OVERCOMING PARTICIPATION CONSTRAINTS

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Overcoming Participation Constraints*

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Abstract

This paper shows that linking a sufficiently large number of independent but unrelated social decisions can achieve approximate efficiency. We provide regularity conditions under which a Groves mechanism amended with a veto game implements an efficient outcome with probability arbitrarily close to one, and satisfies interim participation, incentive and resource constraints.

Keywords: Linking, Participation Constraints, Groves Mechanisms, Veto Power

JEL Classification Number: D61, D82, H41.

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1 Introduction

Collective decision making is often a process in which many issues are resolved jointly. An example is the "Uruguay Round" WTO trade agreement of 1994, which is a staggeringly complex agreement, loaded with special provisions. This complexity has been noted in the literature on international trade, where linking of tariff concessions with other issues, such as labor standards, environmental issues, investment liberalization, and human rights codes, has been discussed extensively in recent years. However, no consensus has emerged. Some, like Bagwell and Staiger [4] argue that giving individual countries *more* sovereignty over "domestic issues" (e.g., to make it harder to link certain issues) would improve bargaining outcomes. Others, such as Copeland and Taylor [12], and Hortsmann et al. [19], and Maskus [23] argue that linking may be beneficial.

Another extreme example of linking a large number of separate issues is federal spending on highway projects: the Transportations Act of 2005, includes more than 3000 earmarks for specific "pet projects" such as new bridges, bike-paths, ramps, parking lots, landscaping enhancements among specific highways etc. While, in principle, the situation is similar to trade negotiations, such bills are overwhelmingly interpreted as the result of log-rolling, and associated with inefficient pork-barrel spending (see Battaglini and Coate [6] and the references therein).

In general, government institutions at all levels fulfill many arguable unrelated functions. For example, local governments provide libraries, museums, public parks, public schools, police, fire protection, and many other goods and services to all tax-paying residents of the community. By contrast, the traditional public finance view is that each problem needs a separate remedy. Fire safety and provision of local library services are thought of as two distinct problems that need two separate solutions. In principle, we understand that income effects and complementarities could rationalize a joint treatment, but how is unclear, so this possibility is usually ignored.

However, an emerging literature is identifying asymmetric information as an explanation for why it may be beneficial to link various problems. An early example is Armstrong [1], who considers a multiproduct monopoly problem. He demonstrates that by charging a fixed fee for the right to purchase any good at marginal cost, the monopolist can extract almost the full surplus if the number of goods is large. The inefficiency associated with monopoly pricing is thereby virtually eliminated by linking sales of many goods, with all gains going to the seller.[1] In the context of (excludable) public goods, several papers have shown that *bundling* (the practice of selling several

[1] Palfrey [27] reaches a different conclusion. The reason is that he considers a setup with the monopolist selling a given number of objects using a Vickrey auction.

goods as a package) can be a useful instrument both for a profit maximizing monopolist (see Bakos and Brynjolfsson [5] and Geng et al [17]) and a welfare maximizing planner (see Bergstrom and Bergstrom [7] and Fang and Norman [15]). In the political economy literature, Casella [10] shows that a voting scheme where votes can be stored allows agents to concentrate their votes where preferences are more intense, which typically leads to ex ante welfare gains. Other examples include a risk-sharing agreement proposed by Townsend [29] and a mechanism for partnership dissolution proposed in McAfee [21].

The applications mentioned in the previous paragraph have one property in common: the underlying inefficiency is generated by informational asymmetries. One may therefore ask whether there is some general principle involved. Recently, Jackson and Sonnenschein [20] answered this question affirmatively. They fix an arbitrary underlying social choice problem, and show that first best efficiency is approximately attainable if a sufficiently large number of independent replicas of the underlying problem is available. Their mechanism constrains agents to report preference profiles where the frequency of any particular "base-problem type" coincides with the true probability distribution. They show that agents seek to be as "truthful as possible" if the social choice function is efficient. With many independent replicas, the realized frequencies are likely to be near the true probabilities. As a result, all equilibria result in approximately efficient allocations with high probability.

A limitation of Jackson and Sonnenschein [20] is that it they can only deal with *identical replicas*. In this paper, we circumvent this shortcoming by considering problems with *transferable utility*. This rules out some applications, such as standard voting problems, but leaves many common applications, such as bilateral bargaining, partnership dissolution, multiproduct monopoly provision, and public good problems. The analytical advantage is that we are able to obtain results for arbitrary sequences of underlying choice problems: public good, congestion, and bargaining problems may be linked into a "grand design problem".

If common value aspects are ruled out (as we do), ex post efficiency is attainable if either interim participation or self-financing constraints are absent, since transfers making agents internalize their externalities exist (Groves [18], d'Aspremont and Gerard-Varet [14] and Arrow [2]). Participation constraints are thus the ultimate source of the inefficiency in our environment.

Our paper contains two results. Proposition 1 establishes that a standard Groves mechanism "almost works", in the sense that violations of the participation constraints are unlikely if many problems are linked. Proposition 2, which is the main contribution of the paper, shows that a

Groves mechanism amended with a veto game satisfies all constraints and generates approximate efficiency if many problems are linked.

More precisely, Proposition 1 states that there is an ex ante budget balancing Groves mechanism for which the probability that a participation constraint fails approaches zero as the number of problems goes out of bounds. The key intuition is that, since Groves mechanisms are efficient, there exist ex ante budget balancing lump sum transfers that give all agents a strict incentive to participate *ex ante.* When problems are linked, the participation constraint can be thought of as stating that the average interim expected utility must exceed the average reservation utility. Standard arguments establish that the average interim utility converges in probability to the average ex ante expected utility. Violations of participation constraints are therefore rare if the number of decision problems is large.

While rare, participation constraints will in general fail with positive probability, regardless of how many issues are linked. We therefore ask whether a nearly efficient budget balancing mechanism that *always* fulfills the participation constraints exists. The subtlety in eliminating the positive, albeit small, probability of the failure of participation constraints under the standard Groves mechanism lies in the possibility of *unravelling.* More specifically, to always satisfy the participation constraints we must allow agents to opt out.[2] But, in general incentives to opt into the mechanism depend on what other players do. Hence, the types that do opt out may upset the participation constraints for other types, which potentially could make the mechanism unravel. Indeed in Section 6.4 we provide an explicit example of such unraveling, where as the number of issues goes to infinity the probability that a participation constraint is violated converges to zero under the standard Groves mechanism, but once we allow for opting out the probability of a *veto* goes to one.

The key result of our paper is Proposition 2, where we establish appropriate regularity conditions under which a mechanism can be constructed so as to rule out unravelling of the participation constraints. Specifically, we amend the Groves mechanism with a veto game, which guarantees that all constraints – including the participation constraints – are satisfied for *all* type realizations. If our regularity conditions hold, the ex post efficient outcome is implemented with probability near one if the number of linked problems is large. In essence these regularity conditions impose uniform bounds on the surpluses related to each issue.

[2] To accomodate pure public goods problems we must equip agents with veto power. In many more specific models, "opting out" could be less extreme measures, such as excluding individuals from usage.

The perturbed mechanism has a sequential interpretation, where in stage one all agents decide whether to cast a veto. In stage two, either a Groves mechanism or a status quo outcome is implemented, depending on whether any agent vetoed the mechanism in stage one. Truth-telling is a conditionally dominant strategy in the second stage, implying that the outcome is efficient if and only if no agent vetoes the mechanism. Interestingly, the use of a Groves mechanism in the second stage (as opposed to a mechanism with Bayesian incentive compatibility) is important for our argument, despite the fact that our result is on weak implementation. The reason is that the conditional dominance makes selection irrelevant for incentive compatibility in the second stage, which in turn gives a tractable characterization of the veto game.

Besides showing the benefits from linking across *different* problems, our paper also contributes to the understanding of the fundamental source of the benefits from linking simply by using a more standard mechanism than Jackson and Sonnenschein [20]. That is, the primary construction in Jackson and Sonnenschein [20] is a rationing scheme for messages, which is designed specifically to overcome the problem to obtain information about *intensity of preferences*.[3] In transferable utility environments, the ultimate source of any efficiency is the interaction between participation, feasibility, and incentive constraints, and there is simply no need to ration any messages. Instead, a variation of a standard pivot mechanism is sufficient. Hence, Jackson and Sonnenschein's [20] mechanism can be seen as a generalization of various rationing schemes, such as a vote storage mechanism proposed in Casella [10], a risk-sharing arrangement proposed by Townsend [29], which limits how often the risk averse agent can claim a loss, and a compromising scheme proposed by Borgers and Postl [9]. In contrast, our mechanism may be thought of as a generalization of mechanisms discussed in Armstrong [1], Bergstrom and Bergstrom [7], and Fang and Norman [15].

In terms of applications, international trade negotiations fit rather nicely into our framework, provided that one takes the view that informational asymmetries are important. In particular, the participation constraints may then be interpreted as national sovereignty, which seems to be a highly relevant consideration in multinational agreements. The obvious limitation is that we abstract from "fundamental" linkages between various issues, such as the fact that incentives for technical barriers of trade obviously hinge on the freedom to set tariffs unilaterally. However, there seems to be no reason that the underlying logic for our argument should disappear in a more realistic

[3] One may indeed interpret Jackson and Sonnenschein [20]'s rationing mechanism as an ingenious way of creating transferrable utility out of a non-transferrable utility environment by adding constraints on the available announcements agents can make.

setup, so we believe that our analysis highlights a potentially important fundamental source for gains from highly multidimensional agreements.

2 Groves Mechanisms

Consider an environment with $n \geq 2$ agents and a set D of possible *social decisions,* where implementing social decision $d \in D$ costs $C(d) \in \mathbb{R}$ units of a numeraire good. Each agent $i \in I \equiv \{1, 2, ..., n\}$ is privately informed about a preference parameter $\theta^i \in \Theta^i$ and has a quasi-linear von Neumann-Morgenstern utility function given by $v^i\left(d, \theta^i\right) - t^i$, where t^i is interpreted as a transfer from agent i in terms of the numeraire good. Let $\Theta = \times_{i=1}^n \Theta^i$ denote the set of all possible type profiles, and denote a generic element of Θ by θ. A *pure direct revelation mechanism* (henceforth *mechanism*) is a pair $\langle x, t\rangle$, where $x : \Theta \to D$ is the *allocation rule* and $t : \Theta \to \mathbb{R}^n$ is the *cost sharing rule,* where $t(\theta) = \left(t^1(\theta), t^2(\theta) ..., t^n(\theta)\right)$, and $t^i(\theta)$ is a transfer *from* agent i when θ is announced.

Many classical implementation results apply in this setup with transferable utility and private values. In particular, absent either ex post budget balance or the combination of interim participation and ex ante budget balancing constraints, any efficient allocation rule can be implemented in dominant strategies by a Groves mechanism. For easy reference, we define this class of mechanisms explicitly.

Definition 1 *A mechanism $\langle x, t\rangle$ is a Groves mechanism if for each θ*

$$x(\theta) \in \arg\max_{d\in D} \sum_{i=1}^n v^i\left(d, \theta^i\right) - C(d) \tag{1}$$

and, where for each $i \in I$,

$$t^i(\theta) = C(x(\theta)) - \sum_{j\neq i} v^j\left(x(\theta), \theta^j\right) + \tau^i\left(\theta^{-i}\right), \tag{2}$$

and $\tau^i : \Theta^{-i} \to \mathbb{R}$ may be arbitrarily chosen.

3 Problems with Many Independent Social Decisions

In this section we describe an economy with many independent social decisions. Our design problem consists of components that in themselves are design problems, and we will refer to these as *single issue economies.* Below, we detail how we generate *many issue economies* from an underlying sequence of single issue economies..

3.1 Single-Issue Economies

Let $e_k \equiv \left\{ \left\langle \Theta_k^i, F_k^i, v_k^i \right\rangle_{i=1}^n, D_k, C_k \right\}$ denote an economy with a single issue k. Here, Θ_k^i denotes the issue-k type-space for agent i, and a generic element is denoted θ_k^i. We assume that types are stochastically independent across agents, implying that the prior distribution over Θ_k^i can be represented by a cumulative distribution F_k^i , which also is the belief over Θ_k^i for other agents and a fictitious planner. Preferences over different ways to resolve issue k are described by the valuation function $v_k^i : \Theta_k^i \times D_k \to \mathbb{R}$, where D_k is the set of possible alternatives for issue k. A type-θ_k^i agent i has a von Neumann-Morgernstern expected utility function with utility index

$$v_k^i \left(\theta_k^i, d_k \right) - t^i \tag{3}$$

for each $d_k \in D_k$, where t^i is a transfer of the numeraire good. Finally, $C_k : D_k \to \mathbb{R}$ describes the cost of implementing each alternative $d_k \in D_k$ in terms of the numeraire.

3.2 Many-Issue Economies

Let $\mathcal{E}_K$ denote an economy consisting of issues $1, ..., K$. A *social decision* is a resolution of each issue, and the set of feasible social decisions is denoted $\mathcal{D}_K = \times_{k=1}^K D_k$. A type realization for individual i is a vector of realized types in single-issue economies $e_1, .., e_K$, and we denote the type space for agent i by $\Theta^i(K) = \times_{k=1}^K \Theta_k^i$. Adopting standard conventions, we write $\Theta(K) \equiv \times_{i=1}^n \Theta^i(K)$ for the space of *type profiles*, and $\Theta^{-i}(K) \equiv \times_{j \neq i} \Theta^j(K)$ for the space of possible type realizations among all agents except for i. To conserve space we suppress K when it cannot cause confusion, and write θ^i, θ, and θ^{-i} for a generic elements of $\Theta^i(K)$, $\Theta(K)$, and $\Theta^{-i}(K)$ respectively. The revelation principle is applicable, so it is without loss of generality to consider direct mechanisms.[4] A direct mechanism in economy $\mathcal{E}_K$ is a pair $\langle x_K, t_K \rangle$, where $x_K : \Theta(K) \to \mathcal{D}_K$ is the allocation rule and $t_K \equiv \left(t_K^1, ..., t_K^n \right) : \Theta(K) \to \mathbb{R}^n$ is the transfer rule.

For each $d \in \mathcal{D}_K$, the cost of implementing d is the sum of the costs of implementing its components,

$$\mathcal{C}_K(d) \equiv \sum_{k=1}^K C_k(d_k). \tag{4}$$

The von Neumann-Morgernstern utility function for agent i of type $\theta^i \in \Theta^i(K)$ is on the same form as (3), but with v^i being replaced by the valuation function $V_K^i : \Theta^i(K) \times \mathcal{D}_K \to \mathbb{R}$ defined

[4] We will, however, add non-type messages in Section 6 for expositional reasons.

as the sum of the single issue valuation functions

$$V_K^i\left(d,\theta^i\right)=\sum_{k=1}^{K}v_k^i\left(d_k,\theta_k^i\right). \tag{5}$$

Next, we assume that the issue-specific types of each agent are independently distributed across issues. That is, for each $\theta^i\in\Theta^i(K)$, the cumulative distribution function is given by

$$\mathcal{F}_K^i\left(\theta^i\right)=\times_{k=1}^{K}F_k^i\left(\theta_k^i\right). \tag{6}$$

To sum up, the primitives of an economy $\mathcal{E}_K$ in terms of the components $\{e_k\}_{k=1}^K$ are

$$\mathcal{E}_K=\left\{\left\langle\Theta^i(K)=\times_1^K\Theta_k^i,V_K^i=\sum_{k=1}^{K}v_k^i,\mathcal{F}_K^i=\times_1^KF_k^i\right\rangle_{i=1}^{n},\mathcal{D}_K=\times_1^KD_k,\mathcal{C}_K=\sum_{k=1}^{K}C_k\right\}. \tag{7}$$

3.3 Reservation Utilities

For each issue k we assume that there is a "status quo outcome" $d_k^0\in D_k$ such that $C_k\left(d_k^0\right)=0$. We let $r_k^i\left(\theta_k^i\right)\equiv v_k^i\left(d_k^0,\theta_k^i\right)$ denote agent $i's$ reservation utility in the economy with the single issue k. For each $\theta^i\in\Theta^i(K)$, the reservation utility in economy $\mathcal{E}_K$, denoted by $R_K^i\left(\theta^i\right)$, is taken to be the utility for agent i in case the status quo outcome is implemented for each issue $1,...,K$

$$R_K^i\left(\theta^i\right)=\sum_{k=1}^{K}r_k^i\left(\theta_k^i\right). \tag{8}$$

3.4 Regularity

Let $\theta_k=\left(\theta_k^1,...,\theta_k^n\right)\in\times_{i=1}^n\Theta_k^i$ denote an issue-k type profile. In an economy with single issue k, denote by $x_k^*(\theta_k)$ an efficient social decision rule, and by $s_k(\theta_k)$ the associated maximized value of surplus. That is,

$$\begin{aligned}x_k^*(\theta_k)&\in\arg\max_{d_k\in D_k}\sum_{i=1}^{n}v_k^i\left(\theta_k^i,d_k\right)-C_k(d_k) \quad (9)\\ s_k(\theta_k)&=\max_{d_k\in D_k}\sum_{i=1}^{n}v_k^i\left(\theta_k^i,d_k\right)-C_k(d_k),\end{aligned}$$

where we assume that appropriate continuity and compactness assumptions hold for the optimization problem in (9) to be well defined. By definition, $s_k(\theta_k)\geq\sum_{i=1}^n r_k^i\left(\theta_k^i\right)$ holds for every $\theta_k\in\times_{i=1}^n\Theta_k^i$. However, for our results we need a bit more, namely that the expected maximized surplus from issue k is uniformly bounded away from the sum of issue-k reservation utilities. We also require the social surplus associated with each issue to be bounded from above and below. Together, we refer to these restrictions as *regularity*. Formally,

Definition 2 *We say that* $\{e_k\}_{k=1}^{\infty}$ *is a* regular sequence *of single-issue economies if*

R1 [Non-vanishing Gains from Trade]. *There exists some* $\delta > 0$ *such that*

$$\mathrm{E}\left[s_k\left(\theta_k\right)\right] \geq \mathrm{E}\left[\sum_{i=1}^{n} r_k^i\left(\theta_k^i\right)\right] + \delta, \text{ for all } k;$$

R2 [Uniform Bounds on Surplus]. *There exist some* $\underline{a} < \bar{a}$ *such that* $\underline{a} < s_k\left(\theta_k\right) < \bar{a}$ *for all* $\theta_k \in \times_{i=1}^{n}\Theta$ *and for all* k*;*

R3 [Uniform Bounds on Reservation Utilities]. *There exist some* $\underline{b} < \bar{b}$ *such that* $\underline{b} < r_k^i\left(\theta_k^i\right) < \bar{b}$ *for all* i*, all* $\theta_k^i \in \Theta_k^i$*, and all* k*.*

3.5 Some Remarks about the Environment

1. We kept the set of agents the same and finite for all problems. Since we can allow agents with trivial roles for any issue k, the restriction can be rephrased as saying that the union of the set of agents over all issues is finite.

2. Throughout the model we made several separability assumptions. Payoffs and costs are additively separable across issues, and, for each agent, the list of issue-specific types are stochastically independent. Moreover, the type parameter θ_k^i is assumed to affect how agent i evaluates issue k, but no other issue. But, if any of these assumptions fail for a pair of issues, we can always bunch these two issues into a single issue. This is possible because there are no dimensionality restrictions on either the issue specific set of alternatives or type spaces. Hence, our results hinge on the *existence of a sufficiently large number of truly independent and separate problems*, but *not* on the absence of any non-separabilities.

3. Our model also assumes that types are stochastically independent across agents. If types are correlated across agents it is usually possible to construct efficient mechanisms (similar to that of Cremer and McLean [13]) that respect both participation and resource constraints, with no role for linking the issues.

4. Our environment is general enough to incorporate both private goods, public goods and other types of externalities. For example, Armstrong's [1] non-linear pricing problem with many products fits nicely in this framework. Consider each issue k as corresponding to a product. The set of alternatives for issue $D_k \in \mathbb{R}^n$ will then correspond to a quantity choice for product

k for each consumer. Agent i's valuation function v_k^i depends on the quantity choice for i, i.e. ith component of d_k (see Section 5.1 for more discussion). Myerson and Satterswaite's [25] bilateral bargaining problem also fits in as an issue k in our setup. A bilateral bargaining problem can be represented by an action space $D_k = [0, 1]$, denoting the probability that agent 1 will obtain the good. Besides agents 1 and 2 – the two agents in the bargaining situation – all other agents' preferences on this issue are simply independent of d_k. Mailath and Postlewaite's [22] many-agent bargaining problem with public goods can similarly be incorporated as an issue in our setup.

5. The setup is related, but distinct from, dynamic mechanism design problems, such as Athey and Segal [3]. The crucial difference is that agents in our setup know the whole type vector at once, as opposed to drawing new realizations sequentially.

4 The Implementation Problem

A mechanism designer seeks to implement an efficient allocation, subject to incentive compatibility, participation and resource constraints. A mechanism is incentive compatible if truth-telling is a Bayesian Nash equilibrium in the incomplete information game induced by $\langle x_K, t_K \rangle$, that is, when

$$\begin{aligned} \mathrm{E}_{-i}\left[V_K^i\left(x_K\left(\theta\right), \theta^i\right) - t_K^i\left(\theta\right)\right] &\geq \mathrm{E}_{-i}\left[V_K^i\left(x_K\left(\theta_i', \theta_{-i}\right), \theta^i\right) - t_K^i\left(\theta_i', \theta\right)\right] \\ \forall i &\in I, \theta_i, \theta_i' \in \Theta^i\left(K\right). \end{aligned} \tag{10}$$

We impose participation constraints in the interim stage as

$$\mathrm{E}_{-i}\left[V_K^i\left(x_K\left(\theta\right), \theta^i\right) - t_K^i\left(\theta\right)\right] \geq R_K^i\left(\theta^i\right) \qquad \forall i \in I, \theta_i \in \Theta^i\left(K\right), \tag{11}$$

where $R_K^i\left(\theta^i\right)$ is defined in (8). Finally, the resource constraint is imposed in the ex ante form,

$$\mathrm{E}\left[\mathcal{C}_K\left(x_K\left(\theta\right)\right)\right] = \mathrm{E}\left[\sum_{i=1}^n t_K^i\left(\theta\right)\right]. \tag{12}$$

A seemingly more stringent condition is to require that the resources balance for every type profile, but, for the setup in this paper, ex ante and ex post resource constraints are equivalent.[5]

[5] This is because of the assumed independence across types and our focus on Bayesian implementation. See Börgers and Norman [8] for details. Applied to a Groves mechanism, this is the well-known AGV implementation result due to d'Aspremont and Gerard-Varet [14].

5 The Groves Mechanism Almost Works

In this section we consider the performance of a traditional Groves mechanism, with a lump sum transfer set to guarantee that all agents would be willing to participate if the participation decision is made behind a veil of ignorance.

Let

$$x_K^*(\theta) \in \arg\max_{d\in\mathcal{D}_K} \sum_{i=1}^{n} V_K^i\left(d,\theta^i\right) - \mathcal{C}_K(d), \tag{13}$$

$$S_K(\theta) = \max_{d\in\mathcal{D}_K} \sum_{i=1}^{n} V_K^i\left(d,\theta^i\right) - \mathcal{C}_K(d) \tag{14}$$

where $V_K^i\left(d,\theta^i\right)$ is defined in (5) and $\mathcal{C}_K(d)$ in (4). Consider a Groves mechanism

$$\left\langle x_K^*, t_{G,K} \equiv \left(t_{G,K}^1,...,t_{G,K}^n\right)\right\rangle, \tag{15}$$

where x_K^* is given by (13) and for each i the transfer $t_{G,K}^i$ is given by

$$t_{G,K}^i(\theta) = V_K^i\left(x_K^*(\theta),\theta^i\right) - S_K(\theta) \underbrace{+\frac{(n-1)}{n}\mathrm{E}\left[S_K(\theta)\right] - \mathrm{E}\left[R_K^i\left(\theta^i\right)\right] + \frac{1}{n}\mathrm{E}\left[\sum_{j=1}^{n} R_K^j\left(\theta^j\right)\right]}_{\text{lump sum transfer independent of } i'\text{s announcement}} \tag{16}$$

All incentive constraints (10) hold since truth-telling is a dominant strategy, and a routine calculation shows that the resource constraint (12) holds. The only issue is thus the participation constraints (11). Let $U_K^i\left(\theta^i\right)$ denote the interim expected utility for agent i given mechanism $\left\langle x_K^*, t_{G,K}\right\rangle$, which allows us to express the participation constraints in (11) compactly as $U_K^i\left(\theta^i\right) \geq R_K^i\left(\theta^i\right)$. We can now show:

Proposition 1 *Suppose that $\{\mathcal{E}_K\}_{K=1}^{\infty}$ is sequence of economies consisting of stochastically independent regular issues (in the sense of Definition 2). Moreover, for each K, let $(x_K^*, t_{G,K})$ be the Groves mechanism as specified in (15). Then, for every $\varepsilon > 0$ there exists some finite $K^*(\varepsilon)$ such that* $\Pr\left[U_K^i\left(\theta^i\right) - R_K^i\left(\theta^i\right) \geq 0\right] \geq 1-\varepsilon$ *for every i and every $K \geq K^*(\varepsilon)$.*

The interpretation of the result is that the probability that all participation constraints in (11) are satisfied (i.e., $(1-\varepsilon)^n$) can be made arbitrarily close to one when we link a sufficiently large number of independent social decisions using a standard Groves mechanism with appropriately chosen lump sum transfers.

A proof of Proposition 1 is in the appendix, but, to fix ideas, an informal argument is instructive. Substituting (16) into the payoff function and taking expectations over θ^{-i} we may express the interim expected utility as

$$U_K^i\left(\theta^i\right) = \mathrm{E}_{-i}\left[S_K\left(\theta\right)\right] - \frac{n-1}{n}\mathrm{E}\left[S_K\left(\theta\right)\right] + \mathrm{E}\left[R_K^i\left(\theta^i\right)\right] - \frac{1}{n}\mathrm{E}\left[\sum_{j=1}^{n} R_K^j\left(\theta^j\right)\right], \tag{17}$$

The objective function in (13) is additively separable across issues, so that

$$S_K\left(\theta\right) = \sum_{k=1}^{K} s_k\left(\theta_k\right), \tag{18}$$

where $s_k\left(\theta_k\right)$ is the maximized issue k surplus defined in (9). By definition $R_K^i\left(\theta^i\right) = \sum_{i=1}^{K} r_k^i\left(\theta_k^i\right)$, so, substituting and rearranging (17), we have that

$$\Pr\left[U_K^i\left(\theta^i\right) - R_K^i\left(\theta^i\right) \geq 0\right] \tag{19}$$

$$= \Pr\left[\underbrace{\mathrm{E}_{-i}\left[\sum_{k=1}^{K}\frac{s_k\left(\theta_k\right)}{K}\right] - \mathrm{E}\left[\sum_{k=1}^{K}\frac{s_k\left(\theta_k\right)}{K}\right]}_{A} + \underbrace{\mathrm{E}\left[\sum_{k=1}^{K}\frac{r_k^i\left(\theta_k^i\right)}{K}\right] - \sum_{k=1}^{K}\frac{r_k^i\left(\theta_k^i\right)}{K}}_{B} + \underbrace{\frac{1}{n}\mathrm{E}\left[\frac{S_K\left(\theta\right) - \sum_1^n R_K^j\left(\theta^j\right)}{K}\right]}_{C} \geq 0\right]$$

Inspecting the three terms we observe that;

Term A: Regularity condition [R2] bounds the variance of $s_k\left(\theta_k\right)$. Together with stochastic independence across issues this implies that $\sum_{k=1}^{K}\frac{s_k(\theta_k)}{K}$, and therefore also $\mathrm{E}_{-i}\left[\sum_{k=1}^{K}\frac{s_k(\theta_k)}{K}\right]$, is close to its expectation with high probability when K is large. Term A in expression (19) is thus close to zero with high probability, provided that K is large.

Term B: Regularity condition [R3] bounds the variance of $r_k^i\left(\theta_k^i\right)$. Together with stochastic independence of the reservation utilities across issues this guarantees that $\sum_{k=1}^{K}\frac{r_k^i\left(\theta_k^i\right)}{K}$ is close to its expectation with high probability, implying that term B in (19) is close to zero with high probability when K is large.

Term C: Finally regularity condition [R1] implies that term C in (19) is strictly positive.

Together, the behavior of these three terms imply that the probability of a participation constraint being violated converges to zero as K goes out of bounds.

As is clear from its formal proof, Proposition 1 actually holds under weaker regularity conditions (with no change in the proof); namely regularity conditions [R2] and [R3] can be weakened to require only a uniform bound on the variance of $s_k\left(\theta_k\right)$. However the stronger regularity conditions are needed for our main result Proposition 2.

5.1 Application: Armstrong's [1] Multiproduct Monopolist Problem

Despite the difference in focus, Proposition 1 can be used to understand Armstrong's [1] analysis of a multiproduct monopolist with K private goods. Armstrong assumes that each good k is produced at constant unit cost c_k. Marginal cost pricing trivially implements the efficient outcome, but monopoly pricing leads to inefficiencies due to informational rents for the consumers. However, when K is large, Armstrong shows that a two-part tariff gives approximate efficiency, with the monopolist extracting almost all consumer surplus.

Due to the constant unit cost assumption, each consumer can be treated separately, so there is no loss in assuming that there is a single consumer. We therefore drop the index i and write

$$V_K(d,\theta) = \sum_{k=1}^{K} v_k(d_k,\theta_k)$$

for the utility function of the consumer, where d_k is now is to be interpreted as the quantity of good k consumed. It can be seen from (13) and (16) that the Groves mechanism in this environment reduces to marginal cost pricing together with a lump sum transfer where

$$x_K^*(\theta) = \arg\max_{d\in\mathbb{R}_+^K} \left[\sum_{k=1}^{K} v_k(d_k,\theta_k) - \sum_{k=1}^{K} c_k d_k\right] \tag{20}$$

$$t_{G,K}(\theta) = \sum_{k=1}^{K} c_k x_k^*(\theta) + T, \tag{21}$$

where T is a lump sum transfer that does not depend on θ. Thus, mechanism (15) reduces to a two-part tariff, where the consumer pays a fixed fee of T for the right to purchase goods at marginal costs. While the budget-balanced Groves mechanism will imply $T = 0$ (obvious from (13) by setting $n = 1$), the approximately profit maximizing two-part tariff is to set $T = (1-\varepsilon)\,\mathrm{E}\left[S_K(\theta)\right]$ where $S_K(\theta) = \left[\sum_{k=1}^{K} v_k(x_k^*(\theta),\theta_k) - \sum_{k=1}^{K} c_k x_k^*(\theta)\right]$ and $\varepsilon > 0$ can be made arbitrarily small. A straightforward application of law of large numbers implies that such a two-part tariff satisfies the consumer's participation constraint almost always, and, since the problem can be solved separately for each consumer the mechanism becomes exactly incentive feasible if types that are not willing to pay T are allowed to opt out. This is exactly the construction Armstrong [1] uses to establish that the monopolist can extract almost the full surplus if there are many goods.

5.2 An Example where Participation Constraints Fail with High Probability

Now we demonstrate the role of the regularity conditions for Proposition 1. We provide examples that do not satisfy the regularity conditions, and show that participation constraints continue to fail with high probability under the standard Groves mechanism even when large number of issues are linked.

To demonstrate the role of regularity condition [R1] – the assumption that the gains from trade are bounded away from zero, we can simply consider an infinite sequence of identical problems with a static inefficiency, and assume that payoffs are discounted. If the discount factor is small enough, only the first (or the earliest) problem matters, so participation constraints fail whenever participation constraints are violated in single issue economy 1.

In this section we will consider a somewhat more subtle example, where [R1] (non-vanishing gains from trade) and [R3] (uniform bounds on reservation utilities) in Definition 2 hold, but where [R2] (uniform bounds on surplus) is violated. In essence, the example consists of a sequence of public goods problems, where the probability of implementing an outcome different from the status quo converges to zero, but where the associated surplus conditional on implementation goes to infinity at a rate that keeps the ex ante surplus to 1 for each problem in the sequence.

Consider a sequence of issues in a 2-person economy where $\Theta_k^i = \{l, h\}$ and $D_k = \{0, 1\}$ for each k. Let

$$C_k(d_k) = \begin{cases} 0 & \text{if } d_k = 0 \\ (k+1)^2 & \text{if } d_k = 0 \end{cases}. \tag{22}$$

Let $\alpha \in \left(0, \frac{1}{2}\right)$ and assume that $v_k^i\left(\theta_k^i, 0\right) = 0$ for $\theta_k^i = l, h$, whereas

$$v_k^i\left(\theta_k^i, 1\right) = \begin{cases} \frac{(k+1)^2}{2} - \left(\frac{1}{2} - \alpha\right)(k+1)^4 & \text{for } \theta_i^k = l \\ (k+1)^2\left[1 + \left(\frac{1}{2} - \alpha\right)k(k+2)\right] & \text{for } \theta_i^k = h \end{cases}. \tag{23}$$

We specify that the reservation payoffs are given by $r_k^i\left(\theta_k^i\right) = 0$ for $\theta_k^i = l, h$ (thus regularity condition [R3] is satisfied). It is then immediate that the efficient mechanism is $x_k^*(ll) = 0$ and $x_k^*(\theta_k) = 1$ for $\theta_k \in \{lh, hl, hh\}$. Some algebra shows that the associated maximized surplus is given by

$$s_k(\theta_k) = \begin{cases} 0 & \text{if } \theta_k = ll \\ \alpha(k+1)^2 & \text{if } \theta_k \in \{lh, hl\} \\ (k+1)^4\left[1 - 2\alpha\frac{k(k+2)}{(k+1)^2}\right] & \text{if } \theta_k = hh \end{cases}.$$

Note that $s_k(\theta_k)$ as specified above violates [R2] because it goes to infinity for all $\theta_k \neq ll$. From this point on, all the analysis can be done in terms of the maximized surplus function (the reason for specifying (22) and (23) is that transferable utility and private values impose some restrictions on the surplus function: not every s_k such that $s_k(\theta_k) \geq r_k^1\left(\theta_k^1\right) + r_k^2\left(\theta_k^2\right)$ is consistent with surplus maximization).

Assuming that $\Pr\left[\theta_k^i = h\right] = \frac{1}{(k+1)^2}$ the unconditional expected maximized surplus is

$$\begin{aligned} \mathrm{E}\left[s_k(\theta_k)\right] &= 2\frac{1}{(k+1)^2}\left[1-\frac{1}{(k+1)^2}\right]\alpha(k+1)^2 + \left[\frac{1}{(k+1)^2}\right]^2(k+1)^4\left[1-2\alpha\frac{k(k+2)}{(k+1)^2}\right] \\ &= 2\alpha\left[1-\frac{1}{(k+1)^2}\right] + \left[1-2\alpha\frac{k(k+2)}{(k+1)^2}\right] = 1+2\alpha\left[1-\frac{1+k^2+2k}{(k+1)^2}\right] = 1. \end{aligned}$$

Thus regularity condition [R1] is also satisfied. Moreover,

$$\mathrm{E}\left[s_k(\theta_k)\,|\theta_k^i = l\right] = \frac{1}{(k+1)^2}\alpha(k+1)^2 = \alpha.$$

Hence, the interim expected payoff for type $\mathbf{l} = \underbrace{(l, ..., l)}_{K}$ in the Groves mechanism with transfer (16) is

$$U^i(\mathbf{l}) = \sum_{k=1}^{K}\mathrm{E}\left[s_k(\theta_k)\,|\theta_k^i = l\right] - \frac{1}{2}\sum_{k=1}^{K}\mathrm{E}\left[s_k(\theta_k)\right] = \alpha K - \frac{1}{2}K < 0,$$

since $\alpha < \frac{1}{2}$. It is easy to check that $\Pr\left[\theta^i = \mathbf{l}\right] = \prod_{k=1}^{K}\frac{k(k+2)}{(k+1)^2} = \frac{K+2}{2(K+1)} > \frac{1}{2}$, implying that the participation constraint is violated with a probability exceeding $\frac{1}{2}$ for each player no matter how many problems are linked.

6 Asymptotic Efficiency

Proposition 1 does not solve the implementation problem as stated in Section 4. Although participation constraints are unlikely to fail if K is large, the probability is in general strictly positive. A natural idea to remedy this problem is to revert to the status quo outcome when a participation constraint is violated. Since failures are rare when K is large, this may appear to create an almost efficient incentive feasible mechanism.

The problem with this idea is that the agents that "opt out" change the interim expected payoff from participation for the agents that "opt in".[6] Hence, if types with interim payoffs below the reservation utilities opt out, this may diminish the value of playing the Groves mechanism for the remaining types, and the best response may be that some additional types opt out due to a negative selection effect. The set of types that agree to play the Groves mechanism must

[6] This issue is not relevant for private goods problems such as Armstrong [1] where the efficient outcome for agent i does not depend on others' types.

therefore be determined through an equilibrium argument. This makes the existence of a nearly efficient equilibrium non-obvious, as the participation constraints can unravel when agents have veto power, even if the probability of a participation constraint being violated, if all types *were* assumed to participate, converges to zero. Indeed, in Section 6.4 we provide an explicit example (which violates the regularity conditions of our main result Proposition 2) where the probability that, as K tends to infinity, a participation constraint is violated converges to zero under the standard Groves mechanism, but where the mechanism is nevertheless vetoed with probability near unity when K is large.

Our main result, Proposition 2, shows that the regularity conditions in Definition 2 are sufficient to rule out such unraveling. Under these conditions, there exists a mechanism satisfying constraints (10), (11), and (12) that is almost efficient when the number of independent issues is sufficiently large.

An analytical difficulty in establishing this result is the fixed point problem discussed above: constructing sequences of direct revelation mechanisms generates a fixed point problem in the set of "non-vetoing types", making direct mechanism hard to work with. Instead, we consider an indirect mechanism that specifies an explicit choice as to whether to veto the Groves mechanism in addition to the type message.

6.1 An Augmented Groves Mechanism

We add the non-type message $m^i \in \{0,1\}$ to the message space, so that each agent reports a pair $\left(m^i, \theta^i\right)$. If all agents report $m^i = 1$, then a Groves mechanism is implemented, whereas if any agent chooses $m^i = 0$, the default outcome d^0 is implemented and all transfers are zero. Hence, all agents have veto power. Notice here the difference with Jackson and Sonnenschein [20]. Our vetoes are cast *before agents know the types of the other agents*, whereas they assume that the vetoing procedure is ex post, implying that their mechanism satisfies *ex post* participation constraints. While ex post participation sometimes is a desirable feature, it is difficult to deal with in our model due to our (possibly) continuous social decisions and type spaces. This is because the conditional dominance solvability of the revelation game breaks down when vetoes are cast ex post. Concretely, the agents would need to form beliefs over the outcomes in the ex post veto game, and the equilibrium veto strategies would affect incentives to report types truthfully. As a result, *exact efficiency with high probability* will be impossible to guarantee. We conjecture that a slightly weaker version of a result would continue to hold with ex post participation constraints, but a proof would involve additional layers of approximations compared with the current proof.

We consider a sequence of mechanisms$\{\widehat{x}_K, \widehat{t}_K\}_{K=1}^{\infty}$, where for each K

$$\begin{aligned}
\widehat{x}_K(\theta, m) &= \begin{cases} x_K^*(\theta) & \text{if } m = (1,1,...,1) \\ d^0 = \left(d_1^0, ..., d_K^0\right) & \text{otherwise} \end{cases} \\
\widehat{t}_K^i(\theta, m) &= \begin{cases} t_{G,K}^i(\theta) + \frac{\delta K}{4n} & \text{if } m = (1,1,...,1) \\ 0 & \text{otherwise,} \end{cases}
\end{aligned} \tag{24}$$

The term $t_{G,K}^i(\theta)$ in the transfer scheme is given by the Groves transfer scheme in (16), and δ is the lower bound on the per issue gains from trade defined in [R1].

While not explicitly written as such, (24) is equivalent to a sequential mechanism, where in stage 1 the agents play a veto game. If any agent casts a veto, then the game is over, while if no veto is cast the agents play a Groves mechanism with *almost* the same transfers as in Section 5. The only difference is that the term $\frac{\delta K}{4n}$ has been added to the transfer scheme, which is to avoid budget deficits. In fact, we will show that

$$\mathrm{E}\left[\mathcal{C}_K\left(\widehat{x}_K(\theta, m)\right)\right] < \mathrm{E}\left[\sum_{i=1}^{n} \widehat{t}_K^i(\theta, m)\right], \tag{25}$$

holds when K is large. A strict budget surplus is obviously inefficient, but, since it can be rebated lump sum (which means that also to types that veto the mechanism get the rebate) without upsetting either participation or incentive constraints, this in turn guarantees existence of a transfer that balances the resource constraint (12) exactly (see Corollary 1 following Proposition 2).

6.2 The Approximate Efficiency Result

We now state our main result, which says that, if the underlying sequence of single-issue economies is regular, then mechanism (24) has equilibria that implement the efficient outcome with probability arbitrarily close to one, provided that sufficiently many issues are linked. Formally,

Proposition 2 *Suppose that $\{\mathcal{E}_K\}_{K=1}^{\infty}$ is sequence of economies consisting of stochastically independent regular issues (in the sense of Definition 2). Then, for every $\varepsilon > 0$ there exists a finite $K^*(\varepsilon)$ such that, for every economy $\mathcal{E}_K$ with $K \geq K^*(\varepsilon)$, there exists an equilibrium in the game induced by mechanism (24) where an efficient outcome is implemented with probability at least $1-\varepsilon$, and where (25) is satisfied.*

If δ is large, then the budget surplus generated by (24) is large, which means that the mechanism is far from efficient. However, this inefficiency is trivial to eliminate by a lump sum rebate of the expected budget surplus:

Corollary 1 *For every $\varepsilon > 0$ and $K \geq K^* (\varepsilon)$ there exists a direct mechanism satisfying (10), (11) and (12), where an efficient outcome is implemented with probability at least $1 - \varepsilon$.*

The key difference with Proposition 2 is that the (potentially large) inefficiency that comes from the strict budget surplus (25) is eliminated by making the mechanism exactly budget balanced (ex ante).[7]

Proof. By the revelation principle, there exists a direct revelation mechanism with an equilibrium that implements the same outcome as mechanism (24) for every K. Hence, (10) is satisfied and (11) holds since each agent has an action ($m^i = 0$) that guarantees the reservation utility. Finally, since (25) is satisfied there exists a lump-sum rebate (where also types that veto the decision receive the rebate) such that (12) is satisfied. This rebate leaves the incentives constraints (10) unaffected, and increases the interim expected utility from participation, implying that every constraint (11) continues to hold. ∎

6.3 The Proof of Proposition 2

An important object in our proof is a set of types, $\overline{\Theta}^i (K)$, who are at least $\frac{\delta K}{2n}$ better off from participating in the Groves mechanism than from the status quo outcome. Formally,

$$\overline{\Theta}^i (K) = \left\{ \theta^i \in \Theta^i (K) | U^i_K \left(\theta^i\right) - R^i_K \left(\theta^i\right) \geq \frac{\delta K}{2n} \right\}, \tag{26}$$

where the interim utility function $U^i_K \left(\theta^i\right)$ is defined in (17), using the Groves mechanism in the previous section. To interpret $\overline{\Theta}^i (K)$, recall that $\delta K/n$ is a lower bound for the expected value for $U^i_K \left(\theta^i\right) - R^i_K \left(\theta^i\right)$, so $\overline{\Theta}^i (K)$ is a set of agent i's types for which her interim expected surplus is at least 1/2 of this lower bound. This set is useful because the probability that θ^i belongs to $\overline{\Theta}^i (K)$ is close to one when K is large, and, because types in $\overline{\Theta}^i (K)$ have a per-issue ex ante expected payoff of at least $\delta/2n$, there is room to consider small deviations from the Groves mechanism without upsetting their participation constraints.

[7]Ex post budget balance can also be guaranteed by a single additional step. See Börgers and Norman [8].

It is convenient to introduce the notation

$$\overline{\Theta}(K) = \left\{\theta \in \Theta(K) \,|\theta^i \in \overline{\Theta}^i(K) \text{ for each } i\right\} \tag{27}$$

$$\overline{\Theta}^{-i}(K) = \left\{\theta^{-i} \in \Theta^{-i}(K) \,|\theta^j \in \overline{\Theta}^j(K) \text{ for each } j \neq i\right\}. \tag{28}$$

Using roughly the same style of argument as in the proof of Proposition 1, we begin by establishing that for large K, the probability that θ^{-i} belongs to $\overline{\Theta}^{-i}(K)$ is near one. This is fairly obvious, since the lump sum transfers from our Groves mechanism are constructed so that

$$\mathrm{E}\left[U_K^i\left(\theta^i\right) - R_K^i\left(\theta^i\right)\right] = \frac{1}{n}\left\{\mathrm{E}\left[S_K(\theta)\right] - \mathrm{E}\left[\sum_{j=1}^{n} R_K^j\left(\theta^j\right)\right]\right\} \geq \frac{\delta K}{n}, \tag{29}$$

by the assumption that the per-issue gains from trade are bounded below by δ in (R1).

Lemma 1 *Suppose that* $\{\mathcal{E}_K\}_{K=1}^{\infty}$ *is sequence of economies consisting of stochastically independent regular issues. Then, for every* $\varepsilon > 0$ *there exists finite* $K'(\varepsilon)$ *such that* $\Pr\left[\overline{\Theta}^{-i}(K)\right] \geq 1 - \varepsilon$ *for all* $K \geq K'(\varepsilon)$.

Define $\mathrm{E}_{-i}\left[S_K(\theta)\,|\Theta^{-i}(K)\right]$ as the interim expectation of the social surplus under an efficient social decision rule for agent i conditional on θ^i *and conditional on the types of the other agents being in the set* $\Theta^{-i}(K)$. Using the regularity assumptions (R2) and (R3) we show that:

Lemma 2 *Suppose that* $\{\mathcal{E}_K\}_{K=1}^{\infty}$ *is sequence of economies consisting of stochastically independent regular issues. Then, for every* $\varepsilon > 0$ *there exists finite* $K''(\varepsilon)$ *such that for all* $K \geq K''(\varepsilon)$,

$$\frac{\mathrm{E}_{-i}\left[S_K(\theta)\,|\Theta^{-i}(K)\right]}{K} \geq \frac{\mathrm{E}_{-i}\left[S_K(\theta)\right]}{K} - \varepsilon, \text{ for all } \theta^i \in \Theta^i(K).$$

Intuitively, the idea is that, since the probability that θ^{-i} lies in $\Theta^{-i}(K)$ can be made arbitrarily close to one by linking sufficiently many problems, the conditional expectation is almost exclusively taken over $\Theta^{-i}(K)$.

Next, we start to characterize equilibrium play in the mechanism. Since the "second stage mechanism" is a Groves mechanism, truth-telling is dominant conditional on no veto being cast. Neither is there any benefit from lying if a veto is cast. There are thus never any gains from misrepresenting the type profile. In terms of the decision when to veto the Groves mechanism, it is therefore just to evaluate the interim expected payoff from not vetoing, which is a computation where selection effects matter for the probabilities, but *not for behavior in the second stage*, and compare this with the reservation utility.

To state this formally we need to introduce notation for the veto rules. Later, we need to allow randomizations to ensure existence of equilibria, so let a *veto rule* in economy $\mathcal{E}_K$ be a map $\psi_K^i : \Theta^i(K) \to [0,1]$, where $\psi_K^i(\theta^i)$ is interpreted as the probability that type θ^i chooses $m^i = 1$, a vote to play the Groves mechanism. Following standard conventions, let $\psi_K = (\psi_K^1, ..., \psi_K^n)$ and $\psi_K^{-i} = (\psi_K^1, .., \psi_K^{i-1}, \psi_K^{i+1}, ..., \psi_K^n)$. Also, let $\widehat{U}^i(m_i, \psi_K^{-i}, \theta^i)$ denote the interim expected payoff for type θ^i from announcement (m^i, θ^i) under the assumption that all other agents report type truthfully and follow an arbitrary veto rule $\psi_K^{-i} : \Theta^{-i}(K) \to [0,1]^{n-1}$.[8] For brevity, we let $\Psi_K^{-i}(\theta^{-i}) \equiv \prod_{j \neq i} \psi_K^j(\theta^j)$ and $F_K^{-i}(\theta^{-i}) = \prod_{j \neq i} F_K^j(\theta^j)$. Given that all agents report their type parameters truth-fully, the interim expected payoff for player i is $\widehat{U}^i(0, \psi_K^{-i}, \theta^i) = R_K^i(\theta^i)$ if i vetoes the mechanism and

$$\widehat{U}^i(1, \psi_K^{-i}, \theta^i) = \int \left(\left[S_K(\theta) + T_K^i \right] \Psi_K^{-i}(\theta^{-i}) + R_K^i(\theta^i) \left[1 - \Psi_K^{-i}(\theta^{-i}) \right] \right) dF_K^{-i}(\theta^{-i}) \tag{30}$$

if i votes in favor, where $S_K(\theta)$ is the surplus defined in (18) and

$$T_K^i \equiv -\frac{n-1}{n} \mathrm{E}\left[S_K(\theta) \right] + \mathrm{E}\left[R_K^i(\theta^i) \right] - \frac{1}{n} \mathrm{E}\left[\sum_{j=1}^{n} R_K^j(\theta^j) \right] - \frac{\delta K}{4n} \tag{31}$$

is the (negative of the) "conditional lump-sum" part of the transfer in (24). It follows that:

Lemma 3 *Suppose that each agent $j \neq i$ reports type truthfully and chooses m^j in accordance to rule $\psi_K^j : \Theta^j(K) \to [0,1]$. Then, it is a best response for agent i to always report truthfully and to follow the rule $\widetilde{\psi}_K^i$, where;*

$$\widetilde{\psi}_K^i(\theta^i) = \begin{cases} 1 & \text{if } \widehat{U}^i(1, \psi_K^{-i}, \theta^i) > R_K^i(\theta^i) \\ 0 & \text{if } \widehat{U}^i(1, \psi_K^{-i}, \theta^i) < R_K^i(\theta^i) \end{cases}$$

We now proceed to the key lemma in the proof, which characterizes the best responses in the veto game.[9] It says that, if K is large enough and if, for all i and $j \neq i$, agent i believes that no type in the set $\overline{\Theta}^j(K)$ will veto the mechanism, then all types θ^i in $\overline{\Theta}^i(K)$ have a strict incentive not to veto the mechanism. This is true regardless of the behavior of types outside of the set $\overline{\Theta}^j(K)$.

[8] At the cost of more complicated notation, we could allow for non-truthful announcements. In the end, this extra "bite" is useless since there is no hope to get dominance in the veto rules.

[9] The reader may note that using a Groves mechanism conditional on no veto, rather than a mechanism with Bayesian incentive compatibility, is crucial for our ability to collapse the problem into a single veto game. That is, the payoff calculation in (30) rests crucially on the second stage being dominance solvable. If this would not be the case, selection in the veto game would affect beliefs, making the problem intractable.

In other words, eventually, incentives are dominated by best responding to $\Theta^{-i}(K)$ and when K is large this rationalizes voting to go ahead with the Groves mechanism for types in $\overline{\Theta}^i(K)$.[10]

Lemma 4 *Suppose that $\{\mathcal{E}_K\}_{K=1}^{\infty}$ is sequence of economies consisting of stochastically independent regular issues. Then, there exists finite K^*such that, for all $K \geq K^*$, if $\psi_K^j(\theta^j) = 1$ for each type $\theta^j \in \overline{\Theta}^j(K), j \neq i$, then it is a best response for agent i to set $\psi_K^i(\theta^i) = 1$ for each $\theta^i \in \overline{\Theta}^i(K)$.*

Recall that the whole exercise would be useless unless we can verify that the mechanism balances the budget. Again, we can only guarantee this if K is large enough. If K is too small the probability that the mechanism is vetoed is non-negligible, and it may be that the types that stay out are those that generate the most revenue for the mechanism designer. However, when K is large the probability of a veto is small and the extra lump sum revenue from the term $\frac{\delta K}{4n}$ in (24) eventually suffices for a budget surplus.

Lemma 5 *Suppose that $\{\mathcal{E}_K\}_{K=1}^{\infty}$ is sequence of economies consisting of stochastically independent regular issues and that, for every K, $\psi_K^i(\theta^i) = 1$ for each $\theta^i \in \overline{\Theta}^i(K)$. Then, there exists finite K^{**}such that (25) is satisfied for all $K \geq K^{**}$.*

The idea is that $\Pr\left[\overline{\Theta}(K)\right]$ converges in probability to unity and so that the mechanism runs a budget surplus that approaches $\frac{\delta K}{4}$ when K is large.

To complete the proof we now combine the five Lemmas above and argue that an equilibrium exists where the efficient outcome is implemented with high probability and the budget is balanced.

Proof of Proposition 2. Consider a game where $m^i = 1$ is the only available action for types in $\overline{\Theta}^i(K)$ and where types in $\Theta^i(K) \backslash \overline{\Theta}^i(K)$ may choose $m^i \in \{0,1\}$, and the interim expected payoffs are given by (30). The action space is finite for each player, so payoffs are equicontinuous in the sense of Milgrom and Weber [24] (see Proposition 1 in Milgrom and Weber [24]). Moreover, stochastic independence implies that the information structure is absolutely continuous (Proposition 3 in Milgrom and Weber [24]). Applying Theorem 1 in Milgrom and Weber the game has an equilibrium in distributional strategies. Since the action space is finite, we can represent this equilibrium as a behavioral strategy $\psi_K^* : \Theta(K) \to [0,1]^n$, where by construction $\psi_K^{i*}(\theta^i) = 1$ whenever $\theta^i \in \overline{\Theta}^i(K)$. But, applying Lemma 4 it follows that that there exists $K^* < \infty$ such that ψ_K^* is an equilibrium also when types in $\overline{\Theta}^i(K)$ have the option to freely pick $m^i \in \{0,1\}$, which by use of Lemma 5 implies that (25) is satisfied for $K \geq \max\{K^*, K^{**}\}$. Moreover, for any $\varepsilon > 0$ Lemma

[10] Trivially, there will always be equilibria where the Groves mechanism is vetoed with probability 1.

1 assures that we may pick $K'(\varepsilon)$ such that such that $\Pr\left[\overline{\Theta}^i(K)\right] \geq 1-\varepsilon$ for each $K > K'(\varepsilon)$. Finally, Lemma 3 guarantees that truth-telling is optimal provided that each agent announce m^i in accordance with ψ_K^{i*}. The result follows by letting $K^*(\varepsilon) = \max\{K^*, K^{**}, K'(\varepsilon)\}$. ■

6.4 An Example where the Participation Constraints Unravel

Let $n = 2$ and assume that, for each k and $i = 1, 2$, the issue k type space is given by $\Theta_k^i = \{l, m, h\}$, and $D_k = \left\{d_k^0, d_k^l, d_k^m, d_k^h\right\}$. In terms of interpretation it is useful to think of d_k^0 as the "status quo" outcome, whereas d_k^l, d_k^m and d_k^h are surplus maximizing alternatives for type profiles ll, mm and hh respectively. Suppose that the cost function is given by,

$$C_k(d_k) = \begin{cases} 0 & \text{if } d_k = d_k^0 \\ \frac{(k+1)^2}{k} & \text{if } d_k = d_k^l \\ \frac{k(k+1)^2}{2} & \text{if } d_k = d_k^m \\ 2(k+1)^3 & \text{if } d_k = d_k^h. \end{cases}$$

Assume that the valuation functions for the three types are given by;

$$\begin{aligned} v_k^i(d_k, l) &= \begin{cases} 0 & \text{if } d_k = d_k^0 \\ \frac{1}{2}\frac{(k+1)^2}{k} & \text{if } d_k = d_k^l \\ 0 & \text{if } d_k = d_k^m \\ \frac{(k+1)^2}{4}\left(k - \frac{1}{k}\right) & \text{if } d_k = d_k^h \end{cases} \\ v_k^i(d_k, m) &= \begin{cases} 0 & \text{if } d_k = d_k^0 \\ \frac{3}{4}\frac{(k+1)^2}{k} + \varepsilon(k+1) & \text{if } d_k = d_k^l \\ \frac{k(k+1)^2}{2} - 2\varepsilon k(k+1)(k+2) & \text{if } d_k = d_k^m \\ \frac{k(k+1)^2}{4} - 3(k+1)^3 & \text{if } d_k = d_k^h \end{cases} \\ v_k^i(d_k, h) &= \begin{cases} 0 & \text{if } d_k = d_k^0 \\ -\varepsilon(k+1) & \text{if } d_k = d_k^l \\ -\frac{k(k+1)^2}{2} & \text{if } d_k = d_k^m \\ 3(k+1)^3 - \frac{k(k+1)^2}{4} & \text{if } d_k = d_k^h, \end{cases} \end{aligned}$$

where $\varepsilon > 0$. Further restrictions on ε will be derived below. Since $C_k\left(d_k^0\right) = 0$ we take d_k^0 as the status quo outcome, implying that $r_k^i\left(\theta_k^i\right) = v_k^i\left(d_k^0, \theta_k^i\right) = 0$ for all θ_k^i.

One may interpret the example as one where there are two public goods and two types of high valuation agents, who mutually dislike the public good preferred by the other high type.

It is easy, but somewhat tedious (see Fang and Norman [16] for the details), to check that if $\varepsilon \leq \frac{1}{24}$ and $k \geq 2$, then an optimal social decision rule is:[11]

$$x_k^*(\theta_k) = \begin{cases} d_k^0 & \text{if } \theta_k \in \{mh, hm\} \\ d_k^l & \text{if } \theta_k \in \{ll, lm, ml\} \\ d_k^m & \text{if } \theta_k = mm \\ d_k^h & \text{if } \theta_k \in \{lh, hl, hh\} \end{cases} . \tag{32}$$

When $k = 1$, d_k^l is the optimal social decision for profile mm. This does not affect anything qualitatively, but derivations are less transparent if this is carried around. We will therefore simply consider a sequence of issues indexed by $\{k\}_{k=2}^{\infty}$. For every $k \geq 2$, the maximized social surplus is

$$s_k(\theta_k) = \begin{cases} 0 & \text{if } \theta_k \in \{ll, mh, hm\} \\ \frac{1}{4}\frac{(k+1)^2}{k} + \varepsilon(k+1) & \theta_k \in \{lm, ml\} \\ \frac{k(k+1)^2}{2} - 4\varepsilon k(k+1)(k+2) & \text{if } \theta_k = mm \\ (k+1)^2\left[\frac{4k(k+1)-1}{4k}\right] & \text{if } \theta_k \in \{lh, hl\} \\ \left[4(k+1) - \frac{k}{2}\right](k+1)^2 & \text{if } \theta_k = hh \end{cases} . \tag{33}$$

Obviously the issues in this example violate regularity condition [R2]. The key asymmetry between m and h is that the surplus when the issue k type profile is lm or ml is of order k, while lh or hl generates a surplus of order k^3. This allows us to make m the "lowest" type in an interim sense, while at the same time making sure that m does contribute sufficiently to the interim expected surplus conditional on $\theta_k^i = l$ for it to affect the participation decision for l (even as the probability of $\theta_k^i = m$ converges rapidly to zero).

Assuming that the probability distribution over Θ_k^i is given by

$$\left(\Pr\left[\theta^i = l\right], \Pr\left[\theta^i = m\right], \Pr\left[\theta^i = h\right]\right) = \left(\frac{k(k+2)}{(k+1)^2}, \frac{1}{2(k+1)^2}, \frac{1}{2(k+1)^2}\right), \tag{34}$$

the relevant expected values of the maximized issue k surplus are

$$\mathrm{E}\left[s_k(\theta_k)\mid l\right] = \frac{k+1}{2} + \frac{\varepsilon}{2(k+1)} \tag{35}$$

$$\mathrm{E}\left[s_k(\theta_k)\mid m\right] = \frac{k+1}{2} - \varepsilon k - \frac{\varepsilon k}{(k+1)} \tag{36}$$

$$\mathrm{E}\left[s_k(\theta_k)\mid h\right] = (k+1)^3 + \frac{(k+1)}{2} \tag{37}$$

$$\mathrm{E}\left[s_k(\theta_k)\right] = k+1 \tag{38}$$

[11] Both d_k^0 and d_k^l are optimal when the type profile is ll. We could adjust the costs and preferences slightly to make the optimal rule unique (without affecting the surplus).

Since $n = 2$ and $R_K^i\left(\theta^i\right) = 0$ for all θ^i the interim expected payoff from participating in the Groves mechanism with transfers (16), (17), simplifies to

$$U_K^i\left(\theta^i\right) = \mathrm{E}_{-i}\left[S_K\left(\theta\right)\right] - \frac{1}{2}\mathrm{E}\left[S_K\left(\theta\right)\right] = \sum_{k=2}^{K}\mathrm{E}_{-i}\left[s_k\left(\theta_k\right)\right] - \frac{1}{2}\sum_{k=2}^{K}\mathrm{E}\left[s_k\left(\theta_k\right)\right] \tag{39}$$

Consider a type of the form $(m, .., m, l, ...l)$. Specifically, assume that $\theta_k^i = m$ for $k = 2, ..., K^*$ and $\theta_k^i = l$ for $k = K^* + 1, ..., K$. For brevity, denote such a type by $(\mathbf{m}_{K^*}, \mathbf{l}_{K-K^*})$. Substituting (35), (36) and (38) into (39) and simplifying the result, the interim expected payoff for such a type is

$$U_K^i\left(\mathbf{m}_{K^*}, \mathbf{l}_{K-K^*}\right) = \varepsilon\left[\sum_{k=2}^{K^*}\left(-k - \frac{k}{k+1}\right) + \frac{1}{2}\sum_{k=K^*+1}^{K}\frac{1}{k+1}\right] \tag{40}$$

Define

$$H\left(K^*, K\right) \equiv \sum_{k=2}^{K^*}\left[-k - \frac{k}{k+1}\right] + \frac{1}{2}\sum_{k=K^*+1}^{K}\frac{1}{k+1} \tag{41}$$

We note that: i) H is strictly decreasing in K^*, and; ii) $H\left(1, K\right) = \frac{1}{2}\sum_{k=2}^{K}\frac{1}{k+1} > 0$, and; iii) $\sum_{k=2}^{K}\left(k + \frac{k}{k+1}\right) < 0$. Hence, there exists a unique integer $K^*\left(K\right)$ such that $H\left(K^*\left(K\right), K\right) \geq 0$ and $H\left(K^*\left(K\right) + 1, K\right) < 0$. The significance of $K^*\left(K\right)$ is that $U_K^i\left(\mathbf{m}_{K^*}, \mathbf{l}_{K-K^*}\right) \geq 0$ (< 0) for $K^* \leq K^*\left(K\right)$ $(K^* > K^*\left(K\right))$. We observe that the positive term in (41) is divergent, that is

$$\sum_{k=K^*+1}^{K}\frac{1}{k+1} = \sum_{k=K^*+1}^{K}\int_{k+1}^{k+2}\frac{1}{k+1}dz > \sum_{k=K^*+1}^{K}\int_{k+1}^{k+2}\frac{1}{z}dz = \int_{K^*+2}^{K+2}\frac{1}{z}dz = \ln\left(K+2\right) - \ln\left(K^*+2\right). \tag{42}$$

It follows that $K^*\left(K\right)$ goes (slowly) to infinity as K goes to infinity. To see this, assume for contradiction that there exists some $\overline{K}$ such that $K^*\left(K\right) < \overline{K} - 1$ for all K. Then,

$$\begin{aligned} H\left(K^*\left(K\right) + 1, K\right) \quad &\geq \quad H\left(\overline{K}, K\right) = \sum_{k=2}^{\overline{K}}\left[-k - \frac{k}{k+1}\right] + \frac{1}{2}\sum_{k=\overline{K}+1}^{K}\frac{1}{k+1} \\ \text{/using (42)/} \quad &> \quad \sum_{k=2}^{\overline{K}}\left[-k - \frac{k}{k+1}\right] + \frac{\ln\left(K+2\right) - \ln\left(\overline{K}+2\right)}{2}. \end{aligned} \tag{43}$$

$\ln\left(K+2\right) \to \infty$ as $K \to \infty$ and the other two terms in (43) are finite, so $H\left(K^*\left(K\right) + 1, K\right) > 0$ if K is large enough, which contradicts the definition of $K^*\left(K\right)$.

Next, consider a type on form $\left(\theta_2^i, ..., \theta_{K^*(K)}^i, l, ..., l\right)$, where the signals for problems $2, ..., K^*\left(K\right)$ are arbitrary, and $\theta_k^i = l$ for $k = K^*\left(K\right), .., K$. For brevity, denote such a type by $\left(\boldsymbol{\theta}_{K^*(K)}^i, \mathbf{l}_{K-K^*(K)}\right)$.

Because $\mathrm{E}\left[s_k(\theta_k)|\, m\right] \leq \mathrm{E}\left[s_k(\theta_k)|\, \theta_k^i\right]$ for all $\theta_k^i \in \{l, m, h\}$ and every k it follows that

$$U^i\left(\boldsymbol{\theta}^i_{K^*(K)}, \mathbf{l}_{K-K^*(K)}\right) = \sum_{k=2}^{K^*(K)} \mathrm{E}\left[s_k(\theta_k)|\, \theta_k^i\right] - \sum_{k=K^*(K)+1}^{K} \mathrm{E}\left[s_k(\theta_k)|\, l\right] - \frac{1}{2}\sum_{k=2}^{K}\mathrm{E}\left[s_k(\theta_k)\right]$$

$$\geq \sum_{k=2}^{K^*(K)} \mathrm{E}\left[s_k(\theta_k)|\, m\right] - \sum_{k=K^*(K)+1}^{K} \mathrm{E}\left[s_k(\theta_k)|\, l\right] - \frac{1}{2}\sum_{k=2}^{K}\mathrm{E}\left[s_k(\theta_k)\right] = \varepsilon H\left(K^*(K), K\right) > 0 \quad (44)$$

We conclude that the participation constraint holds for any such type. Since,

$$\Pr\left[\left(\theta^i_{K^*(K)+1}, ..., \theta^i_K\right) = (l, ..., l)\right] = \prod_{k=K^*(K)+1}^{K} \frac{k(k+2)}{(k+1)^2} = \frac{(K^*(K)+1)(K+2)}{(K^*(K)+2)(K+1)} \to 1 \quad (45)$$

as $K \to \infty$ it follows that the probability that θ^i is on the form $\left(\boldsymbol{\theta}^i_{K^*(K)}, \mathbf{l}_{K-K^*(K)}\right)$ tends to unity as K goes out of bounds. We conclude;

Claim 1 *The probability that all participation constraints hold converges to* 1 *as* $K \to \infty$ *.*

That is, the sufficient conditions used to prove Proposition 1 are not satisfied, but the conclusion in Proposition 1 is nevertheless valid: the Groves mechanism with transfers (16) is almost incentive feasible.

Next, consider type θ^i where all coordinates are l except for $\theta^i_{k^*} = m$. Denote this type $\theta^i = \left(\mathbf{l}|\theta^i_{k^*} = m\right)$ and substitute from (35), (36), and (38) to get

$$\begin{aligned} U\left(\mathbf{l}|\theta^i_{k^*} = m\right) &= \sum_{k \neq k^*}^{K} \frac{k+1}{2} + \frac{\varepsilon}{2(k+1)} + \frac{k^*+1}{2} - \varepsilon k^* - \frac{\varepsilon k^*}{k^*+1} - \frac{1}{2}\sum_{k=2}^{K} k+1 \quad (46) \\ &= \varepsilon\left[\frac{1}{2}\sum_{k=2}^{K}\frac{1}{k+1} - k^* - \frac{k^*}{k^*+1} - \frac{1}{2(k^*+1)}\right] < \varepsilon\left[\frac{1}{2}\sum_{k=2}^{K}\frac{1}{k+1} - k^*\right] \end{aligned}$$

A calculation symmetric to (42) shows that

$$\sum_{k=2}^{K}\frac{1}{k+1} = \sum_{k=2}^{K}\left(\int_k^{k+1}\frac{1}{k+1}dz\right) < \sum_{k=2}^{K}\left(\int_k^{k+1}\frac{1}{z}dz\right) = \int_2^{K+1}\frac{1}{z}dz = \ln\left(\frac{K+1}{2}\right) \quad (47)$$

Hence,

$$U\left(\mathbf{l}|\theta^i_{k^*} = m\right) < \varepsilon\left[\frac{1}{2}\ln\left(\frac{K+1}{2}\right) - k^*\right] \quad (48)$$

Define $\widetilde{k}(K)$ as the integer part of $\frac{1}{2}\ln\left(\frac{K+1}{2}\right) + 1$. For $k^* \geq \widetilde{k}(K)$ the right hand side of (48) is negative, implying that type $\left(\mathbf{l}|\theta^i_{k^*} = m\right)$ is worse off from participating in the Groves mechanism than from the status quo outcome $d^0 = \left(d_2^0, ..., d_K^K\right)$. For brevity, denote this subset of types with a strict incentive to veto the Groves mechanism by Θ_V^i. That is

$$\Theta_V^i = \left\{\theta^i | \theta_k^i = m \text{ for some } k^* \geq \widetilde{k}(K) \text{ and } \theta_k^i = l \text{ for all } k \neq k^*\right\}$$

We note that, for any $k^* \geq \widetilde{k}(K)$

$$\begin{aligned} \Pr\left[\theta^i \in \Theta_V^i | \theta_{k^*}^i = m\right] &= \prod_{k \neq k^*} \Pr\left[\theta_k^i = l\right] > \Pr\left[\theta^i = \mathbf{l}\right] = \prod_{k=2}^{K} \Pr\left[\theta_k^i = l\right] \quad (49) \\ &= \prod_{k=2}^{K} \frac{k(k+2)}{(k+1)^2} = \frac{2}{3}\left[\frac{K+2}{K+1}\right] > \frac{2}{3} \end{aligned}$$

In words, conditional on $\theta_{k^*}^i = m$, the probability that all the other coordinates are ls is (slightly) larger than the unconditional probability that type $(l, ..., l)$ is realized, which is at least $2/3$ for every K. For $k \geq \widetilde{k}(K)$ the expected surplus conditional on $\theta_k^i = l$ and conditional on agent i being aware that any type $\theta^i \in \Theta_V^i$ will veto the mechanism is

$$\mathrm{E}\left[s_k(\theta_k) | l, \text{veto by } \Theta_V^i\right] = \mathrm{E}\left[s_k(\theta_k) | l\right] - \Pr\left[\theta_k^i = m\right] \Pr\left[\theta^i \in \Theta_V^i | \theta_k^i = m\right] s_k(lm). \quad (50)$$

Substituting $\mathrm{E}\left[s_k(\theta_k) | l\right]$ from (35) and $s_k(lm)$ from (33) into (50) and using (49) we get

$$\begin{aligned} \mathrm{E}\left[s_k(\theta_k) | l, \text{veto by } \Theta_V^i\right] &< \underbrace{\frac{k+1}{2} + \frac{\varepsilon}{2(k+1)}}_{=\mathrm{E}[s_k(\theta_k)|l]} - \underbrace{\frac{1}{2(k+1)^2}}_{=\Pr\left[\theta_k^i = m\right]} \underbrace{\frac{2}{3}}_{(49)} \underbrace{\left[\frac{1}{4}\frac{(k+1)^2}{k} + \varepsilon(k+1)\right]}_{=s_k(lm)} \\ &= \frac{k+1}{2} + \frac{\varepsilon}{6(k+1)} - \frac{1}{12k}. \quad (51) \end{aligned}$$

The interim expected payoff of type $\mathbf{l} = (l, ..., l)$ conditional on vetoes from types in Θ_V^i is thus

$$\begin{aligned} U_K^i\left(\mathbf{l} | \text{veto by } \Theta_V^i\right) &= \sum_{k=2}^{\widetilde{k}(K)-1} \mathrm{E}\left[s_k(\theta_k) | l\right] + \sum_{k=\widetilde{k}(K)}^{K} \mathrm{E}\left[s_k(\theta_k) | l, \text{veto by } \theta^i \in \Theta_V^i\right] - \frac{1}{2}\sum_{k=2}^{K} \mathrm{E}\left[s_k(\theta_k)\right] \\ /\text{using (51)}/ &< \frac{\varepsilon}{2}\left[\sum_{k=2}^{\widetilde{k}(K)-1} \frac{1}{k+1}\right] + \frac{\varepsilon}{6}\sum_{k=\widetilde{k}(K)}^{K} \frac{1}{k+1} - \frac{1}{12}\sum_{k=\widetilde{k}(K)}^{K} \frac{1}{k} \quad (52) \end{aligned}$$

From (47) we have that $\sum_2^{\widetilde{k}(K)-1} \frac{1}{k+1} < \ln\left(\frac{\widetilde{k}(K)}{2}\right)$. Similarly,

$$\sum_{\widetilde{k}(K)}^{K} \frac{1}{k+1} = \sum_{k=\widetilde{k}(K)}^{K} \int_k^{k+1} \frac{1}{1+k} dz < \underbrace{\int_{\widetilde{k}(K)}^{K+1} \frac{1}{z} dz}_{=\ln\left(\frac{K+1}{\widetilde{k}(K)}\right)} = \sum_{k=\widetilde{k}(K)}^{K} \int_k^{k+1} \frac{1}{z} dz < \sum_{k=\widetilde{k}(K)}^{K} \int_k^{k+1} \frac{1}{k} dz = \sum_{k=\widetilde{k}(K)}^{K} \frac{1}{k} dz, \quad (53)$$

and substituting these three bounds into (52) we find that

$$U_K^i\left(\mathbf{l}|\text{veto by }\Theta_V^i\right)<\frac{\varepsilon}{2}\ln\left(\frac{\widetilde{k}\left(K\right)}{2}\right)-\frac{\left(1-2\varepsilon\right)}{12}\ln\left(\frac{K+1}{\widetilde{k}\left(K\right)}\right)\tag{54}$$

But, $\widetilde{k}\left(K\right)\leq\frac{\ln(K+1)-\ln 2}{2}+1$, so

$$\ln\left(\frac{K+1}{\widetilde{k}\left(K\right)}\right)=\ln\left(K+1\right)-\ln\left(\widetilde{k}\left(K\right)\right)\geq 2\widetilde{k}\left(K\right)-2+\ln 2-\ln\left(\widetilde{k}\left(K\right)\right)\tag{55}$$

Since $\lim_{x\to\infty}\frac{\ln x}{x}=0$ and since $\lim_{K\to\infty}\widetilde{k}\left(K\right)=\infty$ it follows that the term $\ln\left(\frac{K+1}{\widetilde{k}(K)}\right)$ eventually dominates in expression (54). Consequently, whenever $\varepsilon<\frac{1}{2}$ there exists $\overline{K}$ such that $U^i\left(\mathbf{l}|\text{veto by }\Theta_V^i\right)<0$ for any $K\geq\overline{K}$. Hence:

Claim 2 *There exists $\overline{K}<\infty$ such that type $\mathbf{l}$ and any type in Θ_V^i vetoes the Groves mechanism in any equilibrium for any $K\geq\overline{K}$.*

Now, define the set $\Theta_W^i=\left\{\theta^i|\theta_k^i\in\{l,m\}\text{ for every }k\right\}$. By virtue of Claim 2 types in Θ_V^i and type $\mathbf{l}$ will use their veto power, which, for $k<\widetilde{k}\left(K\right)$, implies that

$$\mathrm{E}\left[s_k\left(\theta_k\right)|m,\text{veto by }\mathbf{l}\text{ and }\Theta_V^i\right]<\mathrm{E}\left[s_k\left(\theta_k\right)|m\right]<\mathrm{E}\left[s_k\left(\theta_k\right)|l\right]=\mathrm{E}\left[s_k\left(\theta_k\right)|l,\text{veto by }\Theta_V^i\right].\tag{56}$$

For $k\geq\widetilde{k}\left(K\right)$ we first note that

$$\mathrm{E}\left[s_k\left(\theta_k\right)|m,\text{veto by }\mathbf{l}\text{ and }\Theta_V^i\right]<\Pr\left[\theta_k^i=l\right]\underbrace{\left[1-\Pr\left[\theta^i=\mathbf{l}|\theta_k^i=l\right]\right]}_{<\frac{1}{3}}s_k\left(ml\right)\tag{57}$$

$$+\Pr\left[\theta_k^i=m\right]\underbrace{\left[1-\Pr\left[\Theta_V^i|\theta_k^i=m\right]\right]}_{<\frac{1}{3}}s_k\left(mm\right)\;<\;\frac{1}{3}\mathrm{E}\left[s_k\left(\theta_k\right)|m\right],$$

and that,

$$\begin{aligned}
&\mathrm{E}\left[s_k\left(\theta_k\right)|l,\text{veto by }\Theta_V^i\right] &(58)\\
=\;&\mathrm{E}\left[s_k\left(\theta_k\right)|l\right]-\Pr\left[\theta_k^i=m\right]\underbrace{\left[\Pr\left[\Theta_V^i|\theta_k^i=m\right]\right]}_{<1}s_k\left(lm\right)>\mathrm{E}\left[s_k\left(\theta_k\right)|l\right]-\Pr\left[\theta_k^i=m\right]s_k\left(lm\right)\\
=\;&\frac{k+1}{2}+\frac{\varepsilon}{2\left(k+1\right)}-\frac{1}{2\left(k+1\right)^2}\left[\frac{1}{4}\frac{\left(k+1\right)^2}{k}+\varepsilon\left(k+1\right)\right]=\frac{k+1}{2}-\frac{1}{8k}\\
>\;&\frac{1}{3}\frac{k+1}{2}>\frac{1}{3}\mathrm{E}\left[s_k\left(\theta_k\right)|m\right]>\mathrm{E}\left[s_k\left(\theta_k\right)|m,\text{veto by }\mathbf{l}\text{ and }\Theta_V^i\right]
\end{aligned}$$

Combining (56) and (58) we have that $\mathrm{E}\left[s_k\left(\theta_k\right)|m,\text{veto by }\mathbf{l}\text{ and }\Theta_V^i\right]<\mathrm{E}\left[s_k\left(\theta_k\right)|l,\text{veto by }\Theta_V^i\right]$ for any k, which implies that $U_K^i\left(\theta^i|\text{veto by }\Theta_V^i\text{ and }\mathbf{l}\right)<U_K^i\left(\mathbf{l}|\text{veto by }\Theta_V^i\right)$ for any type θ^i such that $\theta_k^i\in\{l,m\}$ for every k :

Claim 3 *There exists some $\overline{K}$ such that every type $\theta^i \in \Theta^i_W$ vetoes the Groves mechanism in any equilibrium for any $K \geq \overline{K}$.*

For the final step in the argument consider a type with $\theta^i_k = h$ for $k = 2, .., \widehat{K} - 1$ and $\theta^i_k = l$ for $k = \widehat{K}, .., K$ and denote such a type by $\left(\mathbf{h}_{\widehat{K}-1}, \mathbf{l}_{K-\widehat{K}}\right)$. It is immediate that

$$\mathrm{E}\left[s_k\left(\theta_k\right)|h, \text{veto by } \theta^i \in \Theta^i_W\right] < \mathrm{E}\left[s_k\left(\theta_k\right)|h\right] = (k+1)^3 + \frac{(k+1)}{2}.$$

Moreover, since $\Pr\left[\theta^i \in \Theta^i_W|\theta^i_k = m\right] > \frac{2}{3}$ we can repeat the exact same steps as in calculation (51) to conclude that, for *any* $k \geq 2$,[12]

$$\begin{aligned}\mathrm{E}\left[s_k\left(\theta_k\right)|l, \text{veto by } \Theta^i_W\right] &= \Pr\left[\theta^i_k = m\right]\left[1 - \Pr\left[\theta^i \in \Theta^i_W|\theta^i_k = m\right]\right] s_k(lm) + \Pr\left[\theta^i_k = h\right] s_k(lh) \\ \text{/same steps as in (51)/} &< \frac{k+1}{2} + \frac{\varepsilon}{6(k+1)} - \frac{1}{12k}.\end{aligned}$$

Hence,

$$\begin{aligned}& U^i_K\left(\mathbf{h}_{\widehat{K}-1}, \mathbf{l}_{K-\widehat{K}}|\text{veto by } \Theta^i_W\right) \qquad (59) \\ <& \sum_{k=2}^{\widehat{K}-1}\left[(k+1)^3 + \frac{(k+1)}{2}\right] + \sum_{k=\widehat{K}+1}^{K}\left[\frac{k+1}{2} + \frac{\varepsilon}{6(k+1)} - \frac{1}{12k}\right] - \sum_{k=2}^{K}\frac{k+1}{2} \\ =& \sum_{k=2}^{\widehat{K}-1}(k+1)^3 + \sum_{k=\widehat{K}}^{K}\left[\frac{\varepsilon}{6(k+1)} - \frac{1}{12k}.\right]\end{aligned}$$

But, by the calculations in (53) we know that $\sum_{k=\widehat{K}}^{K}\frac{1}{k+1} < \ln\left(\frac{K+1}{\widehat{K}}\right) < \sum_{k=\widehat{K}}^{K}\frac{1}{k}$, which implies that

$$U^i_K\left(\mathbf{h}_{\widehat{K}-1}, \mathbf{l}_{K-\widehat{K}}|\text{veto by } \theta^i \in \Theta^i_V\right) < \sum_{k=2}^{\widehat{K}-1}(k+1)^3 - \frac{1-2\varepsilon}{12}\ln\left(\frac{K+1}{\widehat{K}}\right) \equiv G\left(\widehat{K}, K\right) \qquad (60)$$

It is easy to check that: i) G is strictly increasing in $\widehat{K}$, and; ii) $G(2, K) < 1$ (assuming that $\varepsilon < \frac{1}{2}$, which we do), and; iii) $G(K+1, K) > 0$. This implies that there exists a unique integer $\widehat{K}(K)$ such that $G\left(\widehat{K}, K\right) < 0$ for every $K \leq \widehat{K}(K)$ and $G\left(\widehat{K}, K\right) \geq 0$ for $K > \widehat{K}(K)$. Moreover, $\widehat{K}(K) \to \infty$ as $K \to \infty$ since otherwise there exists some $\overline{K}$ such that $G\left(\overline{K}, K\right) \geq 0$ for all K, which cannot be the case, since $\lim_{K\to\infty} G\left(\overline{K}, K\right) = -\infty$ for any fixed $\overline{K}$. To complete the argument we observe that if $\left(\boldsymbol{\theta}^i_{\widehat{K}(K)-1}, \mathbf{l}_{K-\widehat{K}(K)}\right)$ is any type profile such that $\theta^i_k = l$ for $k = \widehat{K}(K), ..., K$,

[12] The difference is that (51) is only applicable for $k \geq \widetilde{k}(K)$.

then the right hand side of (60) is an upper bound for the interim expected payoff also for type $\left(\boldsymbol{\theta}^i_{\widehat{K}(K)-1}, \mathbf{1}_{K-\widehat{K}(K)}\right)$ conditional on types in Θ^i_W vetoing the mechanism, implying that

$$U^i_K\left(\boldsymbol{\theta}^i_{\widehat{K}(K)-1}, \mathbf{1}_{K-\widehat{K}(K)} | \text{veto by } \theta^i \in \Theta^i_V\right) < G\left(\widehat{K}(K), K\right) \leq 0.$$

Since $\widehat{K}(K) \to \infty$, calculation (45) implies that $\Pr\left[\left(\theta^i_{\widehat{K}(K)}, ..., \theta^i_K\right) = \mathbf{1}_{K-\widehat{K}(K)}\right] \to 1$ as $K \to \infty$. We can thus conclude that:

Claim 4 *The probability that a veto is cast approaches* 1 *as* $K \to \infty$.

7 Discussion

7.1 Linking in Trade Negotiations

Trade treaties are in some ways a perfect application for our result, provided that one believes that informational asymmetries regarding the relative valuation between, say, a given tariff reduction and improved intellectual property rights is an important consideration. The self-financing constraint (12) then simply says that there is no outside source of resources, and the participation constraint (11) is just national sovereignty, which seems to be an appropriate assumption.

The obvious limitation of our model is that many issues in a trade treaty are fundamentally interrelated. For example, technical barriers of trade become an important issue only after tariffs have been reduced to a level which is below the unilaterally most preferred tariff, providing a direct rationale for linking tariff negotiations with rules governing non-tariff barriers of trade. While this is indeed a shortcoming, we do not think that our result is irrelevant for thinking about real world trade agreements. Instead, we think that our result highlights a fundamental force that works in favor of highly multidimensional agreements that is absent in the previous literature on the topic, which deals exclusively with complete information environments. At present we do not know how generalizable the approximate efficiency result is to more realistic models, but the basic point that one participation constraint is easier to satisfy than many suggests that there should be a role for linking more generally. Our logic then suggests that it may be a mistake to negotiate a treaty for reducing greenhouse gas emissions in separation, since linking with other issues, such as trade concessions, would make a veto more costly.

Unlike our paper, papers in the international trade literature on issue-linking in trade negotiations, such as Bagwell and Staiger [4], Conconi and Perotti [11], Horstmann et al [19], and Spagnolo [28], rely on a given bargaining protocol. The pros and cons of linking in these papers are therefore largely derived from *strategic effects*. Since such effects usually depend on details of the bargaining protocol, it is not that surprising that results are mixed. In contrast, the mechanism design methodology used in this paper has the advantage of ruling out results that are driven by particularities in a non-cooperative bargaining game.

7.2 Governments as Linking Mechanisms

Real world government institutions usually fulfil many and arguably quite unrelated functions, and the results in this paper can be seen as a simple theory explaining this, as far as we know, previously unexplained fact. The existing theory of public finance has identified many potential sources of market failure, and interpreted these as a rationale for "government intervention." However, the existing theory only explains that it may be beneficial to set up some institution to deal with each particular market failure, but does not provide a foundation for why a *single* government institution should be responsible for dealing with all these problem. For example, we understand that there are externalities involved in garbage collection, that public parks would be under-provided by voluntary provisions, and that for-profit policing may be a bad idea, but it seems hard to argue that there are technological reasons for why these services should be provided jointly as part of a local government bundle, as they tend to be. Our results suggest a possible explanation: linking all these seemingly unrelated social decisions via a single government institution helps achieve efficiency by alleviating citizens' participation constraints.

The reader may complain that our mechanism is an unrealistic description of what real world governments do. In particular, it may be argued that a single citizen usually does not have veto power at the local government level, which is the level of government where participation constraints seem the most realistic. We agree, but we also note that *most goods and services provided at the local government level are such that use exclusion is possible.* Veto power is needed for our formal result in order to accommodate pure public goods problems. However, if it is possible to realize the reservation utility for an agent by excluding the agent from usage, then the veto game may be replaced by a game where agents independently choose between opting in and opting out, and where the efficient outcome conditional on whatever agents are opting in is implemented in the second stage. Such a "voting with your feet" participation constraint does not seem unrealistic.

7.3 Excludable versus Non-Excludable Public Goods

The distinction between vetoing and use exclusions is the most transparent in the contexts of an economy with K public goods. Since mechanism (24) rests on veto power, it does not distinguish between excludable and non-excludable public goods. Either all agents opt in and consume all the public goods that are produced, or a veto is cast, in which case none of the public goods is provided. Since the probability of a veto goes to zero as K goes out of bounds, use exclusions are not needed for asymptotic efficiency. This "asymptotic irrelevance of exclusions" depends crucially on the fact that we keep the number of agents n fixed. In contrast, Norman [26] and Fang and Norman [15] demonstrate that the exclusion instrument is a crucial feature of the constrained optimal mechanism when K is fixed (to 1 and 2 respectively) and n is large.

Our proof does not apply to sequences where both K and n tend to infinity. We conjecture that in the case of non-excludable public goods, asymptotic efficiency is impossible if K and n go out of bounds at the same rate. On the other hand, if goods are excludable, there is no need to equip agents with veto power; reservation utilities can be attained by the "milder" exclusion instrument. As a result, asymptotic efficiency is attainable in this case (regardless of the asymptotic behavior of K/n). Indeed, since the efficient provision for good k converges either to "always provide" or "never provide" as n tends to infinity approximate efficiency can be implemented with a mechanism where the provision decisions are made ex ante and a fixed price is charged for access to all public goods, much in the same spirit as Armstrong's [1] two-part tariff scheme.

References

[1] Armstrong, Mark, "Price Discrimination by a Many-Product Firm," *Review of Economic Studies*, 90(1), January 1999, 151-168.

[2] Arrow, Kenneth, "The Property Rights Doctrine and Demand Revelation Under Incomplete Information," in *Economics and Human Welfare*, Academic Press, 1979.

[3] Athey, Susan, and Ilya Segal, "An Efficient Dynamic Mechanism," *unpublished manuscript,* Stanford University, January 2006.

[4] Bagwell, Kyle, and Robert Staiger, "Domestic Policies, National Sovereignty, and International Institutions, "*Quarterly Journal of Economics,* 116, 2001. 519-562.

[5] Bakos, Yannis and Erik Brynjolfsson, "Bundling Information Goods: Pricing, Profits and Efficiency," *Management Science*, 45(12), December 1999, 1613-1630.

[6] Battaglini, Marco, and Stephen Coate, "Inefficiency in Legislative Policy-Making: A Dynamic Analysis," *unpublished manuscript,* Cornell and Princeton, July 2005.

[7] Bergstrom, Carl and Ted Bergstrom, "The Costs and Benefits of Library Site Licenses to Academic Journals," *Proceedings of the National Academy of Sciences*, 101(3), January 2004, 897-902.

[8] Borgers, Tilman and Peter Norman, "Ex-Ante and Ex-Post Budget-Balanced Mechanisms When Signals are Stochastically Independent," *unpublished manuscript*, University College London and University of British Columbia, February 2005.

[9] Borgers, Tilman and Peter Postl, "Efficient Compromising,", *unpublished manuscript,* University College London and Birmingham, June 2004.

[10] Casella, Alessandra, "Storable Votes", *Games and Economic Behavior*, 51, 2005, 391-419.

[11] Conconi, Paula and Carlo Perroni, "Issue Linkage and Issue Tie-In in Multilateral Negotiations," *Journal of International Economics,* 57, 2002, 423-447.

[12] Copeland, Brian, and Scott Taylor, "Trade and Transboundary Pollution," *American Economic Review*, 85, 1995, 716-737.

[13] Cremer, Jacques and Richard P. McLean, "Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions", *Econometrica,* 56, 1988, 1247-57.

[14] d'Aspremont, Claude and Louis-Andre' Gérard-Varet, "Incentives and Incomplete Information", *Journal of Public Economics* 11, 1979, 25-45.

[15] Fang, Hanming and Peter Norman, "An Efficiency Rationale for Bundling of Public Goods" , *Cowles Foundation Discussion Papers*, 1441, November 2004.

[16] Fang, Hanming and Peter Norman, "External Appendix to "Overcoming Participation Constraints"; Detailed Calculations For the Example in Section 6.4, *unpublished manuscript*, Yale and University of British Columbia, February 2006.

[17] Geng, Xianjun, Maxwell B. Stinchcombe and Andrew B. Whinston, "Bundling Information Goods of Decreasing Value," *Management Science*, 51(4), April 2005, 662-667.

[18] Groves, Theodore, "Incentives in Teams," *Econometrica*, 41, 1973, 617-631.

[19] Horstmann, Ignatius J., James R. Markusen, and Jack Robles, "Issue Linking in Trade Negotiations: Ricardo Revisited or No Pain No Gain," forthcoming in *Review of International Economics.*

[20] Jackson, Matt and Hugo Sonnenschein, "Overcoming Incentive Constraints by Linking Decisions," forthcoming, 2005, *Econometrica.*

[21] McAfee, R. Preston, "Amicable Divorce: Dissolving a Partnership with Simple Mechanisms", *Journal of Economic Theory,* 56, no.2, April 1992, 266-93.

[22] Mailath, George and Andrew Postlewaite, "Asymmetric Information Bargaining Problems with Many Agents", *Review of Economic Studies* 57, 1990, 351-368.

[23] Maskus, Keith E. "Should Core Labor Standards Be Imposed Through International Trade Policy," *World Bank Policy Research Working Paper* 1817, 1997.

[24] Milgrom, Paul and Robert Weber, "Distributional Strategies for Games with Incomplete Information," *Mathematics of Operations Research,* 10, 1985.

[25] Myerson, Roger and Mark Satterthwaite, "Efficient Mechanisms for Bilateral Trading", *Journal of Economic Theory* 29, 1983, 265-281.

[26] Norman, Peter, "Efficient Mechanisms for Public Goods with Use Exclusion." *Review of Economic Studies*, 71, 2004, 1163-1188.

[27] Palfrey, Thomas R. "Bundling Decisions by a Multiproduct Monopolist with Incomplete Information." *Econometrica*, 51, 1983, 463-484.

[28] Spagnolo, Giancarlo "Issue Linkage, Credible Delegation, and Policy Cooperation," *CEPR Discussion Paper* No. 2778, 2001.

[29] Townsend, Robert, "Optimal Multi-Period Contracts and the Gain From Enduring Relationships Under Private Information," *Journal of Political Economy* 90, 1166-1186.

A Appendix

A.1 Proof of Proposition 1

Proof. Since $S_K(\theta) = \sum_{k=1}^{K} s_k(\theta_k)$ and $R_K^i(\theta^i) = \sum_{k=1}^{K} r_k^i(\theta_k^i)$, we may express (17) as

$$U_K^i(\theta^i) = \sum_{k=1}^{K} \left\{ \mathrm{E}_{-i}[s_k(\theta_k)] - \frac{n-1}{n}\mathrm{E}[s_k(\theta_k)] + \mathrm{E}\left[r_k^i(\theta_k^i)\right] - \frac{1}{n}\mathrm{E}\left[\sum_{j=1}^{n} r_k^j\left(\theta_k^j\right)\right]\right\} \tag{A1}$$

Define $\phi_k^i(\theta_k^i) \equiv \mathrm{E}_{-i}[s_k(\theta_k)] - r_k^i(\theta_k^i)$. This allows us to express $U_K^i(\theta^i) - R_K^i(\theta^i)$ as

$$U_K^i(\theta^i) - R_K^i(\theta^i) = \sum_{k=1}^{K}\left\{\phi_k^i(\theta_k^i) - \mathrm{E}\left[\phi_k^i(\theta_k^i)\right] + \frac{1}{n}\left(\mathrm{E}[s_k(\theta_k)] - \mathrm{E}\left[\sum_{j=1}^{n} r_k^j\left(\theta_k^j\right)\right]\right)\right\}. \tag{A2}$$

By assumption all issues in economy $\mathcal{E}_K$ are regular, thus by (R1), we have:

$$\mathrm{E}_i\left[U_K^i(\theta^i) - R_K^i(\theta^i)\right] = \frac{1}{n}\sum_{k=1}^{K}\left\{\mathrm{E}[s_k(\theta_k)] - \mathrm{E}\left[\sum_{j=1}^{n} r_k^j(\theta^j)\right]\right\} \geq \frac{K}{n}\delta. \tag{A3}$$

Moreover, since the issues are stochastically independent, $\left\{\theta_k^i\right\}_{k=1}^{K}$ is a sequence of stochastically independent variables, thus $\left\{\phi_k^i(\theta_k^i)\right\}_{k=1}^{K}$ is a sequence of independent variables. Finally, since all issues are regular, $\phi_k^i(\theta_k^i)$ is bounded from above and below by (R2) and (R3). Hence, there exists σ^2 such that $\mathrm{Var}\left[\phi_k^i(\theta_k^i)\right] \leq \sigma^2$ for every i and every k, so

$$\begin{aligned}
\Pr\left[U_K^i\left(\theta^i\right) - R_K^i\left(\theta^i\right) < 0\right] &\overset{(A2)}{=} \Pr\left[\sum_{k=1}^{K}\left(\phi_k^i\left(\theta_k^i\right) - \mathrm{E}\left[\phi_k^i\left(\theta_k^i\right)\right]\right) < -\frac{1}{n}\sum_{k=1}^{K}\left(\mathrm{E}[s_k(\theta_k)] - \mathrm{E}\left[\sum_{j=1}^{n} r_k^j\left(\theta_k^j\right)\right]\right)\right] \\
\text{/By inequality in (A3)/} \quad &\leq \Pr\left[\sum_{k=1}^{K}\left(\phi_k^i\left(\theta_k^i\right) - \mathrm{E}\left[\phi_k^i\left(\theta_k^i\right)\right]\right) < -\frac{\delta K}{n}\right] \\
&\leq \Pr\left[\left|\sum_{k=1}^{K}\left(\phi_k^i\left(\theta_k^i\right) - \mathrm{E}\left[\phi_k^i\left(\theta_k^i\right)\right]\right)\right| > \frac{\delta K}{n}\right] \\
\text{/Chebyshev's inequality/} \quad &\leq \left(\frac{1}{\frac{\delta K}{n}}\right)^2 \mathrm{Var}\left[\sum_{k=1}^{K}\phi_k^i\left(\theta_k^i\right)\right] \leq \frac{n^2\sigma^2}{\delta^2 K}
\end{aligned} \tag{A4}$$

Since $\frac{n^2\sigma^2}{\delta^2 K} \to 0$ as $K \to \infty$, we conclude that for every $\varepsilon > 0$, there exists some finite integer $K_i^*(\varepsilon)$ such that (**??**) holds for agent i. Let $K^*(\varepsilon) = \max_{i \in I} K_i^*(\varepsilon)$ and the result follows. ■

A.2 Proof of Lemma 3

Proof. Assuming that all other agents announce truthfully and that $m = (1, ..., 1) = \mathbf{1}$ is announced, the ex post payoff for agent i of type θ^i from announcement $\widehat{\theta}^i$ is

$$u^i\left(\widehat{\theta}^i, \mathbf{1}, \theta\right) = V_K^i\left(x_K^*\left(\widehat{\theta}^i, \theta^{-i}\right), \theta^i\right) + \sum_{j \neq i} V_K^j\left(x_K^*\left(\widehat{\theta}^i, \theta^{-i}\right), \theta^j\right) + T_K^i,$$

where T_K^i is defined in (31). By construction, $x_K^*(\cdot)$ is maximized at $\widehat{\theta}^i = \theta^i$, resulting in ex post payoff $u^i\left(\theta^i, \mathbf{1}, \theta\right) = S_K(\theta) - T_K^i$. For $m \neq \mathbf{1}$ we have that $u^i\left(\widehat{\theta}^i, m, \theta\right) = R_K^i\left(\theta^i\right)$ for any $\widehat{\theta}^i$. Taking expectations over θ^{-i} gives the result. ■

A.3 Proof of Lemma 1

Proof. Using the expression for $U_K^j\left(\theta^j\right) - R_K^j\left(\theta^j\right)$ in equation (A2), we can proceed just like in that proof, with the only difference being that now the calculation is for a bound on the probability that the payoff is less than half of the lower bound on the expected value of the payoff. That is,

$$\begin{aligned}
1 - \Pr\left[\overline{\Theta}^j(K)\right] &= \Pr\left[U_K^j\left(\theta^j\right) - R_K^j\left(\theta^j\right) < \frac{\delta K}{2n}\right] \\
&= \Pr\left[\sum_{k=1}^{K}\left(\phi_k^j\left(\theta_k^j\right) - \mathrm{E}\left[\phi_k^j\left(\theta_k^j\right)\right] + \frac{\mathrm{E}\left[s_k\left(\theta_k\right)\right] - \mathrm{E}\left[\sum_{j=1}^{n} r_k^j\left(\theta^j\right)\right]}{n}\right) < \frac{\delta K}{2n}\right] \\
\text{/By inequality in (A3)/} \quad &\leq \Pr\left[\sum_{k=1}^{K}\left(\phi_k^j\left(\theta_k^j\right) - \mathrm{E}\left[\phi_k^j\left(\theta_k^j\right)\right]\right) < -\frac{\delta K}{2n}\right] \\
&\leq \Pr\left[\left|\sum_{k=1}^{K}\left(\phi_k^j\left(\theta_k^j\right) - \mathrm{E}\left[\phi_k^j\left(\theta_k^j\right)\right]\right)\right| > \frac{\delta K}{2n}\right] \\
\text{/Chebyshev's inequality/} \quad &\leq \left(\frac{1}{\frac{\delta K}{2n}}\right)^2 \mathrm{Var}\left[\sum_{k=1}^{K}\phi_k^j\left(\theta_k^j\right)\right] \leq \frac{4n^2\sigma^2}{\delta^2 K}
\end{aligned}$$

Since $\frac{n^2\sigma^2}{\delta^2 K} \to 0$ as $K \to \infty$, we conclude that, for every $\varepsilon > 0$, there exists some finite integer $K_j'(\varepsilon)$ such that for every $K \geq K_j'(\varepsilon)$,

$$1 - \Pr\left[\overline{\Theta}^j(K, \delta)\right] \leq 1 - (1 - \varepsilon)^{\frac{1}{n-1}}.$$

Hence, by letting $K'(\varepsilon) = \max_{j \neq i} K_j'(\varepsilon)$ and using stochastic independence we have that

$$\Pr\left[\overline{\Theta}^{-i}(K)\right] = \times_{j \neq i} \Pr\left[\overline{\Theta}^j(K, \delta)\right] \geq 1 - \varepsilon$$

for every $K \geq K'(\varepsilon)$. ■

A.4 Proof of Lemma 2

Proof. By regularity assumption (R2), there is a uniform bound $\bar{a} > 0$ so that $s_k(\theta_k) \leq \bar{a}$ for every θ_k, which in turn implies that $S_K(\theta)/K \leq \bar{a}$ for every K and $\theta \in \Theta(K)$. Thus

$$\begin{aligned}
\frac{\mathrm{E}_{-i}[S_K(\theta)]}{K} &= \Pr\left[\overline{\Theta}^{-i}(K)\right] \frac{\mathrm{E}_{-i}\left[S_K(\theta) | \overline{\Theta}^{-i}(K)\right]}{K} \\
&\quad + \left(1 - \Pr\left[\overline{\Theta}^{-i}(K)\right]\right) \frac{\mathrm{E}_{-i}\left[S_K(\theta) | \theta^{-i} \notin \overline{\Theta}^{-i}(K)\right]}{K} \\
&\leq \frac{\mathrm{E}^{-i}\left[S_K(\theta) | \overline{\Theta}^{-i}(K)\right]}{K} + \left(1 - \Pr\left[\overline{\Theta}^{-i}(K)\right]\right) \bar{a}
\end{aligned}$$

Fix $\varepsilon > 0$. By Lemma 1 there exists $K'\left(\frac{\varepsilon}{\bar{a}}\right)$ such that $\Pr\left[\overline{\Theta}^{-i}(K)\right] \geq 1 - \frac{\varepsilon}{\bar{a}}$ for every $K \geq K'\left(\frac{\varepsilon}{\bar{a}}\right)$, implying that

$$\frac{\mathrm{E}_{-i}\left[S_K(\theta) | \overline{\Theta}^{-i}(K)\right]}{K} \geq \frac{\mathrm{E}^{-i}[S_K(\theta)]}{K} - \left[1 - \left(1 - \frac{\varepsilon}{\bar{a}}\right)\right] \bar{a} = \frac{\mathrm{E}_{-i}[S_K(\theta)]}{K} - \varepsilon,$$

which gives the result for $K''(\varepsilon) = K'\left(\frac{\varepsilon}{\bar{a}}\right)$. ■

A.5 Proof of Lemma 4

Proof. Assuming that $\psi_K^j(\theta^j) = 1$ for all $j \neq i$ and all $\theta^j \in \overline{\Theta}^j(K)$, the interim expected payoff for agent i from setting $m^i = 1$ as defined in (30) can be written as

$$\begin{aligned}
\widehat{U}^i\left(1, \psi_K^{-i}, \theta^i\right) &= \int_{\theta^{-i}} \left(\left[S_K(\theta) + T_K^i\right] \Psi_K^{-i}\left(\theta^{-i}\right) + R_K^i\left(\theta^i\right)\left[1 - \Psi_K^{-i}\left(\theta^{-i}\right)\right]\right) dF_K^{-i}\left(\theta^{-i}\right) \\
&= \Pr\left[\overline{\Theta}^{-i}(K)\right]\left(\mathrm{E}_{-i}\left[S_K(\theta) | \overline{\Theta}^{-i}(K)\right] + T_K^i\right) \qquad \text{(A5)} \\
&\quad + \int_{\theta^{-i} \in \Theta^{-i}(K) \setminus \overline{\Theta}^{-i}(K)} \left(\left[S_K(\theta) + T_K^i\right] \Psi_K^{-i}\left(\theta^{-i}\right) + R_K^i\left(\theta^i\right)\left[1 - \Psi_K^{-i}\left(\theta^{-i}\right)\right]\right) dF_K^{-i}\left(\theta^{-i}\right).
\end{aligned}$$

Since $s_k(\theta_k)$ is uniformly bounded above by $\bar{a}$ (by R2) and $r_k^i\left(\theta_k^i\right)$ is uniformly bounded below by $\underline{b}$ (by R3), we can bound T_K^i as defined in (31) as follows:

$$\begin{aligned}
T_K^i &\equiv -\frac{n-1}{n}\mathrm{E}[S_K(\theta)] + \mathrm{E}\left[R_K^i\left(\theta^i\right)\right] - \frac{1}{n}\mathrm{E}\left[\sum_{j=1}^n R_K^j\left(\theta^j\right)\right] - \frac{\delta K}{4n} \\
\Big/ S_K(\theta) \geq \sum_{j=1}^n R_K^j\left(\theta^j\right) \Big/ \quad &\geq -\mathrm{E}[S_K(\theta)] + \mathrm{E}\left[R_K^i\left(\theta^i\right)\right] - \frac{\delta K}{4n} \geq \left(\underline{b} - \bar{a} - \frac{\delta}{4n}\right) K.
\end{aligned}$$

Moreover $s_k(\theta_k)$ is uniformly bounded below by $\underline{a}$ (by R2), so $S_K(\theta) \geq \underline{a}K$. We conclude that

$$\begin{aligned}
S_K(\theta) + T_K^i &\geq \left(\underline{a} + \underline{b} - \bar{a} - \frac{\delta}{4n}\right) K \\
R_K^i\left(\theta^i\right) &\geq \underline{b}K > \left(\underline{a} + \underline{b} - \bar{a} - \frac{\delta}{4n}\right) K,
\end{aligned}$$

where the strict inequality follows since $\underline{a} < \bar{a}$. Hence,

$$\begin{aligned}
&\int_{\theta^{-i} \in \Theta^{-i}(K) \setminus \overline{\Theta}^{-i}(K)} \left(\left[S_K(\theta) + T_K^i \right] \Psi_K^{-i}\left(\theta^{-i}\right) + R_K^i\left(\theta^i\right) \left[1 - \Psi_K^{-i}\left(\theta^{-i}\right) \right] \right) dF_K^{-i}\left(\theta^{-i}\right) \\
\geq \;& K \left(\underline{a} + \underline{b} - \bar{a} - \frac{\delta}{4n} \right) \left(1 - \Pr\left[\overline{\Theta}^{-i}(K) \right] \right). \qquad \text{(A6)}
\end{aligned}$$

Combining (A5) and (A6) we obtain

$$\begin{aligned}
\widehat{U}^i\left(1, \psi^{-i}, \theta^i\right) \;\geq\;& \Pr\left[\overline{\Theta}^{-i}(K) \right] \left(\mathrm{E}_{-i}\left[S_K(\theta) | \overline{\Theta}^{-i}(K) \right] + T_K^i \right) \\
&+ \left(1 - \Pr\left[\overline{\Theta}^{-i}(K) \right] \right) K \left(\underline{a} + \underline{b} - \bar{a} - \frac{\delta}{4n} \right).
\end{aligned}$$

By Lemma 2, there exists K_1 so that for each $K \geq K_1$,

$$\frac{\mathrm{E}_{-i}\left[S_K(\theta) | \overline{\Theta}^{-i}(K) \right]}{K} \geq \frac{\mathrm{E}_{-i}\left[S_K(\theta) \right]}{K} - \frac{\delta}{8n},$$

where $\delta > 0$ is the uniform bound of the difference between the maximized expected surplus and the sum of the participation utilities (see Definition 2). Hence,

$$\begin{aligned}
\frac{\widehat{U}^i\left(1, \psi^{-i}, \theta^i\right)}{K} \;\geq\;& \Pr\left[\overline{\Theta}^{-i}(K) \right] \left(\frac{\mathrm{E}_{-i} S_K(\theta) + T_K^i}{K} - \frac{\delta}{8n} \right) \\
&+ \left(1 - \Pr\left[\overline{\Theta}^{-i}(K) \right] \right) \left(\underline{a} + \underline{b} - \bar{a} - \frac{\delta}{4n} \right).
\end{aligned}$$

for every $K \geq K_1$. By (17), we know that $\mathrm{E}_{-i}\left[S_K(\theta) \right] + T_K^i + \frac{\delta}{4n}$ is the interim expected payoff for type θ^i in the Groves mechanism. Since $\theta^i \in \overline{\Theta}^i(K)$ it follows from definition (26) that $\mathrm{E}_{-i}\left[S_K(\theta) \right] + T_K^i + \frac{\delta}{4n} \geq R_K^i\left(\theta^i\right) + \frac{\delta K}{2n}$. Hence

$$\begin{aligned}
\frac{\widehat{U}^i\left(1, \psi^{-i}, \theta^i\right)}{K} \;\geq\;& \Pr\left[\overline{\Theta}^{-i}(K) \right] \left(\frac{R_K^i\left(\theta^i\right) + \frac{\delta K}{4n}}{K} - \frac{\delta}{8n} \right) \\
&+ \left(1 - \Pr\left[\overline{\Theta}^{-i}(K) \right] \right) \left(\underline{a} + \underline{b} - \bar{a} \right),
\end{aligned}$$

or

$$\frac{\widehat{U}^i(1, \psi^{-i}, \theta^i) - R_K^i\left(\theta^i\right)}{K} \geq \Pr\left[\overline{\Theta}^{-i}(K) \right] \left(\frac{\delta}{8n} \right) + \left(1 - \Pr\left[\overline{\Theta}^{-i}(K) \right] \right) \left(\underline{a} + \underline{b} - \bar{a} \right),$$

The right hand side converges to $\delta / (8n)$ as $\Pr\left[\overline{\Theta}^{-i}(K) \right]$ approaches one, so there exists $\varepsilon_2 > 0$ such that $\widehat{U}^i(1, \psi^{-i}, \theta^i) - R_K^i\left(\theta^i\right) \geq 0$ if $\Pr\left[\overline{\Theta}^{-i}(K) \right] \geq 1 - \varepsilon_2$. Lemma 1 assures that there exists K_2 such that $\Pr\left[\overline{\Theta}^{-i}(K) \right] \geq 1 - \varepsilon_2$ for every $K \geq K_2$. For $K^* = \max\{K_1, K_2\}$ it therefore follows that $\widehat{U}^i(1, \psi^{-i}, \theta^i) - R_K^i\left(\theta^i\right) \geq 0$. ∎

A.6 Proof of Lemma 5

Let $\Psi_K(\theta) \equiv \prod_{i=1}^{n} \psi_K^i(\theta^i)$ denote the probability that $m = (1, ..., 1)$ given type profile $\theta \in \Theta(K)$. The expected budget tax revenues can then be expressed as

$$
\begin{aligned}
& \mathrm{E}\Psi_K(\theta)\left[\sum_{i=1}^{n} \widehat{t}_K^i(\theta, \mathbf{1})\right] \\
= \ & \mathrm{E}\Psi_K(\theta)\left[\sum_{i=1}^{n} V_K^i\left(x_K^*(\theta), \theta^i\right) - S_K(\theta) + \frac{n-1}{n}\mathrm{E}\left[S_K(\theta)\right] - \mathrm{E}\left[R_K^i\left(\theta^i\right)\right] + \frac{1}{n}\mathrm{E}\left[\sum_{j=1}^{n} R_K^j\left(\theta^i\right)\right] + \frac{\delta K}{4n}\right] \\
= \ & \mathrm{E}\Psi_K(\theta)\left[(n-1)\left(\mathrm{E}\left[S_K(\theta)\right] - S_K(\theta)\right) + \frac{\delta K}{4} + \mathcal{C}_K\left(\widehat{x}(\theta, m)\right)\right]
\end{aligned}
$$

By assumption, $\Psi_K(\theta) = 1$ when $\theta \in \overline{\Theta}(K)$, so the budget surplus/deficit satisfies

$$
\begin{aligned}
& \mathrm{E}\Psi_K(\theta)\left[\sum_{i=1}^{n} \widehat{t}_K^i(\theta, \mathbf{1}) - \mathcal{C}_K\left(\widehat{x}(\theta, m)\right)\right] \\
= \ & (n-1)\int_{\theta \in \overline{\Theta}(K)} \left[\left(\mathrm{E}\left[S_K(\theta)\right] - S_K(\theta)\right) + \frac{\delta K}{4}\right] dF_K(\theta) \\
& + (n-1)\int_{\theta \notin \overline{\Theta}(K)} \Psi_K(\theta)\left[\left(\mathrm{E}\left[S_K(\theta)\right] - S_K(\theta)\right) + \frac{\delta K}{4}\right] dF_K(\theta),
\end{aligned}
$$

But, $\mathrm{E}\left[S_K(\theta)\right] = \int_{\theta \in \overline{\Theta}(K)} S_K(\theta)\, dF(\theta) + \int_{\theta \notin \overline{\Theta}(K)} S_K(\theta)\, dF(\theta)$, so we may rearrange the expression above as

$$
\begin{aligned}
& \frac{\mathrm{E}\Psi_K(\theta)\left[\sum_{i=1}^{n} \widehat{t}_K^i(\theta, \mathbf{1}) - \mathcal{C}_K\left(\widehat{x}(\theta, m)\right)\right]}{n-1} \\
= \ & \Pr\left[\overline{\Theta}(K)\right]\left[\mathrm{E}\left[S_K(\theta)\right] + \frac{\delta K}{4}\right] - \mathrm{E}\left[S_K(\theta)\right] + \int_{\theta \notin \overline{\Theta}(K)} S_K(\theta)\, dF_K(\theta) \\
& + \int_{\theta \notin \overline{\Theta}(K)} \Psi_K(\theta)\left[\left(\mathrm{E}\left[S_K(\theta)\right] - S_K(\theta)\right) + \frac{\delta K}{4}\right] dF_K(\theta) \\
= \ & \Pr\left[\overline{\Theta}(K)\right]\frac{\delta K}{4n} - \left[1 - \Pr\left[\overline{\Theta}(K)\right]\right]\mathrm{E}\left[S_K(\theta)\right] \\
& + \int_{\theta \notin \overline{\Theta}(K)} \left[\Psi_K(\theta)\left(\mathrm{E}\left[S_K(\theta)\right] + \frac{\delta K}{4n}\right) + \left(1 - \Psi_K(\theta)\right) S_K(\theta)\right] dF_K(\theta) \\
> \ & \Pr\left[\overline{\Theta}(K)\right]\frac{\delta K}{4n} + \left(1 - \Pr\left[\overline{\Theta}(K)\right]\right)\left(\underline{a} - \overline{a}\right) K,
\end{aligned}
$$

where the inequality follows since $\mathrm{E}\left[S_K(\theta)\right] + \frac{\delta K}{4} > \underline{a}K$, $S_K(\theta) > \underline{a}K$, and $\mathrm{E}\left[S_K(\theta)\right] < \overline{a}K$ by assumption [R2]. Since $\Pr\left[\overline{\Theta}(K)\right] \to 1$ as $K \to \infty$ it follows that there exists some finite K^{**} such that the expected surplus is positive for any $K \geq K^{**}$. ■

External Appendix to "Overcoming Participation Constraints"; Detailed Calculations For the Example in Section 6.4.

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Abstract

This document provides step by step calculations and explanations that are omitted from the presentation of the example in Section 6.4 of the main paper.

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1 Primitives

The reader may note that there are many sets of primitives that generate the same maximized surplus function. The primitives below are picked mainly to make the calculations convenient. The reader may also take note that once the maximized surplus function has been derived, there is no need to return to the primitives, since interim expected payoffs from participation in the mechanisms depend only on the maximized surplus function. However, it is still important to specify some underlying primitives: not every surplus function $s_k : \Theta_k \to R$ and reservation payoff $r_k : \Theta_k \to R^n$ are consistent with surplus maximization, so we need to demonstrate existence of primitives generating the surplus function and reservation payoff function in the example.

1.1 Costs and Utility Functions

Let $n = 2$ and assume that, for each k and $i = 1, 2$, the single issue k type space is given by $\Theta^i = \{l, m, h\}$. Assume that the set of alternative resolutions of issue k are $D_k = \left\{d_k^0, d_k^l, d_k^m, d_k^h\right\}$. In terms of interpretation it is useful to think of d_k^0 as the "status quo" outcome, whereas d_k^l, d_k^m and d_k^h are surplus maximizing alternatives for type profiles ll, mm and hh respectively.

The cost function is given by

$$C_k\left(d_k\right) = \begin{cases} 0 & \text{if } d_k = d_k^0 \\ \frac{(k+1)^2}{k} & \text{if } d_k = d_k^l \\ \frac{k(k+1)^2}{2} & \text{if } d_k = d_k^m \\ 2\left(k+1\right)^3 & \text{if } d_k = d_k^h \end{cases},$$

Assume that the valuation functions for the three types are given by;

$$v_k^i\left(d_k, l\right) = \begin{cases} 0 & \text{if } d_k = d_k^0 \\ \frac{1}{2}\frac{(k+1)^2}{k} & \text{if } d_k = d_k^l \\ 0 & \text{if } d_k = d_k^m \\ \frac{(k+1)^2}{4}\left[k - \frac{1}{k}\right] & \text{if } d_k = d_k^h \end{cases}$$

for $\theta_k^i = l$,

$$v_k^i\left(d_k, m\right) = \begin{cases} 0 & \text{if } d_k = d_k^0 \\ \frac{3}{4}\frac{(k+1)^2}{k} + \varepsilon\left(k+1\right) & \text{if } d_k = d_k^l \\ \frac{k(k+1)^2}{2} - 2\varepsilon k\left(k+1\right)\left(k+2\right) & \text{if } d_k = d_k^m \\ \frac{k(k+1)^2}{4} - 3\left(k+1\right)^3 & \text{if } d_k = d_k^h \end{cases}$$

for $\theta_k^i = m$, and

$$v_k^i\left(d_k, h\right) = \begin{cases} 0 & \text{if } d_k = d_k^0 \\ -\varepsilon\left(k+1\right) & \text{if } d_k = d_k^l \\ -\frac{k(k+1)^2}{2} & \text{if } d_k = d_k^m \\ 3\left(k+1\right)^3 - \frac{k(k+1)^2}{4} & \text{if } d_k = d_k^h \end{cases}$$

for $\theta_k^i = h$, where $\varepsilon > 0$. Further restrictions on ε will be derived below.

1.2 Reservation Payoffs

Since $C_k\left(d_k^0\right) = 0$ we take d_k^0 as the status quo outcome. Hence, $r_k^i\left(\theta_k^i\right) = v_k^i\left(d_k^0, \theta_k^i\right) = 0$ for all θ_k^i.

2 The Surplus Maximizing Rule

The primitives in the example have been constructed to make sure that: i) d_k^l is optimal given type profiles lm and ml (which generates surplus of order k), and; ii) d_k^m is optimal given type profile mm (surplus of order k^3), and; iii) that d_k^h is optimal for profiles lh, hl and hh (surplus of order k^3, and, which is of some importance, that the surplus from hh is substantially larger than that from mm). As will be seen below, this will make it possible for us to construct sequences where there is a significant effect on interim expected payoffs when types that are realized with an arbitrarily small probability opt out from the mechanism. Below, we provide the details of the derivation of the maximized surplus function.

2.1 Type Profile ll

It is immediate that $v_k^1\left(d_k^0, l\right) + v_k^2\left(d_k^0, l\right) - C_k\left(d_k^0\right) = 0$. If instead $d_k = d_k^l$ we have that

$$v_k^1\left(d_k^l, l\right) + v_k^2\left(d_k^l, l\right) - C_k\left(d_k^l\right) = 2\left[\frac{1}{2}\frac{(k+1)^2}{k}\right] - \frac{(k+1)^2}{k} = 0,$$

whereas, if $d_k = d_k^m$,

$$v_k^1\left(d_k^m, l\right) + v_k^2\left(d_k^m, l\right) - C_k\left(d_k^m\right) = -\frac{k(k+1)^2}{2} < 0$$

and, finally, if $d_k = d_k^h$

$$v_k^1\left(d_k^h, l\right) + v_k^2\left(d_k^h, l\right) - C_k\left(d_k^h\right) = 2\frac{(k+1)^2}{4}\left[k - \frac{1}{k}\right] - 2(k+1)^3 = (k+1)^2\left[\frac{k}{2} - \frac{1}{2k} - k - 2\right] < 0$$

We conclude that an efficient decision is $x_k^*(ll) = d_k^l$ (or d_k^0), which generates a maximized surplus of $s_k(ll) = 0$.

2.2 Type Profiles lm and ml

Again it follows trivially that $v_k^1\left(d_k^0, l\right) + v_k^2\left(d_k^0, m\right) - C_k\left(d_k^0\right) = 0$, whereas the surplus generated by $d_k = d_k^l$ is

$$v_k^1\left(d_k^l, l\right) + v_k^2\left(d_k^l, m\right) - C_k\left(d_k^l\right) = \frac{1}{2}\frac{(k+1)^2}{k} + \frac{3}{4}\frac{(k+1)^2}{k} + \varepsilon(k+1) - \frac{(k+1)^2}{k} = \frac{1}{4}\frac{(k+1)^2}{k} + \varepsilon(k+1).$$

If $d_k = d_k^m$ is chosen

$$\begin{aligned} v_k^1(d_k^m, l) + v_k^2(d_k^m, m) - C_k(d_k^m) &= 0 + \frac{k(k+1)^2}{2} - 2\varepsilon k(k+1)(k+2) - \frac{k(k+1)^2}{2} \\ &= -2\varepsilon k(k+1)(k+2) < 0, \end{aligned}$$

and, for $d_k = d_k^h$

$$\begin{aligned} v_k^1\left(d_k^h, l\right) + v_k^2\left(d_k^h, m\right) - C_k\left(d_k^h\right) &= \frac{(k+1)^2}{4}\left[k - \frac{1}{k}\right] + \frac{k(k+1)^2}{4} - 3(k+1)^3 - 2(k+1)^3 \\ &= (k+1)^2\left[\frac{k}{2} - \frac{1}{4k} - 5k - 5\right] < 0 \end{aligned}$$

Using symmetry, we conclude that the efficient decision is $x_k^*(lm) = x^*(ml) = d_k^l$ and that the associated maximized surplus is

$$s_k(lm) = s_k(ml) = \frac{1}{4}\frac{(k+1)^2}{k} + \varepsilon(k+1)$$

2.3 Type Profile mm

Trivially, $v_k^1\left(d_k^0, m\right) + v_k^2\left(d_k^0, m\right) - C_k\left(d_k^0\right) = 0$. If instead $d_k = d_k^l$ we have that

$$v_k^1\left(d_k^l, m\right) + v_k^2\left(d_k^l, m\right) - C_k\left(d_k^l\right) = 2\left[\frac{3}{4}\frac{(k+1)^2}{k} + \varepsilon(k+1)\right] - \frac{(k+1)^2}{k} = \frac{1}{2}\frac{(k+1)^2}{k} + 2\varepsilon(k+1) > 0$$

and if $d_k = d_k^m$

$$\begin{aligned} v_k^1(d_k^m, m) + v_k^2(d_k^m, m) - C_k(d_k^m) &= 2\left[\frac{k(k+1)^2}{2} - 2\varepsilon k(k+1)(k+2)\right] - \frac{k(k+1)^2}{2} \\ &= \frac{k(k+1)^2}{2} - 4\varepsilon k(k+1)(k+2), \end{aligned}$$

while if $d_k = d_k^h$ we have that

$$v_k^1\left(d_k^h, m\right) + v_k^2\left(d_k^h, m\right) - C_k\left(d_k^h\right) = 2\left[\frac{k(k+1)^2}{4} - 3(k+1)^3\right] - 2(k+1)^3 = (k+1)^2\left[\frac{k}{2} - 8k - 8\right] < 0$$

We conclude that d_k^l and d_k^m are the only remaining candidates that could maximize the social surplus. Define

$$\Delta(k) = \frac{k(k+1)^2}{2} - 4\varepsilon k(k+1)(k+2) - \frac{1}{2}\frac{(k+1)^2}{k} - 2\varepsilon(k+1).$$

If $\Delta(k)$ is strictly positive d_k^m is the surplus maximizing decision, whereas if $\Delta(k)$ is strictly negative, then d_k^l is the unique maximizer.

Claim 1 *Suppose that $\varepsilon \leq \frac{1}{24}$, then $\Delta(\cdot)$ is strictly increasing in k on the interval $[1, \infty)$*

Proof. Rearrange to get

$$\begin{aligned}\Delta(k) &= \frac{k(k+1)^2}{2} - 4\varepsilon k(k+1)(k+2) - \frac{1}{2}\frac{(k+1)^2}{k} - 2\varepsilon(k+1) \\ &= \frac{1}{2}(k+1)^2\left(k-\frac{1}{k}\right) - 4\varepsilon k(k+1)(k+2) - 2\varepsilon(k+1).\end{aligned}$$

Differentiation yields,

$$\Delta'(k) = (k+1)\left(k-\frac{1}{k}\right) + \frac{1}{2}(k+1)^2\left(1+\frac{1}{k^2}\right) - 4\varepsilon\left[(k+1)(k+2)+k(k+2)+k(k+1)\right] - 2\varepsilon.$$

Simplify the bracketed expression to get

$$\begin{aligned}&(k+1)(k+2)+k(k+2)+k(k+1) = \underbrace{(k+1)^2+(k+1)}_{=(k+1)(k+2)} + \underbrace{k(k+1)+k}_{=k(k+2)} + k(k+1) \\ =\; &(k+1)^2+(k+1)+k(k+1)+(k+1)-1+k(k+1) = (k+1)^2+2(k+1)+2k(k+1)-1 \\ =\; &3(k+1)^2-1.\end{aligned}$$

Hence

$$\begin{aligned}\Delta'(k) &= (k+1)\left(k-\frac{1}{k}\right) + \frac{1}{2}(k+1)^2\left(1+\frac{1}{k^2}\right) - 4\varepsilon\left[3(k+1)^2-1\right] - 2\varepsilon \\ &= (k+1)\left(k-\frac{1}{k}\right) + \frac{1}{2}(k+1)^2\left(1+\frac{1}{k^2}-24\varepsilon\right) + 2\varepsilon.\end{aligned}$$

The claim follows since all terms are strictly positive given that $k \geq 1$ and $\varepsilon \leq \frac{1}{24}$. ■

Evaluating we have that

$$\Delta(1) = 2 - 24\varepsilon - 2 - 4\varepsilon = -28\varepsilon < 0$$

for any $\varepsilon > 0$. We conclude that (somewhat unfortunately since it is an additional complication for the example) d_1^l is the surplus maximizing decision for $k = 1$. However

$$\Delta(2) = 9 - 96\varepsilon - \frac{9}{4} - 6\varepsilon$$

Clearly, for ε small enough (the exact bound is $\varepsilon < \frac{27}{408}$, where it may be noted that $\frac{1}{24} = \frac{27}{648} < \frac{27}{408}$) we have that $\Delta(2)$ is strictly positive. Combining with the fact that $\Delta(k)$ is monotonically increasing in k under the condition that $\varepsilon \leq \frac{1}{24}$ we conclude;

Claim 2 *Suppose that $\varepsilon \leq \frac{1}{24}$. Then $x_1^*(mm) = d_1^l$ and $x_k^*(mm) = d_k^m$ for every $k \geq 2$. The associated social surplus is*

$$\begin{aligned}s_1(mm) &= \frac{1}{2}\frac{(k+1)^2}{k} + 2\varepsilon(k+1) \\ s_k(mm) &= \frac{k(k+1)^2}{2} - 4\varepsilon k(k+1)(k+2) \;\; \textit{for } k \geq 2\end{aligned}$$

2.4 Type Profiles lh and hl

As for any other type profile $v_k^1\left(d_k^0, l\right) + v_k^2\left(d_k^0, h\right) - C_k\left(d_k^0\right) = 0$. For the non-trivial alternatives we have that

$$v_k^1\left(d_k^l, l\right) + v_k^2\left(d_k^l, h\right) - C_k\left(d_k^l\right) = \frac{1}{2}\frac{(k+1)^2}{k} - \varepsilon(k+1) - \frac{(k+1)^2}{k} = -\frac{1}{2}\frac{(k+1)^2}{k} - \varepsilon(k+1) < 0,$$

and

$$v_k^1\left(d_k^m, l\right) + v_k^2\left(d_k^m, h\right) - C_k\left(d_k^m\right) = 0 - \frac{k(k+1)^2}{2} - \frac{k(k+1)^2}{2} = -k(k+1)^2 < 0,$$

and

$$\begin{aligned} v_k^1\left(d_k^h, l\right) + v_k^2\left(d_k^h, h\right) - C_k\left(d_k^m\right) &= \frac{(k+1)^2}{4}\left[k - \frac{1}{k}\right] + 3(k+1)^3 - \frac{k(k+1)^2}{4} - 2(k+1)^3 \\ &= \frac{(k+1)^2}{4k} + (k+1)^3 = (k+1)^2\left[\frac{4k(k+1)-1}{4k}\right] \end{aligned}$$

Hence, again using symmetry, $x_k^*(lh) = x_k^*(hl) = d_k^h$ and the resulting maximized surplus is

$$s_k(lh) = s_k(hl) = (k+1)^2\left[\frac{4k(k+1)-1}{4k}\right]$$

2.5 Type Profiles mh and hm

Obviously, $v_k^1\left(d_k^0, m\right) + v_k^2\left(d_k^0, h\right) - C_k\left(d_k^0\right) = 0$. If instead $d_k = d_k^l$

$$v_k^1\left(d_k^l, m\right) + v_k^2\left(d_k^l, h\right) - C_k\left(d_k^l\right) = \frac{3}{4}\frac{(k+1)^2}{k} + \varepsilon(k+1) - \varepsilon(k+1) - \frac{(k+1)^2}{k} = -\frac{1}{4}\frac{(k+1)^2}{k} < 0,$$

and if $d_k = d_k^m$,

$$\begin{aligned} v_k^1\left(d_k^m, m\right) + v_k^2\left(d_k^m, h\right) - C_k\left(d_k^m\right) &= \frac{k(k+1)^2}{2} - 2\varepsilon k(k+1)(k+2) - \frac{k(k+1)^2}{2} - \frac{k(k+1)^2}{2} \\ &= -2\varepsilon k(k+1)(k+2) - \frac{k(k+1)^2}{2} < 0, \end{aligned}$$

whereas if $d_k = d_k^h$, we get that

$$\begin{aligned} v_k^1\left(d_k^h, m\right) + v_k^2\left(d_k^h, h\right) - C_k\left(d_k^h\right) &= \frac{k(k+1)^2}{4} - 3(k+1)^3 + 3(k+1)^3 - \frac{k(k+1)^2}{4} - 2(k+1)^3 \\ &= -2(k+1)^2 < 0 \end{aligned}$$

Hence, $x_k^*(mh) = x_k^*(hm) = d_k^0$, and the maximized surplus is $s_k(mh) = s_k(hm) = 0$

2.6 Type Profile hh

Again, $v_k^1\left(d_k^0, h\right) + v_k^2\left(d_k^0, h\right) - C_k\left(d_k^0\right) = 0$, and

$$v_k^1\left(d_k^l, h\right) + v_k^2\left(d_k^l, h\right) - C_k\left(d_k^l\right) = -2\varepsilon(k+1) - \frac{(k+1)^2}{k} < 0$$

and

$$v_k^1\left(d_k^m,h\right)+v_k^2\left(d_k^m,h\right)-C_k\left(d_k^m\right)=-2\left[\frac{k\left(k+1\right)^2}{2}\right]-\frac{k\left(k+1\right)^2}{2}=-\frac{3k\left(k+1\right)^2}{2}<0$$

and

$$v_k^1\left(d_k^h,h\right)+v_k^2\left(d_k^h,h\right)-C_k\left(d_k^h\right)=2\left[3\left(k+1\right)^3-\frac{k\left(k+1\right)^2}{4}\right]-2\left(k+1\right)^3=4\left(k+1\right)^3-\frac{k\left(k+1\right)^2}{2}>0$$

Hence, $x_k^*\left(hh\right)=d_k^h$ and

$$s_k\left(hh\right)=4\left(k+1\right)^3-\frac{\left(k+1\right)^2}{2k}=\left[4\left(k+1\right)-\frac{k}{2}\right]\left(k+1\right)^2$$

2.7 Summary: An Optimal Decision Rule and the Maximized Surplus

Combining all the cases above and ignoring $k=1$ we have that an optimal social decision rule is[1 2]

$$x_k^*\left(\theta_k\right)=\begin{cases} d_k^0 & \text{if } \theta_k\in\{ll,mh,hm\} \\ d_k^l & \text{if } \theta_k\in\{lm,ml\} \\ d_k^m & \text{if } \theta_k=mm \\ d_k^h & \text{if } \theta_k\in\{lh,hl,hh\} \end{cases}, \tag{1}$$

and the maximized surplus (given $k\geq 2$) is

$$s_k\left(\theta_k\right)=\begin{cases} 0 & \text{if } \theta_k\in\{ll,mh,hm\} \\ \frac{1}{4}\frac{(k+1)^2}{k}+\varepsilon\left(k+1\right) & \theta_k\in\{lm,ml\} \\ \frac{k(k+1)^2}{2}-4\varepsilon k\left(k+1\right)\left(k+2\right) & \text{if } \theta_k=mm \\ \left(k+1\right)^2\left[\frac{4k(k+1)-1}{4k}\right] & \text{if } \theta_k\in\{lh,hl\} \\ \left[4\left(k+1\right)-\frac{k}{2}\right]\left(k+1\right)^2 & \text{if } \theta_k=hh \end{cases}. \tag{2}$$

2.8 Interim Expected Payoffs from Participation in the Groves Mechanism

For $k=2,3...$ we assume that the probability distribution over Θ_k^i be given by

$$\left(\Pr\left[\theta^i=l\right],\Pr\left[\theta^i=m\right],\Pr\left[\theta^i=h\right]\right)=\left(\frac{k\left(k+2\right)}{\left(k+1\right)^2},\frac{1}{2\left(k+1\right)^2},\frac{1}{2\left(k+1\right)^2}\right). \tag{3}$$

[1] There is multiplicity for type profile ll. We could easily get rid of this without affecting the maximized social surplus by adjusting the costs and preferences slightly. This would add some extra terms for the calculations, and, ultimately, only the maximized surplus is relevant, so we have opted to go with the simpler primitives.

[2] Hence, we will start our sequence at $k=2$. Obviously, we could replace every k with $k+1$ and start the sequence at $k=1$, but the formulas get somewhat less transparent, which is why we stick with the current formulation.

With probability distribution (3), the interim expected value of the maximized issue k surplus is

$$
\begin{aligned}
\mathrm{E}\left[s_k\left(\theta_k\right)|l\right] &= \Pr\left[\theta^i = l\right] s_k\left(ll\right) + \Pr\left[\theta^i = m\right] s_k\left(lm\right) + \Pr\left[\theta^i = h\right] s_k\left(lh\right) \\
&= \frac{1}{2\left(k+1\right)^2}\left[\frac{1}{4}\frac{\left(k+1\right)^2}{k} + \varepsilon\left(k+1\right)\right] + \frac{1}{2\left(k+1\right)^2}\left[\left(k+1\right)^2\left[\frac{4k\left(k+1\right)-1}{4k}\right]\right] \\
&= \frac{1}{2\left(k+1\right)^2}\left[\frac{1}{4}\frac{\left(k+1\right)^2}{k} + \varepsilon\left(k+1\right) + \left(k+1\right)^2\left[\frac{4k\left(k+1\right)-1}{4k}\right]\right] \\
&= \frac{1}{2}\left[\frac{1}{4k} + \frac{\varepsilon}{\left(k+1\right)} + \frac{4k\left(k+1\right)-1}{4k}\right] \\
&= \frac{1}{2}\left[\frac{1}{4k} + \frac{\varepsilon}{\left(k+1\right)} + \left(k+1\right) - \frac{1}{4k}\right] = \frac{1}{2}\left[\frac{\varepsilon}{\left(k+1\right)} + \left(k+1\right)\right] = \frac{k+1}{2} + \frac{\varepsilon}{2\left(k+1\right)}
\end{aligned} \tag{4}
$$

for $\theta_i = l$. For $\theta_i = m$ we have that

$$
\begin{aligned}
\mathrm{E}\left[s_k\left(\theta_k\right)|m\right] &= \Pr\left[\theta^i = l\right] s_k\left(ml\right) + \Pr\left[\theta^i = m\right] s_k\left(mm\right) + \Pr\left[\theta^i = h\right] s_k\left(mh\right) \\
&= \frac{k\left(k+2\right)}{\left(k+1\right)^2}\left[\frac{1}{4}\frac{\left(k+1\right)^2}{k} + \varepsilon\left(k+1\right)\right] + \frac{1}{2\left(k+1\right)^2}\left[\frac{k\left(k+1\right)^2}{2} - 4\varepsilon k\left(k+1\right)\left(k+2\right)\right] \\
&= \frac{k+2}{4} + \frac{\varepsilon k\left(k+2\right)}{\left(k+1\right)} + \frac{k}{4} - \frac{2\varepsilon k\left(k+2\right)}{k+1} = \frac{k+1}{2} - \frac{\varepsilon k\left(k+2\right)}{\left(k+1\right)} \\
&= \frac{k+1}{2} - \frac{\varepsilon\left[k\left(k+1\right)+k\right]}{\left(k+1\right)} = \frac{k+1}{2} - \varepsilon k - \frac{\varepsilon k}{k+1}
\end{aligned} \tag{5}
$$

Finally, for $\theta_i = h$, we have that

$$
\begin{aligned}
\mathrm{E}\left[s_k\left(\theta_k\right)|h\right] &= \Pr\left[\theta^i = l\right] s_k\left(hl\right) + \Pr\left[\theta^i = m\right] s_k\left(hm\right) + \Pr\left[\theta^i = h\right] s_k\left(hh\right) \\
&= \frac{k\left(k+2\right)}{\left(k+1\right)^2}\left[\left(k+1\right)^2\left[\frac{4k\left(k+1\right)-1}{4k}\right]\right] + \frac{1}{2\left(k+1\right)^2}\left[4\left(k+1\right) - \frac{k}{2}\right]\left(k+1\right)^2 \\
&= k\left(k+2\right)\left[\frac{4k\left(k+1\right)-1}{4k}\right] + \left[2\left(k+1\right) - \frac{k}{4}\right] \\
&= k\left(k+1\right)\left(k+2\right) + 2\left(k+1\right) - \frac{k+2}{4} - \frac{k}{4} = k\left(k+1\right)\left(k+2\right) + 2\left(k+1\right) - \frac{k+1}{2} \\
&= \left(k+1\right)\left[k^2+2k+1+1\right] - \frac{k+1}{2} = \left(k+1\right)^3 + \left(k+1\right) - \frac{k+1}{2} = \left(k+1\right)^3 + \frac{\left(k+1\right)}{2}
\end{aligned} \tag{6}
$$

The ex ante expected surplus is thus

$$
\begin{aligned}
\mathrm{E}\left[s_k\left(\theta_k\right)\right] &= \frac{k\left(k+2\right)}{\left(k+1\right)^2}\left[\frac{k+1}{2} + \frac{\varepsilon}{2\left(k+1\right)}\right] + \frac{1}{2\left(k+1\right)^2}\left[\frac{k+1}{2} - \frac{\varepsilon k\left(k+2\right)}{\left(k+1\right)}\right] \\
&\quad + \frac{1}{2\left(k+1\right)^2}\left[\left(k+1\right)^3 + \frac{\left(k+1\right)}{2}\right] \\
&= \frac{k\left(k+2\right)}{2\left(k+1\right)} + \varepsilon\left[\frac{k\left(k+2\right)}{2\left(k+1\right)^3}\right] + \frac{1}{4\left(k+1\right)} - \varepsilon\left[\frac{\varepsilon k\left(k+2\right)}{2\left(k+1\right)^3}\right] + \frac{k+1}{2} + \frac{1}{4\left(k+1\right)} \\
&= \frac{k\left(k+2\right)}{2\left(k+1\right)} + \frac{1}{2\left(k+1\right)} + \frac{k+1}{2} = \frac{k^2+2k+1}{2\left(k+1\right)} + \frac{k+1}{2} = \frac{\left(k+1\right)^2}{2\left(k+1\right)} + \frac{k+1}{2} = k+1
\end{aligned} \tag{7}
$$

2.9 Summary of the Expected Surplus Calculations

We have shown that

$$
\begin{aligned}
\mathrm{E}\left[\, s_k\left(\theta_k\right)\right| l] &= \frac{k+1}{2}+\frac{\varepsilon}{2\left(k+1\right)} && (8)\\
\mathrm{E}\left[\, s_k\left(\theta_k\right)\right| m] &= \frac{k+1}{2}-\varepsilon k-\frac{\varepsilon k}{\left(k+1\right)} && (9)\\
\mathrm{E}\left[\, s_k\left(\theta_k\right)\right| h] &= \left(k+1\right)^3+\frac{\left(k+1\right)}{2} && (10)\\
\mathrm{E}\left[s_k\left(\theta_k\right)\right] &= k+1 && (11)
\end{aligned}
$$

3 The Probability that a Participation Constraint is Violated Converges to zero as $K\to\infty$.

Since $R_K^i\left(\theta^i\right)=0$ for all θ^i, the interim expected payoff from participation in the Groves mechanism under consideration simplifies to

$$
U_K^i\left(\theta^i\right)=\mathrm{E}_{-i}\left[S_K\left(\theta\right)\right]-\frac{1}{2}\mathrm{E}\left[S_K\left(\theta\right)\right]=\sum_{k=2}^{K}\mathrm{E}_{-i}\left[s_k\left(\theta_k\right)\right]-\frac{1}{2}\sum_{k=2}^{K}\mathrm{E}\left[s_k\left(\theta_k\right)\right] \tag{12}
$$

Consider a type on the form $(m,....,m,l,....,l)$. Specifically, assume that $\theta_k^i=m$ for $k=2,...,K^*$ and $\theta_k^i=l$ for $k=K^*+1,...,K$ and denote this type by $\left(\mathbf{m}_{K^*},\mathbf{l}_{K^*-K}\right)$. Substituting (8), (9) and (11) into (12) we have that type $\left(\mathbf{m}_{K^*},\mathbf{l}_{K^*-K}\right)$ earns an interim expected payoff of

$$
\begin{aligned}
U_K^i\left(\mathbf{m}_{K^*},\mathbf{l}_{K^*-K}\right) &= \sum_{k=2}^{K^*}\mathrm{E}\left[\, s_k\left(\theta_k\right)\right| m]+\sum_{k=K^*+1}^{K}\mathrm{E}\left[\, s_k\left(\theta_k\right)\right| l]-\frac{1}{2}\sum_{k=2}^{K}\mathrm{E}\left[s_k\left(\theta_k\right)\right] && (13)\\
&= \sum_{k=2}^{K^*}\left[\frac{k+1}{2}-\varepsilon k-\frac{\varepsilon k}{\left(k+1\right)}\right]+\sum_{k=K^*+1}^{K}\frac{k+1}{2}+\frac{\varepsilon}{2\left(k+1\right)}-\frac{1}{2}\sum_{k=2}^{K}\left(k+1\right)\\
&= \varepsilon\left[\sum_{k=2}^{K^*}\left[-k-\frac{k}{k+1}\right]+\frac{1}{2}\sum_{k=K^*+1}^{K}\left[\frac{1}{k+1}\right]\right]
\end{aligned}
$$

Define

$$
H\left(K^*,K\right)=\sum_{k=2}^{K^*}\left[-k-\frac{k}{k+1}\right]+\frac{1}{2}\sum_{k=K^*+1}^{K}\frac{1}{k+1} \tag{14}
$$

We note that;

1. $H\left(K^*,K\right)$ is strictly decreasing in K^*
2. $H\left(1,K\right)=\frac{\varepsilon}{2}\sum_{k=2}^{K}\frac{1}{(k+1)}>0$
3. $H\left(K,K\right)=-\varepsilon\sum_{k=2}^{K}\left[k+\frac{k}{(k+1)}\right]<0$

These three properties imply that for every $K \geq 2$ there exists a unique integer $K^*(K) \in \{1, ..., K\}$ such that

$$
\begin{aligned}
H(K^*(K), K) &> 0 \\
H(K^*(K)+1, K) &< 0
\end{aligned}
\tag{15}
$$

Moreover, $K^*(K)$ is monotonically increasing and goes (slowly) to infinity as K goes to infinity. To see this, we first observe that, for K^* fixed, the positive term in (14) is divergent. That is,

$$
\begin{aligned}
\sum_{k=K^*+1}^{K} \frac{1}{k+1} &= \sum_{k=K^*+1}^{K} \left[\int_{k+1}^{k+2} \frac{1}{k+1} dz\right] > \sum_{k=K^*+1}^{K} \left[\int_{k+1}^{k+2} \frac{1}{z} dz\right] \\
&= \int_{K^*+2}^{K+2} \frac{1}{z} dz = \ln(K+2) - \ln(K^*+2).
\end{aligned}
\tag{16}
$$

For contradiction, assume that there exists some $\overline{K}$ such that $K^*(K) < \overline{K} - 1$ for all K. Then, for any K we have that

$$
\begin{aligned}
H(K^*(K)+1, K) &> H(\overline{K}, K) = -\sum_{k=2}^{\overline{K}} \left[k + \frac{k}{k+1}\right] + \frac{1}{2} \sum_{k=\overline{K}+1}^{K} \left[\frac{1}{k+1}\right] \\
/ \text{ using } (16)/ \quad &> \sum_{k=2}^{\overline{K}} \left[-k - \frac{k}{k+1}\right] + \frac{\ln(K+2) - \ln(\overline{K}+2)}{2}.
\end{aligned}
\tag{17}
$$

Since $\ln(K+2) \to \infty$ as $K \to \infty$ and the other two terms are finite we conclude that $H(K^*(K)+1, K) > 0$ for K sufficiently large, which contradicts the definition of $K^*(K)$.

Next, consider a type on form $\left(\theta_2^i, ..., \theta_{K^*(K)}^i, l, ..., l\right)$, where the signals for problems $2, ..., K^*(K)$ are arbitrary, and $\theta_k^i = l$ for $k = K^*(K)+1, .., K$. We denote such a type $\left(\boldsymbol{\theta}_{K^*(K)}^i, \mathbf{l}_{K^*-K(K)}\right)$. Since $\mathrm{E}[s_k(\theta_k)| m] \leq \mathrm{E}\left[s_k(\theta_k)| \theta_k^i\right]$ for all $\theta_k^i \in \{l, m, h\}$ and every k it follows that

$$
\begin{aligned}
U_K^i\left(\boldsymbol{\theta}_{K^*(K)}^i, \mathbf{l}_{K^*-K(K)}\right) &= \sum_{k=2}^{K^*(K)} \mathrm{E}\left[s_k(\theta_k)| \theta_k^i\right] - \sum_{k=K(K)^*+1}^{K} \mathrm{E}[s_k(\theta_k)| l] - \frac{1}{2}\sum_{k=2}^{K} \mathrm{E}[s_k(\theta_k)] \\
&\geq \sum_{k=2}^{K^*(K)} \mathrm{E}[s_k(\theta_k)| m] - \sum_{k=K(K)^*+1}^{K} \mathrm{E}[s_k(\theta_k)| l] - \frac{1}{2}\sum_{k=2}^{K} \mathrm{E}[s_k(\theta_k)] \\
&= \varepsilon H(K^*(K), K) > 0.
\end{aligned}
\tag{18}
$$

We conclude that the participation constraint holds for any such type. Since,

$$
\Pr\left[\left(\theta_{K^*(K)+1}^i, ..., \theta_K^i\right) = (l, ..., l)\right] = \prod_{k=K^*(K)+1}^{K} \frac{k(k+2)}{(k+1)^2} = \frac{(K^*(K)+1)(K+2)}{(K^*(K)+2)(K+1)} \to 1 \tag{19}
$$

as $K \to \infty$ it follows that the probability that θ^i is on the form $\left(\boldsymbol{\theta}_{K^*(K)}^i, \mathbf{l}_{K^*-K(K)}\right)$ tends to unity as K goes out of bounds. We conclude;

Claim 3 *The probability that all participation constraints hold converges to 1 as K tends to infinity.*

Hence, the Groves mechanism is almost incentive feasible in this example.

4 Unraveling when the Veto Game is Introduced

Note that for type $\mathbf{l} = (l, ..., l)$

$$U_K^i(\mathbf{l}) = \sum_{k=2}^{K} \mathrm{E}\left[s_k(\theta_k) | l\right] - \frac{1}{2}\sum_{k=2}^{K} \mathrm{E}\left[s_k(\theta_k)\right] = \sum_{k=2}^{K}\left[\frac{k+1}{2} + \frac{\varepsilon}{2(k+1)} - \frac{k+1}{2}\right] = \frac{\varepsilon}{2}\sum_{k=2}^{K}\frac{1}{(k+1)} \tag{20}$$

where

$$\sum_{k=2}^{K}\frac{1}{k+1} = \sum_{k=2}^{K}\left[\int_k^{k+1}\frac{1}{k+1}dz\right] < \sum_{k=2}^{K}\left[\int_k^{k+1}\frac{1}{z}dz\right] = \int_2^{K+1}\frac{1}{z}dz = \ln\left(\frac{K+1}{2}\right). \tag{21}$$

Hence,

$$U_K^i(\mathbf{l}) < \frac{\varepsilon}{2}\ln\left(\frac{K+1}{2}\right). \tag{22}$$

However, consider type θ^i where all coordinates are l except for $\theta_{k^*}^i = m$. Denote this type $\theta^i = \mathbf{l}|\theta_{k^*}^i = m$ and note that

$$\begin{aligned}
U\left(\mathbf{l}|\theta_{k^*}^i = m\right) &= \sum_{k\neq k^*}^{K} \mathrm{E}\left[s_k(\theta_k)|l\right] + \mathrm{E}\left[s_{k^*}(\theta_{k^*})|m\right] - \frac{1}{2}\sum_{k=2}^{K}\mathrm{E}\left[s_k(\theta_k)\right] \qquad (23)\\
&= \underbrace{\sum_{k=2}^{K} \mathrm{E}\left[s_k(\theta_k)|l\right] - \frac{1}{2}\sum_{k=2}^{K}\mathrm{E}\left[s_k(\theta_k)\right]}_{=U^i(\mathbf{l})} + \mathrm{E}\left[s_{k^*}(\theta_{k^*})|m\right] - \mathrm{E}\left[s_{k^*}(\theta_{k^*})|l\right]\\
\text{/using (22)/} &\leq \frac{\varepsilon}{2}\ln\left(\frac{K+1}{2}\right) + \frac{k^*+1}{2} - \varepsilon k^* - \frac{\varepsilon k^*}{(k^*+1)} - \left[\frac{k^*+1}{2} + \frac{\varepsilon}{2(k^*+1)}\right]\\
&= \varepsilon\left[\frac{1}{2}\ln\left(\frac{K+1}{2}\right) - k^* - \frac{k^*}{(k^*+1)} - \frac{1}{2(k^*+1)}\right] < \varepsilon\left[\frac{1}{2}\ln\left(\frac{K+1}{2}\right) - k^*\right]
\end{aligned}$$

Define $\widetilde{k}(K)$ as the integer part of $\frac{1}{2}\ln\left(\frac{K+1}{2}\right)+1$. Since $\frac{1}{2}\ln\left(\frac{K+1}{2}\right) - k^*$ is negative for $k^* \geq \widetilde{k}(K)$ this implies that type $\mathbf{l}|\theta_{k^*}^i = m$ is worse off from participating in the Groves mechanism than from the status quo outcome $d^0 = \left(d_2^0, ..., d_K^K\right)$ for every $k^* \geq \widetilde{k}(K)$. For brevity, denote this subset of types with a strict incentive to veto the Groves mechanism by Θ_V^i. That is

$$\Theta_V^i = \left\{\theta^i | \theta_{k^*}^i = m \text{ for some } k^* \geq k^*(K) \text{ and } \theta_k^i = l \text{ for all } k \neq k^*\right\} \tag{24}$$

We note that, for any $k^* \geq k^*(K)$

$$\begin{aligned}
\Pr\left[\theta^i \in \Theta_V^i | \theta_{k^*}^i = m\right] &= \prod_{k\neq k^*}\Pr\left[\theta_k^i = l\right] > \Pr\left[\theta^i = \mathbf{l}\right] = \prod_{k=2}^{K}\Pr\left[\theta_k^i = l\right] = \prod_{k=2}^{K}\frac{k(k+2)}{(k+1)^2} \qquad (25)\\
&= \left(\frac{2\times 4}{3^2}\right)\times\left(\frac{3\times 5}{4^2}\right)\times ... \times\frac{(K-1)(K+1)}{K^2}\times\frac{K(K+2)}{(K+1)^2}\\
&= \frac{2}{3}\left[\frac{K+2}{K+1}\right] > \frac{2}{3}
\end{aligned}$$

In words, conditional on $\theta_{k^*}^i = m$, the probability that all other coordinates are ls is obviously (slightly) larger than the unconditional probability that the type $\mathbf{l}$ is realized. For $k \geq \widetilde{k}(K)$ we can therefore calculate an upper bound on the expected surplus conditional on $\theta_k^i = l$ and conditional on that agent i is aware that any agent $\theta^i \in \Theta_V^i$ will veto the mechanism.

$$\begin{aligned}
\mathrm{E}\left[s_k(\theta_k)\,|l, \text{veto by } \theta^i \in \Theta_V^i\right] &= \Pr\left[\theta_k^i = m\right]\left[1 - \Pr\left[\theta^i \in \Theta_V^i | \theta_k^i = m\right]\right] s_k(lm) + \Pr\left[\theta_k^i = h\right] s_k(lh) \qquad (26) \\
&= \mathrm{E}\left[s_k(\theta_k)\,|l\right] - \Pr\left[\theta_k^i = m\right] \Pr\left[\theta^i \in \Theta_V^i | \theta_k^i = m\right] s_k(lm) \\
&= \underbrace{\frac{k+1}{2} + \frac{\varepsilon}{2(k+1)}}_{=\mathrm{E}[s_k(\theta_k)|l] \text{ by } (8)} - \underbrace{\frac{1}{2(k+1)^2}}_{=\Pr\left[\theta_k^i = m\right]} \Pr\left[\theta^i \in \Theta_V^i | \theta_k^i = m\right] \underbrace{\left[\frac{1}{4}\frac{(k+1)^2}{k} + \varepsilon(k+1)\right]}_{=s_k(lm) \text{ by } (2)} \\
/(25)/ \quad &< \frac{k+1}{2} + \frac{\varepsilon}{2(k+1)} - \frac{1}{2(k+1)^2}\frac{2}{3}\left[\frac{1}{4}\frac{(k+1)^2}{k} + \varepsilon(k+1)\right] \\
&= \frac{k+1}{2} + \frac{\varepsilon}{2(k+1)} - \frac{1}{12k} - \frac{\varepsilon}{3(k+1)} = \frac{k+1}{2} + \frac{\varepsilon}{6(k+1)} - \frac{1}{12k}
\end{aligned}$$

The interim expected payoff of type $\mathbf{l} = (l, ..., l)$ conditional on vetoes from types in Θ_V^i is thus

$$\begin{aligned}
U_K^i\left(\mathbf{l}|\text{veto by } \theta^i \in \Theta_V^i\right) &= \sum_{k=2}^{\mathrm{I}\widetilde{k}(K)-1} \mathrm{E}\left[s_k(\theta_k)|\,l\right] + \sum_{k=\widetilde{k}(K)}^{K} \mathrm{E}\left[s_k(\theta_k)\,|l, \text{veto by } \theta^i \in \Theta_V^i\right] - \frac{1}{2}\sum_{k=2}^{K} \mathrm{E}\left[s_k(\theta_k)\right] \qquad (27) \\
&= \frac{\varepsilon}{2}\sum_{k=2}^{\mathrm{I}\widetilde{k}(K)-1} \frac{1}{k+1} + \frac{\varepsilon}{6}\sum_{k=\widetilde{k}(K)}^{K} \frac{1}{k+1} - \frac{1}{12}\sum_{k=\widetilde{k}(K)}^{K} \frac{1}{k}
\end{aligned}$$

where,

$$\begin{aligned}
\sum_{k=2}^{\mathrm{I}\widetilde{k}(K)-1} \frac{1}{k+1} &= \sum_{k=2}^{\mathrm{I}\widetilde{k}(K)-1}\left[\int_k^{k+1} \frac{1}{k+1}dz\right] < \sum_{k=2}^{\mathrm{I}\widetilde{k}(K)-1}\left[\int_k^{k+1} \frac{1}{z}dz\right] = \int_2^{\mathrm{I}\widetilde{k}(K)} \frac{1}{z}dz = \ln\left(\frac{\mathrm{I}\widetilde{k}(K)}{2}\right) \\
\mathrm{I}\sum_{k=\widetilde{k}(K)}^{K} \frac{1}{k+1} &= \sum_{k=\widetilde{k}(K)}^{K}\left[\int_k^{k+1} \frac{1}{k+1}dz\right] < \sum_{k=\widetilde{k}(K)}^{K}\left[\int_k^{k+1} \frac{1}{z}dz\right] = \int_{\widetilde{k}(K)}^{\mathrm{I}K+1} \frac{1}{z}dz = \ln\left(\frac{\mathrm{I}K+1}{\widetilde{k}(K)}\right) \\
\sum_{k=\widetilde{k}(K)}^{K} \frac{1}{k} &= \sum_{k=\widetilde{k}(K)}^{K}\left[\int_k^{k+1} \frac{1}{k}dz\right] > \sum_{k=\widetilde{k}(K)}^{K}\left[\int_k^{k+1} \frac{1}{z}dz\right] = \ln\left(\frac{\mathrm{I}K+1}{\widetilde{k}(K)}\right) \qquad (28)
\end{aligned}$$

Hence,

$$\begin{aligned}
U_K^i\left(\mathbf{l}|\text{veto by } \theta^i \in \Theta_V^i\right) &= \frac{\varepsilon}{2}\sum_{k=2}^{\mathrm{I}\widetilde{k}(K)-1} \frac{1}{k+1} + \frac{\varepsilon}{6}\sum_{k=\widetilde{k}(K)}^{K} \frac{1}{k+1} - \frac{1}{12}\sum_{k=\widetilde{k}(K)}^{K} \frac{1}{k} \qquad (29) \\
&< \frac{\varepsilon}{2}\ln\left(\frac{\mathrm{I}\widetilde{k}(K)}{2}\right) - \frac{1-2\varepsilon}{6}\ln\left(\frac{\mathrm{I}K+1}{\widetilde{k}(K)}\right) \\
&= \ln\widetilde{k}(K)\left[\frac{1+\varepsilon}{6}\right] - \frac{\varepsilon}{2}\ln 2 - \ln(K+1)\left[\frac{1-2\varepsilon}{6}\right] \qquad (30)
\end{aligned}$$

But, $\widetilde{k}(K) \leq \frac{1}{2}\ln\left(\frac{K+1}{2}\right) + 1$, so

$$\ln(K+1) \geq 2\widetilde{k}(K) + \ln 2 - 2$$

Since $\lim_{x\to\infty}\frac{\ln x}{x}=0$ and since $\lim_{K\to\infty}\widetilde{k}(K)=\infty$ it follows that the term with $\ln(K+1)$ eventually dominates in expression (29). Consequently, whenever $\varepsilon<\frac{1}{2}$ there exists $\overline{K}$ such that $U^i\left(\mathbf{l}|\text{veto by }\theta^i\in\Theta_V^i\right)<0$ for any $K\geq\overline{K}$.[3] In (25) we calculated the probability that $\theta^i=\mathbf{l}$ to be

$$\Pr\left[\theta^i=\mathbf{l}\right]=\frac{2}{3}\left[\frac{K+2}{K+1}\right].$$

We conclude that;

Claim 4 *Suppose that type $\mathbf{l}=(l,....l)$ expects that all types in Θ_V^i will veto play of the Groves mechanism. Then, there exists some $\overline{K}$ such that type $\mathbf{l}=(l,....l)$ will have a strict incentive to veto the Groves mechanism given any economy $K\geq\overline{K}$. Hence, the mechanism is vetoed by each agent with a probability of at least $\frac{2}{3}$ for every $K\geq\overline{K}$.*

The only difference between the Groves mechanism amended with a veto game and the mechanism actually considered in the proof of Proposition 2 is that the latter mechanism has a slightly (on a per problem basis) larger lump sum payment than the underlying budget balancing Groves mechanism. It follows that the conclusion immediately extends to the mechanism under consideration.

The reader may note that any type such that $\theta_k^i\in\{l,m\}$ for all k will have an incentive to cast a veto. The probability that there is one k such that $\theta_k^i\in\{m,h\}$ is less than $\frac{1}{3}$, and, conditional on this event occuring, the probability that the first such draw is m is $\frac{1}{2}$. Moreover, conditional on the first draw different from l being m, the probability that the rest of the sequence is all l is $\frac{2}{3}$, so we can immediately add $\frac{1}{3}\frac{1}{2}\frac{2}{3}=\frac{1}{9}$ to the lower bound on the probability for a veto.

Obviously we can do even better by taking into consideration that the larger k is for the first draw of m, the larger is the probability that the remaining sequence contain only ls. However, qualitatiely, the point with the example is that an outcome that is highly inefficient occurs when the possibility of a veto is introduced, and for that it is sufficient to observe that the (high probability) type $\mathbf{l}$ will cast a veto.

[3] We already have the restriction $\varepsilon\leq\frac{1}{24}$ in order for the surplus in (2) to be valid for every $k\geq 2$.