THE NONPARAMETRIC APPROACH TO APPLIED WELFARE ANALYSIS

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April 2005

COWLES FOUNDATION DISCUSSION PAPER NO. 1507

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The Nonparametric Approach to Applied Welfare Analysis

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Abstract

Changes in total surplus and deadweight loss are traditional measures of economic welfare. We propose necessary and sufficient conditions for rationalizing consumer demand data with a quasilinear utility function. Under these conditions, consumer surplus is a valid measure of consumer welfare. For nonmarketed goods, we propose necessary and sufficient conditions on market data for efficient production , i.e. production at minimum cost. Under these conditions we derive a cost function for the nonmarketed good, where producer surplus is the area above the marginal cost curve.

Keywords: Welfare economics, Quasilinear utilities, Nonmarketed goods, Afriat inequalities

JEL Classification: D11, D12, D21, D60

1 Introduction

Given a finite set of prices and consumption choices, we say that the data set is rationalizable if there exists a concave, continuous and monotonic utility function such that these choices are the maxima of the utility function over the budget sets.[1] Afriat (1967) provided necessary and sufficient conditions for a finite data set to be rationalizable, that is for observable choices to be the result of the consumer maximizing her utility function subject to a budget constraint. In a series of lucid papers Varian (1982), (1983) and (1984) has illuminated our understanding and increased our appreciation of Afriat's seminal contributions to demand theory, Afriat (1967), and the theory of production, Afriat (1972a). As a consequence there is now a growing and vigorous literature on the testable restrictions of strategic and non-strategic behavior of households and firms in market economies —see the survey of Carvajal, et al (2004).

This paper is primarily about the economic welfare of agents. In applied partial equilibrium models we often measure economic welfare in terms of total surplus, i.e., the sum of consumer and producer surplus, and deadweight

[1]Typically this data is obtained from market transactions, but for nonmarketed goods it may have been obtained from stated preference methods, e.g. contingent valuation or contingent choice —see Bockstael and McConnel (1998) for discussion.

loss. As is well known, consumer surplus is a valid measure of consumer welfare only if the consumer's demand derives from maximizing a homothetic or quasilinear utility function subject to her budget constraint —see section 11.5 in Silberberg (1990). Both Afriat (1972b) and Varian (1983) proposed a necessary and sufficient combinatorial condition for rationalizing data sets, consisting of market prices and consumer demands, with homothetic utility functions. This condition is the homothetic axiom of revealed preference or HARP. To our knowledge, there is no comparable result in the literature for quasilinear rationalizations of consumer demand data. In this paper we show that a combinatorial condition introduced in Rockafellar (1970) to characterize the subgradient correspondence for convex real-valued functions on R^n, cyclical monotonicity or CM, is a necessary and sufficient condition for a finite data set to be rationalizable with a quasilinear utility function.[2]

Measuring producer surplus for nonmarketed goods such as health, education or environmental amenities and ascertaining if these goods are produced efficiently, i.e., at minimum cost, are important issues in applied welfare eco-

[2] In the paper by Rochet (1987) "A necessary and sufficient condition for rationalizability in a quasilinear context" published in the *Journal of Mathematical Economics* he defines rationalizability as implementability of an action profile —$X(\cdot)$ from the set of individual characteristics to the set of possible actions— via compensatory transfers, that is if there exist transfer functions $t(\cdot)$ which make the mechanism $(X(\cdot), t(\cdot))$ truthfully implementable in dominant strategies. Therefore the results presented in his paper are of a different nature, despite the title of his paper.

nomics. Our contribution to the welfare literature on nonmarketed goods is the observation that Afriat's combinatorial condition, cyclical consistency or CC, and equivalently Varian's generalized axiom of revealed preference or GARP, are necessary and sufficient conditions for rationalizing a finite data set, consisting of factor demands and factor prices, with a concave, monotone and continuous production function. Hence they constitute necessary and sufficient conditions for nonmarketed goods to be produced at minimum cost for some production function. If these conditions hold, then the supply curve for the nonmarketed good is the marginal cost curve of the associated cost function and producer surplus is well defined. These combinatorial conditions are equivalent to Varian's cost minimizing inequalities where both marginal costs and output levels are unobservable, see Varian (1984).

In the next section we present results characterizing rationalizability of demand data with quasilinear utilities. In the final section of the paper we propose necessary and sufficient conditions on finite data sets of factor demands and prices such that the nonmarketed goods are produced efficiently and we derive the supply curves for such goods.

2 Rationalizing Demand Data with Quasilinear Utilities

Afriat (1967) provides the first non-parametric test for consumer behavior. He provides a necessary and sufficient condition on finite data for it to be rationalizable by a neoclassical utility function.

Definition 1 Let (p_r, x_r), $r = 1, ..., N$ be given. A utility function u rationalizes the data if for all $r = 1, ..., N$ x_r solves:

$$\max_{x \epsilon R^n_{++}} u(x)$$

$$s.t. p_r x \leq I = p_r x_r$$

Theorem 1 (Afriat 1967) The following conditions are equivalent:

1. *There exists a concave, monotone, continuous, non-satiated utility function that rationalizes the data.*

2. *The data* (p_r, x_r), $r = 1, ..., N$ *satisfies Afriat inequalities, that is, there exists* $U_r > 0$ *and* $\lambda_r > 0$ *for* $r = 1, \ldots, N$ *such that*

$$U_r \leq U_l + \lambda_l p_l \cdot (x_r - x_l) \;\; \forall r, l = 1, \ldots, N$$

3. *The data* (p_r, x_r), $r = 1, ..., N$ *satisfies "cyclical consistency", that is,*

$$p_r x_r \geq p_r x_s, \; p_s x_s \geq p_s x_t, \cdots, p_q x_q \geq p_q x_r$$

implies

$$p_r x_r = p_r x_s, \; p_s x_s = p_s x_t, \cdots, p_q x_q = p_q x_r$$

Definition 2 Let (p_r, x_r), $r = 1, ..., N$ be given. The data is *quasilinear rationalizable* if for some $y_r > 0$ and $I > 0$, $\forall r$ x_r solves

$$\max_{x \epsilon R^n_{++}} U(x) + y_r$$

$$s.t. p_r x + y_r = I$$

where U is a concave function and y_r is the numeraire good.

Definition 3 Let (p_r, x_r), $r = 1, ..., N$ be given. The data is *cyclically monotone* if for any given subset of the data $\{(p_s, x_s)\}_{s=1}^{m}$:

$$x_1 \cdot (p_2 - p_1) + x_2 \cdot (p_3 - p_2) + \cdots + x_m \cdot (p_1 - p_m) \geq 0$$

or equivalently:

$$p_1 \cdot (x_2 - x_1) + p_2 \cdot (x_3 - x_2) + \cdots + p_m \cdot (x_1 - x_m) \geq 0$$

Definition 4 If U is concave on R^n, then $\beta \in R^n$ is a subgradient of U at x if for all $y \in R^n : U(y) \leq U(x) + \beta \cdot (y - x)$.

Definition 5 If U is a concave function on R^n, then $\partial U(x)$ is the set of subgradients of U at x.

Theorem 2 The following are equivalent:

1. *The data* (p_r, x_r)*,* $r = 1, ..., N$ *is quasilinear rationalizable by a continuous, concave, strictly monotone utility function* U*.*

2. *The data* (p_r, x_r), $r = 1, ..., N$ *satisfies Afriat's inequalities with constant marginal utilities of income, that is, there exists* $G_r > 0$ *and* $\lambda > 0$ *for* $r = 1, \ldots, N$ *such that*

$$G_r \leq G_l + \lambda p_l \cdot (x_r - x_l) \ \ \forall r, l = 1, \ldots, N$$

or equivalently there exist $U_r > 0$ for $r = 1, \ldots, N$

$$U_r \leq U_l + p_l \cdot (x_r - x_l) \ \ \forall r, l = 1, \ldots, N$$

where $U_r = \frac{G_r}{\lambda}$

3. *The data* (p_r, x_r), $r = 1, ..., N$ *is cyclically monotone.*

Proof:

(1) $\Rightarrow$ (2): From the FOC of the quasilinear utility maximization problem we know:

$$\exists \beta_r \, \epsilon \, \partial U(x), \text{ s.t. } \beta_r = \lambda_r p_r \text{ where } \lambda_r = 1$$

Also, U being concave implies that $U(x_r) \leq U(x_l) + \beta_l (x_r - x_l)$ for $r, l = 1, 2, ..., N$. Since $\beta_l = p_l \ \forall l = 1, \ldots, N$ we get $U(x_r) \leq U(x_l) +$

$p_l(x_r - x_l)\ \forall r, l = 1, ..., N$

(2) $\Rightarrow$ (3): For any set of pairs $\{(x_s, p_s)\}_{s=1}^m$ we need that: $p_0 \cdot (x_1 - x_0) + p_1(x_2 - x_1) + ... + p_m(x_0 - x_m) \geq 0$.

From the Afriat inequalities with constant marginal utilities of income we know:

$$U_1 - U_0 \leq p_0 \cdot (x_1 - x_0)$$

$$U_2 - U_1 \leq p_1 \cdot (x_2 - x_1)$$

$$\cdots$$

$$U_0 - U_m \leq p_m(x_0 - x_m)$$

Adding up these inequalities we see that the left hand sides cancel and the resulting condition defines cyclical monotonicity.

(3) $\Rightarrow$ (1): Let $U(x) = \inf\{p_m \cdot (x - x_m) + \ldots + p_1 \cdot (x_2 - x_1)\}$ where the infimum is taken over all finite subsets of data, then $U(x)$ is a concave

function on R^n and p_r is the subgradient of U at $x = x_r$.[3] Hence if $\lambda_r = 1$ for $r = 1, \ldots, N$ then

$$p_r = \partial U(x_r)$$

constitutes a solution to the first order conditions of the quasilinear maximization problem.

If we require strict inequalities in (2) of Theorem 3, then it follows from Lemma 2 in Chiappori and Rochet (1987) that the rationalization can be chosen to be a C^∞ function. It then follows from Roy's identity that $x(p) = -\frac{\partial V(p)}{\partial p}$. Hence for any line integral we see that $\int_{p_1}^{p_2} x(p)dp = -\int_{p_1}^{p_2} \frac{\partial V(p)}{\partial p} dp = V(p_1) - V(p_2)$. That is, consumer welfare is well-defined and the change in consumer surplus induced by a change in market prices is the change in consumer's welfare.

[3] This construction is due to Rockafellar (1970) in his proof of Theorem 24.8.

3 Rationalizing the Production of Nonmarketed Goods

Health, education and environmental amenities are all examples of nonmarketed goods. To compute producer surplus for such goods, we must derive the supply curve, given only factor demands and prices. A policy issue of some importance is whether these goods are produced efficiently, i.e. at minimum cost, given factor demands and prices. In fact, as we show, there may be no concave, monotone and continuous production function that rationalizes the input data. If one does exist, we can rationalize the data and derive the supply curve for the nonmarketed good.

We provide necessary and sufficient conditions on a finite data set on factor inputs, x_r, and factor prices, p_r, to rationalize the data with a concave, monotone and continuous production function, F. Given F we can derive the cost function, an equivalent representation of the technology for producing the nonmarketed good. Finally from the cost function we derive the supply curve for this good, and producer surplus is well defined.

Definition 6 Let (p_r, x_r), $r = 1, \ldots, N$ be given. A production, F, rationalizes the data if for all $r = 1, \ldots, N$ there exists q_r such that x_r solves:

$$\max_{x \in R^n_{++}} q_r F(x) - p_r \cdot x$$

where F is a concave function.

The rationalization is contained in Theorem 3, where the output price and quantity, represented by q_r and $F(x)$ are unknown.

Theorem 3 The following conditions are equivalent:

1. *There exists a concave, monotone, continuous, non-satiated production function that rationalizes the data.*

2. *The data (p_r, x_r), $r = 1, ..., N$ satisfies Afriat inequalities, that is, there exists $F_r > 0$ and $q_r > 0$ for $r = 1, \ldots, N$ such that*

$$F_r \leq F_l + \frac{1}{q_l} p_l \cdot (x_r - x_l) \;\; \forall r, l = 1, \ldots, N$$

where q_l is the marginal cost of producing F_l.

3. *The data* (p_r, x_r), $r = 1, ..., N$ *satisfies "cyclical consistency", that is,*

$$p_r x_r \geq p_r x_s,\ p_s x_s \geq p_s x_t, \cdots, p_q x_q \geq p_q x_r$$

implies

$$p_r x_r = p_r x_s,\ p_s x_s = p_s x_t, \cdots, p_q x_q = p_q x_r$$

Proof: This is Afriat's (1967) result presented earlier in Theorem 1 where we let $F = U$ and $\lambda_r = \frac{1}{q_r}$.

If we write the cost minimization problem of the firm, $\min_{x \in R^n} p \cdot x$ s.t. $F(x) \geq y$, from the F.O.C. we find $p = \mu F'(x)$, where μ is the Lagrange multiplier associated with the constraint, and therefore is equal to the marginal cost of producing one more unit of output at the optimum. Therefore it is easy to see from the FOC of the profit maximization problem that the output price, q_r, is the marginal cost of production. The inequalities in (2) are the same as those in condition (3) of Theorem 2 in Varian (1984), where he assumes that the production levels F_r are observable.

$F(x) = min_{1 \leq l \leq r}\{F_l + \frac{1}{q_l} p_l(x - x_l)\}$ is Afriat's utility (production) function derived from a solution to the Afriat inequalities. The associated expenditure (cost) function is $c(y; p) = \min_{x \in R^n} p \cdot x$ s.t. $F(x) \geq y$. In the production setting, the supply curve is the marginal cost curve.

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