"An Optimal Fair Job Assignment Problem"

by

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Abstract: We study the problem of how to allocate a set of indivisible objects like jobs or houses and an amount of money among a group of people as fairly and as efficiently as possible. A particular constraint for such an allocation is that every person should be assigned with the same number of objects in his or her bundle. The preferences of people depend on the bundle of objects and the quantity of money they take. We propose a solution to this problem, called a perfectly fair allocation. It is shown that every perfectly fair allocation is efficient and envy-free, income-fair and furthermore gives every person a maximal satisfaction. Then we establish a necessary and sufficient condition for the existence of a perfectly fair allocation. It is shown that there exists a perfectly fair allocation if and only if an associated linear program problem has a solution. As a result, we also provide a finite method of computing a perfectly fair allocation.

Keywords: Perfectly fair allocation, equity, efficiency, indivisibility, multi-person decision, discrete optimization.

JEL-code: D3, D31, D6, D61, D63, D7, D74.

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1 Introduction

The purpose of this study is to define and characterize a solution to a large class of job assignment problems in terms of economic equity, fairness and efficiency. A particular focus of our attention is to identify necessary and sufficient conditions for the existence of this solution. Fair job assignment problems are encountered in various organizations. For example, faculty members are assigned within several committees, a group of students rent a house and decide how to assign rooms among themselves, and workers are asked to do jobs with certain compensations, and parents contemplate a will to give their children various properties and savings.

Formally, we are going to investigate the following general job assignment problem: A set of indivisible objects (or simply objects) like jobs, tasks, houses, offices and a fixed amount of money is to be assigned to a number of people. Indivisible objects are relatively large and fractions of them cannot be assigned to the people. Furthermore, indivisible objects can be desirable or undesirable as well. Here money will be treated as a perfectly divisible commodity, since the value of a unit of money is relatively small compared with one unit of indivisible objects. The amount of money can be positive or negative. A particular constraint for allocations of objects is that every person should be assigned with the same number of objects. This constraint is quite natural and justified if objects are seen to be large and essential. One may interpret this constraint as a guarantee of equal fundamental rights or obligations. For example, the number of committees each faculty member participates is roughly the same. Family members are often assigned with the same number of household chores. The preferences of people depend on the bundle of objects and the quantity of money they take. The objective is to distribute the objects and the money among the people in a way that is both as fair as possible to every individual and is as efficient as possible to the group as a whole.

The concept of fairness is a provocative and widely accepted idea in the literature of economic equity. It is introduced in Foley [4]. An allocation is said to be *fair or envy-free* if no agent prefers any other agent's bundle to his or her own. This idea is quite appealing and interesting because it treats people equally and symmetrically in distributive justice, it is ordinal in nature, and it compares welfare based on individual levels instead of interpersonal levels. From the viewpoint of welfare economics, a major drawback of this concept is that it is inconsistent with the Pareto optimality principle. In other words, a fair allocation may not be (socially) efficient. The income-fairness concept is another important indicator to measure economic equity. This concept is proposed by Pazner and Schmeidler [9]. An allocation is defined to be *income-fair* if there exists a competitive equilibrium price vector so that the allocation is in fact an equilibrium allocation and furthermore the income of every agent is the same. This concept is obviously quite appealing in term of economic equity. A problem with this idea lies in the fact that it is a hard-checking criterion and it is also rather difficult to identify conditions for the existence of such solution. Other kinds of concepts of fairness have also been developed and analyzed; see for example, Varian [16], Pazner and Schmeidler [10], among others. All of this work has dealt with the case of divisible goods. In the present study, we will treat large indivisible objects.

As it is well known, the presence of large indivisibilities poses a serious challenge for economic analysis; see for example, Arrow and Hahn [2], Scarf [11]. Nevertheless, a good deal of progress has been made in understanding several special fair allocation problems associated with indivisiblities. Svensson [15] observes that in a model if there are the same number of people as objects and there is a fixed amount of money, and if each agent consumes exactly one object and a certain amount of money, then every fair allocation is efficient. He further shows that such allocations will exist if all the objects are desirable and there is enough money. See also Maskin [7] for a similar model. Alkan, Demange and Gale [1] have shown that fairness still implies efficiency as long as every agent consumes only one object no matter how many objects are present. They provide several sufficient conditions for the existence of fair allocations. Su [13] and Yang [17] present more general existence conditions.

The model studied in this paper is more general than those mentioned above. In particular, we allow each agent to take more than one object. In the light of complexity theory, the problem of finding just an optimal assignment in the current model is substantially more difficult than the problem of finding an optimal assignment in a model where each agent just takes one object. It is therefore not surprising as it may be somewhat surprising that the present model has not received much attention in the literature. As one easily understands, because each agent is allowed to take more than one object, the various interactions among objects will generate new phenomena which do not arise in a model where every agent is required to take only a single object. In the example to be given in Section 2, we find that the fairness idea is not consistent with the principle of Pareto optimality anymore if each agent is going to take two objects. This is one primary motivation for us to carry out our previous study (see Sun and Yang [14]) and the current one to search for a new solution to this more general problem. In another example to be given at Section 3, we observe that even if there exist fair and efficient allocations, no such allocation is income-fair. This is another major motivation for us to undertake this study. To find a solution which meets the fundamental principles of equity, fairness and efficiency, we propose the concept of perfectly fair allocation. It is shown that every perfectly fair allocation is fair and efficient, income-fair and furthermore gives every person a maximal satisfaction. Then we establish a necessary and sufficient condition for the existence of a perfectly fair allocation. It is shown that there exists a perfectly fair allocation if and only if an associated linear program problem has a solution. As a result, we also provide a finite method of finding a perfectly fair allocation.

The rest of the paper is organized as follows. In Section 2 the general fair job assignment model is introduced, basic concepts are reviewed and the concept of perfectly fair allocation is then defined. In Section 3 we derive several properties of the perfect fairness concept. Those properties include efficiency, fairness, and income-fairness. An illustrative example is also given. Finally, in Section 4 a necessary and sufficient condition is established for the existence of a perfectly fair allocation and a quite general condition is also given for the existence of efficient and fair allocations.

2 The model

We first introduce some notation. Let I_k be the set of the first k positive integers and $\mathbb{R}^k$ the k-dimensional Euclidean space. Given a set T, $|T|$ denotes the cardinality of the set T. Sometimes we also use $\mathbb{R}^T$ to denote the $|T|$-dimensional Euclidean space with coordinates indexed by the elements of the set T. Furthermore, let x_T denote the term $\sum_{h\in T} x_h$ for any set T and any vector x.

Now we describe the general fair job assignment model. There are n agents or people, denoted by $A = I_n = \{1, 2, \cdots, n\}$, and there are γ times objects as many as people, denoted by $O = \{1, 2, \cdots, \gamma n\}$, and there is an amount M of money. The parameter γ can be any positive integer and will be called a *quota*. Objects are inherently indivisible, like jobs, tasks, houses, or offices. Objects can be desirable or undesirable. The amount of money can be positive or negative. The agents' preferences depend on the bundle of objects and the quantity of money they take. The problem is to allocate all the objects and all the money among all the agents in a way that is as fair and as efficient as possible with a constraint that every agent should get the same number of objects.

An assignment of objects is a partition of objects among agents so that each agent gets γ objects. Such an assignment can be expressed as a vector $\pi = (\pi(1), \pi(2), \cdots, \pi(n))$ satisfying $\pi(i) \cap \pi(j) = \emptyset$ for all $i \neq j$, and $|\pi(i)| = \gamma$ for all i, and $\cup_{i=1}^{n} \pi(i) = O$. Let $\Pi(A, O)$ represent the collection of all assignments of objects. A distribution of money is a vector x in $\mathbb{R}^O$ with $\sum_{j\in O} x_j = M$. The j-th componenet of x means that the amount x_j of money is attached to object j. Let $D(M)$ represent the collection of all distribution vectors of money. An allocation (π, x) consists of an assignment π of objects and a distribution x of money. At an allocation (π, x), agent i is assigned with objects $\pi(i)$ and the amount $x_{\pi(i)}$ of money. $(\pi(i), x_{\pi(i)})$ is called the bundle of agent i. In the case $x_{\pi(i)} < 0$, this means agent i pays the amount $|x_{\pi(i)}|$ of money.

Let $\mathcal{O} = \{B \mid B \subset O \text{ with } |B| = \gamma\}$. The preference of each agent $i \in I_n$ over objects and money will be represented by a utility function $u_i : \mathcal{O} \times \mathbb{R} \mapsto \mathbb{R}$. It is very natural to assume that $u_i(D, m)$ is a continuous and nondecreasing function in money (i.e., in m) for all $i \in A$ and all $D \in \mathcal{O}$. This assumption implies that money is always desirable. Money will be treated as a perfectly divisible commodity.

Definition 2.1 *An allocation $(\pi, x) \in \Pi(A, O) \times D(M)$ is fair or envy-free if for every $i \in A$, it holds*

$$u_i(\pi(i), x_{\pi(i)}) \geq u_i(\pi(j), x_{\pi(j)}), \quad \textit{for all } j \in A.$$

The definition says that at a fair allocation, no agent prefers the bundle of any other agent to his own bundle. The normative significance of the fairness idea lies in the fact that it treats people equally and symmetrically, it is ordinal in nature, and welfare comparisons are made only on an individualistic basis instead of interpersonal basis. As shown later, there is, however, one major drawback with this concept. It is inconsistent with the Pareto optimality principle. The Pareto optimality principle is recalled below.

Definition 2.2 *An allocation (π, x) is efficient or Pareto optimal if there is no other allocation (ρ, y) which makes everyone at least as well as at (π, x) and at least one agent strictly better off than at (π, x).*

In the case when there are the same number of objects as people, i.e. $\gamma = 1$, Svensson [15] has shown that the Pareto optimality is implied by envy freeness. This interesting property does not hold anymore if people are allowed to consume more than one object, as indicated by the following example.

Example 1. Consider the case in which there are two agents 1, 2 and there are four houses A, B, C, and D, and total money (say, dollar) M is equal to 20. Each agent is entitled to have two houses. The values assessed by the agents for the pairs of houses are given in Table 1, and agents' utility functions are given by $u_i(T, m) = V_i(T) + m$, $i = 1, 2$.

Table 1: The values of houses for both agents

	AB	AC	AD	BC	BD	CD
Agent 1	3	4	5	3	7	8
Agent 2	6	3	9	4	5	7

In this example the allocation in which agent 1 gets houses B and C and 12 dollars and agent 2 gets houses A and D and 8 dollars is a fair allocation. But this allocation is not Pareto optimal, since it is dominated by the allocation in which agent 1 gets houses C and D and 9 dollars, and agent 2 gets houses A and B and 11 dollars.

Since a fair allocation may not always be efficient, a logical question naturally arises: Is it possible to refine the concept of fair allocation so that the Pareto optimality will be attained? This motivates us to propose the following new solution concept: perfectly fair allocations.

Definition 2.3 *An allocation* $(\pi, x) \in \Pi(A, O) \times D(M)$ *is perfectly fair if it holds*

$$u_i(\pi(i), x_{\pi(i)}) \geq u_i(\rho(i), x_{\rho(i)}), \text{ for all } i \in A \text{ and all } \rho \in \Pi(A, O).$$

Clearly, the concept of perfectly fair allocation still retains symmetric, ordinal and individualistic properties as the fairness concept has. In the subsequent section we will investigate various properties of this new concept which the fairness concept does not have.

3 Several basic properties

In this section we will derive several interesting properties of the concept of perfectly fair allocation. The first lemma says that at a perfectly fair allocation, no agent envies any other agent and what each agent gets is what he likes best. Thus, the concept of perfectly fair allocation is indeed a proper refinement of fair allocation.

Lemma 3.1 *Every perfectly fair allocation is a fair allocation and gives every agent what he likes best.*

Proof: Let (π, x) be a perfectly fair allocation. Then for every $i \in A$ and every set $D \in \mathcal{O}$, we have

$$u_i(\pi(i), x_{\pi(i)}) \geq u_i(D, x_D).$$

The inequality says that every agent gets what he likes best. In particular, for every $i \in A$, we have

$$u_i(\pi(i), x_{\pi(i)}) \geq u_i(\pi(j), x_{\pi(j)}), \quad \text{for all } j \in A.$$

Thus, (π, x) must be a fair allocation. □

The following result shows that every perfectly fair allocation is (socially) efficient. This property indicates that the concept of perfectly fair allocation is also quite appealing and interesting from the viewpoint of welfare economics.

Theorem 3.2 *Every perfectly fair allocation is Pareto optimal.*

Proof: Let (π, x) be a perfectly fair allocation. Now suppose to the contrary that (π, x) is not efficient. Then there would exist another allocation (ρ, y) weakly preferred by all agents and strictly preferred by at least one agent. That is, it holds that

$$\sum_{i \in A} y_{\rho(i)} = M$$

and

$$u_i(\rho(i), y_{\rho(i)}) \geq u_i(\pi(i), x_{\pi(i)}), \forall i \in A; \tag{3.1}$$

and there is some $j \in A$ satisfying

$$u_j(\rho(j), y_{\rho(j)}) > u_j(\pi(j), x_{\pi(j)}). \tag{3.2}$$

Note that since (π, x) is perfectly fair, we have

$$\sum_{i \in A} x_{\pi(i)} = M$$

and

$$u_i(\pi(i), x_{\pi(i)}) \geq u_i(\rho(i), x_{\rho(i)}), \ \forall i \in A. \tag{3.3}$$

It follows from (3.1) and (3.3) that

$$u_i(\rho(i), y_{\rho(i)}) \geq u_i(\rho(i), x_{\rho(i)}), \ \forall i \in A.$$

Furthermore, it follows from (3.2) and (3.3) that

$$u_j(\rho(j), y_{\rho(j)}) > u_j(\rho(j), x_{\rho(j)}).$$

Since $u_i(T, \cdot)$, $i \in A$, $T \in \mathcal{O}$, are nondecreasing in money, we have that for all $i \in A$,

$$y_{\rho(i)} \geq x_{\rho(i)},$$

and

$$y_{\rho(j)} > x_{\rho(j)}.$$

This implies that

$$M = \sum_{h \in A} y_{\rho(h)} > \sum_{h \in A} x_{\rho(h)} = M,$$

yielding a contradiction. Therefore, (π, x) must attain Pareto optimality. □

In addition, we will show that the concept of perfectly fair allocation has yet another remarkable property, namely, it is consistent with income-fairness. The concept of income-fair allocation is suggested by Pazner and Schmeidler [9]. This concept can be reformulated in the present model as follows. Given an allocation (π, x), we construct a pure exchange economy $E(\pi, x)$ in which the bundle $(\pi(i), x_{\pi(i)})$ is viewed as agent i's initial endowment. We say that an allocation (π, x) is *an income-fair allocation* if there exists a vector $(p^1, p^2) \in \mathbb{R}^O \times \mathbb{R}$ such that (π, x) is a competitive equilibrium allocation, (p^1, p^2) is a competitive equilibrium price vector for the economy $E(\pi, x)$, and the potential income is the same for every agent.

Lemma 3.3 *Every perfectly fair allocation is an income-fair allocation.*

Proof: Let (π, x) be a perfectly fair allocation. Now define an economy in which agent i initially owns the bundle $(\pi(i), x_{\pi(i)})$. Let $p^1 = -x$ and $p^2 = 1$. Then the vector (p^1, p^2) is a competitive equilibrium price vector for the economy since for every agent i, perfect-fairness implies that

$$u_i(\pi(i), x_{\pi(i)}) \geq u_i(\rho(i), x_{\rho(i)}), \ \forall \rho \in \Pi(A, O).$$

In the economy, the potential income $I(i) = x_{\pi(i)} + p^1_{\pi(i)} = 0$ for all $i \in A$. Thus, (π, x) is an income-fair allocation. □

In the above analysis, we have noted that, at a perfectly fair allocation, some agent may have to pay a certain amount of money. In some situation, for example, when people do not have any money, we have to require that no agent should pay anything, namely, the amount $x_{\pi(i)} \geq 0$ for all agents. The following lemma states that if the total amount M of money is sufficiently large, every perfectly fair allocation gives each agent a nonnegative amount of money. This condition is similar to that given by Alkan et al. [1] for the case $\gamma = 1$.

Lemma 3.4 *Let the total amount M of money be so large that for every $i \in A$ and every $C, D \subseteq O$ with $|C| = |D| = \gamma$,*

$$u_i(C, \frac{M}{n-1}) \geq u_i(D, 0).$$

Then for every perfectly fair allocation (π, x) we have $x_{\pi(i)} \geq 0$ for all $i \in A$.

Proof: Recall that n in the formula is the number of agents. Let (π, x) be a perfectly fair allocation. Suppose to the contrary that there exists some agent $k \in A$ with $x_{\pi(k)} < 0$. Then we have

$$M = \sum_{i \in A} x_{\pi(i)} < 0 + \sum_{i \neq k} x_{\pi(i)}.$$

It is easy to see from this inequality that there exists some agent $l \in A$ so that $x_{\pi(l)} > \frac{M}{n-1}$. Then by assumption we have

$$\begin{aligned} u_k(\pi(l), x_{\pi(l)}) &> u_k(\pi(l), \tfrac{M}{n-1}) \\ &\geq u_k(\pi(k), 0) \\ &\geq u_k(\pi(k), x_{\pi(k)}). \end{aligned}$$

This means that agent k prefers the bundle of agent l to his own, yielding a contradiction. □

One may wonder if perfectly fair allocations exist. In Example 1 there indeed exists a perfectly fair allocation. It is easily verified that the allocation

$$((\pi(1), \pi(2)), (x_A, x_B, x_C, x_D)) = ((CD, AB), (5, 7, 6, 2))$$

is a perfectly fair allocation. However, perfectly fair allocations may not always exist as shown in the following example.

Example 2. Consider the case in which there are two agents 1, and 2, and there are four houses A, B, C, and D, and total money (say, dollar) M is equal to 20. Agents 1 and 2 each are entitled to have two houses. The values assessed by the agents for the pairs of houses are given in Table 2, and agents' utility functions are given by $u_i(T, m) = V_i(T) + m$, $i = 1, 2, 3$.

Table 2: The values of houses for both agents

	AB	AC	AD	BC	BD	CD
Agent 1	6	8	5	4	2	8
Agent 2	6	11	9	6	5	7

In this example there is a unique Pareto optimal assignment, namely, agent 1 gets houses C and D, agent 2 gets houses A and B. Suppose that this assignment can be constructed as a perfectly fair allocation. Then, we must have

$$9 \leq x_C + x_D \leq 9.5, \tag{3.4}$$

since a perfectly fair allocation is also fair. Furthermore, for agent 2, the following system of inequalities must have a solution.

$$\begin{aligned}
6 + x_A + x_B &\geq 11 + x_A + x_C \\
6 + x_A + x_B &\geq 9 + x_A + x_D \\
6 + x_A + x_B &\geq 6 + x_B + x_C \\
6 + x_A + x_B &\geq 5 + x_B + x_D \\
x_A + x_B &= 20 - (x_C + x_D).
\end{aligned}$$

It follows from the first four inequalities that

$$1 + 2(x_A + x_B) \geq 8 + 2(x_C + x_D).$$

Substituting $x_A + x_B = 20 - (x_C + x_D)$ into the inequality, we obtain

$$x_C + x_D \leq 33/4 < 9,$$

yielding a contradiction to the inequality (3.4). Thus there does not exist any perfectly fair allocation in this example, even though there exists an efficient and fair allocation. This example also shows that fairness together with efficiency does not imply perfect-fairness. The reader can also verify that the unique efficient and fair allocation in this example is not income-fair, either.

4 Existence theorems

In this section we present both a general condition for the existence of an efficient and fair allocation and a necessary and sufficient condition for the existence of a perfectly fair allocation. We will restrict our attention to the case where every agent has quasi-linear utilities in money. The analysis seems to be significantly more difficult without this restriction. When agents have quasi-linear utilities in money, then agents' utility functions can be expressed as $u_i(T, x) = V_i(T) + x$ for all $i \in A$, $T \in \mathcal{O}$, and $x \in \mathbb{R}$. In this case we will use $\mathcal{E} = ((V_i, i \in A), O, M)$ to represent the model. Given a model $\mathcal{E} = ((V_i, i \in A), O, M)$, we call an assignment $\pi \in \Pi(A, O)$ an *optimal assignment* if $\sum_{i \in I_n} V_i(\pi(i)) \geq \sum_{i \in I_n} V_i(\rho(i))$ for every $\rho \in \Pi(A, O)$. The problem of finding an optimal assignment for $\gamma \geq 2$ is a very difficult one from the computational complexity viewpoint, because it is an NP-hard problem even for $\gamma = 2$. See for example Papadimitriou and Steiglitz [8] and Fujishige [5]. However, the problem of finding an optimal assignment for $\gamma = 1$ is an easy one. This is the classical job assignment problem to be discussed shortly.

To show the existence of an efficient and fair allocation, we first recall the classical job assignment problem; see Dantzig [3]. In this model there are n people and n tasks. Each person is going to be assigned with one task. Let $t(i, j)$ denote the profit for person i to perform task j and let $T = [t(i, j)]$ denote the n by n matrix. The goal is to find an assignment of tasks among people so that the total profit is maximal. In this model, an assignment is a permutation of the n elements of I_n. Let Θ be the collection of all assignments. An assignment $\pi \in \Theta$ is an *optimal assignment* if $\sum_{i \in I_n} t(i, \pi(i)) \geq \sum_{i \in I_n} t(i, \tau(i))$ for every $\tau \in \Theta$.

Recall the following duality theorem from linear programming (see Dantzig [3]), which has been used by Koopmans and Beckman [6], Shapley and Shubik [12], and Alkan et al. [1] for related models.

Lemma 4.1 *Let $T = [t(i, j)]$ be an $n \times n$ matrix. If $\pi \in \Theta$ is an optimal assignment, there exist two n-vectors v and w such that*

$$v_i + w_j \geq t(i, j), \ \forall\, i,\ j \in I_n$$

and

$$v_i + w_{\pi(i)} = t(i, \pi(i)),\ \forall i \in I_n.$$

Theorem 4.2 *Given a model $\mathcal{E} = ((V_i, i \in A), O, M)$, then there exists at least one optimal assignment. For each optimal assignment π, there exists a distribution $2n$-vector x of money M such that (π, x) is an efficient and fair allocation.*

Proof: The first statement is obvious, since there are only a finite number of assignments. Now we prove the second part. Let $\pi \in \Pi(A, O)$ be such an optimal assignment. We will view the set $\pi(i)$ of objects as task i and $V_i(\pi(j))$ as the profit for agent i to perform task j. Define $t(i, j) = V_i(\pi(j))$ and $T = [t(i, j)]$ the n by n matrix. Thus, we have defined the associated job assignment problem. Clearly, $\rho = (\rho(1), \rho(2), \cdots, \rho(n)) = (1, 2, \cdots, n)$ is an optimal job assignment. Since ρ is an optimal assignment, it follows from Lemma 4.1 that there exists v and w such that

$$v_i + w_j \geq t(i, j),\ \forall\, i,\ j \in I_n$$

and

$$v_i + w_{\rho(i)} = t(i, \rho(i)),\ \forall i \in I_n.$$

From the above inequalities we obtain

$$t(i, \rho(i)) - w_{\rho(i)} \geq t(i, j) - w_j,\ \forall\, i,\ j\ \in I_n.$$

Let $y_i = -w_i$, $\delta = (M - \sum_{i \in I_n} y_i)/n$, and $z_i = y_i + \delta$ for each $i \in I_n$. Define $z = (z_1, \cdots, z_n)$. Then we have

$$t(i, \rho(i)) + z_{\rho(i)} \geq t(i, j) + z_j,\ \forall\, i,\ j \in I_n$$

and

$$\sum_{i \in I_n} z_i = M.$$

Let $x \in D(M)$ so that $x_{\pi(i)} = z_i$ for all $i \in I_n$. Clearly, (π, x) is an efficient and fair allocation. □

As Example 2 indicates that perfectly fair allocations may not always exist, this motivates a natural question: Under what circumstance does a perfectly fair allocation exist? The remaining section is to present a necessary and sufficient condition for the existence of a perfectly fair allocation.

Given $\emptyset \neq S \subset A$ and $\emptyset \neq T \subset O$, we say that S and T are *compatible* if there exists some $\pi \in \Pi(A, O)$ satisfying $\cup_{i \in S} \pi(i) = T$. Obviously, A and O are compatible. Let S and T be compatible and define

$$\Pi(S, T) = \{\bar{\pi} \mid \bar{\pi} = (\pi(i) \mid i \in S) \text{ for some } \pi \in \Pi(A, O) \text{ with } \cup_{i \in S} \pi(i) = T\},$$

and

$$V(S, T) = \text{maximize}_{\pi \in \Pi(S,T)} \{\sum_{i \in S} V_i(\pi(i))\}.$$

We will associate the fair job assignment model $\mathcal{E}$ with the following linear program problem:

$$\begin{aligned} &\text{minimize } \textstyle\sum_{i \in A} v_i \\ \text{subject to } &\textstyle\sum_{i \in S} v_i - \sum_{j \in T} x_j \geq V(S, T), \quad \text{for all compatible sets } S \subset A, \ T \subset O \\ &\textstyle\sum_{j \in O} x_j = M. \end{aligned}$$

Observe from the linear program that

$$v_i \geq \frac{M + \max_{\pi \in \Pi(A,O)} \sum_{h \in A} V_i(\pi(h))}{n}$$

for all $i \in A$. This means that each v_i is bounded below. Since we minimize the objective function $\sum_{i \in A} v_i$, the problem will always have an optimal solution. It is also easy to see that the optimal value of the linear program is no less than the value of $M + V(A, O)$.

We are now ready to introduce the main existence result of this paper which states a necessary and sufficient condition for the existence of a perfectly fair allocation.

Theorem 4.3 *Given a model $\mathcal{E} = ((V_i, i \in A), O, M)$, there exists a perfectly fair allocation if and only if the linear program has an optimal solution with its value equal to $M + V(A, O)$.*

Proof: Let (v, x) be an optimal solution of the linear program with its value equal to $M + V(A, O)$. Then there exists an element $\pi \in \Pi(A, O)$ so that

$$\sum_{i \in A} v_i = \sum_{j \in O} x_j + \sum_{i \in A} V_i(\pi(i)) = M + V(A, O).$$

We can rewrite the equality as

$$\sum_{i \in A} v_i - \sum_{i \in A} x_{\pi(i)} = \sum_{i \in A} V_i(\pi(i)). \tag{4.5}$$

Furthermore, it follows from the linear program that for all pairs (S, T) of compatible sets

$$\sum_{i \in S} v_i - \sum_{j \in T} x_j \geq V(S, T).$$

In particular, it holds that for all $i \in A$ and all $\rho \in \Pi(A, O)$

$$v_i - x_{\rho(i)} \geq V_i(\rho(i)).$$

Of course, it is true that

$$v_i - x_{\pi(i)} \geq V_i(\pi(i))$$

for all $i \in A$. This together with equation (4.5) implies that

$$v_i - x_{\pi(i)} = V_i(\pi(i))$$

for all $i \in A$. Therefore we have

$$x_{\pi(i)} + V_i(\pi(i)) \geq x_{\rho(i)} + V_i(\rho(i))$$

for all $i \in A$ and all $\rho \in \Pi(A, O)$. By definition (π, x) is a perfectly fair allocation.

On the other hand, let (π, x) be a perfectly fair allocation. By definition, for all $i \in A$ and all $\rho \in \Pi(A, O)$, it holds

$$V_i(\pi(i)) + x_{\pi(i)} \geq V_i(\rho(i)) + x_{\rho(i)}.$$

Let $v_i = V_i(\pi(i)) + x_{\pi(i)}$ for each $i \in A$. Note that $\sum_{j \in O} x_j = M$. Let ρ be an arbitrary element in $\Pi(A, O)$. Then it holds that

$$V_i(\pi(i)) + x_{\pi(i)} \geq V_i(\rho(i)) + x_{\rho(i)}$$

for all $i \in A$. It follows that $\sum_{i \in A} V_i(\pi(i)) \geq \sum_{i \in A} V_i(\rho(i))$. This implies that $\sum_{i \in A} v_i = \sum_{i \in A}(V_i(\pi(i)) + x_{\pi(i)}) = M + V(A, O)$. We still have to show that (v, x) also satisfies the first constraints of the linear program. Suppose to the contrary that some constraint is violated. Then there would be a nonempty subset S of A and a subset T of O compatible so that

$$\sum_{i \in S} v_i - \sum_{j \in T} x_j < V(S, T).$$

This implies that there exists some $\bar{\rho} \in \Pi(S, T)$ so that

$$\sum_{i \in S}(v_i - x_{\bar{\rho}(i)}) < \sum_{i \in S} V_i(\bar{\rho}(i)).$$

It is clear from the inequality that there is some $i \in S$ so that

$$v_i - x_{\bar{\rho}(i)} < V_i(\bar{\rho}(i)).$$

But this is impossible, since we have

$$v_h \geq V_h(\rho(i)) + x_{\rho(h)}$$

for all $h \in A$ and all $\rho \in \Pi(A, O)$. This demonstrates the theorem. □

Given a perfectly fair allocation (π, x), we call $V_i(\pi(i)) + x_{\pi(i)}$ *the maximal attainable share value* of agent i. As shown in the theorem, agent i's maximal attainable share value is equal to v_i, a solution of the linear program if the linear program has an optimal solution (v, x) with its value equal to $M + V(A, O)$. Knowing each agent's maximal attainable share value and the money distribution vector x, we can easily find an assignment π of objects and so we find a perfectly fair allocation (π, x). It is somehow surprising that the linear program turns out to be such a bliss to the resolution of the fair job assignment problem.

We will illustrate the above theorem with Example 1 and leave Example 2 to the interested reader. With respect to Example 1 to see whether there is a perfectly fair allocation or not we only need to check the following linear program problem:

$$
\begin{aligned}
\text{minimize} \quad & v_1 + v_2 \\
\text{subject to} \quad & v_1 - x_A - x_B \geq 3 \\
& v_1 - x_A - x_C \geq 4 \\
& v_1 - x_A - x_D \geq 5 \\
& v_1 - x_B - x_C \geq 3 \\
& v_1 - x_B - x_D \geq 7 \\
& v_1 - x_C - x_D \geq 8 \\
& v_2 - x_A - x_B \geq 6 \\
& v_2 - x_A - x_C \geq 3 \\
& v_2 - x_A - x_D \geq 9 \\
& v_2 - x_B - x_C \geq 4 \\
& v_2 - x_B - x_D \geq 5 \\
& v_2 - x_C - x_D \geq 7 \\
& v_1 + v_2 - x_A - x_B - x_C - x_D \geq 14 \\
& x_A + x_B + x_C + x_D = 20.
\end{aligned}
$$

The value of $V(A, O)$ is 14 and $M = 20$, and the linear program problem has an optimal solution $(v_1, v_2, x_A, x_B, x_C, x_D) = (16, 18, 5, 7, 6, 2)$ with its value $34 = M + V(A, O) = 20 + 14 = 34$. According to Theorem 4.3, there exists a perfectly fair allocation.

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