STRATEGIC FREEDOM, CONSTRAINT AND SYMMETRY IN ONE-PERIOD MARKETS WITH CASH AND CREDIT PAYMENT

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Strategic freedom, constraint, and symmetry in one-period markets with cash and credit payment

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In order to explain in a systematic way why certain combinations of market, financial, and legal structures may be intrinsic to certain capabilities to exchange real goods, we introduce criteria for abstracting the qualitative functions of markets. The criteria involve the number of strategic freedoms the combined institutions, considered as formalized strategic games, present to traders, the constraints they impose, and the symmetry with which those constraints are applied to the traders. We pay particular attention to what is required to make these "strategic market games" well-defined, and to make various solutions computable by the agents within the bounds on information and control they are assumed to have. As an application of these criteria, we present a complete taxonomy of the minimal one-period exchange economies with symmetric information and inside money. A natural hierarchy of market forms is observed to emerge, in which institutionally simpler markets are often found to be more suitable to fewer and less-diversified traders, while the institutionally richer markets only become functional as the size and diversity of their users gets large.

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I. INTRODUCTION: A FOUNDATION FOR ANALYSIS OF MARKETS

The empirical diversity of markets, and the financial and legal institutions that support them, lends itself to two types of analysis. The first is essentially descriptive and frequently historical, aiming at an account of the detailed structure and operation of the many instruments and institutions in any real market system. The second could be called "causal", aiming to explain how differently-structured markets compare to each other in abstract function, and how a functional distinction may lead to preference for certain specific institutional structures.

Nothing mysterious is intended here by the term "abstract function". One long-range goal of a causal theory of markets should be to determine whether definite institutional structures can be sensibly selected and sustained according to what they enable people to do from day to day, somewhat independently of the particular histories through which any given instance may have come about. Since the operation of real markets is complex, and the diversity among their histories enormous, it is a nontrivial matter to separate those details that may be history dependent from those that could not be otherwise and still allow the surrounding society to function as it does. This paper and its companion [1] provide a set of fundamental criteria for generalizing the function of markets, and a taxonomy based on it that includes all of the distinguishable forms of one-period exchange with inside money [2].

The function of markets can be generalized in many domains and at many levels. The physical existence of marketplaces, and the effects of physical structure on costs and time efficiency, are important and must be quantitatively analyzed at some stage. However, it is the informational structure of markets that is most easily categorized prior to specifying the larger context of social life, and the one that will be of concern here. Markets and their supporting institutions define algorithms for distributed computation and realization of resource re-allocations, by agents with limited spheres of control and often of knowledge. The most basic criteria for the informational categorization of all markets involve the choices of actions they offer to agents, the constraints they impose to allow the agents to arrive at consistent and agreeable allocations, and the symmetry with which agents are subjected to those constraints. Classification according to these serves as a foundation on which finer distinctions may be drawn, or other domains of function projected.

The real rules of bidding and exchange, which may make realizable markets nonequivalent, are often obviated by the assumptions of general equilibrium theory. The non-cooperative equilibria of finite, playable market games [3] are generally not the same as the competitive equilibria in the same systems, due to the strategic considerations agents have about the effects of their own actions on prices. Markets that remove subsets of such actions from agents' strategy sets can differ from those where the strategies remain, in the agents' computations of optimality and ultimately in their final allocations. This distinction is completely lost in the general equilibrium representation of agents as price takers.

As a corollary of the absence of price impacts in the competitive equilibrium concept, the possibility that different markets may impose the constraints of consistency asymmetrically on agents is also overlooked. Agents who are permitted by the market's "rules of the game" to exert control on prices have qualitatively different strategic freedoms than those who are not, and these can lead to biased outcomes even under conditions not warranted by their initial allocations or preferences. This consideration becomes important in assessing the relative value of using a commodity as money versus a non-consumable financial instrument. Thus the concept of Pareto optimality in the context of price-taking is a good beginning notion of the quality of solutions a market makes available, but it leaves many important distinctions among realizable markets unaddressed.

The goal of these two papers [1] is twofold. It is first to present a principled set of criteria for grouping markets and generalizing their functions, and an associated approach to instantiating the distinctions with minimal representative models. It is second to define a notion of causality, whereby a given institutional structure may be recognized as necessary to achieve allocations with specified properties of symmetry, feasibility, or robustness under constraints that may befall agents' endowments. The position will be adopted that markets must be represented as playable games, and that the definition of market function must be separated from the choice of agents' initial conditions and also from their definitions of what constitute acceptable solutions.

Three concepts – dimensionality, counting of strategic freedoms and constraints, and symmetry – will be argued in Sec. II to be the foundation for an informational theory of markets, and for the use of models to derive its claims. Dimensional analysis, though not new to economics [4] will be very briefly reviewed as the conceptual tool requiring nontrivial minimal structures. The indexing of freedoms will then be extended into a more general concept of simplicity and complexity within the constraints forced by dimensioning. Symmetry (considered in General Equilibrium Theory mostly through the dubious device of social welfare utilities) will then be shown to be the next most fundamental concept after dimensions and degrees of freedom, in qualitatively distinguishing markets as strategic games. Using test utilities to categorize market *rules* by symmetry appears on the surface only subtly different from assigning a meaning to the exchange symmetry of the utilities themselves. However, a symmetry of the rules remains a well-defined property at general agent specification, and so may be discussed apart from the question of how agents can or should be compared. It also respects better the spirit of welfare analysis in economics, which is not to declare that people are interchangeable, but to determine whether markets inflict unnecessary or counterproductive biases on them.

The conventions used in this paper for generating minimal models according the the proposed categorization will then be given in Sec. III, and a short treatment of the associated natural measure of allocative efficiency given in Sec. IV. These are then followed by a list of the market structures in Sec. V. The examples will extend only through one-period exchange, with payment in cash or credit. Fiat money will not be considered here, because the additional function it bestowes is first defined in the context of multiperiod models, or with more explicit treatment of trust and its role in boundedly rational solution concepts.

II. THE ROLE OF MODELING IN ANALYSIS AND INTERPRETATION

A theory of money and markets must ultimately draw its validation from predictive, quantitative models. However, it is an error to equate the *notion* of theory with nothing more than a particular model or class of models. An even greater fallacy is to assign significance to models which happen to be numeric, if one has not first taken care to show that the standards of measurement defining the numbers are relevant to the economic phenomena described.

To present a taxonomy of markets here that cannot be taken for just a collection of *ad hoc* models, it is necessary to introduce standards of logical necessity and minimality. Under these, the collection of models becomes a theory in a larger sense than just a set of examples.

The first step in defining both simplicity and necessity is to require that real markets, which may differ in their instantiation, be categorizable according to the types of allocations they make accessible to agents with specified control, knowledge, and preferences. Such categorization may be performed at many levels of refinement; here the counting of agents' degrees of strategic freedom and the symmetry of their final allocations will be used.

The second step is to be specific about the criteria by which one good is substitutable for another to the agents or institutions, and to recognize that ideal substitutability (or fungibility, formalized in the notion of dimensionality) places constraints on the forms of prices, market-clearing rules, and legal actions. These constraints imply necessary minimal levels of structure and differentiation.

Once the resulting category criteria are known, one must refine these to criteria of simplicity within categories, according to temporal dependence, common knowledge, preference, and rationality. By these criteria, there will be a hierarchy of models within a given category. The simplest model within a category then serves as a minimal, or "most generic" form that generates the function of the category, and the least that can be assumed in order to produce that function. In comparing any aspect of function, features separating the simplest models in the compared categories may be recognized as "necessary" to the distinction between them. If the models are hierarchical, in the sense that one contains a superset of the institutions in the other, then minimality ensures that every institution added by the larger category is indispensable if the extra functionality associated with it is to be achieved.

It is important to note, however, that the taxonomy of function is not generally hierarchical in any obvious way. There can be functions provided by institutionally more specialized market systems that are not provided by the less specialized ones, but other functions of which the converse is true.

Because the defining constraints on the minimal models within any category are induced by dimensionality and strategic freedom, more complex models within the category can only differ from the minimal cases by the addition of either structures or parameters that are not required by the categorization. Since the categorization is what elevates the set of its minimal models to an interpretive theory, one can avoid mistaking all the other possible structure in non-minimal models as being "theoretically" motivated. If one then wishes to account for their empirical existence as more than frozen accidents of history, that requires a refinement of the functional categorization, which can produce those features in its minimal representations.

This approach to analysis could be called "effective market theory", by association with the same mode of analysis in physics. The physical version, called "effective field theory" [5], removed a long-standing mystery about why models of fundamental processes were found to work. It showed that, for every process identified by its symmetries and degrees of freedom, there was a minimally complex mathematical model instantiating those properties, whose features could properly be understood as *causing* that process.

A. The necessary constraints on minimal models

The minimal structure a market system can have is one that recognizes all of the distinctions among types of goods acted on in a rationalizable way by the collection of agents. (Alternatively put, the structure of the market determines which distinctions are offered to the agents to act on strategically.)

Dimensions: Ultimately it is the imposition of criteria of *fungibility* that determines what is to be indistinguishable from what else, and there can be considerable freedom in choosing these. However, once they are chosen, it is adherence to them that makes the antecedents of economic numbers (objects in the world) into quantities that can be sensibly added, subtracted, and so forth.

The idea of fungibility is captured in the concept of a *dimension*. It may be that one bushel of corn at a particular time and place can be substituted in essentially all respects for another like bushel at the same time and place, while corn cannot be substituted in all respects for a paper dollar, since the dollar is not good to eat, and perhaps the corn cannot be used in exchange at some markets. The category of things thus fungible by some specified set of criteria is said to define a *dimension*. Equations must be homogeneous in dimension to even admit of making sense [4].

Degrees of freedom: The criterion of strategic freedom is captured in the set of decision variables that the rules of the game allow agents to choose. Each such variable is called a *degree of freedom*, and the number of these may be counted. Constraints that restrict decision variables to include open intervals do not change the number of the degrees of freedom in a system, but those that restrict them to points do, and must be countable as some definite negative number of degrees of freedom.

Symmetry: We will choose to require that market function be specified and well-defined, prior to commitment to allocations for the participants, or to a solution concept for them. However, in cases where the allocations or solution concepts may be preserved under some permutation of the agents, there arises a discrete *symmetry* in the description of their state prior to their interaction with the trade and financial apparatus. It then becomes a sensible (and relevant) question, whether a given market structure has the flexibility to produce solutions that have the full symmetry of the initial conditions.

Careful use of dimensions will imply nontrivial minimal levels of structure for prices and market-clearing rules, and the requirement for models capable of distinguishing symmetries of inputs and outcomes will motivate minimal forms for allocations and utilities in a subset of cases. When a complexity hierarchy is chosen, particular minimal forms will be selected as those whose *only* content is that required by the dimensions and symmetry, and these will in general be unique. Models that are more complex than the minimal forms will necessarily require addition of either dimensional factors beyond those required to define the functions of trade or clearing, or additional nondimensional constants that are not simply unity.

B. Complexity hierarchy for information and clearing rules

Without digressing into a formal definition of the complexity of algorithms, it is sufficient for a theory of markets to commit to any definite and reasonable ordering, by which one algorithm will be called more or less complex than another. We will use the following rules:

- Simultaneous actions are simpler than serial actions.
- Fewer decision variables for agents in their interactions with markets are simpler than more variables. The fewest nontrivial variables is one per agent-market pairing, and it is natural to make this a choice of a bid for goods. Markets at this level, to be nontrivial, must require that agents put their

entire initial allocations up for sale, removing the choice of how much to offer. Such cases, called "sell-all" markets, are treated thoroughly in Ref. [6], and will not be reconsidered here.

The next simplest structure creates two variables per agent-market pairing, and it is natural to make these bids and amounts of supply. These markets include the sell-all case, and allow a broader and more interesting set of possible means of payment. The examples in the following taxonomy will all be of this type.

The next simplest market structure admits four decision variables, which may be taken as bids and quantities offered, and prices for these whose constraining function must be specified. At this level of complexity, though, the simplest clearing mechanism not overconstrained by these inputs involves contingency and rules to handle incomplete clearing of bids and offers. These imply a qualitatively more complex class of games, which will not be treated here. (Examples in each of these categories have been treated in Ref. [2], and references therein.)

- Within the realm of solution concepts, non-cooperative concepts are simpler than cooperative ones.

- Finally, among non-cooperative solution algorithms, pure-strategy Nash equilibria with maximal possible knowledge of other-agent utilities and endowments are the simplest possible. While the mechanics of the computation may be more involved than that for more knowledge-local algorithms, the full-knowledge Nash equilibrium is *theoretically* simpler than a bounded-rationality concept, because it does not require the additional distinction between what is shared and what is local knowledge.

C. Separating games, situations, and solution concepts

It is natural, in representing markets as games in strategic form, to discipline the building of models by enforcing a strict separation of the three essential concepts of game specification, agent situation, and solution concept. Here, the three stages will also be given a prioritization:

1. The agent strategy sets, the rules for price formation, and their effect on allocations, must be specified without regard to the initial conditions affecting the agents who play the game, or the solution concept by which they choose strategies.

2. Once a market is specified in this way, the situations of agents, such as initial allocations or constraints on cash, must be specified without regard to the solution concept assumed for the agents, or how it will constrain the solutions.

3. A solution concept must contain a complete specification of what each agent wants and what he knows, about himself and possibly other agents interacting with the markets. For a market to generate a solution, it must be possible for all of the agents to compute their solution strategies within the bounds of their knowledge.

Enforcing this separation will make clear that the *function* of markets is to put the computation of consistent allocation decisions within reach of agents who have limited spheres of control (and more generally of knowledge). Even under conditions of the most rationality sensibly definable, some market structures will make more solutions computable than others.

The prioritization chosen here is not the only one that could be used. We emphasize it because it will sometimes be the case that, in markets that do not generally make solutions computable, some fine-tuning of the rules of the game (say, its bankruptcy penalties or a credit supply) in explicit relation to agent utility scales would solve the problem. This kind of fine-tuning violates the whole point, though, that markets are supposed to be aggregating devices specified more generally than any instance or subset of the population they serve. It is therefore inappropriate to the kind of theory sought here, and this hierarchy offers a clean way to rule it out.

D. Computability

In general equilibrium theory, prices are a pure accounting tool, and their normalization is arbitrary. While there is thus no way to determine absolute price scale, it also has no effect on the definition of competitive equilibrium; all information about the equilibria is resident in agents' preferences and endowments, and the Walrasian definition of wealth preservation. Even in this very special case, when the dynamics of attaining equilibrium are considered, algorithms proven to find some equilibrium cannot in general find all equilibria, or select among them.

Common knowledge of preferences and endowments is part of complete specification of the strategic form of market games, and like competitive equilibria, their noncooperative solutions will in general be indeterminate [7]. One form of indeterminacy in some of the one-period models arises from the inability of the noncooperative solution concept to specify a unique degree of wash sales. NE exist at all such degrees, and yet the best response of each agent requires as input the degrees of wash sales engaged in by all others. Such forms of indeterminacy most closely resemble those of CE (though here they can be continuous rather than discrete), in that they do not involve the overall scale of prices, and merely reflect an incompleteness in the specification of the solution concept by the endowments and preferences.

In market games, though, assumptions about the scale of prices become a nontrivial part of agents' strategic choices, and are not included in the common knowledge of the game specification. While the overall scale of prices remains unimportant as in general equilibrium, in market games the ability of agents to coordinate the scales they use in bidding is essential to arriving at any NE. When independent price scale assumptions can be hidden within agent strategy sets, this induces a severe form of indeterminacy *not* present in general equilibrium theory. Resolving the indeterminacy within the one-period context is one of the major driving forces behind the credit-evaluating institutions derived here. Explaining the function of these institutions is another way in which the process orientation of the game description of markets exposes qualitative omissions in the perfect-competition idealization.

III. CONVENTIONS FOR REPRESENTING MARKET STRUCTURES

To apply any of these principles quantitatively, of course, one must actually build models, and this requires conventions and notation. We will use the framework standard for strategic market games [3], assuming primitive notions of agents with strategy sets, consumable goods of which they have initial and final allocations (possibly zero), and preferences represented by ordinal utilities of the final allocations. From any cardinalization of these (as long as boundary parameters such as bankruptcy penalties are made sufficiently large), pure-strategy Nash equilibria may be computed, in which the final allocations do not depend on the cardinalization.[20]

A. Depth of treatment

It is important to appreciate that all of the models generated as instantiations have more depth and interpretive power than will be developed here. It is not the goal of this paper to pursue all of the consequences of each model, and many particular cases have been thoroughly treated elsewhere, as is noted below.

Thus, for instance, the importance of constraints on cash or credit, or how these relate to which commodity is used to guarantee promissory notes, will not be considered if they are not so tight as to change the number of degrees of freedom. Similarly, while there are many possible relations between the number of distinguishable goods and the number of distinct or redundant agents as identified by their preferences, only the minimal combinations that express consequences of the symmetry or asymmetry of rules will be developed.

B. Agents, goods, and allocations

Agents come in a number m of *types*, with a number r replicas of each type (replication allows treatment of those aspects of competition not arising from specialization). Each agent is indexed by a subscript $i \in (1, \ldots, mr)$. It will frequently be convenient to arrange the replicas serially, so that agent indices $i \in (1, \ldots, m)$ may also be taken to represent types.

Consumable goods also come in m types, for the convenience that later each agent type may be made the sole provider of one good, as an illuminating special case. Goods are indexed with superscript $j \in (1, \ldots, m)$. (More general cases, in which agent types may be distinguished by preferences, yet be sole producers of a common good, are regarded as hybrid from the simplified forms shown here.)

Market trade takes place in a single period, offering each agent a set of strategic choices, and computing from the collection of values chosen a transformation from initial to final allocations of the goods for all agents. The initial allocation of good j to agent i will be called an *endowment*, and denoted a_i^j. The final allocations resulting from interaction with the markets are denoted A_i^j. A utility for agent i is a function of the final allocations of his consumable goods $\mathcal{U}_i\left(A_i^1, \ldots, A_i^m\right)$.

Markets function by aggregating and disaggregating both consumable goods offered for sale and *bids*, which may be consumable or may be essentially financial in nature. Even nonconsumable bids are goods, in the sense that they enter and leave the system according to specified rules, and are preserved in the act of exchange. However, such goods are distinguished from consumable goods in conferring no utility on the holder.

If the bids are of a separate type from the consumable goods, each agent i is characterized by an additional "monetary" state variable, with initial value m_i (possibly zero) and final value M_i.

C. Trading posts, clearing houses, courts, and credit evaluators

Markets are realized in the form of *trading posts*, with a single post receiving all offers of one good to be sold. Each agent i strategically chooses a set of offer quantities q_i^j for all posts j, and in the models below the restriction $0 \le q_i^j \le a_i^j$ will be imposed.

The trading posts are defined by an algorithm that clears all bids and offers, and defines a unique price at each post for all exchanges of its good. The institutional guarantee of a law of one price distinguishes market trading from barter, even in cases where independent posts provide direct exchange of any good for any other.

The unique clearing rule constrained by the dimensions of price, synchronous strategy choice by each agent without knowledge of other agents' choices, and the absence of any other dimensional or dimensionless constants, is

$$p^j \equiv \frac{\sum_i b_i^j}{\sum_i q_i^j} \equiv \frac{B^j}{Q^j}. \quad (1)$$

The final allocation for agent i resulting from trade in the market for good j is

$$A_i^j = a_i^j - q_i^j + \frac{b_i^j}{p^j}, \quad (2)$$

representing delivery of a fraction of the total goods at post j, proportional to i's fraction of the total bids at that post. The final allocation of i's means of payment is

$$M_i = m_i - \sum_j b_i^j + \sum_j q_i^j p^j, \quad (3)$$

representing a disbursement from each post of its total bids, in proportion to i's fraction of its goods delivered. (In Markets 1 and 2 below, where each good serves both as commodity sold and means of payment, this rule will require special definition, though it will take the same general form.)

In cases where the bids are promissory notes, disbursements may be made at a central clearinghouse, rather than at the trading posts. In such cases, the clearing rule for the means of payment becomes

$$M_i = \max \left[m_i + \sum_j \left(q_i^j p^j - b_i^j \right), 0 \right]. \quad (4)$$

In addition, a penalty may be assessed, representing well-defined laws for bankruptcy. For convenience, the penalty will be evaluated as a direct change of utilities external to the market

$$\mathcal{U}_i \rightarrow \mathcal{U}_i + \Pi \min \left[m_i + \sum_j \left(q_i^j p^j - b_i^j \right), 0 \right]. \quad (5)$$

The penalty constant Π formally induces a relation between market structure and the scale for cardinal agent utilities, violating the prioritization of model building stipulated above. Therefore it will only be used here to define a boundary for interior solutions, where neither the magnitude of Π nor the cardinal representation of $\mathcal{U}_i$ affects the allocations or ordinal utility. (It is shown in Ref. [6], that as long as Π is sufficiently harsh to preclude strategic bankruptcy by some limiting agent, such interior solutions are assured.) The market and clearinghouse function remains well defined even for solutions sampling the magnitude of Π; we simply resist assigning that magnitude a theoretical significance of the sort the rest of the results are given.

Courts provide the penalty function (5), in cases where all clearing of goods and notes happens at the trading posts. It is unimportant here whether the action of the clearinghouse is represented as a combination of a pure clearing activity with a court activity, or the court is represented as a degenerate clearinghouse with no requirement to handle goods.

It will be found that while clearinghouses and courts may lead agents to "prefer" consistent solutions without bankruptcy, these institutions alone may not place such solutions within their ability to calculate. This limitation will motivate introducing a degenerate form of credit evaluation agency as it applies to systems without exogenous uncertainty. Surprisingly, such an agency has nontrivial function, though it is intuitively closer to that of a foreign-exchange pricing body than a risk-assessing credit evaluator. Nonetheless, from the function that credit evaluation is designed to enable in general, that designation will prove the logically proper, if degenerate, one here.

D. Agent symmetry and the issue of utilities

As we have required the description of markets to decompose – into game, situation, and solution concept – utilities arise only in the last and most contingent part of specifying the problem. This is good, since utilities are also the most fragile and debatable part in the representation of economic decision making. Even weaker than the defense of utilities is any abstract comparison of preferences across agents, as would be assumed in a social welfare function.

Nonetheless, it is a natural problem in market analysis to ask what a given market would make possible *if* agents were in some sense substitutable, and more defensibly, if goods and their associations with agents were to be permuted. A market is somehow qualitatively different (and embeds differently in larger games), if it prohibits agents from arriving at symmetric final states even if *in principle* they are the same by some measure, than if it can generate a symmetric solution for such agents.

The counting of a market's strategic freedoms at general allocations requires no special assumptions about utilities, though as usual if analytic interior solutions are desired, sufficient ranges may be required of the allocations, and regularities of the utilities [6]. Examining the capabilities of markets to respect symmetry, however, involves restrictions on both allocations and utilities, for which it is helpful to choose specific forms. As in the specification of prices and clearing rules, this can be done in a most-informative, minimally-complex form.

For permutation of agents, or goods among agents, to be meaningful, the agents or good distributions must somehow be distinguishable. For it to distinguish among markets, it must also imply the permuting of some restrictions. Both of these properties arise most simply in a generalization of Jevons' failure of double coincidence of wants [8]. The failure occurs when agents' allocation constraints remove the ability to place either bids or offers in markets where the agents would benefit from respectively buying or selling goods.

If the general form of the endowments is written as a matrix of components

$$a_{\text{gen}} = \left[a_i^j \right], \tag{6}$$

(the a_i^j not generally assumed equal, but for convenience assumed sufficient to produce interior solutions in the markets considered) the case of maximal, and maximally symmetric, failure of the double-coincidence of wants corresponds to the matrix (written for the first m agents, one per type, and then replicated r times)

$$a_{\text{spec}} = aI \tag{7}$$

where I is the $m \times m$ identity matrix.

(Eq. (7) is an abuse of notation, since obviously distinct goods are supposed to have distinct dimensions, and should not be representable by a common constant. The abuse is tolerable because the rules of market formation will be specified at general a_i^j and thus correctly dimensionally constrained. The correct relation of dimensions to symmetry is obtained by recognizing that the value of each final allocation in the utility must be nondimensionalized by some reference quantity, and that it is the endowment *scaled* by the utility reference that can be set to the common, dimensionless, constant a.)

Once agent types have been distinguished in a symmetric way by making each type the sole supplier ("specialist") of one of the goods, maximal symmetry under agent permutation is attained if all agents share the same utility function, and if that function values all goods equally. Technically expressed, the property of utilities $\mathcal{U}_i\left(A_i^1, \ldots, A_i^m\right)$ that will define interior solutions is

$$\frac{\partial \mathcal{U}_i / \partial A_i^j}{\partial \mathcal{U}_i / \partial A_i^{j'}} \equiv - \left. \frac{\partial A_i^{j'}}{\partial A_i^j} \right|_{\mathcal{U}_i}, \tag{8}$$

the relative price of any two goods j and j'.

The unique form of relative price that is

- Symmetric in all goods
- Constrained by the dimensions of relative price
- Involves no other dimensional or dimensionless parameters

is

$$- \left. \frac{\partial A_i^{j'}}{\partial A_i^j} \right|_{\mathcal{U}_i} = \frac{A_i^{j'}}{A_i^j}, \tag{9}$$

and the most general utilities producing this form are

$$\mathcal{U}_i = f_i \left(\prod_j A_i^j \right). \tag{10}$$

The f_i may be any monotone, concave functions, and it may be desirable to let f_i be unbounded below as the argument $\prod_j A_i^j \rightarrow 0$. The price relation (9) corresponds as a minimal form, for purposes of sampling market symmetry through allocations and utilities, to the price relation (1), for purposes of enforcing market clearing.

E. Dimensions and operators that act on them

The commonly recognized functions of money are as numéraire, means of payment, and store of value, descriptions which also apply in different combinations to other financial goods. From the perspective of symmetry, the choice of a money can also impose a form of agent selection or identification, which may or may not be desirable.

While the store-of-value function of money will not be explored within one-period models, it will be helpful to specify the meaning of the other functions more precisely by listing their dimensions, and introducing two operators that act on dimensional quantities as representations of the test for substitutability, and clearing of promissory notes.

- Each consumable good is given a dimension, and the *name* given to the dimension of the jth good will be g^j (think: "apples", "oranges", "pears").
- There will in general be a numéraire, which is a specific quantity of a specific good (think "one orange"). The numéraire therefore has both quantity and dimension (in this case, the name of the dimension is "oranges"). The numéraire will be denoted N.
- Personal credit takes the form of promissory notes denominated in the numéraire. Specification of the game must include defining who may produce these, when they are to be paid, how they are exchanged for the denominated goods, and when or if they may be destroyed. Promissory notes may be written by individual agents or institutions like central banks, and in general these are not substitutable and thus have different dimensions. The name given to the dimension of a promissory note written by agent i in numéraire N will be $I_i\left(N\right)$. Promissory notes from a central bank will be given dimension $I_{CB}\left(N\right)$.
- The notation $[\;]$ will be used for the operator on a dimensional quantity that gives the name of its dimension (example: ["one orange"] = "oranges").
- The notation $C\left(\;\right)$ will be used for the *clearing operator* that maps a promissory note for a good to an amount of that good. The operator represents the actions of a clearinghouse, and the need for it becomes clear once one demands dimensional homogeneity of equations. As the example, if b_i^j is

a promissory note from agent i for one unit of the numéraire N, $\left[b_i^j\right] = I_i\,(N)$, and $C\left(b_i^j\right) = N$.

F. Graphical representations

A complete specification of a minimal market may now be done easily, by listing the dimensions of the bids, offers, and numéraire, and diagrammatically representing which goods are directly exchangeable for which others. Only goods, penalties, and the institutional relations between them require graphical representation, since the interaction of agents with the institutions is uniformly applied according to the clearing rules given above, and an event sequence described in the next subsection.

- A good, whether consumable or financial, is represented by a filled dot ●.

- The penalty variable applied by the courts is an open circle ○.

- A trading post that takes in two goods is represented by a solid line between the dots representing the goods ●——●. The goods may be consumables directly exchanged, or consumables offered and notes used in payment. They may be specifically-assigned roles as bids and offers, or there may be freedom in this assignment, as in Market 2 below. When there is more than one market between the same two goods, as in Market 1 below, the specification of which is bid and which offer distinguishes the posts, in which case the line between them is made into a directional arrow ●——➝●.

- A clearinghouse relation between goods is a dashed line between the dots representing the goods ●- - - -●. In all the models built here, one of the goods will be a promissory note, and the other the underlying consumable.

- A court relation between a balance of goods and a penalty assignment is also a dashed line ●- - - -○. The court and the clearinghouse share a linestyle that is different from that of the trading posts, as the relation to the agents is essentially coercive within the rules of these games, whereas the market relation is essentially voluntary.

- A credit-evaluation relation between two forms of promissory notes is given the same graphical representation as a market relation. Since they are both goods, and the action of the credit evaluator is linear (involves no threshold relations), it functions the same way as a different type of market with a different clearing rule.

G. The sequence of events and who does what

Markets, clearinghouses and courts, and credit evaluators, all participate in trade with a definite sequence, which may be stylized by referring to periods of an imagined trading day. Offers and bids are delivered by agents to the trading posts (markets) "in the morning". If bids are in promissory notes, and a credit evaluation agency is used to evaluate their worth, this takes place through an exchange between the trading posts and the evaluation agency "at noon" (if not, noon is an irrelevant time). Bought goods and disbursements of payment for sold goods are delivered from the markets to the agents "in the afternoon". If bids are in promissory notes, and these notes must be cleared, an exchange takes place between the agents and clearinghouses "in the evening", when promissory notes are returned to their creators, and if necessary penalties are assessed (if payment is in consumables, evening is an irrelevant time). Agents consume their final allocation bundles entirely "at night". Passage of one day defines one complete period of exchange.

Agent ignorance of each other's strategies is formalized by supposing that agents choose their q_i^j and b_i^j sets before going to market in the morning. If bids are in promissory notes, agents also write these *de novo* at the time of determining b_i^j.

The goal of clearinghouses is to exchange promissory notes, possibly at an exchange rate determined by a credit evaluator, so as to allow each agent to "buy back" his own promissory notes by paying with the notes of others. Agents are then free to destroy notes of their own writing at night, together with consuming their final allocations. The rules of the game prohibit agents from destroying any promissory notes besides their own.

IV. QUANTIFYING ALLOCATIVE EFFICIENCY

The noncooperative equilibria of all of the one-period strategic market games will be found to produce final allocations that are are non-uniform in goods at $a_{\rm spec}$ endowments. Those that at least capture the symmetry of the agents relate to the competitive equilibrium in the manner shown (for two agents and two goods) in Fig. 1. (Those that do not lie off the hyperplane $\sum_j A_i^j = \sum_j a_i^j = a,\ \forall i$.) While one measure of the importance of the symmetry or asymmetry of allocations is the incentive to veto they could create if these one-period markets were embedded in an institution-choosing game, another is the overall efficiency of the trading solutions they generate. The goal of an efficiency description should be a monetary value of exchange, in order to determine under what circumstances market structures can "pay their way" from the functions they make available to the traders.

The CE is generally considered a standard of efficiency in neoclassical economics, in that it both inherits the op-

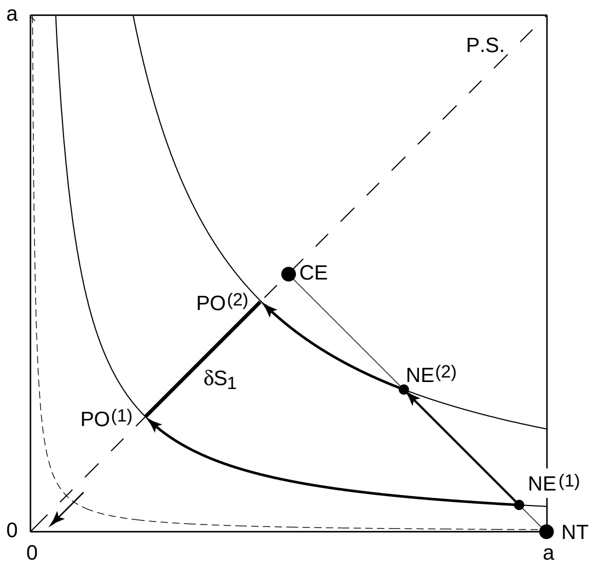


FIG. 1: Edgeworth box construction for two specialist producers with utilities of the form (10). Hyperbolae are preference surfaces for agent 1, whose origin is in the lower left corner. NT identifies the initial allocation, which is also the no-trade solution for the final allocation. $\mathrm{NE}^{(1)}$ and $\mathrm{NE}^{(2)}$ are two final allocations as might arise from non-cooperative equilibria in two different market situations (if this pair is replicated $r > 1$ times). The dashed line P. S. is the Pareto set, and $\mathrm{PO}^{(1)}$ and $\mathrm{PO}^{(2)}$ are points (also Pareto optima) to which a speculator external to the system could move agent 1 by voluntary trading, from the corresponding NE allocations. CE is the competitive equilibrium for this system. The limit of the thin dashed hyperbola indicated by the small arrow suggests a regularization of the speculator's actions from the initial endowments. In this limit, agent 1 is indifferent between his initial endowment and the origin, in which case the speculator can extract a. δS_1 is the profit the speculator would lose from such external manipulations, if the agents were somehow able to attain $\mathrm{NE}^{(2)}$ by internal trading, instead of $\mathrm{NE}^{(1)}$.

timality properties of the Pareto set, and respects the endowments according to a certain (Walrasian) prescription for accounting. Such a qualitative notion of efficiency is weaker than one would like, however, in a principled theory of markets, by several standards.

First, "efficiency" should mean something more than optimality if it is to deserve a separate name, and in particular it should indicate how to quantitatively compare general outcomes to each other and to the CE. A weakness in the nontechnical use of efficiency is that it contains no specification of how it should be quantified. This results in independent notions of "allocative" and "informational" efficiency, with no sense of whether these share a name because they should have some relation to each other. It even permits multiple uses within a single mode of analyses, since it is unclear whether efficiency should be constructed from an individual or economy-wide point of reference [9].

Second, even within neoclassical theory, it is well understood that the CE is not special if agents are allowed any trading programme, and that in general they can arrive at any point on the contract curve by infinite recontracting of infinitesimal trades [10]. Thus if the CE is to be a standard of efficiency, some better justification than arbitrary fixation on Walrasian auction as a definition of "wealth preservation" should be provided. Alternatively, if the sensible definition of efficiency makes no special use of the CE, that should be made clear by the theory, in keeping with the approach that it should be possible to omit anything not "explained" from a minimal instantiation.

Third, since the emphasis on playable games is that their noncooperative solutions are in general not even Pareto optima, a measure of allocative efficiency is *only* useful to the extent that it indexes these. It must do so, however, in a way that makes no special use of particular games, or even of the use of games to arrive at final allocations.

A. Efficiency from economic notions of work

An unambiguous definition of the efficiency of outcomes can be derived from the methodological correspondence of utility theory to physical thermodynamics [11], where the term has a particular quantitative meaning. Efficiency in physics and engineering (from which the economic usage was drawn [12]) refers to the ratio of work extracted from a system, to an idealized greatest amount of work extractable in principle. Mapping that usage into utility theory requires only constructing the potential functions whose change is the economic equivalent of "work".

The details of that construction will be carried out in App. B, but the result is easy to understand for the a_{spec} endowments and utilities (10) from Fig. 1. A given allocation, such as a no-trade solution (NT) or a noncooperative equilibrium (NE) is less than perfectly efficient if agents have potential welfare that remains unrealized at that allocation. A natural measure of the unrealized welfare for each agent is some bundle of goods that an external speculator could extract from him by a sequence of infinitesimal trades along his indifference surface from that allocation (the worst kind of trading sequence he could voluntarily be induced to accept). If the agent is left at some point on the contract curve by the speculative extraction of goods, the speculator can then decouple (for these models), and even if all agents are allowed subsequently to trade internally, *by any algorithm*, no further advantageous trade will be possible. Since the price vector for goods at the final allocation will be common to all the agents, it defines (up to a normalization) the value of the bundle of goods any collection of speculators can extract together from all the agents.

One then defines the welfare gain of each agent, resulting from internal trade, as the value of the goods-bundle that he has kept a speculator from extracting, relative to that extractable from the NT solution. Since in the a_{spec} case, every agent's endowment is indifferent to an all-zero allocation bundle, from the NT solution a collection of speculators could extract all of $\sum_j \sum_i a_i^j = rma$ from the collection of agents (valued according to prices normalized to $p^j = 1$, $\forall j$ on the contract curve). In comparison, if the agents traded internally to *any* point on the contract curve, by *any* path, the sum of their final allocations would be the entire endowment. Hence their gains from trade, expressed in money-metric, are rma. (For each agent this is just the length of the segment of contract curve from the origin to the Pareto optimum, obtained by dotting the allocation vector with the normalized price vector.) While this result is immediately clear for the CE by symmetry, it is also clearly true for any other Pareto optimum, because relative to the CE these differ only by exchange among the agents of a uniform bundle $\delta A_i^j = \delta A_i^{j'}, \forall j, j'$, for each i. Thus we observe that the CE has no special role in the welfare definitions relevant to allocative efficiency.

The value of a bundle extractable by an external speculator is the economic equivalent of the physical notion of *work*, extracted by a load from a thermal system. The agents' utility gain in trading to the contract curve is measured by just the reduction in work they could do on external speculators (profit they could voluntarily surrender). While it is clear that the reduction for any speculator coupled to a single agent will depend on which Pareto optimum the collection of agents finally reaches, is it also clear that the reduction in work (speculative profit) from combined exploitation of all the agents is an invariant. It is with respect to this invariant shared gain that efficiency may be non-arbitrarily measured. Though it is a utility gain, because it is expressed in a money-metric induced by voluntary trade, it sensibly aggregates to a measure of social welfare. The welfare gain by the agent shown in Fig. 1, in going from $NE^{(1)}$ to $NE^{(2)}$, is just the length of the dark segment labeled δS_1. This quantity happens also to be the increase in the utility-version of agent 1's *entropy* S_1 from the transition $NE^{(2)} \rightarrow NE^{(1)}$ (see Ref. [11] for motivation, and App. B for derivation of this case).

It is natural, then, to define the *efficiency* η of any allocation as the ratio of the sum of utility gains by all the agents from the NT solution, to the gain rma that they could realize upon trading to the contract curve. This gives an intrinsic measure to the fraction of the contract curve for the endowments captured by any other allocation bundle. It is shown in App. B (though it is nothing more than the translation of the verbal description here) that

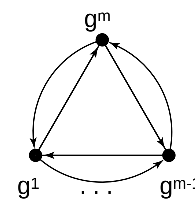
$$\eta \equiv \frac{\sum_i m \left(\prod_j A_i^j \right)^{1/m}}{rma} = \frac{1}{rm} \sum_i \frac{m}{a} \left(\prod_j A_i^j \right)^{1/m} . \tag{11}$$

Clearly $\eta = 0$ at the NT solution, and unity anywhere on the contract curve. The index η will be computed for each distinct allocation bundle that arises in the analysis below. While for agent-symmetric solutions the result will be unsurprising (that allocations further from the contract curve are less efficient), this index also provides a way to compare these results with the more complex agent-nonsymmetric solutions produced by some forms of commodity money.

V. THE ONE-PERIOD SIMPLE-MARKET TAXONOMY

Market 1 (All-for-all, directed)

$$\begin{aligned} \left[q_i^{jk} \right] &= g^j && \forall j, k \\ [N] &= \text{various} \\ \left[b_i^{jk} \right] &= g^k && \forall j, k \\ \left[p^{jk} \right] &= \tfrac{g^k}{g^j} && \forall j, k \end{aligned}$$



$m(m-1)$ markets. Payment in goods; no short sales, no credit. NE computable, symmetric, and robust to allocation constraints.

This is the market structure minimally distinguished from barter, by enforcing a law of one price. Direct-exchange trading posts exist between all commodities j and k, and are indexed superscript jk. Offers q_i^{jk} in these markets are thus of dimension g^j and bids of dimension g^k. Each commodity serves as means of payment in $m-1$ markets, and is offered for sale in $m-1$ others. No credit is offered in this market system, meaning that $0 \le \sum_{k \ne j} \left(q_i^{jk} + b_i^{kj} \right) \le a_i^j$, for all agents i and all goods j.

At general endowments a_{gen}, each agent i has $2m(m-1)$ strategic DOF, the $\left\{ q_i^{jk}, b_i^{jk} \right\}$ in all markets. Markets accepting bids and offers in the same quantity (would-be self-loops in the diagram) are not considered, because the only sensible clearing rule would simply return to each agent what he had delivered. The total number of strategic DOF for all m agents is therefore $2m^2(m-1)$.

The price for the jk market is defined by the notational generalization of Eq. (1),

$$p^{jk} \equiv \frac{\sum_i b_i^{jk}}{\sum_i q_i^{jk}}, \tag{12}$$

and is not forced by the institutional structure to satisfy the dimensionally allowed relation $p^{jk} = 1/p^{kj}$, though some solution concepts may lead to solutions with this property. The allocation rule for agent i as a result of trades in all markets combines Eq. (2) and Eq. (3):

$$A_i^j = a_i^j + \sum_{k \ne j} \left(-q_i^{jk} + \frac{b_i^{jk}}{p^{jk}} \right) + \sum_{k \ne j} \left(q_i^{kj} p^{kj} - b_i^{kj} \right). \tag{13}$$

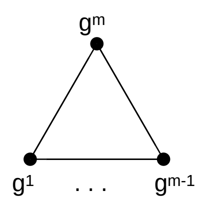
Since bids are consumables, there are no separate monetary state variables, and each good changes as a sold item and a money.

Note that failure of the double coincidence of wants a_{spec} reduces each agent to $2(m-1)$ DOF, the $\left\{q_i^{jk} \neq 0\right\}$ in the $(m-1)$ markets for his endowed good, and the $(m-1)$ $\left\{b_i^{jk} \neq 0\right\}$ in markets for other goods, for which the endowment is used as means of payment. The resulting total number of DOF is $2m(m-1)$, and the system leads to symmetric exchange solutions for specialists, which are computable at any m, as derived in App. A 1. The efficiency of these allocations is less than unity, but by the smallest margin the one-period noncooperative solutions can achieve.

Market 1 is thus agent-permutation symmetric and robust in allowing nonzero-trade solutions under allocation constraints. If the costs of operating trading posts were considered, though, it would also be the most costly of the structures considered here at large m, because the costs to maintain $m(m-1)$ trading posts must be paid, even if agents can only use $2m(m-1)$ of the degrees of freedom they provide.

Market 2 (All-for-all, undirected)

$$
\begin{aligned}
\left[q_i^{jk}\right] &= g^j \qquad \forall j,k \\
[N] &= \text{various} \\
\left[b_i^{jk}\right] &= g^k \qquad \forall j,k \\
\left[p^{jk}\right] &= \tfrac{g^k}{g^j} \qquad \forall j,k
\end{aligned}
$$



$m(m-1)/2$ markets. Payment in goods; no short sales, no credit. NE computable, symmetric, but limited by allocation constraints for $m \geq 3$.

This market allows all-for-all exchange, like Market 1, but reduces the number of trading posts needed in a natural way. Markets are again indexed superscript jk, but there is now a unique trading post for direct exchange of any pair of goods.

The cost of this simplification is that bids and offers, rather than having separate roles in every market, become defined by context. All goods of type j delivered to post jk are interpreted as q-values, and all goods of type k at that post are interpreted as b-values. Since the name jk or kj of the post is arbitrary within this convention, this implies the identity $q_i^{jk} \equiv b_i^{kj}$. At general endowments a_{gen}, each agent is thus reduced to $m(m-1)$ strategic DOF, the $\left\{q_i^{jk}, b_i^{jk}\right\}$ in the symmetrized markets, and the total number of DOF is $m^2(m-1)$.

Price is computed just as in Eq. (12), with the context-dependent definition of the bs and qs, but now by definition $p^{jk} \equiv 1/p^{kj}$. There is a natural modification of the disaggregation rules for both goods, from Equations (2) and (3): each agent gets a fraction of the total both of the j and the k good, equal to his contribution to the value at the jk post, (which may be measured relative to any denomination). The value fraction for an agent i may be written

$$
\frac{q_i^{jk} p^{jk} + b_i^{jk}}{\sum_{i'} \left(q_{i'}^{jk} p^{jk} + b_{i'}^{jk}\right)} = \frac{1}{2} \left(\frac{q_i^{jk}}{\sum_{i'} q_{i'}^{jk}} + \frac{b_i^{jk}}{\sum_{i'} b_{i'}^{jk}} \right), \quad (14)
$$

and the allocation rules that follow from it (written with either market naming order) is

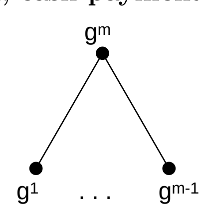
$$
\begin{aligned}
A_i^j &= a_i^j + \frac{1}{2} \sum_{k \neq j} \left(-q_i^{jk} + \frac{b_i^{jk}}{p^{jk}} \right) \\
&= a_i^j + \frac{1}{2} \sum_{k \neq j} \left(q_i^{kj} p^{kj} - b_i^{kj} \right) \\
&= a_i^j - \frac{1}{2} \sum_{k \neq j} \left(q_i^{jk} - q_i^{kj} \frac{Q^{jk}}{Q^{kj}} \right). \qquad (15)
\end{aligned}
$$

In the first line of Eq. (15) good j is treated as the bought and sold consumable, while in the second line it is regarded as the means of payment for the other $k \neq j$ bought and sold consumables. In the last line (where $Q^{jk} \equiv \sum_i q_i^{jk}$), the order of jk is permuted in order to express all quantities as qs, a convenience when maximizing utilities in App. A.

Under failure of the coincidence of wants a_{spec}, each agent still has $(m-1)$ DOF for a total of $m(m-1)$ DOF, because there are precisely $(m-1)$ markets for any single endowed good in exchange for other goods. The solutions are still symmetric and computable at all m, and the collection of markets is half as costly as those in Market 1, in exchange for offering half the strategic freedoms. However, the clearing rule leads to large fractions of returned goods, so if there is no short sale, for $m \geq 3$ there is a shortage of either offers or means of payment to attain the interior solution. This is demonstrated in App. A 2. Thus agents would be better off spontaneously breaking the symmetry of solutions in Market 1, and abandoning one of the two markets for each good, but continuing to use the conventional clearing rules. While one might have expected that thinning of the markets in the directed case would affect final allocations, this turns out not to be the case in either the symmetric or fully broken solutions, since only the replication index affects the impact assessments.

Market 3 (Commodity standard, cash payment)

$$
\begin{aligned}
\left[q_i^{j}\right] &= g^j \quad \forall j \neq m,\ q_i^m = 0 \\
[N] &= g^m \\
\left[b_i^{j}\right] &= g^m \quad \forall j \neq m,\ b_i^m = 0 \\
\left[p^{j}\right] &= \tfrac{g^m}{g^j} \quad \forall j \neq m,\ p^m \text{ undef}
\end{aligned}
$$



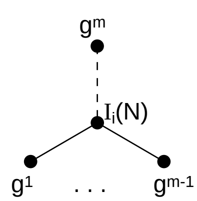
> $m-1$ markets. Payment in a preselected good; no short sales, no credit. NE computable, but approach the no-trade solution under allocation constraints.

This structure has $m-1$ trading posts indexed simply by the good $j \neq m$ sold at them. Payment is in good m and, comparable to the exclusion in the previous cases, there is no market exchanging m for m. At a_{gen} there are $2\,(m-1)$ strategic DOF per agent – the $\left\{ q_i^{j\neq m}, b_i^{j\neq m} \right\}$ – for a total of $2m\,(m-1)$ DOF. A version of this market is treated in Ref. [2], ch. 7, where the role of the m^{th} good as money is given precedence over its consumable status.

Under failure of the coincidence of wants a_{spec}, the $m-1$ agents of type $i \neq m$ are reduced to one offer q_i^i each, while agents of type m have $m-1$ bids $\left\{ b_m^{j\neq m} \right\}$. There are thus $2\,(m-1)$ total DOF. However, the solutions are highly asymmetric because only one agent type may bid in the commodity money, leading to solutions that approach the no-trade allocation at large m. As the efficiency of no-trade is zero, the efficiency of the solutions to this game decay as $\mathcal{O}\,(1/m)$.

Market 4 (Commodity standard, personal credit)

$$
\begin{aligned}
\left[q_i^j\right] &= g^j \qquad \forall j \neq m,\ q_i^m = 0 \\
[N] &= g^m \\
\left[b_i^j\right] &= I_i\,(N) \quad \forall j \neq m,\ b_i^m = 0 \\
\left[p^j\right] &= \tfrac{I_i(N)}{g^j} \quad \forall j \neq m,\ p^m \text{ undef}
\end{aligned}
$$



> $m-1$ markets for goods. Payment in promissory notes for a preselected good, with one clearinghouse for notes and that good. NE computable at large m and survive under allocation constraints, but are not symmetric.

This structure has the same asymmetry as Market 3, but softens the impact of the failure of coincidence of wants by adding credit in the form of personal promissory notes to deliver the m^{th} good. All notes are given credit by the market price and allocation mechanisms (1 - 3) at face value, so promissory notes are substitutable and the various dimensions $I_i\,(N)\,, i \in 1, \ldots, mr$ may be regarded as a single dimension. With the addition of credit, bankruptcy rules are formally required to handle the possibility of default in the promissory notes. However, for the interior solutions which it is the purpose of credit to make possible, the bankruptcy constraint is never binding, because there is net flow of good m to all of the type $i \neq m$ agents, motivated by its consumption value.

While a “money market” for the m^{th} good is now definable, bid with promissory notes in that good, this model is distinguished by not including it, giving each agent at a_{gen} $2\,(m-1)$ of the $\left\{ q_i^{j\neq m}, b_i^{j\neq m} \right\}$ DOF, for total of $2m\,(m-1)$. The allocation rule for the m^{th} good replaces Eq. (2) with the clearinghouse value

$$
A_i^m = a_i^m + \sum_{j\neq m} C\left(q_i^j p^j - b_i^j \right). \tag{16}
$$

(When this interior solution is impossible because a_i^m is too small, the bankruptcy penalties modify Eq. (16), and induce an endogenous rate of interest, which is the shadow price of the capacity constraint [2], ch. 6-7.)

The importance of credit is that at a_{spec}, each agent of type $i \neq m$ is reduced only to m DOF, the q_i^i for his own endowment and $m-1$ $\left\{ b_i^{j\neq m} \right\}$ (versus only 1 DOF in Market 3). Agents of type m retain the $m-1$ $\left\{ b_m^{j\neq m} \right\}$ that they have in Market 3, for a total of m^2-1 DOF in the system.

Final allocations at a_{spec}, derived in App. A 3, while not converging to the no-trade solution at large m as in Market 3, are still not symmetric, however. Though at $m \to \infty$ and $r \to \infty$ they converge to the competitive equilibrium [13], at large m and fixed r, they converge at $\mathcal{O}\left((1/m)^0 \right)$ to a relation of the form

$$
\prod_j A_m^j \to \left(1 - \frac{1}{r} \right) e^{1/r} \prod_j A_{i\neq m}^j \to e^{-\frac{1}{r(r-1)}} \prod_j A_{CE}^j, \tag{17}
$$

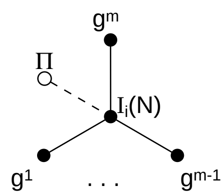
where A_{CE}^j is the final allocation (the same here for all agents and all goods) that would be attained at a competitive equilibrium. At a level of resolution where there are many goods in the world produced by specialists, but relatively few truly equivalent producers of any one good (small r), the agents called on to provide the standard of value are penalized relative to the rest. Ultimately this arises because, whereas all other agents have some strategic freedom to impact the prices of their own goods, the provider of the numéraire loses this freedom. Though he has reduced purchasing power relative to other agents, in other respects he effectively becomes a price taker (as borne out by the symmetry of A_m^j among j, a property of the CE allocations but not the other $\left\{ A_{i\neq m}^j \right\}$ in the NE. The reduction in allocative efficiency due to the penalty on type-m agents is not compensated by the slight windfall given to the other types, so the efficiency of this Market is the lowest of those that retain finite levels of trade at large m.

The practical relevance of the asymmetry in this Market arises if one considers embedding its function into a larger game involving the agents’ *choice* to become specialist producers versus generalists, and then embedding that in the larger game of adopting a market structure as well. If the rules of the larger games give agents “veto” power over the adoption of a market structure, the requirement of Market 4 that the money-providers always be disadvantaged could in plausible games prohibit its adoption.

The solutions in this case exist at all m, but are formally uncomputable due to an unspecified degree of wash sales by each producer. The formal ambiguity occupies an interval of relative size $1/m$ in price and final allocation, though, and so can practically always be placed below some threshold of severity at large m. In this practical sense, NE are "computable" in a limit of sufficiently many types.

Market 5 (Personal credit with bankruptcy law)

$$\begin{aligned} \left[q_i^j\right] &= g^j \quad \forall j \\ [N] &= \text{free} \\ \left[b_i^j\right] &= I_i\left(N\right) \ \ \forall j \\ \left[p^j\right] &= \tfrac{I_i(N)}{g^j} \quad \forall j \end{aligned}$$



m markets for goods. Payment in personal promissory notes for an arbitrary numéraire, credited at face value, with a court imposing bankruptcy penalties. NE are not computable.

One can attempt to patch up the asymmetry in Market 4 by adding a trading post for good m, in effect a commodity money-market. At this point, since bids for all goods are in terms of the same set of promissory notes credited at face value, their denomination, and hence the numéraire, becomes arbitrary up to its interpretation in the bankruptcy laws. Thus with the adoption of a money market the previously-money commodity is reduced to just another consumable good, at interior solutions.

Each agent naïvely has $2m$ DOF at a_{gen}, in the form of $\left\{q_i^j, b_i^j\right\} \forall j$, giving $2m^2$ total DOF for the system. However, correct counting of the strategic DOF requires more careful treatment of the bankruptcy penalty as it applies to interior solutions.

The min operator in Eq. (5), having undefined derivative at zero argument, is unsuited to evaluating the gradients required by Nash equilibria. The discontinuous derivative may be regularized for each agent by replacing Eq. (5) with

$$\mathcal{U}_i \to \mathcal{U}_i + \kappa_i M_i \tag{18}$$

where M_i is defined in Eq. (3), and since there is no source of initial debts, $m_i = 0, \forall i$. The discontinuous derivative is replaced in Eq. (18) with a set $\{\kappa_i\}$ of Kuhn-Tucker multipliers, one for each agent i. One represents bankruptcy as a game between each agent and the courts, where the agent tries to maximize Eq. (18) with respect to the $\left\{q_i^j, b_i^j\right\}$, and the courts try to minimize it with respect to κ_i. It differs from a simple Kuhn-Tucker constraint in that the courts are only permitted the variation $0 \le \kappa_i \le \Pi$.

At nonzero values of M_i, the derivative is manifestly that of Eq. (5), while at $M_i = 0$, the derivative of the penalty term with respect to M_i is κ_i. This gradient is set equal to the gradient of $\mathcal{U}_i$ to obtain the first-order conditions on the final allocations. Symmetric, interior solutions exist, and in them no agent goes bankrupt, but by $m_i = 0, \forall i$ and the accounting identity $\sum_i M_i = \sum_i m_i$, they satisfy exactly $M_i = 0, \forall i$, and $0 < \kappa_i < \Pi, \forall i$.

If the Kuhn-Tucker expression of the min were nothing more than a regularization method for derivatives, it would say that agents sample all b_i^j values, and then compare the relative merits of going bankrupt to buy a little more of good j to that for a little more of good $j^{'}$. Obviously this is the wrong interpretation for interior values of κ_i, which are set by the utilities and not the legal value Π. Interior values are a device for comparing the relative prices of all pairs j and $j^{'}$ of goods. They thus represent a strategy evaluation in which bankruptcy is *not* considered, and the relative values of the goods are compared directly. In other words κ_i, like an ordinary binding Kuhn-Tucker or Lagrange multiplier, is one *negative* DOF per agent, leading to $-m$ DOF relative to the naïve count of $2m^2$ for the whole system.

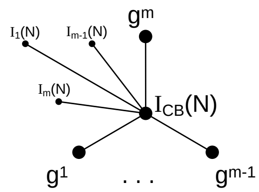
$2m^2 - m$ is still not the correct DOF count for this system, though, because there is an overall scaling freedom for the promissory notes $b_i^j \to \Lambda b_i^j$, where the same Λ is applied by all agents. This scaling freedom reflects the arbitrariness of the numéraire familiar in competitive equilibria. It obviously does not affect a strategic ability of agents to change either allocations or utilities, so the correct number of total DOF is $2m^2 - m - 1 = (2m+1)(m-1)$. However, this negative strategic degree of freedom is *distributed* over the agents, with two consequences. First, showing it explicitly as a constraint cannot be done within this model, because all constraints have already been incorporated. Second, more important, and an expression of the same fact, it signals that *no selection process exists* to arrive at a solution, even given maximal rationality, if one makes the commonsense restriction that rationality should not be defined to include lucky guessing.

The way to see the uncomputability of solutions is to recognize that, even knowing everything about all agent utilities and endowments, there is no way any agent can know the scale other agents will use for all of *their* b_i^j, because he cannot know their strategies. Since this scale factor is not constrained by the NE solution itself, there is *nothing* knowable outside the strategies that can solve this dilemma.

Apart from the fact that agents have no way to find NE solutions, a continuum of them indexed by the scale factor Λ exists, both at a_{gen}, and a_{spec}, and in the latter case they have the symmetry that the allocations of Market 4 lack. Allocation constraints remove $m-1$ offer DOF $\left\{q_i^{j\neq i}\right\}$ from each agent's strategies, or $m\left(m-1\right)$ from the total, leaving $m^2 - 1$. They are the same DOF as in Market 4 except for two: Agent m can now offer q_m^m in a market for his endowment, but this localized freedom is offset by the distributed constraint that overall rescaling by Λ has no consequences for interior outcomes.

Market 6 (Personal credit monetized)

$$\begin{aligned} \left[q_i^j\right] &= g^j && \forall j \\ [N] &= \text{free} && \\ \left[b_i^j\right] &= I_{CB}(N) && \forall j \\ \left[p^j\right] &= \frac{I_{CB}(N)}{g^j} && \forall j \end{aligned}$$



m markets for goods. Initial payment in personal promissory notes for an arbitrary numéraire, exchanged at a credit evaluator for central-bank promissory notes, at a computed rate of exchange. NE computable at large m, symmetric, and robust under allocation constraints.

Market 5 attempted to recover symmetry and introduced a new, severe form of uncomputability. The reason for this failure is that it replaced a localized constraint on agents of type m in Market 4, with a global scaling symmetry that was under the control of no-one, and not deducible by anyone. The institution that retains symmetry while restoring computability is effectively an exchange-rate service, here considered as one function of a Central Bank. (In Ref. [14], exactly the process used here was modeled, but rather than considering exchange-rate computation to define a new institution, Sorin described it as a modification to the clearing rule of the posts, which were then necessarily regarded together as a single centralized trading mechanism.)

Agents again offer and bid in the mornings at markets for all m goods, in q_i^j and personal promissory notes, and as before, the denomination of the notes is arbitrary. Now, though, the personal notes are not evaluated at face value, so the markets do not deliver goods or disbursements right away. Rather, they bring the notes along with a record of the received $\left\{q_i^j\right\}$ to a Central Bank, which computes a set of *exchange rates* λ_i, and returns Central-Bank promissory notes on behalf of i to the trading posts according to a formula

$$b_{CB(i)}^j = \lambda_i b_i^j, \tag{19}$$

which notes become the effective deposit by agent i at post j. The dimensions of λ_i are thus

$$[\lambda_i] = \frac{I_{CB}(N)}{I_i(N)}. \tag{20}$$

It is clear that with notes not credited at face value, the b_i^j at different i are no longer directly interchangeable, and must be regarded as separate goods with distinct dimensions, as in the Market 6 diagram.

Equations (1 - 3) are then evaluated by the markets in the afternoon, as before, but with the bids for each agent i now represented in the uniformly denominated $\left\{b_{CB(i)}^j\right\}$. Disbursements of Central-Bank notes to the agents are made, and the Central Bank takes the agents to a clearinghouse in the evening, where promissory notes are exchanged back to their originators at the rates λ_i defined earlier in the day.

The rule that the Central Bank uses to compute the λ_i is that all promissory notes will clear exactly at the end of the day, $M_i = 0, \forall i$, so that there is no need for bankruptcy penalties. The $\{M_i\}$ are linear functions of the $\{\lambda_{i'}\}$, with coefficients $\left\{M_i^{i'}\right\}$ that are functions of the $\left\{q_i^j, b_i^j\right\}$. Assembling the constraints and the λs into column vectors, the condition of perfect clearing may be written

$$[M_i] \equiv \left[M_i^{i'}\right][\lambda_{i'}] = [0] \tag{21}$$

From Eq. (3) it is straightforward to compute the diagonal values

$$M_i^i = -\sum_j \left(1 - \frac{q_i^j}{Q^j}\right) b_i^j, \tag{22}$$

and the off-diagonal

$$M_i^{i' \neq i} = \sum_j \frac{q_i^j}{Q^j} b_{i'}^j. \tag{23}$$

It is then an elementary accounting check that

$$\sum_{i'} M_{i'}^i = 0, \; \forall i. \tag{24}$$

Thus $\left[M_i^{i'}\right]$ is degenerate, and so has at least one eigenvector $[\lambda_{i'}]$ of zero eigenvalue. (If there is more than one null eigenvector, this indicates that the bids and offers allow the market to break up into more than one independently-valued subsystems of exchange.)

The null eigenvector(s) corrects any $\left\{b_i^j\right\}$ that are consistent with an interior NE, up to a uniform but independent rescaling by each agent i, to the NE with a single consistent scale factor. Since $[\lambda_{i'}]$ is a null eigenvector, its overall scale is of course undetermined, and this is equivalent, up to notation, with precisely the ambiguity of the numéraire common to competitive equilibria. The Bank may choose any convention it likes to make the computation of $[\lambda_{i'}]$ procedurally well-defined, but nothing about the final allocations depends on this choice. Note that the Central Bank is *not* required to compute a general equilibrium solution, like a Walrasian auctioneer, and that the much simpler, linear note-clearing condition (21) remains exact whether or not agents' bids are consistent with a NE.

The agents' strategies are now well-defined, and the NE are computable within the context of Nash optimization. Each agent is rational to compute his $\left\{b_i^j\right\}$ *as if* all other agents were generating $\left\{b^j_{i' \neq i}\right\}$ with the same assumed numéraire and scale, and *as if* all promissory notes were to be credited at face value. In general, of course, the scales assumed by different agents will have no relation, but because each agent's bids will have the correct ratios *internally*, the scale scale factors chosen will not matter.

The scale-freedom of the $\left\{b^j_i\right\}$, fixed in Market 5 by the Kuhn-Tucker multiplier but restored as a bidding freedom in Market 6, is not a new degree of freedom, because it is fixed by the clearing rule (21) fiscally, rather than by threat from the courts. Thus one DOF per agent has been added and subtracted, and in this respect the the markets are the same. The only new feature in Market 6 is that the previously untraceable strategic symmetry, the overall scale, is now explicitly fixed by any choice of normalization made by the Central Bank, which can be counted as the required -1 DOF. The counting of DOF in both the a_{gen}, and a_{spec} is thus the same as argued in Market 5. Further, all of the NE are the same, and the only consequence of the institutional difference is to make them computable within a sensible rationality concept, without requiring that this extend to include impossibly lucky guessing. In particular, though, this means that the solutions retain an uncomputability of the degree of wash sales, which is only removed practically – that is, restricted to price and allocations consequences in an interval of order $1/m$ – in the large-m limit. Calculations showing that the solutions achieved by this market are the same as those in the all-for-all markets are provided in App. A 4.

A. Foreign exchange or credit evaluation?

The evaluation service institutionalized here in a Central Bank has two interpretations, both correct, but both specializations of different general concepts. One could be called Foreign Exchange evaluation, and the other credit evaluation. Both interpretations are forms of *monetization of personal credit*, which generalizes in a different way when issues of trust and visibility are introduced to solution concepts to compensate for incomplete rationality or information.

In Market 6, every agent effectively produces his own currency. The exchange rate evaluation occurs as if each were a country bringing goods and otherwise-valueless notes for trade to a set of international markets. The real goods offered and bid for, together with the distribution of the currency, can be used by a "world bank" to determine exchange rates, at which no country is left holding another country's valueless notes at the end of the period. Values of real goods offered play an essential part in the relative valuation of currencies.

Alternatively, exchange rate evaluation makes sense as a form of *credit evaluation* in the absence of exogenous uncertainty, and it is under this logical heading that it is more usefully generalized. The function of credit evaluation in the monetization of credit is to exchange unreliable notes for reliable ones, at a rate that precludes only some preventable defaults, in exchange for enabling trades that would not be possible with cash payment alone [15]. When default can result from exogenous uncertainty in the world, as well as strategically, this trade-off is nontrivial and depends on some marginal valuation of more trade versus more default.

With or without uncertainty about the world, there can also be endogenous uncertainty about consistency of agent strategies, as arises in Market 5 from the uncomputability of a uniquely defined scale factor for b^j_i at different i. The purely endogenous component is special, though, in that it is *entirely* preventable with no sacrifice in trade, because it requires only a coordination mechanism. The mechanism for removing default solutions is the same for both forms of uncertainty – the imposition of exchange rates between personally- and centrally-generated credit – but whereas this exchange rate generates an interest rate to balance default under exogenous uncertainty, it reduces to a pure accounting tool in the cases here, because complete repayment is always possible.

B. The gauge structure of monetized credit

The relation of Markets 5 and 6 happens to be precisely that between two well-understood classes of dynamical freedom in physics. As shown elsewhere [11] in the context of competitive equilibria, the constrained maximization of utilities corresponds mathematically to the problem of identifying the energetically and probabilistically preferred state of a macroscopic physical system (in this case, at zero temperature, where it is call the "ground state") by minimization of certain thermodynamic potentials. It is a theorem in physics that, when the underlying micro-dynamics of a physical system possesses a continuous symmetry that any single ground state is unable to express, there must be a continuous set of such states, degenerate under the thermodynamic potentials. The impossibility of distinguishing one such state from another by its thermodynamic properties corresponds to the impossibility of selecting a preferred equilibrium based on utilities.

The interesting physical content of such undeterminable cases is that the propagation of the microscopically defined symmetries, from the local dynamics to a relation among global states, may be done in two ways. If *only* the global symmetry exists, the problem of coordination among the local elements to find a single ground state can only be carried out by dynamics. The excitations propagating this information are known as *Goldstone particles*, and the problem of identifying a scale for prices from the dynamics of expectations has already been formulated as one of exchanging these particles [16]. The consequence of this fact for one-period models is that, if no time elapses to make dynamics definable, the coherent best solution cannot be formed, as has been shown in Market 5.

A physically different origin for symmetric ground states arises from what is called *gauge symmetry*. A gauge field is a globally-defined excitation whose value serves as a reference for the scale of each of the locally defined symmetry transformations.

Gauge fields create a new class of *local* symmetries, which consist of changing the "gauged" value of the local variable, while adjusting the reference value (the "gauge") of the field together with it to indicate no change in the actual physical state. In the example of Market 6, such a so-called *gauge transformation* is the combination of

$$b_i^j \to e^{\eta_i} b_i^j, \ \forall j \text{ at some } i \tag{25}$$

with the corresponding adjustment that the Central Bank will make:

$$\lambda_i \to e^{-\eta_i} \lambda_i, \tag{26}$$

which any agent is free to induce independently of what the others do. This leads to no change in the physical outcomes of the market activities, and in a market where the NE bids were computable from the utilities, like Market 4, creating an additional layer of paper in this form would be a pointless thing to do. The addition of the local rescaling symmetry is accompanied by a set of rules for determining the gauge, here the constraints (21), so the total number of DOF in the system, and its symmetry, is preserved.

Gauge fields become nontrivial when, as here, there is a global symmetry of the ground states that can be absorbed as one of the physically consequenceless degrees of freedom in the gauge variation. Then the remaining, physically meaningful values of the ground state DOF can be specified instantaneously and in a symmetric way by the gauge conditions, as was done in Market 6.

It is important to appreciate that a physical theory with gauge fields is different from one without, in principle both in the instantaneous specification of its *kinematics*, or state space, which corresponds to the one-period models, and in the larger specification of its *dynamics*, which become definable over multiple periods. The distinction may or may not be visible from the restricted kinematic perspective of the selection of equilibrium ground states, depending on whether they can or cannot express the global symmetry that is gauged. If there is no degeneracy of the one-period NE, as in Market 4, then the addition of a layer of paper will not affect either the solution or its Nash computability. If, on the other hand, the ground states are degenerate due to the expression of some underlying global symmetry, the coupling of that symmetry to a gauge field absorbs the degeneracy into the gauge freedom, removing it from the spectrum of uncomputables, and allowing it to be computed even in a single period. [21]

An interesting issue when one comes to dynamics is that the evolution of gauge fields (financial instruments) and the underlying (commodities) is in general independently specified. While one must admit fluctuations about the ground states for this dynamics to have any consequence, such fluctuations are expected when one weakens ideal rationality to limited information or computational capacity (introduces "trembles" of whatever sort). Then the multiperiod solution states with financial instruments can be different from those without, even if under perfect rationality they would be specified to be the same at every instant.

C. Hybrid cases

The six cases listed here are pure forms, maximally symmetrized among those goods or agent types that are not distinguished by some institution. In real economies, subsets of goods can be connected to each other according to different paradigms, resulting in a hybrid graph with elements drawn from more than one pure form. Historically interesting cases, such as Bimetallism, can arise in this way, though they also involve considerations not tractable within the one period context, such as how durable goods with modest short-term utility-of-holding can become moneys to mediate exchange of other goods with shorter lifetimes. Some examples of this type are discussed in more detail in Ref. [1], but will not be pursued further in this paper, because the interpretive scope of these calculations is too coarse to resolve what makes them interesting.

VI. SUMMARY OF DEGREES OF FREEDOM AND SYMMETRIES OF THE ONE-PERIOD MARKETS

Table I summarizes the various indices computed for the six market cases considered in Sec. V. It is noteworthy that there are a number of overlaps across columns in the table as institutions are added or changed, in either the specialist or generalist allocation situations.

Market 3 and Market 4 have the same DOF at a_{gen}, because this is the case of "enough cash, properly distributed" [17], in which the additional availability of credit is irrelevant to the Nash solutions. The models differ in their robustness against boundary solutions, but this does not affect the DOF count. It is, however, re-

Prop.\Mkt.	1	2	3	4	5	6
Num. Mkts.	$m\,(m-1)$	$m\,(m-1)\,/2$	$m-1$	$m-1$	m	m
a_{gen} DOF	$2m^2\,(m-1)$	$m^2\,(m-1)$	$2m\,(m-1)$	$2m\,(m-1)$	$(2m+1)\,(m-1)$	$(2m+1)\,(m-1)$
a_{spec} DOF	$2m\,(m-1)$	$m\,(m-1)$	$2\,(m-1)$	m^2-1	m^2-1	m^2-1
Sym.	Yes	Yes	No	No	Yes	Yes
Comp.	any m	$m<3$	NT	large m	N/C	large m

TABLE I: Summary of the properties of the one-period markets. First line is the number of trading posts, which would be input for costs if larger institution-choosing games were being considered. Second line is DOF for unconstrained bids and offers; third line is for the case of specialist producers. Fourth line indicates whether the NE have the same symmetry as the agents in the specialist case. Last line is computability, where NT means degeneration to the no-trade solution, and N/C means not computable. Large m denotes a limit not specifiable within the setting posed, but defined qualitatively in the solutions of App. A. The value qualifying as "large" in a larger game would depend on a tradeoff between costs of maintaining trading posts, and either cardinality of the utility or the costs of bankruptcy.

flected in the collapse of the cash market under a_{spec}, as reflected in the reduced DOF.

On the other hand, Markets 4 - 6 are indistinguishable for a_{spec}, and indeed all provide finite-trade solutions under allocation constraints. Their difference is in the computability of these solutions.

VII. CONCLUSIONS

A. Relations of symmetry to efficiency

The one-period exchange mechanisms summarized in Table I produce three qualitative types of solutions, which become more efficient as they become more symmetric, both under permutation of each individual's goods, and under permutation of individuals. There is an intrinsic asymmetry of endowments in the generalized Jevons failure a_{spec}, while the preferences are assumed completely symmetric. The efficiency of any market system considered here turns out to measure how effectively it erases the endowment asymmetry as a property of the allocations, and replaces it with the symmetry of the preferences, preferably without introducing any new asymmetry under permutation of agents. The difference in efficiency between any two market types, multiplied by the initial wealth rma of the system, is the money-measure of utility gain available to pay for any additional institutions required by the more efficient system.

If one assigns the pure no-trade solution as the consequence of strategic uncertainty of Market 5, this solution has efficiency $\eta_{NT} \equiv 0$. Barely better, the severely allocation-constrained solution of Market 3, in which total exchange is limited by the endowments of the cash holders, produce efficiencies $\eta \sim 1/m$. Market 2, for which symmetric interior solutions are limited by boundary constraints, are a more complex intermediate case in which efficiency should asymptotically approach a limit associated with trade by half the desired amount at large m.

In contrast, as soon as credit is introduced, most goods are traded, and efficiencies approach unity at large m and r. The least imprint of the endowment asymmetry on the final allocations is produced by Markets 1 and 6, which preserve the joint permutation symmetry of agents together with their endowed goods, and attain the result (A11): $\eta \sim 1 - 1/2m(r-1)^2$ (though the latter does so only at large m).

When the joint permutation symmetry of agents with their goods is violated by the institutional structure itself, as in Market 4, even though most goods are traded, the cost is a slightly reduced efficiency (A37): $\eta \sim 1 - 1/2m(r-1)^2 - 1/2mr^2$. At large r, the reduction in welfare of the type-m agents, relative to the others, lowers efficiencies as much as the entire goods-asymmetry for all agents combined in Markets 1 or 6.

B. The path from barter to credit evaluation

As efficiency gains measure what wealth is available to pay for institutions that facilitate trade, the multiplicity of institutions measures what must be paid for. If one were to embed this analysis of one-period markets in larger games, where trading posts have costs of operation, and the choice to specialize was accompanied by productive economies of scale, a natural association of market types with scales of trade could naturally be made to arise.

For sufficiently infrequent or specialized trade, there would probably be only barter, because the cost of sustaining even one post per good could not be supported by the value of variance reduction accrued from a law of one price. With more trade, but of limited types of goods and with relatively little redundancy by suppliers, all-for-all markets could achieve the best noncooperative equilibria possible, without guessing and at any m. It would not be necessary for the more centralized credit markets to have gone undiscovered, in order for them not to exist. Their non-existence would in a sense be determined by their inability to make optimal NE computable for small m.

As scales of trade grow, however, the cost of the all-for-all trading posts grows quadratically in the number of goods, even as the errors from mis-estimation of wash sales in the credit markets declines. Given any structure of costs per post and per error, there will be some scale above which all-for-all markets become unsustainable and centralized credit markets dominate.

These conclusions can be qualitatively drawn without reference to exogenous uncertainty or the other roles of credit evaluation, money as a surrogate for trust, or the heterogeneity of volumes and market types that affects real economies. Under addition of these complicating factors, the general associations of market-system types with scales of trade would then serve as an organizing principle or tendency from which to recognize the deviations induced by combination.

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APPENDIX A: ALLOCATION-CONSTRAINED SOLUTIONS

Noncooperative equilibria for all of the nontrivial market models are computed here at a_{spec} endowments. Only solutions with the maximal symmetry respected by both the endowments and markets will be computed. In all cases considered these exist, but they are unique only where so noted. No attempt will be made to prove nonexistence of solutions that spontaneously break the symmetry of the endowments or markets, though in many cases, such as replica-asymmetric solutions in the all-for-all markets, it is elementary to show that these do not exist as long as preferences are convex and C^1-differentiable.

Because symmetric allocations are of interest, utilities will be restricted to the form (10). The cardinalization functions f_i will never matter, and to factor them out of the optimization equations, it is convenient to introduce the notation for the log-derivatives

$$f_i^{(l)} \equiv \frac{d}{d \log \left(\prod_j A_i^j \right)} f_i \left(\prod_j A_i^j \right). \tag{A1}$$

The *arguments* of the f_i, though, will be objects of recurring interest, as this is the meaningful function of consumables that measures the efficacy of markets, both relatively among users in a single economy, and across economies. For reference, as both the most symmetric outcome possible, and the one leading to the largest utility all agents can simultaneously have, the allocation product produced by a competitive equilibrium is (by symmetry)

$$\prod_j A_i^j \Big|_{CE} = \left(\frac{a}{m} \right)^m, \; \forall i. \tag{A2}$$

In the derivations below, other allocations will be compared to the solution (A2), and found at large m to differ by factors of the form

$$\frac{1+x}{e^x} < 1, \; \forall x \neq 0, \tag{A3}$$

where x will be various functions of the replication index.

1. Market 1

The final allocations are given as a function of the strategic variable choices by Eq. (13). Varying these, and adopting the notation $B^{jk} \equiv \sum_i b_i^{jk}$, $Q^{jk} \equiv \sum_i q_i^{jk}$, gives

$$\begin{aligned} \delta A_i^j \; = \; & \sum_{k \neq j} \left(\frac{\delta b_i^{jk}}{p^{jk}} - \delta q_i^{jk} \right) \left(1 - \frac{b_i^{jk}}{B^{jk}} \right) \\ & - \sum_{k \neq j} \left(\delta b_i^{kj} - p^{kj} \delta q_i^{kj} \right) \left(1 - \frac{q_i^{kj}}{Q^{kj}} \right). \end{aligned} \tag{A4}$$

The parenthesized factors involving B^{jk} and Q^{jk} arise from the agents' own assessment of their impacts on prices. It it through these factors that the NE differ from the CE in all cases.

The condition of maximal utility for any agent i follows from Eq. (A4) as

$$\frac{\delta \mathcal{U}_i}{f_i^{(l)}} = \sum_j \sum_{k \neq j} \left[\frac{1}{p^{jk} A_i^j} \left(1 - b_i^{jk} / B^{jk} \right) - \frac{1}{A_i^k} \left(1 - q_i^{jk} / Q^{jk} \right) \right] \left(\delta b_i^j - p^j \delta q_i^j \right). \tag{A5}$$

Maximally symmetric solutions permitted by the markets are those in which all agents offer the same amount $q \equiv q_i^{ik}$ and bid the same amount $b \equiv b_i^{ki}$, for any $i \neq k$. The price at any trading post in terms of these is $p \equiv p^{jk} = b/q$, for any $j \neq k$. The only nonzero impact factors, respectively for bids and offers, are

$$\left(1-\frac{b_i^{ki}}{B^{ki}}\right) = \left(1-\frac{1}{r}\right),$$
$$\left(1-\frac{q_i^{ik}}{Q^{ik}}\right) = \left(1-\frac{1}{r}\right). \quad \text{(A6)}$$

The notation A^+ may be introduced for the final allocation of any agent in terms of his endowed good, and A^- for his final allocation in any of the non-endowed goods, by symmetry all the same. In terms of b and q, these are

$$\mathcal{A}^+ \equiv \frac{A_i^i}{a} = 1-(m-1)\frac{q+b}{a},$$
$$\mathcal{A}^- \equiv \frac{A_i^{j\neq i}}{a} = \frac{q+b}{a}. \quad \text{(A7)}$$

(As noted, the declaration of $a_{\rm spec}$ by the matrix 7 is bad notation, since different goods are supposed to have different dimension, and as such should not be denoted with the same quantity a. A similar crime is the use of q to denote agent offers by different types. Those abuses are fixed in Eq. (A7), because $\mathcal{A}^+$ and $\mathcal{A}^-$ are properly nondimensionalized, even if offers and endowments are given different dimensions for each i. The proper definition of the symmetric allocation constraint is really obtained by setting these nondimensional quantities to take type-symmetric values, as is done in Eq.(A7).)

The intrinsic symmetry of the endowments, since it is respected by the market structure, implies that for symmetric solutions $p = 1 \Rightarrow b = q$. Substituted, with the reduced notations (A6-A7) into Eq. (A5), give as the only nontrivial relation

$$\frac{1}{\mathcal{A}^+} = \frac{\left(1-\frac{1}{r}\right)}{\mathcal{A}^-}. \quad \text{(A8)}$$

It is exactly solvable and unique for both A^+ and A^- at all $m > 1$, but is useful to expand as a power series in $1/m$ for comparison to nonunique solutions of later markets:

$$\mathcal{A}^+ = \frac{1+\frac{1}{r-1}}{m+\frac{1}{r-1}} = \frac{1}{m}\left[1+\frac{1}{r-1}\right]+\mathcal{O}\left(\frac{1}{m^2}\right),$$
$$\mathcal{A}^- = \frac{1}{m+\frac{1}{r-1}} = \frac{1}{m}\left[1-\frac{1}{(r-1)\,m}\right]+\mathcal{O}\left(\frac{1}{m^3}\right). \quad \text{(A9)}$$

The cost of this asymmetry of agents, in terms of reduction in the allocation product which is the argument of the utilities, takes a particularly convenient form at large m and any $r > 1$,

$$\prod_j A_i^j \rightarrow \frac{\left(1+\frac{1}{r-1}\right)}{\exp\left(\frac{1}{r-1}\right)}\left(\frac{a}{m}\right)^m, \ \forall i. \quad \text{(A10)}$$

The allocative efficiency (11) corresponding to Eq. (A10) is

$$\eta = \frac{\left(1+\frac{1}{r-1}\right)^{1/m}}{\exp\left(\frac{1}{m(r-1)}\right)}. \quad \text{(A11)}$$

This solution, in this costliest of market structures (in terms of number of trading posts) turns out to be the best that one-period exchange among non-cooperating agents who account for their own price impacts can achieve. This market and the next are unique, in making this solution computable for any m, a feature that even the institutionally more complex credit markets will not have. However, only this market makes the solution accessible without short selling at finite $m \geq 3$.

2. Market 2

It is most convenient notationally to use the line from Eq. (15) for the final allocations, in which all submissions are treated as q_i^{jk}, and the market-index order rather than a q/b distinction is used to identify which good is being submitted. Then the variation takes the simple form

$$\delta A_i^j = -\frac{1}{2}\sum_{k\neq j}\left(\delta q_i^{jk}-\frac{Q^{jk}}{Q^{kj}}\delta q_i^{kj}\right)\left(1-\frac{q_i^{kj}}{Q^{kj}}\right), \quad \text{(A12)}$$

with Q^{jk} defined as for Market 1. The variation of any $\mathcal{U}_i$ is then

$$\frac{\delta \mathcal{U}_i}{f_i^{(l)}} = -\frac{1}{2}\sum_j\sum_{k\neq j}\left[\frac{1}{A_i^j}\left(1-q_i^{kj}/Q^{kj}\right)-\frac{Q^{kj}}{Q^{jk}A_i^k}\left(1-q_i^{jk}/Q^{jk}\right)\right]\delta q_i^{jk}. \quad \text{(A13)}$$

Agents can offer only their endowed goods, and for symmetric solutions, the offers will be the same, allowing the notation $q \equiv q_i^{ik}$, for any k. This is the same as the $b = q$ property of the all-for-all-directed market, and implies $Q^{kj}/Q^{jk} = 1$. The offer impact in the second line of Eq. (A6) is as before, and now includes the bid impact up to the renaming of bids and offers at a post. The expressions for final allocations of the endowed and non-endowed goods are also much the same, with the only meaningful differences being in factors of 2:

$$\begin{aligned}\mathcal{A}^{+} &\equiv \frac{A_i^i}{a} = 1 - (m-1)\,\frac{q}{2a},\\ \mathcal{A}^{-} &\equiv \frac{q}{2a}.\end{aligned} \tag{A14}$$

Again Eq. (A8) follows from maximization, so that the final allocations for all i are just Eq. (A9), and the argument of the utility remains as in Eq. (A10).

It is worth noting that $q/2$ in this market has the same value as $q+b$ in Market 1, so nominally each undirected market is "thicker" than its directed counterpart. The importance of thickness in this market, though, is that the best the agents can do under their allocation constraints is reach a boundary solution approximating the NE (often not closely) at $m > 3$. Combining Eq. (A14) and the second line of Eq. (A9) gives the total outlay

$$(m-1)\,\frac{q}{a} = \frac{2\,(m-1)}{m + \frac{1}{r-1}}, \tag{A15}$$

a fraction greater than one for $m \geq 3$. More than half of this, of course, agents have returned to them by the clearing rule, but by having to put it up in the absence of credit, the rule has made the optimal solution inaccessible to them. Market 1, meanwhile, has no such fragility.

3. Market 4

The allocations (2) and (3) finally become simple with this market, and the notation for total bids and offers reduces to $B^j \equiv \sum_i b_i^j$, $Q^j \equiv \sum_i q_i^j$. The variation of the consumable goods takes the simple form

$$\delta A_i^j = -\left(\delta q_i^j - \frac{\delta b_i^j}{p^j}\right)\left(1 - \frac{b_i^j}{B^j}\right). \tag{A16}$$

There is also, for the first time in a nontrivial market, a separate monetary state variable, in the form of a final balance of promissory notes. Its variation is

$$\delta M_i = -\sum_j \left(\delta b_i^j - p^j \delta q_i^j\right)\left(1 - \frac{q_i^j}{Q^j}\right). \tag{A17}$$

Using the bankruptcy-modified utility (5) within a range where the the min is identically zero, the optimization condition is

$$\frac{\delta \mathcal{U}_i}{f_i^{(l)}} = \sum_{j \neq m}\left[\frac{1}{p^j A_i^j}\left(1 - b_i^j/B^j\right) - \frac{1}{A_i^m}\left(1 - q_i^j/Q^j\right)\right]\left(\delta b_i^j - p^j \delta q_i^j\right). \tag{A18}$$

with $i \neq j \neq m$,

$$\begin{aligned}\left(1 - \frac{b_i^i}{B^i}\right) &= \left(1 - \frac{b^+}{rmb}\right)\\ \left(1 - \frac{b_i^j}{B^j}\right) &= \left(1 - \frac{b^-}{rmb}\right)\\ \left(1 - \frac{b_m^j}{B^j}\right) &= \left(1 - \frac{b^0}{rmb}\right).\end{aligned} \tag{A19}$$

The offer impact is then relevant only for type i:

$$\left(1 - \frac{q_i^i}{Q^i}\right) = \left(1 - \frac{1}{r}\right) \tag{A20}$$

The final allocations for agents of type i are named as before,

$$\mathcal{A}^{+} \equiv \frac{A_i^i}{a} = 1 - \frac{q}{a}\left(1 - \frac{b^+}{mb}\right),$$

This market structure has the least symmetry of those considered. It admits a single offer variable $q \equiv q_i^i$, and as before requires two bid variables: one $b^+ \equiv b_i^i$ for the endowed good, and another $b^- \equiv b_i^{j \neq i,m}$ for the non-endowed non-money good, for any $i \neq m$. Now, however, there is a third bidding scale $b^0 \equiv b_m^{j \neq m}$ by the provider of the $m^{\rm th}$ good, for all other goods, which for him are not endowed. The total bids on any market may be given the notation $mb \equiv B^j = b^+ + (m-2)\,b^- + b^0$, in terms of which the price of each non-money good is $p \equiv p^j = mb/q$, for any $j \neq m$.

The absence of a trading post for the quantity denominating the promissory notes removes an offer-impact factor for agents of type m, separating goods into three categories: endowed, non-endowed, and the commodity money. It is convenient to define the nondimensional variables of the most-symmetric solutions by momentarily letting i be the index of a non-m agent and his endowed good, and j any other non-money index, so that

$$\mathcal{A}^- \equiv \frac{A_i^j}{a} = \frac{q}{a}\frac{b^-}{mb},$$
$$\mathcal{A}^m \equiv \frac{A_i^m}{a} = \frac{b^0}{a}, \tag{A21}$$

except that now there are three of them, while the allocations for agents of type m must be given separate names (as a mnemonic)

$$\mathcal{M}^- \equiv \frac{A_m^j}{a} = \frac{q}{a}\frac{b^0}{mb},$$
$$\mathcal{M}^m \equiv \frac{A_m^m}{a} = 1-(m-1)\frac{b^0}{a}. \tag{A22}$$

The two nontrivial relations arising from Eq. (A18) for type i are now

$$\frac{\left(1-\frac{b^+}{rmb}\right)}{\left(1-\frac{1}{r}\right)\mathcal{A}^+} = \frac{\left(1-\frac{b^-}{rmb}\right)}{\mathcal{A}^-} = \frac{p}{\mathcal{A}^m}, \tag{A23}$$

while the one relation for agents of type m is

$$\frac{\left(1-\frac{b^0}{rmb}\right)}{\mathcal{M}^-} = \frac{p}{\mathcal{M}^m}. \tag{A24}$$

In terms of these, the allocation product for the agents of any type $i \neq m$ are

$$\prod_j A_{i\neq m}^j = a^m \mathcal{A}^+ \left(\mathcal{A}^-\right)^{(m-2)} \mathcal{A}^m, \tag{A25}$$

while the allocations for the money-providers are

$$\prod_j A_m^j = a^m \left(\mathcal{M}^-\right)^{(m-1)} \mathcal{M}^m. \tag{A26}$$

As it follows from the definition of p and Equations (A21-A22) that $p\mathcal{M}^- = \mathcal{A}^m$, the two allocations satisfy the relation

$$\prod_j \frac{A_{i\neq m}^j}{A_m^j} = \frac{\left(1-\frac{b^+}{mb}\right)\left(1-\frac{b^-}{rmb}\right)^{(m-2)}\left(1-\frac{b^0}{rmb}\right)}{\left(1-\frac{1}{r}\right)}$$
$$= \frac{\exp\left(-\frac{1}{r}\right)}{\left(1-\frac{1}{r}\right)}\left[1+\mathcal{O}\left(\frac{1}{m}\right)\right], \tag{A27}$$

where the second expansion is appropriate at large m as long as $b^+/b, b^-/b, b^0/b \sim \mathcal{O}\left(m^0\right)$ (as will be demonstrated momentarily).

Also from the price definition and Eq. (A24), it follows that

$$\frac{b^0}{b^-} = \left(1+\frac{b^0}{rmb}\right), \tag{A28}$$

which has a unique solution for b^0/b^-, if $b^+/b \to 0$. There are thus unique solutions with no wash sales, which can serve as a basis for the analysis of more general NE.

The final allocation $\mathcal{M}^m$ (A22) may also be computed in terms of the ratio b^0/rmb at general b values, as

$$1-(m-1)\frac{b^0}{a} = \frac{1}{1+(m-1)\left(1-\frac{b^0}{rmb}\right)} \tag{A29}$$

Plugging this relation into Eq. (A22) for $\mathcal{M}^-$, the offer constraint of no short sales $q \leq a$ implies the bound

$$0 \leq \frac{b^+}{b^-} \leq 1 + \frac{1}{r-1} + \mathcal{O}\left(\frac{1}{m}\right). \tag{A30}$$

Together with the relation (A29), this implies the asymptotically m-independent scaling of all components of b_i^j required to satisfy Eq. (A27).

The nature of the NE parametrized by b^+/b is easy to understand, but implies a new form of uncomputability for all solutions of this kind. It is clear that, given any interior NE with $b^+/b \to 0$ (which the above equations yield), there is a neutral freedom for any single agent i to engage in wash sales. Because at $b^+/b \to 0$ he as offered less than all of his endowment (solution is interior in A_i^i), he may always offer some additional increment of q_i^i, and increase his b_i^i, without incurring bankruptcy penalties, as long as his wash sales do not change the price.

Of course, if other agents knew he were going to thicken the market, it would change their own impact assessments, and so their bids would not remain the same, by Eq. (A28). Thus the neutral curve for any agent given $b^+/b \to 0$ does not coincide at general m with the one-parameter family of NE obtainable by the group as they vary b^+ collectively. Since any b_i^i is subject to a neutral variation, nothing about the NE condition can fix it for any agent, and so cannot allow him to guess the value that will be chosen by other agents either.

If one were to alter the solution concept, such as by ranking the NE globally, the $b^+/b \to 0$ solution would be the least preferred, an algebraically tedious result, but one whose origin is easy to understand. The reason the allocation products of NE are smaller than those of CE is that suppliers hoard their endowed good to some extent, because of $1-1/r$ decrease in the marginal value of increasing offers, due to price impact. Any allowed increase in wash selling stabilizes prices to some degree, placing each agent in a situation slightly closer to that of a price taker, and so reduces the hoarding tendency and moves every agent's allocations closer to the symmetry of CE. Thus the maximum wash sale possible is the globally Pareto superior NE. At the same time, any agent who chooses that outcome is maximally vulnerable to bankruptcy, with its more severe penalty than a mere reduction in allocations, and so any other NE is risk-dominant over the maximal-wash-sale solution. Rather than pursue the resulting choice among more refined solution concepts for this one structure, we prefer simply to acknowledge the uncomputability of the standardly defined NE solution with the tools provided by bankruptcy institutions alone.

The problem of uncomputability is regulated at large m by Eq. (A30), because no provider can impact either the price or the total amount of his own bids at order larger than $1/m$, by engaging in wash sales. With vanishing price impact, the corrections by other agents also disappear, and so the neutral curve for each agent independently converges to coincidence with the collective change of b^+ in the NE, implying that both become strategically null changes. The entire range of NE then correspond to the allocations for type $i \neq m$ of

$$\begin{aligned}
\mathcal{A}^+ &= \frac{1}{m}\left[1+\frac{1}{r-1}\right]+\mathcal{O}\left(\frac{1}{m^2}\right), \\
\mathcal{A}^- &= \frac{1}{m}\left[1-\frac{1}{(r-1)\,m}\right]+\mathcal{O}\left(\frac{1}{m^3}\right), \\
\mathcal{A}^m &= \frac{1}{m}+\mathcal{O}\left(\frac{1}{m^2}\right),
\end{aligned} \tag{A31}$$

and for type m of

$$\begin{aligned}
\mathcal{M}^- &= \frac{1}{m}\left[1-\frac{1}{r\,(r-1)\,m}\right]+\mathcal{O}\left(\frac{1}{m^3}\right), \\
\mathcal{M}^m &= \frac{1}{m}+\mathcal{O}\left(\frac{1}{m^2}\right).
\end{aligned} \tag{A32}$$

Thus, at large m but only there, for each agent it becomes rational to engage in any degree of wash sale within the bounds imposed by the NE, and converging as $1/m$, the product of final allocations that is the argument of his utility

$$\prod_j A^j_{i\neq m} \rightarrow \frac{\left(1+\frac{1}{r-1}\right)}{\exp\left(\frac{1}{r-1}\right)}\left(\frac{a}{m}\right)^m, \tag{A33}$$

the same for agents of type $i \neq m$ as in Markets 1 and 2, except that in Market 2 the absence of short sales made this equilibrium unattainable.

Meanwhile, with the same order of convergence, the allocation product for the money providers takes the limit

$$\prod_j A^j_m \rightarrow \frac{\left(1-\frac{1}{r}\right)}{\exp\left(-\frac{1}{r}\right)}\prod_j A^j_{i\neq m} \rightarrow e^{-\frac{1}{r(r-1)}}\left(\frac{a}{m}\right)^m. \tag{A34}$$

It is indeed smaller than that for the other types by the factor in Eq. (A27), which is in turn smaller by a factor of the same basic form (A3) than the CE solution.

The final allocation from this market is the only one in the one-period exchange taxonomy that falls off the hyperplane $\sum_j A^j_i = \sum_j a^j_i = a$. From Eq. (A32) follows

$$\sum_j A^j_m = a\left[1-\frac{1}{r\,(r-1)\,m}+\mathcal{O}\left(\frac{1}{m^2}\right)\right], \tag{A35}$$

and though Eq. (A31) is not evaluated to the order to determine it, it follows by symmetry that

$$\sum_j A^j_{i\neq m} = a\left[1+\frac{1}{r\,(r-1)\,m\,(m-1)}+\mathcal{O}\left(\frac{1}{m^3}\right)\right]. \tag{A36}$$

These corrections to the type-$i \neq m$ allocations only perturb the efficiency at $\mathcal{O}\left(1/m^2\right)$, while the order-unity reduction for type m versus the rest, in Eq. (A34), perturbs the efficiency at $\mathcal{O}\left(1/m\right)$. As a result, the overall efficiency satisfies

$$\eta = \frac{\left(1+\frac{1}{r-1}\right)^{1/m}}{\exp\left(\frac{1}{m(r-1)}\right)} - \frac{1}{m}\left[1-\frac{\left(1-\frac{1}{r}\right)}{\exp\left(-\frac{1}{r}\right)}\right]+\mathcal{O}\left(\frac{1}{m^2}\right). \tag{A37}$$

The allocations become progressively less efficient as they become less symmetric.

4. Markets 5 and 6

The NE for both symmetric credit markets are identical, and it is more direct to compute them for the elementary Kuhn-Tucker form of the bankruptcy constraint in Market 5. The solutions will therefore be computed in that case, without regard for the fact that agents cannot achieve them due to the scale ambiguity of bids. Since the rational solution concept in Market 6 is to assume a shared scale of bidding and continued bankruptcy limits, the operation is the same, with the scale-factor corrections between different agents' solutions merely supplied by the evaluation agency after the fact.

The variations of allocations and moneys are the same in Market 5 as Equations (A16-A17), but because the denomination of the promissory notes is now arbitrary relative to any of the consumables, the bankruptcy multiplier enters in a meaningful way into the Nash optimization (interior solutions for the consumables no longer make the bankruptcy constraint slack). The variation of the Kuhn-Tucker form (18) of the utility then becomes

$$\frac{\delta\mathcal{U}_i}{f^{(l)}_i} = \sum_j\left[\frac{1}{p^j A^j_i}\frac{\left(1-b^j_i/B^j\right)}{\left(1-q^j_i/Q^j\right)}-\frac{\kappa_i}{f^{(l)}_i}\right]\left(1-\frac{q^j_i}{Q^j}\right)\left(\delta b^j_i - p^j\delta q^j_i\right), \tag{A38}$$

where the magnitude of the penalty variable Π does not matter as long as the cardinalization of f_i makes $f_i^{(l)}$ sufficiently small.

There are now solutions entirely symmetric in the offer variable $q \equiv q_i^i$, and in terms of only two bid variables $b^+ \equiv b_i^i$, $b^- \equiv b_i^{j \neq i}$, for any i. The total bids on any post may be denoted almost as before, $mb \equiv B^j = b^+ + (m-1)\, b^-$, and in that notation $p \equiv p^j = mb/q$, for any j.

The first two lines of Eq. (A19), and Eq. (A20), again describe the only nonzero impact factors, and the final allocations $\mathcal{A}^+ \equiv A_i^i/a$ and $\mathcal{A}^- \equiv A_i^{j\neq i}/a$ again take the form in the first two lines of Eq. (A21), in terms of the present definition of mb. The first equality of Eq. (A23) remains true, as the only nontrivial relation. In these markets it defines a unique solution for $b^+/b \to 0$.

To obtain a bound on wash sales, one can introduce the parameter $\zeta \equiv b^-/b$, and show that

$$\begin{aligned} \frac{q}{a}\zeta &= \frac{\left(1 - \frac{\zeta}{rm}\right)}{1 + \frac{\zeta}{r^2 m}\left(1 - \frac{1}{m}\right)/\left(1 - \frac{1}{r}\right)} \\ &= 1 - \frac{\zeta}{(r-1)\, m} + \mathcal{O}\left(\frac{1}{m^2}\right). \end{aligned} \tag{A39}$$

As before, $q/a \leq 1$ implies a bound on ζ at order $1 - 1/m$, showing that b^+/b lies in a range around unity of size $\leq \left(m^0\right)$. Thus again, at large m but only there, the neutral curve of independent wash sales for each agent converges to the joint curve of b^+-parametrized NE, for which the allocations become degenerate.

To estimate the allocations simply, one observes that $b^+/b \to 0 \Rightarrow \zeta \to m/(m-1)$, in which limit

$$\begin{aligned} \mathcal{A}^+ &= \frac{1}{m}\left[1 + \frac{1}{r-1}\right] + \mathcal{O}\left(\frac{1}{m^2}\right), \\ \mathcal{A}^- &= \frac{1}{m}\left[1 - \frac{1}{(r-1)(m-1)}\right] + \mathcal{O}\left(\frac{1}{m^3}\right) \end{aligned} \tag{A40}$$

The allocations (A40), correct to the stated order at all allowed b^+/b, agree with Eq. (A9) and the relevant lines of Eq. (A31) to those orders, and for all agents i. The allocation product, and with it the efficiency, is thus the same as the result (A10) produced by the all-for-all markets, though only as a limit.

APPENDIX B: THE THERMAL POTENTIALS FOR SPECIALIST TRADERS

The mapping from the potential theory of physical thermodynamics [18] to a similar potential formulation of utility theory [11] is based on two quantities: the expenditure function [19] $e_i(p, \mathcal{U}_i)$ for each agent, and a dual *contour money-metric* utility $\mu_i(x_i)$, defined from an allocation contour in the Pareto set. In this notation p is an arbitrary vector of normalized prices, $\mathcal{U}_i$ any ordinal utility representation for agent i, and x_i a vector of his allocations. There is some freedom in how to assign these values for the utility problem at $a_{\rm spec}$, but it does not affect the outcome of the derivation. The simplest choice is to work only with the allocations $x_i \equiv \left\{A_i^j\right\}$ internal to the system, and impose an arbitrary normalization on prices to assign values to bundles. The main derivation in the following subsections will be done this way, and only at the end will a more general "cash valuation" be considered, to show the detailed relation to the constructions of Ref. [11].

1. The expenditure function and the contour money-metric utility

Using the utility representation $\mathcal{U}_i$ of Eq. (10), and any normalization convention for prices, the expenditure function has the usual definition [19]

$$e_i(p, \mathcal{U}_i) \equiv \sum_j p^j A_i^j, \tag{B1}$$

where $p \equiv \left[p^j\right]$ is a vector of given prices, and the A_i^j are Hicksian demands, at which agent i would be in equilibrium at those prices and utility scale $\mathcal{U}_i$. A scale factor $\prod_j A_i^j = f_i^{-1}(\mathcal{U}_i)$ is imposed on demands by Eq. (10), and the directionality by the set of equilibrium relations

$$\frac{p^j}{\left(\prod_{j'} p^{j'}\right)^{1/m}} = \frac{\left(\prod_{j'} A_i^{j'}\right)^{1/m}}{A_i^j}, \tag{B2}$$

expressed in a form that is independent of both price normalization and utility scale.

Where the expenditure function (B1) defines a money-metric utility relative to a vector of prices and an ordinal $\mathcal{U}_i$, the dual contour money-metric utility [11] is a function of demands, and amounts to a particular choice of cardinalization f_i. Because the Pareto set for the sample cases $a_{\rm spec}$ is characterized by constant prices, the two forms will coincide on the Pareto optima.

Prices on the Pareto set satisfy

$$p^j = p^{j'}, \ \forall j, j', \tag{B3}$$

and we may as well adopt any of the number of normalizations that set these $p^j = 1$, $\forall j$. Just as price components are equal, by permutation symmetry among the goods, the "best-exploitation" allocation to which an external speculator could induce agent i to trade satisfy

$$B_i^j = B_i^{j'}, \ \forall j, j'. \tag{B4}$$

These are points on the Pareto set such as $\text{PO}^{(1)}$ and $\text{PO}^{(2)}$ in Fig. 1.

For the $\left\{B_i^j\right\}$ to satisfy Eq. (B4), and also be in the indifference set of some allocations $\left\{A_i^j\right\}$ (such as those produced by a NE of one of the market cases),

$$\left(B_i^j\right)^m = \prod_{j'} A_i^{j'}, \tag{B5}$$

for any j. The contour money-metric utility μ_i at any bundle $\left\{A_i^j\right\}$ is then defined as the value of the best-exploitation bundle (B5) at its equilibrium price:

$$\mu_i\left(\left\{A_i^j\right\}\right) = \sum_j B_i^j = m\left(\prod_j A_i^j\right)^{1/m}. \tag{B6}$$

The economic work potentials defined in Ref. [11] are essentially various surpluses, of the expenditure function (B1) over the contour utility (B6), scaled to some common reference price.

2. A symmetric value index for measuring wealth

The idea of an intrinsic, monetary welfare measure for the allocations of an economy is based on the ability of an external speculator, by offering differential pricing among agents, to extract a maximal bundle of goods from them collectively, after which they will have no more desirable trades to pursue. To give a money value to the extracted bundle, there is usually assumed to be a common price for at least one good, which for convenience is held constant during the extraction. In general there is more than one such final allocation from the point of view of the agents. To distinguish among them the one that defines the intrinsic welfare measure, it is assumed that the speculator has has no way to value goods *other than* the one at the fixed price, and so chooses a final allocation in which he holds none of those goods.

In this problem, using the internal variables $\left\{A_i^j\right\}$ only, there is no single relative price that can be held constant in passing from the endowments to any Pareto optimum, and still be symmetric among the agents. There is, however, a natural symmetric bundle of goods, and extracting any maximal amount of it leaves all agents somewhere within the original Pareto set, where that bundle is valued at the unique price $p^j = 1$, $\forall j$. It is thus natural to let the Pareto value of the extracted symmetric bundle be the money measure of welfare even when there is no common price shared by all agents during its extraction. This can also be shown formally to be the correct choice by adding a "cash" market with a shadow price for the symmetric bundle, but since that increases notation and adds nothing to the solution, it will only be summarized briefly at the end.

The symmetric bundle for each agent i will be given the designation [22] $U_i \equiv \sum_j A_i^j$. With this choice it becomes clear that the appropriate normalization convention for prices is $\sum_j p^j = m$, so that U_i functions as a numéraire. Introducing a vector notation $A_i \equiv \left[A_i^j\right] \equiv \left(U_i, \vec{A}_i\right)$, $p \equiv (1, \vec{p})$ for prices and allocations, and performing an orthogonal transformation on allocations and prices, one can make the separation

$$\begin{aligned} p \cdot A_i &\equiv \sum_j p^j A_i^j \\ &\equiv \left(\frac{1}{m}\sum_j p^j\right)\left(\sum_j A_i^j\right) + \vec{p}\cdot\vec{A}_i \\ &= U_i + \vec{p}\cdot\vec{A}_i, \end{aligned} \tag{B7}$$

where the last line follows from the normalization of p.

From the definition (B6) and the directionality relation (B2), it follows that under any differential change of final allocations $\delta A_i \equiv \left[\delta A_i^j\right]$,

$$p \cdot \delta A_i = \left(\prod_j p^j\right)^{1/m} \delta\mu_i. \tag{B8}$$

Combining this with Eq. (B7), one gets the conservation law for U_i in terms of extracted non-numéraire goods $-\delta\vec{A}_i$ and gained contour utility

$$\delta U_i \equiv \delta\left(\sum_j A_i^j\right) = -\vec{p}\cdot\delta\vec{A}_i + \left(\prod_j p^j\right)^{1/m} \delta\mu_i. \tag{B9}$$

Eq. (B9) corresponds to the physical conservation of internal energy for some i'th thermal subsystem in a collection [18]:

$$\delta U_i = -p\delta V_i + T\delta S_i, \tag{B10}$$

where p in physics is pressure, V_i is the subsystem volume, and $p\delta V_i$ is the work done by the subsystem i on its environment (here the speculator). The economic allocation A_i generalizes volume to a vector-valued extensive state variable, and p (price) generalizes pressure to the conjugate vector-valued intensive variable [11]. In physics T is temperature and S entropy, prompting the identification $T \leftrightarrow \left(\prod_j p^j\right)^{1/m}$, and $S_i \leftrightarrow \mu_i$. These identifications are in fact the same as those made in Ref. [11] apart from the absence of any exchangeable "heat" component to the entropy, because for the moment the agents are not coupled, even formally, to a source of fixed prices.

3. The work potentials of markets

For the reversible legs of the extraction process (the transformations along indifference surfaces in Fig. 1), the profit delivered to the speculator by each agent comes from Eq. (B9) in the $\delta\mu_i \to 0$ form

$$\delta U_i = -\vec{p} \cdot \delta \vec{A}_i. \tag{B11}$$

Meanwhile, from any allocation-preserving initial condition, the total $\vec{A}$ accumulated by the speculator from taking the agents to the contract curve is zero: $\sum_i \int \delta \vec{A}_i = 0$. If the speculator starts from the initial endowments, he can extract

$$\sum_i \int \delta U_i = rma, \tag{B12}$$

leaving each agent i at the origin, where

$$U_i\left(0\right) = 0. \tag{B13}$$

(This transformation can be regularized by taking the limit of transformations along the dashed hyperbola in Fig. 1. As the initial allocations approach $a_{\rm spec}$, the speculator's best-extraction bundle for agent i approaches the origin.)

When utility does change, it follows from the conservation relation (B9) that at fixed $T \leftrightarrow \left(\prod_j p^j\right)^{1/m}$, the work $-\int \vec{p} \cdot \delta \vec{A}_i$ extracted from each agent i equals the maximum change in the so-called *Helmholtz potential*

$$\begin{aligned} \mathbf{A}_i &\equiv U_i - T S_i \\ &= \sum_j A_i^j - m \left(\prod_j p^j A_i^j \right)^{1/m} . \end{aligned} \tag{B14}$$

$\mathbf{A}_i$ is a potential for prices because it satisfies the relation

$$\left. \frac{\partial \mathbf{A}_i}{\partial A_i^j} \right|_T = -p^j, \tag{B15}$$

for either reversible or irreversible changes.

In particular, this applies to comparisons among the final Pareto optima at which the speculator could leave each agent i. Then, $T \equiv 1$, and all $\delta \vec{A}_i \equiv 0$, so $\delta U_i = \delta S_i \equiv \delta \mu_i$, and from Eq. (B13),

$$U_i\left(\left\{B_i^j\right\}\right) = S_i\left(\left\{A_i^j\right\}\right), \tag{B16}$$

for $\left\{B_i^j\right\}$ satisfying Eq. (B5).

It is clear that if agents can trade internally to the contract curve, by whatever means, endowments are preserved, and at any Pareto optimum

$$\sum_i \left. U_i \right|_{\rm P.O.} = \sum_i \sum_j a_i^j = rma. \tag{B17}$$

Thus it is natural to define the efficiency η of any final state to which the agents could be brought by a speculator as the ratio of their resultant holdings $\sum_i U_i\left(\left\{B_i^j\right\}\right)$ to the value at the Pareto optima:

$$\eta = \frac{\sum_i U_i\left(\left\{B_i^j\right\}\right)}{\sum_i \left. U_i \right|_{\rm P.O.}} = \frac{S_i\left(\left\{A_i^j\right\}\right)}{rma} \tag{B18}$$

This is precisely the ratio of actual "work" denied by the agents to the speculator, to the best they could do. Evaluating Eq. (B18) by means of the definition (B6) then gives the result (11) in the text.

4. Formal use of reservoirs

The inability to hold any nontrivial price fixed (meaning, other than the normalization) during an extractive process can be overcome by formally introducing an external market for some form of "cash". This fulfills the role of the "world market" that assigns values to internal goods in Ref. [11]. Incorporating such an external market extends agent i's allocation bundle to include a new "money" variable: $x_i \equiv \left(M_i, \left\{A_i^j\right\}\right)$, and a fixed price for M_i is introduced if the world market accepts only exchanges

$$\sum_j \delta A_i^j + \pi \delta M_i = \delta U_i + \pi \delta M_i = 0. \tag{B19}$$

The external money may be considered intrinsically valueless to the agents, thus not changing the preference structure we have assumed for them, but still be given a formal presence in the utility, by assuming the world market accepts only cash payments, and the agents have no initial endowments of M_i. The first constraint implies $M_i \geq 0$ always, while the intrinsic valuelessness implies that agents will never surrender any of the consumable U_i to acquire a finite stock of it. The two considerations are satisfied simultaneously by a utility of the form

$$\mathcal{U}_i\left(x_i\right) \equiv m \left(\prod_j A_i^j \right)^{1/m} + \kappa_i M_i, \tag{B20}$$

where κ_i is now a proper Kuhn-Tucker multiplier, taken as minimizing $\mathcal{U}_i$ on the range $[0, \infty)$, and the contour money-metric form (B6) has been used to define f_i to avoid the step of a later translation.

Prices are identified by varying the utility (B20) extended by the budget constraint with yet another Kuhn-Tucker multiplier λ_i:

$$\delta \left[\mathcal{U}_i - \lambda_i \left(\sum_j p^j A_i^j + \pi M_i \right) \right] = 0. \tag{B21}$$

Solving Eq. (B21) gives the marginal utility of M_i, which is a sort of shadow price,

$$\kappa_i = \frac{\pi}{\left(\prod_j p^j\right)^{1/m}} \leftrightarrow \frac{\pi}{T} \tag{B22}$$

The expenditure function becomes

$$e_i(\pi, p, \mathcal{U}_i) = \pi M_i + \sum_j p^j A_i^j \equiv \pi M_i + U_i + \vec{p} \cdot \vec{A}_i, \quad \text{(B23)}$$

and the contour money-metric utility μ_i is simply $\mathcal{U}_i$ of Eq. (B20). The Helmholtz potential has the relation to e_i and μ_i given in Eq. (6) of Ref. [11]:

$$\mathbf{A}_i = e_i - \left(\prod_j p^j\right)^{1/m} \mu_i - \vec{p} \cdot \vec{A}_i, \quad \text{(B24)}$$

in which the terms involving M_i cancel exactly, recovering the form (B14). The entropy is the same function, but now relates to the contour utility as

$$S_i = \mu_i - \frac{\pi}{\left(\prod_j p^j\right)^{1/m}} M_i = m \left(\prod_j A_i^j\right)^{1/m}, \quad \text{(B25)}$$

as in Eq. (B23) of Ref. [11]. Nominally, the entropy can change by exchange δM_i without utility change, apparently at a well-defined price. However, because agents start with no money and are permitted no credit, $M_i \equiv 0$, $\forall i$, under all circumstances. The world-market price π only serves formally to give an M-value to the good U_i, since in the utilities it has no real function but to fix a Kuhn-Tucker multiplier.

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[20] The commitment to pure strategies follows from a wish to specify preferences only at the level of ordinal utilities. For the problems of complete common knowledge considered here, this will imply no loss of generality, though with more complex information sets, sensible mixed-strategy solutions could become relevant, in which case the cardinalization needed to compute them would require justification.
[21] For the physicists: the computability of a solution comes from non-vanishing of the eigenvalues of the Hessian of the constrained utility at its maxima, which correspond to the squared masses of their eigenvectors, considered as particle excitations. Vanishing of an eigenvalue due to a global symmetry creates a Goldstone mode, which is a massless degree of freedom. Gauge fields coupled to massive degrees of freedom correspond to cases where no global symmetry is an index of the ground state, meaning that all fundamental symmetries are expressed, not hidden. This is the case of an additional layer of paper added to a system in which the NE are not indexed by a global scale factor of the numéraire. While gauge fields coupled to massive excitations remain massless due to the different, local, gauge symmetry, a gauge field *coupled to a Goldstone mode* absorbs the Goldstone symmetry into the gauge conditions, removing both massless degrees of freedom from the system, and in this language making the ground state computable from the utilities.
[22] U_i is the internal energy state variable in physics, not to be confused with the utility $\mathcal{U}_i$. This unfortunate notation may be due in part to a belief by Walras that utility would be the economic counterpart to physical internal energy, which is now known not to be the case, as detailed in Ref. [11]