whose reputation distribution evolves, eventually concentrating around 0 and 1. Thus, older cohorts earn larger informational rents (on average), which tend to increase as that cohort ages.

Empirical evidence supports this life-cycle relationship between wages and productivity (see Kotlikoff and Gokhale (1992) and Hutchens (1987)), and ours is not the only model consistent with this evidence — but it is novel in its pure focus on reputation. Lazear (1981) shows that paying wages below a worker's marginal product early in his tenure with the firm, and above his marginal product later, can partially offset moral hazard problems. Inasmuch as age and tenure with one's current firm are positively correlated, our models offer the same predictions for life cycle profiles. Harris and Holmstrom (1982) develop a model where workers' gradually abilities are revealed by observing stochastic output. They also show that equilibrium wage contracts produce the wage productivity gap increasing with age.

6 Conclusion

Just two changes: Dynamics, and uncertainty. Separately, neither affects the classic PAM conclusion of Becker (1973). Jointly, they induce a reputational concern that unravels PAM. We have created literally the *simplest possible dynamic model* of this economically important concern with no inessential structure: just two types of agents, all long-lived and patient, and publicly observed stochastic output.

Despite such parsimony, we discovered that contrary to Becker (1973), production complementarity no longer implies global PAM. We instead find a conflict between productive and informational efficiency. With enough patience, PAM cannot arise in a stylized two period model. One could thus argue that Bayesian experimenters usually choose a non-myopically optimal strategy. But that critique would be wrong. For we have shown by a robust example that PAM may well be informationally efficient for high or low agents. We have argued, however, that it cannot *globally* be informationally best when agents are patient enough: *Either* high *or* low reputation agents will match non-assortatively, but often not both. What matters is a new statistically-based condition on the production technology alone that is completely unrelated to supermodularity.

Our finding is further subtle because the underlying two-period logic essentially falls apart with no last period to rely on: In an infinite horizon world, value functions flatten out with patience, and an unqualified PAM failure no longer holds. Still, we prove that PAM fails for high reputation agents if production is sufficiently high skill — as with the Beatles. Our proof relies on what turned out to be a knife-edge trade-off between informational and productive efficiency, as δ races up to 1. The skill conditions are so weak that they almost always hold as the number of outputs N grows — and in simulations, almost always at very low N, such as 3 or 4.

Optimal experimentation models with a continuum of heterogeneous agents are hard, and so as yet unexplored. In this setting, we proved existence and a welfare theorem. We then developed a proof by counterfactual that we hope others can use. Apart from the PAM failure characterization, there are two positive results: wage profile jumps, and the efficiency of break-ups among matched stars in creative industries. Matches efficiently dissolve as reputations are established. This is an interesting potential area for empirical work.

On the wage profile, intuitively, agents who have been around longer should have more focused reputations then agents at the beginning of their careers. Our paper thus predicts that older workers will, on average, match with younger workers. We would expect to see such a pattern in environments where information about agents' abilities is incomplete, output contributions are not fully identified, production is stochastic, and future reputations are important. Our reputation and information rent story fundamentally differs from explanations owing to job-training, where matching older workers (the skilled) with younger workers (the unskilled) may be efficient.

In summary, Becker (1973) clearly identified the single most important economic force in matching, focusing on the static productive concern. We believe that learning about types is an important secondary consideration in dynamic matching environments. Inasmuch as PAM is an imperfect metaphor, and regressions have an imperfect fit, we think that our information rents may be relevant. We believe that this is novel, and the simplicity of our model attests to how primal is the force. We further think it sheds new light on the economic forces behind match instability in many industries.

A Existence, Welfare Theorems, and Values

For our formal proofs, we assume that the social planner chooses the measure over matches μ, where $\mu(X,Y)$ is the measure of matches of (x,y), where $x \in X$ and $y \in Y$, for measurable sets $X, Y \subset [0,1]$. Thus, the conditional cdf is $F(x|y) \equiv \int_{x' \leq x} \mu(dx', y)$.

A.1 Steady-State Pareto Optima: Proof of Theorem 1

Equip[13] $Z \equiv \mathcal{L}_\infty([0,1])$ and $X \equiv \mathcal{L}_\infty([0,1]^2)$ with the standard norm topology. The dual X^* of X is the space of bounded measures on $[0,1]^2$. We endow X^* with the weak* topology. Let $\Phi : Z \to X^*$ be the correspondence that captures constraint (2):

$$\Phi(g) = \{\mu \in X^* : \lambda_g(A) \geq \mu(A \times [0,1]) \geq 0 \ \forall A \text{ measurable}\}, \ g \in Z \tag{16}$$

where $\lambda_g(A) \equiv \int_A g d\lambda$, and λ is Lebesgue measure.

Claim 1 *The correspondence* Φ *given by* (16) *is continuous and compact valued.*

Proof: Alaoglu's Theorem (see Royden (1988), §10.6) states that if $\Phi(g) \subset X^*$ is weak* closed, bounded, and convex then $\Phi(g)$ is weak* compact. Convexity and boundedness are immediate. For weak-* closed, let $\mathbb{I}_Y$ be the characteristic function of the set Y. Let $\mathbb{B}^A_{[a,b]} = \{\mu : a \leq \int \mathbb{I}_{A\times[0,1]} d\mu \leq b\}$, and likewise define notation for open intervals and half-open and half-closed intervals. Note that $\Phi(g) = \bigcap_A \mathbb{B}^A_{[0,\lambda_g(A)]}$ and that $\mathbb{I}_{A\times[0,1]} \in X$. By definition, $\mathbb{B}^A_{[0,\lambda_g(A)]}$ is weak* closed, and therefore $\Phi(g)$ is weak* closed.

We now show that this correspondence is upper and lower hemi-continuous (u.h.c. and l.h.c.). Now Φ is point closed, and we can assume without loss of generality that it maps into a compact subset of X^*, say with upper bound $M < \infty$. Thus, u.h.c. follows if Φ has the closed graph property — i.e. if for any $g \in Z$: $\mu \notin \Phi(g)$ implies that there exists an open set $\mathcal{O}$ that contains μ such that $\mathcal{O} \cap \Phi(g) = \emptyset$. But $\mu \notin \Phi(g)$ if $\mu > \lambda_g(A)$ for some A. The result follows from continuity of $\lambda_g(A) = \int \mathbb{I}_{A\times[0,1]} d\mu$ in μ.

To show l.h.c., WLOG we only consider (basis) open sets of the form $\mathcal{O} = \bigcap_{k=1}^m \mathbb{B}^{A_k}_{(a_k,b_k)}$. Pick $\mu \in \mathcal{O} \cap \Phi(g)$, and let $\mu_\epsilon(A \times [0,1]) \equiv \mu(A \times [0,1]) - \epsilon\lambda(A)$ for all A. We claim that there exists $\epsilon, \eta > 0$ such that $\mu_\epsilon \in \mathcal{O} \cap \Phi(\hat{g})$ for all $\hat{g} \in Z$ with $\|\hat{g} - g\|_\infty < \eta$. Pick

[13] We are indebted to Ennio Stacchetti for providing the key insights for this proof. Any errors and preference for hemi- over semi-continuity are, of course, our responsibility.

any such $\hat{g}$. For ϵ small enough, $\mu_\epsilon \in \mathcal{O}$. To show $\mu_\epsilon \in \Phi(\hat{g})$, i.e. $\lambda_{\hat{g}}(A) \geq \mu_\epsilon(A \times [0,1])$ for all A, first note that $|\lambda_g(A) - \lambda_{\hat{g}}(A)| < \eta\lambda(A)$ for all A, as shown below:

$$|\lambda_g(A) - \lambda_{\hat{g}}(A)| = |\textstyle\int_A g d\lambda - \int_A \hat{g} d\lambda| \leq \int_A |g - \hat{g}| d\lambda \leq \int_A \sup_{x\in A} |g(x) - \hat{g}(x)| d\lambda$$
$$= \sup_{x \in A} |g(x) - \hat{g}(x)| \lambda(A) \leq \|g - \hat{g}\|_\infty \lambda(A) < \eta\lambda(A)$$

Thus, $\lambda_{\hat{g}}(A) > \lambda_g(A) - \eta\lambda(A)$, so that $\lambda_g(A) - \eta\lambda(A) \geq \mu(A \times [0,1]) - \epsilon\lambda(A)$ suffices. Since $\lambda_g(A) \geq \mu(A \times [0,1])$, it is enough that $\eta\lambda(A) \leq \epsilon\lambda(A)$, or $\sigma \leq \epsilon$. □

Continuing with our development, define the Bayes operator $B : X^* \to Z$ by

$$B(\mu)(z) = \textstyle\int \rho(z, x, y) d\mu(x, y), \ \mu \in X^*, \ z \in [0,1] \tag{17}$$

where $\rho(z, x, y)$ is the easily computed probability that x updates to z when matched with y plus the probability that y updates to z when matched with x. Let

$$W = \{\mathcal{V} : Z \to \mathbb{R} : \mathcal{V} \text{ is homogeneous of degree 1, continuous, and } \|\mathcal{V}\| < \infty\} \tag{18}$$

where W has norm $\|\mathcal{V}\| = \sup_{\|g\|\leq 1} |\mathcal{V}(g)|$. Define the Bellman operator T on W by:

$$T\mathcal{V}(g) = \max_{\mu \in \Phi(g)} \Gamma(\mathcal{V}, \mu) \quad \text{where } g \in Z \text{ and } \quad \Gamma(\mathcal{V}, \mu) = (1 - \delta) \int Q d\mu + \gamma \mathcal{V}(\overline{g} + \sigma B(\mu))$$

Claim 2 *$T : W \to W$, where W is given by* (18).

Proof: The mapping clearly preserves boundedness and homogeneity. We now show that T preserves continuity. First, $\Gamma(\mathcal{V}, \mu)$ is weak* continuous in $(\mathcal{V}, \mu) \in W \times X^*$. Indeed, $\mu \mapsto \int Q d\mu$ is weak* continuous on X^*. Similarly $\rho \in X$ yields $\mu \mapsto B(\mu)$ weak* continuous on X^*. For each $\mathcal{V} \in W$, the composition $\mathcal{V}(B(\mu))$ is continuous in μ. Thus, $\Gamma(\mathcal{V}, \mu)$ is continuous. Also, the constraint correspondence is continuous and compact valued by Claim 1. Then $T\mathcal{V}$ is continuous by Robinson and Day (1974) — a generalization of Berge's Theorem of the Maximum. □

Claim 3 *For any density g, a Pareto optimum value $\mathcal{V}$ and matching measure μ exists.*

Proof: First, the Bellman operator T is a contraction. Indeed, T is monotonic and $T(\mathcal{V} + c) = T\mathcal{V} + \gamma c$, where $0 < \gamma < 1$ and c is real. Thus, T is a contraction by

$$g \xrightarrow[\text{u.h.c., convex- and closed-valued}]{T^* \text{ (maximization)}} \mu \xrightarrow[\text{continuous \& linear}]{\Theta \text{ (Bayes' Rule)}} g$$

Figure 6: **Existence.** This schematic illustrates the proof of steady state existence.

Blackwell's Theorem, has a unique fixed point $\mathcal{V}$ in W by the Banach Fixed Point Theorem. So $\mathcal{V}$ is continuous, as is the composition $\mathcal{V}(B(\mu))$. Thus, the maximizer μ of the continuous function $\Gamma(\mathcal{V}, \mu)$ on the compact constraint set $\Phi(g)$ exists. $\square$

Claim 4 *There exists a density g and a matching measure μ that is a steady-state PO.*

Proof: Define the correspondence $T^* : Z \to X^*$ by $T^*(g) = \arg\max_{\mu \in \Phi(g)} \Gamma(\mathcal{V}, \mu)$, where $g \in Z$. Let $\Theta : X^* \to Z$ be the function capturing the transition equation: $\Theta(\mu) = \overline{g} + \sigma B(\mu)$, for $\mu \in X^*$. If the map $T^* \circ \Theta : X^* \to X^*$ has a fixed point $g^* = T^*(g^*)$, then we can assume a constant optimal matching measure $\mu^* \in \Phi(g^*)$.

By an extension of the Kakutani Fixed Point Theorem in Istratescu (1981), §10, it suffices that $T^* \circ \Theta$ be nonempty, convex-valued, closed-valued, and u.h.c. Claim 3 yields non-emptiness. Now, T^* is u.h.c. and closed-valued by Robinson and Day (1974). It is convex-valued, as Γ is linear in μ and the constraint set $\Phi(g)$ is convex. Claim 2 proved Θ continuous. As a composition of a continuous function and a u.h.c. correspondence, $T^* \circ \Theta$ is u.h.c. Also, Θ is linear in μ, and preserves closedness and convexity; thus, $T^* \circ \Theta$ is convex-valued and closed-valued. It has a fixed point $g^* = T^*(g^*)$. $\square$

A.2 Welfare Theorems: Proof of Theorem 2

First Welfare Theorem. Let $\langle f, F \rangle \equiv (1-\delta) \sum_0^\infty \gamma^t \int_{[0,1]^2} f(x,y) dF(x|y) dy$ for measurable functions f. Assume that (F, v, w) is a CE, but F is not a PO. Thus, there exists feasible F' with $\langle Q, F' \rangle > \langle Q, F \rangle$. Define $w^y(x,y) = w(y|x)$. By definition of a CE and (3), we have $\langle w + w^y, F \rangle = \langle Q, F \rangle$ and $\langle w + w^y, F' \rangle = \langle Q, F' \rangle$. Hence, $\langle w + w^y, F' \rangle > \langle w + w^y, F \rangle$. By symmetry, $\langle w, F' \rangle = \langle w^y, F' \rangle$, and so $\langle w, F' \rangle > \langle w, F \rangle$.

Let $\hat{g}$ be the density associated with the matching $\hat{F}$, and let $\rho_t(z, x, F)$ be the chance that agent x at time 0 updates to z at time t. By worker maximization (10),

$$\textstyle\sum_t \delta^t \int \int_{z \in supp\, g} w(z|y) \rho_t(z, x, F) \frac{dF(z|y)}{g(z)} dy \geq \sum_t \delta^t \int \int_{z \in supp\, g} w(z|y) \rho_t(z, x, \hat{F}) \frac{d\hat{F}(z|y)}{\hat{g}(z)} dy$$

Multiply both sides by $g(x)$, and integrate over x. This yields $\langle w, F\rangle \geq \langle w, \hat{F}\rangle$, which contradicts $\langle w, \hat{F}\rangle > \langle w, F\rangle$. Thus, F is a PO.

To prove that v is a multiplier in the planner's problem for the given (efficient) F, we show that (F, v) satisfies the planner's FOC. Take any matched pair (x, y). If we sum the worker maximization conditions (10) for x and y we obtain:

$$v(x) + v(y) = (1-\delta)(w(x|y) + w(y|x)) + \delta\Psi^v(x, y)$$

Since $w(x|y)+w(y|x) = Q(x, y)$, the planner's FOC (9) is satisfied for this matched pair. Now take any (x, y) (not necessarily matched). Worker maximization (10) implies:

$$v(x) \geq (1-\delta)w(x|y) + \delta\psi^v(x|y) \qquad \text{and} \qquad v(y) \geq (1-\delta)w(y|x) + \delta\psi^v(y|x)$$

Summing these two inequalities and applying (3) yields:

$$v(x) + v(y) \geq (1-\delta)(w(x|y) + w(y|x)) + \delta\Psi^v(x, y) = (1-\delta)Q(x, y) + \delta\Psi^v(x, y). \qquad \square$$

Second Welfare Theorem. Let (F, v) be a PO. Assume that the pairs (x, y) and $(\hat{x}, \hat{y})$ are matched in the PO, but there does not exist a CE in which these pairs are matched. Let $V(x, y) \equiv (1-\delta)Q(x, y) + \delta\Psi^v(x, y)$. By definition of PO, we have:

$$V(x, y) + V(\hat{x}, \hat{y}) \geq V(x, \hat{y}) + V(\hat{x}, y)$$

As this holds for *any* matched pairs, output shares w exist so that (F, v, w) is a CE. $\square$

A.3 Strict Convexity of Values: Proof of Lemma 1

Claim 5 *The value function v is convex.*

Proof: If information is revealed about agents' true types in an ε-mass ball around x, then their reputations experience a mean preserving spread, as beliefs are a martingale. The first order change in the planner's value equals the sum of the changes in shadow values, or $\varepsilon E[v(\tilde{x}) - v(x)]$. Since the planner may ignore this signal, this cannot be negative; this reverse direction of Jensen's inequality proves weak convexity. $\square$

Claim 6 *The value v is linear on no interval, and so is strictly convex.*

Proof: Fix $x > 0$. Since $Q(x,x) > 0$, and the planner can always "self-match" any x for whom $g(x) > 0$, it cannot be optimal to leave x unmatched; thus, for all x there must exist $y \in \text{supp}\{F(\cdot|x)\}$. Along with the FOCs (9) this implies that:

$$v(x) = \max_y[(1-\delta)Q(x,y) + \delta\Psi^v(x,y) - v(y)] \tag{19}$$

If v is linear, then Ψ^v is modular (i.e. $\Psi^v_{12} = 0$). As Q is strictly SPM, match values are strictly SPM, so that PAM obtains. But then $v(x) = (1-\delta)Q(x,x)/2 + \delta\Psi^v(x,x)$. As Q is strictly SPM, $Q(x,x)$ is strictly convex, which contradicts v globally linear.

Next, being convex (Lemma 1), v is continuous. Any maximal interval of linearity in $[0,1]$ is closed, say $[\underline{x}, \bar{x}]$. We now argue by contradiction that no such interval can exist, so that v is strictly convex. By continuity of $z_i(x,y)$ and Assumption 2, $\exists \varepsilon \in (0, \bar{x} - \underline{x})$, such that $\forall\ y\ \exists\, i$ s.t. $z_i(\bar{x}-\varepsilon, y) > \bar{x}$. The logic of Lemma 2 yields Ψ^v strictly convex at $\bar{x}-\varepsilon$ for all y. But since $v(\bar{x}-\varepsilon) = \max_y[(1-\delta)Q(\bar{x}-\varepsilon,y) + \delta\Psi^v(\bar{x}-\varepsilon,y) - v(y)]$, where the maximand is strictly convex at $\bar{x}-\varepsilon$, this contradicts v linear on $[\underline{x}, \bar{x}]$. □

A.4 Convexity of Continuations: Proof of Lemma 2

Since v is convex, it is twice differentiable a.e., as is $\psi^v(x|y) = E[v(z(\tilde{q},x,y))]$ in x or y. Let $u = x$ or y. Now $p_i(x,y)$ and $p_i(x,y)z_i(x,y)$ are both bilinear in (x,y). So $p_{iuu}(x,y) = 0$. Also, $p_i(x,y)z_i(x,y)$ is linear in x or y given (4); thus, its second derivative vanishes:

$$p_{iuu}z_i + 2p_{iu}z_{iu} + p_i z_{iuu} = 0 \tag{20}$$

This yields $\psi^v_{uu}(x,y) = \sum_i p_i(x,y)z_{iu}(x,y)^2 v''(z_i(x,y)) > 0$, whenever v'' exists. □

A.5 Convexity of Expected Continuations: Derivation of (6)

To derive the expression (6), note that

$$\psi^v(x|y) = \pi \textstyle\sum_i p_i(x,y)z_i(x,y)^2/2 + (M-L)x + L/2$$

and so, using (20),

$$\psi^v_{yy}(x|y) = \pi \textstyle\sum_i \left[p_i z_{iy}^2 + \frac{z_i}{2}(p_{iyy}z_i + 2p_{iy}z_{iy} + p_i z_{iyy})\right] = \pi \sum_i p_i z_{iy}^2.$$

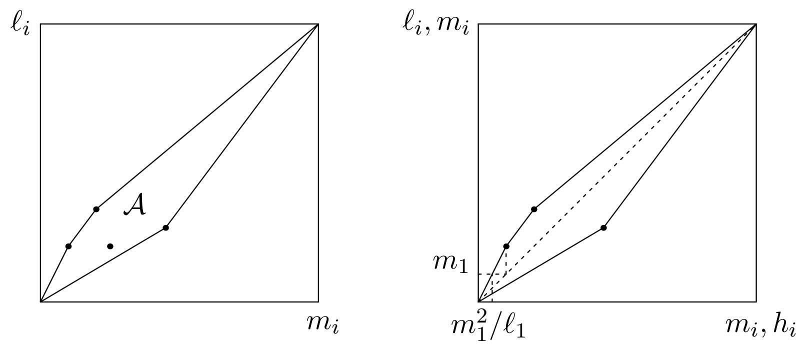

Figure 7: **Example of Dominance Region.** On the left, we illustrate the convex hull $\mathcal{A}$ of the points $(0,0)$, $(1,1)$, and (m_i, ℓ_i) across all i. For $(x,0)$ to dominate (x,x), $x(h_i, m_i) + (1-x)(m_i, \ell_i) \in \mathcal{A}$ for all i. The right hand panel illustrates how $h_1 < m_1^2/\ell_1$ implies that $(x,0)$ cannot dominate (x,x).

B PAM is Not A Priori Informationally Inferior

We firstly derive a simple necessary condition for $(x,0)$ to *dominate* (x,x), in the sense of Blackwell (i.e. statistical sufficiency). We then show that this requirement is not generically satisfied — to wit, PAM is not necessarily informationally inferior. Our approach is graphical and thus loose, as our intention is to motivate.

Let $\mathcal{P}_{(x,y)}$ be the conditional distribution of signals from an (x,y) match, a $2 \times N$ matrix with columns $((p_i(1,y), p_i(0,y))$. Then $(x,0)$ dominates (x,x) *iff* there exists an $N \times N$ Markov matrix Λ (non-negative with rows sums 1) so that $\mathcal{P}_{(x,x)} = \mathcal{P}_{(x,0)}\Lambda$:

$$x \begin{pmatrix} h \\ m \end{pmatrix} + (1-x) \begin{pmatrix} m \\ \ell \end{pmatrix} = \begin{pmatrix} m \\ \ell \end{pmatrix} \Lambda$$

Put $\lambda_j \equiv \sum_i \lambda_{ij}$. Since $\sum_i h_i = \sum_i m_i = \sum_i \ell_i = 1$, this can be simplified to:

$$x(h_j, m_j) + (1-x)(m_j, \ell_j) = \lambda_j \sum_i \frac{\lambda_{ij}}{\lambda_j}(m_i, \ell_i) \quad \forall\ j = 1, \ldots, N-1 \tag{21}$$

Thus, $(x,0)$ dominates (x,x) *iff* (21) holds for some $\{\lambda_{ij} \geq 0\}$ with $\sum_j \lambda_{ij} = 1$.

Since $\lambda_{ij} \leq 1$, we have $\lambda_j \leq N$. Thus, imposing $\lambda_{ij} \geq 0$ and only the weaker constraint that $\lambda_j \leq N$ yields a necessary condition for $(x,0)$ to dominate (x,x). Let's illustrate the $N = 4$ example seen in the left panel of Figure 7. Examining (21), we first take the convex hull of all pairs (m_i, ℓ_i). Then by scaling these pairs by some $\lambda_i \in [0, N]$, we trace out the region $\mathcal{A}$ equal to the convex hull of $\{(0,0), (m_1, \ell_1), \ldots, (m_N, \ell_N), (1,1)\}$.

Thus, our necessary condition is restated as

$$x(h_j, m_j) + (1-x)(m_j, \ell_j) \in \mathcal{A} \quad \forall\, j = 1, \ldots, N-1$$

Order the m_i and ℓ_i so that $m_1/\ell_1 = \max_i m_i/\ell_i$. Then (right panel in Figure 7):

$$h_1 < m_1^2/\ell_1 \Rightarrow x(h_1, m_1) + (1-x)(m_1, \ell_1) \notin \mathcal{A} \qquad \text{for any } x \in (0,1).$$

Thus we see that $(x,0)$ may dominate (x,x) for low x, but this is not generically true.

C S as a Measure of Statistical Distance

Our triple distance measure admits a statistical interpretation. The Kullback-Leibler distance between probability density vectors a and b is denoted by:

$$KL(a,b) = \textstyle\sum_i a_i \log(a_i/b_i).$$

This is the most commonly used (albeit *asymmetric*) distance measure between two conditional signal distributions in the information literature outside of economics. It obeys $KL(a,b) \geq 0$ with $KL(a,b) = 0$ iff $a = b$. We can write $S(h,m,\ell) = KL(m,\ell) + KL(h,m) - KL(h,\ell) + \sum(m_i^2 - h_i\ell_i)/\ell_i$. The latter term is positive since,

$$\textstyle\sum_i (m_i^2 - h_i\ell_i)/\ell_i = \sum_i (m_i^2 - m_i\ell_i)/\ell_i = \sum_i (m_i - \ell_i) m_i/\ell_i \tag{22}$$

and $\sum_i (m_i - \ell_i) = 0$, while $m_i - \ell_i$ is weighted by $m_i/\ell_i \gtreqless 1$ for $m_i - \ell_i \gtreqless 0$. We see that $S(h,m,\ell) > 0$ surely when $KL(m,\ell) + KL(h,m) > KL(h,\ell)$. Now KL does not obey this triangle inequality as a rule, but it is intuitively true when m is far enough from ℓ and h compared with how far h is from ℓ. In this sense, S is a distance measure.

D Non-Assortative Matching

D.1 Asymptotic Analysis for Lemma 3 and Proposition 2

We proceed by contradiction, assuming PAM and using the implied value function v_δ. Indeed, given $v_\delta'(0) = M - L$ and $v_\delta'(1) = H - M$ fixed, $\int_0^1 v_\delta''(x)dx$ is constant in δ. Since the value function flattens ($\lim_{\delta\to 1} v''_\delta(x) = 0$) for any interior $x \in (0,1)$, convexity must accumulate at 0 and 1, as in Figure 3. Curiously, we now show that $\delta \sum_i m_i^2/\ell_i > 1$ suffices near $x = 0$, so that the convexity explosion may obtain far from $\delta = 1$: for instance, if $\ell = (0.01, 0.99)$ and $m = h = (0.99, 0.01)$, then $\delta > (\sum_i m_i^2/h_i)^{-1} \approx 1/98$ works.

Claim 7 *Assume PAM and $\delta \sum_i m_i^2/\ell_i > 1$. Define $\alpha_\delta \in (0,1)$ by (13). Then:*

(i) Asymptotically, $v''_\delta(x) \sim c_\delta x^{-\alpha_\delta}$ as $x \to 0$ where $c_\delta > 0$ (i.e. $\lim_{x\to 0} x^{\alpha_\delta} v''_\delta(x)/c_\delta = 1$).

(ii) $\lim_{\delta\uparrow 1} c_\delta/(1-\alpha_\delta) = c$ where $c > 0$.

Likewise, for $\delta < 1$ big enough that $\delta \sum_i m_i^2/h_i > 1$, we have $v''_\delta(1-x) \sim \hat{c}_\delta x^{-\beta_\delta}$ near $x = 0$ for $\beta_\delta \in (0,1)$ given by (13); further, $\hat{c}_\delta/(1-\beta_\delta) \to \hat{c}$ as $\beta \uparrow 1$, for some $\hat{c} > 0$.

Part (i) provides a functional form for the second derivative near $x = 0$ at fixed δ. Essentially, v'' geometrically blows up as fast as it can and still leave v' integrable.

Proof: By Anderson and Smith (2004) [AS02],[14] for any twice continuously differentiable policy function $a(x)$, the second derivative $v''(x)$ obeys part (i) as long as $v(x)$ is convex, the static output function $Q(x, a(x))$ is strictly convex, and $\delta \sum_i m_i^2/\ell_i > 1$ holds. These assumptions obtain, as $a(x) = x$, and $Q(x,x)$ is strictly convex by SPM.

Part (ii) also follows from AS02. Beyond the assumptions necessary for part (i), we need the derivative of the policy function finite at $x = 0$, and $|p_{iy}(x,y)| < \infty$. Since $a(x) = x$ and $p_i(x,y)$ is linear, these assumptions are easily satisfied. □

Define $R(\alpha) \equiv \sum_i (h_i - m_i)(m_i/\ell_i)^{1-\alpha} + (1-\alpha)\sum_i [(h_i\ell_i - m_i^2)/\ell_i](m_i/\ell_i)^{1-\alpha}$.

Claim 8 $\Psi^\delta_{12}(x,x) \sim c_\delta R(\alpha_\delta) x^{1-\alpha_\delta}/(1-\alpha_\delta)$ *near $x = 0$*

Proof: Computing $\Psi^\delta_{12}(x,x)$ we have:

$$\kappa_\delta(x) \equiv \Psi^\delta_{12}(x,x) = \textstyle\sum_i (\pi_i v_\delta(z_i) + a_i(x) v'_\delta(z_i) + b_i(x) v''_\delta(z_i)), \tag{23}$$

where $\pi_i = h_i - 2m_i + \ell_i$, $a_i(x) = h_i - m_i + O(x)$, $b_i(x) = [(h_i\ell_i - m_i^2)m_i/\ell_i^2]x + O(x^2)$.[15]

[14] AS02 studies a generic Bayesian experimentation problem, with two possible states of the world and a distribution over signals for each action a. We can translate to the AS02 framework by letting the action $a(x)$ be the reputation of the agent to match with x, and $h_i y + m_i(1-y)$ and $m_i y + \ell_i(1-y)$ be the probabilities of realizing signal q_i given action y in the high and low state of the world, respectively.

[15] $\phi(x) = o(g(x))$ iff $\lim_{x\to 0} \phi(x)/g(x) = 0$ while $\phi(x) = O(g(x))$ if $\limsup_{x\to 0} |\phi(x)/g(x)| < \infty$.

We can use our asymptotic approximation $v''_\delta(x) = c_\delta x^{-\alpha_\delta}(1 + o(1))$ from Claim 7 to approximate (23), and integrate to yield approximations for v_δ and v'_δ. Making this substitution and using $z_i(x) = (m_i/\ell_i)x + O(x^2)$, we find that the terms in $\kappa_\delta(x)$ are:

$$\begin{aligned}
\sum_i \pi_i v_\delta(z_i) &= \sum_i \pi_i \left[\frac{c_\delta}{(1-\alpha_\delta)(2-\alpha_\delta)} z_i^{2-\alpha_\delta} + v'_\delta(0) z_i + v_\delta(0) + o(x^{2-\alpha_\delta}) \right] \\
&= \left(\textstyle\sum_i \pi_i (m_i/\ell_i)\right) v'_\delta(0) x (1 + o(1)) \\
\sum_i a_i(x) v'_\delta(z_i) &= \sum_i (h_i - m_i + O(x)) \left[v'_\delta(0) + \frac{c_\delta}{1-\alpha_\delta} z_i^{1-\alpha_\delta}(1 + o(1)) \right] \\
&= \frac{c_\delta}{1-\alpha_\delta} \sum_i (h_i - m_i) \left(\frac{m_i}{\ell_i} \right)^{1-\alpha_\delta} x^{1-\alpha_\delta}(1 + o(1)) \\
\sum_i b_i(x) v''_\delta(z_i) &= c_\delta \sum_i \left[\frac{(h_i \ell_i - m_i^2) m_i}{\ell_i^2} x + O(x^2) \right] \left(\frac{m_i}{\ell_i} x \right)^{-\alpha_\delta} (1 + o(1)) \\
&= c_\delta \sum_i \frac{h_i \ell_i - m_i^2}{\ell_i} \left(\frac{m_i}{\ell_i} \right)^{1-\alpha_\delta} x^{1-\alpha_\delta}(1 + o(1))
\end{aligned}$$

Thus,

$$\kappa_\delta(x) = \frac{c_\delta}{1-\alpha_\delta} R(\alpha_\delta) x^{1-\alpha_\delta}(1 + o(1)).$$

We need to know that this term is negative and bounded away from zero as $\delta \to 1$. We have already shown that $c_\delta > 0$ for $\delta < 1$ (Claim 7 part (i)). Thus, we need only show now that $R(\alpha) < 0$ for large enough $\alpha < 1$, since $\alpha_1 = 1$ and α_δ is increasing in δ near 1. Since $R(1) = 0$, we are done if $R'(1) > 0$. For then $\kappa_\delta(x) < 0$ for δ near 1 for x small enough. Differentiation then reveals that $R'(1) = S(h, m, \ell)$. $\square$

D.2 PAM Almost Always Fails: Proof of Proposition 3

We need the measure of parameters (h, m, ℓ) for which $S(h, m, \ell) > 0$ fails (the opposite weak inequality holds) to vanish as n increases. That is, if we define $s_n^1 = (1/n)\sum_i \log(m_i^n/\ell_i^n)(m_i^n - h_i^n)$ and $s_n^2 = (1/n)\sum_i (m_i^n)^2/\ell_i^n$, then $Pr[s_n^1 + s_n^2 > 1] \to 1$.

For s_n^1, assume h, m, ℓ are each independently generated by a uniform measure on the simplex, denoted Δ_N. The map $(m, h) \mapsto (m - h)\log(m/\ell)$ is strictly convex in (m, h). By Jensen's inequality and $Em_i = Eh_i$, if $u_i \equiv \log(m_i/\ell_i)(m_i - h_i)$ then $E(u_i) > \log[(Em_i)/\ell_j](Em_i - Eh_i) = 0$, for all fixed ℓ_j. We are tempted to apply a Law of Large Numbers, but the summands are dependent (eg. $\sum_i h_i = 1$), and the domain

$(\Delta_N)^3$ changes in N. (In fact, this expectation vanishes in the limit, by Claim 9.)

Instead, with an atomless assumption, we assume WLOG that $h^n = (h_1^n, h_2^n, \ldots, h_n^n)$ is uniformly distributed on Δ_n. The joint density $(n-1)!$ has marginals (λ is Lebesgue measure) $d\lambda_i(h_i^n) = (n-1)(1-h_i^n)^{n-2}$ and $d\lambda_{ij}(h_i^n, h_j^n) = (n-1)(n-2)(1-h_i^n-h_j^n)^{n-3}$.

Claim 9 *The sum s_n^1 vanishes in probability: $s_n^1 \Rightarrow 0$.*

Proof: First $Es_n^1 \to 0$. Indeed, straightforward calculation using the densities given above yields that $Es_n^1 = ((n-1)/n^2)(\varsigma + \Gamma'(n)/\Gamma(n))$. Here ς is Euler's constant and $\Gamma(n) \equiv \int_0^\infty s^{n-1}e^{-s}ds$ is the Gamma function, i.e. $\Gamma'(n) = \int_0^\infty \log(s)s^{n-1}e^{-s}ds$. It is known that (the 'ψ function') $\Gamma'(n)/\Gamma(n) \sim \log n$ as $n \to \infty$. So $Es_n^1 > 0$ but $Es_n^1 \to 0$.

Then by routine calculations, $\text{var}(s_n^1) \to 0$. Thus, $Pr[|s_n^1 - Es_n^1| \geq \varepsilon] \to 0\ \forall\varepsilon$. Finally, Chebyshev's inequality states that $Pr[|s_n^1 - Es_n^1| \geq \varepsilon] \leq \text{var}(s_n^1)/\varepsilon^2\ \forall\varepsilon$. $\square$

The second sum is positive, as we have shown in (22). More strongly, we have:

Claim 10 *The sum s_n^2 converges in probability to some constant ≥ 2.*

Proof: Define $\tilde{\ell}_i^n = \ell_i^n$ if $\ell_i^n \geq 1/n^2$ and $1/n^2$ otherwise. Let $\tilde{s}_n^2 = s_n^2$ with ℓ_i^n replaced with $\tilde{\ell}_i^n$, and note that $s_n^2 \geq \tilde{s}_n^2$. It suffices to prove $\tilde{s}_n^2 \Rightarrow 2$.

We first claim that $E\tilde{s}_n^2 \to 2$. Indeed, if $\rho^n = 1-(1-1/n^2)^{n-1}$ is the chance that $\ell_i \leq 1/n^2$, then $E(1/\tilde{\ell}_i^n) = n^2\rho^n + \int_{1/n^2}^1 (n-1)(1-s)^{n-2}/s ds \leq n^2$. Letting $\rho^n \to 1$ yields $E(1/\tilde{\ell}_i^n) = n^2 + o(n^2)$. Finally, $E(m_i^n)^2 = 2/n(n+1)$, so that $E\tilde{s}_n^2 \to 2$.

It thus suffices that $Pr[|\tilde{s}_n^2 - E\tilde{s}_n^2| \geq \varepsilon] \to 0\ \forall\varepsilon$. By similar reasoning as above, $E(1/(\tilde{\ell}_i^n)^2) = n^4 + o(n^4)$, and $E((1/\tilde{\ell}_i^n)(1/\tilde{\ell}_j^n)) = n^4 + o(n^4)$. Also, $E((m_i^n)^4) = 24/(n(n+1)(n+2)(n+3))$, and $E((m_i^n)^2(m_j^n)^2) = 4/(n(n+1)(n+2)(n+3))$. Then

$$\text{var}(\tilde{s}_n^2) = \frac{1}{n}E\frac{(m_i^n)^4}{(\tilde{\ell}_i^n)^2} + \frac{n-1}{n}E\frac{(m_i^n m_j^n)^2}{\tilde{\ell}_i^n\tilde{\ell}_j^n} - \left(E\frac{(m_i^n)^2}{\tilde{\ell}_i^n}\right)^2 = \frac{4n^2(n-1)}{(n+1)^2(n+2)(n+3)} + o(n)$$

using the independence of m^n from $\tilde{\ell}^n$ — i.e., $\text{var}(\tilde{s}_n^2) \to 0$. Apply Chebyshev. $\square$

D.3 Long Run Match Dynamics: Proof of Proposition 4

We shall assume that no agent ever gets stuck at any reputation.

Assumption 2 *It is not true that* $\dfrac{\ell_i - m_i}{h_i + \ell_i - 2m_i} = c\ \forall i$ *for some constant $c \in [0,1]$.*

This housekeeping condition follows from setting $z_i(x, y) \equiv x$, and is generically valid. Also, its failure yields $L - M = y(H - 2M + L) > 0$, or non-monotonic output.

Case (a). By Easley and Kiefer (1988), the reputation may only converge to a potentially confounding point, where it is unchanged. If $x \neq 0$, then $z_i(x, y) = x$ *iff* $p_i(1, y) = p_i(x, y)$. Since $\partial p_i(x, y)/\partial x$ is constant in x, this requires $\partial p_i(x, y)/\partial x = (h_i + \ell_i - 2m_i)y + m_i - \ell_i = 0$, or $y = (m_i - \ell_i)/(h_i + \ell_i - 2m_i)\ \forall\, i$, contrary to Assumption 2. As there are no interior potentially confounding beliefs, the long run density g_∞ has support $\{0, 1\}$. Reputation being a martingale, the weights are $(1 - x_0, x_0)$. $\square$

Case (b). Let $P_{GG}, P_{GB}, P_{BG}, P_{BB}$ be the current beliefs over the possible state space $\{(G, G), (G, B), (B, G), (B, B)\}$. Let $\hat{P}_{GG} \equiv h_i P_{GG}/(h_i + m_i(P_{GB} + P_{BG}) + \ell_i P_B B)$ be the updated chance that the state is (G, G), and define similar updated beliefs $\hat{P}_{GB}, \hat{P}_{BG}, \hat{P}_{BB}$. Note that $\hat{P}_{ii} = P_{ii}$ iff $P_{ii} = 1$, while $\hat{P}_{ij} = P_{ij}$ iff $P_{ij} = 1$ or $P_{ij} = P_{ji} = 1/2$. Thus, these are the only potentially confounding beliefs. $\square$

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