DEMOGRAPHY AND THE LONG-RUN PREDICTABILITY OF THE STOCK MARKET

By

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1. Introduction

The secular movement of the US stock market in the postwar period has been characterized by three distinct twenty-year episodes of sustained increase or decrease of real stock prices: the bull market of 1945–64, the subsequent bear market of the seventies and early eighties, and the bull market of the late eighties and nineties. For the most recent and spectacular bull market, typical explanations[1] have been based on the advent of a "new economy" in which innovations create a permanently higher rate of growth and an accompanying increase in the intangible capital of the corporate sector (McGrattan–Prescott (2001), Hall (2001)), on the substantial increase in agents' participation in the market, and on the apparent decrease in the Baby Boomers' risk aversion (Heaton–Lucas (1999)). Similar arguments, based on the "new economy" technical innovations of the immediate postwar period, and increased participation in the stock market, have also been used to justify the bull market of the fifties (see Malkiel (1990) and Shiller (2000)). The declining phase, from 1965 to 1982, has spawned fewer rationales, as documented by the well-known paper of Modigliani and Cohn (1979). They argue that real earnings and interest rates cannot account for the 60% decline in the real S&P index between 1965 and 1977, and find themselves forced to conclude that the only way of explaining the sustained decrease in stock prices is that investors, at least in the presence of unaccustomed and fluctuating inflation, are unable to free themselves from certain forms of "money illusion," using the nominal rather than the real rate of interest to value equity. While these explanations probably capture important elements underlying the behavior of stock prices in each of the episodes, they cannot readily be pieced together to form a coherent explanation of the stock market over the whole sixty-year period.

The idea motivating this paper is that demography is a common thread which might provide a single explanation for the alternating bull and bear markets over the whole sixty-year period. Since the turn of the century, the livebirths in the US have also gone through alternating twenty-year phases of baby booms and baby busts: for example, the low birth rate during the Great Depression was followed by the Baby Boom of the fifties and the subsequent Baby Bust of the seventies. These birth waves have resulted in systematic temporal changes in the age composition of the population in the postwar period, roughly corresponding to the twenty-year phases of the stock market.

[1] These explanations, while couched in the language of analytical models, are essentially the same as those given by Irving Fisher (1929) for the stock market boom of the 1920's.

Agents have distinct financial needs at different periods of their life, typically borrowing when young, investing for retirement when middle-aged and disinvesting in retirement. The stock market (along with other assets like real estate and bonds) is a vehicle for the savings of agents preparing for their retirement. It seems plausible that a large middle-aged cohort seeking to save for their retirement will push up the prices of these securities, while prices will be depressed in periods where the middle-aged cohort is small. We find that this is indeed the case in our model, whether agents are myopic or fully aware of demography and its implications. Poterba (2001) has argued that if agents were rational, they would anticipate any demography-induced rise in stock prices twenty years earlier, bidding up the prices at that time, thereby negating much of the demographic effect on stock prices. We show that if agents were myopic, blindly plowing savings into stocks when middle-aged, then stock prices would be proportional to the size of the middle-aged cohort. When agents fully anticipate demographic trends, their rational response actually reinforces the effect on stock prices, making its rise more than proportional to the size of the middle-aged cohort.

To test how much variation in security prices can be explained by the combination of life-cycle behavior and changing demographic structure, we study the equilibria of a cyclical, stochastic, overlapping-generations exchange economy, calibrated to the stylized facts on agents' lifetime income patterns, the payoffs of the securities and the demographic structure in the US for the post-war period. We derive three predictions from our model, which we then compare with data on stock and bond returns. The first prediction is that PE (price earnings) ratios should be proportional to the ratio of middle-aged to young (the MY ratio). The second is that real rates of return on equity and bonds should be an increasing function of the *change* in the MY ratio. Lastly, we show in our model that the equity premium should covary with the YM ratio (the reciprocal of the MY ratio), even though the young are more risk tolerant than the old.

The fact that the most recent stock market boom coincided with the period in which the generation of post World War II Baby Boomers reached middle age has led Wall Street participants and the financial press to attribute part of the rise in prices to the investment behavior of the Baby Boomers preparing for their retirement. Professional economists, on the other hand, have been sceptical of the connection between demography and stock prices. Although Bakshi and Chen (1994) documented a striking relation between the average age of the US population over 20 and the movement of the real S&P since 1945, a systematic literature studying the relation between demography and prices of financial assets emerged only recently. On the empirical side, Macunovich (1997, 2002) found a relation between the (smoothed) rate of change of the real Dow Jones index and the rate of change of cohort sizes, while Poterba (2001) tested the relation between various indicators of demography and the prices and returns of equity, concluding that the retiring of the baby boom generation would have a small effect on asset prices. On the theoretical side, Brooks (1998, 2001) and Abel (2001, 2003) pioneered the use of equilibrium models to study the effect of demography. They both used a Diamond model with random birth rates. Brooks found that demography had a small effect on real rates of return, and that the equity premium shrinks when the population is relatively young. Abel's model was not calibrated, but a calibrated version of it was studied by Bütler and Harms (2001), who concluded that the variation of the labor supply could smooth out some of the effects of a demographic shock like a Baby-Boom. Bakshi and Chen had used an infinite horizon representative-agent pricing model to account for the behavior of security prices, the representative agent having an age which is the average age of the population. A key assumption was that the relative risk aversion of the representative agent is an increasing function of the average age. The existing literature has been admirably summarized by Young (2002).

Our paper and conclusions differ in several respects. First, we study a cyclical model in which high birth cohorts are deterministically followed by low birth cohorts, as has been the case for the past century in the US, rather than a stochastic birth model in which a large cohort might be followed by an even larger cohort. Second we assume utilities for which savings are relatively insensitive to interest rates. Third we take as our reference point a model in which a fixed quantity of land produces a fixed output per period, and then move to models with endogenous capital and adjustment costs. As a result we find that the demographic effect on PE ratios is larger than our predecessors have suggested. Finally, in contrast to Brooks and Bakshi and Chen, we find that the equity premium is smaller when the population of savers is older, thus reinforcing the demographic effect, as has been the case historically.

Section 2 studies the equilibria of a simple deterministic model in which generations are alternately large and small, periods last for twenty years, and equity in a fixed asset ("land"or "trees") yields a constant stream of dividends each period. The sizes of the generations, and the dividends and wages for the young and middle-aged are chosen in accordance with historical averages for the US. This certainty model gives the order of magnitude of the change in security prices which can be attributed to demographic change: even though output increases by only 7% when the large generation is in its peak earning years, and cohort sizes fluctuate by 50%, PE ratios increase by 130% when the large cohort is middle-aged. We show that the lower the intertemporal elasticity of substitution in preferences, the greater the fluctuation in equity prices.

In Section 3 we show that the qualitative behavior of the equilibrium is not significantly changed when the model is enriched to accommodate more realistic features such as kids, Social Security, or bequests. Kids and Social Security both reinforce the demographic effect on asset prices, while bequests attenuate it, but taken together at levels calibrated to fit American data, there is not much difference. The equilibria of our model can also be related to the equilibria of the standard Diamond model with production. By introducing adjustment costs for capital we obtain a parametrized family of models which includes at one extreme the Diamond model, with zero adjustment costs, and at the other extreme models with progressively higher adjustment costs whose equilibria converge to the equilibrium of the land economy. The possibility that savings can go into new capital instead of pushing up the price of existing capital reduces the demographic variation in rates of return and in equity prices. However, since there is a lag between physical investment and increased output, the demographic variation in price-dividend ratios can be as high in the Diamond model as in the exchange model with fixed land.

In Section 4 we show how shortening the time periods exhibits the relation between demographic structure and security prices in its most striking form: in the stationary equilibrium equity prices are precisely in phase with the demographic structure, attaining a maximum when the number of middle-aged agents is at a maximum and the number of young agents is at a minimum, and a minimum when the numbers are inverted. Rates of returns on the other hand are not in phase with the demographic cycle. The maximum for the rate of return occurs in the middle of the ascending phase of the equity prices, when the increase in the MY ratio is at its maximum, inducing a large capital gain. The minimum rate of return occurs in the middle of the descending phase of equity prices, when the decrease in MY ratio and the capital loss are the greatest. Thus in the absence of shocks to the economy, a cyclical birth process translates into a cyclical behavior of equity prices and interest rates, with short-term interest rates "leading" equity prices by half a phase, because equity prices move with the ratio of middle-aged to young while short-term interest rates move with the change in this ratio.

In Section 5 we add uncertainty to the model, in wages and in dividends. In the postwar period in the US, bull markets have had peak to trough ratios of the order of 5 or 6, while the pure demographic model delivers increments of the order of 2 or 3. Thus "other forces" must contribute a factor of order 2.5 to 3 to the changes in stock prices. The periods in which the middle-aged agents were numerous relative to the young (the fifties and early sixties and the late eighties and nineties) were also periods in which the economy was subject to positive shocks, while the period of the seventies when the young Baby Boomers were plentiful was marked by negative shocks (oil shortages, inflation). Thus in the second part of the paper we add business cycle shocks to incomes and dividends and calculate the stationary Markov equilibrium of the resulting economy by a method similar to that recently used by Constantinides–Donaldson–Mehra (2002). With the shocks our model can deliver variations in price-earnings ratios of 5 or 6.

The equity premium (the excess return stocks earn over the riskless interest rate) is the new variable of interest in the stochastic economy. Previous work has suggested that the equity premium observed historically is difficult to reconcile with a rational expectations model on two counts. First, the historical equity premium is too large to be rationalized by reasonable levels of risk aversion (Mehra–Prescott (1985)). Second, and most important for us, the observation that young people are more risk tolerant than old people (Bakshi–Chen (1994)) suggests that the equity premium should be lowest when the proportion of young people is highest, exactly contrary to the historical record.

Our stochastic model sheds some light on the second problem. If there is a strong demographic effect, then the numerous young (and contemporaneous few middle aged) should rationally anticipate that investment returns will be relatively high. Since wages and dividends do not vary so dramatically with demographic shifts, they should anticipate that a relatively high fraction of their future wealth will come from risky equity capital values. Though the average risk tolerance is higher, the average exposure to risk is also higher, and so we find that in our model the equity premium is higher when the stock market is low, consistent with the historical record.

As for the problem that the historically observed equity premium in the US is above the *ex ante* equity premium generated by standard models, we have little new to contribute. We impose limited participation in equity markets (confining such participation to 50% of the population, consistent with recent history), and we find that the equity premium rises in our model, while preserving the demographic effect on equity prices. As is now standard, we attribute the larger historical *ex post* equity premium to luck (see for example Brown, Goetzmann, and Ross (1995)).

In Section 6 we compare the results of the model with the stylized facts on the bond and equity markets for the period 1910–2002. The variables which most closely fit the predictions of the model are the price-earnings ratio and the rate of return on equity. Since 1945 the price-earnings ratio has strikingly followed the cyclical pattern of the medium-young ratio in the population while the rate of return on equity has a significant relation with the changes in the medium-young ratio, as predicted by the model. The behavior of real interest rates departs much more from the predictions of the model, and only after 1965 does the real interest rate have a significant relation with the change in the medium-young ratio. Moreover interest rate variations have been smaller than in the calibrated model, with the result that the level and variability of the equity premium is higher in the data than in the model. Section 6 also briefly presents some evidence on equity markets and demography for Germany, France, UK and Japan. Section 7 concludes with cautionary remarks on the use of the model for predicting the future course of prices due to the globalization of equity markets.

2. Simple Model with Demographic Fluctuation

Consider an overlapping generations exchange economy with a single good (income) in which the economic life of an agent lasts for three periods: young, middle-age and retired. All agents have the same preferences and endowments and only differ by the date at which they enter the economic scene. Their preferences over lifetime consumption streams are represented by a standard discounted sum of expected utilities

$$U(c) = E\Big(u(c^y) + \delta u(c^m) + \delta^2 u(c^r)\Big), \qquad \delta > 0 \tag{1}$$

where $c = (c^y, c^m, c^r)$ denotes the random consumption stream of an agent when young, middle-aged and retired. For the calibration, u will be taken to be a power utility function

$$u(x) = \frac{1}{1-\alpha} x^{1-\alpha}, \quad \alpha > 0$$

where α is the coefficient of relative risk aversion (and $1/\alpha$ is the elasticity of substitution over time). Since a "period" in the model represents 20 years in the lifetime of an agent, we take the discount factor to be $\delta = 0.5$ (corresponding to a annual discount factor of 0.97).

In this Section we outline the basic features of the model and explain how we choose average values for the calibration: these average values can be taken as the characteristics of a deterministic exchange economy whose equilibrium is easy to compute, and this provides a first approximation for the effect of demographic fluctuations on the stock market.

Each agent has an endowment $w = (w^y, w^m, 0)$ which can be interpreted as the agent's labor income when young and middle-aged, and income in retirement is zero. There are two financial instruments—a riskless bond and an equity contract—which agents can trade to redistribute income over time (and, in the stochastic version of the model, to alter their exposure to risk). The (real) bond pays one unit of income (for sure) next period and is in zero net supply; the equity contract is an infinite-lived security in positive supply (normalized to 1), which pays a dividend each period. Agents own the financial instruments only by virtue of having bought them in the past: they are not initially in any agent's endowment. In this section the dividends and wage income are nonstochastic, so that the bond and equity are perfect substitutes; in Section 5, where we introduce random shocks to both dividends and wages, bond and equity cease to be perfect substitutes.

Since we want to study the effect of the fluctuations in the age composition of the population on capital market prices rather than the effect of a general growth of the population, we assume that the model has been "detrended" so that the systematic sources of growth of dividends and wages arising from population growth, capital accumulation and technical progress are factored out. The sole source of variation in total output comes from the cyclical change in the demographic structure, to which we now turn, and from the random "business cycle" shocks introduced in Section 5.

Demographic Structure. Livebirths induce the subsequent age structure of the population: the annual livebirths for the US during the 20th century[2] are shown in Figure 1. If the livebirths for a sequence of twenty adjoining years are grouped together into a cohort, then the number of births can be approximated by five twenty-year periods which create the alternatively large and small cohorts known as the 10's, 30's, 50's, 70's and 90's generations.

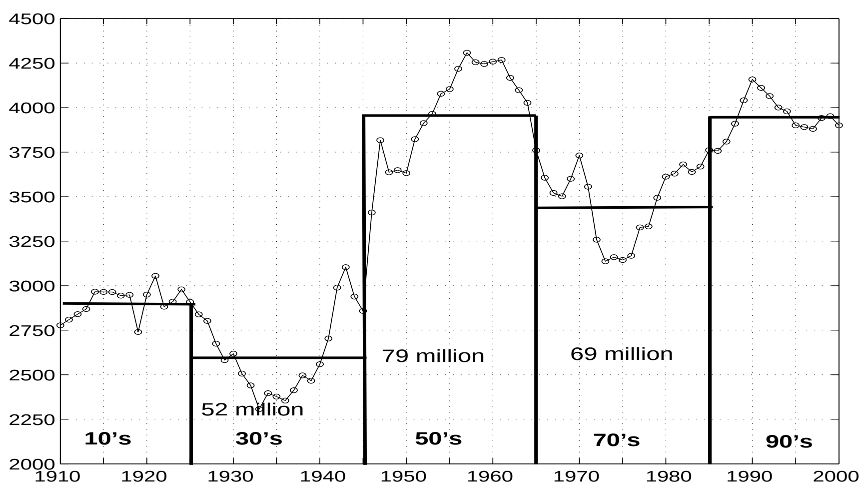

Figure 1: *US livebirths and the 5 cohorts of the* 20^{th} *century*

We seek the simplest way of modeling this alternating sequence of generation sizes: time is divided into a sequence of 20-year periods. An individual's "biological life" is divided into 4 periods, 0–19 (child), 20–39 (young) 40–59 (middle age), 60–79 (retired); the agent's "economic life" (earning income, trading on financial markets) begins when the agent is young. We assume that in each odd period a large cohort (N) enters the economic scene as young, while in each even period a small cohort (n) enters. Thus there are N young, n middle-aged, and N old in every odd period, and n young, N middle-aged, and n old in every even period.

Because the typical lifetime income of an individual is small in youth, high in middle age and

[2]Historical Statistics of the United States, Series B1, and Bureau of the Census.

small or nonexistent in retirement, agents typically seek to borrow in youth, invest in equity and bonds in middle age, and live off this middle-age investment in their retirement. As we shall see, this life-cycle portfolio behavior implies that the relative size of the middle and young cohorts plays an important role in determining the behavior of equilibrium bond and equity prices. For the above alternating cohort structure, the *medium-young cohort ratio* (MY, for short) alternates between $n/N < 1$ in odd periods and $N/n > 1$ in even periods.

The demographic structure shown in Figure 1 is not perfectly stationary. There were 52 million livebirths in the Great Depression generation from 1925–1944, and 79 million born in the Baby Boom from 1945–1964; these two generations traded as medium and young in the period 1965–1984. In the Baby Bust (Xer) generation between 1965 and 1984 births fell, but only to 69 million; the Baby Boom and Xer generations have traded with each other from 1985–2004. The Echo Baby Boom generation born between 1985 and the present seems headed for the same order of magnitude as the Baby Boom generation; the Echo Baby Boom generation and the Xer generation will trade with each other from 2005–2024.

In order to mimic the actual history with a stationary economy, we are thus led to study two cases: in the first $n = 52$ and $N = 79$ which are the relative sizes of the Great Depression and the Baby Boom generations. This is the case for which the demographic effect is the strongest and whose equilibrium is studied in the body of the paper. We also compute the equilibrium for a second case in which N is kept at 79, and the smaller cohort size is $n = 69$. The equilibrium values for this case are given in Appendix C.

Calibrating the age pyramid using the number of livebirths neglects immigration, which plays an important role in the demography of the US. However we show in Appendix A that taking the immigrants into account essentially leaves unchanged the ratio of the cohorts of middle-aged to young for the periods 1965–1984 and 1985–2004 which we have taken as reference values for the calibration.

Wage Income. The exchange economy is viewed as an economy with "fixed production plans." Equity in land or trees yields a steady stream of dividends D each period, and each young and middle-aged worker produces output w^y and w^m respectively. To calibrate the relative shares of wage income going to young and middle-aged agents, we draw on data from the Bureau of the Census shown in Figure 2: the maximum ratio of the average annual real income of agents in the age-groups 45–54 and 25–34 is 1.54: we round this to 1.5 and calibrate the model on the basis of a wage income of $w^y = 2$ for each young agent and $w^m = 3$ for each middle-aged agent. Since the agents have homothetic (CES) preferences, the absolute levels of endowments and dividends do not influence the relative prices or relative consumption levels, which will be the primary focus of the study.

Since the wage income of middle-aged agents is greater than that of the young, the total wage is greater in even periods when the middle-aged generation is large than in odd periods when the young generation is large. Since the active population is constant, this increase in wages has to be interpreted as coming from an increase in the average productivity of labor: implicitly the model presumes that middle-aged agents are more experienced and productive than the young since they are paid higher wages.

When $(N, n) = (79, 52)$, the total wages alternate between $341 = 79 \times 3 + 52 \times 2$ and $314 = 79 \times 2 + 52 \times 3$. When the demographic structure is less skewed, as in the economy with $(N, n) = (79, 69)$, the total wage income alternates between $375 = 79 \times 3 + 69 \times 2$ and $365 = 79 \times 2 + 69 \times 3$.

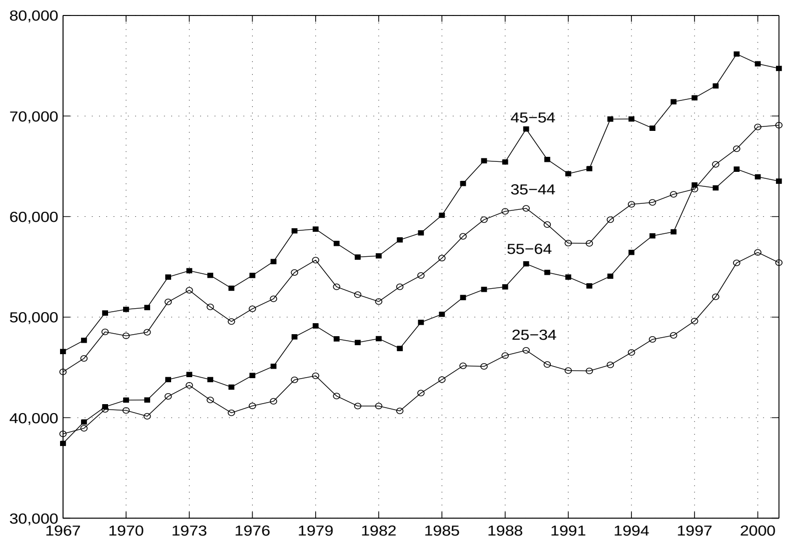

Figure 2: *Real wage income of different age cohorts over time (2001dollars)*

Dividends. Land produces output which is distributed as dividends to the equity holders. We take the ratio of dividends to wages to be of the same order of magnitude as the ratio of (generalized) dividends to (generalized) wages in the National Income and Products Accounts.[3] More precisely we define as generalized wages the sum of the categories “Compensation to Employees,” “Supplements to Wages and Salaries,” and half of the “Proprietors' Income.” The rationale for this is that

[3]The reference data set is the annual National Income by Type of Income from 1959 to 1999, Bureau of Economic Analysis.

the "Proprietor's Income" category contains the net income of unincorporated business (farmers, doctors, lawyers, partners, small business proprietors) which is just wage income[4] from the perspective of our model. We define generalized dividends as the sum of "Rental Income," "Dividends Paid by Corporations," "Net Interest," and half the "Proprietors' Income." These are the payments to capital which are priced in long-lived securities. We postulate that the retained earnings of corporations are used to finance growth and, since our model does not have growth and investment, we do not take them into account. On average the ratio of generalized dividends to generalized wage is 0.19. Thus in the economy in which $(N, n) = (79, 52)$ we take $D = 0.19(341 + 314)/2 \simeq 62$ and when $(N, n) = (79, 69)$ we take $D = 0.19(375 + 365)/2 \simeq 70$.

For the demographic structure $(N, n) = (79, 52)$ in which there is a large variation in the cohort ratio, total income (wages plus dividends) is on average 7.2% higher in even periods than in odd periods: for the case $(N, n) = (79, 69)$ with its smaller variation in the cohort ratio, the output difference is 2.3%.

Pure Demographic Equilibrium. When the only source of change in the economy comes from fluctuations in the demographic structure, it is straightforward to describe and solve for the stationary equilibrium. Let q_t^b be the price of the bond at time t, that is, the amount of good that is required in period t to buy one unit of output in the next period; then $q_t^b = 1/(1 + r_t)$, where r_t is the interest rate from period t to $t + 1$. It is easy to show that there is an equilibrium in which $q_t^b = q_1$ whenever t is odd, and $q_t^b = q_2$ whenever t is even. Since agents can use the bond or the equity to transfer income across the different periods of their life, they can equalize the present value of their consumption to the present value of their income. Agents in the large cohort who are young in odd periods choose a consumption stream (C^y, C^m, C^r) so as to maximize the utility function (1) subject to the budget constraint

$$C^y + q_1 C^m + q_1 q_2 C^r = w^y + q_1 w^m + q_1 q_2 w^r = 2 + q_1 3 \tag{2}$$

while agents in the small cohort who are young in even periods choose (c^y, c^m, c^r) so as to maximize (1) under the budget constraint

$$c^y + q_2 c^m + q_1 q_2 c^r = w^y + q_2 w^m + q_1 q_2 w^r = 2 + q_2 3 \tag{3}$$

In equilibrium we must have

$$\begin{aligned} NC^y + nc^m + NC^r &= N \times 2 + n \times 3 + D \\ nc^y + NC^m + nc^r &= n \times 2 + N \times 3 + D \end{aligned} \tag{4}$$

[4]McGrattan and Prescott (2001) attribute 80% of the Proprietor's Income to wages and 20% to dividends.

Since there is no uncertainty, the bond and equity must be perfect substitutes in each period. From the no-arbitrage property of equilibrium, the rate of return on the equity market and on the bond market must be the same. Thus if bond prices alternate between q_1 and q_2, then the price of equity must alternate between q_1^e and q_2^e, where

$$\frac{D+q_2^e}{q_1^e} = \frac{1}{q_1} = 1 + r_1, \quad \frac{D+q_1^e}{q_2^e} = \frac{1}{q_2} = 1 + r_2$$

If $q_1 < q_2$, or equivalently if $r_1 > r_2$, then it must be that $q_1^e < q_2^e$: thus interest rates are high when equity prices are rising, and low when equity prices are decreasing. Solving the rate-of-return equations yields the relation between bond and equity prices

$$q_1^e/D = (q_1 q_2 + q_1)/(1 - q_1 q_2) \quad \text{and} \quad q_2^e/D = (q_1 q_2 + q_2)/(1 - q_1 q_2) \tag{5}$$

Note that the same result could have been obtained by expressing that the price of equity is equal to the discounted value of its dividends

$$\begin{aligned} q_1^e &= Dq_1 + Dq_1q_2 + Dq_1q_2q_1 + Dq_1q_2q_1q_2 + ... \\ q_2^e &= Dq_2 + Dq_2q_1 + Dq_2q_1q_2 + Dq_2q_1q_2q_1 + ... \end{aligned}$$

A convenient way of assessing the level of equity prices is to compute the price-dividend ratio (PD) defined by $PD(k) = q_i^e/(D/20),\ i = 1, 2$, where the dividends are expressed on a yearly basis. To compare the results of the model with the well publicized price-earnings ratios of corporate equity, a good rule of thumb is to divide them by 2, since on average corporate firms distribute half their earnings in dividends.[5] In the text we will often refer to PE $\equiv$ PD/2 as the "price-earning ratio." In the same way, rather than reporting the interest rate on a twenty-year period, we report the annualized interest rate r_i^{an} defined by $(1 + r_i^{an})^{20} = 1 + r_i$, for $i = 1, 2$.

Properties of Equilibrium. If the bond prices were to coincide with the consumer discount rate, $q_1 = q_2 = 0.5$, then individuals would attempt to completely smooth their consumption, demanding the stream $(c^y, c^m, c^r) = (2, 2, 2)$. But then, in the case where the population structure is $(N, n) = (79, 52)$, in odd periods the aggregate excess demand for consumption would be $79(2 - 2) + 52(2 - 3) + 79(2 - 0) - 62 = 44$, while in even periods it would be $52(0) + 79(-1) + 52(2) - 62 = -37$. Thus in odd periods there is excess demand for consumption as the retired agents consume beyond their income more than the middle-aged save for their retirement, while in even periods when the middle-aged cohort is large, there is excess demand for savings as those households seek to invest

[5] For the period 1951–2000 the average payout ratio for the firms in the S&P500 Index was 0.51.

for their retirement. To clear markets, the interest rates must adjust, discouraging consumption (stimulating savings) in odd periods, and discouraging saving (stimulating consumption) in even periods: as a result equilibrium bond prices must be below 0.5 in odd periods, and above 0.5 in even periods. By no-arbitrage, land prices must be higher in even periods than in odd periods. How far the interest rates and land prices must adjust depends on how big a price change is required to move consumers away from equal consumption in each period of their lives, which in turn is connected to the relative strength of income and substitution effects, as we shall see in a moment.

Here is another way of understanding how the demographic effect on equity prices can be so large, and how it is reinforced by rational optimization. Suppose agents myopically consume 2 when young (thus saving nothing), consume 2 again when middle-aged (investing all their savings in land), finally selling all their land in old age to finance their retirement consumption. The price of land would then be 79 in even periods with the big middle-aged population, and 52 in odd periods with the small middle-aged population. Myopic behavior in which the middle-aged do all the saving explains a $50\% = 79/52 - 1$ variation in equity values, even though total output varies by 7%.

Rationality boosts the effect: for rational agents would perceive that in following the myopic strategy, the large generations would end up with consumption approximately $(2, 2, 1.3)$ in youth, middle-age, and old age, respectively, while small generations would end up with consumption approximately equal to $(2, 2, 2.5)$. Anticipating this jump in old-age consumption twenty years ahead, and assuming sufficient aversion to drastic jumps in consumption, the large generations would save more in middle-age, and the small generations less, reinforcing the demographic effect.

If agents foresaw the demographics forty years ahead (which is possible, since knowing the size of the current child cohort gives a good idea of the middle-aged cohort in forty years), the large generations would also tend to save more when young, buying say 30% of the land, with the purpose of holding it until old age. If they did not use the land to increase their middle-aged consumption, this would still further reinforce the demographic effect: 30% of the land would be removed from the market in both periods, and their middle-aged savings would rise by 30% of land dividends.

The only damper on the demographic effect on equity prices is that rational agents will anticipate that the return on land between odd and even periods will be greater than the return between even and odd periods, rendering middle-aged consumption relatively cheap for the big generations, and relatively expensive for the small generations. If their utilities provide for a large substitution effect, middle-aged consumption for the large generation will increase, thus partially reducing their middle-aged savings, mitigating the demographic effect. When the risk aversion parameter in the utility function is $\alpha = 4$, the elasticity of substitution of consumption across time is $1/4$ and the substitution effect is small.

Since agents are always saving (for their retirement years) when middle-aged, the high returns to land in odd periods and the low returns from even periods favors agents born in small cohorts (who are middle-aged when returns are high) relative to those born in large cohorts. We call this the *favored cohort effect*. This income effect just offsets the substitution effect when $\alpha = 4$: large and small cohorts have the same middle-age consumption.

Calculating the stationary equilibrium for the economy with $(N, n) = (79, 52)$, and utility function parameter $\alpha = 4$, gives equity prices, annual interest rates, and PE ratios

$$(q_1^e, q_2^e,\ r_1^{an}, r_2^{an},\ PE_1,\ PE_2) = (52, 120,\ 6.4\%, -0.3\%, 8.4, 19.4)$$

consumption streams $C = (C^y, C^m, C^r), c = (c^y, c^m, c^r)$, for large and small generations, and utilities

$$(C, c,\ U, u) = ((1.8, 2, 1.7), (2.4, 2, 2.3),\ (-.1, -.05))$$

As expected, when the large cohort is young and the small cohort middle-aged, the equity price is low, with a price-earnings ratio around 8; when the large cohort moves into middle-age and seeks to save for retirement, the equity price is more than 2 times higher ($q_2^e/q_1^e = 2.3$), the PE ratio increasing to 19. The variation of equity prices (or equivalently, the variation of PE ratios) is roughly equal to the variation in the MY ratio, namely $2.3 \approx (79/52)/(52/79)$. When the equity price is low and is anticipated to increase, the interest rate is high (6.4%), and it falls to -0.3% when the equity price is high and is going to decrease. As predicted by the favored cohort effect, the smaller generation is better off ($-.05 > -0.1$).

Table 1: Effect of intertemporal elasticity parameter and cohort sizes on equilibrium

	$N = 79,\ n = 52$						$N = 79,\ n = 69$					
	Pyramid Δ_1			Pyramid Δ_2			Pyramid Δ_1			Pyramid Δ_2		
	q_e	r^{an}	PE	q_e	r^{an}	PE	q_e	r^{an}	PE	q_e	r^{an}	PE
$\alpha = 2$	55	5.3%	8.8	91	1.3%	14.6	68	4%	9.8	81	2.7%	11.6
$\alpha = 4$	52	6.4%	8.4	120	-0.25%	19.4	67	4.4%	9.5	89	2.2%	12.6
$\alpha = 6$	51	7.3%	8.2	147	-1.3%	23.8	66	4.7%	9.4	94	1.8%	13.5

When the demographic structure (N, n) gets less skewed, the disequilibrium implied when the bond prices are equal $q_1 = q_2$ is less pronounced, so that bond and equity prices do not need to fluctuate as much to establish equilibrium. With $(N, n) = (79, 69)$, equity prices are again roughly proportional to MY ratios, that is $89/67 = 12.6/9.5 \approx (79/69)/(69/79)$. For a given demographic structure, if the aversion to consumption variability is lower (i.e., if the intertemporal substitution is higher), the variation in prices needed to establish an equilibrium is also lower. Table 1 shows the effect on equilibrium prices of decreasing the difference in cohort sizes and of varying the coefficient α determining the elasticity of substitution of consumption across periods (equal to $1/\alpha$). The rule that equity prices are proportional to MY ratios holds very closely when $\alpha = 4$, but only approximately for $\alpha \neq 4$.

3. Robustness of Pure Demographic Equilibrium

Family, Bequests, and Social Security. The model of the previous section can be viewed as the simplest model for studying the consequences for the stock market of fluctuations in demographic structure. However it abstracts from a number of important features which alter agents' needs to redistribute income over time. In particular, the presence of bequests, of social security payments in retirement, or the fact that young agents have to provide for their children, alters the need for intertemporal savings. In this section we study how the predictions of the basic model are modified by the introduction of these factors.

Implicit in the model is that parents from a large cohort have on average small families — each agent of a large cohort has $\nu_1 = n/N$ children — while parents from a small cohort have large families, i.e., they have on average $\nu_2 = N/n$ children. The Easterlin hypothesis (Easterlin (1987)) provides an explanation for such fluctuations in the fertility ratio, which can be rephrased in the setting of our model as follows. The young of any generation form their material aspirations as children in the households of their parents: in deciding their family size they compare the material prospects they can offer to their children with the aspirations they have formed as children in their parents' households. Since the young in a small cohort have greater lifetime income than their parents who come from a large cohort, they feel that they can offer their children material conditions that exceed their aspirations, and are led to choose a large family size. Conversely the young of a large cohort facing difficult conditions but having formed high aspirations choose a small family size. This suggests a simple, albeit highly stylized, way of linking the choice of family size (fertility) to the underlying economic conditions.

Let us now take into account the fact that parents provide for the consumption of their children. If ν denotes the number of children, then the utility of a young parent is $\nu\lambda u(c^k) + u(c^p)$ where c^k

denotes the consumption of a child (kid), c^p the parent's consumption, and λ is the weight given by the parent to a child's utility.[6] Assume that agents give bequests to their children and let b denote the bequest transferred by retired parents to their middle-aged children. We take the utility in the retired period to be $u(c^r, b) = (c^r)^{1-\beta} b^\beta$, $0 < \beta < 1$. In practice individuals end up with wealth at the time of their death, both because they hold precautionary balances against the uncertain time of death and because they derive direct utility from the bequests they leave to their children.[7] We model the combination of these two motives by assuming that the utility is a function of the bequest and not of the bequest per child. The utility function of the representative agent, which replaces (1), is given by

$$U(c,b) = \frac{1}{1-\alpha}\left[\nu\lambda(c^k)^{1-\alpha} + (c^p)^{1-\alpha} + \delta(c^m)^{1-\alpha} + \delta^2\Big((c^r)^{1-\beta}b^\beta\Big)^{1-\alpha}\right] \tag{6}$$

To complete the model we add the transfers to an agent's lifetime income arising from a pay-as-you-go social security system. We assume that each retired agent (regardless of cohort size) receives a transfer $\theta \geq 0$ and that the labor income received in pyramid Δ_1 (resp. Δ_2) is taxed at the rate τ_1 (resp. τ_2), where τ_1 and τ_2 are chosen so that the social security budget is balanced. The life-time budget constraint of an agent young in pyramid Δ_i, $i = 1, 2$ can then be written as

$$\nu c_i^k + c_i^p + q_i c_i^m + q_i q_{i+1}\Big(c_i^r + b_i\Big) = w^y(1-\tau_i) + q_i\left(w^m(1-\tau_{i+1}) + \frac{b_{i+1}}{\nu_{i+1}}\right) + q_i q_{i+1}\theta, \quad i = 1,2 \tag{7}$$

where $i+1$ is taken modulo 2 $(1+1 = 2, 2+1 = 1)$. In a stationary equilibrium with kids, bequests and social security, young agents in cohort Δ_i maximizes (6) subject to (7), the market clearing equations (4) hold and the social security tax satisfies the balance-budget equations

$$(Nw^y + nw^m)\tau_1 = N\theta \qquad (nw^y + Nw^m)\tau_2 = n\theta$$

The equilibrium equity prices are then given in terms of (q_1, q_2) and D by (5).

Since the first-order conditions imply that $\lambda(c^k)^{-\alpha} = (c^p)^{-\alpha}$, the weight $\lambda^{1/\alpha}$ determines the ratio of the consumption of a child to the consumption of the adult parent which, in the literature, is called the child-equivalent consumption. Since we can find estimates for this ratio in the empirical literature, it is convenient to parameterize the model by the child-equivalent consumption η, and choose $\lambda = \eta^\alpha$. The equilibrium depends on three new coefficients (η, β, θ) which parameterize the child-equivalent consumption, the strength of the bequest motive, and the magnitude of the social security transfer. By setting two of these coefficients to zero we can study how each parameter

[6] This is the specification used by Brooks (2001).

[7] See Modigliani (1986, 1988) for a discussion and estimation of the proportion of wealth transferred through bequests.

affects the equilibrium; by choosing a representative value for each parameter we can examine their combined effect on the equilibrium. We follow Deaton (1997) and take the consumption of a child to be half the consumption of an adult parent ($\eta = 0.5$); we take $\beta = 0.3$ to generate a ratio of bequests to aggregate income between 15.5% and 18.5% of aggregate income which is the "consensus estimate" reported by Modigliani (1988). At the end of the 90's the ratio of social security transfers and medicare benefits to national income was of the order of 8%: by choosing $\theta = 0.5$ as the per-capita social security transfer we obtain a ratio of social security transfers to total income of 10.5% in pyramid Δ_1 and 6.45% in pyramid Δ_2. The table that follows shows the separate and combined effects of the three parameters on the equilibrium. The preference coefficient is set to $\alpha = 4$ and the demographic parameters are $(N, n) = (79, 52)$.

Table 2: Bequests, Family and Social Security

	$\beta = 0, \eta = 0, \theta = 0$		$\beta = 0.3, \eta = 0, \theta = 0$		$\beta = 0, \eta = 0.5, \theta = 0$		$\beta = 0, \eta = 0, \theta = 0.5$		$\beta = 0.3, \eta = 0.5, \theta = 0.5$	
	Δ_1	Δ_2	Δ_1	Δ_2	Δ_1	Δ_2	Δ_1	Δ_2	Δ_1	Δ_2
q^e	52	120	97	172	19	55	26	73	19.8	45
PE	8.5	19	15.5	28	3	9	4.2	12	3.2	7.3
r^{an}	6.4%	-0.3%	4.5%	-0.4%	9.4%	1.9%	8.6%	0.9%	8.8%	3%
b	0	0	0.7	1	0	0	0	0	0.8	1
τ	0	0	0	0	0	0	12.6%	7.6%	12.6%	7.6%
q_2^e/q_1^e		2.3		1.8		2.9		2.8		2.3

Poterba (2001) has argued that the presence of bequests will attenuate — if not cancel — the decrease in security prices that is expected when the Baby Boomers go into retirement, since they will not attempt to sell all their securities. However if all generations transferred the same fraction of their wealth as bequests, it still implies that a large generation will need to sell the share of its wealth that it needs as retirement income to a smaller generation of middle-aged savers. Abel (2001) has shown that in his model with production and two-period-lived agents, the presence of bequests does not change the equilibrium. In our model adding a bequest motive does lower the ratio of equity prices, but it does not cancel the effect: the main effect is to lower the interest rate since agents in both cohorts have more income in middle age by virtue of the bequests from their parents, and thus save more for retirement. The smaller ratio of equity prices comes partly from the fact that the small generation when middle-aged receives a higher per-capita bequest ($0.7/\nu_1 = 1.06$) than the large generation ($1/\nu_2 = 0.66$), the larger income tending to compensate for the smaller size of the cohort in the aggregate savings function.

The other parameters, the child-equivalent consumption η and the social-security benefit θ have the reverse effect, increasing the ratio of equity prices and increasing interest rates. The need to provide for the children tends to increase the demand for borrowing or equivalently to decrease the savings rate in each pyramid. Since small generations have more children, their savings drop more, thereby increasing the demographic effect on equity prices. Introducing social-security benefits decreases the income of agents when they are working and increases their income when retired, thus also decreasing the savings rate. When the three effects are combined the forces causing interest rates to be high prevail, lowering the price-earnings ratios. But the ratio of equity prices is the same order of magnitude as in the simple model. If this more detailed institutional model was chosen as the reference model, we would need to increase the discount factor to obtain more realistic interest rates and PE ratios. For example, with $\delta = (0.99)^{20} = 0.82$ and the same parameters as in the last two columns of Table 2, the equilibrium is

$$(q_1^e, q_2^e, r_1^{an}, r_2^{an}, PE_1, PE_2) = (45, 81, 6\%, 1.4\%, 7.2, 13)$$

The relatively low discount factor $\delta = (0.97)^{20} = 0.5$ used in the simple exchange model, can then be viewed as a convenient proxy for these more realistic institutional features which are left out of the model and which lower the savings rate.

Comparing Equilibrium of Exchange and Production Economy. In this section we study the effect of replacing the assumption that the asset is in fixed supply with the assumption that the asset is producible “capital.” Variations in savings can now be channeled into changes in the capital stock, reducing the demographic variation in interest rates and equity values. However since there is a lag between the moment when savings occur and the time when output and dividends are generated, the price-dividend ratio is as sensitive to demography as it was before.[8] Finally, we show that in the presence of adjustment costs — which permit equity prices to differ from the capital stock — the equilibria of the production economy become similar to those of the exchange economy, and essentially coincide when the adjustment costs are sufficiently high.

Consider an economy with the same consumer side as in the exchange economy, but in which wages and dividends are endogenous. Each agent is endowed with one unit of labor when young

[8] In this section we assume that the representative firm has no debt and finances investment by retained earnings. For an arbitrary financial policy the ratio that we compute is the ratio of the market value of the firm to its “net” dividend, i.e., the sum of what is paid to shareholders and bondholders minus the new borrowing from bondholders or new shareholders.

and middle-aged, and supplies labor inelastically. The efficiency of a unit of young labor is 2/3 the efficiency of a unit of middle-aged labor. The effective labor supply in pyramid Δ_1 or Δ_2 is thus

$$L_1 = 2/3N + n, \qquad L_2 = 2/3n + N \tag{8}$$

There is a single (representative, infinitely-lived) firm which uses capital and labor to produce the single output with the production function $F(K, L) = AK^a L^{1-a}$. At the beginning of period t the firm has K_t units of capital, inherited from period $t-1$. It hires L_t units of (effective) labor paid at the wage rate w_t, and after producing $F(K_t, L_t)$ units of output is left with $(1-\mu)K_t$ units of capital, where μ, with $0 \leq \mu \leq 1$, is the depreciation rate. The firm then decides to spend I_t on investment, where investment is subject to convex adjustment costs[9]

$$K_{t+1} = (1-\mu)K_t + I_t - \gamma(K_{t+1} - K_t)^2 \tag{9}$$

with $\gamma \geq 0$. The cost of replacing the depreciated units μK_t of capital is equal to μK_t, but if the firm wants to change its capital stock then an adjustment cost, convex in the change $|K_{t+1} - K_t|$, has to be incurred. If $\gamma = 0$ there is no adjustment cost and the model is the standard Diamond model.

After paying for wages and investment the firm distributes the rest of its output as dividend

$$D_t = F(K_t, L_t) - w_t L_t - I_t \tag{10}$$

The stock market opens and agents buy and sell shares of the firm at price q_t^e. For simplicity we assume that the bond is not used,[10] and define the rate of interest as the rate of return on equity

$$1 + r_t = \frac{D_{t+1} + q_{t+1}^e}{q_t^e} \tag{11}$$

Let (w_i, q_i^e, r_i) denote the wage, equity price and interest rate in pyramid Δ_i, $i = 1, 2$; similarly let $(c_i, z_{e,i}) = (c_i^y, c_i^m, c_i^r, z_{e,i}^y z_{e,i}^m)$ denote the consumption stream and equity holdings of an agent who is young in pyramid Δ_i, and let (K_i, I_i, D_i) denote the capital inherited by the firm, the investment

[9] We introduce a cost to modifying the level of capital to capture the fact that altering firm size by introducing new plant or introducing more capital intensive technology involves a cost over and above the cost of the materials involved, while the maintenance of depreciated capital involves no additional cost. We make the cost symmetric in increases or decreases of capital since it is typically costly to uninstall used capital which is not worth maintaining. Equation (9) is different from the equation of evolution of capital $K_{t+1} = G(K_t, I_t)$ introduced by Basu (1987) and adopted by Abel (2003) where G is a Cobb–Douglas function. This latter equation expresses decreasing returns to investment, but does not necessarily involve a cost for changing the level of capital.

[10] Introducing borrowing and lending on the bond only induces indeterminacy in portfolios and does not change the market value (equity plus debt of the firm).

undertaken to form the capital next period and the dividend distributed in pyramid Δ_i, $i = 1, 2$. $(c_i, z_{e,i})$ maximizes the utility function (1) subject to the sequence of budget constraints

$$\begin{aligned} c_i^y &= 2/3 w_i - q_i^e z_{e,i}^y \\ c_i^m &= w_{i+1} + (D_{i+1} + q_{i+1}^e) z_{e,i}^y - q_{i+1}^e z_{e,i}^m \\ c_i^r &= (D_i + q_i^e) z_{e,i}^m \end{aligned} \tag{12}$$

where $i+1$ is taken modulo 2 (1+1=2, 2+1=1). Note that these sequential budget constraints are equivalent to the single lifetime budget constraint

$$c_i^y + q_i c_i^m + q_i q_{i+1} c_i^r = 2/3\, w_i + q_i w_{i+1}$$

with present-value prices $q_i = 1/(1 + r_i)$ which can be taken as equilibrating variables.

The firm is assumed to maximize its market value — the present value of its dividends — with perfect foresight of future prices. Thus at each date t the choice of labor L_t must maximize $F(K_t, L_t) - w_t L_t$ given K_t, and the choice of capital K_{t+1} must maximize

$$-I_t(K_{t+1}, K_t) + \frac{1}{1 + r_t}(F(K_{t+1}, L_{t+1}) - w_{t+1} L_{t+1} - I(K_{t+2}, K_{t+1}))$$

given K_t, L_{t+1}, and K_{t+2}, where $I_t(K_{t+1}, K_t)$ is given by (9). This leads to the first-order conditions which define the optimal production plan of the firm in the stationary equilibrium: for $i = 1, 2$

$$\begin{aligned} F'_L(K_i, L_i) &= w_i \\ F'_K(K_{i+1}, L_{i+1}) &= r_i + \beta + 2\gamma(2 + r_i)(K_{i+1} - K_i) \end{aligned}$$

where we use the fact that $K_{i+2} = K_i$, and where L_1 and L_2 are given by (8), so that the labor market clears. The market clearing conditions for the consumption good market are

$$N c_1^y + n c_2^m + N c_1^r + I_1 = F(K_1, L_1)$$

$$n c_2^y + N c_1^m + n c_2^r + I_2 = F(K_2, L_2)$$

where $I_i = I(K_{i+1}, K_i)$. The simplest approach is to find the equilibrium $(\bar{c}_i, \bar{K}_i, \bar{r}_i, i = 1, 2)$, with the interest rates, or equivalently the present-value prices $(\bar{q}_1, \bar{q}_2)$, as equilibrating variables, and to deduce the financial variables $(\bar{D}_i, \bar{q}_i^e, \bar{z}_i)$ using (10), (11) and (12). As in the exchange economy the equity price is the present value of the dividends, which are now endogenous and vary between pyramids Δ_1 and Δ_2

$$\begin{aligned} \bar{q}_i^e &= \bar{D}_{i+1}\bar{q}_i(1+\bar{q}_1\bar{q}_2+(\bar{q}_1\bar{q}_2)^2+\ldots)+\bar{D}_i\bar{q}_1\bar{q}_2(1+\bar{q}_1\bar{q}_2+(\bar{q}_1\bar{q}_2)^2+\ldots) \\ &= \frac{\bar{D}_{i+1}\bar{q}_i+\bar{D}_i\bar{q}_1\bar{q}_2}{1-\bar{q}_1\bar{q}_2} \end{aligned} \tag{13}$$

By varying the adjustment cost parameter γ in the above model we can now compare the equilibrium outcomes of a family of models of the stock market (see Table 3), starting with the Diamond equilibrium $\gamma = 0$, and ending with $\gamma = 0.1$, for which the equilibrium is close to that of the simple exchange economy analyzed above, which is shown in the last two columns. In this family of models the coefficient of relative risk aversion is fixed at $\alpha = 4$, the demography parameters at $(N, n) = (79, 52)$, and the production parameters A, a, μ are chosen so that the depreciation parameter is $\mu = 0.5$ (depreciation of the order of 3% per year), and so that the Diamond equilibrium generates wages, dividends and output close to that of the exchange economy: this leads to the choice $A = 4.2$, $a = 0.24$.

To compare the equilibria, let $s_t^y = q_t^e z_{e,t}^y$ (resp. $s_t^m = q_t^e z_{e,t-1}^m$) denote the saving of the representative young (resp. middle-aged) agent trading at date t. Since the total demand for equity must be equal to the one unit that exists, in equilibrium the total savings of the active agents of the economy must be equal to the price of the equity, which itself is equal to the present value of the dividends. The steady-state equilibrium on the savings market in the two pyramids can then be written as

$$Ns^y(\bar{r}_1,\bar{r}_2)+ns^y(\bar{r}_1,\bar{r}_2) = q^e(\bar{r}_1,\bar{r}_2) = \frac{\bar{D}_2\bar{q}_1+\bar{D}_1\bar{q}_1\bar{q}_2}{1-\bar{q}_1\bar{q}_2} \tag{14}$$

$$ns^y(\bar{r}_2,\bar{r}_1)+Ns^y(\bar{r}_2,\bar{r}_1) = q^e(\bar{r}_2,\bar{r}_1) = \frac{\bar{D}_1\bar{q}_2+\bar{D}_2\bar{q}_1\bar{q}_2}{1-\bar{q}_1\bar{q}_2} \tag{15}$$

In pyramid Δ_1 (equation (14)), the young and middle-aged agents receive the current interest rate r_1 on their savings; r_2 affects the young because it is the rate of return that they will obtain on their future savings in middle age, and r_2 affects the current middle-aged because it is the rate of return that they have obtained on the (possibly negative) savings that they have done when young. The same holds for equation (15) with the role of r_1 and r_2 reversed.

In the exchange equilibrium with fixed land, greater savings can only be accommodated by adjustments in the interest rate. In the Diamond model, new investment can instead be channeled into new capital, reducing the variation in interest rates, as seen in Table 3. Furthermore, since the investment appears as capital one generation later, the large middle-aged cohort will earn lower

Table 3: Comparing Production and Exchange Economy Equilibrium								
	Diamond equilibrium $\gamma = 0$		*Adjustment cost equilibrium* $\gamma = 0.01$		$\gamma = 0.1$		*Exchange equilibrium*	
	Δ_1	Δ_2	Δ_1	Δ_2	Δ_1	Δ_2	Δ_1	Δ_2
K	74	53	76	65	86	84		
Y	405	397	406	417	419	444	376	403
I	16	48	28	44	42	44		
D	81	47	69	55	59	62	62	62
w	2.9	2.7	3	2.8	3	3	3	3
r^{an}	4.3%	3%	5.4%	1.3%	6.1%	0%	6.5%	-0.3%
q^e	53	74	52	94	55	117	52	120
PD	13	31	15	34	18	37	17	39
$\frac{q_2^e}{q_1^e}$		1.4		1.8		2.1		2.3
c	1.84 1.9 1.86	1.9 1.86 1.93	1.79 1.96 1.76	2.12 1.90 2.08	1.79 2.02 1.69	2.37 1.98 2.24	1.77 2.03 1.69	2.38 1.98 2.28

wages since it will work with the smaller capital stock bequeathed by the previous small generation. This reduces their middle-aged savings, and we see that the variation in equity values falls from 130% in the exchange economy to 40% in the pure Diamond model.

On the other hand, the dividends D_1 and D_2 differ in a way that reinforces the effect of the difference in rates of return on PD ratios: a lower r_2 induces a higher investment I_2: the savings of the large middle-aged cohort result in high investment for building the capital stock of the following period. The high capital stock of pyramid Δ_1 leads to a high dividend D_1, both because the economy is productive and because I_1 is low. The ratio PD of equity price to dividend is thus affected even more by demographics in the Diamond model than it is in the land model.

Introducing a convex adjustment cost tends to reduce the difference between K_1 and K_2 and to limit investment to the replacement of the depreciated capital. The dividends are then almost equal in the two pyramids and the rates of returns must vary more widely, as in the exchange economy, to establish equilibrium.

Finally, the less variable the rate of return, the less marked the favored cohort effect. The large cohort is young and retired in states where the effective labor supply is lower: in the Diamond equilibrium there is more capital in these states and the output is the same as in the states where there is a large productive middle-aged cohort. There is still an adverse effect of numbers, but it is much less marked than in the equilibria where the capital is constant and the output varies.

4. Equilibrium with Shorter Time Periods

An objection commonly presented to the idea that the increase in equity prices during the 90's was partly due to the saving behavior of the Baby Boomers reaching middle age is that the interest rates in the 90's were not historically low (see Poterba (2001)). The argument is that if the increase in prices was due to a higher than usual propensity to save due to the presence of a large generation in its saving years, then this high propensity to save should have forced interest rates down. The model which we have studied so far (with 3-period lived agents) supports this argument, since the equity price alternates between high values (when the large cohort is middle-aged) and low values (when the small cohort is middle-aged), with the result that the rate of return — and hence the interest rate — alternates between low and high values. High equity prices coincide with low interest rates and conversely.

However the joint dynamics of interest rates and equity prices in a model with shorter time periods is, as we shall now show, more subtle. We study how security prices behave when the three active 20-year periods of an agent are each divided into 5 periods of 4 years, i.e., the time period is now 4 years, and the economic life of an agent lasts for 15 periods. Adopting a 4-year time period as the basic unit of time keeps the calculation manageable and suffices to show how a more detailed statement of the changing sizes of age cohorts over time carves itself precisely into a cyclical pattern for equity prices and interest rates, with a phase shift in the path of interest rates relative to equity prices.

We continue to assume that the population cycle repeats itself every 40 years, or 10 periods, i.e., the number of agents entering the economy in period $t+10$ is the same as the number of agents entering in period t. Since the age composition in period t is the same as in period $t+10$, there are 10 different pyramids $\Delta_1, \dots, \Delta_{10}$ which keep repeating themselves. For $i = 1, \dots, 10$, let n_i denote the size of the cohort beginning its economic life in pyramid Δ_i. The sequence $n_1, \dots, n_{10}$ can better approximate the progressive increase and decrease in livebirths shown in Figure 1. The choice of $n_1, \dots, n_{10}$ which approximates the Great Depression and the Baby Boom generations is shown in

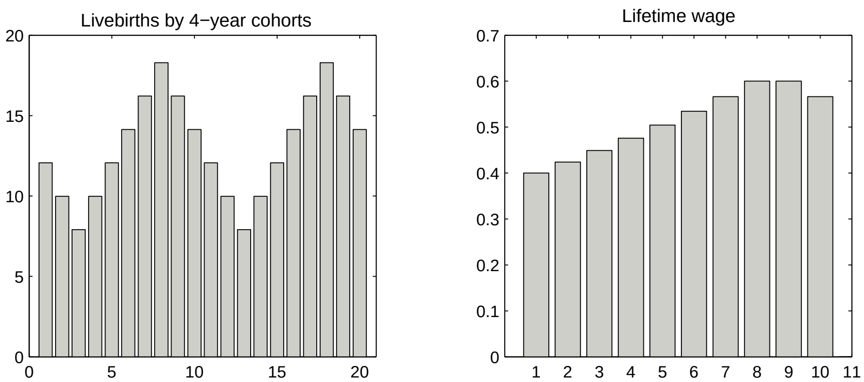

Figure 3: *Cohort sizes and wage profile for 4-year intervals*

Figure 3: during the first 5 periods (20 years) the small cohort enters, with $n_1+n_2+n_3+n_4+n_5 = 52$, and in the next five periods the large cohort enters, with $n_6 + n_7 + n_8 + n_9 + n_{10} = 79$. The cycle then repeats itself.

To keep the structure of the economy comparable and consistent with the previous calibration, we assume that the wage schedule increases by regular percentage increases from $w^1 = 2/5$ (wage of the 20-23 cohort) to $w^8 = 1.5w^1$ (wage of the 48-51 cohort), stays the same in the 9^{th} period of life ($w^9 = w^8$) and decreases to $w^{10} = w^7$ in the last period of work. The forty-year work phase ends at age 60 (the agent entered the workplace at age 20) and the agent receives no wage income during the last five periods (20 years) of his life. Figure 3 shows the representative agent's wage income during the work phase $(w^1, \dots, w^{10})$.

Agents trade the equity contract which pays a constant dividend D each period, where D is 19% of the average total wage income over the 10 pyramids ($D = 12.74$). Agents can also borrow and lend at the riskless one-period interest rate r_t, and since the bond and the equity are perfect substitutes, the sequence $(q_t^b)_{t\geq 1}$ with $q_t^b = 1/(1+r_t)$ and the sequence of equity prices $(q_t^e)_{t\geq 1}$ must satisfy

$$\frac{D + q_{t+1}^e}{q_t^e} = 1 + r_t = \frac{1}{q_t^b}$$

As in the three-period case there is a stationary equilibrium: let $c_i = (c_i^1, \dots, c_i^{15})$ denote the equilibrium consumption stream, during the fifteen periods of life, of the representative agent of a cohort entering the economy in pyramid i, and let (q_i^b, q_i^e) denote the equilibrium prices of the securities in pyramid Δ_i. Using k for the index of age (an agent in his k^{th} period of economic activity is called an agent of "age" k, i.e., an agent of "age" 2 is between 24 and 27 years old), the consumption stream c_i must maximize $\sum_{k=1}^{15} \delta^{k-1} u(c_i^k)$ (with $u(c) = c^{1-\alpha}/(1-\alpha)$) subject to the budget constraint

$$c_i^1 - w^1 + q_i(c_i^2 - w^2) + \ldots + q_i q_{i+1} \ldots q_{i+14}(c_i^{15} - w^{15}) = 0 \qquad (16)$$

where to simplify $q_i^b = q_i$, and all indices are taken modulo 10. Let Δ_i^k denote the number of agents of age k in pyramid Δ_i. Since these agents entered the economy $k-1$ periods earlier, their number is n_{i-k+1}, where again the indices must be taken modulo 10. The equilibrium prices must be such that in each pyramid Δ_i markets clear, i.e.,

$$\sum_{i=1}^{10} \Delta_i^k (c_{i-k+1}^k - w^k) = D, \qquad k = 1, \ldots, 15$$

The equilibrium interest rates and equity prices for the case $\alpha = 4$ are shown in the upper graphs of Figure 4 and are shown as functions of the index i of the population pyramid Δ_i as it runs through 2 cycles ($i = 1, \ldots, 20$). The third graph shows a convenient index of the age composition of pyramid Δ_i reflecting the number of middle-aged relative to young agents: for pyramid Δ_i, we take the ratio MY_i to be defined by

$$MY_i = \frac{\Delta_i^6 + \ldots + \Delta_i^{10}}{\Delta_i^1 + \ldots + \Delta_i^5}, \quad i = 1, \ldots, 10$$

i.e., the ratio of the number of agents aged 40–59 to the number of agents aged 20–39.

It is remarkable that the price of equity is exactly in phase with a simple summary statistic of the age pyramid — the medium-young ratio — despite the fact that agents at the different phases of their youth, middle-age and retirement periods, have different levels of income and different propensities to save.

On the other hand, as shown by Figure 4, in equilibrium the short-term interest rate is out of phase with the cycle of equity prices and the MY ratio. The interest rate, which coincides with the rate of return on equity, is the sum of the dividend yield and the capital gain yield. The dividend yield is inversely proportional to the equity price and thus comoves negatively with it. However the capital gain yield depends on the rate of change of the equity price and, because of the cyclical form of the birth rate, this rate of change is maximal (minimal) in the middle of the ascending (descending) phase of the equity prices: because of these capital gain terms, the turning points in the interest rate occur in the middle of the ascending and descending phases of the equity prices. Short-term interest rates begin to increase before equity prices have bottomed out, and begin to decrease before equity prices have peaked. This synchronous behavior of equity prices and nonsynchronous behavior of rates of return with the MY ratio may help to explain one of the empirical findings reported by Poterba (2001): while certain summary demographic

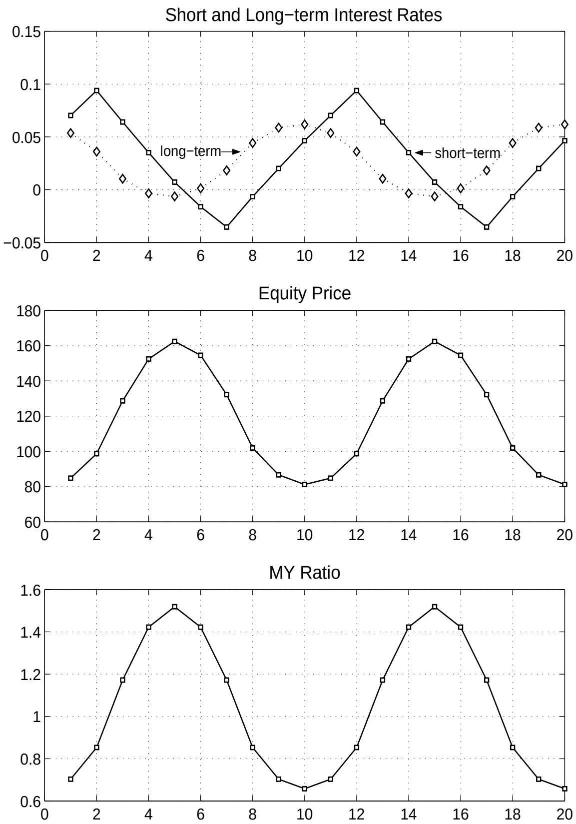

Figure 4: *Short and long-term interest rates, equity price and MY ratio on equilibrium trajectory.*

statistics (similar to the MY ratio) correlate relatively well with the level of equity prices, they have essentially no significant correlation with rates of return on equity.

Figure 4 also shows the behavior of the long-term (real) interest rate, defined as the interest rate on the 20-year (5 period) bond, namely the geometric mean of the short-term (real) rates of return 5 periods into the future. The long-term interest rate is in (reverse) phase with the equity prices and the MY ratio. Thus the result of the model with 3-period-lived agents—low interest rates associated with high equity prices and conversely—holds true for the long-term real interest rate, which unfortunately is difficult to obtain from the data. The model also implies a changing term-structure of (real) interest rates with the long-term rate below (above) the short rate on the ascending (descending) phase of equity prices.

5. Introducing Business Cycle Shocks

If the real S&P 500 is used as an approximate proxy for the level of stock prices, then the trough to peak variations observed over the past 50 years are more than twice those predicted by the simple demographic model of the previous sections (see Figure 6). Demography cannot explain everything, nor should it. The long trends in equity prices over this period coincided not only with demographic trends, but also with runs of luck: in the seventies and beginning eighties there were mainly negative shocks — oil shortages, bursts of high inflation followed by restrictive monetary policy, leading to unemployment and low productivity — while the period of the nineties was characterized by aggregate shocks which were mainly positive — low inflation and energy prices, rapid technological progress resulting in low employment and high productivity. We thus add to the demographic model of the previous section the possibility of random shocks to income to study the combined effect for asset prices of demographic and business cycle fluctuations.

Once uncertainty is introduced, risky equity and the riskless bond cease to be perfect substitutes. The equity must earn a risk premium relative to the bond to induce agents to hold it, and the model permits us to study the effect of the changing demographic structure on the risk premium.

The certainty model of the previous section showed that the qualitative results of the simplest model with three-period lived agents, exogenous dividends and no bequest are robust to the introduction of more realistic features. We thus revert to this simplest model, adding the possibility of random wages and dividends, to study the combined effect for asset prices of demographic and business cycle fluctuations.

Risk Structure. We model the risk structure of the economy by assuming that the wages and dividends on equity are subject to shocks. We use a highly simplified structure, assuming that at each date there are four possible states of nature (shocks), s_1=(high wage, high dividend), s_2 = (high wage, low dividend), s_3 = (low wage, high dividend), s_4 = (low wage, low dividend). Given the nature of the risks and the very extended length of time represented by a period (20 years), we have chosen not to a invoke a Markov structure, but rather to assume that the shocks are i.i.d. To reflect the fact that aggregate income and dividends are positively correlated we assume that s_1 and s_4 are more likely (probability 0.4 each), than s_2 and s_3 (probability 0.1 each). This gives rise to a correlation between dividends and wages of 0.6.

Figure 2 shows that the maximum variability of the real annual wage income of the 45–54 cohort is about 4%: in the recession of 1990-91 the mean wage of this cohort went from 65 to 60 (thousands of 1999 dollars), a variability of $(2.5/62.5) = 0.04$; the variability of the wage income of the 25–34 cohort is somewhat lower. To take into account that some periods, like 1970–1983, experienced a sequence of negative shocks, in the calibration we increase the coefficient of variation (CV) of the wage income of the middle-aged to 20% and that of the young to 15%. Since the fluctuations of real (generalized) dividends are of the same order as those of wages, we take a CV of 19% for dividends. This leads to a coefficient of variation of about 16% for aggregate income. In short, we assume four possible shocks with probabilities $(0.4, 0.1, 0.1, 0.4)$, and wage income and dividends across the four states given by $w^y = (2.3, 2.3, 1.7, 1.7)$, $w^m = (3.6, 3.6, 2.4, 2.4)$, and $D = (74, 50, 74, 50)$.

Equilibrium. Since the financial markets are incomplete — each date-event is followed by four possible income/dividend shocks, and agents can trade only two securities (equity and the bond) — the equilibrium cannot be solved (as in the previous section) in terms of the consumption variables with a single present-value budget constraint for each agent. We need to explicitly introduce the asset trades, portfolio optimization, and market clearing asset prices. Let $z_t = (z_t^y, z_{t+1}^m) = (z_{b\,t}^y, z_{e\,t}^y, z_{b\,t+1}^m, z_{e\,t+1}^m)$ denote the lifetime portfolio of an agent born at date t, namely the holdings of the bond and equity $z_t^y = (z_{b\,t}^y, z_{e\,t}^y)$ in youth, and in middle age $z_{t+1}^m = (z_{b\,t+1}^m, z_{e\,t+1}^m)$. Let $c_t = (c_t^y, c_{t+1}^m, c_{t+2}^r)$ denote the agent's lifetime consumption in youth, middle age and retirement. Both z_t and c_t are stochastic, depending on the past history of shocks and on the shocks to wages and dividends during the agent's lifetime. The agent's consumption and portfolio holdings must satisfy his budget constraints in each state, given by

$$\begin{aligned} c_t^y &= w_t^y - q_t z_t^y \\ c_{t+1}^m &= w_{t+1}^m + V_{t+1} z_t^y - q_{t+1} z_{t+1}^m \\ c_{t+2}^r &= V_{t+2} z_{t+1}^m \end{aligned} \tag{17}$$

where $q_t = (q_t^b, q_t^e)$ denotes the vector of bond and equity prices at date t and $V_{t+1} = [1, D_{t+1} + q_{t+1}^e]$ denotes the payoff of the bond and equity at date $t+1$. An equilibrium on the bond and equity markets is then a sequence $(z_t, q_t)_{t \geq 0}$ of portfolios and prices such that the representative agent born at date t maximizes lifetime expected utility (1), subject to the budget equations (17), and such that the bond and equity markets clear at each date $t \geq 0$ for each state

$$\left\{ \begin{aligned} N z_{b\,t}^y + n z_{b\,t}^m &= 0 \\ N z_{e\,t}^y + n z_{e\,t}^m &= 1 \end{aligned} \right. \quad t \text{ odd} \qquad \left\{ \begin{aligned} n z_{b\,t}^y + N z_{b\,t}^m &= 0 \\ n z_{e\,t}^y + N z_{e\,t}^m &= 1 \end{aligned} \right. \quad t \text{ even}$$

Our objective is to study how the alternating cohort sizes of young and medium influence the equilibrium on the financial markets. In view of alternating cohort structure and the assumption that the wage income and dividends are i.i.d., it is natural to look for a stationary equilibrium of the economy: in the appendix we define such an equilibrium and explain how it can be calculated.

Calibration Results. To study the properties of the equilibrium trajectories, we consider an economy with cohort sizes $(N, n) = (79, 52)$, and risk aversion parameter $\alpha = 4$. The characteristics of the prices on equilibrium trajectories are shown in Table 4, while the characteristics of the consumption and portfolio strategies are shown in Table 5. A less detailed description is given in the Appendix (Table 10) for an economy with a smaller variation in cohort sizes $(N, n) = (79, 69)$ — calibrated to the sizes of the cohorts born over the periods 1945–1964 and 1965–1984 — for three different parameters of risk aversion ($\alpha = 2, 4, 6$).

As explained in the Appendix, in order to find a Markov equilibrium, an endogenous state variable—the portfolio income that the middle-aged bring over from their youth—needs to be added to the exogenous state (k, s) where k is the population pyramid state ($k = 1, 2$)), depending on whether the period is even or odd, and s is one of the four income/dividend shocks. Along every path, each pyramid-shock state (k, s) will occur infinitely often: in Table 4 the standard deviation of prices (the numbers between parentheses) about their means (the numbers without parentheses) are given for each pyramid-shock state (k, s), averaged over all paths. An interesting feature of the equilibrium trajectories is that the standard deviations are very small, meaning that prices essentially depend only on the exogenous state (k, s). Thus the average values of the equity price (q^e) and of the interest rate (r^{an}) in the different states (k, s) give a rather precise description of the prices on the equilibrium trajectories. Table 4 also shows the price-dividend ratio for each state, which we have divided by two to make it comparable with the more familiar price-earnings ratio, commonly used for evaluating the level of prices on the stock market.

A new variable which enters when uncertainty is introduced is the equity premium — namely the amount by which the expected return on equity exceeds the return on bonds. The (annualized) equity premium is calculated on a trajectory as

$$rp^{an} = \text{average}(r_e^{an} - r^{an})$$

where

$$r_{e,t}^{an} = \left(\frac{q_{t+1}^e + D_{t+1}}{q_t^e}\right)^{\frac{1}{20}} - 1$$

is the (annualized) *ex post* rate of return on equity at date t. The (*ex ante*) equity premium is thus defined as the mean *ex post* equity premium and is given in Table 4. The high variance of the *ex post* equity premium, even for a given pyramid-shock state (k, s), is natural since the realized equity premium is large when a favorable state follows state s, and is small when an unfavorable state follows state s.

As is well known the *ex ante* risk premia predicted by standard rational expectations models are significantly smaller than those obtained ex post from the data — at least for the US. Several approaches have been proposed to obtain models with higher risk premia. One is to take into account the fact that agents face individual risks which make their consumption significantly more variable than aggregate consumption. We cannot take into account individual risks without unduly complicating the model and, to compensate, we have been generous in the calibration with the aggregate risk. Other solutions involve entering as constraints some observed deviations of the behavior of agents from that predicted by the model. One prediction of the model is that agents make use of all the available instruments to redistribute income and share risks. However, even though the proportion of US households investing in the stock market has increased significantly over the last fifty years,[11] it still remains less than 50%. To take this into account we solve for the equilibrium under the restriction that 50% of the agents in any cohort do not trade on the equity market and restrict their financial transactions to the bond market (Case B in Tables 4–5).

An alternative approach recently proposed by Constantinides–Donaldson–Mehra (2002) is to impose a borrowing constraint on the young: as shown in Table 5 without such a constraint the young typically borrow and use much of the proceeds to invest in the equity market, to take advantage of the equity premium. As Constantinides–Donaldson–Mehra argue, this is not especially realistic. While young agents can and do borrow significantly to buy houses (which serve as collateral) they do not typically borrow to invest on the stock market. The simplest way of preventing the young from taking leveraged positions on the equity market is to impose a borrowing constraint. Such a constraint on the young decreases the demand for the risky security and tends to increase the risk premium. However in the simple model that we study, preventing every young agent from borrowing closes the bond market — and the interest rate is not longer well defined. To avoid this, while studying the effect on prices of reducing the demand for equity by the young, we solve for the equilibrium assuming that 90% of the young have borrowing constraints and the remaining 10% are unconstrained (Case C in Tables 4–5).

In addition to the intrinsic interest, and potentially greater realism of these two cases with restricted participation, they are also useful for checking the robustness of the results predicted by the standard model (Case A in Tables 4–5) to different assumptions on market participation.

Cyclical Fluctuations of Security Prices. The general principle which underlies the certainty model — namely that aggregate demand for savings is high in even periods when there is a large middle-aged and a small young cohort, while it is low in odd periods where there is a small middle-

[11] Vissing–Jorgenson (1999) estimates the participation rate in the stock market at around 6% in the early 1950's and around 40 % in 1995.

Table 4: Prices in Markov Equilibrium

$N = 79,\ n = 52,\ \alpha = 4,\ \ w^y = (2.3, 2.3, 1.7, 1.7),\ w^m = (3.6, 3.6, 2.4, 2.4),\ D = (74, 50, 74, 50)$

	pyramid 1: MY ratio=0.66				**pyramid 2: MY ratio=1.5**			
	q^e	$PD/2$	r^{an}	rp^{an}	q^e	$PD/2$	r^{an}	rp^{an}
Case A: Standard Equilibrium								
s_1	103 (3)	14 (.4)	2.1% (.05)	1.1 % (2.8)	292 (27)	39.5 (3.6)	-5% (.19)	.9% (1.65)
s_2	97.5 (2.5)	19.5 (.5)	2.5% (.04)	1.13% (2.8)	250 (22)	50 (4)	-4.3% (.17)	.87% (1.65)
s_3	37 (.9)	5 (.13)	7.9% (.05)	1.14% (3.1)	80 (7)	11 (1)	1.2% (.2)	.96% (1.69)
s_4	34 (.6)	7 (.13)	8.6% (.03)	1.15% (3.1)	61 (5)	12 (1)	2.6% (.2)	.93% (1.74)
Average	68 (34)	10.7 (5)	5.4% (3.1)	1.13% (2.95)	175 (112)	27 (15)	-1.3% (3.7)	.91% (1.69)
	ratio of av. prices: 2.6, *peak / trough*: 8.5, *trough / peak* 0.12							
Case B: 50% Participation in Equity Market								
s_1	117 (1.7)	15.8 (.23)	.75% (.1)	2.18% (2.5)	297 (35)	40 (4.7)	-5.1% (.55)	1.25% (1.6)
s_2	110 (1)	22 (.2)	1.1% (.06)	1.96% (2.57)	259 (28)	52 (5.5)	-4.5% (.5)	1.16% (1.64)
s_3	43 (0.7)	5.8 (.09)	6.2% (.13)	2.44% (2.73)	93 (10)	12.5 (1.3)	0.3% (.5)	1.4% (1.74)
s_4	40 (.8)	8.1 (.15)	6.7% (.2)	2.5% (2.9)	74.5 (7)	14.9 (1.35)	1.4% (.5)	1.63% (1.8)
Average	79 (37)	12.3 (5)	3.7% (2.9)	2.3% (2.7)	186 (109)	28.7 (15)	-2% (3.2)	1.4% (1.7)
	ratio of av. prices: 2.4, *peak / trough*: 7.3, *trough / peak* 0.14							
Case C: 90% Young with Borrowing Constraints								
s_1	101 (.4)	13.7 (.06)	2.75% (.03)	1.4% (1.76)	259 (10)	35 (1.3)	-4.5% (.2)	1.12% (1.42)
s_2	101 (.35)	20 (.07)	2.8% (.03)	1.31% (1.76)	240 (8)	48 (1.7)	-4.2% (.2)	1.12% (1.43)
s_3	42.7 (.15)	5.8 (.02)	7.5% (.04)	1.35% (1.85)	125 (5)	16.9 (.7)	-1% (.2)	1.06% (1.48)
s_4	42 (.1)	8.4 (.06)	7.6% (.03)	1.46% (1.88)	110 (3.6)	22 (.7)	-.35% (.16)	1.13% (1.49)
Average	72 (30)	11.4 (4)	5.2% (2.4)	1.41% (1.8)	184 (72)	29.2 (9.2)	-2.5% (2)	1.12% (1.45)
	ratio of av. prices: 2.6, *peak / trough*: 6.1, *trough / peak* .16							

Table 5: Consumption and Portfolio Strategies in Markov Equilibrium								
Case A: Standard Equilibrium								
	Pyramid 1 ($r = 5.4,\ q^e = 68$)				**Pyramid 2** ($r = -1.3,\ q^e = 175$)			
	number	consumption	equity holding	bond holding	number	consumption	equity holding	bond holding
young	N	1.73 (.21)	0.70 (0.33)	-0.44 (0.25)	n	2.41 (0.49)	1.38 (0.94)	-1.78 (1.13)
middle-aged	n	2.02 (0.4)	0.24 (0.14)	0.66 (0.38)	N	2 (0.33)	1.31 (0.8)	1.17 (0.74)
retired	N	1.69 (0.33)	0	0	n	2.31 (0.4)	0	0
Case B: 50% Participation in Equity Market								
	Pyramid 1 ($r = 3.7,\ q^e = 78.5$)				**Pyramid 2** ($r = -2,\ q^e = 186$)			
constrained young	$.5N$	1.65 (0.21)	0	.35 (0.09)	$0.5n$	2.35 (0.4)	0	-.35 (.14)
unconstrained young	$.5N$	1.82 (.24)	1.50 (0.7)	-1.32 (.6)	$0.5n$	2.44 (0.4)	3.16 (1.6)	-3.6 (1.8)
constrained middle-aged	$.5n$	1.79 (.15)	0	.98 (0.46)	$.5N$	1.69 (0.17)	0	2.06 (0.78)
unconstrained middle-aged	$.5n$	2.24 (0.67)	.71 (0.44)	.49 (0.32)	$.5N$	2.33 (0.78)	2.6 (1.7)	.53 (.49)
constrained retired	$.5N$	1.32 (0.33)	0	0	$.5n$	1.8 (0.11)	0	0
unconstrained retired	$.5N$	2.08 (.7)	0	0	$.5n$	2.82 (1.04)	0	0
Case C: 90% Young with Borrowing Constraints								
	Pyramid 1 ($r = 5.2,\ q^e = 72$)				**Pyramid 2** ($r = -2.5,\ q^e = 184$)			
constrained young	$.9N$	1.65 (.17)	0.22 (.00)	0	$.9n$	2 (.3)	0	0
unconstrained young	$.1N$	1.7 (.19)	0.79 (.00)	-.61 (.00)	$.1n$	2.49 (.36)	1.59 (.00)	-2.12 (.00)
middle-aged constrained in youth	$.9n$	2.14 (.24)	.39 (.00)	.09 (.00)	$.9N$	1.9 (.15)	1.53 (.00)	.14 (.00)
middle-aged unconstrained in youth	$.1n$	2.18 (.6)	.31 (.00)	.07 (.00)	$.1N$	2.17 (.6)	1.59 (.00)	.14 (.00)
retired constrained in youth	$.9N$	1.66 (.44)	0	0	$.9n$	2.74 (.8)	0	0
retired unconstrained in youth	$.1N$	1.86 (.61)	0	0	$.1n$	2.73 (.9)	0	0

aged and a large young cohort, carries over to the economy with uncertainty. In an economy with both demographic and business cycle shocks, the stochastic sequence of equilibrium security prices (q_t^e, q_t^b) co-moves with the medium-young cohort ratio, being higher (lower) than average when the MY cohort ratio is high (low). Thus long-run fluctuations in demographic structure lead to long-run cyclical fluctuations in security prices over time. The order of magnitude of the demographic effect is indicated in Table 4 by the ratio of the average prices in the two pyramid states, and this is approximately the same as in the certainty model.

Note that the average interest rate is high in odd periods, in which equity prices are low and rising, and low in even periods in which the equity prices are high and falling, and it is precisely this simultaneous adjustment of interest rates and equity prices which prevents arbitrage opportunities from arising.

Since, for a given population structure, an increase in income increases the demand for savings, equity prices covary positively with aggregate income. Thus adding shocks to income opens the possibility of greater variations in equity prices: the greatest increase occurs when the economy moves from (Δ_1, s_4) to (Δ_2, s_1), namely from a period with a large young cohort and negative income shocks to a period with a large middle-aged cohort and favorable shocks. The ratio of these prices is given on Table 4 by the peak-to-trough ratio (and its inverse the trough to peak ratio), where we see that 6 or 7 is attained.

Equity Premium. The striking feature of the risk premium in the equilibria that we compute is that it is higher in pyramid Δ_1 than in pyramid Δ_2. At their initial endowment, the risk aversion of young agents is smaller than that of the middle-aged: they have the prospect of income in middle age, while the middle-aged have no income in retirement to help smooth the risk associated with buying a risky security. As a result the young hold a higher percentage of stock in their portfolio (actually borrowing to hold equity). One might have thought that the equilibrium risk premium would therefore be lower in pyramid Δ_1 where there are many young and few middle-aged. Indeed this is the standard prediction in the literature.

There are two reasons why we get the opposite conclusion. First, the risks are not the same. Agents investing in pyramid Δ_1 face a more risky — if more favorable — market than agents investing in pyramid Δ_2 because the return $D_{t+1} + Q_{t+1}$ depends more on the capital value term Q_{t+1} when the price/dividend ratio is expected to be high, and more on the dividend D_{t+1} when the price/dividend ratio is expected to be low. Dividends are less variable than capital values (in Table 4 the CV of equity prices is always more than 40% while the CV of dividends is 19%), so the return on equity is more variable for agents investing in odd periods and expecting high equity prices next period than for those investing in even periods and expecting low prices. This can be seen from the standard deviation of the risk premium in Table 4, which is essentially the same as the standard deviation of the rate of return, and is higher in pyramid Δ_1 than in pyramid Δ_2. The increase in risk from another dollar of equity is thus higher for the small generation of middle-aged than for the large generation of middle-aged.

Second, agents become more averse to additional risk as their consumption becomes riskier. The middle-aged are buyers of equity in every generation. Their risk aversion on the margin depends on how much risk they face in old age. The variability of consumption of the old agents in large generations is smaller than that of the old in small generations, precisely because their stock returns are less variable. Thus the middle-aged in the large generations may face less risk and be more risk tolerant than the middle-aged of the small generations. This is sure to be the case if the young are prevented from holding much stock, as they are in cases B and C.

As can be seen from Table 4, restricting the participation on the equity market to 50% of the agents (Case B) is the most effective way of increasing the risk premium because the risk of the equity is divided among a smaller number of agents,[12] Roughly speaking the agents who are trading on the equity market (the unconstrained agents in Table 5) hold twice the amount of equity of their counterpart in Case A, and expose themselves to more than twice the volatility of consumption. As a result the equilibrium risk aversion is higher. Since the risk of equity is of the same order of magnitude, the risk premium is higher.

The last case where most young (90%) cannot borrow (Case C), is perhaps more realistic in terms of portfolio behavior — although the borrowing constraint is too extreme since it is not uncommon for a young agent to borrow to buy a house while at the same time investing a fraction of his wage income in equity in a retirement account — but it is less effective at increasing the risk premium than Case B. There are two reasons for this: the first is that the risk of equity decreases — because of the reduced participation of the young on the equity market, the variability of their income impinges less on the market, reducing the variability of equity prices. The second is that this reduced risk is shared among more agents than in Case B.

Favored Cohort Effect. As in the model of Section 2, the long-run cyclical fluctuations in the demographic structure imply that agents in small cohorts receive more favorable equilibrium lifetime consumption streams than agents in large cohorts. The lifetime equilibrium consumption

[12] This is consistent with the findings of Heaton and Lucas (1999) who explore — in an OLG model with two-period lived agents — the idea of using restricted participation as a way of increasing the equity premium: however in our model participation has a bigger impact on the premium.

Table 6: Lifetime Consumption and the Favored Cohort Effect in Markov Equilibrium

$N = 79,\ n = 52,\ \alpha = 4,\ w^y = (2.3, 2.3, 1.7, 1.7),\ w^m = (3.6, 3.6, 2.4, 2.4),\ D = (74, 50, 74, 50)$

Case A: Standard Equilibrium

	large cohort			*small cohort*		
	consumption	prices		consumption	prices	
young	17,300 (2,100)	68 (34)	5.4% (3.1)	24,100 (4,900)	175 (112)	-1.3% (3.7)
middle	20,000 (2,300)	175 (112)	-1.3% (3.7)	20,200 (4,000)	68 (34)	5.4% (3.1)
retired	16,900 (3,300)	68 (34)	$U = -1$	23,100 (4,000)	175 (112)	$U = -0.6$

Case B: 50% Participation in Equity Market

	large cohort				*small cohort*			
	consumption		prices		consumption		prices	
	constrained	unconstrained			constrained	unconstrained		
young	16,500 (2,100)	18,200 (2,400)	79 (37)	3.7% (2.9)	23,500 (4,000)	24,400 (4,000)	186 (109)	-2% (3.2)
middle-aged	16,900 (1,700)	23,300 (7,800)	186 (109)	-2% (3.2)	17,900 (1,500)	22,400 (6,700)	79 (37)	3.7% (2.9)
retired	13,200 (3,300)	20,800 (7,000)	79 (37)	$U^c = -1.5$ $U^u = -.95$	18,000 (1,100)	28,200 (10,400)	186 (109)	$U^c = -.69$ $U^u = -.56$

Case C: 90% Young with Borrowing Constraints

	large cohort				*small cohort*			
	consumption		prices		consumption		prices	
	constrained	unconstrained			constrained	unconstrained		
young	16,500 (1,700)	17,000 (1,900)	72 (30)	5.2% (2.4)	20,000 (3,000)	24,900 (3,600)	184 (72)	-2.5% (2)
middle-aged	19,000 (1500)	21,700 (6,000)	184 (72)	-2.5% (2)	21,400 (2,400)	21,800 (6,000)	72 (30)	5.2% (2.4)
retired	16,600 (4,400)	18,600 (6,100)	72 (30)	$U^c = -1.2$ $U^u = -1.1$	27,400 (8,000)	27,300 (9,000)	184 (72)	$U^c = -.65$ $U^u = -.51$

streams of agents born into the small (respectively, large) cohorts is shown in Table 6 — they have been multiplied by 10,000 to make the comparison of the consumption streams more intuitive. Even though all agents begin with the same average lifetime wage income (20,000, 30,000, 0), the average lifetime consumption stream of an agent born into a small cohort is significantly greater than that of an agent in a large cohort. This difference arises from the cyclical fluctuations in the security prices: the two columns to the right of the average consumption stream show the average prices (the equity price and the interest rate) that the corresponding agent faces during his lifetime and the last entry in the interest rate column gives the expected utility (averaged over the possible income shocks when young) of an agent born in a large or small cohort. In the constrained participation cases, U^c and U^u denote the utility of the constrained and unconstrained agents respectively.[13]

Case B in Table 6 shows the loss to their average lifetime consumption stream, incurred by agents who are assumed not to participate on the equity market — as usual the loss incurred by Boomers is greater than that for Xers. While there is a gain in terms of reduced variability of consumption, the loss to average consumption is substantial, especially in middle age and retirement. As a result, agents who for whatever reason — by ignorance or by fear — do not participate on the stock market, do so at considerable cost to their lifetime consumption and utility.

The cost of nonparticipation is less marked in Case C where agents face borrowing constraints in youth. Constrained Xers only lose when they are young because they cannot take advantage of the favorable terms for borrowing, while constrained Boomers lose throughout their life, since they cannot exploit the favorable terms for saving in youth, giving them less wealth in middle age and hence less consumption in both middle age and retirement.

Other authors, in particular Easterlin (1987), have pointed out that the Baby Boomers, being a large cohort, face more competition on the labor market and thus should be expected to receive lower wages than the small cohort which preceded them: this labor-market cohort effect, which has been somewhat controversial,[14] is absent from our model since we assume that agents have the same lifetime wage profile in both cohorts. Our model shows however that large cohorts face a second curse from the financial markets: by being so numerous they drive the terms of trade against themselves, favoring the small cohort on the other side of the market which follows or precedes them.

[13] It can be shown that the extent to which the small cohort is favored depends on the magnitude of the fluctuations in security prices: the greater the difference in the cohort sizes, the greater the degree of relative risk aversion, or the greater the variability of agents' endowment streams, the greater the fluctuations in security prices, and the greater the extent to which capital markets favor the small cohort.

[14] Welch (1979) found evidence that wages depend on cohort sizes for the period preceeding 1980: for the period after 1980, as Macunovich (2002) has shown, additional variables are needed to explain the movements in wages.

6. Comparing Calibration with Observations

The model studied in the previous sections predicts relationships between demographic variables and asset prices. In this section we analyze in a stylized way if the predictions of the model are consistent with data during the last century for the US. The key demographic hypothesis of the model is that the birthrate is cyclical, with a period of 40 years, which is a simplification of the observed birthrate in the US during the 20th century. As we saw in Section 4, leaving aside output shocks, the cyclical birthrate implies that the equilibrium prices and quantities can be expressed as a function of a simple statistic of the population pyramid—the medium-young (MY) ratio. The MY ratio (shown in Figure 5) is taken as the ratio of the size of the cohort 40–49 to the size of the cohort 20–29 for the US population.[15] Note that the use of the MY ratio as a summary statistic of the population pyramid is justified only in the context of an intertemporal equilibrium of an economy with a cyclical birth rate: the MY ratio indicates where in the pyramid cycle the economy is located, but does not imply that the young and the medium cohorts which serve to define the ratio are the only cohorts whose trade influences the equilibrium — all cohorts trade, and all influence the equilibrium outcome.[16] The very weak cyclical movement in the MY ratio until 1945 indicates that there was only a weak cyclical component in the birthrate (and immigration rate) at the end of the 19th and beginning of the 20th century: thus in the period 1910–1945 we should expect a less systematic relation between asset prices and the MY ratio, than during the period 1945–2002.

Equity Prices. Using the real Standard and Poors (S&P) index[17] expressed in dollars of 2000 as the index of equity prices (see Figure 6), consider in broad outline the joint behavior of the MY ratio and equity prices. Up to the late 1940's there were no significant variations in the MY ratio, and this corresponds roughly with the lack of systematic long-run movement in the S&P index around its trend over this period. To be sure there were ten-year fluctuations in the 30's and 40's, and the ten-year boom of the Roaring 20's, which we think of as shorter run business cycle fluctuations. Starting in the late 40's and continuing all through the 50's and early 60's the ratio of middle-aged to young agents was rising: the middle-aged agents were born at the turn of the

[15] The MY ratio obtained by using the size of the cohort 40–59 to the size of the cohort 20–39 is approximately the same as the ratio we have chosen, with a phase shift (advance) of 4 years. The ratio chosen is slightly better related to the asset price data, but both indices give very similar results. The cohort data is derived from Series A 33-35 in *Historical Statistics of the US* and Bureau of the Census data.

[16] Empirical studies which have studied the influence of demography on asset markets without an equilibrium model have either considered several summary statistics of the population pyramid (Poterba (2001), Ang and Maddaloni (2003)) or the influence of all age cohorts on the financial variables (Poterba (2001), Macunovich (2002)).

[17] We are grateful to Robert Shiller for making the data set for the Standard and Poors index available.

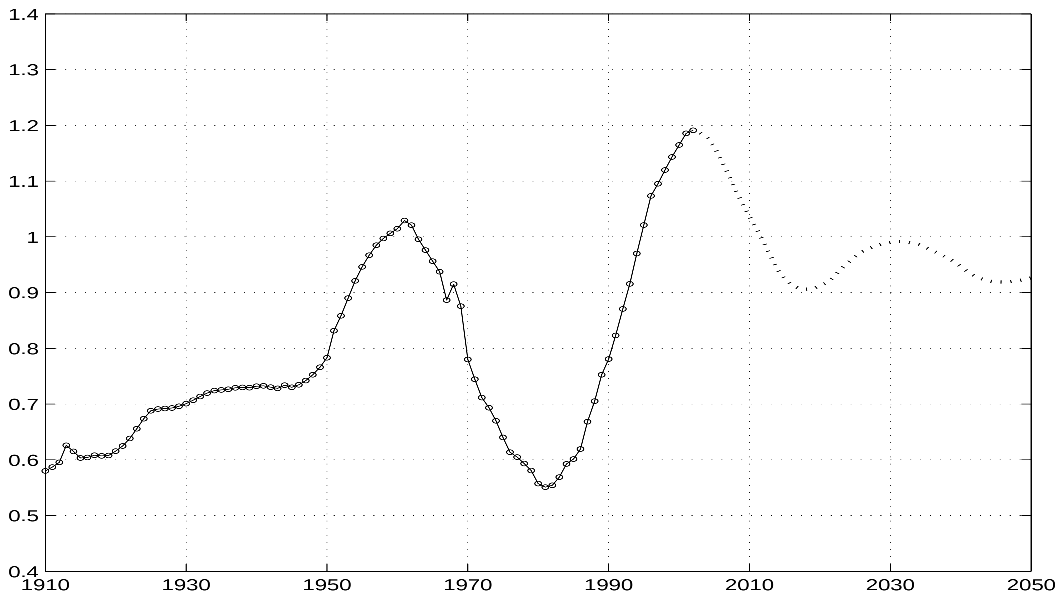

Figure 5: *The middle-young (MY) cohort ratio.*

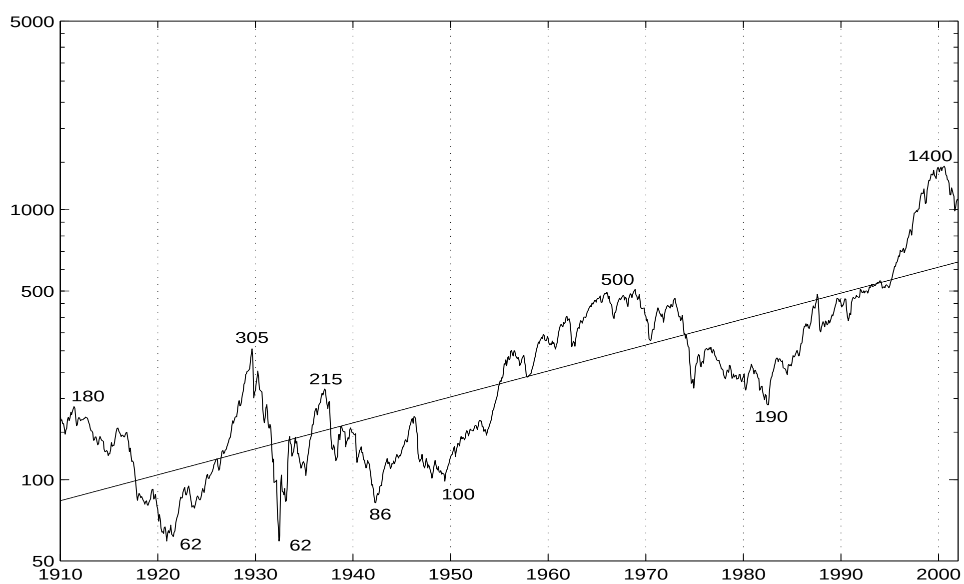

Figure 6: *Real Standard and Poors Index of Common Stock Prices 1910-2002.*

century, a period of relatively high birth rates (see Figure 1) and immigration, while the young were the small generation born during the Great Depression. During this same period equity prices were steadily rising. Stock market prices declined in real terms at the end of the 60's and during the 70's, during which the MY cohort ratio also declined significantly: the small Great Depression cohort became the middle-aged, while the large cohort of Baby Boomers entered their active life. In the early 80's equity prices began their remarkable ascent to their peak in 2000, and it was during this period that the plentiful Baby Boomers moved into middle-age, while the small cohort of Xers born in the 70's entered their economic life, creating the equally dramatic surge in the MY ratio.

The price-earnings ratio is a normalized measure of level of equity prices which has the advantage of factoring out growth and is thus more directly comparable with the results of our model. As Figure 7 shows[18], the PE ratio follows roughly the same pattern as the real S&P index and corresponds well with the long-run fluctuations in the MY cohort ratio. The PE ratio increases from a low of 8 to around 20 in the 60's, decreases in the seventies and early 80's to around 8, after which it increases to around 30 in 2000. These numbers correspond well with the predictions of Tables 4 and 10 (with $\alpha = 4$) with PE ratios (or half price-dividend ratios in the tables) varying between 7–8 in the bad state s_4 of pyramid Δ_1 and 25–40 in the good state s_1 of Δ_2.

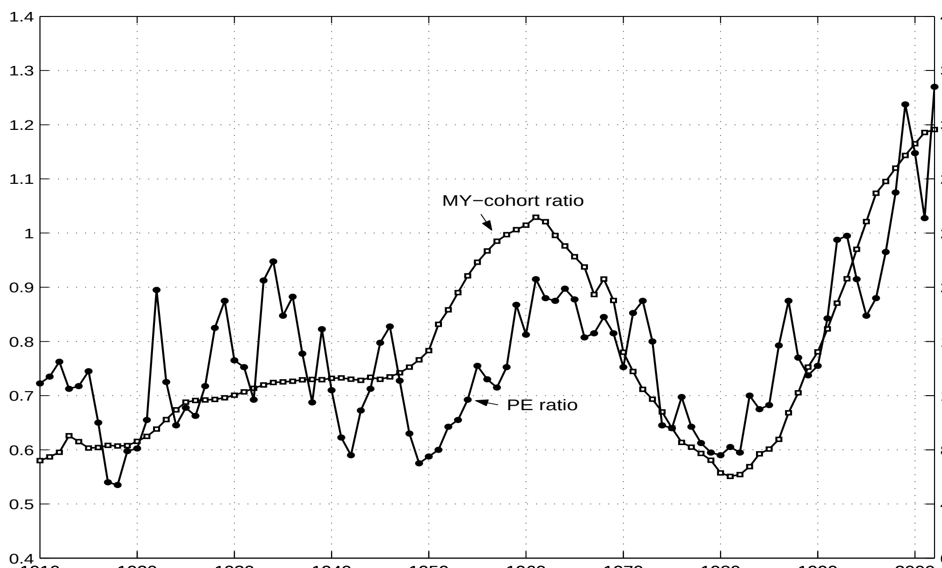

Figure 7: *S&P price-earnings ratio and MY cohort ratio.*

[18] The PE ratio that we consider is obtained by dividing the S&P price index of each month by an average of the earnings of the past 12 months and anualizing the series.

Table 7 shows the results of the regression of the PE ratio on the MY-cohort ratio

$$PE_t = c + \beta MY_t + \epsilon_t$$

Table 7: Regression of PE Ratio on MY Ratio

1910-2002				1945-2002				1965-2002			
c	β	R^2	ADF t-stat	c	β	R^2	ADF t-stat	c	β	R^2	ADF t-stat
-3.5 (3.2)	23.5 (4.4)	0.48	-4.1	-5.5 (3.7)	25.4 (4.7)	0.55	-2.8	-7.1 (2.6)	29.7 (3.3)	0.78	-4.8

for different time periods. The standard errors, corrected for heteroskedasticity and autocorrelation (Newey–West (1987)) are indicated in parentheses below the coefficients. Since the series are slow moving and there is a danger of finding spurious correlations, we report the t-statistics of the Augmented Dickey–Fuller unit root test on the residuals of the regression.[19] The regression tends to support the assumption of a systematic relation between the PE ratio and the MY ratio: the regression coefficients are significant, stable, and the probability of a unit root in the residuals is low on the largest sample 1910–2002. These results are consistent with the results of Poterba (2001, Table 9) who finds a significant relation between the price-dividend ratio and demographic variables.

Rates of Return. A defect of the stochastic model with 20 year time periods is that it cannot give insight into short-run rates of return. We were able to study short-run rates of return only in the deterministic model in which the rate of return on equity coincides with the interest rate. There we found that the rate of return (and hence the interest rate) is not synchronized with the MY ratio because it is importantly influenced by the capital gains or losses which depend on the change in the equity price, and hence on the change (and not the level) of the MY ratio. This suggests studying how annual interest rates and rates of return on equity covary with the differenced MY ratio. The result of the regression

$$X_t = c + \beta D(MY)_t + \epsilon_t$$

[19] All the ADF t-statistics of residuals reported in this section are derived from the regression of the differenced residual on the residual without a constant and with one lagged variable. A critical value smaller than –3.39 leads to a rejection of the null hypothesis of a unit root in the residuals at the 1% confidence level. The critical levels for the 2.5% and 5% confidence levels are –3.05 and –2.76 respectively (Phillips–Ouliaris (1990)).

is shown in Table 8 for different time periods, where X_t is either the rate of return on the S&P or the real short-term interest rate, and $D(MY)_t = MY_t - MY_{t-1}$. The Newey–West standard errors are shown in parentheses below the coefficients.

The results for the rate of return on equity are as expected: the rate of return on equity is much more variable than the change in the MY ratio and is clearly affected by other shocks (to output). Nevertheless demographic changes account for 14% of the variability of the rate of return between 1945 and 2002, which is non-negligible. Figure 8 shows the relationship: there is a tendency for rates of return to be higher in the late 40's and 50's, and in the mid 80's and 90's when the MY ratio was increasing, and lower than average in the late 60's and 70's when the MY ratio was decreasing.

Table 8: Regression of Rate of Returns on Differenced MY Ratio

	Real Rate of Return on S&P			Real Short-Term Interest Rate		
	c	β	R^2	c	β	R^2
1910-2002	6.73 (1.9)	206 (43)	0.07	0.76 (0.7)	20.23 (12.8)	0.01
1945-2002	7.42 (1.6)	197 (41)	0.14	0.53 (0.6)	12.6 (11.4)	0.02
1965-2002	5.9 (2)	186 (40)	0.16	1.28 (0.5)	22.8 (10.6)	0.16

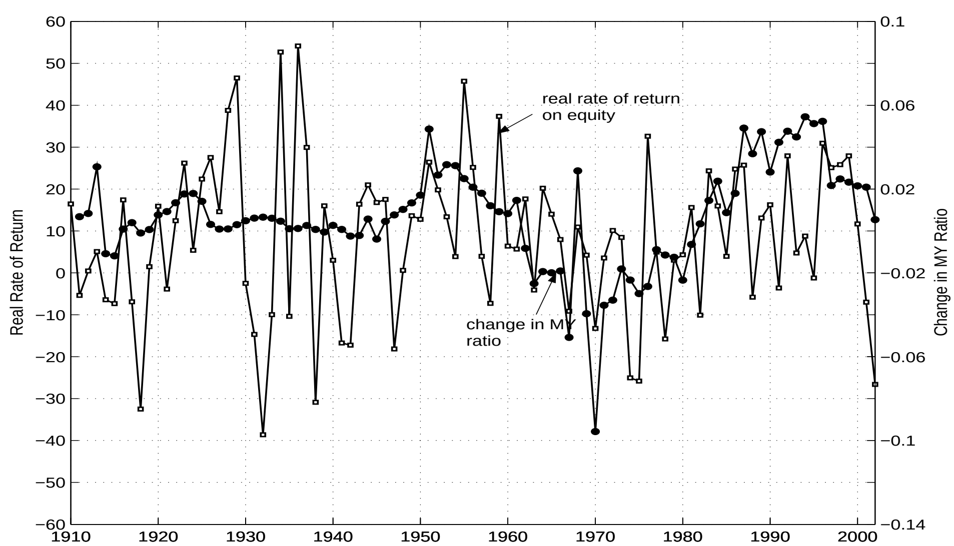

Figure 8: *Real rate of return on S&P index and change in MY ratio.*

On the other hand the relation between the short-term interest rate and the change in the MY

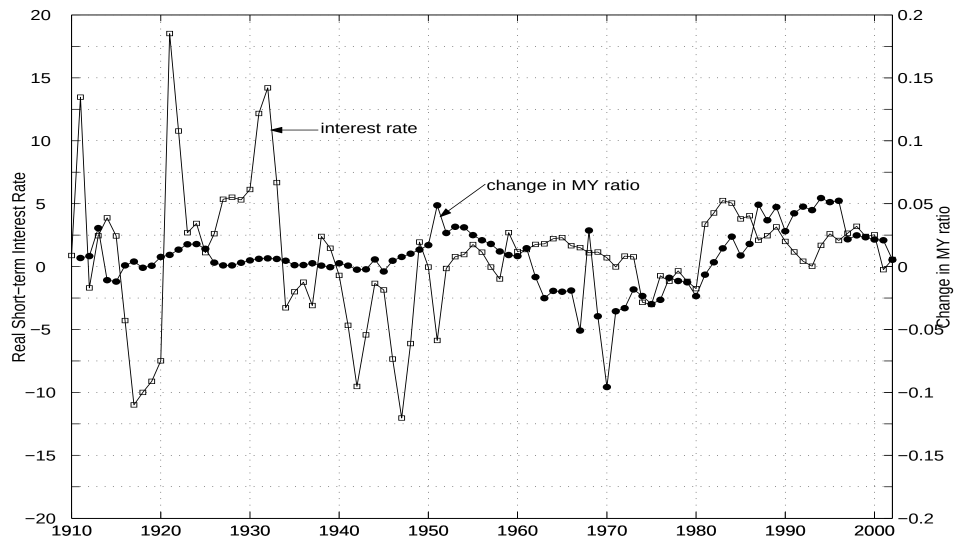

Figure 9: *Real short-term interest rate and change in MY ratio.*

ratio is weaker than expected during the period 1945–2002. The regression has to be restricted to the period 1965–2002 to obtain a significant relation between the interest rate and the differenced MY ratio: indeed as shown by Figure 9 during this period[20] the behavior of the interest rate is roughly compatible with the equilibrium behavior shown in Figure 4: real interest rates declined after 1965 and were very low in the mid seventies when the MY ratio and real equity prices were declining rapidly. The turn in interest rates occurred in 1980, before the turn in equity prices, and interest rates were high in the early 80's at the beginning of the rise in stock prices. They stayed relatively high up to 2000, with a small intermission before and during the fall in equity prices accompanying the Gulf War recession. The period 1945–1965 does not however fit the predictions of the model: the return on equity was consistently high during the bull market of the 1950's and early 1960's while the interest rate was low, especially at the beginning of the rise in the late 1940's and early 1950's: this is difficult to reconcile with rational expectations. One hypothesis is that many investors, scared by the enormous losses incurred on the stock market during the Great Depression, fled to the relative safety of the bond market, leading to a period of low interest rates. As we have seen, restricted participation in the equity market decreases interest rates.

Equity Premium. In the different equilibria that we calculated, interest rates were in the interval

[20] The series shown in Figure 9 is obtained by pasting the three-month Treasury Bill rate for the period 1939–2002 (Economic Report of the President) with the short-term commercial paper rate (Historical Statistics) for the period 1910–1938, decreased by 0.55%, which is the average premium on the commercial paper for the period when the two series are available.

$[-5\%, 9\%]$. Although, as seen from Figure 9, this interval is not exceptional by historical standards — prior to the fifties the real interest rate fluctuated between –12% and 18% — the fluctuation in interest rates in the postwar period, in which the significant demographic changes occurred, have been smaller — between –3% and 5% — in part because the change of regime from a gold standard to fiat money has increased the effectiveness of monetary policy which was aimed at reducing the variability of inflation and stabilizing real interest rates.

The smaller than predicted adjustment of interest rates to movements of equity prices implies that the high values of the risk premium are much higher than that predicted by the model. The equity premium in Figure 10 is calculated by taking the geometric mean rate of return on the S&P twenty years forward at each date, and subtracting the geometric mean of the short-term interest rate over the same period; this gives the average equity premium that agents could expect if they invested at this date with perfect foresight. The maximum occurred in the early to mid 1940's reflecting the fact that the excess return on equity was high during the twenty years of the rising market from 1945 to 1965. The minimum occurred around 1965, which means that the premium was low during the declining market of the 1970's and early 1980's. Then there is a local maximum in 1980 arising from the high rate of return on equity from the beginning 1980's up to 2000.[21]

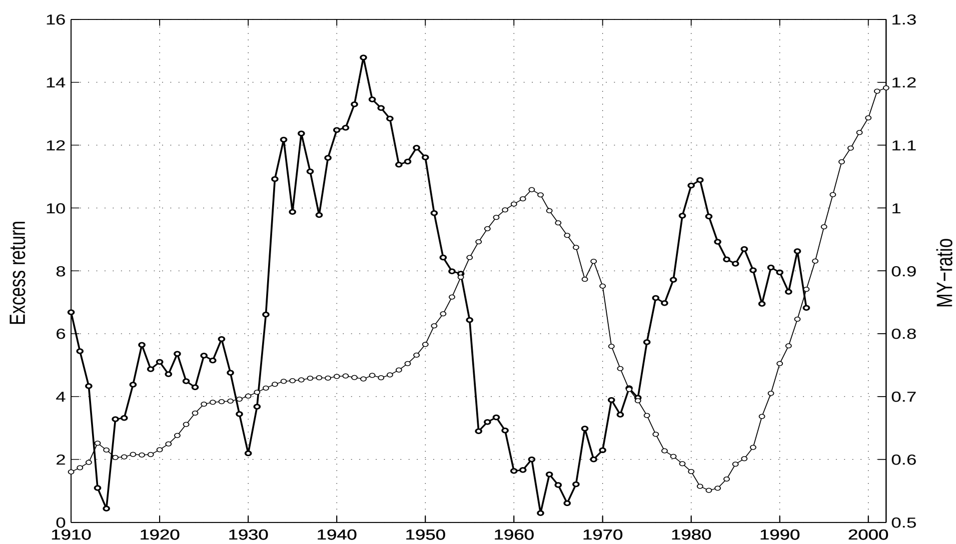

Figure 10: *Equity premium on S&P index and MY ratio.*

The qualitative behavior of the equity premium fits the predictions of the model well: in equilibrium the excess return is higher on average for the agents who buy at low prices (when the MY

[21] The series is continued after 1983 which is the last year for which 20 observations forward are available, up to 1993 by taking the forward geometric means over the available observations.

ratio is low) than for those who buy at high equity prices and expect a low return (when the MY ratio is high). The equilibrium results on the equity premium are driven by the fact that returns are more variable when prices go up than when they go down. This is only partially supported by the data: with yearly data there is no marked change in variability of the S&P index between ascending and descending phases.[22] However at the higher frequency of daily data, the market has been substantially riskier in the recent ascending phase (1982–2000) than in the preceding declining phase (1965–1982): for these periods the standard deviation of the daily rate of change in the price index went from 0.83% to 1.1%, and the number of days with more than 2% change in prices went from 121 to 207. The bull market of the fifties on the other hand did not exhibit more volatility than the ensuing bear market of the seventies.

Note that, given the small variability of the short-term interest rate, the behavior of the average (geometric) excess return 20 years forward, is close to that of the average (geometric) rate of return 20 years forward. This long-term rate of return on equity thus exhibits a cyclical behavior with a 20-year phase shift from the MY ratio, which roughly fits the prediction of the deterministic and stochastic models.

International Evidence. The three alternating twenty-year episodes of increasing and decreasing equity prices in the US constitute a rather small sample for checking whether demographic forces were a significant causal element in these price changes. The experience of countries other than the US may help to increase the number of observations for testing the demographic hypothesis. This section studies whether there is a relation between equity prices and demography for Germany, France, UK and Japan.

The model that led to the tests for the US rests on two assumptions — a cyclical livebirth process and a closed economy — which may not be appropriate for other countries. The cyclical livebirth process comes directly from the observation of the US livebirth process, and justifies taking the MY ratio as a proxy for the composition of the population. Since the livebirths of the other countries mentioned above are less clearly cyclical, we study two proxies for the composition of the population, the MY ratio defined as for the US and the size of the cohort 35–59, which is a direct measure of the middle-aged group.

The assumption of a closed economy has been made to explain the asset prices in the US by

[22] If we compute for each year the standard deviation of the rate of return on the S&P during the following twenty years, the most obvious result is that, because of the Great Depression, the volatility of the rate of returns experienced by the investors at the beginning of the century was much larger than the volatility experienced after the Second World War. For example the standard deviation of the twenty-year-forward rate of return was between 24% and 28% from 1914 to 1932, while it has varied between 13% and 17% after 1940.

its own demographic structure. This assumption seems reasonable for studying the past, if not the future, behavior of the US stock market, since until recently US equity has been largely owned by US investors: up to 1975 foreigners held less than 4% of US equity, and despite the increase during the 80's foreigners still hold less than 11%.[23] The home bias phenomenon has been documented for other countries, but the closed economy assumption may nevertheless be more appropriate for the US and Japan which have the two largest stock markets in the world, than for the smaller European markets which seem to follow the US market.

Table 9: Regression of Real Price Index on Demographic Variable 1950-2001

	MY ratio				Cohort 35-59			
	c	β	R^2	ADF t-stat	c	β	R^2	ADF t-stat
Germany: Real CDAX	89 (75)	65 (80)	0.01	-1.03	-671 (109)	0.033 (0.004)	0.53	-2.47
France: Real SBF	-2200 (706)	3958 (810)	0.32	-2.86	-2400 (669)	0.23 (0.04)	0.37	-0.84
U.K: Real FTS	349 (695)	1087 (747)	0.04	-0.77	-8600 (894)	0.57 (0.05)	0.72	-3.14
Japan: Real Nikkei 225	-16481 (2656)	33555 (3116)	0.70	-3.3	-14493 (1880)	0.75 (0.08)	0.63	-2.86

Table 9 presents the results of the regression

$$RP_t = c + \beta D_t + \epsilon_t$$

where RP_t is the real stock price index of the country considered[24] and D_t is the demographic index: in the first block D_t is the MY ratio for the cohorts 40–49 to 20–29, while in the second block D_t is the size of the cohort 35–59. The regression is limited to the period 1950–2001 since the population data comes from the United Nations[25] data which is available only since 1950.

The results are mixed: Germany shows little sign of a relation between equity prices and demography — the R^2 is small and the ADF t-statistic does not support cointegration. For France the real SBF index has a relatively significant relation with the MY ratio, but no convincing relation with the 35–59 cohort, and conversely the UK real FTS index has no relation with the MY ratio, but a relatively strong relation with the 35–59 cohort. All the results improve significantly when the regression is restricted to 1980–2001: each of the European countries had a baby boom after

[23] Flow of Funds Accounts of the US, 1945–2002, Board of Governors of the Federal Reserve System.
[24] Source: Global Financial Data.
[25] World Population Prospects: The 2000 Revision, United Nations, Population Division, February 2001.

1945, giving rise to a large and growing middle-age cohort from 1980–2001, and each, like the US, experienced a stock market boom over this period.

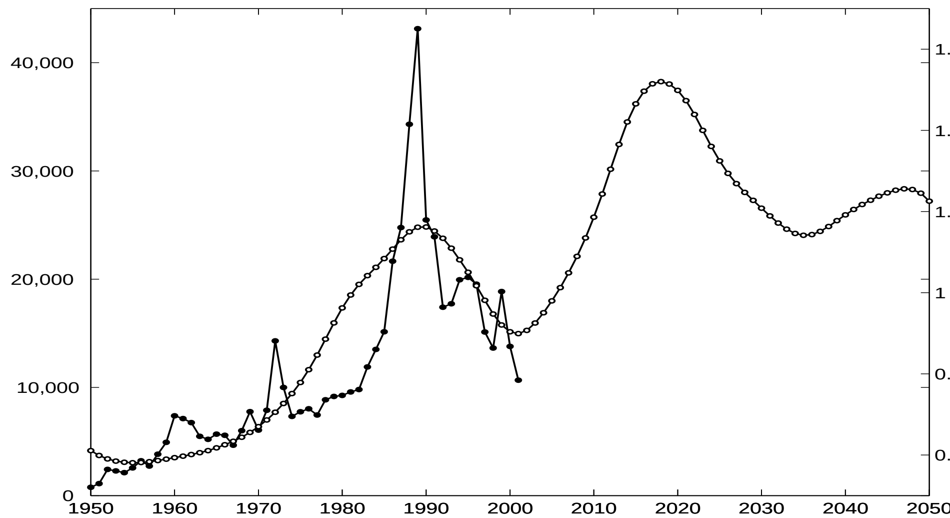

Figure 11: *Japan Nikkei 225 real price index and MY ratio.*

The most convincing evidence for the demographic hypothesis is provided by Japan. The Japanese market does not seem to follow the US market: it increased in the mid 60's and 70's, and decreased during the 90's when the US market was booming. The livebirth process has some of the cyclical aspects of the US, but with different dates for the peaks and troughs. Figure 11 shows the Nikkei price index and the MY ratio. It seems remarkable that the turning point of the Nikkei index has coincided almost exactly with the turning point of the medium-young ratio.

7. Concluding Remarks

The model studied in this paper has combined a demographic structure tailored to the demographic experience of the US during the last century with a life-cycle behavior of the representative agent for each generation. The calculation of equilibrium shows that fluctuating cohort sizes induce substantial changes in equity prices, resulting in the predictability of the rate of return on equity: high price-earnings ratios are followed on average by low rates of return and conversely. The changes in equity prices are accompanied by changes in rates of return and interest rates which are linked to the change, rather than the level, of the MY ratio. The equilibrium also exhibits some predictability of excess returns. When compared to data, the model does not do too badly on equity prices and on rates of return on the stock market. However the predictions on interest rates and excess returns are less satisfactory. On the whole the fact that the turning points of stock prices and price-earning ratios are well synchronized with the demographic cycle as measured by the medium-young ratio seems to go in favor of the demographic hypothesis.

Contrary to the conclusion of Poterba (2001), given the predicted future behavior of the MY ratio (Figure 5) our model predicts a decline in the PE ratio in the US equity market over the next twenty years: this conclusion is similar to that of Campbell–Shiller (2001) based on the historical mean reversion of the PE ratio process. The predictions of our model should however be interpreted with caution in view of the ongoing globalization of equity markets. This study has been based on national (mainly US) data for equity markets and demography, which is justified by the strong and well-documented "home bias" toward national equity issues (French and Poterba (1991), Tesar and Werner (1995)). However a model placed directly in an international setting predicts that agents diversify across the equity issues of other countries. This discrepancy between theory and observation tends to disappear with the decrease in transactions and informational costs, and the development of financial markets,[26] so that the future evolution of US equity prices may well depend more on the joint demography of the countries with participation in the US equity market than on the US demography. Most developed countries have similar demographic perspectives for the next thirty years, with a Baby Boom generation going into retirement, low birthrates, and lengthening of life expectancy, all factors leading a high dependency ratio (ratio of retired to working agents). The only real prospect for offsetting the effect of a small generation of middle-aged agents buying the equity of a large retired generation, comes from an increased participation in the US security market by investors from the developing countries.

[26]Recent papers have documented an important decrease in the home bias: see Lane and Milesi-Ferretti (2003) and Ahmadi (2003).

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Appendix

Appendix A: Correcting for Immigration. Annual data for immigration can be obtained from the Historical Statistics of the US and from the Statistical Yearbook of the Immigration and Naturalization Service (INS), which has become the US Citizenship and Immigration Service (USCIS). ¿From this data we find that the number of immigrants was approximately 2.4 million for the period 1925-1944, 4.5 million for the period 1945–1964, 8.8 million for the period 1965–1984 and 16.8 million for the period 1985–2004. The USCIS statistics indicate that between 1994 and 1996 on average 21% of the immigrants were below age 15, 33.3% were in the age group 15–29, 26.3% were in the age group 30–44, 14.7 in the age group 45–64, and 4.7% were over 65 years old. These age groups do not correspond exactly to the age cohorts that we consider; we estimate at 25% the number of immigrants below age 20, 50% the number in the age group 20–39, 20% in the age group 40–59 and 5% over 60. We use the formulae

$$Y_t = LB_{t-1} + C_{t-1}^{im} + Y_t^{im}, \quad M_t = Y_{t-1} + M_t^{im}$$

to correct the size of the cohorts, where Y_t, M_t denote the number of young and middle-aged at period t, LB_t the number of livebirth, and $C_t^{im}, Y_t^{im}, M_t^{im}$ the number of children, young and middle-aged immigrants.

The immigration-adjusted number of young (Baby-Boomers) and middle-aged (Depression Generation) for the period 1965-1984 becomes

$$Y_{65-84} = 79 + 0.25 \times 4.6 + 0.5 \times 8.8 = 85, \quad M_{65-84} = 52 + 0.25 \times 2.4 + 0.5 \times 4.6 + 0.2 \times 8.8 = 57$$

with a ratio $85/57 = 0.67$ instead of the ratio $79/52 = 0.66$ adopted in the paper. Similarly for the period 1985–2004, the corrected number of young (Xers) is $Y_{85-04} = 79.6$ and the corrected number of middle-aged (Baby-Boomers) is $M_{85-04} = 88.4$, leading to a ratio $88.4/79.6 = 1.11$ instead of the ratio $79/69 = 1.14$ adopted in the paper.

Appendix B: Markov Equilibrium. Since agents' (economic) lives span 3 periods, it can be shown that a Markov equilibrium which depends on the exogenous states — the pyramid and shock states — does not exist. What is needed is an endogenous variable which summarizes the dependence of the equilibrium on the past—the income which the middle-aged agents inherit from their portfolio decision in their youth. Thus we study equilibria with a state space $\Xi = G \times K \times S$

where G is a compact subset of $\mathbb{R}_+$, $K = \{1, 2\}$ is the set of pyramid states (indexed by $k \in \{1, 2\}$) and $S = \{s_1, s_2, s_3, s_4\}$ is the set of shock states: we let $\xi = (\gamma, k, s)$ denote a typical element of the state space Ξ, γ denoting the portfolio income inherited by the middle-aged agents from their youth. The pyramid state k determines the age pyramid $\Delta_k = \left(\Delta_k^y, \Delta_k^m, \Delta_k^r\right)$. If k is the population state at date t, let k^+ (resp. k^-) denote the pyramid state at date $t+1$ (resp. $t-1$). Since the pyramid states alternate, if $k = 1$ then $k^+ = k^- = 2$. The output shock $s \in S$ determines the incomes $w^y = (w_s^y, s \in S)$ and $w^m = (w_s^m, s \in S)$ of the young and middle-aged agents, as well as the dividend $D = (D_s, s \in S)$ on the equity contract.

To find a Markov equilibrium, we note that the security prices only need to make the portfolio trades of the young and middle-aged agents compatible: the retired agents have no portfolio decision to make — they collect the dividends and sell their equity holdings. Thus we are led to study the portfolio problems of the young and the middle-aged agents, the latter inheriting the income γ, and to look for security prices which clear the markets. This problem can be reduced to the study of a family of two-period portfolio problems in which middle-aged agents anticipate the consequences of their decisions for their retirement — they need to anticipate the next period equity price Q^e — and young agents anticipate the portfolio income they will transfer into middle age (which also depends on Q^e) and the saving decision F that they will make next period to provide income for their retirement. A correct expectations equilibrium then has the property that the agents' expectations are fulfilled in the next period. Given that an equilibrium involves both current and anticipated variables we introduce the convention that current variables are denoted by lower case letters, while anticipated variables are denoted by capitals. A stationary Markov equilibrium will be a function $\Phi : \Xi \longrightarrow \mathbb{R}^4 \times \mathbb{R}_+^2 \times \mathbb{R}_+^8$ with $\Phi = (z, q, Q^e, F)$, where $z = (z^y, z^m) = (z_b^y, z_e^y, z_b^m, z_e^m)$ is the vector of bond and equity holdings of the young and middle-aged agents respectively, $q = (q^b, q^e)$ is the vector of current prices for bond and equity, $Q^e = (Q_s^e, s \in S)$ is the vector of anticipated next period equity prices and $F = (F_s, s \in S)$ is the vector of anticipated next period savings of the young. To express the condition on correct expectations we need the following notation: if in state ξ young agents choose a portfolio $z^y(\xi)$ and anticipate equity prices $Q^e(\xi)$, then the income $\Gamma(\xi) = \left(\Gamma_s(\xi), s \in S\right)$ which they anticipate transferring into middle age is given by

$$\Gamma(\xi) = V(\xi) z^y(\xi), \quad \xi \in \Xi$$

where $V(\xi) = (\mathbf{1}, D + Q^e(\xi))$, $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^4$ denoting the sure payoff on the bond and $D = (D_s, s \in S)$ the random dividend on equity. We let $f(\xi)$ denote the actual savings chosen by middle-aged agents when the state is ξ, thus

$$f(\xi) = q(\xi) z^m(\xi), \quad \xi \in \Xi$$

Definition. A function $\Phi = (z, q, Q^e, F) : \Xi \longrightarrow \mathbb{R}^4 \times \mathbb{R}^2_+ \times \mathbb{R}^8_+$ is a stationary (Markov) equilibrium of the economy $E(u, w, D, \Delta)$ if, $\forall\ \xi = (\gamma, k, s) \in \Xi$,

(i) $z^y(\xi) = \arg\max_{z^y \in \mathbb{R}^2} \left\{ u(c^y) + \delta \sum_{s' \in S} \rho_{s'} u(C^m_{s'}) \;\middle|\; \begin{array}{l} c^y = w^y_s - q(\xi) z^y \\ C^m = w^m + V(\xi) z^y - F(\xi) \end{array} \right\}$

(ii) $z^m(\xi) = \arg\max_{z^m \in \mathbb{R}^2} \left\{ u(c^m) + \delta \sum_{s' \in S} \rho_{s'} u(C^r_{s'}) \;\middle|\; \begin{array}{l} c^m = w^m_s + \gamma - q(\xi) z^m \\ C^r = V(\xi) z^m \end{array} \right\}$

(iii) $\Delta^y_k\, z^y_b(\xi) + \Delta^m_k\, z^m_b(\xi) = 0, \quad \Delta^y_k\, z^y_e(\xi) + \Delta^m_k\, z^m_e(\xi) = 1$

(iv) $Q^e_{s'}(\xi) = q^e\Big(\Gamma_{s'}(\xi), k^+, s'\Big),\ \forall\ s' \in S, \qquad F_{s'}(\xi) = f\Big(\Gamma_{s'}(\xi), k^+, s'\Big),\ \forall\ s' \in S$

(i) and (ii) are the conditions requiring maximizing behavior on the part of young and middle-aged agents who anticipate the equity prices $Q^e(\xi)$ and, in the case of the young agents, anticipate the savings $F(\xi)$. Note that the vector of consumption $C^m \in \mathbb{R}^4_+$ which a young agent anticipates for middle age (hence the capital letter) must be distinguished from $c^m(\xi) \in \mathbb{R}$ which is the current consumption of a middle-aged agent. (iii) requires that the aggregate demands of the two cohorts for the bond and equity clear the markets. (iv) is the condition requiring the agents' expectations be correct. In choosing their portfolio $z^y(\xi)$ in state ξ, young agents anticipate transferring the income $\Gamma(\xi) = V(\xi) z^y(\xi)$ to the next period — where $V(\xi)$ is the anticipated payoff of the securities depending on $Q^e(\xi)$. In order that $Q^e_{s'}(\xi)$ be a correct expectation, it must coincide with the price $q^e\Big(\Gamma_{s'}(\xi), k^+, s'\Big)$ which is realized in output state s' when middle-aged agents receive the portfolio income $\gamma' = \Gamma_{s'}(\xi)$ and the pyramid state is k^+; in the same way the saving $F_{s'}(\xi)$ that the young anticipate doing in their middle age must coincide with the actual savings of a middle-aged agent with asset income $\gamma' = \Gamma_{s'}(\xi)$.

For given anticipation functions

$$(Q^e, F) : \Xi \to \mathbb{R}^4_+ \times \mathbb{R}^4_+$$

(i), (ii), and (iii) in the Definition of a stationary equilibrium in Section 5, define a family of two-period equilibria indexed by $\xi = (\gamma, k, s) \in \Xi$. Assuming uniqueness of the equilibria, let

$$\Big(z_{(Q^e,F)}(\xi), q_{(Q^e,F)}(\xi), \Gamma_{(Q^e,F)}(\xi), f_{(Q^e,F)}(\xi)\Big)$$

denote the equilibrium portfolios, prices, anticipated income transfers by the young, and the actual savings of the middle aged, for each $\xi \in \Xi$. Finding a recursive equilibrium amounts to finding functions (Q^e, F) such that (iv) is satisfied i.e.,

$$\text{(E)} \qquad \begin{bmatrix} Q^e_{s'}(\xi) \\ \\ F_{s'}(\xi) \end{bmatrix} = \begin{bmatrix} q^e_{(Q^e,F)}(\Gamma_{(Q^e,F)s'}(\xi), k^+, s') \\ \\ f_{(Q^e,F)}(\Gamma_{(Q^e,F)s'}(\xi), k^+, s') \end{bmatrix} \quad \forall s' \in S,\ \forall \xi = (\gamma, k, s) \in \Xi$$

Assuming that the anticipation functions as well as the equilibrium functions are continuous, an equilibrium is a fixed point on the space of continuous functions $C(\Xi, \mathbb{R}^8_+)$ of the form $(Q^e, F) = \psi(Q^e, F)$ where $\psi(Q^e, F)$ is defined by the RHS of (E). We look for an approximate equilibrium in the space of piecewise linear functions on $G \times K \times S$, calculating "as if" ψ was a contraction.

We begin by choosing an interval $G = [\underline{\gamma}, \bar{\gamma}]$ and a grid $G_m = \{g_1 \ldots, g_m\}$ on this interval, and then choose arbitrary initial anticipation functions $(Q^{e,0}, F^0)$ on $G_m \times K \times S$. By solving a sequence of two-period equilibrium problems we can then compute the family of associated two-period equilibria $(z^0(\xi), q^0(\xi), \Gamma^0(\xi), f^0(\xi),\ \xi \in G_m \times K \times S)$, possibly modifying the interval G so that $\Gamma^0_s(\xi) \in G$ for all s and all $\xi \in G_m \times K \times S$. Then by recursion we define for $n \geq 1$ the anticipation functions $(Q^{e,n}, F^n)$ by

$$\begin{bmatrix} Q^{e,n}_{s'}(\xi) \\ \\ F^n_{s'}(\xi) \end{bmatrix} = \text{Lin} \begin{bmatrix} q^{e,n-1}(\Gamma^{n-1}_{s'}(\xi), k_+, s') \\ \\ f^{n-1}(\Gamma^{n-1}_{s'}(\xi), k_+, s') \end{bmatrix} \quad \forall s' \in S,\ \forall \xi \in G_m \times K \times S$$

where $(z^{n-1}, q^{n-1}, \Gamma^{n-1}, f^{n-1})$ is the family of two-period equilibria associated with $(Q^{e,n-1}, F^{n-1})$, and Lin denotes the linear interpolation

$$\text{Lin}\, q^{e,n-1}(\Gamma^{n-1}_{s'}(\xi), k_+, s') = \lambda q^{e,n-1}(g_j, k^+, s') + (1-\lambda) q^{e,n-1}(g_{j+1}, k^+, s')$$

if $\Gamma^{n-1}_{s'}(\xi) = \lambda g_j + (1-\lambda) g_{j+1}$. At each step we modify G if necessary so that $\Gamma^n_s(\xi) \in G$ for all s and all $\xi \in G_m \times K \times S$. Although it seems difficult to prove formally that the properties of uniqueness and continuity of the two-period equilibria are satisfied, and that ψ is a contraction, in practice the algorithm converges in less than 1000 iterations.

Appendix C: Table 10. Low Cohort Ratio: Prices in Markov Equilibrium

$N = 79,\ n = 69,\ w^y = (2.3, 2.3, 1.7, 1.7),\ w^m = (3.6, 3.6, 2.4, 2.4),\ D = (83, 57, 83, 57)$

		pyramid 1: MY ratio=0.87				**pyramid 2: MY ratio=1.14**			
		q^e	$PD/2$	r^{an}	rp^{an}	q^e	$PD/2$	r^{an}	rp^{an}
$\boldsymbol{\alpha = 2}$	s_1	97 (.9)	11.4 (.1)	1.96 % (.01)	0.27 % (1.3)	117 (1.9)	14 (.23)	.6 % (.01)	0.28 % (1.2)
	s_2	95 (.8)	16.7 (.3)	2. 07% (.01)	0.25 % (1.3)	113 (1.6)	20 (.6)	.8 % (.01)	0.26 % (1.2)
	s_3	51 (.5)	6.5 (.13)	5.3 % (.01)	0.23 % (1.3)	61 (1)	7.3 (.3)	3.96 % (.01)	0.27 % (1.2)
	s_4	50 (.4)	8.7 (.08)	5.5 % (.01)	0.26 % (1.3)	58 (.9)	10.1 (.15)	4.2 % (.01)	0.26 % (1.2)
	Average	73 (23)	10.5 (2.3)	3.7 % (1.8)	0.26 % (1.3)	86 (28)	12.3 (3.3)	2.5 % (1.8)	0.27 % (1.2)
		ratio of av. prices: 1.2, *peak / trough*: 2.3, *trough / peak* : 0.4							
$\boldsymbol{\alpha = 4}$	s_1	145 (2.5)	17.5 (.3)	-.16 % (.01)	1.04 % (2.3)	207 (10)	25 (1.3)	-2.5 % (.07)	.99 % (2)
	s_2	134 (2)	23.4 (.7)	.34 % (.01)	1.06 % (2.4)	183 (8.4)	32.2 (1.5)	-1.9 % (.06)	0.87 % (1.95)
	s_3	46 (.7)	5.5 (.09)	5.8 % (.01)	.96 % (2.5)	60 (3)	7.2 (.7)	3.6 % (.07)	1.09 % (2)
	s_4	40 (.5)	7.1 (.2)	6.8 % (.01)	1.13% (2.6)	49 (2)	8.6 (.4)	4.8 % (.06)	0.99 % (2)
	Average	92 (51)	12.7 (6.2)	3.3 % (3.3)	1.07 % (2.45)	127 (77)	17.4 (9)	1 % (3.5)	0.98 % (2)
		ratio of av. prices: 1.4 , *peak / trough*: 5.1, *trough / peak* : 0.19							
$\boldsymbol{\alpha = 6}$	s_1	236 (8.3)	28.4 (1)	-2.6 % (.04)	2.16% (3.5)	301 (40)	47 (5)	–5.7 % (.2)	1.87 % (2.7)
	s_2	202 (6.3)	35.5 (1.1)	-1.8 % (.03)	1.9 % (3.6)	319 (30)	56 (11)	-4.8 % (.2)	1.93 % (2.8)
	s_3	48 (1.3)	5.8 (.15)	5.6 % (.03)	2.14% (3.9)	64 (5.7)	7.7 (.7)	2.97 % (.2)	2 % (2.87)
	s_4	37 (.8)	6.5 (.1)	7.4 % (.03)	2.3 % (4.2)	43 (3.3)	7.5 (.8)	5.2 % (.2)	2.7 % (3)
	Average	134 (96)	18 (12)	2.3 % (4.8)	2.2 % (3.8)	211 (169)	28 (21)	-.4 % (5.2)	2 % (2.9)
		ratio of av. prices: 1.6 , *peak / trough*: 10.6, *trough / peak* : 0.09							