COMMODITY MONEY AND THE VALUATION OF TRADE

By

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Abstract

In a previous essay we modeled the enforcement of contract, and through it the provision of money and markets, as a production function within the society, the scale of which is optimized endogenously by labor allocation away from primary production of goods. Government and a central bank provided fiat money and enforced repayment of loans, giving fiat a predictable value in trade, and also rationalizing the allocation of labor to government service, in return for a fiat salary. Here, for comparison, we consider the same trade problem without government or fiat money, using instead a durable good (gold) as a commodity money between the time it is produced and the time it is removed by manufacture to yield utilitarian services. We compare the monetary value of the two money systems themselves, by introducing a natural money-metric social welfare function. Because labor allocation both to production and potentially to government of the economy is endogenous, the only constraint in the society is its population, so that the natural money-metric is labor. Money systems, whether fiat or commodity, are valued in units of the labor that would produce an equivalent utility gain among competitive equilibria, if it were added to the primary production capacity of the society.

1 Introduction

In a previous essay we formalized one aspect of the observation that the economy is embedded in the polity and its society. The rules of society such as the commercial code, contract law, and the bankruptcy law are all necessities that must be paid for. It costs to create them, but it also costs to enforce them. The rules of the game for the market economy are not immutable; they are part of the creation of the rules of the economic game within the political and societal games. They will be more or less obeyed as a function of many societal, bureaucratic, and political factors. The honesty and efficiency of the bureaucracy, peer pressure, customs and conventions of society all play a role. The supply

*We adopt the convention that in joint work the order of appearance of names on the publication should be selected randomly unless there is a specific stated reason otherwise. We have acted accordingly.

and maintenance of the institutional structure that permits the functioning of the economy is, for the most part, a given to the short-run economic agent. The economic system is open to its political and societal environment, but their variables are the parameters for economic activity. The time scales of these systems are different and generally much longer than those internal to the economy.

In the previous essay, by adding as parameters of the economic system taxation, the salaries of the bureaucracy, the default rules, and the enforcement technology, we obtain an endogenously determined optimal size for the bureaucracy, as a function of their values. In the interplay among the economy, polity, and society over a longer period, the parameters may be modified. In the highly simplified model presented here and previously, these longer-term modifications are ignored. Here the labor costs of producing commodity money as a substitute for the role of the bureaucracy is examined.

1.1 Previous results and some new questions

In the previous essay a model was constructed with fiat money mediating the signals of commitment needed by agents to determine the natural scale of government as a support for anonymous markets. No sensitivity analysis was given of the consequences of our particular choice of money. Although essentially all modern economies utilize fiat, a reasonable question motivated by historical succession of moneys is what relation different stages have to each other. Is gold with a king's stamp a pure commodity money? Is fiat intrinsically a debt contract for gold, or is it an instrument within a system defined by quite independent rules, and do debt contracts merely provide a mechanisms for the orderly transition from a material substrate. Here we show that the use of a commodity money is strategically very different from that of a fiat money. Among other factors, the control options of a government over the economy are fewer than in a fiat economy.

As observed in our first essay, the strategic market game with fiat introduces an explicit institutional cost from the creation of government, whose input is labor and whose output is determined by technologies for enforcing default penalties on the supervising of central-bank loans and their repayment. Thus contracts have a specific operational meaning, and the extra-social cost of their enforcement is institutionally represented. Fiat money emerges as the source of the basic contracts in a society, without itself being a contract for anything.

In this essay, a second strategic market game is presented. This game omits the institutional cost of government by using the durable commodity as money, between the time when it is produced and the time when it is consumed by manufacture to yield utilitarian services. The essential element of pre-commitment that gives fiat its signaling value has a counterpart for commodity money, which is the cost of acquiring it and the demand for its utility, which agents know. Idealized commodities, however, do not require separate and costly institutions to give them this signal, because it is assumed that they possess a consumption or production worth that can be judged by individuals independently at the point of trade. In the act of exchange, however, commodities are a physical "store of value", which imprints any discounting of the future on the spot interest rate for all goods, inducing trade inefficiencies that have labor-equivalent costs.

2 The gold standard for fiat: commodity money

An apparent tendency of societies to progress from commodity moneys toward pure systems of account has sometimes led to a misapprehension that the *meaning* of fiat derives solely from its origin as a debt contract for gold or some other imagined ideal commodity. In this view, fiat that goes "off the gold standard" is a breach of contract (perhaps motivated by its regulatory role) rather than a regulatory device within its own rule system, which remains poorly understood. The process structure of strategic market games shows that understanding fiat requires modeling the actions of government [1], and the model of the first essay draws the functions of government from the labor budget of the society. In emphasizing that fiat makes agents predictable by tying a prior commitment to a default penalty to their bids at market, it shows how fiat can introduce the first contracts, without itself being a debt contract. In this essay we consider commodity money as an alternative rule system for making agents' actions predictable from their preferences and productive capabilities, and compare its costs and capabilities as a trust substitute to those of fiat[1].

2.1 Origins of salvage value

From the first essay we reproduce the form of intertemporal utility of consumption that we assume for all agents in a society:

$$\mathcal{U}_i = \sum_{t=0}^{\infty} (\rho_D t_0)\, \beta^{t/t_0} \mathcal{U}_{i,t}, \tag{1}$$

with per-period utility

$$\mathcal{U}_{i,t} = \Upsilon \left\{ s \log\left(\frac{C_{i,t}\rho_D}{e^0}\right) + \sum_{j=1}^{m} \log\left(\frac{A^j_{i,t}}{a^j}\right) \right\}. \tag{2}$$

As long as the period discount factor $1/\beta \equiv 1 + \rho_D t_0$, we may consistently take $t_0 \to 0$, or small and finite relative to any other timescales in the problem such as the utility discount horizon $1/\rho_D$, in which limit the period of trade does not affect the allocations of consumables. The allocations $A^j_{i,t} t_0$ are the bundles of nondurable goods (j) consumed at the end of each trading period (t) by agent i, and $C_{i,t}$ is i's holding of a depreciating capital stock replaced

[1]Note, as mentioned in the introduction, that fiat is only a trust substitute from the point of view of the markets. In more complete treatments of government it merely displaces trust. The important issue raised in discussions of the gold standard is that displaced trust is limited by the ability of society to control the government, a potential weakness to which some commodity monies are less readily subject.

by the input of gold[2], held at the beginning of period t, which yields utilitarian services through the period. The extensive form of the labor-allocation and buy/sell trading game are given in the first essay, and in App. C, we list all notation associated with the model as implemented with commodity money.

Idealized commodity money escapes the Hahn paradox if it produces real consumption value, as gold does in these models at $s > 0$. It is free of explicit institutional costs if it exploits pre-existing biological or social means for judging its value at the point of trade. A stream-of-service model is a more natural model for the salvage value of gold than point consumption, and the durability $1 - \Delta t_0$ that we have assumed for the capital stock can be chosen to continuously vary the cost of replacement. In any model with discounting, the store of value in commodity money is only as efficient as the agents' concern for the future, and the discount rate is inevitably impressed on the spot rate for trade in all goods.

The extensive form for the game with commodity money defines repeated periods, each with the same structure as the previous essay, except that there are no bureaucrats. Gold can now be carried across periods as money, and is invested in capital stock at the beginning of the period, reducing the amount available for bidding at the trading posts. As in the fiat model, its service does not begin to be realized until the next period. The only endogenized division of labor in the current model is between farmers (producers of nondurable consumables) and prospectors (extractors of gold). Two other differences are also assumed: As gold can be injected into the market by prospectors as bids for consumable goods, we omit the additional trading post for gold, which was required in the fiat model if agents were to have a means of obtaining utilitarian stocks. We also remove any equivalent to a central bank in the form of a lending market for gold, as its full definition requires conditions for default and leads us into hybrid models that re-instate a role for government or an equivalent enforcement institution.

In addition to carry-forward equations for capital stock $C_{i,t}$, we now introduce carry-forward equations for gold[3]. For a farmer indexed i the initial stock of money in the period $\mu_{i,t}$ is the limit for combined investment in capital stock and bids at trading posts. Revenues from sales are carried forward to the next period, so that

$$\mu_{i,t+1} = \mu_{i,t} - \sigma_{i,t} t_0 - \sum_j b^j_{i,t} + \sum_j q^j_{i,t} p^j_t, \tag{3}$$

with the constraint $\mu_{i,t} - \sigma_{i,t} t_0 - \sum_j b^j_{i,t} \geq 0$. $\sigma_{i,t}$ is the rate of conversion of gold to capital stock, dimensionally comparable to its rate of extraction e^0. If i is a prospector the continuity equation includes the rate of primary production

[2] We find it less ephemeral to model the industrial services rendered by gold than its aesthetic services. Such a model naturally also provides an exit pathway for gold, other than wear, theft, or loss of the money supply itself.

[3] These formally exist also in the fiat money example, but are not used there in noncooperative equilibria, so we omitted them from the strategy space in the first essay to reduce notation.

but no longer revenues from trading posts:

$$\mu_{i,t+1} = \mu_{i,t} + \left(e^0 - \sigma_{i,t}\right) t_0 - \sum_j b^j_{i,t}. \tag{4}$$

with the constraint $\mu_{i,t} + \left(e^0 - \sigma_{i,t}\right) t_0 - \sum_j b^j_{i,t} \geq 0$. Farmers acquire gold only through revenue from sales of consumable goods, and everyone acquires capital stock only by individual conversion of gold.

Trading posts and period-end allocations of nondurable consumable goods are the same as in the fiat model, except that gold denominates bids. The per-period utilities are simpler than those for producers embedded in fiat economies, because they lack default constraints. For farmers

$$\mathcal{U}_{i,t} = \Upsilon \left\{ s \log \left(\frac{C_{i,t} \rho_D}{e^0} \right) + \sum_{j=1}^{m} \log \left(\frac{A^j_{i,t}}{a^j} \right) \right\} + \lambda_{i,t} \left(\mu_{i,t} - \sigma_{i,t} t_0 - \sum_j b^j_{i,t} \right), \tag{5}$$

and for prospectors

$$\mathcal{U}_{i,t} = \Upsilon \left\{ s \log \left(\frac{C_{i,t} \rho_D}{e^0} \right) + \sum_{j=1}^{m} \log \left(\frac{A^j_{i,t}}{a^j} \right) \right\} + \lambda_{i,t} \left(\mu_{i,t} + \left(e^0 - \sigma_{i,t}\right) t_0 - \sum_j b^j_{i,t} \right). \tag{6}$$

Steady-state solutions to the trading subgame and the labor allocation to prospecting are derived in App. A. The same patterns of hoarding occur as with fiat, with the discount rate ρ_D replacing ρ, and a simpler population structure. Whereas a tax rule based on the labor allocation was necessary in the fiat model to allow labor allocation to be determined endogenously, the reduced complexity of the labor allocation problem removes the need for any such structure with gold money. The optimal balance of prospecting to farm production is an elementary equilibrium of supply and demand, with any under-represented type having superior bidding position in the markets, and inducing a migration of individuals to adopt that type.

3 Money-metric values and costs of money systems

We now show that there is a natural measure of the value of any allocation of goods, which answers the question "what is the money value of the money system itself", either relative to another money system or relative to no-trade or autarchy. Our construction is similar to one used to assign value to non-CE allocations from one-period trade with exogenous market structure [2]. It is based on the observation that any initial endowment, in a society with given preferences, defines an uncaptured segment of the Pareto set to which an intrinsic money-measure can be assigned [3]. Pareto optima, including but not limited to the CE, distribute among the society the value of capturing the whole segment. Other allocations leave sub-segments uncaptured, and their efficiency relative to the Pareto optima is the ratio in money metric of the segments they capture to the available gain from Pareto-optimal trade.

The essential feature of the model, exploited quantitatively in App. B, is the separation of timescales, between long periods over which the money system is assumed fixed, and short repeated trading periods in which it is used. The money system specifies a rational trading strategy for repeated use until it can be changed. Variations in the money system entail variations in repeated trade outcomes, over which the multi-period utility for individual rounds of trade induces an effective utility for money systems represented as pre-committed repeated strategies. Three basic observations emerge from the analysis of App. B:

First, the CE of the independently chosen intertemporal strategies correspond to the CE of the effective utility for repeated strategies, but the two utilities are generally not of the same form. In this model at $s \neq 0$, the utility for committed strategies is equivalent to a simple Cobb-Douglas form, in which the exponents on consumables and gold reflect their actual weight in the first-order conditions. In the intertemporal model, only the combination of utility variation with the integrated carry-forward equation (Eq. (7) of the first essay) produces these first-order conditions.

Second, as the identification of Pareto optima is not constrained by process, the problem is equivalent to a General Equilibrium model in which a single input (labor) is converted through a set of production functions to diverse utilitarian goods. Both the single-period and the accumulated (effective) utilities we have chosen are Gorman aggregatable, so that the Pareto set is an $(n-1)$-dimensional linear space defined by redistributions of a single commodity bundle among the n agents. The natural numéraire for that bundle is the unique scarce resource in the society: labor counted as agent-periods[4].

Third, the measure of the efficiency of a money system is a ratio of two endogenously-defined money-metric social welfare functions. It is a sum over agents i of terms $\exp\left[-D\left(\mathcal{P} \parallel \mathcal{Q}_i\right)\right]$ weighted by the fraction of whole-society wealth held by i. $D\left(\mathcal{P} \parallel \mathcal{Q}_i\right)$ is an information-theoretic pseudodistance called the Kullback-Leibler divergence [4], defined and derived in this context in App. B.2. It measures the divergence of i's distribution of consumptions relative to a Pareto-optimal distribution (both denominated in labor-metric).

3.1 Results for fiat and commodity moneys

The money value assigned to Pareto optima in this scheme is mr, the number of agents contributing to a social welfare function at full employment of the primary production technologies for goods, with optimal distribution. Commodity money preserves full primary production, and the fraction of total wealth held by each agent is the fraction of the society's labor accounted in the agent's consumed bundle. Because labor-allocation in the strategic market games precedes

[4]The natural emergence of a labor metric for value, along with the unrealism of such a metric as a basis for economic theory, is a consequence of our assumption of efficient, endogenously optimized labor allocation among linear production functions.

per-period trade, and is based on anticipated per-period utilities, equal utilities do not in general result in equal weights in labor-metric. This effect is only a budget redistribution, though, and does not appear in the whole-society efficiency. The latter results only from non-CE allocation of consumption by each agent, and thus scales as $(\rho_D t_0)^2/m$ (45).

Fiat money similarly induces non-CE distribution from the taxation and interest $\tau + \rho t_0$, and a reduction in efficiency $\sim (\tau + \rho t_0)^2/m$. As $\tau + \rho t_0$ is comparable to $\bar{r}/r$, at small bureaucracy this allocation inefficiency is less important than the lost of a fraction $\bar{r}/r$ of the society from primary production. The combination $\tau + \rho t_0$ in the fiat economy is entirely equivalent in its effects on allocation, to the monetary interest $\rho_D t_0$ in the gold economy created by utilitarian discounting.

4 Discussion

4.1 Fiat, gold and credit

In our previous essay we considered an economy with both fiat money and credit. The government was required expicitly to enforce debt contracts, and implicitly to help provide the basis for the acceptance of its fiat. In this essay we have concentrated on an economy with gold as money. In order to separate out difficulties we omitted the credit market. Although government will still nevertheless be needed for many other purposes, including the enforcement of the commercial code and accounting standards there no longer a need for a bureaucracy to produce and police the money supply or to enforce loan contracts. Thus, in comparison to the economy with fiat and credit, a bureaucry is not needed. At the expense of somewhat more calculation we could have included a credit market for individuals to be able to borrow gold. This would have created a need for the enforcing bureaucracy as it did in the instance of an economy with fiat and a credit market. However, as we have already illustrated these conditions for the endogenous bureaucracy, here we limit our concern to illustrating the costs of private production of a commodity money. The bureaucrats with a cheaply produced fiat with strategic control over the money supply are replaced by a competitive manufacturing industry with associated costs.

4.2 Governments and markets within open systems

A critical observation is that one function provided by government is to output a self-referential definition of its own structure and strategy set. Another is the publication of interfaces to the decentralized labor pool (as well as other markets), and with it the distribution of benefits to the members who make up the coalitions.

We have argued descriptively that the theory of decentralized coordination cannot be sensibly defined within a closed system, and that it operates within the structure of rules output by institutions. These in turn act within the social norms of society, so in fact each sector is an open system coupled to a larger source of institutional supports.

A glance at economic history shows that the economic structures of society shape the trust habits of their members in ways that affect their play of games isolated from any of those economies [5]. Thus rule systems over time alter the habit structures of people, in ways that the rules can be designed to exploit[5]. An important dynamical question is whether any self-referential cascade of government and markets can function without depending for most of its inputs on social norms that it does not in fact create. It is known that within biology many coordination devices would be ruled out as perpetuum mobili in closed systems, which function robustly as catalysts for substrate-level driven processes. It would be interesting to formalize the question whether the stability of government and market systems is possible only when they are catalysts of large-scale coordination activities driven mostly by social norms and habits. In the narrow sense that governments must more than pay for the cost of their maintenance to be created in these games, we have asserted a very weak version of that result.

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A Interior solutions to specialization and trade with commodity money

This appendix solves the trading subgame and identifies the interior solution for labor allocation in the commodity-money markets of Sec. 2.1.

A.1 Intertemporal variations with carry-over

The variation of a farmer's intertemporal utility (5) with commodity money, (neglecting impacts by passing to infinite r, and using the conservation relations (Eq. (7) of the first essay) and (3) to accumulate the consequences of strategies over succeeding times) is

$$\begin{aligned}\delta\mathcal{U}_i &= \sum_{t=0}^{\infty}\beta^{t/t_0}\left\{\sum_j \frac{\Upsilon}{p_t^j A_{i,t}^j t_0}\left(\delta b_{i,t}^j - p_t^j \delta q_{i,t}^j\right) + \frac{\Upsilon s}{C_{i,t}}\sum_{t'=0}^{t-t_0}\left(1-\Delta t_0\right)^{\frac{t-t'-t_0}{t_0}}\delta\sigma_{i,t'}t_0\right.\\ &\quad - \lambda_{i,t}\left[\sum_j\left(\sum_{t'=0}^{t}\delta b_{i,t'}^j - \sum_{t'=0}^{t-t_0}p_{t'}^j\delta q_{i,t'}^j\right) + \sum_{t'=0}^{t}\delta\sigma_{i,t'}t_0\right] + \left.\left(\mu_{i,t} - \sigma_{i,t}t_0 - \sum_j b_{i,t}^j\right)\delta\lambda_{i,t}\right\},\end{aligned} \tag{7}$$

[5] In a society with habits of rule-following, a far smaller force can be allocated to sustain a rule than would be necessary in cases of mass defection

where now $j \in 1, \ldots, m$ because there is no trading post for gold, and $\sigma_{i,t} t_0$ competes with bids rather than being drawn from purchases. Rearranging the sums over t and t' in Eq. (7) applies the "shadow of the future" to all present decisions, in the form

$$\delta \mathcal{U}_i = \sum_{t=0}^{\infty} \beta^{t/t_0} \left\{ \sum_j \left(-\frac{\Upsilon}{p_t^j A_{i,t}^j t_0} + \sum_{t'=t+t_0}^{\infty} \beta^{\frac{t'-t}{t_0}} \lambda_{i,t'} \right) p_t^j \delta q_{i,t}^j + \left(\frac{\Upsilon}{p_t^j A_{i,t}^j t_0} - \sum_{t'=t}^{\infty} \beta^{\frac{t'-t}{t_0}} \lambda_{i,t'} \right) \delta b_{i,t}^j \right.$$
$$\left. + \left(\sum_{t'=t+t_0}^{\infty} \beta^{\frac{t'-t}{t_0}} (1 - \Delta t_0)^{\frac{t'-t-t_0}{t_0}} \frac{\Upsilon s}{C_{i,t'}} - \sum_{t'=t}^{\infty} \beta^{\frac{t'-t}{t_0}} \lambda_{i,t'} \right) \delta \sigma_{i,t'} t_0 + \left(\mu_{i,t} - \sigma_{i,t} t_0 - \sum_j b_{i,t}^j \right) \delta \lambda_{i,t} \right\}. \tag{8}$$

The sums of discounted $\lambda_{i,t'}$ for q and b variations are equivalent to the combinations of λ and η constraints in the fiat model: they impose a condition of no wash selling at any $\beta < 1$. The shadow of the future looms one period larger on bids than offers, so the hoarding depends on Δt_0 as well as β. The prospector variation can be similarly rearranged, and is somewhat simpler:

$$\begin{aligned} \delta \mathcal{U}_i \quad = \quad & \sum_{t=0}^{\infty} \beta^{t/t_0} \left\{ \sum_j \left(\frac{\Upsilon}{p_t^j A_{i,t}^j t_0} - \sum_{t'=t}^{\infty} \beta^{\frac{t'-t}{t_0}} \lambda_{i,t'} \right) \delta b_{i,t}^j \right. \\ & + \left(\sum_{t'=t+t_0}^{\infty} \beta^{\frac{t'-t}{t_0}} (1 - \Delta t_0)^{\frac{t'-t-t_0}{t_0}} \frac{\Upsilon s}{C_{i,t'}} - \sum_{t'=t}^{\infty} \beta^{\frac{t'-t}{t_0}} \lambda_{i,t'} \right) \delta \sigma_{i,t'} t_0 \\ & \left. + \left(\mu_{i,t} + \left(e^0 - \sigma_{i,t} \right) t_0 - \sum_j b_{i,t}^j \right) \delta \lambda_{i,t} \right\}. \end{aligned} \tag{9}$$

A.2 Stationary, type-symmetric trade solutions

A subset of the reduced notation for type-symmetric solutions of the trading subgame with fiat describes allocations with commodity money. To the terms defined we must add μ_0 for the stationary carry-over money supply of prospectors, and μ for farmers. Summing the geometric series of discount factors and solving for the K-T multipliers λ, the discounted value of future service from capital stock replaces the penalty constraint as the limit on spot consumption. Only nondurable goods are priced, and their prices are all equal, so the marginal utilities reduce to the set

$$\frac{1}{pA_0} = \frac{t_0}{b_0} = \frac{\alpha^0}{\sigma_0} \tag{10}$$

for prospectors,

$$\frac{1}{pA_{\|}} = \frac{t_0}{p\,(at_0 - q)} = \frac{1}{(1 + \rho_D t_0)} \frac{\alpha^0}{\sigma} \tag{11}$$

for farmers on their own goods, and

$$\frac{1}{pA_{\perp}} = \frac{t_0}{b} = \frac{\alpha^0}{\sigma} \tag{12}$$

on other's consumable goods, where

$$\alpha^0 = \frac{s\beta\Delta t_0}{1-\beta\left(1-\Delta t_0\right)} = \frac{s\Delta}{\rho_D+\Delta} \tag{13}$$

replaces s as the scale for marginal utilities in the presence of intertemporal discounting. Comparison against the fiat first-order conditions (Equations (38-44) of the first essay) shows that the effective per-period interest from discounting, $\rho_D t_0 = 1/\beta - 1$, now appears also as the spot rate replacing $\tau + \rho t_0$. All K-T multipliers are nonzero, so farmers either bid or convert all their gold every period, $\mu = \sigma t_0 + (m-1)\, b$, and prospectors carry nothing forward $\mu_0 = 0$. Their extraction endowment covers bids and use in capital stock $e^0 t_0 = \sigma_0 t_0 + m b_0$. Unlike the fiat case, with commodity money the bids like the offers scale as t_0 rather than with a fixed money supply, so that the velocity of circulation goes to a constant as $t_0 \to 0$.

With commodities as with fiat there is hoarding of endowed goods $A_{\|}/A_{\perp} = 1+\rho_D t_0$. Capital stocks $\sigma_0/\sigma = b_0/b$ replace the K-T multipliers of the penalty constraint, and the condition of equal utility between prospectors and farmers becomes

$$(m+s)\log\frac{b_0}{b} = \log\left(1+\rho_D t_0\right), \tag{14}$$

The absence of a gold market keeps prospectors from hoarding, so that their overall level of consumption is reduced to a utility equivalent to the suboptimal utility of hoarding farmers. All money is used each period, and returned to farmers from the posts to begin the next period, so $pq = \mu \Rightarrow m b_0 r^0/\left(r - r^0\right) = \sigma t_0 \Rightarrow$

$$m\frac{r^0}{r-r^0}\frac{b_0}{b} = \alpha^0. \tag{15}$$

The bidding strength of prospectors versus farmers is inverse, through Eq. (15), to their numbers in the population, so the neutrality condition (14) defines a stable equilibrium the livelihood distribution. Combining solutions of Eq. (11) for q with Eq. (12) to remove the $1/\beta$, and making use of Eq. (15), consumptions of nondurable goods are

$$\left(\frac{A_0}{a}, \frac{A_{\|}}{a}, \frac{A_{\perp}}{a}\right) = \left(\frac{1}{m+\rho_D t_0+\alpha^0}\right)\left(\frac{b_0}{b}, \left(1+\rho_D t_0\right), 1\right), \tag{16}$$

The budget constraint and equal marginal utilities of prospectors set $e^0 t_0 = \left(m+\alpha^0\right) b_0$, from which follows the gold investment distribution

$$\left(\frac{\sigma_0}{e^0}, \frac{\sigma}{e^0}\right) = \left(\frac{\alpha^0}{m+\alpha^0}\right)\left(1, \frac{b}{b_0}\right). \tag{17}$$

B Money-metric utilities for committed decisions

The adoption of a money system commits a society to a specific rational labor allocation and trade strategies for as long as they use it, both of which are in general different from those of a CE. It is thus sensible to associate variations in the money system with the joint variations in all one-period strategies, corresponding to stationary solutions of the varying system. The per-period preferences then induce preferences for these joint variations, which are not generally of the same form. The CE of the intertemporal utility is also the CE of the corresponding joint-variation problem, but that latter generally has a larger Pareto set, corresponding to repeated non-CE trade strategies, that are not required by the money system alone. We consider these because they restrict the full set of Pareto-optimal trade strategies (most of which are nonstationary) to a stationary one-period equivalent set that contains the non-optimal strategies associated with stationary NE of general money systems. This restricted Pareto set is characterized by a constant money-metric social welfare function, and the effective preferences then define efficiency measures relative to it for general stationary allocations.

B.1 Effective preferences for repeated decisions

Consider, for any agent, an intertemporal utility of the form:

$$\mathcal{U}_i = \sum_{t=0}^{\infty} (\rho_D t_0)\, \beta^{t/t_0} \Upsilon \left\{ s \log\left(\frac{C_{i,t}\rho_D}{e^0}\right) + \sum_{j=1}^{m} \log\left(\frac{A^j_{i,t}}{a^j}\right) \right\}, \tag{18}$$

in which the capital stock decays by a fraction Δt_0 per period, so that

$$C_{i,t} = (1-\Delta t_0)^{t/t_0} C_{i,0} + \sum_{t'=0}^{t-t_0} (1-\Delta t_0)^{\frac{t-t_0-t'}{t_0}} \sigma_{i,t'} t_0. \tag{19}$$

In this appendix only, because the goal is to identify the structure of Pareto optima, i is an index representing all agents, whose types need not be distinguished in the notation. However, we continue to denote by r^j the number of agents producing nondurable good j in steady state, and by mr^0 the number producing gold. The variation purely in terms of per-period allocations is

$$\delta\mathcal{U}_i = \sum_{t=0}^{\infty} (\rho_D t_0)\, \beta^{t/t_0} \Upsilon \left\{ s\beta\delta\sigma_{i,t} t_0 \sum_{\tau=0}^{\infty} \frac{\left[\beta(1-\Delta t_0)\right]^{\tau/t_0}}{C_{i,t+t_0+\tau}} + \sum_{j=1}^{m} \frac{\delta A^j_{i,t}}{A^j_{i,t}} \right\}. \tag{20}$$

Expanding around any stationary solution $A^j_{i,t} \equiv A^j_i$, $\sigma_{i,t} = \sigma_i$, in which $C_{i,t} = \sigma_i/\Delta, \forall t$, Eq. (20) becomes

$$\delta\mathcal{U}_i = \sum_{t=0}^{\infty} (\rho_D t_0)\, \beta^{t/t_0} \Upsilon \left\{ \alpha^0 \frac{\delta\sigma_{i,t}}{\sigma_i} + \sum_{j=1}^{m} \frac{\delta A^j_{i,t}}{A^j_i} \right\}, \tag{21}$$

where α^0 is as in Eq. (13).

Commitment to a repeated strategy sets $\delta A^j_{i,t} \equiv \delta A^j_i$, $\delta\sigma_{i,t} \equiv \delta\sigma_i, \forall t$. The resulting utility change under committed variation,

$$\delta\mathcal{U}_{i,\mathrm{comm}} = \frac{\Upsilon\rho_D t_0}{1-\beta}\left\{\alpha^0\frac{\delta\sigma_i}{\sigma_i} + \sum_{j=1}^{m}\frac{\delta A^j_i}{A^j_i}\right\}, \tag{22}$$

is the variation of the *effective utility* for a committed strategy

$$\mathcal{U}_{i,\mathrm{comm}} = \frac{\Upsilon}{\beta}\left\{\alpha^0\log\left(\frac{\sigma_i}{e^0}\right) + \sum_{j=1}^{m}\log\left(\frac{A^j_i}{a}\right)\right\}, \tag{23}$$

where we use $\rho_D t_0/\left(1-\beta\right) = 1/\beta$.

Eq. (23) is equivalent, under monotone transformation, to the Cobb-Douglas utility (normalized here so that the exponents sum to unity)

$$\mathcal{U}^{CD}_{i,\mathrm{comm}} = \left[\left(\frac{\sigma_i}{e^0}\right)^{\alpha^0}\left(\prod_{j=1}^{m}\frac{A^j_i}{a}\right)\right]^{\frac{1}{m+\alpha^0}}. \tag{24}$$

The trade constraint on agents who have chosen a social allocation is

$$\sum_i \sigma_i = mr^0e^0 \tag{25}$$

for gold, and

$$\sum_i A^j_i = r^j a \tag{26}$$

for nondurable consumables.

The first-order conditions for a Pareto optimum of trade under these whole-society endowments are

$$\left(p^0, p^{j\neq 0}\right) = \frac{\varphi_i}{m+\alpha^0}\left(\frac{\alpha^0}{\sigma_i}, \frac{1}{A^j_i}\right), \tag{27}$$

corresponding to a budget for agent i in any period

$$\left(p^0\sigma_i + \sum_{j=1}^{m}p^jA^j_i\right)t_0 = \varphi_i t_0. \tag{28}$$

The rate of wealth input to the society as a function of its labor allocation is

$$\sum_i \varphi_i = p^0mr^0e^0 + \sum_{j=1}^{m}p^jar^j. \tag{29}$$

At any Pareto optimum of trade, therefore,

$$\left(\frac{\sigma_i}{e^0}, \frac{A_i^j}{a}\right) = \frac{\varphi_i}{\sum_{i'} \varphi_{i'}} \left(mr^0, r^j\right). \tag{30}$$

Even at fixed budget, agents can individually alter the labor allocation, each subject to the constraint of a fixed size of the society: $\sum_j \delta r^j + \delta\left(mr^0\right) = 0$. Enforcing this constraint on Eq. (22) (the Lagrangian multiplier is suppressed from the notation here to simplify the presentation), and using Eq. (30) to relate variations in A_i^j and σ_i to r^j and mr^0, gives the property of prices and labor allocations at any Pareto optimum of these preferences subject to a fixed labor force and production technology:

$$p^0 e^0 = p^j a \;\Rightarrow\; mr^0 = \alpha^0 r^j, \; j \in 1, \ldots, m. \tag{31}$$

If we normalize prices by the condition $p^0 e^0 = 1$,[6] the society wealth (29) on the Pareto set becomes

$$\sum_{i'} \varphi_{i'} = mr. \tag{32}$$

B.2 Direct money-metric utility for repeated strategies

Cobb-Douglas utilities are Gorman aggregatable [6], implying Pareto optima defined by a single price system, dependent only on the whole-society endowments. The only constraint in the committed-choice problem is whole-society labor, which both accounts for the single price system (31), and implies that money-metric at equilibrium prices is ultimately *labor-metric*.

The direct money-metric utility [6] for any agent i at equilibrium prices is the expenditure function of an equilibrium consumption bundle indifferent to i's actual bundle $\left(\sigma_i, A_i^j\right)$. With the labor-normalization for prices (32), the equilibrium-price direct money-metric utility corresponding to Eq. (24) is

$$\varphi_i \equiv \mathcal{U}_{i,\mathrm{comm}}^{DMM} = \left(\frac{\sigma_i}{e^0} + \sum_{j=1}^{m} \frac{A_i^j}{a}\right) e^{-D(\mathcal{P} \| \mathcal{Q}_i)}, \tag{33}$$

where

$$D\left(\mathcal{P} \parallel \mathcal{Q}_i\right) \equiv \mathcal{P} \log \frac{\mathcal{P}}{\mathcal{Q}_i} \tag{34}$$

is the Kullback-Leibler divergence [4] of the actual distribution of labor cost over i's consumption bundle,

$$\mathcal{Q}_i \equiv \frac{1}{\sigma_i/e^0 + \sum_{j=1}^{m} A_i^j/a} \left(\frac{\sigma_i}{e^0}, \frac{A_i^1}{a}, \ldots, \frac{A_i^m}{a}\right), \tag{35}$$

[6] The straightforward normalization $\rho_D t_0 / (1 - \beta) = 1/\beta$, corresponding to the infinite sum of discounted values of one producer's gold in each period of the game, differs from unity only because we have scaled utilities with $\rho_D t_0$ rather than $1 - \beta$, using Euler's formula to normalize the utility weights to one at $t_0 \to 0$.

from a target distribution adopted by any agent in the Pareto set:

$$\mathcal{P} \equiv \frac{1}{m+\alpha^0}\left(\alpha^0, 1, \ldots, 1\right). \tag{36}$$

$D\left(\mathcal{P} \parallel \mathcal{Q}_i\right)$ is positive semidefinite and Hausdorf, vanishing only for $\mathcal{Q}_i = \mathcal{P}$.

The direct money-metric utility at equilibrium prices coincides for Gorman economies with the dual *contour money-metric utility* [3], an intrinsic measure of the length of a contour in the Pareto set terminating in the indifference surface of the bundle $\left(\sigma_i, A_i^j\right)$. The contour money-metric utility has the property that its sum over agents in the economy is an invariant of all Pareto optima in the contour, which for Gorman economies extends to the entire Pareto set. The sum of utilities (33) is thus an intrinsic money measure (here, a labor measure) of the part of the Pareto-set contour inside the initial indifference surfaces captured by the agents. Its maximum mr, attained everywhere in the Pareto set, defines a reference for optimal welfare, and the ratio

$$\frac{1}{mr}\sum_i \mathcal{U}_{i,\mathrm{comm}}^{DMM} = \sum_i \mathcal{W}_i e^{-D(\mathcal{P}\parallel\mathcal{Q}_i)} \tag{37}$$

is an intrinsic measure of the efficiency of any allocation produced by a stationary labor and trade strategy. $\mathcal{W}_i$ in Eq. (37) are the weight functions

$$\mathcal{W}_i \equiv \frac{1}{mr}\left(\frac{\sigma_i}{e^0} + \sum_{j=1}^{m}\frac{A_i^j}{a}\right) \tag{38}$$

measuring the fraction of labor cost represented by i's consumption bundle, and satisfying $\sum_i \mathcal{W}_i = 1$ for all outcomes in which every agent adopts one of the primary production technologies. Note that Eq. (37) is a function entirely of *rates* of consumption relative to rates of production, but that Eq. (38) converts these ratios into the fractions of a stock variable (population) provisioning any individual i.

B.3 Solutions for ideal commodity money

Returning to the allocations for pure commodity money from App. A it is straightforward to compute the weight functions and labor distributions for farmers and prospectors. Define a parameter

$$\xi \equiv \frac{b}{b_0}\left(\frac{m+\rho_D t_0+\alpha^0}{m+\alpha^0}\right). \tag{39}$$

Solving progressively from Eq. (16), the ratio of distributions of wealth that determines the K-L divergence of farmers from Pareto optima is

$$\frac{\mathcal{Q}}{\mathcal{P}} = \frac{m+\alpha^0}{m+\rho_D t_0+\alpha^0\xi}\left(\xi, 1, \ldots, 1, \left(1+\rho_D t_0\right), 1, \ldots, 1\right), \tag{40}$$

where $(1 + \rho_D t_0)$ appears in the jth entry corresponding to the good produced. A farmer's weight is

$$\mathcal{W} = \frac{1}{mr} \left\{ \frac{m + \rho_D t_0}{m + \rho_D t_0 + \alpha^0} + \left(\frac{r^0}{r - r^0} \right) \frac{m}{m + \alpha^0} \right\}. \tag{41}$$

The labor distribution ratio for prospectors is

$$\frac{\mathcal{Q}_0}{\mathcal{P}} = \frac{m + \alpha^0}{m + \alpha^0 \xi} \left(\xi, 1, \ldots, 1 \right), \tag{42}$$

and a prospector's weight is

$$\mathcal{W}_0 = \frac{1}{mr} \left\{ \left(\frac{r - r^0}{r^0} \right) \frac{\alpha^0}{m + \rho_D t_0 + \alpha^0} + \frac{\alpha^0}{m + \alpha^0} \right\}. \tag{43}$$

Manifestly $m \left(r - r^0 \right) \mathcal{W} + m r^0 \mathcal{W}_0 = 1$, and $\mathcal{Q}$ and $\mathcal{Q}_0$ differ from $\mathcal{P}$ by terms $\sim \rho_D t_0, (\xi - 1)$.

To illustrate while maintaining tolerable algebra, we suppose $\alpha^0/m \ll 1$, which can be achieved at large m, small salvage value s, or durable capital stock $\Delta/\rho_D \ll 1$. $\mathcal{W} = 1 - \mathcal{O}\left(\alpha^0/m\right)$, $\mathcal{W}_0 = \mathcal{O}\left(\alpha^0/m\right)$, and $D\left(\mathcal{P} \parallel \mathcal{Q}_0\right) = \mathcal{O}\left(\alpha^0 (\rho_D t_0)^2 / m^3\right)$. The efficiency is then $\exp\left\{-D\left(\mathcal{P} \parallel \mathcal{Q}\right)\right\} + \mathcal{O}\left(\alpha^0/m\right)$, where

$$\begin{aligned} D\left(\mathcal{P} \parallel \mathcal{Q}\right) &= \frac{1}{m + \alpha^0} \log \left[\frac{\left(1 + \rho_D t_0 / \left(m + \alpha^0\right)\right)^{m + \alpha^0}}{1 + \rho_D t_0} \right] \times \left\{1 + \mathcal{O}\left(\alpha^0/m^2\right)\right\} \\ &\rightarrow \frac{1}{m + \alpha^0} \log \left(\frac{e^{\rho_D t_0}}{1 + \rho_D t_0} \right) \times \left\{1 + \mathcal{O}\left(\alpha^0/m^2\right)\right\} \end{aligned} \tag{44}$$

at large $m + \alpha^0$. Translating the efficiency measure (44) into actual cost in terms of lost labor to leading order in α^0/m,

$$mr \left(1 - e^{-D(\mathcal{P} \parallel \mathcal{Q})} \right) \rightarrow r \log \left(\frac{e^{\rho_D t_0}}{1 + \rho_D t_0} \right) \approx \frac{r}{2} (\rho_D t_0)^2. \tag{45}$$

In other words, the utilitarian cost to each set of agents sufficient to make up a full suite of specialist production, from the imprinting of the discount horizon onto a monetary interest rate, is equivalent to having lost the output of $\sim (\rho_D t_0)^2/2$ agents'-worth of labor deployed with the same durable capital stock but without interest.

B.4 Solutions for fiat and markets in all goods

The distributions resulting from ideal fiat money (Eq. (58,59) in App. A of the first essay) are simple functions of a variable

$$x \equiv \frac{b}{b_0} \tag{46}$$

which may be compared with Eq. (39) for ideal commodity money. The ratio of distributions determining the K-L divergence for prospectors is

$$\frac{\mathcal{Q}^0}{\mathcal{P}} = \frac{m + \alpha^0}{m + \alpha^0 \left(1 + \tau + \rho t_0\right) x} \big(\left(1 + \tau + \rho t_0\right) x, 1, \ldots, 1 \big). \tag{47}$$

For farmers it is

$$\frac{\mathcal{Q}}{\mathcal{P}} = \frac{m + \alpha^0}{m + \tau + \rho t_0 + \alpha^0 x} \big(x, 1, \ldots, 1, \left(1 + \tau + \rho t_0\right), 1, \ldots, 1 \big), \tag{48}$$

where $(1 + \tau + \rho t_0)$ appears in the jth entry corresponding to the good produced. For bureaucrats the ratio is

$$\frac{\bar{\mathcal{Q}}}{\mathcal{P}} = \frac{m + \alpha^0}{m + \alpha^0 x} \big(x, 1, \ldots, 1, \big). \tag{49}$$

The corresponding weight functions for the three are

$$\mathcal{W}^0 = \frac{1}{mr} \left(\frac{b_0}{b} \frac{1}{1 + \tau + \rho t_0} \right) \frac{m + \alpha^0 \left(1 + \tau + \rho t_0\right) x}{m + \alpha^0}, \tag{50}$$

$$\mathcal{W} = \frac{1}{mr} \left(\frac{1}{1 + \tau + \rho t_0} \right) \frac{m + \tau + \rho t_0 + \alpha^0 x}{m + \alpha^0}, \tag{51}$$

$$\bar{\mathcal{W}} = \frac{1}{mr} \left(\frac{\bar{b}}{b} \frac{1}{1 + \tau + \rho t_0} \right) \frac{m + \alpha^0 x}{m + \alpha^0}. \tag{52}$$

It follows from the no-default conditions (Eq, (51,53) in the first essay) that $m r^0 \mathcal{W}^0 + m \hat{r} \mathcal{W} + m \bar{r} \bar{\mathcal{W}} = (1 - \bar{r}/r)$, the fraction of society producing consumer goods. The distributions (48,49) correspond to those in the gold economy (40,42), under $\xi \to x$ and $\rho_D t_0 \to \tau + \rho t_0$. The reason the bureaucrat in the fiat model allocates goods as the prospector does in the gold economy is that neither has a money-market trading post, and both serve as the primary injectors of money into their respective economies. In the fiat model, Eq. (47) for prospectors is a new form because they have the only non-permutation-symmetric good.

It follows that at small $\tau + \rho t_0$, the K-L divergence-corrections from misallocation by all agents are $\mathcal{O}\left(\left(\tau + \rho t_0\right)^2, \left(x - 1\right)^2 \right)$. Meanwhile, from Eq. (64,67) of the first essay, we may extract the leading source of lost efficiency, from the total labor in primary production, $m \left(r - \bar{r}\right) / mr$, as

$$\frac{\bar{r}}{r} \approx \frac{m + \alpha^0 - 1}{m + \alpha^0} \left(\tau + \rho t_0\right). \tag{53}$$

The average money-metric welfare lost per replica is simply the lost labor from primary production diverted to the bureaucracy. Fiat is a more valuable technology than gold money only if $\bar{r}/r \approx (\tau + \rho t_0) < \left(\rho_D t_0\right)^2/2$. Since the sales tax fraction τ is fixed at $t_0 \to 0$, fiat is only preferred if gold introduces a sufficiently large necessary trading period t_0 relative to the discount horizon $1/\rho_D$.

C Definition and condensed notation for variables and parameters

Notation used in modeling of the gold economy.

Parameters	
m	Number of types of consumption goods
$n \equiv mr$	Number of agents in the society
t_0	Time interval for a cycle of production, trade, and consumption
a	Allocation rate of consumption goods to farmers
e^0	Allocation rate of gold to prospectors
Δ	Rate of decay of capital stock
Υ	Scale factor for utilities
ρ_D	Temporal utility discount rate
β	Per-period discount fraction
Labor, price, and allocation variables (period notation $\rightarrow$ stationary notation)	
$\nu_t^0 \rightarrow mr^0$	Number of prospectors at any time t
$\nu_t^j \rightarrow \hat{r}$	Number of farmers of any consumption good at any time t
$Q_t^j \rightarrow Q$	Total quantity offered of a single consumption good
$B_t^j \rightarrow B$	Total gold bid on a single consumption good
$p_t^j \rightarrow p$	Price of any consumption good in period t
$\lambda_{i,t} \rightarrow \lambda$	Kuhn-Tucker multipliers
$A_{i,t}^j \rightarrow A_0$	Prospector's final allocation rate of any consumption good
$A_{i,t}^j \rightarrow A_\parallel$	Farmer's final allocation rate of self-produced consumption good
$A_{i,t}^j \rightarrow A_\perp$	Farmer's final allocation rate of other-produced consumption good
$\sigma_{i,t} \rightarrow \sigma_0$	Prospector's conversion rate of gold to capital stock in a period
$\sigma_{i,t} \rightarrow \sigma$	Farmer's conversion rate of gold to capital stock in a period
$\mu_{i,t} \rightarrow \mu_0$	Prospector's initial gold carried into period
$\mu_{i,t} \rightarrow \mu$	Farmer's initial gold carried into period
$C_{i,t} \rightarrow \sigma_0/\Delta$	Capital stock of prospectors
$C_{i,t} \rightarrow \sigma/\Delta$	Capital stock of farmers
Individual decision variables(period notation $\rightarrow$ stationary notation)	
$q_{i,t}^j \rightarrow q$	Self-produced consumption good offered by any farmer per period
$b_{i,t}^j \rightarrow b_0$	Bid by prospector on any consumption good
$b_{i,t}^j \rightarrow b$	Bid by any farmer on any other-produced consumption good

Table 1: Reduced notation for the decision, price, and allocation variables of the gold economy in case of stationary solutions.

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