FIAT MONEY AND THE NATURAL SCALE OF GOVERNMENT

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Abstract

The competitive market structure of a decentralized economy is converted into a self-policing system treating the bureaucracy and enforcement of the legal system endogenously. In particular we consider money systems as constructs to make agents' economic strategies predictable from knowledge of their preferences and endowments, and thus to support coordinated resource production and distribution from independent decision making. Diverse rule systems can accomplish this, and we construct minimal strategic market games representing government-issued fiat money and ideal commodity money as two cases. We endogenize the provision of money and rules for its use as productive activities within the society, and consider the problem of transition from generalist to specialist production of subsistence goods as one requiring economic coordination under the support of a money system to be solved. The scarce resource in a society is labor limited by its ability to coordinate (specifically, calling for the expenditure of time and effort on communication, computation, and control), which must be diverted from primary production either to maintain coordinated group activity, or to provide the institutional services supporting decentralized trade. Social optima are solutions in which the reduced costs of individual decision making against rules (relative to maintenance of coalitions) are larger than the costs of the institutions providing the rules, and in which the costs of the institutions are less than the gains from the trade they enable to take place.

1 Introduction: institutional foundations of the decentralized economy

The general equilibrium system is usually presented without institutional or strategic context. Here a game theoretic structure is provided that converts the competitive model into a self-policing system where a bureaucracy polices the laws of the economy to limit the strategic manipulation of the agents in the

*We adopt the convention that in joint work the order of appearance of names on the publication should be selected randomly unless there is a specific stated reason otherwise. We have acted accordingly.

economy. The economy is modeled as a strategic market game. As there is considerable notation, a table is provided in App. B for referential convenience.

The deepest formal distinction one can make in the modeling of a society is whether the primitive strategic actors are individuals or groups[1]. Since Adam Smith's emphasis on the surprising coordinating effects of independent, self-interested action [1], the mathematical formalization of economic theory has progressively narrowed to the problem of associating algorithms for disaggregated decision making with observed or desired distributions of resources. Yet the actions of governments and firms are essentially strategies carried out by the managers of institutions, or aggregates of agents, or coalitions[2], and these strategies implement the rules against which individuals act as members of the decentralized economy. The basic structure of economic life, as well as many specific observable regularities, thus cannot emerge solely from the action of individuals against rule systems, but must be understood from the interactions of the decentralized economy with its centralized supporting institutions.

1.1 Problems that require the treatment of institutions

In this and a companion paper we seek to understand the following specific questions, as well as to define a functional framework within which the institutional origins and economic roles of structurally different moneys can be understood and compared:

First, what sets the natural scale of government as a support for anonymous markets? A recognized weakness of General Equilibrium analysis is its somewhat amorphous assumption of "costlessly enforceable contracts" [2]. In economies with fiat money, the government is the institutional enforcer of contracts, and a large part of the cost of contract enforcement is the economic cost of running the government. More fundamentally, government is the *definer* of many forms of contract in the economy, and of other non-contractual "rules of the game" [3]. Presumably there are criteria for optimizing its size and acceptable cost in terms of what those rules make possible.

Second, what determines the value of fiat money in trade for real goods, and what is the interpretation of that value within its own rule system? A progression from commoditiy moneys (such as salt or tea), to metals by weight or in ingot form, to coins with a state's seal, selling over commodity value, to fiat [4], and even to abstract systems of account, can be seen in culturally diverse societies. Possible reasons why such a progression is favored are that fiat frees materials of utilitarian value from being sequestered as moneys rather than used otherwise [5], or that it provides additional regulatory controls over the macroeconomy. These features depend on its intrinsic near-valuelessness, but are only realized if its valuation in real goods is stable and predictable. Expectations models of fiat valuation [6] suggest some short-term bounds on its change, but longer-term stability must be institutional in origin.

[1] In our society two types of legal persons are considered, natural and corporate persons.

[2] These are coalitions within governments or firms, with the institution as a whole acting as the grand coalition that establishes the rules for the games that define its particular function.

A third question motivated by historical succession of moneys is what relation different stages have to each other. Is gold with a king's stamp a pure commodity money? Is fiat intrinsically a debt contract for gold, or is it an instrument within a system defined by quite independent rules, and do debt contracts merely provide a mechanism for the orderly transition from a material substrate? Answering these conceptual questions requires an understanding of the barriers to trade, and of the independent ways structurally different money systems define protocols to overcome the same barriers. The physical characteristics that make a commodity a "good" commodity money need not have anything intrinsic to do with the nature of money; they need only be tailored to requirements for commodity-mediated signals.

1.2 The functional roles of money and its supports

In this and a companion essay we introduce two strategic market games [4] constructed to provide the strategies and signals needed to make specialization rational. In the first, an endogenously formed government distributes fiat money, so that its flow from government salaries to taxation supports trade in private markets for all goods, while simultaneously providing a rational strategy for a subset of agents to man the government rather than produce privately. The enforcement of tax liabilities ties fiat to government-induced penalties, making "tax relief" a utilitarian service purchased with fiat, and thus stabilizing the numéraire relative to consumables, along the basic lines argued by Knapp [7]. Rather than leave implicit how agents are subject to tax liabilities, in the spirit of implementing a fully decentralized system we introduce central-bank lending as the only source of fiat in the private sector before trade has begun, and initiate taxation on trade together with interest charges on borrowing. The interest mechanism to stabilize fiat value has been used in models without endogenous government [4], where the distribution of money was not tied to any individual's action and was a pure device to support trade.

The strategic market game with fiat introduces an explicit institutional cost from the creation of government, whose input is labor and whose output is determined by technologies for enforcing default penalties on the supervising of central-bank loans and their repayment. Thus contracts have a specific operational meaning, and the extra-social cost of their enforcement is institutionally represented. Fiat money emerges as the source of the basic contracts in society, without itself being a contract for anything [3] .

In the second second essay, a second strategic market game is presented. This game omits the institutional cost of government by using the durable commodity as money, between the time when it is produced and the time when it is consumed by manufacture to yield utilitarian services. The essential element of pre-commitment that gives fiat its signaling value has a counterpart for commodity money, which is the cost of acquiring it and the demand for its utility, which agents know[4]. Idealized commodities, however, do not require separate

[3]We can argue that it is a null contract. It promises to pay $1 for $1.

[4]or at least, in real societies can often estimate.

and costly institutions to give them this signal, because it is assumed that their worth can be judged by individuals independently at the point of trade[5]. In the act of exchange, however, commodities are a physical "store of value", which imprints any discounting of the future on the spot interest rate for all goods, inducing trade inefficiencies that have labor-equivalent costs.

1.3 The costs of centralized and decentralized coordination

The rules for trading, apart from those related to stabilizing money, are essentially dummy institutions in these models. They are formal rules not costing labor to provide and are given as invariants of all the games. The government is modeled only as the aggregation of agents who adopt contract enforcement as a livelihood. In this form, the problem of optimal labor allocation and trade could almost be viewed as a process-oriented elaboration of a General Equilibrium problem of optimal production and consumption (apart from the absence of contracts to enforce in Equilibrium analysis). In the second essay, we use such a reduction in the goods sector to define natural efficiency measures for the distribution of labor and subsequently of goods in each of the market games, which assigns value to the trading systems themselves.

However, this highly encapsulating representation of government should not be understood to mean that the problems of cooperative action have been replaced with noncooperative ones in a closed system. Rather, the costs of forming coalitions and obtaining coherent strategic action from the individuals that are their members are so high that they would be unaffordable for direct economic coordination throughout the society, and this cost is the barrier to large-scale coordination through cooperative mechanisms[6]. These same costs are present in microcosm in the internal workings of government, and are responsible for its consumption of labor. The task for more elaborate models is to derive them explicitly. The important feature of the government/economy interface is that the bureaucratic, institutional or coalition strategies within the government build *generative* rule and signaling systems for low-cost independent decision making; their generative character is the "invisible hand" sent forth from the visible institutional interface. The government itself is thus only affordable if it is a small fraction of the economy's size[7], and innovation in government (as in firms)

[5]It is something of a misnomer to call this "costless". Rather, commodity money exploits mechanisms of verification that have evolved within the society for other reasons. Their availability is an externality of other quite costly processes, but ones whose costs would be paid in the course of exchange of the commodities as goods anyway, and have no separate institutional account.

[6]As an example consider tax evasion. Tax compliance takes place by noncooperative decisions against the tax laws, and the transaction itself is a small economic burden. Tax evasion (in the sense of complete defection), while feasible on sufficiently large scale as a coalition strategy, is not served by any similar rules for coordinated decentralized action. This is presumably an important part of the reason it does not occur.

[7]In year 2000 government employment in the United States was around 15.7% of the work force (SAUS, 2001, Table 607).

consists of the transfer of successively more functions from coalition-based mechanisms to decentralized market-supporting rule systems[8].

In this framework of modeling and interpretation, the succession of moneys is "explained" if it produces a synergy: government and fiat survive when they can improve the utility of trade or the efficiency of production by more than their cost of providing rules and signals as outputs of an internal coalition structure. Both the costs that make non-market coordination infeasible and the costs to implement markets have many of the same underlying causes.

1.4 Layout of the papers

In Sec. 2 we discuss the problem of transition to specialist production, and the game-theoretic representation of barriers to trade. In Sec. 3 we define the requirements on money systems in terms of signaling and predictability, and distinguish the ways these are provided by different institutional forms. Sec. 4 begins formal modeling, with the characterization of society and the invariants from which different moneys will be compared. Sec. 5 defines a game for the formation of government, and analyzes the problems of limiting strategic default in endogenously optimized markets, and their consequences for the natural size of government. Sec. 1 of the second essay defines an alternative game for use of commodity money, and Sec. 2 introduces natural money-metric valuation for the money system, comparing the models in both essays.

2 Specialization as a quintessential coordination problem

The ability of a society to specialize in its production of goods is a paradigm coordination problem within which to define the economic guidance goals of government and to study the functions of moneys. It is sufficiently general to be abstracted across societies and periods of history, and sufficiently central to survival and growth to be robust against quibbles about utilitarian representation [8] of people's needs[9]. The utilitarian representation of specialization as a robust problem is illustrated in Fig. 1.

Specialization in the production of subsistence goods potentially captures economies of scale, increasing the level of consumption for everyone. However, specialization realistically requires a livelihood choice, with a committed time period between the abandonment of generalist production and the output of surplus of a single good. Historically the transformation from autarchy or near

[8]The other activity of coalitions within government is the definition, in a self-referential way, of the structure and strategies of the other coalitions in government. How this can be done is an important and deep problem at the interface of government with the social and cognitive substructures of society.

[9]Much of nutrition can be reduced to shared human biochemistry, and the growth or attrition of populations provides evolutionary measures of their success which may in some cases be substituted for intentionality.

autarchy has been an emergent process occuring over a long run. The commitment to specialize creates a risk for the potential specialist, who forfeits subsistence production of many goods, and can only benefit from surplus of one good in a society with complementary specialists and markets that efficiently support their redistribution. The problem for the specialist is predicting the production and trade strategies of the other individuals in society, which in general he can neither learn nor enforce.

Game theoretically the commitment of specialization is represented by concurrency and independence in the strategic choice of livelihood. If a mechanism for trade is not specified, the absence of an associated post-production economic strategy set is equivalent to a cost for all trading strategies greater than any in the production move of the game. Possible costs of non-money coordination mechanisms can be those of negotiating division of labor and ensuring non-defection, which are essentially costs of communication and control. Alternatively, as in the Walrasian auction – the closest representation of a process model whose output is the General Equilibrium allocation – the costs are for communication of whole demand curves from all agents, computation of equilibria (when those are computable [11]), and delivery of goods (control). Interestingly, the nominally institution-free General Equilibrium algorithm is associated with one of the more costly process models. (Empirical estimates of the cost-limitations on the scale of trade that can be managed in something like the Walrasian manner can perhaps be inferred from modern state-managed centralized economies, accounting for differences if agents cannot define their own preferences).

When money-metric values are assigned to outcomes with trade relative to those without, the time and effort needed for communication, computation, and control become scarce resources on an equal footing with factors of production of consumable goods, whose marginal utility is the marginal value of trade. Models of economic organization that incorporate centralizing bodies generalize the idea of optimal distribution under conditions of scarcity to include the allocation of this effort, and consistently represent trade as part of the productive apparatus of the society.

3 Money and predictability

The essential requirement for a post-production trade algorithm is that agents be able to estimate each other's rational strategies for livelihood choice and market activity, based on knowledge of production technologies, labor supply, and agent preferences. Bounds on available or rational strategies are represented in decentralized markets as bounds on the use of money. Therefore the defining requirement of a money and credit system is that its use in the markets by each agent be tied by some form of pre-commitment to either the production activities or utilities of that agent, which are generally known or can be estimated.

Mechanistically, commodity moneys and fiat separate according to the type of precommitment they employ. Commodities to be used as money are obtained at the expense of spot consumption of other goods, and a knowledge of what stock of goods other agents are capable of producing (and of their preferences) allows an estimation of their money supply and strength in the markets. While consumption utility provides the salvage value for commodities to resolve the Hahn paradox for their intermediate money value [12], part of the marginal utility of commodities at the point of purchase is the shadow price of their release of constraints on future trade[10], and the other part is the discounted future utility of consumption.

Government issue ties the posession of fiat to penalties such as tax or interest liability, and through this, ties the marginal posession of fiat to the marginal utility of default relative to consumption[11]. The ability of government to rule out (or at least limit) strategic default is the necessary condition to tie the money supply and hence the scale of bids to an initial numéraire, such as the injection through government salaries.

3.1 Trust, displaced trust, or trust substitutes

The degree to which an agent's strategic action supposes predictability of other strategies that are not directly controlled could be called a measure of *trust*[12]. Institutional supports for decentralized markets can be characterized as either augmenting or displacing trust, or creating trust substitutes.

Monetization of personal credit is an instance of trust augmentation, through mechanisms of reputation frequently supported by coercion [13], which are potentially costly and limited in scope, and which we do not consider. Fiat money displaces trust from agent intentions onto the government's ability to stabilize the supply and value of money. This displacement is a network effect, arising from greater ease of information-gathering for fewer and more visible agencies.

Trust substitutes are provided by institutional rules or social norms like cash payment, which limit agents' strategy sets to their available offers of money. Ideal commodity money is a pure trust substitute, while fiat appears as a trust substitute within the trading subgame, but a mechanism to displace trust in the larger game of adopting monetary and fiscal policy.

3.2 Minimal models

As in the functional taxonomy of one-period markets [10], our concern is with minimal models defined by criteria of strategic freedom, constraint, and symmetry, which formalize barriers to trade and the trust mechanisms that overcome

[10] In a basic way the economics of finance and trust calls for the mathematics of inequalities which is generally more difficult than the mathematical economics of equalities primarily encountered.

[11] Thus we see that when the value of commodities cannot actually be judged costlessly at the point of trade, and a king's stamp ensures weight and composition – along with threat of punishment for defacement – the money is not pure commodity but a hybrid of elements of commodity and fiat. See also discussion in Ref. [4].

[12] This use of trust concerns only predictability, and does not include other connotations of good faith.

them. We define hierarchical extensive game forms, in which the outer move is the adoption of a money system and the commitment to production strategies, and the inner move is trade of surplus. The hierarchical structure of the game is recapitulated in its temporal structure: livelihood choices are fixed over episodes, within which a large number of production and trade periods take place. We define models that allow an arbitrary separation of these two timescales, to reflect the fact that the adoption of government structure and money, or of livelihood, is a more costly process than trade itself, happening much less frequently, and generating information about the structure of society which is available to agents as input to their trading strategies. In deterministic models with stationary solutions the ratio of scales does not affect underlying "physical" observables such as prices or optimal allocations, but in models with stochastic endowments and money markets, it would be expected to affect the ability of either government or social norms to stabilize the values of either fiat or commodity moneys, probably transferring much of the determination of short-term valuation onto expectations [6].

We idealize the problem of specialization to produce an extreme m-dimensional version of Jevons's failure of the double coincidence of wants [14], and make preferences symmetric under permutation of non-durable consumption goods[13], to study the symmetry properties of the money systems themselves. In each model, we minimize the number of strategic degrees of freedom representing each necessary institution. We formalize a general problem of intertemporal utilities with durable as well as non-durable consumption goods, so that fiat and commodity monies can be compared quantitatively, but the durable commodity is defined in such a way that a regular limit decouples it from strategies concerning point consumables[14]. In this limit the time periods become independent, and the models of fiat may be compared under an identical efficiency measure to models in the one-period taxonomy [10], in which the rules of the game were exogenous and their costs were not considered.

4 Game-theoretic formulation: the invariants of society

We generally utilize strategic market games to ensure a well-defined process model with specified strategies and consequences, irrespective of endowments or other factors that may constrain agents' access to them, or preferences that may bias their use. Implicit in this prioritization is a presumption that endowments are more variable, and preference models more questionable, than the observable institutional structure of a society [15, 10].

Our concern here, though, is how the stability or succession of money systems may be explained by their costs and satisfaction of needs for trade, in societies

[13] We consider a non-durable good to be a point-consumption good.

[14] We consider that a consumable is consumed at a point in time, whereas durables provide a stream of services over an interval.

otherwise constrained by productive potential, and having enough stability of preferences for basic needs of life to make different institutional structures comparable, as suggested in Fig. 1. This requires casting preferences and certain technologies that lead to endowment as primary, and the strategy spaces that mediate those as subject to variation. Such a view of society is probably not applicable to most aspects of luxury consumption or very long timescales, but we suppose it is an acceptable description of basic subsistence needs over time scales shorter than those of biological or deep social adaptation. We therefore introduce a minimal structure of agents, time, goods, production technology, and preferences, within which the rules of various institutions make domains of strategic action available, and from which the institutions themselves may be assigned utilitarian values.

4.1 Agents, time, goods, technology, preferences

We introduce a collection S of n initially indistinguishable agents called a *society*, $m+1$ primary production goods indexed $j \in 0, \ldots, m$, and time within an episode having discrete periods of length t_0. We assume the money system and rules of government are fixed for the episode, and at time $t = 0$ agents commit to livelihoods. In each period indexed $t = kt_0$ production, trade, and consumption occur. Good $j = 0$ is a durable which we think of as gold, and goods $j \in 1, \ldots, m$ are nondurable consumables (corn, beans, squash) that expire if not consumed within the period in which they are produced. Gold is not of utilitarian value in itself, but may be used as an input to the creation of some form of capital stock (tooth fillings, braclets or electronic devices, etc.), which yields services of utilitarian value in proportion to the amount of the capital stock agents hold at the beginning of each period. This modeling choice allows gold to be an ideal durable which may or may not be used as a commodity money. The combination of utilitarian value of capital stock, and the finite labor cost of replacing gold that has exited the system, provides the salvage value for gold as a commodity money.

Associated with the jth primary production good is a specialist production technology, which any agent may adopt and which yields an endowment $a^j t_0$ of that good to the agent in each period, and none of any other goods. Thus we think of the production *rate* a^j as primary, and the periodization of production, trade, and consumption as independently determinable. For notational simplicity we suppose that the nondurable consumables are measurable in some equivalent unit (bushels), and to study the symmetry properties of money systems we take all production rates $a^j = a, j \in 1, \ldots, m$ as in Ref. [10], and $a^0 \equiv e^0$ (an “extraction” rate for gold from nature). The specialist production functions are mutually exclusive (each agent can adopt at most one), and as a null model for a society without money or markets, we nominally consider a generalist production technology yielding some equal rates of production $\epsilon \ll a/m$ of each nondurable consumable, and ϵ^0 of gold.

We make the formal simplification of ruling out the individual choice for generalist (or autarkic) production in the games studied because in an advanced economy within a social structure maximal specialization always characterizes solutions to games offering both, if specialization appears at all[15].

In addition to specialist technologies for primary production, we consider a specialist technology for the production of contract enforcement, which yields some measure π per individual per period per numéraire fiat, whose meaning is made precise within the context of the games where it appears. The membership n of S represents a labor constraint, which together with the $m+1$ primary production technologies and the technology for contract enforcement define the productive potential of society. For arithmetic convenience we set $|S| = n = mr$, where r is some integer. In type-symmetric solutions, r will be a replication index for full suites of production of the nondurable consumable goods.

In all games, the first move of the extensive form, occuring once per episode, will be the adoption by each agent of a production technology (a livelihood), and we denote by ν^j number of individuals who elect to produce the jth good in that episode, and by $\bar{\nu}$ the number who specialize in contract enforcement. It is convenient to name type $j = 0$ producers *prospectors*, types $j \in 1, \ldots, m$ *farmers*, and the contract enforcers *bureaucrats*, and to distinguish all bureaucrat parameters and variables with overbars.

As in Ref. [10], all agents are taken to have identical preferences, expressed as utility functions of their allocations of the nondurable consumables at the end of each period's trade, and of the utility of capital stock held at the beginning of the period (so that it effectively delivers its services through the period)[16]. This final allocation of good j to any producer i in period t is denoted $A^j_{i,t} t_0$, and to any bureaucrat k in period t is denoted $\bar{A}^j_{k,t} t_0$. Initial endowments $a^j_{i,t} t_0$ are defined for all producers to make equations uniform, but take only values zero or $a^j t_0$ as appropriate. The capital stock held by i at the beginning of period t is denoted $C_{i,t}$, and that held by k is denoted $\bar{C}_{k,t}$.

4.2 Stock and flow variables

We are careful to distinguish *stock* from *flow* variables because, in the dimensional analysis of these systems, defining a quantity in terms of a flow carries with it scaling consequences as we take the period under which the flow is accumulated to zero (in this case, the trading day length t_0). In particular, we represent utilities as functions of *rates of consumption*[17], in which ρ_D is a utilitarian rate of discounting:

$$\mathcal{U}_i = \sum_{t=0}^{\infty} (\rho_D t_0)\, \beta^{t/t_0} \mathcal{U}_{i,t} \tag{1}$$

[15] Historically there have been hermits and communes who have opted out of the greater specialist economy for the "simpler life", but these are of minor significance.

[16] This modeling choice is not interpreted as an assertion that all individuals in a real society are interchangeable, but as a diagnostic of the ability of the markets to produce trade without structurally introducing asymmetries.

[17] One can snack all day or eat a few large meals; the criterion of being well-fed depends on the rates of intake relative to metabolic needs not set by the economy.

for producer i, and

$$\bar{\mathcal{U}}_k = \sum_{t=0}^{\infty} (\rho_D t_0)\, \beta^{t/t_0} \bar{\mathcal{U}}_{k,t} \tag{2}$$

for bureaucrat k.

$\rho_D \mathcal{U}_{i,t}$ and $\rho_D \bar{\mathcal{U}}_{k,t}$ may be thought of as *flow-valued* utility rates. As long as they are defined in terms of rates of consumption $A^j_{i,t}$, $\bar{A}^j_{k,t}$, and production $a^j_{i,t}$, and we define the period discount factor $1/\beta \equiv 1 + \rho_D t_0$, the accumulations (1,2) remain well-behaved at $t_0 \to 0$, where the combination $\sum_{t=0}^{\infty} t_0 \rho_D \beta^{t/t_0} \to \sum_{i \equiv t/t_0 = 0}^{\infty} t_0 \rho_D 1/\left(1 + \rho_D t/i\right)^i \to \int_0^\infty dt \rho_D e^{-\rho_D t} = 1$ (by the Euler formula for the exponential as $i \to \infty$ for any fixed time t). For stationary solutions the discounted utilities $\mathcal{U}_i$, $\bar{\mathcal{U}}_k$ approach the values of $\mathcal{U}_{i,t}$ or $\bar{\mathcal{U}}_{k,t}$ attained in steady state. A characteristic of commodity money is that if it is in sufficient supply, ρ_D also becomes the spot rate, while for fiat moneys the spot rate ρ is specified independently.

We choose per-period utilies in Eq. (1) in the form[18]

$$\mathcal{U}_{i,t} = \Upsilon \left\{ s \log\left(\frac{C_{i,t}\rho_D}{e^0}\right) + \sum_{j=1}^{m} \log\left(\frac{A^j_{i,t}}{a^j}\right) \right\}, \tag{3}$$

and in Eq. (2) take the form

$$\bar{\mathcal{U}}_{k,t} = \Upsilon \left\{ s \log\left(\frac{\bar{C}_{k,t}\rho_D}{e^0}\right) + \sum_{j=1}^{m} \log\left(\frac{\bar{A}^j_{k,t}}{a^j}\right) \right\}. \tag{4}$$

Υ is a dimensional tracking parameter for utility[19], whose meaning will be defined when we introduce default penalties. We reference consumption rates to production rates, and stocks $C_{i,t}$ and $\bar{C}_{k,t}$ of the durable good to amounts produced at rate e^0 within a discount horizon $1/\rho_D$.

The parameter s scales the rate of marginal utility obtained from services from capital stock relative to that from rates of consumption. When $s \to 0$ there is no salvage value to resolve the Hahn paradox for commodity money, but with fiat money the periods are decoupled and the time structure may be ignored. In general we solve for $s \neq 0$ in order to compare the costs related to discounting and spot interest rates under different money systems.

[18]This cardinalization of the Cobb-Douglas utility has the feature that the algebraic first-order conditions exactly match their dimensional estimates in continuum limits. This form was used also in the minimal models of Refs. [10].

[19]In other words, to claim that cardinal utilities may be compared for scale against some quantities (particularly bankruptcy penalties as we define them below) but not others, it is necessary to assign a scaling dimension to the comparable quantities, which we will name "util". Υ is a carrier of that dimension, such as a formal scale factor "1 util", which relates the (dimensionless) logarithms to other quantities that set scales for utility.

4.3 Trade, investment, and decay of capital stock

We impose as further invariants on all models the mechanism of one-period trade, and a specification for carry-over and decay of capital stock across periods. These will permit comparison of fiat and commodity moneys, without consideration of changes in market structure that could be invented to accompany them.

All trade uses a one-period bid-offer game [4, 10] with cash payment and no credit. Any producer i offers good j in amount $q^j_{i,t}$, and cash bids in amount $b^j_{i,t}$, at a unique trading post for good j at the beginning of trade in period t. Offers satisfy $0 \leq q^j_{i,t} \leq a^j_{i,t} t_0$, (hence zero unless i has adopted the production technology for good j). Any bureaucrat k by construction is not a primary producer, and is limited to bids $\bar{b}^j_{k,t}$. Constraints on bids depend on the money system, and are specified together with the extensive form of each game as it is defined. The aggregated bids for good j are $B^j_t \equiv \sum_i b^j_{i,t} + \sum_k \bar{b}^j_{k,t}$, and aggregated offers are $Q^j_t \equiv \sum_i q^j_{i,t}$. Bids scale with the money supply, which is a stock variable, while rates of offer scale with rates of production, which are flows. This causes prices and the velocity of money to depend on the trade period t_0, though the stationary equilibrium allocation of real goods does not.

Bids and offers are simultaneous and independent among agents within each period, and each trading post j clears immediately at price $p^j_t = B^j_t / Q^j_t$. The allocation of any good j to producer i resulting from trade in period t is then

$$A^j_{i,t} t_0 = a^j_{i,t} t_0 - q^j_{i,t} + \frac{b^j_{i,t}}{p^j_t}, \tag{5}$$

and for bureaucrat k it is

$$\bar{A}^j_{k,t} t_0 = \frac{\bar{b}^j_{k,t}}{p^j_t}. \tag{6}$$

Cash in the amount $q^j_{i,t} p^j_t$ is returned to producer i from each post j, and nothing is returned from posts to bureaucrats.

Gold can be converted to capital stock, at various times depending on the game considered. In all cases we denote by $\sigma_{i,t} t_0$ the amount converted by producer i within period t, and by $\bar{\sigma}_{k,t} t_0$ the amount converted by bureaucrat k. (Thus $\sigma_{i,t}$ and $\bar{\sigma}_{k,t}$ are rates of conversion.) Converted gold does not appear as utilitarian capital stock until the beginning of the next period, and a fraction $\Delta t_0 \leq 1$ of the beginning capital stock from the previous period also vanishes[20]. (Δ is a continuum rate of decay.) The carry-forward equation for i from these two effects is

$$C_{i,t+t_0} = (1 - \Delta t_0)\, C_{i,t} + \sigma_{i,t} t_0, \tag{7}$$

and for k is

$$\bar{C}_{k,t+t_0} = (1 - \Delta t_0)\, \bar{C}_{k,t} + \bar{\sigma}_{k,t} t_0. \tag{8}$$

[20] Our industrial exit route is a simplified proxy for diverse processes, such as theft, depreciation, or export.

Capital stock is thus measured in terms of the amount of gold required to create it[21].

5 Endogenously formed government and fiat money

5.1 Extensive form and money supply

We associate with the technology of contract enforcement some (for convenience, durable, costlessly produced) fiat money[22], which the government can distribute to and collect from agents, denominated in some unit for the fiat. Contract enforcement is represented as some method of control by which government can directly influence the utility of agents, defining a form of generalized penalty.

Specifically, we represent the power of the bureaucracy as a ratio π mapping any level of default in fiat to a disutility directly comparable to the utilities $\mathcal{U}_{i,t}$, $\bar{\mathcal{U}}_{k,t}$ of rates of consumption. The dimensionality and scale of π only become meaningful when we know the money supply that sets the scale for potential default by any agent. Alternatively, the meaning and dimensions of "utility" are operationally defined through the disutility associated with the default penalty. Government is defined by the existence of contract enforcement and spot markets for all goods (including gold) accepting bids in fiat. An identical multiperiod game structure has the following moves:

1. At the beginning of the episode, each agent chooses a production function, so that jointly they set $\left\{\nu^j, \bar{\nu}\right\}$, which are then made common knowledge. A taxation *function* of the livelihood distribution is specified as part of the rules of the game, which agents therefore also know. Its specific form is given in terms of trade below.

2. A sequence of periods, which for convenience we take to be infinite[23] are then repeated, subordinate to the labor allocation. In each period each bureaucrat is issued $\bar{\mu}t_0$ units of fiat, and each producer receives the endowment $a^j t_0$ of the appropriate good. $\bar{\mu}$ is the salary rate of bureaucratic employment, defined independent of the length of trading period.

3. Once endowments have been assigned, all agents are eligible to borrow fiat from a central bank at a prespecified rate of interest ρ, with principle and interest $\propto \rho t_0$ repayable to the bank in the last move of the period. $g_{i,t}$ denotes the amount borrowed by producer i, and $\bar{g}_{k,t}$ by bureaucrat k. (Any of $g_{i,t}$ or $\bar{g}_{k,t}$ may be negative[24], though this option is not used in the solutions found here.)

[21]In industrial uses gold may be lost, or subject to wear. If the economy is deemed to be open, i.e. set in the context of international trade, gold may flow out or in. We do not consider the latter possibility here.

[22]In fact in 2004 it cost 4.5 cents per note for \$1 and \$2 bills and 8.7 cents per note for the new color currency (Treasury website information).

[23]Practically speaking, this requires only that the largest $t \gg 1/\rho_D$.

[24]i.e., deposit instead of borrowing

4. Fiat and goods are then traded by the rules of Sec. 4.3. The constraint of no credit implies $\sum_j b^j_{i,t} \leq g_{i,t}$ for agent i and $\sum_j \bar{b}^j_{k,t} \leq \bar{\mu} t_0 + \bar{g}_{k,t}$ for k. A tax liability $\tau \sum_j b^j_{i,t}$ is assessed to agent i for the period, and $\tau \sum_j \bar{b}^j_{k,t}$ to k. τ, a sales tax fraction, will be defined functionally in terms of $\left\{ \nu^j, \bar{\nu} \right\}$ to permit agents to optimize trade within each period, and then as a function of the resulting distribution, to optimize the labor allocation in the outer move of the game.

   Trading-post disbursements are the revenue of the producers, and it is natural to regard taxes collected as the revenue of the bureaucrats, distributed equally among them. Each bureaucrat k thus has income $(\tau / \bar{\nu}) \sum_{j=0}^{m} B^j$.

5. We avoid an unnecessary inequality (always saturated in solutions to games with this preference structure and fiat money) and its associated Kuhn-Tucker multiplier, by writing into the rules of the game that all gold purchased in the markets is converted to capital stock in the same period. Thus the rates $\sigma_{i,t} = A^0_{i,t}$ and $\bar{\sigma}_{k,t} = \bar{A}^0_{k,t}$ in each period.

6. In the last move of the period, agents repay principle and interest to the central bank, and taxes to the government, and are assessed any penalties for underpayment. We have all agents return all fiat in their posessions (again to simplify the algebra by removing an unnecessary additional repayment variable and inequality constraint). Thus all fiat issued through either salaries or loans is recollected at the end of the period (though that may not be all that is owed, if agents borrow and spend into default). The total liability of a producer i is $g_{i,t} (1 + \rho t_0) + \tau \sum_{j=0}^{m} b^j_{i,t}$, and for a bureaucrat k it is $\bar{g}_{k,t} (1 + \rho t_0) + \tau \sum_{j=0}^{m} \bar{b}^j_{k,t}$.

   Producers collectively recover all principle they borrow through the trading posts, while bureaucrats collectively recover all taxes they owe through taxation revenue. The balance between bureaucrat spending into the private sector to cover its taxes and interest, and the flow back of tax revenue to cover bureaucrat principle, determines the labor equilibrium between the private and public sectors. Ultimately it is possible to find strategies in which each individual's principle liability is covered by market revenue at any level of borrowing, so that only the net flow from government salaries to interest (both proportional to t_0 in any period) determines the money supply.

This game is a straightforward process-oriented extension of the way one might incorporate contract enforcement into a General Equilibrium model, as an output of production whose input is labor. Provision of the money supply, like primary production of durable and nondurable goods, consumes the labor of an endogenously optimized fraction of the population.

In stationary solutions, carry-forward of capital stock reduces its first-order conditions to effectively independent decisions in the variables $\sigma_{i,t}$ and $\bar{\sigma}_{k,t}$, which appear simply as additional consumables. In the limit $s \rightarrow 0$ gold production is not adopted, permutation becomes a symmetry of all remaining goods and periods of trade decouple, so the trade efficiency of the society can be compared to the minimal models in a one-period taxonomy without government. Alternatively, at $s \neq 0$, utilitarian capital stock provides the salvage value to resolve the Hahn paradox for gold, and the origins and magnitudes of cost of a money system may be compared for commodity money and government fiat.

5.2 Budget constraint, default, and enforcement

The budget constraint on trade can be enforced by adding respectively to $\mathcal{U}_{i,t}$ and $\bar{\mathcal{U}}_{k,t}$ Kuhn-Tucker terms

$$\mathcal{U}_{i,t} \to \mathcal{U}_{i,t} + \lambda_{i,t} \left(g_{i,t} - \sum_{j=0}^{m} b_{i,t}^{j} \right), \tag{9}$$

$$\bar{\mathcal{U}}_{k,t} \to \bar{\mathcal{U}}_{k,t} + \bar{\lambda}_{k,t} \left(\bar{\mu} t_0 + \bar{g}_{k,t} - \sum_{j=0}^{m} \bar{b}_{k,t}^{j} \right), \tag{10}$$

where the multipliers $\lambda_{i,t}$ and $\bar{\lambda}_{k,t}$ run from $[0, \infty)$, and are varied to minimize $\mathcal{U}_{i,t}$ and $\bar{\mathcal{U}}_{k,t}$ by an imaginary adversary (the adversary has the interpretation of a social norm such as cash payment enforcing the constraint, which we need not model as an embodied institution or source of excess cost).

Producer i's fiat at the end of borrowing and trading is

$$M_{i,t} = g_{i,t} - \sum_{j=0}^{m} b_{i,t}^{j} + \sum_{j=0}^{m} q_{i,t}^{j} p_{t}^{j}, \tag{11}$$

while bureaucrat k's is

$$\bar{M}_{k,t} = \bar{\mu} t_0 + \bar{g}_{k,t} - \sum_{j=0}^{m} \bar{b}_{k,t}^{j} + \frac{\tau}{\bar{\nu}} \left(\sum_{j=0}^{m} B_{t}^{j} \right). \tag{12}$$

The money supply $\mathcal{M}_t$ over the period is the sum of salaries and borrowings

$$\mathcal{M}_t = \sum_{k} \left(\bar{\mu} t_0 + \bar{g}_{k,t} \right) + \sum_{i} g_{i,t}. \tag{13}$$

The liability for repayment of fiat at the end of the period is enforced by adding to each utility a penalty function proportional to the unpaid debt and to some intensity of enforcement. For producers, this may be written

$$\mathcal{U}_{i,t} \to \mathcal{U}_{i,t} + \Pi \min \left[\left(M_{i,t} - g_{i,t} \left(1 + \rho t_0 \right) - \tau \sum_{j=0}^{m} b_{i,t}^{j} \right), 0 \right], \tag{14}$$

where Π defines a linear map from default to utility, like the technology parameter π. The corresponding relation for bureaucrats is

$$\bar{\mathcal{U}}_{k,t} \to \bar{\mathcal{U}}_{k,t} + \Pi \min \left[\left(\bar{M}_{k,t} - \bar{g}_{k,t} \left(1 + \rho t_0\right) - \tau \sum_{j=0}^{m} \bar{b}^j_{k,t} \right), 0 \right]. \qquad (15)$$

We represent the labor limitation of the bureaucracy by specifying, as a rule of the game, that the penalty Π is defined in terms of the expected equilibrium money supply, the size of the bureaucracy, and the technology parameter π, by

$$\mathcal{M}_{\mathrm{eq}} \Pi \equiv \frac{\pi \bar{\mu} \bar{\nu}}{\rho_D}. \qquad (16)$$

The total expected default possible in the society, $\mathcal{M}_{\mathrm{eq}}\Pi$, can only be penalized at a rate proportional to the number $\bar{\nu}$ of bureaucrats, and we set the meaning of the technology parameter π in terms of the salary rate $\bar{\mu}$. ρ_D is an arbitrary scale factor allowing Π and π to have the same dimensions, and assigning physical meaning to penalties in terms of the discount horizon.

As in Ref. [10], we regularize the discontinuous derivative of the min functions with Kuhn-Tucker multipliers $\eta_{i,t}$ and $\bar{\eta}_{k,t}$ respectively for specialists and bureaucrats, both in the range $[0, \Pi]$, and varied by an adversary to minimize the constrained utilities. The initial (society) utilities of consumption, modified by the constraints of cash payment and acceptance of the debt penalties, define the utilities for agents as economic actors. With some algebraic condensation to remove strategy-independent cancellations, these take the forms respectively for producers and bureaucrats:

$$\mathcal{U}_{i,t} \to \mathcal{U}_{i,t} + \left(\lambda_{i,t} + \eta_{i,t}\right) \left[g_{i,t} - \sum_{j=0}^{m} b^j_{i,t} \right] + \eta_{i,t} \left[\sum_{j=0}^{m} q^j_{i,t} p^j_t - g_{i,t} \left(1 + \rho t_0\right) - \tau \sum_{j=0}^{m} b^j_{i,t} \right] \equiv \mathcal{U}^{\mathrm{econ}}_{i,t}, \qquad (17)$$

$$\bar{\mathcal{U}}_{k,t} \to \bar{\mathcal{U}}_{k,t} + \left(\bar{\lambda}_{k,t} + \bar{\eta}_{k,t}\right) \left[\bar{\mu} t_0 + \bar{g}_{k,t} - \sum_{j=0}^{m} \bar{b}^j_{k,t} \right] + \bar{\eta}_{k,t} \left[\frac{\tau}{\bar{\nu}} \sum_{j=0}^{m} B^j_t - \bar{g}_{k,t} \left(1 + \rho t_0\right) - \tau \sum_{j=0}^{m} \bar{b}^j_{k,t} \right] \equiv \bar{\mathcal{U}}^{\mathrm{econ}}_{k,t}. \qquad (18)$$

5.3 Dimensional analysis

A complete listing of all variables and parameters is given as App. B, and the NE for trade in the markets are derived in App. A, subordinate to the rules of the buy/sell subgame, at general values of the parameters $\bar{\mu}$, τ, Π. The utilitarian outcome of trade, as a function of these parameters, is then the input to the labor-allocation decisions in the outer game. The existence and forms of interior solutions for these two nested games will determine which tax rules permit the size of government to be optimized endogenously.

We notice, however, that in the scaling limit of continuum trading, t_0 ceases to be a controlling parameter of the model, in which case there is a small subset of parameters Υ, π, $\bar{\mu}$, and ρ_D, whose dimensions must determine those in the possible forms of solutions. There are additionally the numbers n of agents and m of consumable goods, which become scale factors for permutation symmetries[25].

The combination of scale independence and a limited number of dimension-carrying factors responsible for a set of phenomena opens the possibility of predicting the sizes of observables that also carry dimension on the basis of their dimensional content alone. The method, called *dimensional analysis*, is often the first form of estimation of complex problems in the natural sciences, because direct algebraic solutions are often not known, and one reliable constraint is that all equations must be homogeneous in dimension. The notion of dimension is in some respects similar to that of fungibility, in that it is defined by criteria of subsitutability. Dimensionality differs from fungibility in that the subsitution relations are limited to those that do not depend on the scale of the inputs or outputs; thus dimensionality entails a scaling invariance at least in the measurement of the system, if not in its dynamics directly. There is no limit a priori to the number of dimensions that may structure a problem, but whether any exist at all does depend on the set of scale-invariant homogeneity requirements that can be imposed on equations. Thus in this problem utility, money, time, and various types of goods have naturally assigned dimensions, and the requirement of homogeneity in the algebraic solutions provides constraints on the scales of the money supply or allocation variables, more general than the algebraic solutions associated with particular choices of utility.

Using dimensional analysis, we predict here the scaling behavior and order of magnitude of all relevant quantities, as a precursor to algebraic solution. For simplicity we take $s \to 0$ and omit gold from the goods space, as s is an additional nondimensional parameter that can correct any dimensional prediction. Several limits in which gold behaves like an ordinary good are readily recovered in the exact solutions. App. A verifies the dimension-driven predictions, and adds corrections from dimensionless factors ρt_0, ρ_D / Δ, etc., which vanish in continuum limits and at $s \to 0$. A feature of the minimal models we have introduced is that they require no additional dimension-carrying parameters, with the consequence that the algebraic model solutions are exactly those of the dimensional analysis.

5.3.1 Dimensions

We use square brackets around a variable to denote the name of its dimension. For brevity we call the unit of time "days", and the numéraire of fiat "dollars". In this notation $[\Upsilon] \equiv$ util[26].

[25] In scaling limits $n \to \infty$, $m \to \infty$, we could have created a more elaborate dimensioning in which n and m have units as well, but for this model the simpler dimensioning system and use of symmetry will accomplish the same ends.

[26] An impediment to the use of dimensional analysis in neoclassical economics is an insistence that utility be unmeasurable except as an ordinal relation. Any cardinal form, however, operationally defines units for utility in terms of measurable actions, and it is necessary to accept this and understand its consequences for scaling.

Salaries have dimension $[\bar{\mu}] \equiv$ dollar/day, which we will expand slightly to say $[\bar{\mu}] \equiv$ dollar/ (day · bureaucrat), treating the size of the bureaucracy as a scaling variable. All pure rates in the problem have dimension $[\rho] = [\rho_D] = [\Delta] \equiv 1/\text{day}$. We may distinguish ρ as defined by the rules of the game, from ρ_D, a property of agent preferences, when asking which controls various quantities.

The penalties have dimension $[\Pi] = [\pi] \equiv$ util/dollar. As Π is a derived quantity, π is the dimensional controlling parameter.

5.3.2 Scaling predictions

We can now anticipate the scaling behavior of the two principle derived quantities in this model: the money supply and the size of the bureaucracy. Dimensionally $[\mathcal{M}] \equiv$ dollar, and it must be determined from the parameters specifying the game $\bar{\mu}$, π and the parameters of the macroeconomy $\bar{\nu}$, ρ. The only combination of these quantities with correct dimensions is

$$[\mathcal{M}] = \left[\frac{\bar{\mu}\bar{\nu}}{\rho}\right] = \left(\frac{\text{dollar}}{\text{day} \cdot \text{bureaucrat}}\right) (\text{bureaucrats}) \left(\frac{1}{1/\text{day}}\right) \tag{19}$$

On grounds of dimensional constraint we then expect the money supply to scale as this combination, possibly by proportionality factors of order unity[27]. This relation is denoted $\mathcal{M} \sim \bar{\mu}\bar{\nu}/\rho$. We note as a corollary that the velocity of money, which is a rate and hence does depend on the trading period length (and not on the discount horizon), should scale as $\mathcal{M} \sim \bar{\mu}\bar{\nu}/\rho t_0$. From the scaling of $\mathcal{M}$ and Eq. (16) we obtain the scaling of Π, the micro-penalty variable, in terms of π, the macroscopically specified technology parameter:

$$\Pi \sim \pi \frac{\rho}{\rho_D}. \tag{20}$$

As all agents and all goods are symmetric *a priori* with respect to permutation, the scale for any bid determined by the rules of the game should be

$$b \approx \frac{\mathcal{M}}{nm} \sim \frac{\bar{\mu}\bar{\nu}}{mn\rho}. \tag{21}$$

There is also a scaling for bids predicted from the dimensional content of preferences, specifically the penalty and discount horizon,

$$b \sim \frac{\Upsilon}{\Pi} \approx \frac{\Upsilon \rho_D}{\pi\rho}. \tag{22}$$

At equilibria the microscopic penalties define the macroscopic rates of borrowing and the global distribution of goods. Setting equal Eq. (21) and Eq. (22) we

[27]For example, algebraic relations such as first-order conditions potentially introduce factors of 2, and differential equations frequently produce factors of Euler's constant or pi (the constant), which appear as prefactors in scaling relations. While these are not predicted on dimensional grounds by the scaling relations themselves, experience has shown that for structurally simple equations they do not deviate by orders of magnitude from unity. Therefore the scaling relations frequently also provide good predictions of quantity, to order of magnitude.

obtain the expected scaling for the fraction of society in the bureaucracy

$$\frac{\bar{\nu}}{n} \sim \frac{\Upsilon \rho_D}{\pi \bar{\mu}} m. \tag{23}$$

The analysis we used to obtain Eq. (23) is more complex than needed, and strictly should be corrected by arbitrary functions of the dimensionless ratio ρ/ρ_D, by the assumptions we have stated. It could have been obtained directly by recalling that a defining feature of fiat is its intrinsic valuelessness, outside the scope of its rules, and the proposition of this model that the only purpose of the rules is to facilitate as nearly optimal trade as possible. Thus, the scale of the money supply is irrelevant to allocation of consumables, and the macroeconomic parameter that controls it (ρ) is irrelevant as a scale-controlling variable. Then Eq. (23) follows uniquely on dimensional grounds, up to the factor of m, which we have not treated as a scaling variable with assigned dimensions.

Finally, we note the operational definition of the penalty technology and cardinal utility. The combination

$$[\pi \bar{\mu}] = \left(\frac{\text{util}}{\text{dollar}} \right) \left(\frac{\text{dollars}}{\text{day} \cdot \text{bureaucrat}} \right) = \left(\frac{\text{util}}{\text{day}} \right) / \left(\text{bureaucrat} \right). \tag{24}$$

Default penalties are only meaningful in combination with the numéraire, and we may think of Eq. (24) as the *disutility rate* associated (for example) with living in debtor's prison versus living free[28]. Then as $\Upsilon \rho_D$ is a utility rate per agent associated with ongoing rates of consumption, $\pi \bar{\mu} / \Upsilon \rho_D$ has the interpretation of a number of individuals who can be imprisoned per bureaucrat enforcing the default laws, where the value measure of prison time per individual is expressed in terms of the relative utilities of rates of consumption. The combination $\pi \bar{\mu} \bar{\nu} / \Upsilon \rho_D$, a number of people the courts can maintain in debtor's prison at any time, is a stock variable, even though the utilities and disutilities associated with the imprisoned states are properly defined in terms of utility rates.

6 Concluding remarks

6.1 Institutions, self-policing and Pareto optimality

The default and bankruptcy penalites of a society are variables in the long run, but they are parameters from the viewpoint of enforcement. In an economy with strategic agents and credit arrangements default may be an optimizing choice if penalties are not sufficient. Without exogeneous uncertainty there will always be a minimal default penalty that is sufficient to prevent strategic default[16]. The enforcment of the rules concerning default come at a cost. A reasonable way to consider the cost is that it involves primarily labor cost such as those of judges, lawyers, accountants, bookkeepers and possibly police and

[28] Thus the rate-valued nature of the disutility of debtor's prison is a consequence of the rate-valued nature of living, and is properly comparable to other utilities of rates of consumption.

others. The government bureaucracy has a production function which produces enforcement. Given this institutional structure it is possible to define Pareto optimality conventionally.

6.2 Coalitions, time and institutions

Much of economic and financial life involves decisions by groups of fiduciaries running other peoples' money. In formal cooperative game theory a coalition is not an institution. It is an abstraction where communication and coordination is free. Here some stucture of government has been implicitly provided. An individual who chooses to become a bureaucrat is assumed to enter into an enforcement production structure that is implicitly given; the corporate coodination problem is assumed to have been solved. The output, however depends on the number of bureaucrats recruited. In our analysis we have assumed that enforcement depends in a linear manner on the number of bureaucrats. The actual shape of the enforcement calls for empirical investigation and may differ from country to country.

6.3 Finance and inequalities

In our investigation we have been concerned with both interior and boundary solutions. A key feature in finance as contrasted with general equilibrium theory is that cash flow constraints and credit matter. When there is not enough money or credit, the cash flow constraints become binding.

6.4 The control problem

In essence, we suggest that the microeconomic abstraction of general equilibrium, which served as an excellent platform from which to study the static existence of an efficient price system for nonstrategic agents is not adequate to include the possibility of strategic agents. When strategic agents are considered the natural structure that supplants the n-person general equilibrium model is the $(n+1)$-person strategic market game, where the $n+1$st player is the government. It is large relative to the others. We attempt to provide this enlarged model with minimal financial institutions. In doing so we observe that the introduction of taxes, the rate of interest, government salaries, borrowing and contract enforcement appear naturally as control variables. Thus the embedding of a market system with strategic agents in a larger model which provides enforcement and the possibility for coordination at a cost presents a natural link between the microeconomic concerns with markets and price and the macroeconomic concerns with money, coordination and control.

In macroeconomic analysis a distinction is often made between fiscal or monetary policy. For may practical purposes the distinction is reasonably clear, but if the cost of administration is significant and enforcement may be regarded as a public good, pure monetary policy involves a fiscal component.

6.5 Gross checks of the scaling estimates

Our models are chosen for minimality, not directly for quantitative prediction in any particular economy. A check of the dimensional estimates of our primary quantities, however, shows both that they are real and relevant observables in the characterization of government and the macroeconomy, and that the dimensionally induced estimates can be reasonable to order of magnitude.

Most immediately we consider Eq. (19) for the money supply. The size of the federal bureaucracy in 2003 was $\bar{\nu} \approx 2.7$ million employees (SAUS 2003). Supposing a bureaucrat's salary to average $\bar{\mu} \approx \$40,000$ per year, and an average monetary rate of interest to be $\rho \approx 0.06$ per year, gives an estimated money supply of $\mathcal{M} \approx \$1.8$ trillion. Estimated M1 in 2003 was \$1.1 trillion, for comparison. From Eq. (21) we may consider bm any individual's spending constraint at any time, as a share of the national limitation in the money supply (irrespective of how many effectively symmetric goods m it buys). For a population of 288 million (SAUS 2003), and continuing within the dimensional estimate for $\mathcal{M}$, we obtain $bm \approx \$6,250$, comparable to consumer credit limits that for most Americans define their primary spending constraint.

The size of the bureaucracy itself is based on a more artificial construction (23) mediated by utility, but still one that can be given a pragmatic interpretation. The combination $\Upsilon\rho_D$ characterizes the change in the "rate" of utility associated with changes in consumption by fractions[29] $\sim 1/3$, while living otherwise free. In the spirit of the stylized model of government-imposed disutility as confinement in debtor's prison, if we consider a reduction by 1/3 in all m goods as characterizing the transition to starvation, we may as well set $\Upsilon\rho_D m$ as the disutility of consumption at which agents have nothing to lose by being imprisoned. Then a pragmatic interpretaion of $\pi\bar{\mu}\bar{\nu}$ as the total disutility imposable by the bureaucracy is that $\pi\bar{\mu}/\Upsilon\rho_D m$ is the number of people the bureaucracy can maintain in prison per bureaucrat employed. With a total US bureaucracy (federal + state) ≈ 20.7 million, and a prison population ≈ 1.3 million, the ratio $\Upsilon\rho_D m/\pi\bar{\mu} \approx 16$. While neither the bureaucracy nor the prison population is primarily concerned with default in these statistics, they attach a number to the technology parameter which serves to define what one means by it. By Eq. (23) we then estimate $\bar{\nu}/n \approx 16$, a bureaucracy 16 times the size of the society it serves.

While colorful metaphor, this illustrates the failure of scaling estimates, in a way that can be traced explicitly to the assumption in Eq. (16) that the bureaucracy stands ready to enforce penalties on default of the entire money supply. Since at interior solutions none of the money supply is in default, the actual requirement for enforcement is something more like

$$\frac{\pi\bar{\mu}\bar{\nu}}{\rho_D} \to f\mathcal{M}_{\mathrm{eq}}\Pi, \tag{25}$$

where $0 < f \ll 1$ is the fraction the bureaucracy can enforce. With Eq. (25),

[29] taking 3 to approximate Euler's constant $e \approx 2.718$ for which ratio any logarithm in the utility changes by unity

Eq. (23) becomes

$$\frac{\bar{\nu}}{n} \sim f \frac{\Upsilon \rho_D m}{\pi \bar{\mu}}. \tag{26}$$

Taking actual $\bar{\nu}/n \approx 2.7\,\text{million}/288\,\text{million} \approx 0.01$ (federal bureaucracy per population), we generate a value for $f \approx 1/1600$.

6.6 Open problems

The scaling relations reified in this model provide no way to estimate a dimensionless fraction like f, and the main augmentation needed to estimate f meaningfully is uncertainty leading to non-strategic default. Then f would arise from the balance between discouraging strategic default and the support of trade by moderating penalties on accident, in which legislation and lawyers enter as essential new players. We leave this as an open problem at this time.

Two other problems are considered in a companion second essay. We investigate an economy that has a sufficiency of gold to use as its money. We observe that if a loan market is not needed the Hahn Paradox is not encountered. We consider the concept of efficiency under these circumstances.

Our last problem is to consider if it is feasible to construct a money-metric to measure the relative efficiencies of different monetary systems.

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A Spot markets for gold and nondurable consumables

Table 1 in App. B provides an index to the labor allocation and other variables of the model, used in this appendix.

The variation in the allocation of good j to producer i resulting from the trade rules of Equations (5,6) is

$$\delta A^j_{i,t} t_0 = -\left(\delta q^j_{i,t} - \frac{\delta b^j_{i,t}}{p^j_t}\right)\left(1 - \frac{b^j_{i,t}}{B^j_t}\right), \tag{27}$$

while for bureaucrat k it is

$$\delta \bar{A}^j_{k,t} t_0 = \frac{\delta \bar{b}^j_{k,t}}{p^j_t}\left(1 - \frac{\bar{b}^j_{k,t}}{B^j_t}\right). \tag{28}$$

The variation of i's degree of default in fiat is

$$\delta\left[\sum_{j=0}^{m}\left(p_t^j q_{i,t}^j - \tau b_{i,t}^j\right) - g_{i,t}\left(1+\rho t_0\right)\right] = \sum_{j=0}^{m}\left[p_t^j \delta q_{i,t}^j\left(1-\frac{q_{i,t}^j}{Q_t^j}\right) - \delta b_{i,t}^j\left(\tau - \frac{q_{i,t}^j}{Q_t^j}\right) - \delta g_{i,t}\rho t_0\right]. \tag{29}$$

The variation for k is slightly simpler:

$$\delta\left[\tau\sum_{j=0}^{m}\left(\frac{B_t^j}{\bar{\nu}} - \bar{b}_{k,t}^j\right) - \bar{g}_{k,t}\left(1+\rho t_0\right)\right] = -\left[\tau\sum_{j=0}^{m}\delta\bar{b}_{k,t}^j\left(1-\frac{1}{\bar{\nu}}\right) + \delta\bar{g}_{k,t}\left(1+\rho t_0\right)\right]. \tag{30}$$

Applying the variations (27,29) to Eq. (17), and using $\sigma_{i,t} = A_{i,t}^0$, gives

$$\begin{aligned}
\delta\mathcal{U}_{i,\text{econ}} = \\
&\sum_{t=0}^{\infty}\left(\rho_D t_0\right)\beta^{t/t_0}\left\{\left[\Upsilon s\sum_{t'=t+t_0}^{\infty}\frac{\beta^{\frac{t'-t}{t_0}}\left(1-\Delta t_0\right)^{\frac{t'-t-t_0}{t_0}}}{p_t^0 C_{i,t'}}\left(1-\frac{b_{i,t}^0}{B_t^0}\right) - \eta_{i,t}\left(1-\frac{q_{i,t}^0}{Q_t^0}\right)\right]\left(\delta b_{i,t}^0 - p_t^0\delta q_{i,t}^0\right)\right. \\
&+\sum_{j=1}^{m}\left[\frac{\Upsilon}{p_t^j A_{i,t}^j t_0}\left(1-\frac{b_{i,t}^j}{B_t^j}\right) - \eta_{i,t}\left(1-\frac{q_{i,t}^j}{Q_t^j}\right)\right]\left(\delta b_{i,t}^j - p_t^j\delta q_{i,t}^j\right) \\
&-\left(\lambda_{i,t}+\tau\eta_{i,t}\right)\sum_{j=0}^{m}\delta b_{i,t}^j + \left(\lambda_{i,t}-\rho t_0\eta_{i,t}\right)\delta g_{i,t} \\
&\left.+\delta\lambda_{i,t}\left(g_{i,t}-\sum_{j=0}^{m}b_{i,t}^j\right) + \delta\eta_{i,t}\left[\sum_{j=0}^{m}\left(q_{i,t}^j p_t^j - b_{i,t}^j\left(1+\tau\right)\right) - g_{i,t}\rho t_0\right]\right\}.
\end{aligned} \tag{31}$$

Applying variations (28,30) to Eq. (18), and using $\bar{\sigma}_{k,t} = \bar{A}_{k,t}^0$, gives

$$\begin{aligned}
\delta\bar{\mathcal{U}}_{k,\text{econ}} = \\
&\sum_{t=0}^{\infty}\left(\rho_D t_0\right)\beta^{t/t_0}\left\{\left[\Upsilon s\sum_{t'=t+t_0}^{\infty}\frac{\beta^{\frac{t'-t}{t_0}}\left(1-\Delta t_0\right)^{\frac{t'-t-t_0}{t_0}}}{p_t^0 \bar{C}_{k,t'}}\left(1-\frac{\bar{b}_{k,t}^0}{B_t^0}\right) - \bar{\lambda}_{k,t} - \bar{\eta}_{k,t}\left(1+\tau\left(1-\frac{1}{\bar{\nu}}\right)\right)\right]\delta\bar{b}_{k,t}^0\right. \\
&+\sum_{j=1}^{m}\left[\frac{\Upsilon}{p_t^j \bar{A}_{k,t}^j t_0}\left(1-\frac{\bar{b}_{k,t}^j}{B_t^j}\right) - \bar{\lambda}_{k,t} - \bar{\eta}_{k,t}\left(1+\tau\left(1-\frac{1}{\bar{\nu}}\right)\right)\right]\delta\bar{b}_j^{k,t} + \left(\bar{\lambda}_{k,t}-\rho t_0\bar{\eta}_{k,t}\right)\delta\bar{g}_{k,t} \\
&\left.+\delta\bar{\lambda}_{k,t}\left(\bar{\mu}t_0+\bar{g}_{k,t}-\sum_{j=0}^{m}\bar{b}_{k,t}^j\right) + \delta\bar{\eta}_{k,t}\left[\bar{\mu}t_0+\sum_{j=0}^{m}\left(\frac{\tau}{\bar{\nu}}B_t^j - \left(1+\tau\right)\bar{b}_{k,t}^j\right) - \bar{g}_{k,t}\rho t_0\right]\right\}.
\end{aligned} \tag{32}$$

The variations in $g_{i,t}$, $\bar{g}_{k,t}$, $\lambda_{i,t}$, and $\bar{\lambda}_{k,t}$ are on unbounded intervals, so their coefficients must vanish. Hence the bidding constraint is always tight ($g_{i,t} = \sum_{j=0}^{m} b_{i,t}^j$, $\bar{\mu}t_0 + \bar{g}_{k,t} = \sum_{j=0}^{m}\bar{b}_{k,t}^j$), and we may also set $\lambda_{i,t} = \rho\eta_{i,t}$, $\bar{\lambda}_{k,t} = \rho\bar{\eta}_{k,t}$ in what follows. As $\eta_{i,t}$ and therefore $\lambda_{i,t}$ are nonzero, it is impossible for the

$\delta b^j_{i,t}$ and $\delta q^j_{i,t}$ variations both to cancel in Eq. (31). Hence one of $b^j_{i,t}$ and $q^j_{i,t}$ takes the boundary condition 0 for any good j, with the interpretation that there is no wash selling in the private sector. Borrowed fiat is always better used to bid on non-endowed goods, than to engage in wash selling, at a finite rate of interest.

We look only for stationary solutions, symmetric under permutation of agents of a given type, and under permutations of the types that produce non-durable consumables. A reduced notation for offers is $q^0_{i,t} \equiv q^\sigma$ for prospectors, $q^j_{i,t} \equiv q$ for farmer i of good j, and zero otherwise. Bids $b^j_{i,t} \equiv b_0$ from prospectors on goods $j \in 1, \ldots, m$ and zero otherwise, $b^0_{i,t} \equiv b^\sigma$ from farmers on gold, and $b^j_{i,t} \equiv b$ for farmer i on goods *other than* $j \in 1, \ldots, m$, and zero otherwise. Bids from bureaucrats are $\bar{b}^0_{k,t} \equiv \bar{b}^\sigma$ on gold, and $\bar{b}^j_{k,t} \equiv \bar{b}$ on all nondurable goods. The K-T multiplier for default by prospectors is denoted $\eta_{i,t} = \eta_0$, for farmers it is $\eta_{i,t} = \eta$, and for bureaucrats $\bar{\eta}_{k,t} = \bar{\eta}$. Finally, we denote $\bar{g}_{k,t} \equiv \bar{g}$ for stationary prospector borrowing, $g_{i,t} \equiv g_0$ for i a prospector, or $g_{i,t} \equiv g$ for i a farmer.

Type symmetry presumes equal numbers of each type of farmer, which we denote $\nu^j_t = \hat{r}, j \in 1, \ldots, m$, and we further introduce notations $\nu^0_t \equiv mr^0$, $\bar{\nu}_t \equiv m\bar{r}$, so that r^0 and $\bar{r}$ are respectively the number of agents from each replica who become prospectors and bureaucrats, and $r^0 + \hat{r} + \bar{r} \equiv r$. Total bids on gold are then denoted $B^0_t \equiv B^\sigma = m\hat{r}b^\sigma + m\bar{r}\bar{b}^\sigma$, and on nondurables $B^j_t \equiv B = mr^0 b_0 + (m-1)\,\hat{r}b + m\bar{r}\bar{b}$. Offers of gold are $Q^0_t \equiv Q^\sigma = mr^0 q^\sigma$, and of nondurables are $Q^j_t \equiv Q = \hat{r}q$. The price of gold is then

$$p^0_t \equiv p^\sigma = \frac{B^\sigma}{Q^\sigma} = \frac{\hat{r}}{r^0}\frac{b^\sigma}{q^\sigma} + \frac{\bar{r}}{r^0}\frac{\bar{b}^\sigma}{q^\sigma}. \tag{33}$$

The price of the nondurables $j \in 1, \ldots, m$ is

$$p^j_t \equiv p = \frac{B}{Q} = (m-1)\,\frac{b}{q} + m\frac{r^0}{\hat{r}}\frac{b_0}{q} + m\frac{\bar{r}}{\hat{r}}\frac{\bar{b}}{q}. \tag{34}$$

The final allocations of gold are denoted as the investments in capital stock they produce: to prospectors $A^0_{i,t} = e^0 - q^\sigma \equiv \sigma_0$, to farmers $A^0_{i,t} = b^\sigma/p^\sigma \equiv \sigma$, and to bureaucrats $\bar{A}^0_{k,t} = \bar{b}^\sigma/p^\sigma \equiv \bar{\sigma}$. In stationary solutions the capital stock itself has the value $C_{i,t} = \sigma_0/\Delta$ for prospectors, $C_{i,t} = \sigma/\Delta$ for farmers, and $\bar{C}_{k,t} = \bar{\sigma}/\Delta$ for bureaucrats. The variations in utilities (31,32) involving $C_{i,t}$ and $\bar{C}_{k,t}$ then reduce to forms equivalent to those from Cobb-Douglas preferences, in which the exponent for all nondurable goods is $\alpha^j = 1, j \in 1, \ldots, m$, and for investment in gold is

$$\alpha^0 = \frac{s\beta\Delta t_0}{1 - \beta\,(1 - \Delta t_0)} = \frac{s\Delta}{\rho_D + \Delta}. \tag{35}$$

In type-symmetric solutions, we introduce a condensed notation $A_0, A_\parallel, A_\perp, \bar{A}$ for the final allocation rates to different agents in their own or other types' consumable goods. Final allocations of nondurable goods to prospectors are denoted

$$A_i^j t_0 \equiv A_0 t_0 = \frac{b_0}{p}, \tag{36}$$

to farmer i of good j are

$$A_i^j t_0 \equiv A_\parallel t_0 = a t_0 - q, \tag{37}$$

and for farmers i of all types $\neq j$

$$A_i^j t_0 \equiv A_\perp t_0 = \frac{b}{p}. \tag{38}$$

Final allocation to all bureaucrats in nondurable goods is

$$\bar{A}_k^j t_0 \equiv \bar{A} t_0 = \frac{\bar{b}}{p}. \tag{39}$$

A consequence of finite replication index r (together with finitely many goods $m+1$) is that variations in agent bids and offers impact the prices at which the goods on which they are bidding clear. Such "price impact factors" represent finite-economy hoarding effects, and cause the NE of strategic market games to differ from the CE at equivalent preferences, even in the absence of other inefficiencies like nonzero spot interest rate. These are the terms $1-q_{i,t}^j/Q_t^j$ affecting default, and $1-b_{i,t}^j/B_t^j$ and $1-\bar{b}_{k,t}^j/B_t^j$ affecting consumption in Eq. (31,32). In the interest of space we suppress these terms by considering the infinite-replica limit $r \to \infty$ (interior solutions have fixed r^0/r and $\bar{r}/r$ in this limit), in which they differ from unity by terms $\mathcal{O}(1/r)$ or smaller. The same limit removes the term $\bar{\eta}_{k,t}/\bar{\nu}_t$ from Eq. (32) induced by the subsidy.

With these simplifications, the $q_{i,t}^0$ variation in Eq. (31) from prospectors requires

$$\frac{\Upsilon \alpha^0}{p^\sigma \sigma_0 t_0} = \frac{\Upsilon \alpha^0}{p^\sigma \left(e^0 t_0 - q^\sigma\right)} = \eta_0. \tag{40}$$

Their $b_{i,t}^j$ variations on $j \neq 0$ require

$$\frac{\Upsilon}{p A_0 t_0} = \frac{\Upsilon}{b_0} = \eta_0 \left(1 + \tau + \rho t_0\right). \tag{41}$$

The $b_{i,t}^0$ variation from farmers gives

$$\frac{\Upsilon \alpha^0}{p^\sigma \sigma t_0} = \frac{\Upsilon \alpha^0}{b^\sigma} = \eta \left(1 + \tau + \rho t_0\right). \tag{42}$$

Their $q_{i,t}^j$ variation requires

$$\frac{\Upsilon}{p A_\parallel t_0} = \frac{\Upsilon}{p \left(a t_0 - q\right)} = \eta, \tag{43}$$

and their $b_{i,t}^{j}$ variation on their non-endowed nondurable goods requires

$$\frac{\Upsilon}{pA_{\perp}t_0} = \frac{\Upsilon}{b} = \eta\left(1+\tau+\rho t_0\right). \tag{44}$$

The $\bar{b}_{k,t}^{0}$ variation in Eq. (32) from bureaucrats gives

$$\frac{\Upsilon\alpha^0}{p^{\sigma}\bar{\sigma}t_0} = \frac{\Upsilon\alpha^0}{\bar{b}^{\sigma}} = \bar{\eta}\left(1+\tau+\rho t_0\right), \tag{45}$$

while their $\bar{b}_{k,t}^{j}$ variations on $j \neq 0$ gives

$$\frac{\Upsilon}{p\bar{A}t_0} = \frac{\Upsilon}{\bar{b}} = \bar{\eta}\left(1+\tau+\rho t_0\right). \tag{46}$$

A.1 Interior solutions

We consider here solutions in which the default constraint is tight but not violated. These are the solutions in the well-functioning economy where the laws support trade without requiring agents to strategically violate them.

The one-period utilities of Eq. (3) for prospectors at $r \to \infty$ stationary solutions, in the reduced notation, are

$$\mathcal{U}_{i,\mathrm{econ}}/\Upsilon \to s\log\left(\frac{\sigma_0\rho_D}{e^0\Delta}\right) + m\log\left(\frac{A_0}{a}\right), \tag{47}$$

while for farmers they are

$$\mathcal{U}_{i,\mathrm{econ}}/\Upsilon \to s\log\left(\frac{\sigma\rho_D}{e^0\Delta}\right) + \log\left(\frac{A_{\parallel}}{a}\right) + (m-1)\log\left(\frac{A_{\perp}}{a}\right). \tag{48}$$

The one-period utilities for bureaucrats in Eq. (4) are

$$\bar{\mathcal{U}}_{k,\mathrm{econ}}/\Upsilon \to s\log\left(\frac{\bar{\sigma}\rho_D}{e^0\Delta}\right) + m\log\left(\frac{\bar{A}}{a}\right). \tag{49}$$

The condition for an interior solution is that in the move where agents choose livelihoods, there is no incentive to change from the current choice; thus Equations (47) - (49) must be equal.

Comparing Eq. (40) and Eq. (42) to Eq. (41) and Eq. (44), we find that $\sigma_0/\sigma = \left(1+\tau+\rho t_0\right) b_0/b = \left(1+\tau+\rho t_0\right) A_0/A_{\perp}$. Similarly from Eq. (43) and Eq. (44), $A_{\parallel}/A_{\perp} = (1+\tau+\rho t_0)$, and from Equations (42,45,44,46), we find $\bar{\sigma}/\sigma = \bar{b}^{\sigma}/b^{\sigma} = \bar{b}/b$. Thus the condition for neutrality between prospectors and farmers is

$$(m+s)\log\frac{b_0}{b} = (1-s)\log\left(1+\tau+\rho t_0\right), \tag{50}$$

while the condition for neutrality between bureaucrats and farmers is

$$(m+s)\log\frac{\bar{b}}{b} = \log\left(1+\tau+\rho t_0\right). \tag{51}$$

A useful constraint of equal marginal utility from Eq. (42) and Eq. (44) gives $b^\sigma / b = \alpha^0$. Corollary relations of from the cash payment constraint are: $mb_0 = g_0$, $\left(\alpha^0 + m - 1\right) b = g$, $\left(\alpha^0 + m\right) \bar{b} = \bar{g} + \bar{\mu} t_0$.

The no-default condition for prospectors is that revenues must pay principle and interest on borrowings, plus tax on total bids (which equal the amount borrowed). Expressed as an inequality:

$$(1 + \tau + \rho t_0)\, g_0 \leq \frac{\hat{r}}{r^0} b^\sigma + \frac{\bar{r}}{r^0} \bar{b}^\sigma. \tag{52}$$

When equality holds (the bound is tight), this converts to

$$\frac{r^0 b_0}{\hat{r} b} = \left(\frac{\alpha^0}{m}\right) \frac{1}{1 + \tau + \rho t_0} \left(1 + \frac{\bar{r}\bar{b}}{\hat{r} b}\right). \tag{53}$$

Farmers must pay taxes on bids and interest on borrowings (for which they reclaim the principle spent on other farm goods through revenues from the trading posts), plus the principle they bid for gold (which they do not recover), out of revenues from bureaucrats and prospectors. As an inequality,

$$(\tau + \rho t_0)\, g + b^\sigma \leq \frac{r^0}{\hat{r}} m b_0 + \frac{\bar{r}}{\hat{r}} m \bar{b}. \tag{54}$$

When the bound is tight, with Eq. (53), this reduces to

$$\frac{\bar{r}\bar{b}}{\hat{r} b} = (\tau + \rho t_0) \left[\frac{\left(\alpha^0 + m - 1\right)(1 + \tau + \rho t_0) + \alpha^0}{m\,(1 + \tau + \rho t_0) + \alpha^0}\right]. \tag{55}$$

Bureaucrats, on the other hand, reclaim taxes on their bids as part of revenue from taxation, and must only pay principle and interest on borrowings out of the taxes on private-sector bids (which consume all borrowings). The inequality is

$$(1 + \rho t_0)\, \bar{g} \leq \tau \left(\frac{\hat{r}}{\bar{r}} g + \frac{r^0}{\bar{r}} g_0\right) \tag{56}$$

When combined with Eq. (52) and Eq. (54) and made tight, this yields a scale for bureaucrat bids

$$\bar{b} = \frac{\bar{\mu}}{\rho\left(\alpha^0 + m\right)} \cdot \frac{(1 + \rho t_0)(\tau + \rho t_0)}{(1 + \tau + \rho t_0)}. \tag{57}$$

Equivalently, the total money supply at an interior equilibrium is

$$\mathcal{M}_{\mathrm{eq}} = \frac{\bar{\mu}\bar{\nu}}{\rho}(1 + \rho t_0) \tag{58}$$

For the purposes of this argument, it is sufficient to choose the tax law for the convenience of its expression in terms of the population structure. From

Eq. (55) and Eq. (51), the following ensures optimal distribution at interior equilibria:

$$\frac{\bar{r}}{\hat{r}} = \frac{(\tau + \rho t_0)}{(1 + \tau + \rho t_0)^{1/(m+s)}} \left[\frac{\left(\alpha^0 + m - 1\right)(1 + \tau + \rho t_0) + \alpha^0}{m\,(1 + \tau + \rho t_0) + \alpha^0} \right]. \tag{59}$$

In Sec. A.3 we show that as $\rho t_0 \to 0$, this form ensures stability arbitrarily close to the no-default boundary.

Equality of all the no-default bounds gives allocations whose forms directly express the influences on marginal valuation:

$$\left\{ \frac{\sigma_0}{e^0}, \frac{\sigma}{e^0}, \frac{\bar{\sigma}}{e^0} \right\} = \left(\frac{\alpha^0}{m + \alpha^0} \right) \frac{1}{1 + \tau + \rho t_0} \left\{ (1 + \tau + \rho t_0)\,, \frac{b}{b_0}, \frac{\bar{b}}{b_0} \right\}, \tag{60}$$

for gold, and

$$\left\{ \frac{A_0}{a}, \frac{A_\parallel}{a}, \frac{A_\perp}{a}, \frac{\bar{A}}{a} \right\} = \left(\frac{1}{m + \alpha^0} \right) \frac{1}{1 + \tau + \rho t_0} \left\{ \frac{b_0}{b}, (1 + \tau + \rho t_0)\,, 1, \frac{\bar{b}}{b} \right\}, \tag{61}$$

for consumables.

The relative amounts borrowed by different agent types also have direct expressions in terms of their relative bids, parameters, and the tax laws:

$$\frac{\bar{g}}{g} = \frac{\bar{b}}{b} \left(\frac{\alpha^0 + m}{\alpha^0 + m - 1} \right) \frac{1}{(1 + \rho t_0)\,(1 + \rho t_0/\tau)}, \tag{62}$$

where we use Eq. (51) for $\bar{b}/b$, and

$$\frac{g_0}{g} = \frac{b_0}{b} \left(\frac{m}{\alpha^0 + m - 1} \right), \tag{63}$$

where we use Eq. (50) for b_0/b.

A.2 Penalty scaling for interior solutions

Note that the money supply (58) is proportional to $\bar{\nu}$, and since the borrowing and bids are comparable among all agents (at large m), the money supply per agent is $\sim \mathcal{M}_{\mathrm{eq}}/n$. Hence the scale for the penalty $\sim \Pi \mathcal{M}_{\mathrm{eq}}/n \sim \bar{\nu}/n$.

Agents optimize toward tight bounds on default by reallocating labor in the outer game into the private sector (in a sort of tatônnement to improve total production), reducing the bureaucracy and hence the money supply, and driving the K-T multipliers in equations (40 - 46) toward their limits Π. From Eq. (50) and Eq. (51), the multipliers at any interior solution have ratios

$$\frac{\eta_0}{\bar{\eta}} = (1 + \tau + \rho t_0)^{\frac{s}{m+s}}. \tag{64}$$

$$\frac{\eta}{\bar{\eta}} = (1 + \tau + \rho t_0)^{\frac{1}{m+s}}. \tag{65}$$

Hence either prospectors or farmers default first, and if we take $s < 1$ we may make it the farmers, for convenience. Thus first default penalty is reached at $\eta = \Pi$, giving the scale for the tax rate (evaluated at the interior function of population structure) at first default,

$$\frac{(\tau + \rho t_0)}{(1 + \tau + \rho t_0)^{1/(m+s)}} = \frac{\Upsilon \rho_D \left(\alpha^0 + m\right)}{\pi \bar{\mu}}. \tag{66}$$

An expression for the population structure at these solutions, not involving the tax rate explicitly, is

$$\frac{\bar{r} g}{\hat{r} g + r^0 g_0} = \frac{\Upsilon \rho_D}{\pi \bar{\mu}} \left(\alpha^0 + m - 1\right), \tag{67}$$

which approaches $\bar{r}/\left(r - \bar{r}\right)$ at large m, by Eq. (63).

Alternatively, we can convert Eq. (67) to an expression for the sector borrowing, expressed in terms of the self-consistently determined tax rule, in a form that becomes simple in the continuum trading limit $\rho t_0 \to 0$, $\bar{\mu} t_0 \to 0$:

$$\frac{\bar{\nu} \bar{g}}{\mathcal{M}_{\mathrm{eq}} - \bar{\nu} \bar{g}} \to \frac{\Upsilon \rho_D}{\pi \bar{\mu}} \left(\alpha^0 + m\right) (1 + \tau)^{\frac{1}{m+s}}. \tag{68}$$

At large m, this simplifies to the dimensional analysis result

$$\frac{\bar{\nu} \bar{g}}{\mathcal{M}_{\mathrm{eq}} - \bar{\nu} \bar{g}} \to \frac{\Upsilon \rho_D}{\pi \bar{\mu}} \left(\alpha^0 + m\right). \tag{69}$$

A.3 Stability of the labor allocation problem

We now ask what happens if Π is insufficient for Eq. (44) to hold b in the no-default region, as a result of the labor allocation in the outer game. Since this is known to all agents, we compute the optimal allocation in the regime of strategic default by farmers, and we take as a bound the condition of no-default for the other two types (even if they could improve utility slighty by defaulting). As long as the no-default solution by bureaucrats is superior to the default condition by farmers, a labor allocation that leads to default by underpopulating the bureaucracy is not an optimum of the outer game. We suppose that farmers deviated from the no-default condition by a small fraction ε; specifically

$$b = (1 + \tau + \rho t_0)^{-1/(m+s)} \bar{b} \left(1 + \varepsilon\right). \tag{70}$$

Eq. (53) still holds, and the specialists can internally allocate $r^0/\hat{r}$ so that b_0/b satisfies Eq. (50). On this surface, the monetary amount of farmer default is

$$(\tau + \rho t_0) \, g + b^\sigma - \frac{r^0}{\hat{r}} m b_0 - \frac{\bar{r}}{\hat{r}} m \bar{b} = \varepsilon \frac{\bar{r}}{\hat{r}} \left(m + \frac{\alpha^0}{1 + \tau + \rho t_0}\right) \bar{b}. \tag{71}$$

At the same time, the overall scale for borrowing is set by the bureaucrat no-default condition, which changes from Eq. (57) to

$$\bar{b} = \frac{\bar{\mu}}{\rho} \cdot \frac{(1+\rho t_0)(\tau+\rho t_0)}{(1+\tau+\rho t_0)(\alpha^0+m) - (\tau\varepsilon/\rho t_0)\left(m + \frac{\alpha^0}{1+\tau+\rho t_0}\right)}. \tag{72}$$

Bureaucrats need only pay interest on borrowings, allowing them to escalate the scale of bidding by $1/\rho t_0$ relative to the degree of farmer default[30]. From the utilities (14,15), with the former only in default, the penalty definition (16) and equilibrium money supply (58), and approximating $\log(1+\varepsilon) \approx \varepsilon$, the approximate utility difference resulting from farmer-only default is

$$\mathcal{U}_{i,\text{econ}}/\Upsilon - \bar{\mathcal{U}}_{k,\text{econ}}/\Upsilon \approx \varepsilon \left[s + m - \left(\frac{\pi\bar{\mu}}{\Upsilon\rho_D}\right) \frac{(\tau+\rho t_0)\,\bar{r}/\hat{r}}{\frac{(1+\tau+\rho t_0)(m+\alpha^0)}{m+\alpha^0/(1+\tau+\rho t_0)} - \frac{\tau\varepsilon}{\rho t_0}}. \right] \tag{73}$$

Using $\pi\bar{\mu}/\Upsilon\rho_D$ to approximate the tax function at first default through Eq. (66), we simplify Eq. (73) to

$$\mathcal{U}_{i,\text{econ}}/\Upsilon - \bar{\mathcal{U}}_{k,\text{econ}}/\Upsilon \approx \varepsilon \left[s + m - \left(\frac{\bar{r}}{\hat{r}}\right) \frac{\left(\alpha^0+m\right)(1+\tau+\rho t_0)^{1/(m+s)}}{\frac{(1+\tau+\rho t_0)(m+\alpha^0)}{m+\alpha^0/(1+\tau+\rho t_0)} - \frac{\tau\varepsilon}{\rho t_0}}. \right] \tag{74}$$

At nonzero ρt_0, for $\bar{r} < \hat{r}$, there is generally a range of $\varepsilon > 0$ for which farmers benefit from excess consumption, inducing strategic default and with it defection from bureaucracy to the private sector in the outer optimization, relative to $\varepsilon = 0$. However, τ and the equilibrium allocation change only by $\mathcal{O}(\varepsilon)$ from equilibrium values, and Π not at all under these rules. Meanwhile, at any nonzero ε there is a bound on ρt_0 below which Eq. (74) is clearly zero or negative, making farmers worse off by default than non-defaulting bureaucrats, who in turn are not optimizing by an allowed default of their own that would reclaim some consumption $\propto \varepsilon$. When farmers are worse off, the equilibrium of the outer game has population allocation closer to the interior (no-default) solution. Thus at sufficiently small ρt_0 we may place the equilibrium of labor allocation arbitrarily close to the no-default boundary, and $\varepsilon \propto \rho t_0 \to 0$ in the continuum trading limit.

Details aside, the reason this tax law creates a stable solution with nonzero bureaucracy is that bureaucrats reclaim the (finite) taxes on their own bids, and need pay only interest on borrowing, while specialists in the private sector pay a fraction of the money supply per period in taxes. The leveraging effect when the velocity of money is large (finite $\mathcal{M}_t$ at $\rho t_0 \to 0$) makes it impossible for specialists to gain a large advantage on bureaucrats, even though a linear default penalty is strictly a *protection* relative to suffering the concave disutility of decreased consumption, because bureaucrats can always ultimately out-borrow specialists at a comparable or lesser penalty.

[30] Note that when this bid-up is severe, Π will also be insufficient to prevent bureaucrat default. The point is that the amount they borrow in a default equilibrium is only $\mathcal{O}(\varepsilon)$ larger than the no-default limit that is our bound.

B Definition and condensed notation for variables and parameters

Notation used in the paper.

References

[1] A. Smith, *An inquiry into the nature and causes of the wealth of nations. Books 1-3*, (Viking Penguin, New York, 1986).

[2] A. Mas-Collel, M. D. Whinston, and J. R. Green, *Microeconomic Theory* (Oxford U. Press, New York, 1995).

[3] D. C. North, *Institutions, Institutional Change, and Economic Performance* (Cambridge U. Press, Cambridge, U. K., 1990).

[4] M. Shubik, *The Theory of Money and Financial Institutions* (MIT Press, Cambridge, Mass., 1999).

[5] M. Shubik, and E. Smith, "The physics of time and dimension in the economics of financial control", special issue of Physica A in memory of Per Bak (2004).

[6] P. Bak, S.F. Nørrelykke, and M. Shubik, "Money and Goldstone Modes", *Quantitative Finance* **1** (2001) 186-190.

[7] G. F. Knapp, *Staatliche Theorie des Geldes* (Duncker and Humblot, Leipzig, 1905).

[8] Ref. [4], Ch. 3-4.

[9] E. Smith, and D. K. Foley, "Classical thermodynamics and economic general equilibrium theory", submitted.

[10] E. Smith and M. Shubik, "Strategic Freedom, Constraint, and Symmetry in One-Period Markets with Cash and Credit Payment", *Economic Theory.*

[11] A. Lewis "On Effectively Computable Realisation of Choice Functions", Mathematical Social Sciences, **10** (1985) 43-80.

[12] T. Quint, and M. Shubik 2004, "Gold, fiat, and credit: An Elementary Discussion of Commodity Money, Fiat Money and Credit. Part2." Cowles Foundation Discussion paper 1460

[13] J. F. Padgett, "Modeling Florentine Republicanism", Santa Fe Institute Working paper # 01-02-008; "Elite Transformation and Organizational Invention in Renaissance Florence", Santa Fe Institute Working paper # 05-02-001

[14] W. S. Jevons, *Money and the Mechanism of Exchange* (Appleton and Co., New York, 1875).

[15] M. Shubik, *Game theory in the social sciences* (MIT Press, Cambridge, Mass., 1999).

[16] M. Shubik and C. Wilson, "The optimal bankruptcy rule in a trading economy with fiat money" *Zeitschrift für Nationalökonomie* **37** (1977) 3-4.

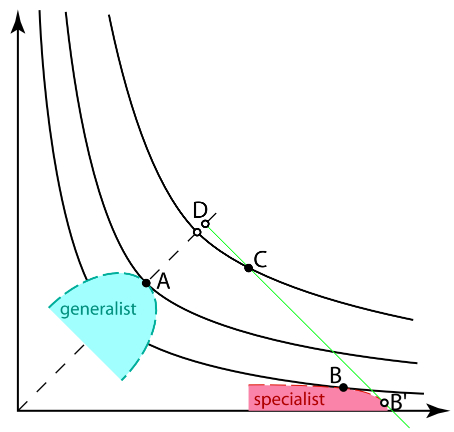

Figure 1: Qualitative features of the transition to specialization represent regions in strategy space, not sensitive to perturbations in preference representation. Convex preferences in two goods are represented here by solid lines. Possible endowments achievable within the generalist or specialist production technologies represented as shaded regions with dashed boundaries. Solid dots are outcomes realizable by certain combinations of production decision and trade technology, and open circles are reference points in the analysis. A is the best outcome one can achieve as a generalist, and B the (Pareto-inferior) best outcome as a specialist without trade. $B^{'}$ is the endowment that produces a best outcome (C) with the NE of some trading game, and D for reference is the CE at the same endowments. Utility in money-metric, measured by integration along the Pareto Set (diagonal dashed line) [9, 10], defines a natural measure of the relative commodity value of the different outcomes. The value of C (open circle on the Pareto set indifferent to C) stably dominates A but is dominated by D, capturing the gains to trade as well as inefficiencies associated with finite population size. (Other inefficiencies associated with costs of markets not shown in this figure, for simplicity.) In minimal models the endowment difference between B and $B^{'}$ is suppressed, as it depends sensitively on representations of both utility and specialist production at small arguments.

Parameters	
m	Number of types of consumption goods
$n \equiv mr$	Number of agents in the society
t_0	Time interval for a cycle of production, trade, and consumption
a	Allocation rate of consumption goods to farmers
e^0	Allocation rate of gold to prospectors
$\bar{\mu}$	Bureaucrat salary rate
π	Penalty technology of the bureaucracy
Δ	Rate of decay of capital stock
ρ	Rate of interest on borrowing
τ	Fraction of gross receipts from sales demanded in taxes
Υ	Scale factor for utilities
ρ_D	Temporal utility discount rate
β	Per-period discount fraction
Labor, price, and allocation variables (period notation → stationary notation)	
$\nu_t^0 \to mr^0$	Number of prospectors at any time t
$\nu_t^j \to \hat{r}$	Number of farmers of any consumption good at any time t
$\bar{\nu}_t \to m\bar{r}$	Number of bureaucrats at any time t
$Q_t^j \to Q$	Total quantity offered of a single consumable
$Q_t^0 \to Q^\sigma$	Total quantity of gold offered
$B_t^j \to B$	Total fiat bid on a single consumable
$B_t^0 \to B^\sigma$	Total fiat bid on gold
$p_t^j \to p$	Price of any consumption good at any time t
$p_t^0 \to p^\sigma$	Price of gold at any time t
$\lambda, \bar{\lambda}, \eta, \bar{\eta}$	Kuhn-Tucker multipliers
$A_{i,t}^j \to A_0$	Prospector's final allocation rate of any consumption good
$A_{i,t}^j \to A_\parallel$	Farmer's final allocation rate of self-produced consumption good
$A_{i,t}^j \to A_\perp$	Farmer's final allocation rate of other-produced consumption good
$\bar{A}_{k,t}^j \to \bar{A}$	Bureaucrat's final allocation rate of consumption good
$A_{i,t}^0 \to \sigma_0$	Prospector's final allocation rate of gold in each period
$A_{i,t}^0 \to \sigma$	Farmer's final allocation rate of gold in each period
$\bar{A}_{k,t}^0 \to \bar{\sigma}$	Bureaucrat's final allocation rate of gold in each period
$C_{i,t} \to \sigma_0/\Delta$	Capital stock of prospectors
$C_{i,t} \to \sigma/\Delta$	Capital stock of farmers
$\bar{C}_{k,t} \to \bar{\sigma}/\Delta$	Capital stock of bureaucrats
Individual decision variables(period notation → stationary notation)	
$q_{i,t}^0 \to q^\sigma$	Gold offered by any prospector per period
$q_{i,t}^j \to q$	Self-produced consumption good offered by any farmer per period
$g_{i,t} \to g_0$	Borrowing by a prospector per period
$g_{i,t} \to g$	Borrowing by a farmer per period
$\bar{g}_{k,t} \to \bar{g}$	Borrowing by a bureaucrat per period
$b_{i,t}^j \to b_0$	Bid by prospector on any consumption good
$b_{i,t}^0 \to b^\sigma$	Bid by any farmer on gold
$b_{i,t}^j \to b$	Bid by any farmer on any other-produced consumption good
$\bar{b}_{k,t}^0 \to \bar{b}^\sigma$	Bid by any bureaucrat on gold
$\bar{b}_{k,t}^j \to \bar{b}$	Bid by any bureaucrat on any consumable good

Table 1: Reduced notation for the decision, price, and allocation variables of the fiat economy in case of stationary solutions.