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HAC Estimation by Automated Regression*

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Abstract

A simple regression approach to HAC and LRV estimation is suggested. The method exploits the fact that the quantities of interest relate to only one point of the spectrum (the origin). The new estimator is simply the explained sum of squares in a linear regression whose regressors are a set of trend basis functions. Positive definiteness in the estimate is therefore automatically enforced and the technique can be implemented with standard regression packages. No kernel choice is needed in practical implementation but basis functions need to be chosen and a smoothing parameter corresponding to the number of basis functions needs to be selected. An automated approach to making this selection based on optimizing the asymptotic mean squared error is derived. The limit theory of the new estimator shows that its properties, including the convergence rate, are comparable to those of conventional HAC estimates constructed from quadratic kernels.

Key words and Phrases: Asymptotic mean squared error, automation, bias, HAC estimation, long run variance, trend regression, trigonometric polynomial.

JEL Classification: C22

1. Introduction

Attempts to robustify inference in econometrics have led to the systematic development of techniques that take into account potential heterogeneity and autocorrelation in the data. Two major practical applications of this work involve HAC (heteroskedasticity and autocorrelation consistent) covariance matrix estimation and long-run variance (LRV) matrix estimation. All HAC and LRV estimators that are commonly used in econometric work are based on kernel methods. These estimators

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inherit their form and their asymptotic properties from work in the earlier literature of spectral density estimation, where kernel methods are again dominant.

Automated versions of these kernel methods have also been developed. Automation removes the need for discretionary bandwidth choice in kernel estimation by implementing data-determined bandwidth selection rules that are commonly based on asymptotic mean squared error formulae. Bandwidth selection rules have been built into some popular econometric software programs and users may implement them without having to make any discretionary decisions. This convenience has helped promote the use of the methods in empirical research. Automated techniques of this genre, just like the kernel methods on which they are used, themselves belong to a longer pedigree of related work in statistics.

The present contribution suggests a novel approach to HAC/LRV estimation that does not involve the direct use of kernels. To the author's knowledge, the approach is new and has not been suggested before in any earlier work in statistics or econometrics. Unlike conventional procedures the method is not based on kernel estimation, either by way of a lag kernel of weighted autocovariances or by kernel smoothing of the periodogram in the frequency domain. However, we shall see that the approach may be interpreted as producing an asymptotic form of kernel estimate.

The idea is motivated by the fact that the quantities of interest in HAC/LRV estimation relate to only one point on the spectrum and that this point (the origin) refers to long-run behavior. This feature is exploited by designing a linear regression of the variable of interest on a set of regressors designed to represent long-run behavior directly. The regressors form a set of trend basis functions. Any set of basis functions may be used, but in the development given here (and in empirical work) it is generally convenient to use an orthonormal set of trigonometric polynomials. Several examples are given. The new HAC estimator is simply part of the output of this regression and is given by the explained sum of squares in a linear regression on the trend basis. It can be implemented by standard regression packages. Positive definiteness in the estimate is automatically enforced by its construction as a sum of squares and this is so whatever the choice of basis functions. This property is important, being one of the main concerns in Newey and West (1987) and playing a significant role in Andrews (1991) regarding the selection of suitable kernel functions.

Curiously, this is an example of a regression that would conventionally be regarded in econometrics as misspecified (or even spurious) because the regressors are in fact irrelevant to the determination of the dependent variable. Nevertheless, the coefficients in this regression produce, upon straightforward normalization, a consistent HAC estimator.

The approach has the advantage of the simple convenience of least squares regression and no kernel choice is needed in its implementation. However, the trend basis functions need to be chosen and a 'smoothing' parameter corresponding to the number of the trend functions actually used in the regression also needs to be selected. The smoothing parameter choice can be automated based on the behavior of the asymptotic mean squared error of the estimator and a rule for such automated implementation is developed in the paper. As far as the choice of trend basis functions is concerned, it is often convenient to use trigonometric functions and it turns out that the asymptotic results are invariant to the choice of basis within the class of trigonometric polynomials.

The fact that consistent estimation is possible using apparently irrelevant regressors may appear somewhat magical. However, projecting a stationary time series onto a space of trends, even when there is no trend in the data, has the effect of isolating the long-run behavior in the time series and this is what enables the direct regression estimation of the long-run parameter. The idea has extensions and many other applications which are not discussed in the present paper. Some of these are considered by the author (2004) in other work.

2. Trend Regression of Untrended Time Series

The following development concentrates on the scalar case. Only minor modifications are required to extend the results to the vector case and matrix HAC estimation. Accordingly, let u_t be a weakly dependent time series satisfying

$$u_t = C(L)\,\varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}, \quad \sum_{j=0}^{\infty} j^a |c_j| < \infty, \quad C(1) \neq 0,\ a > 3, \tag{$\mathbf{L}$}$$

where $\varepsilon_t = iid\left(0, \sigma^2\right)$ and $E(|\varepsilon_t|^v) < \infty$, for some $v > 2$.

The time series u_t is stationary with variance $\sigma_u^2 = \sum_{j=0}^{\infty} c_j^2 \sigma^2$, autocovariance function $\gamma_u(h) = E\left(u_t u_{t+h}\right) = \sigma^2 \sum_{j=0}^{\infty} c_j c_{j+h}$, finite v'th absolute moment $E|u_t|^{\nu} \leq \left(\sum_{j=0}^{\infty} |c_j|\right)^v E|\varepsilon_t|^v < \infty$, spectrum $f_u(\lambda) = \left(\sigma^2/2\pi\right)\left|C\left(e^{i\lambda}\right)\right|^2$, and long-run variance $\omega^2 = 2\pi f_u(0) = \sigma^2 C(1)^2$. The summability condition in $\mathbf{L}$ ensures that

$$\sum_{h=-\infty}^{\infty} h^3 |\gamma_u(h)| < \infty, \tag{1}$$

which is helpful in some technical derivations below and means that $f_u(\lambda)$ has continuous second derivative $f_u^{(2)}(\lambda) = -\frac{\sigma^2}{2\pi}\sum_{h=-\infty}^{\infty} h^2 \gamma_u(h)\, e^{-i\lambda h}$. Allowance for heterogeneity in ε_t and u_t can be made in the usual way with minor modifications to $\mathbf{L}$ (c.f. Phillips and Solo, 1992). without affecting the procedures or the properties discussed below in an essential way.

Under $\mathbf{L}$, partial sums $S_t = \sum_{i=1}^{t} u_i$ satisfy the functional law (e.g., Phillips and Solo, 1992)

$$B_n(\cdot) := \frac{S_{\lfloor n\cdot \rfloor}}{\sqrt{n}} = \frac{\sum_{i=1}^{\lfloor n\cdot \rfloor} u_i}{\sqrt{n}} \Rightarrow B(\cdot) \tag{2}$$

where $\lfloor a \rfloor$ signifies the integer part of a, $\Rightarrow$ is weak convergence, and $B(\cdot)$ is Brownian motion with variance ω^2.

Let $\{\varphi_k\}_{k=1}^{\infty}$ be a complete orthonormal system in $L_2[0,1]$. Later, we will work with the explicit sequence (8) but it is sufficient to assume that the functions φ_k are twice continuously differentiable on $[0,1]$. We propose a regression of u_t on a collection of K deterministic regressors $\{\varphi_k\left(\frac{t}{n}\right)\}_{k=1}^{K}$ formed by taking the first K members of this orthonormal sequence evaluated over $t = 1,..,n$. Write this regression in the form

$$u_t = \sum_{k=1}^{K} \widehat{b}_k \varphi_k(\frac{t}{n}) + e_t := \widehat{b}'_K \varphi_{Kt} + e_t, \quad t = 1, ..., n; \tag{3}$$

where $\varphi_{Kt} = (\varphi_1(\frac{t}{n}), ..., \varphi_K(\frac{t}{n}))'$. In observation format (3) can be written as $u = \Phi_K \widehat{b}_K + e_K$, with $\widehat{b}_K = \left(\Phi'_K \Phi_K\right)^{-1} \Phi'_K u$.

Let $P_K = \Phi_K \left(\Phi'_K \Phi_K\right)^{-1} \Phi'_K$ and construct the estimate

$$\hat{\omega}_K^2 = \frac{1}{K} u' P_K u = \left(\frac{u' \Phi_K}{\sqrt{n}}\right) \left(\frac{\Phi'_K \Phi_K}{n}\right)^{-1} \left(\frac{\Phi'_K u}{\sqrt{n}}\right). \tag{4}$$

As shown in Lemma A in the Appendix, $n^{-1} \sum_{t=1}^{n} \varphi_{Kt} \varphi'_{Kt} = I_K + O\left(\frac{1}{n}\right)$ and $(n^{-1} \sum_{t=1}^{n} \varphi_{Kt} \varphi'_{Kt})^{-1} = I_K + O\left(\frac{1}{n}\right)$. Standard functional limit arguments and Wiener integration reveal that for fixed K as $n \to \infty$ we have

$$n^{-1/2} \sum_{t=1}^{n} \varphi_{Kt} u_t \to_d \int_0^1 \varphi_K(r)\, dB(r) := \xi_K =_d N\left(0, \omega^2 I_K\right).$$

It follows immediately that

$$\frac{1}{K} u' P_K u \to_d \frac{1}{K} \xi'_K \xi_K =_d \omega^2 \frac{\chi_K^2}{K}, \tag{5}$$

where χ_K^2 is chi-squared with K degrees of freedom. For fixed K, the asymptotic mean and variance of (5) are

$$E\left(\frac{1}{K} u' P_K u\right) = \omega^2 + o(1), \quad \text{Var}\left(\frac{1}{K} u' P_K u\right) = \frac{2\omega^4}{K} + o(1), \tag{6}$$

as $n \to \infty$. These results motivate the long-run variance estimator

$$\hat{\omega}_K^2 = \frac{1}{K} u' P_K u = \frac{1}{K} u'_{P_K} u_{P_K}, \text{ where } u_{P_K} = P_K u, \tag{7}$$

which, in view of (6), can be expected to be consistent for ω^2 when $K \to \infty$ as $n \to \infty$.

The estimate $\hat{\omega}_K^2$ is simply the sample variance of u_{P_K}, the data projected onto the space spanned by the regressors Φ_K. Thus, $\hat{\omega}_K^2$ is that part of the sample variance of u_t explained by the regression of u_t onto a deterministic trend basis. This explained sum of squares may be regarded as another way of thinking about a long-run variance - the contribution to the variation of u_t that comes from long-run (or trend-like) behavior in the series. Thus, there would seem to be a strong heuristic motivation for considering estimates like (7).

Note that $\hat{\omega}_K^2$ is a nonnegative definite quadratic form in the data, whose matrix is the projection P_K. Thus, $\hat{\omega}_K^2$ belongs to the general class of quadratic estimators of

ω^2. General quadratic estimators were considered in the early spectral analysis literature but have received little subsequent attention relative to kernel estimates. One reason is that, for every such quadratic estimator, one can find a corresponding lag kernel estimator with smaller mean squared error (Grenander and Rosenblatt, 1957, p.129). Interestingly, as we will show below, $\hat{\omega}_K^2$ turns out itself to be asymptotically equivalent to a lag kernel estimator and to have nice asymptotic properties analogous to those of quadratic kernel estimates. Thus, there is no need to adapt $\hat{\omega}_K^2$ into kernel form and, of course, $\hat{\omega}_K^2$ is positive by construction.

To develop a consistent estimation technique, we need to allow for the number of regressors K to pass to infinity with the sample size n in such a way that the regression (3) remains feasible. Accordingly, we impose the following rate condition on K

$$\frac{n}{K^2} + \frac{K}{n} \to 0, \tag{\textbf{R}}$$

which requires K to go to infinity faster than $\sqrt{n}$ but slower than n. To establish a central limit theorem for $\hat{\omega}_K^2$, we need the further condition $K = o\left(n^{4/5}\right)$, which controls the expansion rate so that there is no bias in the limit.

For an explicit limit theory, including an explicit expression for the limiting bias of $\hat{\omega}_K^2$, it is convenient to use the orthonormal sequence

$$\varphi_k(r) = \sqrt{2}\sin\left\{\left(k - \frac{1}{2}\right)\pi r\right\}, \quad k = 1, 2, ... \tag{8}$$

The functions (8) are the eigenvectors of the covariance kernel of Brownian motion (c.f., Phillips, 1998) and form an orthonormal system for $L_2[0,1]$. Of course, other orthonormal sequences can be used. However, it turns out that the asymptotic results given below are invariant to the choice of the orthonormal sequence within the class of trigonometric polynomials because the estimates are asymptotically equivalent to the same lag kernel estimator. This point is discussed below in section 4.

Under this set-up, we give formulae for the limiting bias, variance and mean squared error and a limit distribution theory for $\hat{\omega}_K$. The details of the proofs are different from those of the conventional literature on HAC estimation and are of some independent interest, so they are provided here. But the final results end up being qualitatively similar, as the following result shows.

Theorem *Under conditions* **L** *and* **R**

(a) $\lim_{n\to\infty}\left(\frac{n}{K}\right)^2 E\left(\hat{\omega}_K^2 - \omega^2\right) = -\frac{\pi^2}{6}\sum_{h=-\infty}^{\infty} h^2\gamma_u(h) := D$;

(b) *If* $K = o\left(n^{4/5}\right)$, *then* $\sqrt{K}\left(\hat{\omega}_K^2 - \omega^2\right) \Rightarrow N\left(0, 2\omega^4\right)$;

(c) *If* $K^5/n^4 \to 1$, *then* $\lim_{n\to\infty}\left(\frac{n}{K}\right)^4 E\left(\hat{\omega}_K^2 - \omega^2\right)^2 = D^2 + 2\omega^4$.

Part (a) shows that $\hat{\omega}_K^2$ has bias of order K^2/n^2 of the form

$$E\left(\hat{\omega}_K^2\right) = \omega^2 + \frac{K^2}{n^2}D\left[1 + o(1)\right], \quad \text{where } D = -\frac{\pi^2}{6}\sum_{h=-\infty}^{\infty} h^2\gamma_u(h) := -\frac{\pi^2}{6}\omega_{(2)}^2.$$

From (b), the variance of $\hat{\omega}_K^2$ is of $O\left(K^{-1}\right)$. So, increases in the number of regressors K increase bias and reduce variance. The situation is analogous to bandwidth choice in kernel estimation.

The mean squared error of $\hat{\omega}_K^2$ has the form

$$\text{MSE}\left(\hat{\omega}_K^2\right) = \text{Bias}^2 + \text{Var} = \frac{K^4}{n^4}D^2 + \frac{2\omega^4}{K}.$$

Optimization of this quantity with respect to K leads to the first order condition $\frac{4K^3}{n^4}D^2 - \frac{2\omega^4}{K^2} = 0$, which gives the following formula for the optimal value of K

$$K = n^{\frac{4}{5}}\left[\frac{2\omega^4}{4D^2}\right]^{\frac{1}{5}}. \tag{9}$$

Of course, this is analogous to conventional MSE optimization formulae for bandwidth choice in kernel estimation (e.g. Grenander and Rosenblatt, 1957).

Formula (9) can be used to implement a data-determined choice of K in a conventional way. One approach is to use nonparametric estimates of D^2 and ω^4 in (9) as, for example, in Newey and West (1994). The most common and convenient method in practice is a simple plug-in estimator based on the use of a parametric model for preliminary estimation of ω^4 and D^2 in (9). In the case of a first order autoregression with fitted coefficient $\hat{a}$ and error variance s^2, the standard formulae give $\hat{\omega}^2 = s^2/\left(1-\hat{a}\right)^2$ and $\hat{D} = -\frac{\pi^2}{6}2\hat{a}s^2/\left(1-\hat{a}\right)^4$. Some modifications to these formulae may be desirable in cases where $\hat{a}$ is close to unity. In the context of prewhitening, for example, Andrews and Monahan (1992) proposed a 0.97 rule in which $\hat{a}$ be replaced by 0.97 whenever $\hat{a}$ exceeds this value. It is known that this particular rule seriously interferes with power in some cases, especially stationarity testing (c.f. Lee, 1996). An alternative boundary restriction that seems to improve the size and power properties of procedures based on HAC estimates is the sample-size-dependent rule given in Sul, Phillips and Choi (2003), where $\hat{a}$ is replaced by $1-1/\sqrt{n}$ whenever it exceeds that value.

3. Asymptotic Form as a Kernel Estimator

Lemma B(c) in the Appendix shows that the orthogonal sequence $\varphi_k(r)$ in (8) satisfies the following summation formula

$$\sum_{k=1}^{K}\varphi_k\left(\frac{t}{n}\right)\varphi_k\left(\frac{s}{n}\right) = \frac{1}{2}\frac{\sin\left\{\frac{K\pi(t-s)}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi(t-s)}{n}\right\}} - \frac{1}{2}\frac{\sin\left\{\frac{K\pi(t+s)}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi(t+s)}{n}\right\}}. \tag{10}$$

Using this formula and (27), (31) and (36) from the proof of part (b) of the Theorem, we find that the HAC estimate $\hat{\omega}_K^2$ has the following asymptotic form

$$\begin{aligned}\hat{\omega}_K^2 &= \frac{1}{K}\sum_{h=-n+1}^{n-1}\frac{1}{2}\frac{\sin\left\{\frac{K\pi h}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi h}{n}\right\}}\left(\frac{1}{n}\sum_{1\le t,t+h\le n}u_t u_{t+h}\right)+O_p\left(\frac{\sqrt{K}}{n}+\frac{K^2}{n^2}+\frac{\sqrt{\log n}}{K}\right)\\ &= \sum_{h=-n+1}^{n-1}k_{Kn}\left(\frac{h}{n}\right)\hat{\gamma}_u(h)+O_p\left(\frac{\sqrt{K}}{n}+\frac{K^2}{n^2}+\frac{\sqrt{\log n}}{K}\right), \qquad (11)\end{aligned}$$

where

$$k_{Kn}\left(\frac{h}{n}\right)=\frac{1}{2K}\frac{\sin\left\{\frac{K\pi h}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi h}{n}\right\}}=\frac{1}{K}\cos\left\{\frac{1}{2}\frac{K\pi h}{n}\right\}\frac{\sin\left\{\frac{K\pi h}{2n}\right\}}{\sin\left\{\frac{\pi h}{2n}\right\}} \qquad (12)$$

may be regarded as a lag kernel function and $\hat{\gamma}_u(h)=\frac{1}{n}\sum_{1\le t,t+h\le n}u_t u_{t+h}$ is the sample autocovariance. The dominant term in (11) has the usual form of a kernel estimate of ω^2 and is dependent on K, which serves the role of a smoothing parameter. Thus, $\hat{\omega}_K^2$ behaves asymptotically like a kernel estimate.

Let $n = KM$. For h/n small we can write the lag kernel $k_{Kn}\left(\frac{h}{n}\right)$ as a function of h/M in approximate form as follows

$$k_M\left(\frac{h}{M}\right)=\frac{1}{2K}\frac{\sin\left\{\frac{\pi h}{M}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi h}{KM}\right\}}\sim\frac{\sin\left\{\frac{\pi h}{M}\right\}}{\frac{\pi h}{M}},$$

with which we may associate the function

$$k(x)=\frac{\sin \pi x}{\pi x},$$

which is the lag kernel for the Daniell estimate (e.g., Priestley, 1981, p. 441). This lag kernel is a smoothed periodogram estimate with a rectangular spectral window. Here, $2\pi/M$ is the width of the frequency band over which the periodogram is being smoothed. The regression based estimate $\hat{\omega}_K^2$ is therefore closely related to this well known kernel estimate of ω^2 and has the same bias, variance and mean squared error as the Daniell estimate. Evidently, therefore, the asymptotic mean squared error of $\hat{\omega}_K^2$ is dominated by that of the Bartlett - Priestley quadratic spectral window (Priestley, 1981; Andrews, 1991).

4. The Effect of Different Trend Bases

We may choose to use other sequences of orthogonal functions on $[0, 1]$ in the regression (3) and it is interesting to explore the effects of such alternate choices on the asymptotic kernel form of estimate $\hat{\omega}_K^2$. In the following discussion, we confine our attention to sequences of trigonometric polynomials.

Parts (a) and (b) of Lemma C in the Appendix give the following summation formulae for the orthonormal sine and cosine trigonometric polynomials $\varphi_k(r) =$

$\sqrt{2}\sin\{k\pi r\}$ and $\varphi_k(r) = \sqrt{2}\cos\{k\pi r\}$, respectively,

$$\sum_{k=1}^{K}\varphi_k(\frac{t}{n})\varphi_k(\frac{s}{n}) = \frac{\sin\left\{\frac{\left(K-\frac{1}{2}\right)\pi(t-s)}{n}\right\}}{2\sin\left\{\frac{\pi(t-s)}{2n}\right\}} \mp \frac{\sin\left\{\frac{\left(K-\frac{1}{2}\right)\pi(t+s)}{n}\right\}}{2\sin\left\{\frac{\pi(t+s)}{2n}\right\}}. \tag{13}$$

As in the calculation leading to (11), the second component of (13) turns out to be of smaller order as $K, n \to \infty$ and (11) holds with

$$\begin{aligned} k_{Kn}\left(\frac{h}{n}\right) &= \frac{1}{2K}\frac{\sin\left\{\frac{\left(K-\frac{1}{2}\right)\pi h}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi h}{n}\right\}} = \frac{1}{2K}\frac{\sin\left\{\frac{K\pi h}{n}\right\}\cos\left\{\frac{\pi h}{2n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi h}{n}\right\}} - \frac{\cos\left\{\frac{K\pi h}{n}\right\}}{2K} \\ &= \frac{1}{2K}\frac{\sin\left\{\frac{K\pi h}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi h}{n}\right\}} + O\left(\frac{1}{K}\right), \end{aligned}$$

which is asymptotically equivalent to the lag kernel (12). Thus, these different orthonormal trigonometric sequences all lead to HAC estimates that are asymptotically equivalent.

If the complex orthonormal sequence $\varphi_k(r) = e^{-2\pi i k r}$ is used, then the regression coefficients in (3) are themselves complex and have the form $\widehat{b}_K = \left(\Phi_K^*\Phi_K\right)^{-1}\left(\Phi_K^* u\right)$, where Φ_K is the matrix of observations of the K regressors $\{\varphi_k\left(\frac{t}{n}\right) : k = 1, ..., K\}$ and $*$ signifies complex conjugation and transposition. In place of (3) we can write

$$u_t = \sum_{k=1}^{K}\widehat{b}_k e^{-\frac{2\pi i k t}{n}} + e_t, \tag{14}$$

which may be regarded as a fitted empirical version of the Cramér representation of the stationary process u_t, which was noticed earlier in Phillips (1996, Remark 5.2a). Note that $\lambda_k = \frac{2\pi k}{n} \to 0$ for all $k = 1, .., K$ since $\frac{K}{n} \to 0$. Thus, the regression (14) focuses attention on the zero frequency (or long-run) component of the Cramér representation of u_t.

The matrix Φ_K satisfies $\Phi_K^*\Phi_K = nI_K$ and is a scaled unitary matrix. So the k'th element of $\widehat{b}_K$ is simply the (standardized) discrete Fourier transform, $\frac{1}{n}\sum_{t=1}^{n} e^{\frac{2\pi i k t}{n}} u_t$, of u_t, which is well-known to satisfy a central limit theorem (e.g. Hannan, 1970, p. 224) upon rescaling. Additionally, as shown in Phillips (1999, theorem 3.2), the (asymptotically infinite) collection of K such elements have the following limit as $n \to \infty$ and are asymptotically independent provided $K \to \infty$ but not too fast relative to n. In particular

$$\zeta_{nk} = \frac{1}{\sqrt{n}}\sum_{t=1}^{n} e^{\frac{2\pi i k t}{n}} u_t \Rightarrow \int_0^1 e^{2\pi i k r} dB(r), \quad k = 1, 2, ...$$

As is easily seen, the limit variates $\zeta_k = \int_0^1 e^{2\pi i k r} dB(r)$ are independent complex Gaussian $N_c\left(0, \omega^2\right)$. Letting $P_K = \Phi_K\left(\Phi_K^*\Phi_K\right)^{-1}\Phi_K^*$, we then have

$$\hat{\omega}_K^2 = \frac{1}{K}u' P_K u = \frac{n}{K}\widehat{b}_K^*\widehat{b}_K = \frac{1}{K}\sum_{k=1}^{K}\left|\zeta_{nk}\right|^2 \to_p E\left(\left|\zeta_k\right|^2\right) = \omega^2. \tag{15}$$

As such, the estimate $\hat{\omega}_K^2$ may be interpreted as the sample variance of the empirical estimates (obtained from the fitted regression (14)) of the orthogonal process (at the zero frequency) that appears in the Cramér representation of u_t.

Observe that $|\zeta_{nk}|^2$ is $2\pi I_u(\lambda_k)$, where $I_u(\lambda)$ is the periodogram of u_t and $\lambda_k = \frac{2\pi k}{n}$ are the fundamental frequencies. Thus we can write

$$\hat{\omega}_K^2 = 2\pi \frac{1}{K} \sum_{k=1}^{K} I_u(\lambda_k), \tag{16}$$

which corresponds to a smoothed periodogram estimate of the spectrum at the zero frequency (given that $\frac{K}{n} \to 0$). The spectral window in the estimate $\hat{\omega}_K^2$ is clearly rectangular, just as that of the Daniell window (e.g., Hannan, 1970, p. 279). Again the results are asymptotically equivalent to those of the estimate (7) based on the sinusoidal sequence (8).

Of course, this set-up easily permits the use other spectral windows. Let $W_K = diag\{W(\lambda_1), ..., W(\lambda_K)\}$ be a diagonal matrix prescribing a particular weighting sequence based on the window function $W(\lambda)$. Then the HAC estimate

$$\hat{\omega}_{KW}^2 = \frac{n}{K} \widehat{b}_K^* W_K \widehat{b}_K = 2\pi \frac{1}{K} \sum_{k=1}^{K} W(\lambda_K) I_u(\lambda_k),$$

has the usual form of a smoothed periodogram estimate with spectral window $W(\lambda)$. In this way, the present approach accommodates all conventional kernel-based HAC estimates.

5. Discussion

The estimate $\hat{\omega}_K^2$ is straightforward to compute, being one of the outputs of a linear regression and it has a simple heuristic motivation. The fact that regression on trend produces this consistent estimate indicates that trend coefficients carry information about the long-run features of the data, even though the 'true' trend coefficients in this regression of a stationary time series are zero. In fact, as discussed in the preceeding section, the estimate $\hat{\omega}_K^2$ may be interpreted as the sample variance of the coefficients in a fitted regression that is an empirical version of the long-run part of the Cramér representation of a stationary process.

Simulations (not reported here) indicate that $\hat{\omega}_K^2$ performs as well as current industry-standard methods using quadratic kernels, automated bandwidth selection and prewhitening (Andrews, 1991; Andrews and Monahan, 1992; Den Haan, W.J., and A. Levin, 1997; Lee and Phillips, 1993; Newey and West, 1994). This may be expected given the asymptotic relationship between $\hat{\omega}_K^2$ and the Daniell kernel estimate. The regression approach developed here may be extended to estimate the spectrum of u_t at points other than the origin, although we do not pursue that possibility here.

6. Additional Lemmas and Proofs

6.1 Lemma A *Under* **R**, $n^{-1}\sum_{t=1}^{n}\varphi_{Kt}\varphi_{Kt}' = I_K + O\left(\frac{1}{n}\right)$, and $\left(n^{-1}\sum_{t=1}^{n}\varphi_{Kt}\varphi_{Kt}'\right)^{-1} = I_K + O\left(\frac{1}{n}\right)$, *as* $n \to \infty$.

6.2 Proof We first provide a direct calculation when the elements of φ_{Kt} are given by the trigonometric functions (8). In this case, the diagonal elements of $n^{-1}\sum_{t=1}^{n}\varphi_{Kt}\varphi_{Kt}'$ are

$$\begin{aligned}
\frac{1}{n}\sum_{t=1}^{n}\varphi_k^2\left(\frac{t}{n}\right) &= \frac{2}{n}\sum_{t=1}^{n}\sin^2\left\{\left(k-\frac{1}{2}\right)\frac{\pi t}{n}\right\} = \frac{2}{n}\sum_{t=1}^{n}\frac{1-2\cos\left\{(2k-1)\frac{\pi t}{n}\right\}}{2} \\
&= 1-\frac{2}{n}\sum_{t=1}^{n}\cos\left\{(2k-1)\frac{\pi t}{n}\right\} = 1-\frac{2}{n}\mathrm{Re}\left\{\sum_{t=1}^{n}e^{i(2k-1)\frac{\pi t}{n}}\right\} \\
&= 1-\frac{2}{n}\mathrm{Re}\left\{e^{i(2k-1)\frac{\pi}{n}}\frac{e^{i(2k-1)\pi}-1}{e^{i(2k-1)\frac{\pi}{n}}-1}\right\} = 1+\frac{4}{n}\mathrm{Re}\left\{\frac{e^{i(2k-1)\frac{\pi}{n}}}{e^{i(2k-1)\frac{\pi}{n}}-1}\right\}
\end{aligned}$$

$$\begin{aligned}
&= 1+\frac{4}{n}\mathrm{Re}\left\{\frac{1-e^{i(2k-1)\frac{\pi}{n}}}{\left|e^{i(2k-1)\frac{\pi}{n}}-1\right|^2}\right\} = 1+\frac{4}{n}\frac{1-\cos\left\{(2k-1)\frac{\pi}{n}\right\}}{2-2\cos\left\{(2k-1)\frac{\pi}{n}\right\}} \\
&= 1+\frac{2}{n},
\end{aligned}$$

and the off-diagonals are

$$\begin{aligned}
\frac{1}{n}\sum_{t=1}^{n}\varphi_k\left(\frac{t}{n}\right)\varphi_\ell\left(\frac{t}{n}\right) &= \frac{2}{n}\sum_{t=1}^{n}\sin\left\{\left(k-\frac{1}{2}\right)\frac{\pi t}{n}\right\}\sin\left\{\left(\ell-\frac{1}{2}\right)\frac{\pi t}{n}\right\} \\
&= \frac{1}{n}\sum_{t=1}^{n}\left[\cos\left\{(k-\ell)\frac{\pi t}{n}\right\}-\cos\left\{(k+\ell-1)\frac{\pi t}{n}\right\}\right] \\
&= \frac{1}{n}\mathrm{Re}\left\{\sum_{t=1}^{n}e^{i(k-\ell)\frac{\pi t}{n}}-\sum_{t=1}^{n}e^{i(k+\ell-1)\frac{\pi t}{n}}\right\} \\
&= \frac{1}{n}\mathrm{Re}\left\{e^{i(k-\ell)\frac{\pi}{n}}\frac{e^{i(k-\ell)\pi}-1}{e^{i(k-\ell)\frac{\pi}{n}}-1}-e^{i(k+\ell-1)\frac{\pi}{n}}\frac{e^{i(k+\ell-1)\pi}-1}{e^{i(k+\ell-1)\frac{\pi n}{n}}-1}\right\} \\
&= \begin{cases} \frac{1}{n} & k-\ell \text{ odd, } k+\ell-1 \text{ even} \\ -\frac{1}{n} & k-\ell \text{ even, } k+\ell-1 \text{ odd} \\ 0 & k-\ell, k+\ell-1 \text{ both odd or even} \end{cases}.
\end{aligned}$$

It follows that $n^{-1}\sum_{t=1}^{n}\varphi_{Kt}\varphi_{Kt}' = I_K + O\left(\frac{1}{n}\right)$ and $\left(n^{-1}\sum_{t=1}^{n}\varphi_{Kt}\varphi_{Kt}'\right)^{-1} = I_K + O\left(\frac{1}{n}\right)$.

In the general case, if $\varphi_k(s)$ is twice continuously differentiable on $[0,1]$ by Euler summation we have

$$\begin{aligned}
\frac{1}{n}\sum_{t=1}^{n}\varphi_k^2\left(\frac{t}{n}\right) &= \frac{1}{n}\int_1^n \varphi_k^2\left(\frac{t}{n}\right)dt + \frac{1}{2n}\left\{\varphi_k^2\left(\frac{1}{n}\right)+\varphi_k^2\left(\frac{n}{n}\right)\right\} \\
&\quad + \frac{2}{n}\int_1^n\left\{t-[t]-\frac{1}{2}\right\}\varphi_k\left(\frac{t}{n}\right)\varphi_k'\left(\frac{t}{n}\right)\frac{dt}{n} \\
&= \int_{1/n}^{1}\varphi_k^2(s)\,ds + O\left(\frac{1}{n}\right) = 1 + O\left(\frac{1}{n}\right),
\end{aligned}$$

since $\varphi_k\left(\frac{t}{n}\right)$ and $\varphi_k'\left(\frac{t}{n}\right)$ are uniformly bounded on $[0,1]$. Similarly, $n^{-1}\sum_{t=1}^{n}\varphi_k\left(\frac{t}{n}\right)\varphi_\ell\left(\frac{t}{n}\right) = O\left(\frac{1}{n}\right)$ uniformly for $k\neq\ell$.

6.3 Lemma B *For* $\varphi_k(r)=\sqrt{2}\sin\left\{\left(k-\frac{1}{2}\right)\pi r\right\}$, *we have:*

(a) $\Delta\varphi_k\left(\frac{t}{n}\right) = 2\sqrt{2}\cos\left\{\left(k-\frac{1}{2}\right)\frac{\pi\left(t-\frac{1}{2}\right)}{n}\right\}\sin\left\{\left(k-\frac{1}{2}\right)\frac{\pi}{2n}\right\}$;

(b) $\sum_{k=1}^{K}\varphi_k\left(\frac{t}{n}\right) = \sqrt{2}\frac{\sin^2\left(\frac{K\pi t}{2n}\right)}{\sin\left(\frac{\pi t}{2n}\right)}$;

(c) $\sum_{k=1}^{K}\varphi_k\left(\frac{t}{n}\right)\varphi_k\left(\frac{s}{n}\right) = \frac{1}{2}\frac{\sin\left\{\frac{K\pi(t-s)}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi(t-s)}{n}\right\}} - \frac{1}{2}\frac{\sin\left\{\frac{K\pi(t+s)}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi(t+s)}{n}\right\}}$;

(d) $\frac{\sigma^4 C(1)^4}{Kn}\sum_{h=1}^{n-1}\frac{\sin^2\left\{\frac{K\pi h}{n}\right\}}{\sin^2\left\{\frac{1}{2}\frac{\pi h}{n}\right\}} = 2\sigma^4 C(1)^4\left[1+o(1)\right]$,

where Δ *is the differencing operator* $\Delta\varphi_k\left(\frac{t}{n}\right)=\varphi_k\left(\frac{t}{n}\right)-\varphi_k\left(\frac{t-1}{n}\right)$.

6.4 Proof of Lemma B For part (a)

$$\begin{aligned}
\Delta\varphi_k\left(\frac{t}{n}\right) &= \sqrt{2}\left[\sin\left\{\left(k-\frac{1}{2}\right)\frac{\pi t}{n}\right\}-\sin\left\{\left(k-\frac{1}{2}\right)\frac{\pi(t-1)}{n}\right\}\right] \\
&= 2\sqrt{2}\left[\cos\left\{\left(k-\frac{1}{2}\right)\frac{\pi\left(t-\frac{1}{2}\right)}{n}\right\}\sin\left\{\left(k-\frac{1}{2}\right)\frac{\pi}{2n}\right\}\right].
\end{aligned}$$

For part (b)

$$\begin{aligned}
\sum_{k=1}^{K}\varphi_k\left(\frac{t}{n}\right) &= \sqrt{2}\sum_{k=1}^{K}\sin\left\{\left(k-\frac{1}{2}\right)\frac{\pi t}{n}\right\} = \sqrt{2}\,\mathrm{Im}\left\{\sum_{k=1}^{K}e^{i\left(k-\frac{1}{2}\right)\frac{\pi t}{n}}\right\} \\
&= \sqrt{2}\,\mathrm{Im}\left\{e^{i\frac{\pi t}{2n}}\frac{e^{iK\frac{\pi t}{n}}-1}{e^{i\frac{\pi t}{n}}-1}\right\} = \sqrt{2}\,\mathrm{Im}\left\{e^{i\frac{K\pi t}{2n}}\frac{\sin\left(\frac{K\pi t}{2n}\right)}{\sin\left(\frac{\pi t}{2n}\right)}\right\} \\
&= \sqrt{2}\frac{\sin^2\left(\frac{K\pi t}{2n}\right)}{\sin\left(\frac{\pi t}{2n}\right)}.
\end{aligned}$$

For part (c)

$$\begin{aligned}
\sum_{k=1}^{K}\varphi_k(\frac{t}{n})\varphi_k(\frac{s}{n}) &= 2\sum_{k=1}^{K}\sin\left\{\left(k-\frac{1}{2}\right)\frac{\pi t}{n}\right\}\sin\left\{\left(k-\frac{1}{2}\right)\frac{\pi s}{n}\right\} \\
&= \sum_{k=1}^{K}\left[\cos\left\{\left(k-\frac{1}{2}\right)\frac{\pi\,(t-s)}{n}\right\}-\cos\left\{\left(k-\frac{1}{2}\right)\frac{\pi\,(t+s)}{n}\right\}\right] \\
&= \mathrm{Re}\left\{\sum_{k=1}^{K}\left[e^{i\left(k-\frac{1}{2}\right)\frac{\pi(t-s)}{n}}-e^{i\left(k-\frac{1}{2}\right)\frac{\pi(t+s)}{n}}\right]\right\} \\
&= \mathrm{Re}\left\{e^{\frac{1}{2}\frac{\pi(t-s)}{n}i}\frac{e^{\frac{K\pi(t-s)}{n}i}-1}{e^{\frac{\pi(t-s)}{n}i}-1}-e^{\frac{1}{2}\frac{\pi(t+s)}{n}i}\frac{e^{\frac{K\pi(t+s)}{n}i}-1}{e^{\frac{\pi(t+s)}{n}i}-1}\right\}
\end{aligned}$$

$$\begin{aligned}
&= \mathrm{Re}\left\{e^{\frac{K\pi(t-s)}{2n}i}\frac{\frac{1}{2i}\left(e^{\frac{K\pi(t-s)}{2n}i}-e^{-\frac{K\pi(t-s)}{2n}i}\right)}{\frac{1}{2i}\left(e^{\frac{1}{2}\frac{\pi(t-s)}{n}i}-e^{-\frac{1}{2}\frac{\pi(t-s)}{n}i}\right)}-e^{\frac{K\pi(t+s)}{2n}i}\frac{\frac{1}{2i}\left(e^{\frac{K\pi(t+s)}{2n}i}-e^{-\frac{K\pi(t+s)}{2n}i}\right)}{\frac{1}{2i}\left(e^{\frac{\pi(t+s)}{n}i}-e^{-\frac{1}{2}\frac{\pi(t+s)}{n}i}\right)}\right\} \\
&= \mathrm{Re}\left\{e^{\frac{K\pi(t-s)}{2n}i}\frac{\sin\left\{\frac{K\pi(t-s)}{2n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi(t-s)}{n}\right\}}-e^{\frac{K\pi(t+s)}{2n}i}\frac{\sin\left\{\frac{K\pi(t+s)}{2n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi(t+s)}{n}\right\}}\right\} \\
&= \cos\left\{\frac{1}{2}\frac{K\pi\,(t-s)}{n}\right\}\frac{\sin\left\{\frac{K\pi(t-s)}{2n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi(t-s)}{n}\right\}}-\cos\left\{\frac{1}{2}\frac{K\pi\,(t+s)}{n}\right\}\frac{\sin\left\{\frac{K\pi(t+s)}{2n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi(t+s)}{n}\right\}} \\
&= \frac{1}{2}\frac{\sin\left\{\frac{K\pi(t-s)}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi(t-s)}{n}\right\}}-\frac{1}{2}\frac{\sin\left\{\frac{K\pi(t+s)}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi(t+s)}{n}\right\}}.
\end{aligned}$$

For part (d), we use Fejér's integral giving

$$\begin{aligned}
\frac{\sigma^4 C\,(1)^4}{Kn}\sum_{h=1}^{n-1}\frac{\sin^2\left\{\frac{K\pi h}{n}\right\}}{\sin^2\left\{\frac{1}{2}\frac{\pi h}{n}\right\}} &= \frac{\sigma^4 C\,(1)^4}{K}\int_0^1\frac{\sin^2\left\{\frac{2K\pi r}{2}\right\}}{\sin^2\left\{\frac{1}{2}\pi r\right\}}dr\,[1+o\,(1)] \\
&= \frac{\sigma^4 C\,(1)^4}{K}2K\,[1+o\,(1)] = 2\omega^2\,[1+o\,(1)]\,,
\end{aligned}$$

as required.

6.5 Lemma C

(a) *If* $\varphi_k(r)=\sqrt{2}\sin\{k\pi r\}\,,$

$$\sum_{k=1}^{K}\varphi_k(\frac{t}{n})\varphi_k(\frac{s}{n})=\frac{\sin\left\{\frac{\left(K-\frac{1}{2}\right)\pi(t-s)}{n}\right\}}{2\sin\left\{\frac{\pi(t-s)}{2n}\right\}}-\frac{\sin\left\{\frac{\left(K-\frac{1}{2}\right)\pi(t+s)}{n}\right\}}{2\sin\left\{\frac{\pi(t+s)}{2n}\right\}};$$

(b) *If* $\varphi_k(r) = \sqrt{2}\cos\{k\pi r\}$,

$$\sum_{k=1}^{K}\varphi_k(\frac{t}{n})\varphi_k(\frac{s}{n}) = \frac{\sin\left\{\frac{\left(K-\frac{1}{2}\right)\pi(t-s)}{n}\right\}}{2\sin\left\{\frac{\pi(t-s)}{2n}\right\}} + \frac{\sin\left\{\frac{\left(K-\frac{1}{2}\right)\pi(t+s)}{n}\right\}}{2\sin\left\{\frac{\pi(t+s)}{2n}\right\}};$$

6.6 Proof of Lemma C For part (a)

$$\begin{aligned}
\sum_{k=1}^{K}\varphi_k(\frac{t}{n})\varphi_k(\frac{s}{n}) &= 2\sum_{k=1}^{K}\sin\left\{k\frac{\pi t}{n}\right\}\sin\left\{k\frac{\pi s}{n}\right\} \\
&= \sum_{k=1}^{K}\left[\cos\left\{k\frac{\pi\,(t-s)}{n}\right\} - \cos\left\{k\frac{\pi\,(t+s)}{n}\right\}\right] \\
&= \mathrm{Re}\left\{\sum_{k=1}^{K}\left[e^{ik\frac{\pi(t-s)}{n}} - e^{ik\frac{\pi(t+s)}{n}}\right]\right\} \\
&= \mathrm{Re}\left\{e^{\frac{\pi(t-s)}{n}i}\frac{e^{\frac{K\pi(t-s)}{n}i}-1}{e^{\frac{\pi(t-s)}{n}i}-1} - e^{\frac{\pi(t+s)}{n}i}\frac{e^{\frac{K\pi(t+s)}{n}i}-1}{e^{\frac{\pi(t+s)}{n}i}-1}\right\}
\end{aligned}$$

$$\begin{aligned}
&= \mathrm{Re}\left\{\frac{e^{\frac{K\pi(t-s)}{n}i}-1}{1-e^{-\frac{\pi(t-s)}{n}i}} - \frac{e^{\frac{K\pi(t+s)}{n}i}-1}{1-e^{-\frac{\pi(t+s)}{n}i}}\right\} \\
&= \mathrm{Re}\left\{\frac{\left(e^{\frac{K\pi(t-s)}{n}i}-1\right)\left(1-e^{-\frac{\pi(t-s)}{n}i}\right)}{2-2\cos\left\{\frac{\pi(t-s)}{n}\right\}} - \frac{\left(e^{\frac{K\pi(t+s)}{n}i}-1\right)\left(1-e^{-\frac{\pi(t+s)}{n}i}\right)}{2-2\cos\left\{\frac{\pi(t+s)}{n}\right\}}\right\} \\
&= \frac{-1+\cos\left\{\frac{\pi(t-s)}{n}\right\}+\cos\left\{\frac{K\pi(t-s)}{n}\right\}-\cos\left\{\frac{(K-1)\pi(t-s)}{n}\right\}}{2-2\cos\left\{\frac{\pi(t-s)}{n}\right\}} \\
&\quad -\frac{-1+\cos\left\{\frac{\pi(t+s)}{n}\right\}+\cos\left\{\frac{K\pi(t+s)}{n}\right\}-\cos\left\{\frac{(K-1)\pi(t+s)}{n}\right\}}{2-2\cos\left\{\frac{\pi(t+s)}{n}\right\}} \\
&= -\frac{1}{2}+\frac{2\sin\left\{\frac{\left(K-\frac{1}{2}\right)\pi(t-s)}{n}\right\}\sin\left\{\frac{\pi(t-s)}{2n}\right\}}{2-2\cos\left\{\frac{\pi(t-s)}{n}\right\}} \\
&\quad +\frac{1}{2}-\frac{2\sin\left\{\frac{\left(K-\frac{1}{2}\right)\pi(t+s)}{n}\right\}\sin\left\{\frac{\pi(t+s)}{2n}\right\}}{2-2\cos\left\{\frac{\pi(t+s)}{n}\right\}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sin\left\{\frac{\left(K-\frac{1}{2}\right)\pi(t-s)}{n}\right\}\sin\left\{\frac{\pi(t-s)}{2n}\right\}}{1-\cos\left\{\frac{\pi(t-s)}{n}\right\}} - \frac{\sin\left\{\frac{\left(K-\frac{1}{2}\right)\pi(t+s)}{n}\right\}\sin\left\{\frac{\pi(t+s)}{2n}\right\}}{1-\cos\left\{\frac{\pi(t+s)}{n}\right\}} \\
&= \frac{\sin\left\{\frac{\left(K-\frac{1}{2}\right)\pi(t-s)}{n}\right\}\sin\left\{\frac{\pi(t-s)}{2n}\right\}}{2\sin^2\left\{\frac{\pi(t-s)}{2n}\right\}} - \frac{\sin\left\{\frac{\left(K-\frac{1}{2}\right)\pi(t+s)}{n}\right\}\sin\left\{\frac{\pi(t+s)}{2n}\right\}}{2\sin^2\left\{\frac{\pi(t+s)}{2n}\right\}} \\
&= \frac{\sin\left\{\frac{\left(K-\frac{1}{2}\right)\pi(t-s)}{n}\right\}}{2\sin\left\{\frac{\pi(t-s)}{2n}\right\}} - \frac{\sin\left\{\frac{\left(K-\frac{1}{2}\right)\pi(t+s)}{n}\right\}}{2\sin\left\{\frac{\pi(t+s)}{2n}\right\}}.
\end{aligned}
$$

For part (b)

$$
\begin{aligned}
\sum_{k=1}^{K}\varphi_k\left(\frac{t}{n}\right)\varphi_k\left(\frac{s}{n}\right) &= 2\sum_{k=1}^{K}\cos\left\{k\frac{\pi t}{n}\right\}\cos\left\{k\frac{\pi s}{n}\right\} \\
&= \sum_{k=1}^{K}\left[\cos\left\{k\frac{\pi(t-s)}{n}\right\}+\cos\left\{k\frac{\pi(t+s)}{n}\right\}\right],
\end{aligned}
$$

which differs from part (a) only in the sign of the second term. The stated result therefore follows by the same calculations.

6.7 Proof of Theorem 1

Part (a) In view of Lemma A

$$
\begin{aligned}
E\left(\hat{\omega}_K^2\right) &= \frac{1}{K}\mathrm{tr}\left\{\left(\frac{\Phi_K'\Phi_K}{n}\right)^{-1}\sum_{h=-n+1}^{n-1}\frac{1}{n}\sum_{1\le t,t+h\le n}\varphi_{Kt}\varphi_{Kt+h}'\gamma_u(h)\right\} \\
&= \frac{1}{K}\mathrm{tr}\left\{\sum_{h=-n+1}^{n-1}\frac{1}{n}\sum_{1\le t,t+h\le n}\varphi_{Kt}\varphi_{Kt+h}'\gamma_u(h)\right\}\left[1+O\left(\frac{1}{n}\right)\right] \\
&= \frac{1}{K}\sum_{k=1}^{K}\sum_{h=-n+1}^{n-1}\frac{1}{n}\sum_{1\le t,t+h\le n}\varphi_k\left(\frac{t}{n}\right)\varphi_k\left(\frac{t+h}{n}\right)\gamma_u(h)\left[1+O\left(\frac{1}{n}\right)\right],
\end{aligned}
$$

and, since $\omega^2 = \sum_{h=-\infty}^{\infty} \gamma_u(h)$,

$$
\begin{aligned}
& E\left(\hat{\omega}_K^2 - \omega^2\right) \\
= & \sum_{h=-n+1}^{n-1} \frac{1}{K} \sum_{k=1}^{K} \left\{ \frac{1}{n} \sum_{1 \le t, t+h \le n} \varphi_k\left(\frac{t}{n}\right) \varphi_k\left(\frac{t+h}{n}\right) - 1 \right\} \gamma_u(h) \left[1 + O\left(\frac{1}{n}\right)\right] \\
& - \sum_{|h| \ge n}^{\infty} \gamma_u(h) \\
= & \sum_{h=-n+1}^{n-1} \frac{1}{K} \sum_{k=1}^{K} \left\{ \frac{1}{n} \sum_{1 \le t, t+h \le n} \varphi_k\left(\frac{t}{n}\right) \varphi_k\left(\frac{t+h}{n}\right) - 1 \right\} \gamma_u(h) \left[1 + O\left(\frac{1}{n}\right)\right] \\
& + o\left(\frac{1}{n^a}\right), \qquad (17)
\end{aligned}
$$

for $a > 3$ because

$$
\left| \sum_{|h| \ge n}^{\infty} \gamma_u(h) \right| \le \sum_{|h| \ge n}^{\infty} |\gamma_u(h)| \le \frac{1}{n^a} \sum_{|h| \ge n}^{\infty} |h|^a |\gamma_u(h)| = o\left(\frac{1}{n^a}\right),
$$

by condition **L**. Also for any positive integer $L_n < n$ we have

$$
\begin{aligned}
& \sum_{h=-n+1}^{n-1} \frac{1}{K} \sum_{k=1}^{K} \left\{ \frac{1}{n} \sum_{1 \le t, t+h \le n} \varphi_k\left(\frac{t}{n}\right) \varphi_k\left(\frac{t+h}{n}\right) - 1 \right\} \gamma_u(h) \left[1 + O\left(\frac{1}{n}\right)\right] \\
= & \sum_{h=-L_n}^{L_n} \frac{1}{K} \sum_{k=1}^{K} \left\{ \frac{1}{n} \sum_{1 \le t, t+h \le n} \varphi_k\left(\frac{t}{n}\right) \varphi_k\left(\frac{t+h}{n}\right) - 1 \right\} \gamma_u(h) \left[1 + O\left(\frac{1}{n}\right)\right] \\
& + \sum_{L_n < |h| < n} \frac{1}{K} \sum_{k=1}^{K} \left\{ \frac{1}{n} \sum_{1 \le t, t+h \le n} \varphi_k\left(\frac{t}{n}\right) \varphi_k\left(\frac{t+h}{n}\right) - 1 \right\} \gamma_u(h) \left[1 + O\left(\frac{1}{n}\right)\right],
\end{aligned}
\qquad (18)
$$

and since the elements $\varphi_k\left(\frac{t}{n}\right)$ are bounded uniformly in t we have

$$
\begin{aligned}
& \left| \sum_{L_n < |h| < n} \frac{1}{K} \sum_{k=1}^{K} \left\{ \frac{1}{n} \sum_{1 \le t, t+h \le n} \varphi_k\left(\frac{t}{n}\right) \varphi_k\left(\frac{t+h}{n}\right) - 1 \right\} \gamma_u(h) \right| \\
\le \ & C \sum_{L_n < |h| < n} |\gamma_u(h)| \le \frac{C}{L_n^3} \sum_{L_n < |h| < n} h^3 |\gamma_u(h)| = o\left(\frac{1}{L_n^3}\right).
\end{aligned}
$$

Now choose L_n such that

$$
\frac{n}{L_n^{3/2} K} + \frac{L_n K}{n} = o(1) \qquad (19)
$$

as $n \to \infty$. Then, $\frac{1}{L_n^3} = o\left(\frac{K^2}{n^2}\right)$ and

$$\left| \sum_{L_n < |h| < n} \frac{1}{K} \sum_{k=1}^{K} \left\{ \frac{1}{n} \sum_{1 \le t, t+h \le n} \varphi_k\left(\frac{t}{n}\right) \varphi_k\left(\frac{t+h}{n}\right) - 1 \right\} \gamma_u(h) \right| = o\left(\frac{K^2}{n^2}\right). \quad (20)$$

For the first term of (18) we note that

$$\begin{aligned} & \frac{1}{n} \sum_{1 \le t, t+h \le n} \varphi_k\left(\frac{t}{n}\right) \varphi_k\left(\frac{t+h}{n}\right) \\ = & \frac{1}{n} \sum_{1 \le t, t+h \le n} \varphi_k\left(\frac{t}{n}\right)^2 + \frac{1}{n} \sum_{1 \le t, t+h \le n} \varphi_k\left(\frac{t}{n}\right) \left[\varphi_k\left(\frac{t+h}{n}\right) - \varphi_k\left(\frac{t}{n}\right)\right] \\ = & 1 + \frac{1}{n} \sum_{1 \le t, t+h \le n} \varphi_k\left(\frac{t}{n}\right) \left[\varphi_k\left(\frac{t+h}{n}\right) - \varphi_k\left(\frac{t}{n}\right)\right] + O\left(\frac{1}{n}\right), \end{aligned} \quad (21)$$

uniformly in $|h| \le L_n$. It therefore follows from (17) - (21) that

$$\begin{aligned} & E\left(\hat{\omega}_K^2 - \omega^2\right) \\ = & \sum_{h=-L_n}^{L_n} \frac{1}{K} \sum_{k=1}^{K} \left\{ \frac{1}{n} \sum_{1 \le t, t+h \le n} \varphi_k\left(\frac{t}{n}\right) \left[\varphi_k\left(\frac{t+h}{n}\right) - \varphi_k\left(\frac{t}{n}\right)\right] \right\} \gamma_u(h) \\ & + o\left(\frac{K^2}{n^2}\right) + O\left(\frac{1}{n}\right). \end{aligned} \quad (22)$$

Next consider

$$\begin{aligned} & \frac{1}{n} \sum_{1 \le t, t+h \le n} \varphi_k\left(\frac{t}{n}\right) \left[\varphi_k\left(\frac{t+h}{n}\right) - \varphi_k\left(\frac{t}{n}\right)\right] \\ = & \frac{2}{n} \sum_{1 \le t, t+h \le n} \sin\left\{\left(k - \frac{1}{2}\right)\frac{\pi t}{n}\right\} \left[\sin\left\{\left(k - \frac{1}{2}\right)\frac{\pi(t+h)}{n}\right\} - \sin\left\{\left(k - \frac{1}{2}\right)\frac{\pi t}{n}\right\}\right] \\ = & \frac{1}{n} \sum_{1 \le t, t+h \le n} \left\{ \cos\left\{\left(k - \frac{1}{2}\right)\frac{\pi h}{n}\right\} - \cos\left\{(2k-1)\frac{\pi\left(t + \frac{1}{2}h\right)}{n}\right\} \right\} \\ & - \frac{1}{n} \sum_{1 \le t, t+h \le n} \left\{ 1 - \cos\left\{(2k-1)\frac{\pi t}{n}\right\} \right\} \\ = & \left(1 - \frac{|h|}{n}\right) \left[\cos\left\{\left(k - \frac{1}{2}\right)\frac{\pi h}{n}\right\} - 1\right] \\ & - \frac{1}{n} \sum_{1 \le t, t+h \le n} \left[\cos\left\{(2k-1)\frac{\pi\left(t + \frac{1}{2}h\right)}{n}\right\} - \cos\left\{(2k-1)\frac{\pi t}{n}\right\}\right] \end{aligned} \quad (23)$$

Taking the first term of (23), averaging over k, and using the fact that $|h| \le L_n$ and

L_n satisfies (19) so that $\frac{L_n K}{n} = o(1)$, we get

$$\begin{aligned}
&\frac{1}{K}\sum_{k=1}^{K}\cos\left\{\left(k-\frac{1}{2}\right)\frac{\pi h}{n}\right\}-1 \\
=\ &\frac{1}{K}\sum_{k=1}^{K}\left\{1-\frac{1}{2}\left(k-\frac{1}{2}\right)^2\left(\frac{\pi h}{n}\right)^2+o\left(\frac{K^2h^2}{n^2}\right)\right\}-1 \\
=\ &-\frac{K^2\pi^2h^2}{6n^2}\left[1+o(1)\right]. \qquad (24)
\end{aligned}$$

For $h \geq 0$ we can write the second term of (23) as

$$\begin{aligned}
&\frac{1}{n}\sum_{1\leq t,t+h\leq n}\left[\cos\left\{(2k-1)\frac{\pi\left(t+\frac{1}{2}h\right)}{n}\right\}-\cos\left\{(2k-1)\frac{\pi t}{n}\right\}\right] \\
=\ &\int_{1/n}^{1-h/n}\cos\left\{(2k-1)\pi\left(r+\frac{h}{2n}\right)\right\}dr\left[1+O\left(\frac{1}{n}\right)\right] \\
&-\int_{1/n}^{1-h/n}\cos\left\{(2k-1)\pi r\right\}dr\left[1+O\left(\frac{1}{n}\right)\right] \\
=\ &\left[\frac{\sin\left\{(2k-1)\pi\left(r+\frac{h}{2n}\right)\right\}}{(2k-1)\pi}\right]_{1/n}^{1-h/n}\left[1+O\left(\frac{1}{n}\right)\right] \\
&-\left[\frac{\sin\left\{(2k-1)\pi r\right\}}{(2k-1)\pi}\right]_{1/n}^{1-h/n}\left[1+O\left(\frac{1}{n}\right)\right]
\end{aligned}$$

$$\begin{aligned}
=\ &\frac{1}{(2k-1)\pi}\left[\sin\left\{(2k-1)\pi\left(1-\frac{h}{2n}\right)\right\}-\sin\left\{(2k-1)\pi\left(\frac{h+2}{2n}\right)\right\}\right]\left[1+O\left(\frac{1}{n}\right)\right] \\
&-\frac{1}{(2k-1)\pi}\left[\sin\left\{(2k-1)\pi\left(1-\frac{h}{n}\right)\right\}-\sin\left\{(2k-1)\pi\left(\frac{1}{n}\right)\right\}\right]\left[1+O\left(\frac{1}{n}\right)\right] \\
=\ &\frac{1}{(2k-1)\pi}\left[\sin\left\{(2k-1)\pi\left(1-\frac{h}{2n}\right)\right\}-\sin\left\{(2k-1)\pi\left(1-\frac{h}{n}\right)\right\}\right]\left[1+O\left(\frac{1}{n}\right)\right] \\
&-\frac{1}{(2k-1)\pi}\left[\sin\left\{(2k-1)\pi\left(\frac{h+2}{2n}\right)\right\}-\sin\left\{(2k-1)\pi\left(\frac{1}{n}\right)\right\}\right]\left[1+O\left(\frac{1}{n}\right)\right] \\
=\ &\frac{2}{(2k-1)\pi}\left[\cos\left\{(2k-1)\pi\left(1-\frac{3h}{4n}\right)\right\}\sin\left\{(2k-1)\pi\frac{h}{4n}\right\}\right]\left[1+O\left(\frac{1}{n}\right)\right] \\
&-\frac{2}{(2k-1)\pi}\left[\cos\left\{(2k-1)\pi\left(\frac{h+4}{4n}\right)\right\}\sin\left\{(2k-1)\pi\left(\frac{h}{4n}\right)\right\}\right]\left[1+O\left(\frac{1}{n}\right)\right] \\
=\ &2\left\{\cos\left\{(2k-1)\pi\left(1-\frac{3h}{4n}\right)\right\}\frac{h}{4n}-\frac{h}{4n}\right\}\left[1+O\left(\frac{1}{n}\right)\right], \qquad (25)
\end{aligned}$$

for $|h| \le L_n$ and L_n satisfying (19). Averaging over k we find

$$\begin{aligned}
& -\frac{1}{K}\sum_{k=1}^{K}\cos\left\{(2k-1)\pi\left(1-\frac{3h}{4n}\right)\right\}\frac{h}{4n}-\frac{h}{4n} \\
= & -\frac{1}{K}\sum_{k=1}^{K}\cos\left\{-\pi+(2k-1)\frac{3h}{4n}\right\}\frac{h}{4n}-\frac{h}{4n} \\
= & \frac{1}{K}\sum_{k=1}^{K}\cos\left\{(2k-1)\frac{3h}{4n}\right\}\frac{h}{4n}-\frac{h}{4n} \\
= & \frac{1}{K}\sum_{k=1}^{K}\left\{1-\frac{1}{2}(2k-1)^2\pi^2\left(\frac{3h}{4n}\right)^2\right\}\frac{h}{4n}-\frac{h}{4n} \\
= & \frac{4K^2\pi^2 9h^2}{16n^2}\frac{h}{4n}+o\left(\frac{K^2h^3}{n^3}\right)=o\left(\frac{K^2h^2}{n^2}\right)=o\left(\frac{K^2L_n^2}{n^2}\right). \qquad (26)
\end{aligned}$$

uniformly in $0 \le h \le L_n$. Without going through the calculation, the same rate result holds when $-L_n \le h < 0$.

We deduce from (23), (24), (25) and (26) that

$$\begin{aligned}
& \frac{1}{K}\sum_{k=1}^{K}\frac{1}{n}\sum_{1\le t,t+h\le n}\varphi_k\left(\frac{t}{n}\right)\left[\varphi_k\left(\frac{t+h}{n}\right)-\varphi_k\left(\frac{t}{n}\right)\right] \\
= & -\left(1-\frac{|h|}{n}\right)\frac{K^2\pi^2h^2}{6n^2}+o\left(\frac{K^2h^2}{n^2}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
& E\left(\hat{\omega}_K^2-\omega^2\right) \\
= & \sum_{h=-L_n}^{L_n}\frac{1}{K}\sum_{k=1}^{K}\left\{\frac{1}{n}\sum_{1\le t,t+h\le n}\varphi_k\left(\frac{t}{n}\right)\left[\varphi_k\left(\frac{t+h}{n}\right)-\varphi_k\left(\frac{t}{n}\right)\right]\right\}\gamma_u(h) \\
& +o\left(\frac{K^2}{n^2}\right)+O\left(\frac{1}{n}\right) \\
= & -\frac{K^2\pi^2}{6n^2}\sum_{h=-L_n}^{L_n}\left(1-\frac{|h|}{n}\right)h^2\gamma_u(h)+o\left(\frac{K^2}{n^2}\right)+O\left(\frac{1}{n}\right) \\
= & -\frac{K^2\pi^2}{6n^2}\sum_{h=-\infty}^{\infty}h^2\gamma_u(h)\left[1+o(1)\right],
\end{aligned}$$

since $\frac{1}{n}=o\left(\frac{K^2}{n}\right)$ for K satisfying **R.** Thus

$$\lim_{n\to\infty}\left(\frac{n}{K}\right)^2 E\left(\hat{\omega}_K^2-\omega^2\right)=-\frac{\pi^2}{6}\sum_{h=-\infty}^{\infty}h^2\gamma_u(h),$$

as stated.

Part (b) From Lemma A

$$
\begin{aligned}
\hat{\omega}_K^2 &= \frac{1}{K}u'P_K u = \frac{1}{K}\left(\frac{u'\Phi_K}{\sqrt{n}}\right)\left(\frac{\Phi_K'\Phi_K}{n}\right)^{-1}\left(\frac{\Phi_K' u}{\sqrt{n}}\right) \\
&= \frac{1}{K}\left(\frac{u'\Phi_K}{\sqrt{n}}\right)\left(\frac{\Phi_K' u}{\sqrt{n}}\right)\left[1+O\left(\frac{1}{n}\right)\right] \\
&= \frac{1}{K}\sum_{k=1}^{K}\frac{1}{n}\sum_{t,s=1}^{n}\varphi_k(\frac{t}{n})\varphi_k(\frac{s}{n})u_t u_s\left[1+O\left(\frac{1}{n}\right)\right] \\
&= \frac{1}{K}\sum_{k=1}^{K}\sum_{h=-n+1}^{n-1}\frac{1}{n}\sum_{1\le t,t+h\le n}\varphi_k(\frac{t}{n})\varphi_k(\frac{t+h}{n})u_t u_{t+h}\left[1+O\left(\frac{1}{n}\right)\right]. \qquad (27)
\end{aligned}
$$

Using the device in Phillips and Solo (1992) we have the decomposition

$$
u_t = C(1)\varepsilon_t + \tilde{\varepsilon}_{t-1} - \tilde{\varepsilon}_t, \quad \text{for } \tilde{\varepsilon}_t = \sum_{j=0}^{\infty}\tilde{c}_j\varepsilon_{t-j},\ \tilde{c}_j = \sum_{j+1}^{\infty}c_s, \qquad (28)
$$

where $\sum_{j=0}^{\infty}|\tilde{c}_j| < \infty$ under **L**. Then, (28), partial summation and Lemma B(a) yield

$$
\begin{aligned}
\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\varphi_k(\frac{t}{n})u_t &= \frac{C(1)}{\sqrt{n}}\sum_{t=1}^{n}\varphi_k(\frac{t}{n})\varepsilon_t - \frac{C(1)}{\sqrt{n}}\sum_{t=1}^{n}\varphi_k(\frac{t}{n})\Delta\tilde{\varepsilon}_t \\
&= \frac{C(1)}{\sqrt{n}}\sum_{t=1}^{n}\varphi_k(\frac{t}{n})\varepsilon_t - \left\{\frac{1}{\sqrt{n}}\varphi_k(1)\tilde{\varepsilon}_n - \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\Delta\varphi_k(\frac{t}{n})\tilde{\varepsilon}_t\right\} \\
&= \frac{C(1)}{\sqrt{n}}\sum_{t=1}^{n}\varphi_k(\frac{t}{n})\varepsilon_t - \varphi_k(1)\frac{\tilde{\varepsilon}_n}{\sqrt{n}} \\
&\quad +2^{\frac{3}{2}}\left[\sum_{t=1}^{n}\frac{\tilde{\varepsilon}_t}{\sqrt{n}}\cos\left\{\left(k-\frac{1}{2}\right)\frac{\pi\left(t-\frac{1}{2}\right)}{n}\right\}\right]\frac{1}{n}\left(k-\frac{1}{2}\right)\frac{\pi}{2}\frac{\sin\left\{\left(k-\frac{1}{2}\right)\frac{\pi}{2n}\right\}}{\left(k-\frac{1}{2}\right)\frac{\pi}{2n}} \\
&= \frac{C(1)}{\sqrt{n}}\sum_{t=1}^{n}\varphi_k(\frac{t}{n})\varepsilon_t + O_p\left(\frac{1}{\sqrt{n}}\right) + O_p\left(\frac{k}{n}\right) \\
&= \frac{C(1)}{\sqrt{n}}\sum_{t=1}^{n}\varphi_k(\frac{t}{n})\varepsilon_t + O_p\left(\frac{1}{\sqrt{n}}+\frac{K}{n}\right),
\end{aligned}
$$

uniformly in $k \leq K$. Thus,

$$\begin{aligned}
\hat{\omega}_K^2 &= \frac{1}{K}\sum_{k=1}^{K}\left\{\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\varphi_k(\frac{t}{n})u_t\right\}^2\left[1+O\left(\frac{1}{n}\right)\right] \\
&= \frac{1}{K}\sum_{k=1}^{K}\left\{\frac{C(1)}{\sqrt{n}}\sum_{t=1}^{n}\varphi_k(\frac{t}{n})\varepsilon_t + O_p\left(\frac{K}{n}\right)\right\}^2\left[1+O\left(\frac{1}{n}\right)\right] \\
&= \frac{1}{K}\sum_{k=1}^{K}\left\{\frac{C(1)}{\sqrt{n}}\sum_{t=1}^{n}\varphi_k(\frac{t}{n})\varepsilon_t\right\}^2\left[1+O\left(\frac{1}{n}\right)\right] \\
&\quad + O_p\left(\frac{K}{n}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{1}{K}\sum_{k=1}^{K}\varphi_k(\frac{t}{n})\varepsilon_t\right) + O_p\left(\frac{K^2}{n^2}\right). \qquad (29)
\end{aligned}$$

From Lemma B(b), $\sum_{k=1}^{K}\varphi_k(\frac{t}{n}) = \sqrt{2}\frac{\sin^2\left(\frac{K\pi t}{2n}\right)}{\sin\left(\frac{\pi t}{2n}\right)}$, and so

$$\begin{aligned}
\mathrm{Var}\left\{\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\left[\sum_{k=1}^{K}\varphi_k(\frac{t}{n})\right]\varepsilon_t\right\} &= \frac{2\sigma^2}{n}\sum_{t=1}^{n}\frac{\sin^4\left(\frac{K\pi t}{2n}\right)}{\sin^2\left(\frac{\pi t}{2n}\right)} \\
&\leq \frac{2\sigma^2}{n}\sum_{t=1}^{n}\frac{\sin^2\left(\frac{K\pi t}{2n}\right)}{\sin^2\left(\frac{\pi t}{2n}\right)} \\
&= O\left(\int_0^1\frac{\sin^2\left(\frac{K\pi r}{2}\right)}{\sin^2\left(\frac{\pi r}{2}\right)}dr\right) \\
&= O(K),
\end{aligned}$$

by Lemma B(d). Hence,

$$\frac{1}{\sqrt{K}\sqrt{n}}\sum_{t=1}^{n}\left[\sum_{k=1}^{K}\varphi_k(\frac{t}{n})\right]\varepsilon_t = O_p(1), \qquad (30)$$

and it follows from (29) and (30) that

$$\begin{aligned}
\hat{\omega}_K^2 &= \frac{1}{K}\sum_{k=1}^{K}\left\{\frac{C(1)}{\sqrt{n}}\sum_{t=1}^{n}\varphi_k(\frac{t}{n})\varepsilon_t\right\}^2\left[1+O\left(\frac{1}{n}\right)\right] \\
&\quad + O_p\left(\frac{\sqrt{K}}{n}+\frac{K^2}{n^2}\right). \qquad (31)
\end{aligned}$$

Thus,

$$\sqrt{K}\left(\hat{\omega}_K^2-\omega^2\right) = \frac{C(1)^2}{\sqrt{K}}\sum_{k=1}^{K}\left[\left\{\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\varphi_k(\frac{t}{n})\varepsilon_t\right\}^2-\sigma^2\right] + O_p\left(\frac{K}{n}+\frac{K^{\frac{5}{2}}}{n^2}\right) \quad (32)$$

Write

$$
\begin{aligned}
& \frac{C(1)^2}{\sqrt{K}} \sum_{k=1}^{K} \left[\left\{ \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \varphi_k(\frac{t}{n}) \varepsilon_t \right\}^2 - \sigma^2 \right] \\
= & \frac{C(1)^2}{\sqrt{K}} \sum_{k=1}^{K} \left[\left(\frac{1}{n} \sum_{t=1}^{n} \varphi_k(\frac{t}{n})^2 \varepsilon_t^2 - \sigma^2 \right) + \frac{2}{n} \sum_{t>s}^{n} \varphi_k(\frac{t}{n}) \varphi_k(\frac{s}{n}) \varepsilon_t \varepsilon_s \right]. \quad (33)
\end{aligned}
$$

In view of Lemma A, we have

$$
\begin{aligned}
& \frac{C(1)^2}{\sqrt{K}} \sum_{k=1}^{K} \left\{ \frac{1}{n} \sum_{t=1}^{n} \varphi_k(\frac{t}{n})^2 \varepsilon_t^2 - \sigma^2 \right\} \\
= & \frac{C(1)^2}{\sqrt{K}} \sum_{k=1}^{K} \left\{ \frac{1}{n} \sum_{t=1}^{n} \varphi_k(\frac{t}{n})^2 \left(\varepsilon_t^2 - \sigma^2\right) + \left(\frac{1}{n} \sum_{t=1}^{n} \varphi_k(\frac{t}{n})^2 - 1 \right) \sigma^2 \right\} \\
= & \frac{C(1)^2}{\sqrt{K}} \sum_{k=1}^{K} \left\{ \frac{1}{n} \sum_{t=1}^{n} \varphi_k(\frac{t}{n})^2 \left(\varepsilon_t^2 - \sigma^2\right) \right\} + O\left(\frac{\sqrt{K}}{n} \right) \\
= & O_p\left(\frac{\sqrt{K}}{\sqrt{n}} + \frac{\sqrt{K}}{n} \right). \quad (34)
\end{aligned}
$$

Thus, for $K = o\left(n^{4/5}\right)$ we have from (32) - (34)

$$
\begin{aligned}
\sqrt{K}\left(\hat{\omega}_K^2 - \omega^2\right) = & \frac{C(1)^2}{\sqrt{K}} \sum_{k=1}^{K} \frac{2}{n} \sum_{t>s}^{n} \varphi_k(\frac{t}{n}) \varphi_k(\frac{s}{n}) \varepsilon_t \varepsilon_s + o_p(1) \\
= & \frac{C(1)^2}{\sqrt{K}} \frac{2}{n} \sum_{t>s}^{n} \left\{ \sum_{k=1}^{K} \varphi_k(\frac{t}{n}) \varphi_k(\frac{s}{n}) \right\} \varepsilon_t \varepsilon_s + o_p(1) \\
= & \frac{C(1)^2}{\sqrt{K}} \frac{1}{n} \left\{ \sum_{t>s}^{n} \frac{\sin\left\{\frac{K\pi(t-s)}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi(t-s)}{n}\right\}} - \frac{\sin\left\{\frac{K\pi(t+s)}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi(t+s)}{n}\right\}} \right\} \varepsilon_t \varepsilon_s + o_p(1) \\
= & \frac{C(1)^2}{\sqrt{K}\sqrt{n}} \sum_{h=1}^{n-1} \frac{\sin\left\{\frac{K\pi h}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi h}{n}\right\}} \left(\frac{1}{\sqrt{n}} \sum_{1 \le t, t+h \le n} \varepsilon_t \varepsilon_{t+h} \right) \\
& + \frac{C(1)^2}{\sqrt{K}} \frac{1}{n} \sum_{t>s}^{n} \frac{\sin\left\{\frac{K\pi(t+s)}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi(t+s)}{n}\right\}} \varepsilon_t \varepsilon_s + o_p(1). \quad (35)
\end{aligned}
$$

We show that the second term in the final expression (35) is negligible. Note that

$$
\frac{C(1)^2}{\sqrt{K}} \frac{1}{n} \sum_{t>s}^{n} \frac{\sin\left\{\frac{K\pi(t+s)}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi(t+s)}{n}\right\}} \varepsilon_t \varepsilon_s
$$

has mean zero and variance

$$
\begin{aligned}
& \frac{1}{K}\frac{\sigma^4}{n^2}\sum_{t=2}^{n}\sum_{s=1}^{t-1}\frac{\sin^2\left\{\frac{K\pi(t+s)}{n}\right\}}{\sin^2\left\{\frac{1}{2}\frac{\pi(t+s)}{n}\right\}} \\
= \; & O\left(\frac{\sigma^4}{K}\int_{\frac{2}{n}}^{1}\int_{\frac{1}{n}}^{r-\frac{1}{n}}\frac{\sin^2\left\{K\pi\left(r+p\right)\right\}}{\sin^2\left\{\frac{1}{2}\pi\left(r+p\right)\right\}}dpdr\right) \\
= \; & O\left(\frac{\sigma^4}{K}\int_{\frac{2}{n}}^{1}\int_{\frac{1}{n}}^{r-\frac{1}{n}}\frac{1}{\sin^2\left\{\frac{1}{2}\pi\left(r+p\right)\right\}}dpdr\right) \\
= \; & O\left(\frac{\sigma^4}{K}\int_{\frac{2}{n}}^{1}\left[-\frac{2}{\pi}\frac{\cos\left\{\frac{\pi}{2}\left(r+p\right)\right\}}{\sin\left\{\frac{\pi}{2}\left(r+p\right)\right\}}\right]_{\frac{1}{n}}^{r-\frac{1}{n}}dr\right) \\
= \; & O\left(\frac{\sigma^4}{K}\int_{\frac{2}{n}}^{1}\left\{\frac{2}{\pi}\frac{\cos\left\{\frac{\pi}{2}\left(2r-\frac{1}{n}\right)\right\}}{\sin\left\{\frac{\pi}{2}\left(2r-\frac{1}{n}\right)\right\}}+\frac{2}{\pi}\frac{\cos\left\{\frac{\pi}{2}\left(r+\frac{1}{n}\right)\right\}}{\sin\left\{\frac{\pi}{2}\left(r+\frac{1}{n}\right)\right\}}\right\}dr\right) \\
= \; & O\left(\frac{\log n}{K}\right).
\end{aligned}
$$

Hence

$$
\frac{C\left(1\right)^2}{\sqrt{K}}\frac{1}{n}\sum_{t>s}^{n}\frac{\sin\left\{\frac{K\pi(t+s)}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi(t+s)}{n}\right\}}\varepsilon_t\varepsilon_s = O_p\left(\sqrt{\frac{\log n}{K}}\right) = o_p\left(1\right), \tag{36}
$$

and so

$$
\sqrt{K}\left(\hat{\omega}_K^2-\omega^2\right) = \frac{C\left(1\right)^2}{\sqrt{K}\sqrt{n}}\sum_{h=1}^{n-1}\frac{\sin\left\{\frac{K\pi h}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi h}{n}\right\}}\left(\frac{1}{\sqrt{n}}\sum_{1\le t,t+h\le n}\varepsilon_t\varepsilon_{t+h}\right)+o_p\left(1\right).
$$

Next note from Lemma B(d) that

$$
\frac{\sigma^4 C\left(1\right)^4}{Kn}\sum_{h=1}^{n-1}\frac{\sin^2\left\{\frac{K\pi h}{n}\right\}}{\sin^2\left\{\frac{1}{2}\frac{\pi h}{n}\right\}} = 2\sigma^4 C\left(1\right)^4\left[1+o\left(1\right)\right].
$$

Finally, using a martingale central limit argument along the same lines as that in Phillips, Sun and Jin (2003), we may establish that

$$
\frac{C\left(1\right)^2}{\sqrt{K}\sqrt{n}}\sum_{h=1}^{n-1}\frac{\sin\left\{\frac{K\pi h}{n}\right\}}{\sin\left\{\frac{1}{2}\frac{\pi h}{n}\right\}}\left(\frac{1}{\sqrt{n}}\sum_{1\le t,t+h\le n}\varepsilon_t\varepsilon_{t+h}\right)\rightarrow_d N\left(0,2\sigma^4C\left(1\right)^4\right)\equiv N\left(0,2\omega^4\right),
$$

giving the stated CLT for $\sqrt{K}\left(\hat{\omega}_K^2-\omega^2\right)$.

Part (c) Part (a) shows that the bias of $\hat{\omega}_K^2$ is given by

$$E\left(\hat{\omega}_K^2 - \omega^2\right) = -\frac{K^2}{n^2} D\left[1 + o\left(1\right)\right], \quad \text{where } D = -\frac{\pi^2}{6} \sum_{h=-\infty}^{\infty} h^2 \gamma_u\left(h\right),$$

while arguments as in part (b) show the variance of $\hat{\omega}_K^2$ to be

$$\text{Var}\left\{\hat{\omega}_K^2\right\} = \frac{2\omega^4}{K}\left[1 + o\left(1\right)\right].$$

It follows that

$$\text{MSE}\left(\hat{\omega}_K^2\right) = E\left(\hat{\omega}_K^2 - \omega^2\right)^2 = \left\{\left(\frac{K^2}{n^2} D\right)^2 + \left(\frac{2\omega^4}{K}\right)\right\}\left[1 + o\left(1\right)\right]$$

and, if $K^5/n^4 \to 1$, we get

$$\lim_{n\to\infty}\left(\frac{n}{K}\right)^4 \text{MSE}\left(\hat{\omega}_K^2\right) = \lim_{n\to\infty}\left(\frac{n}{K}\right)^4 E\left(\hat{\omega}_K^2 - \omega^2\right)^2 = D^2 + 2\omega^4,$$

as stated.

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