A NOTE ON A "SQUARE-ROOT RULE" FOR REINSURANCE

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"Square-Root Rule" for Reinsurance

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Abstract

In previous work, the current authors derived a mathematical expression for the optimal (or "saturation") number of reinsurers for a given number of primary insurers (see Powers and Shubik, 2001). In the current paper, we show analytically that, for large numbers of primary insurers, this mathematical expression provides a "square-root rule"; i.e., the optimal number of reinsurers in a market is given asymptotically by the square root of the total number of primary insurers. We note further that an analogous "fourth-root rule" applies to markets for retrocession (the reinsurance of reinsurance).

Keywords: Primary insurance, reinsurance, market size, square-root rule.

JEL Classification:

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1. Introduction

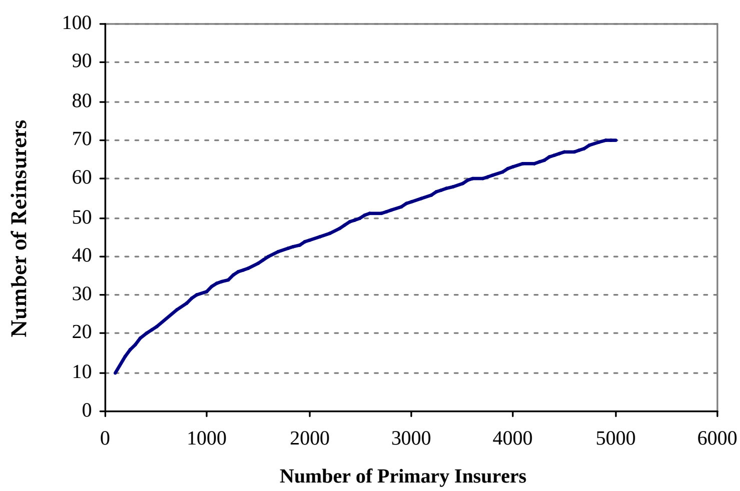
In previous work, the current authors derived a mathematical expression for the optimal (or "saturation") number of reinsurers for a given number of primary insurers (see Powers and Shubik, 2001). Specifically, we identified the optimal number of reinsurers as the maximum value of $n_1 \in \{2,3,\ldots,n_0\}$ such that the price per unit of primary insurance in a market with n_0 primary insurers and n_1 reinsurers remains less than the price of insurance in a market with n_0+1 primary insurers and n_1-1 reinsurers. This marginal analysis yields

$$n_1{*} = Max\left\{n_1 : \left(\frac{n_1}{n_1-1}\right)\left(\frac{n_0}{n_0-1}\right)\left(\frac{n_0}{n_0-n_1}\right) < \left(\frac{n_1-1}{n_1-2}\right)\left(\frac{n_0+1}{n_0}\right)\left(\frac{n_0+1}{n_0-n_1+2}\right)\right\}, \qquad (1)$$

and the existence of a unique solution $n_1{*} \in \{2,3,\ldots,n_0-1\}$ is guaranteed by the fact that the inequality in (1) is satisfied for $n_1 = 2$, but not for $n_1 = n_0$.

Having computed $n_1{*}$ for values of n_0 in the interval $[10, 5000]$, we presented a graph of these results, which is reproduced in Figure 1.

Figure 1. Insurance/Reinsurance Market



From a tabular display of the same results (see Table 1), it is easy to see that the solution to (1), $n_1{}^*$, is approximately equal to the square root of n_0 (although this observation was not made in Powers and Shubik, 2001).

n_0 (Number of Primary Insurers)	$n_1{}^*$ (Optimal Number of Reinsurers)
10	3
20	4
30	5
40	6
50	7
100	10
200	14
300	17
400	20
500	22
1000	31
2000	44
3000	54
4000	63
5000	70
10,000	101

Table 1. Optimal Numbers of Reinsurers for Selected Numbers of Primary Insurers

2. A Square-Root Rule

In the current research, we show analytically that, for large numbers of primary insurers, the solution to problem (1) is indeed a "square-root rule"; i.e., the optimal number of reinsurers in a market is given asymptotically by the square root of the total number of primary insurers, as stated formally in the following result.

Theorem 1: For sufficiently large n_0, there exists a unique solution $n_1{*} = n_1{*}(n_0) \in \{2, 3, \ldots, n_0 - 1\}$ to problem (1), where

$$n_1{*}(n_0) \sim \sqrt{n_0}$$

as $n_0 \to \infty$.

Proof: First, we extend problem (1) from the two-dimensional integer grid $\{(n_0, n_1) : 2 \le n_1 \le n_0\}$ to the corresponding two-dimensional real space $\{(a, x) : 2 \le x \le a\}$ by considering the inequality

$$\left(\frac{x}{x-1}\right)\left(\frac{a}{a-1}\right)\left(\frac{a}{a-x}\right) < \left(\frac{x-1}{x-2}\right)\left(\frac{a+1}{a}\right)\left(\frac{a+1}{a-x+2}\right). \tag{2}$$

Apart from its points of unboundedness, (2) is equivalent to the cubic polynomial inequality

$$\begin{aligned} f(x) = {} & \left(3a^2 - 3a + 1\right)x^3 - \left(5a^3 + 3a^2 - 5a + 2\right)x^2 \\ & + \left(9a^3 - 3a^2 - a + 1\right)x + \left(a^4 - 3a^3 + 3a^2 - a\right) > 0. \end{aligned} \tag{3}$$

Thus, the unique solution specified by the theorem—if it exists—is given by $n_1{*} = \lfloor x{*} \rfloor$, where $x{*} = x{*}(a) \in (2, a)$ is a positive real root of $f(x)$ such that $f'(x) < 0$.

For large a, one can rewrite (2) as

$$\left(\frac{x}{x-1}\right)\left(\frac{x-2}{x-1}\right)\left(\frac{a-x+2}{a-x}\right)\left(\frac{a}{a+1}\right) < \left(\frac{a-1}{a}\right)\left(\frac{a+1}{a}\right) = 1 + O\left(a^{-2}\right),$$

from which it follows that we seek the roots $x(a)$ of

$$\begin{aligned} f(x) = {} & \left[1 + (x-1)^2 O\left(a^{-2}\right)\right]a^2 - \left[x^2 - x - 1 + (x-1)^3 O\left(a^{-2}\right)\right]a \\ & - \left[x^3 - 2x^2 + x + x(x-1)^2 O\left(a^{-2}\right)\right] = 0. \end{aligned} \tag{4}$$

Solving (4) for a as a function of x yields

$$a=\frac{\left[x^2-x-1+(x-1)^3 O\left(a^{-2}\right)\right]}{2\left[1+(x-1)^2 O\left(a^{-2}\right)\right]}$$

$$\pm\frac{\sqrt{\left[x^2-x-1+(x-1)^3 O\left(a^{-2}\right)\right]^2+4\left[1+(x-1)^2 O\left(a^{-2}\right)\right]\left[x^3-2x^2+x+x(x-1)^2 O\left(a^{-2}\right)\right]}}{2\left[1+(x-1)^2 O\left(a^{-2}\right)\right]}. \qquad (5)$$

Anticipating that there exists at least one positive root $x(a)=o\left(a^{2/3}\right)$ to (4), we find exactly one solution to (5); namely,

$$a=\frac{\hat{x}^2+O(\hat{x})}{2[1+o(1)]}+\frac{\sqrt{\hat{x}^4+O\left(\hat{x}^3\right)}}{2[1+o(1)]}\sim\hat{x}^2,$$

which implies

$$\hat{x}(a)\sim\sqrt{a}.$$

For *sufficiently large* a, it is clear that $\hat{x}(a)\in(2,a)$.

To confirm that $\hat{x}(a)$ is the desired root of $f(x)$, we consider the local extrema of this polynomial, given by

$$f'(x)=3\left(3a^2-3a+1\right)x^2-2\left(5a^3+3a^2-5a+2\right)x+\left(9a^3-3a^2-a+1\right)=0,$$

or equivalently,

$$x=\frac{\left(5a^3+3a^2-5a+2\right)\pm\sqrt{\left(5a^3+3a^2-5a+2\right)^2-3\left(3a^2-3a+1\right)\left(9a^3-3a^2-a+1\right)}}{3\left(3a^2-3a+1\right)}. \qquad (6)$$

As $a\to\infty$, the two solutions to (6) are

$$x_L(a)=\frac{\left(5a^3+3a^2-5a+2\right)-\sqrt{\left[5a^3+3a^2-5a+2\right]^2+\left[-81a^5+O\left(a^4\right)\right]}}{9a^2+O(a)}$$

$$= \frac{-\frac{\left[-81a^5 + O\left(a^4\right)\right]}{2\left(5a^3 + 3a^2 - 5a + 2\right)} + O(a)}{9a^2 + O(a)} = \frac{\frac{81}{10}a^2 + o\left(a^2\right)}{9a^2 + O(a)} \sim \frac{9}{10}$$

and

$$x_U(a) = \frac{5a^3 + O\left(a^2\right) + \sqrt{25a^6 + O\left(a^5\right)}}{9a^2 + O(a)} \sim \frac{10}{9}a,$$

respectively. For *sufficiently large* a, it follows that $\hat{x}(a)$ lies between $x_L(a)$ and $x_U(a)$, implying $f'(\hat{x}) < 0$. Therefore, $x^*(a) = \hat{x}(a)$. ■

3. Extension to Retrocession

Beyond the world of ordinary reinsurance lie the misty realms of retrocession (second-order reinsurance), second-order retrocession (third-order reinsurance), and so on. Although the model in our 2001 paper was extended to an arbitrary number of reinsurance levels, we acknowledged that such clearly defined levels are not reflective of the real world. While a few specialized purveyors of retrocession do exist, higher-order reinsurance is typically provided by ordinary reinsurers through the packaging and repackaging of risk through various types of pooling arrangements.

It may be argued that the absence of distinct higher-order reinsurance markets is consistent with the analytical results of our game-theoretic model. Assuming that there exist at least two second-order reinsurers in the market, and again using the minimization of price in the primary insurance market as the optimality criterion, our expression for the optimal number of second-order reinsurers is given by:

$$n_2{*} = Max\left\{ n_2 : \left(\frac{n_2}{n_2-1}\right)\left(\frac{n_1}{n_1-1}\right)\left(\frac{n_1}{n_1-n_2}\right)\left(\frac{n_0}{n_0-n_1}\right) < \left(\frac{n_2-1}{n_2-2}\right)\left(\frac{n_1+1}{n_1}\right)\left(\frac{n_1+1}{n_1-n_2+2}\right)\left(\frac{n_0}{n_0-n_1-1}\right)\right\}. \quad (7)$$

By methods analogous to those employed in the proof of Theorem 1, it is quite straightforward to show the following result.

Theorem 2: For sufficiently large n_0, if $n_1 \sim \sqrt{n_0}$, then there exists a unique solution $n_2{*} = n_2{*}(n_0) \in \{2,3,\ldots,n_1-1\}$ to problem (7), where

$$n_2{*}(n_0) \sim \sqrt[4]{n_0}$$

as $n_0 \to \infty$.

In short, the number of retrocessionaires in a market should be approximately equal to the fourth-root of the number of primary insurers, which for most national insurance markets (other than that of the U.S.) is rather small. Thus, according to the model, one should not expect to see distinct significant retrocession markets, except perhaps in the U.S. This result agrees with empirical observation.

4. A Comment on Paper on Paper

More generally, for all financial instruments involving risk and transaction costs, the principles dictating how many levels of paper are economically optimal need to be considered. We suspect that four or five is an extreme upper bound, and our result here conforms to this.

Reference

Powers, Michael R. and Shubik, Martin, 2001, "Toward a Theory of Reinsurance and Retrocession," *Insurance: Mathematics and Economics*, 29, 2, 271-290.