Believe and Let Believe:

Axiomatic Foundations for Belief Dependent Utility Functionals

By

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Believe and Let Believe: Axiomatic Foundations for Belief Dependent Utility Functionals.

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Abstract

A large body of experimental data demonstrates that people's beliefs influence their well-being beyond the indirect effect through the actions taken. I present a model that incorporates beliefs into an agent's utility function. The paper provides axiomatic foundations for a special class of non-additive utility indices defined over infinite streams of beliefs and actions. I assume that: 1. there exists a (null) belief that does not have any effect on future preferences; 2. the agent has finite memory - only finite histories have an effect on current preferences; and 3. if the agent knows she will not be getting any additional future information, she prefers today's beliefs to be consistent with her past choices. Preferences satisfying these assumptions admit a generalized discounted utility representation in which all terms depend on both actions and beliefs. Experimental testability of the proposed framework is also discussed.

Key words: Axiomatic foundations, beliefs, time preference, utility.

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"I don't believe you! Flowers are weak creatures. They are naïve. They reassure themselves as best as they can. They believe that their thorns are terrible weapons."

- The Little Prince, Antoine de Saint-Exupery.

1 Introduction

In most decision models in which uncertainty prevails, the fundamental atom for calculation of well-being is action rather than beliefs about the germane parameter realizations. That is, the utilities we commonly use contain states and actions as arguments. Beliefs enter the decision making process only via expected utility calculations of agents contemplating different feasible actions.

The psychology literature identifies an abundance of phenomena hinting that beliefs per-se affect people's sense of well-being. To mention a few, *cognitive dissonance* states that after having taken an action people tend to change their beliefs about the relative agreeableness of this action. *Confirmatory bias* refers to the phenomenon of people interpreting new evidence in ways that confirm their current beliefs (see [13, 14, 17]). Theories of self-esteem concentrate on people's beliefs about themselves, and their search for information that weakens or reinforces these beliefs (see [4]). *Hindsight bias*, a special memory distortion, occurs when people make a judgment or choice and are later asked to recall their judgment. If, in the interim, they are told what the correct judgment would have been, their memory of their own judgment may become biased toward the new information (see, [10]). In social contexts, people exhibit different behaviors for different beliefs about their opponents' intentions or types, even when the material incentives are maintained constant (see [6]).

The current paper suggests a generalization of the standard discounted utility model incorporating beliefs as additional arguments. More specifically, I provide an axiomatic foundation for considering utility functions that depend directly on beliefs. I assume that agents have a preference ranking over sequences of beliefs and actions. Hence, the starting point is one in which beliefs have a direct effect on agents' perception of well-being.

Psychological observations are used to motivate axioms on the agents' preferences over such sequences. In rough terms, I assume that: 1. there exists a (null) belief that does not have any effect on future preferences, no matter what action the agent takes when holding that belief; 2. only finite histories have an effect on current preferences. That is, the agent has bounded memory and is aware of it; and 3. if the agent knows she will not be getting any additional information from tomorrow on, she prefers today's beliefs to be consistent with yesterday's action and belief. Under these assumptions any von Neumann - Morgenstern utility representation depends on both action and belief. Moreover, this dependency takes quite a specific form. In the standard (exponential) discounted utility the t'th term is a function of action (consumption) at time t (see, e.g., Epstein [8]). Here, the utility is a discounted sum where the t'th term is a function of a triplet: the action at time t and the beliefs of time t and $t+1$. This function is the same for all t. Thus, the presentation theorem, Proposition 1, provides a foundation for a generalized form of discounted utility functions in which the beliefs enter the utility function directly.

One of the corollaries of such a representation is that rates of time preference may depend on agents' beliefs and not solely on their consumption levels. In particular, agents with the same history of consumption decisions may exhibit different rates of time preference if their beliefs about the relevant states of the world diverge.

In Section 4 I consider an equivalent axiomatic construction to that presented in Sections 2 and 3, in which the preference ranking is over streams of triplets of (signal, signal accuracy, action). The use of (signal, signal accuracy) substitutes the term corresponding to beliefs in the original formulation and is intended to provide a more natural framework for bringing the current theory to the experimental lab. Once the subject's prior beliefs are elicited (using, e.g., Anscombe and Aumann [3]), the subject would be required to compare streams of signals and actions, which are more common objects of choice than future beliefs. The equivalence of such a construction to that presented in the first part of the paper is summarized in Proposition 3.

There have been several attempts in the literature to study the implications of incorporating beliefs directly into the agent's utility function. Geanakoplos, Pearce, and Stacchetti [9] introduced the notion of psychological games, games in which players' utilities depend on their hierarchies of beliefs. They provide an existence theorem, as well as a number of specific examples.

Akerlof and Dickens [1] propose a simple model of cognitive dissonance relevant to workers in hazardous occupations. They look at a two period model in which cost-effective safety equipment is available only in the second period and consider agents who exhibit a cognitive dissonance in that they convince themselves in the first period that they are not unsafe when they are not protected. Formally, agents change their beliefs depending on the action they themselves took (use safety equipment or not), even when no additional information is given to them. As a consequence, agents purchase too little of the safety equipment in the second period.

Bodner and Prelec [7] and Akerlof and Kranton [2] introduce the ideas of self-signaling and identity. These authors consider agents who value their beliefs about themselves and thus choose actions that not only maximize some instrumental utility, but also a utility that reflects their self-perceptions.

Benabou and Tirole [5] and Koszegi [12] propose models that explain overconfidence. They assume very specific functional forms that include beliefs about the agents' type (or ability). Furthermore, the agents' beliefs are a choice variable in the sense that the agents choose when to stop receiving information.

Yariv [17] considers a finite learning model in which: 1. beliefs serve as arguments in the specific utility function studied and 2. agents choose both actions and beliefs at each stage of the process. Such a framework helps organize some observations in information economics such as under and overconfidence, excess persistence in action choices, and preferences for less accurate information.

To summarize, the link between all of these papers is the acknowledgement of the direct effect beliefs have on agents' sense of well-being. The current paper provides an axiomatic foundation for these types of utilities.

The structure of the paper is as follows. Section 2 contains the general setup, while Section 3 provides the main representation results. Section 4 contains a modification of the axiomatization to an experimental setting. Section 5 concludes. Proofs of the main results of the paper are relegated to the Appendix.

2 Basic Setup and Results

I consider an axiomatic structure similar in spirit to that used for consumption streams (see, Epstein [8] and references therein).

Assume there is a finite set of states of the world Ω.

A belief over Ω is a vector in the $|\Omega|-1$ dimensional simplex, $\Delta_{|\Omega|}$.[1]

I assume that at each stage t the agent holds a belief $\mu_t \in \Delta_{|\Omega|}$. These beliefs can naturally arise in some learning process in which at each stage a signal realization and its accuracy are reported, as will be discussed later in the paper. In this section I take the beliefs as given.

The agent has an (arbitrary) set of actions A endowed with a topology $\Im$.

Let $B = \Delta_{|\Omega|} \times A$. An element $\Theta \in B$ is a pair $\Theta = (\mu, a)$. That is, B denotes the set of pairs (belief, action).

B is endowed with the product topology (Euclidean for $\Delta_{|\Omega|}$, $\Im$ for A).

At any stage t, having held the belief μ_{t-1} and taken the action a_{t-1} at stage $t-1$, I assume a preference relation $\succsim_t$ over sequences of (belief, action) pairs. So $\succsim_t$ is defined on sequences of the form $(\Theta_t, \Theta_{t+1}, \Theta_{t+2}, ...)$.

I will further assume that these preferences are stationary, so that $\succsim \equiv \succsim_t = \succsim_{t'}$ for all t, t'.

At each stage t we assume that the agent remembers only one stage backwards and so the preference $\succsim$ depends on all current and future pairs Θ_i, $i \geqslant t$, and also on Θ_{t-1}. I will assume a

[1] I assume Ω is finite for the sake of exposition simplicity. In fact, all of the results would carry through with arbitrary topological space Ω. We would then define the topological space of additive measures of $(\Omega, R(\Omega))$ where $R(\Omega)$ is the Borel $\sigma-$field of Ω.

one stage memory, so in fact the time t preferences will depend only on Θ_{t-1}. At time t, the agent is considering objects of the form $(\omega, \Theta_{t-1}; \Theta_t, \Theta_{t+1}, ...)$ where ";" separates between the prior and the sequence of belief and action pairs from period t and on. In fact, we could assume a memory for any fixed and finite number of stages and the analysis, while more cumbersome, would have remained qualitatively the same.

Let B^∞ denote the set of sequences $(\Theta_1, \Theta_2, \Theta_3, ...)$ where $\Theta_i \in B, \quad \forall i \geqslant 1$.

Endow B^∞ with the product topology. $R(B^\infty)$ denotes the Borel σ-field of B^∞.

$M(B^\infty)$ is the space of all lotteries over sequences of pairs $(\Theta_1, \Theta_2, \Theta_3, ...)$. Formally, $M(B^\infty)$ is the set of all probability measures (countably additive) on the measure space $(B^\infty, R(B^\infty))$.

B^∞ is embedded in $M(B^\infty)$ via the isomorphism defined by $b \to p_b \in M(B^\infty)$, where p_b is the element that gives probability 1 to $\{B^\infty\}$. I will often abuse notation and identify $b \in B^\infty$ with p_b.

Assume that each belief μ induces a von Neumann-Morgenstern preference relation over actions $\succsim_\mu$. Let v_μ represent the corresponding utility index (unique up to a linear transformation). Intuitively, given a belief, there are corresponding utilities from taking different actions, and hence an ordering of actions may be achieved via a belief.

v_μ induces a preference relation over beliefs. That is, we can think of each action a as inducing a preference relation over beliefs $\succsim_a$: $\mu \succsim_a \mu'$ iff $v_\mu(a) \geqslant v_{\mu'}(a)$. This relation satisfies the von Neumann-Morgenstern axioms and I can therefore write $v_a(\mu) = \alpha v_\mu(a) + \beta$ for $\alpha > 0$ (in particular, if we can write down v_μ by experimental observations, we can deduce v_a).

I make the following assumptions on the preference relation $\succsim$.

The first axiom states that there exists a belief such that, combined with any action, has no effect on the ranking of future (belief, action) streams. For example, if $\Omega = \{$*highest demand for economists, highest demand for aerobics instructors*$\}$ then believing today that these two events are equally probable (in our notation, belief $\mu = \frac{1}{2}$) is likely not to affect future preferences. If beliefs are tilted one way or the other, confirmatory bias or cognitive dissonance, psychological needs for consistency between beliefs and between beliefs and actions, may arise. Stated formally,

Axiom 1 (existence of null history). There exist μ_- such that for all $a_- \in A$, $\omega \in \Omega$, $\Theta = (\mu, a) \in B$, and $p, q \in M(B^\infty)$,

$$(\omega, \mu_-, a_-; \Theta, p) \succsim (\omega, \mu_-, a_-; \Theta, q) \Leftrightarrow (\omega, \Theta; p) \succsim (\omega, \Theta; q).$$

The second axiom poses limited memory. For the sake of presentational convenience I assume an agent remembers beliefs from the previous stage only and is forward looking in the sense that she is aware of this limit. The essence of Axiom 2 is that yesterday's beliefs and actions do not have a direct effect on tomorrow's preferences. At each time period, the cognitive dissonance and confirmatory bias effects are captured by last period's belief and action.

Axiom 2 (one-stage memory). For all $\Theta_0, \Theta_0', \Theta_1 \in B$, $\omega \in \Omega$, and for all $p, q \in M(B^\infty)$,

$$(\omega, \Theta_0; \Theta_1, p) \succsim (\omega, \Theta_0; \Theta_1, q) \Leftrightarrow (\omega, \Theta_0'; \Theta_1, p) \succsim (\omega, \Theta_0'; \Theta_1, q).$$

Note: Throughout the paper it is assumed that the agent has a one-stage memory block. While the current axiomatic structure could be worked out in a completely analogous manner to a similar framework in which the agent has a memory block of size longer than one, the psychology literature dealing with memory, hints that considering longer streams of past events may confound such phenomena as selective memory and peak-end memory (see [11, 16]). Consideration of these issues is left for future research.

Note that Axioms 1 and 2 both deal with the agent's treatment of far histories, but do not imply one another. Indeed, the following is an example of a preference relation satisfying Axiom 2 but not Axiom 1.

Example 1 (Axiom 2 does not imply Axiom 1). Consider the following preference ordering:

$$(\omega, \mu_0, a_0; p) \succsim (\omega, \mu_0', a_0'; q) \Leftrightarrow$$

$$E_p \mu_1 \geqslant E_q \mu_1.$$

That is, the expectation of the belief at $t = 1$ with respect to p is no lower than the expectation of the belief at $t = 1$ with respect to q. In paricular, for atomic distributions:

$$(\omega, \mu_0, a_0; \mu_1, a_1, \mu_2, a_2, ...) \succsim (\omega, \mu_0', a_0'; \mu_1', a_1', \mu_2', a_2', ...) \Leftrightarrow$$

$$\mu_1 \geqslant \mu_1'$$

Axiom 2 is satisfied generically (under the conditions of the axiom there is indifference on both sides). However, Axiom 1 is not satisfied. Indeed, for any μ_-,μ there exist $\mu_2 > \mu_2'$ such that for all $a_0, a_1, a_2, a_2' \in A$, $p, q \in M(B^\infty)$ so that:

$$\begin{aligned}(\omega, \mu_-, a_0; \mu_1, a_1, \mu_2, a_2, p) &\simeq (\omega, \mu_-, a_0; \mu_1, a_1, \mu_2', a_2', q) \text{ but} \\ (\omega, \mu_1, a_1; \mu_2, a_2, p) &\succ (\omega, \mu_1, a_1; \mu_2', a_2', q).\end{aligned}$$

The third axiom is rather technical. It states the existence of a standard von Neumann-Morgenstern utility function over sequences in our space B^∞.

Axiom 3 (continuity). There exists a von Neumann-Morgenstern utility index U for $\succsim$, which is a continuous function on its domain B^∞.

The existence of U may be proven from assumptions on $\succsim$.

Axiom 4 formalizes the taste for consistency that has been observed in psychological experiments. It postulates that, ceteris paribus, the stream is perceived preferable when the beliefs are more justifying of the actions. Note that we assume here that only the previous stage action is taken into consideration (equivalently, one *memorizes* only one stage). This is a technical qualification that is made for the sake of presentational simplicity and can be generalized. Formally,

Axiom 4 (consistency). For all $\omega, \Theta_0 = (\mu_0, a), \Theta_1 = (\mu_1, a), \Theta_1' = (\mu_1', a)$, preferences satisfy:

$$(\omega, \Theta_0; \Theta_1, \Theta_1, \Theta_1, ...) \succsim (\omega, \Theta_0; \Theta_1', \Theta_1', ...) \text{ when } v_a(\mu_1) \geqslant v_a(\mu_1').$$

3 Main Results

Axioms 1 and 2 together imply separability of any von Neumann Morgenstern utility over time. Intuitively, limited memory implies that the agent foresees her future inability to distinguish between the actual belief and action, and the null history two periods back. This in turn implies that at each stage $t+1$ the pair of belief and action two stages back, Θ_{t-1}, cannot have an effect on preferences. The sort of linearity inherent in any von Neumann-Morgenstern utility implies that the utility at time t is a sum of some instantaneous term and a term that represents the preferences from time $t+1$ on. It follows that while the instantaneous term may depend on Θ_{t-1}, the next term will not. Hence we get separability. Proposition 1 formalizes this intuition by providing a utility representation that relies only on Axioms 1-3.

Proposition 1 (utility representation): $\succsim$ *satisfies Axioms 1-3 if and only if* U *takes the form:*

$$U(\omega, \mu_0, a_0; \mu_1, a_1, \mu_2, a_2, ...) = \sum_{i=0}^{\infty} u_\omega(\mu_i, a_i, \mu_{i+1}) \exp(-\sum_{j=0}^{i-1} \rho_\omega(\mu_j, a_j, \mu_{j+1}))$$

(under the convention that $\sum_{j=0}^{-1} = 0$*).*

$\succsim$ is not represented uniquely. As in von Neumann-Morgenstern theory or standard discounted utility theory (Epstein [8]), it is possible to identify the equivalence class of indices (u_ω, ρ_ω) that represent $\succsim$.

Indeed, we say that the pair (u, ρ) represents $\succsim$ if for all ω

$$U(\omega, \mu_0, a_0; \mu_1, a_1, \mu_2, a_2, ...) = \sum_{i=0}^{\infty} u_\omega(\mu_i, a_i, \mu_{i+1}) \exp(-\sum_{j=0}^{i-1} \rho_\omega(\mu_j, a_j, \mu_{j+1}))$$

is a von Neumann-Morgenstern utility index of $\succsim$.

The following corollary describes the relation that links pairs representing $\succsim$.

Corollary (uniqueness): *Assume $\succsim$ is such that for all $\omega \in \Omega, (\mu_0, a_0) \in [0,1] \times A$ there exist $b, b' \in B$ for which $(\omega, \mu_0, a_0; b) \succ (\omega, \mu_0, a_0; b')$. (u, ρ) and $(\widetilde{u}, \widetilde{\rho})$ both represent $\succsim$ satisfying Axioms 1-3 if and only if there exist $\alpha_\omega, \beta_\omega \in R, \beta_\omega > 0$ such that, over $[0,1] \times A \times [0,1]$,*

$$\widetilde{u}_\omega = \alpha_\omega(1 - \exp(-\rho_\omega)) + \beta_\omega u_\omega \quad \text{and} \quad \widetilde{\rho}_\omega = \rho_\omega.$$

Axiom 4 provides more restrictions on the class of preferences the agent may possess. Proposition 2 provides the restricted utility representation when Axiom 4 is satisfied in addition to Axioms 1-3.

Proposition 2 (refined representation): *Assume $\succsim$ satisfies Axioms 1-4 and, using the notation of Proposition 2, $\rho_\omega(a, \mu) \equiv \rho$ is a constant for all $\omega \in \Omega$. Define $\delta \equiv \exp(-\rho)$. Then U takes the form:*

$$U(\omega, \mu_0, a_0; s_1, q_1, a_1, s_2, q_2, a_2, ...) =$$

$$= \sum_{i=0}^{\infty} [\alpha^*_{\omega 1}(\mu_i, a_i) + \alpha^*_{\omega 2}(\mu_{i+1}, a_i) + \beta^*_\omega(\mu_i, a_i) v_{a_i}(\mu_{i+1})]\delta^i.$$

(under the convention that $\sum_{j=0}^{-1} = 0$).

Hence, Axiom 4 specifies some separability between terms that contain consecutive beliefs.

Example (directional confidence). Assume the agent's task is to try and guess the state of the world $\omega \in \Omega = \{L, R\}$. She gets one util for making a correct guess and 0 otherwise. Hence, an action here is $a \in \{L, R\}$ and if μ denotes the probability of $\omega = L$ then $v_\mu(a) = \mu I_L(a) + (1 - \mu) I_R(a)$. In this case, if a_t is chosen to maximize v_{μ_t} at each stage t then

$$v_{\mu_t}(a_t) = \max\{\mu_t, 1 - \mu_t\},$$

$$v_{\mu_{t+1}}(a_t) = \mu_{t+1} I_L(a_t) + (1 - \mu_{t+1}) I_R(a_t) = \begin{cases} \mu_{t+1} & \mu_t > \frac{1}{2} \\ 1 - \mu_{t+1} & \mu_t < \frac{1}{2} \end{cases}$$

If $\alpha^*_1 + \alpha^*_2 \equiv 0$, and $\beta^* \equiv 1$, we get:

$$E_\omega U(\omega, \mu_0, a_0; s_1, q_1, a_1, s_2, q_2, a_2, ...) = \sum_{i=0}^{\infty} [v_{\mu_i}(a_i) + w(\mu_i, \mu_{i+1})]\delta^i$$

where

$$w(\mu_i, \mu_{i+1}) = \begin{cases} \mu_{i+1} - \mu_i & \mu_i > \frac{1}{2} \\ \mu_i - \mu_{i+1} & \mu_i < \frac{1}{2} \end{cases} .$$

(for $\mu_i = \frac{1}{2}$, the value of $w(\mu_i, \mu_{i+1})$ depends on the specification of the corresponding a_i).

This specification corresponds to what Yariv [17] calls a *taste for directional confidence.* If $\mu_t > \frac{1}{2}$, the agent believes at time t that $\omega = L$ is more likely. $\mu_{t+1} > \mu_t$ then means that the agent believes at time $t+1$ that $\omega = L$ is more likely, and moreover, she is stronger in her convictions. More formally, the variance in her probability estimate is lower and she is more confident. In that scenario, the agent receives a positive (belief) utility. If $\mu_{t+1} < \mu_t$ then the agent is less confident (or has lower variance, but changed her mind to $\omega = R$ being more likely) and her (belief term) utility decreases. Similarly for $\mu_t < \frac{1}{2}$.

Note (the effective rate of time preference): Consider $A = [0, \overline{a}]$, endowed with the Euclidean topology. Acions may then serve as metaphors for consumption levels. The effective rate of time preference can then be calculated by marginal rates of substitution between actions.

Assuming Axioms 1-3 and using Proposition 1,

$$\begin{aligned} \frac{\partial U}{\partial a_1} &= \frac{\partial u_\omega(\mu_1, a_1, \mu_2)}{\partial a} \exp(-\rho_\omega(\mu_0, a_0, \mu_1)) - \\ &\quad - \frac{\partial \rho_\omega(\mu_1, a_1, \mu_2)}{\partial a} \sum_{i=2}^{\infty} u_\omega(\mu_i, a_i, \mu_{i+1}) \exp(-\sum_{j=0}^{i-1} \rho_\omega(\mu_j, a_j, \mu_{j+1}). \end{aligned}$$

$$\begin{aligned} \frac{\partial U}{\partial a_2} &= \frac{\partial u_\omega(\mu_2, a_2, \mu_3)}{\partial a} \exp(-\rho_\omega(\mu_0, a_0, \mu_1) - \rho_\omega(\mu_1, a_1, \mu_2)) - \\ &\quad - \frac{\partial \rho_\omega(\mu_2, a_2, \mu_3)}{\partial a} \sum_{i=3}^{\infty} u_\omega(\mu_i, a_i, \mu_{i+1}) \exp(-\sum_{j=0}^{i-1} \rho_\omega(\mu_j, a_j, \mu_{j+1}). \end{aligned}$$

Looking at streams with $\mu_i \equiv \mu$, $a_i \equiv a$ for all i, and let $r_\omega(\mu, a)$ be defined by:

$$\frac{1}{1 + r_\omega(\mu, a)} = \frac{\partial U/\partial a_2(\omega, \mu, a; \mu, a, \mu, a, ...)}{\partial U/\partial a_1(\omega, \mu, a; \mu, a, \mu, a, ...)}.$$

Solving for the relevant geometric series we get that:

$$r_\omega(\mu, a) = e^{\rho_\omega(\mu, a, \mu)} - 1$$

$r_\omega(\mu, a)$ is the effective time preference.

It follows from this analysis that agents' effective time preference may depend on their stream of beliefs. In particular, agents' effective time preference may vary across agents even when past actions coincide, if agents' past beliefs vary.

4 Taking an Experimental Outlook

In this section I provide a suggestion for an experimental approach. So far, the fundamental entity dealt with in the axioms was comprised of a prior belief and action and a sequence of (future) beliefs and actions. I deal with each of these separately.

Eliciting a prior action and belief can be done via a questionnaire using, e.g., the Anscombe and Aumann [3] algorithm.

Testing the suggested axioms would also require asking a subject to compare lists of hypothetical beliefs. A direct way to go about testing this would be to ask subjects to imagine holding certain beliefs, treating the beliefs as objects.

An alternative and indirect way to test the axioms could be done by giving subjects sequences of signal realizations and their accuracies. This method, while more conventional experimentally, requires accommodation of the axioms to sequences of triplets of the form (signal realization, accuracy, action). In what follows, I discuss this modification.

I consider agents holding prior beliefs and having a history of actions they have taken in the past. Given any past action and prior the agent holds today (for example, "I entered a Ph.D. program in Economics" and I believe Economics is my destiny over other professions), we look at preferences over sequences of triplets. Each triplet is comprised of a signal realization ("at time t, I get a B- in IO and advised to consider other directions in life"), the signal's accuracy (50%?), and the action taken ("I go into behavioral theory"). The advantage of looking at such triplets is that they can be easily induced in the lab.

In this section I present a modification of the axiomatic structure proposed in Section 3 to sequences of triplets, each of which is comprised of a signal realization, signal accuracy, and action. The spirit is indeed experimental. While it is possible to pose agents with sequences of future beliefs, it seems somewhat more natural, at least based on existing experiments, to provide agents with signals and accuracies, and allow them to deduce their beliefs.

The basic setup and notation is a restriction of that introduced in Section 2.

$\Omega = \{L, R\}$ is the set of states of the world and a belief over Ω is a number $0 \leqslant \mu \leqslant 1$ denoting the probability that the state of the world $\omega \in \Omega$ is L. That is, $\mu = \Pr(\omega = L)$.

At each stage t the agent receives a signal $s_t \in \Omega$ with accuracy $q_t \in [\frac{1}{2}, 1]$: $q_t = \Pr(s_t = \omega)$.

The agent has a finite set of actions $A = \{L, R\}$.

I view the choice between sequences of signals and accuracies not totally farfetched. In day to day life, signal realizations are exogenously determined and their accuracies, when chosen, are determined before the signals are actually realized. Nonetheless, agents often have some leeway in interpreting the information they receive. For example, suppose you read in the paper that

"Microsoft is pushing ahead on a variety of fronts to move into new markets and consolidate existing ones."[2] Does it mean that Apple computers will have new Microsoft WORD editors with a 30% probability next year? 15% probability next year? In a sense, there's room for interpretation concerning the signal and it's accuracy. Even if one insists that agents always update using Bayes' rule when the signal and its accuracy are pinned down completely, freedom over choice of signals and accuracies is mathematically equivalent to freedom over choice of posterior beliefs. Such agents behave *as if* they are choosing beliefs. The following Lemma formalizes this equivalence.

Denote by $\mu^B(\mu, s, q)$ the Bayesian update when the prior is μ and a signal s of accuracy q. That is,

$$\mu^B(\mu, s, q) = \begin{cases} \frac{\mu q}{\mu q + (1-\mu)(1-q)} & s = L \\ \frac{\mu(1-q)}{\mu(1-q)+(1-\mu)q} & s = R \end{cases}.$$

Lemma 1: *For any μ_1, $\mu_2 \in (0,1)$, $\mu_1 \neq \mu_2$ there exists a unique pair $(s,q) \in \Omega \times (\frac{1}{2}, 1]$ such that $\mu_2 = \mu^B(\mu_1, s, q)$.*

Remark: If $\mu_1 = \mu_2$ then $\mu_2 = \mu^B(\mu_1, s, q)$ for $q = \frac{1}{2}$ and any $s \in \Omega$.

Proof: For all $q > \frac{1}{2}$,

$$\mu^B(\mu_1, s, q) \begin{cases} > \mu_1 & s = L \\ < \mu_1 & s = R \end{cases}.$$

So, define

$$s = \begin{cases} L & \mu_2 > \mu_1 \\ R & \mu_1 > \mu_2 \end{cases} \text{ and } q = \begin{cases} \frac{\mu_2(1-\mu_1)}{\mu_1+\mu_2-2\mu_1\mu_2} & s = L \\ \frac{\mu_1(1-\mu_2)}{\mu_1+\mu_2-2\mu_1\mu_2} & s = R \end{cases}.$$

By construction, indeed s and q satisfy (uniquely) $\mu_2 = \mu^B(\mu_1, s, q)$. ■

Denote by $B = \Omega \times [\frac{1}{2}, 1] \times A$. An element $\Theta \in B$ is a triplet $\Theta = (s, q, a)$.

At any stage t, having had the belief μ_{t-1} and taken the action a_{t-1} at stage $t-1$, I assume a stationary preference relation $\succsim$ over streams of triplets of (signal, accuracy, action). I.e., $\succsim$ is defined on sequences of the form $(\Theta_t, \Theta_{t+1}, \Theta_{t+2}, ...)$.

Note that in an experimental setting, the experimenter can provide the signals and their accuracies and see the actions that the agent prefers. Hence, sequences $(\Theta_t, \Theta_{t+1}, \Theta_{t+2}, ...)$ are obeservable in the lab.

In the same spirit, prior beliefs can also be elicited using the Anscombe and Aumann [3] algorithm. Therefore, at each stage t of a dynamic decision making process carried out in the lab, the experimenter can deduce the prior belief μ_{t-1}.

[2] John Markoff, "At Microsoft, It's Still Business According to Strategy and Plan," *The New York Times*, 9.27.2000.

As in Section 2, assume that each belief μ induces a von Neumann Morgenstern preference relation over actions $\succsim_\mu \Rightarrow v_\mu$ is a corresponding utility index (unique up to a linear transformation). These preference can also be elicited in the lab. As in Section 3, v_μ induces a preference relation over beliefs for actions that corresponds to the utility index $v_a(\mu)$, which is an affine transformation of v_μ. In particular, if we can write down v_μ by experimental observations, we can deduce v_a.

As before, I assume the agent has a one stage memory for beliefs and actions.

Let B^∞ denote the set of streams $(\Theta_1, \Theta_2, \Theta_3, ...)$ where $\Theta_i \in \Omega^* \quad \forall i \geqslant 1$. I endow B with the product topology and denote by $R(B^\infty)$ the Borel $\sigma-$field of B^∞.

I look at the space $M(B^\infty)$ of all additive probability measures over the measure space $(B^\infty, R(B^\infty))$.

In a similar spirit to Section 2, I make the following assumptions on the preference relation $\succsim$.

Axiom 1* (existence of null history). There exists μ_- s.t. for all $a_- \in A$, all $\omega \in \Omega$, $\Theta = (s, q, a) \in B$, and $x, y \in M(B^\infty)$,

$$(\omega, \mu_-, a_-; \Theta, x) \succsim (\omega, \mu_-, a_-; \Theta, y) \Leftrightarrow (\omega, \mu^B(\mu_-, s, q), a; x) \succsim (\omega, \mu^B(\mu_-, s, q), a; y).$$

Axiom 2* (one-stage memory). For all μ_0, μ_0', a_0, a_0' and $\Theta_1 = (s_1, q_1, a_1), \Theta_1' = (s_1', q_1', a_1) \in B$ s.t. $\mu_1^B(\mu_0, s_1, q_1) = \mu_1^B(\mu_0', s_1', q_1')$, for all $\omega \in \Omega$,and for all $x, y \in M(B^\infty)$,

$$(\omega, \mu_0, a_0; \Theta_1, x) \succsim (\omega, \mu_0, a_0; \Theta_1, y) \Leftrightarrow (\omega, \mu_0', a_0'; \Theta_1', x) \succsim (\omega, \mu_0', a_0'; \Theta_1', y).$$

Axiom 3* (continuity). There exists a von Neumann-Morgenstern utility index U for $\succsim$,which is a continuous function on its domain B^∞.

Axiom 4* (consistency). For all parameters, preferences satisfy:

$$(\omega, \mu_0, a_0; s, \frac{1}{2}, a_0, s_1, q_1, a_1, s, \frac{1}{2}, a_1, s, \frac{1}{2}, a_1, ...) \succsim (\omega, \mu_0, a_0; s, \frac{1}{2}, a_0, s_1', q_1', a_1, s, \frac{1}{2}, a_1, s, \frac{1}{2}, a_1, ...)$$

$$\text{when } v_{a_0}(\mu^B(\mu_0, s_1, q_1)) \geqslant v_{a_0}(\mu^B(\mu_0, s_1', q_1')).$$

Proposition 3 (utility representation for streams of signals and accuracies): *If $\succsim$ satisfies Axioms 1*-3* then U takes the form:*

$$U(\omega, \mu_0, a_0; s_1, q_1, a_1, s_2, q_2, a_2, ...) = \sum_{i=0}^{\infty} u_\omega(\mu_i^B, a_i, \mu_{i+1}^B) \exp(-\sum_{j=0}^{i-1} \rho_\omega(\mu_j^B, a_j, \mu_{j+1}^B))$$

(under the convention that $\sum_{j=0}^{-1} = 0$).

Where μ_i^B *is defined inductively:* $\mu_0^B = \mu_0$ *and* $\mu_i^B = \mu^B(\mu_{i-1}^B, s_i, q_i)$ *for all* $i > 0$.

Assume $\rho(a, \mu) \equiv \rho$ *is a constant and define* $\delta \equiv \exp(-\rho)$.

If $\succsim$ *satisfies Assumption 4*, then* U *takes the form:*

$$U(\omega, \mu_0, a_0; s_1, q_1, a_1, s_2, q_2, a_2, ...) =$$

$$= \sum_{i=0}^{\infty} [\alpha_{\omega 1}^*(\mu_i^B, a_i) + \alpha_{\omega 2}^*(\mu_{i+1}^B, a_i) + \beta_\omega^*(\mu_i^B, a_i) v_{a_i}(\mu_{i+1}^B)] \delta^i.$$

Uniqueness can be expressed in an analougous manner to the corollary in Section 3.

Note: Eliciting prior beliefs when the agent knows she will be facing a choice between streams of the sort described above would potentially lead to distorted reports. For example, suppose an experimenter tries to elicit my beliefs as to whether I'll be better off as an economist or as an aerobics instructor. Furthermore, assume now that I know that I will be receiving information (from the experimenter) about my grades in economics and my athletic leadership abilities in future months. I might prefer to report an attenuated belief having in mind that "I'd better give an estimate around 50-50 since I have more information arriving next period." In fact, it is exactly the psychological biases that provide motivation for this line of research (e.g., confirmatory bias and cognitive dissonance, that were briefly discussed in the introduction) that may contaminate such an experiment. Hence, an experimental design structured around the current should have subjects being questioned about their priors *without knowing* that they will be receiving additional information of any sort related to the issue they're asked about.[3]

5 Concluding Comments

The main contribution of this paper is an axiomatic foundation for a generalized discounted utility functional form in which both actions and beliefs enter as arguments.

Looking at preference orderings over sequences of (belief, action) pairs, then in order to achieve a generalized general discounted utility representation, one needs to assume that: 1. there exists a (null) belief that does not have any effect on future preferences, no matter what action the agent takes when holding that belief; 2. only finite histories have an effect on current preferences. That is, the agent has bounded memory and is aware of it; and 3. there exist a von Neumann - Morgenstern utility representation of the preference ordering. In the generalized discounted utility

[3] This would not require deceiving the subjects if the experiment is done in two parts, even on two different dates.

representation, each term depends on the action taken at an instant of time as well as the beliefs held at that period and the period that preceded it.

If, in addition, I assume that when the agent knows she will not be getting any additional information from tomorrow on, she prefers today's beliefs to be consistent with yesterday's action and belief, then the representation takes a special form. Namely, terms that include past and present beliefs are separable.

One of the immediate implications of such a utility representation is that the rate of time preference, as measured by marginal rates of substitution between consecutive actions, depends on the beliefs the agent holds.

The axiomatic foundations can be translated into sequences of triplets of the form (signal, accuracy, action), which may be easier to use in a realistic experimental setting.

In terms of future research, Bodner and Prelec [7] and Yariv [17] both look at special cases of Proposition 1 in which the the terms depending on beliefs and actions are separable (see Example in Section 3). They embed this special case in a standard learning framework in which an agent's action is a guess of a random state of the world. The agent observes a new signal before each choice. In the context of the present paper, it would be useful to have an axiomatic foundation for the special cases discussed in the literature. More importantly, the underlying *process of choice* has no foundations in the literature, axiomatic, evolutionary, or other.

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7 Appendix - Proofs

Proof of Proposition 1: Sufficiency follows directly from the specification of the axioms. I thus concentrate on proving necessity. From Axiom 2 and 3, it follows that for all $\omega \in \Omega$,

$$U(\omega, \mu_0, a_0; \mu_1, a, p) = \widetilde{A}_\omega(\mu_0, a_0, \mu_1) + \widetilde{B}_\omega(\mu_0, a_0, \mu_1) U(\omega, \mu_-, a_0; \mu_1, a, p)$$

where $\widetilde{B}_\omega(\mu_0, a_0, \mu_1) > 0$.

Similarly, from Axiom 1,

$$U(\omega, \mu_-, a_0; \mu_1, a, p) = \widetilde{\widetilde{A}}_\omega(a_0, \mu_1) + \widetilde{\widetilde{B}}_\omega(a_0, \mu_1) U(\omega, \mu_1, a_1; p)$$

where $\widetilde{\widetilde{B}}_\omega(a_0, \mu_1) > 0$.

Combining the last two equalities,

$$U(\omega, \mu_0, a_0; \mu_1, a, p) = u_\omega(\mu_0, a_0, \mu_1) + r_\omega(\mu_0, a_0, \mu_1) U(\omega, \mu_1, a; p) \tag{1}$$

where $r_\omega > 0$.

Define

$$\rho_\omega(\mu_0, a_0, \mu_1) \equiv -\ln r_\omega(\mu_0, a_0, \mu_1) \Rightarrow r_\omega(\mu_0, a_0, \mu_1) = \exp(-\rho_\omega(\mu_0, a_0, \mu_1))$$

Applying equation (1) many times I get:

$$U(\omega, \mu_0, a_0; \mu_1, a_1, \mu_2, a_2, ...) = \sum_{i=0}^{\infty} u_\omega(\mu_i, a_i, \mu_{i+1}) \exp(-\sum_{j=0}^{i} \rho_\omega(\mu_i, a_i, \mu_{i+1}))$$

(under the convention that $\sum_{i=0}^{0} = 0$). ■

Proof of Proposition 2: Using Proposition 1, assume that $\rho_\omega(\mu, a, \mu') \equiv \rho$ is a constant and define $\delta \equiv \exp(-\rho)$.

$$U(\omega, \mu, a; \mu', a, \mu', a, \mu', a, ...) = u_\omega(\mu, a, \mu') + \sum_{i=0}^{\infty} \delta u_\omega(\mu', a, \mu') \delta^i =$$

$$= u_\omega(\mu, a, \mu') + \frac{\delta u_\omega(\mu', a, \mu')}{1-\delta}.$$

When Axiom 4 holds,

$$\begin{aligned} U(\omega, \mu, a; \mu', a, \mu', a, \mu', a, ...) &= u_\omega(\mu, a, \mu') + \frac{\delta u_\omega(\mu', a, \mu')}{1-\delta} \\ &= \alpha_\omega(\mu, a) + \beta_\omega(\mu, a) v_a(\mu'). \end{aligned}$$

In particular, when $\mu = \mu'$,

$$\frac{u_\omega(\mu, a, \mu)}{1-\delta} = \alpha_\omega(\mu, a) + \beta_\omega(\mu, a) v_a(\mu).$$

Combining these two equalities we get:

$$u_\omega(\mu, a, \mu') = \alpha^*_{\omega 1}(\mu, a) + \alpha^*_{\omega 2}(\mu', a) + \beta^*_\omega(\mu, a) v_a(\mu'). \blacksquare$$

Proof of Corollary: Assume these equalities hold. Then, dropping ω indices,

$$\widetilde{U}(\mu_0, a_0; \mu_1, a_1, \mu_2, a_2, ...) = \sum_{i=0}^{\infty} \widetilde{u}(\mu_i, a_i, \mu_{i+1}) \exp(-\sum_{j=0}^{i-1} \widetilde{\rho}(\mu_j, a_j, \mu_{j+1})) =$$

$$= \sum_{i=0}^{\infty} [\alpha(1 - \exp(-\rho(\mu_i, a_i, \mu_{i+1}))) + \beta u(\mu_i, a_i, \mu_{i+1})] \exp(-\sum_{j=0}^{i-1} \rho(\mu_j, a_j, \mu_{j+1})) =$$

$$= \alpha + \beta \sum_{i=0}^{\infty} u(\mu_i, a_i, \mu_{i+1}) \exp(-\sum_{j=0}^{i-1} \rho(\mu_j, a_j, \mu_{j+1})) =$$

$$= \alpha + \beta U(\mu_0, a_0; \mu_1, a_1, \mu_2, a_2, ...).$$

Conversely, suppose that $\widetilde{U} = \alpha + \beta U$. As in the proof of Proposition 1, let $r = \exp(-\rho)$ and $\widetilde{r} = \exp(-\widetilde{\rho})$. Take any (μ_0, a_0), and $b = (\mu_1, a_1, \mu_2, a_2, ...) \in B$.

$$\widetilde{U}(\mu_0, a_0; b) = \alpha + \beta U(\mu_0, a_0; b) = \widetilde{u}(\mu_0, a_0, \mu_1) + \widetilde{r}(\mu_0, a_0, \mu_1)(\alpha + \beta U(\mu_1, a_1; \mu_2, a_2, ...)).$$

Since $U = \frac{\widetilde{U}}{\beta} - \frac{\alpha}{\beta}$,

$$U(\mu_0, a_0; \mu_1, a_1, ...) = -\frac{\alpha}{\beta} + \frac{\widetilde{u}(\mu_0, a_0, \mu_1)}{\beta} + \frac{\alpha \widetilde{r}(\mu_0, a_0, \mu_1)}{\beta} + \widetilde{r}(\mu_0, a_0, \mu_1) U(\mu_1, a_1; \mu_2, a_2, ...)).$$

Combining the last equality with

$$U(\mu_0, a_0; \mu_1, a_1, \mu_2, a_2, ...) = u(\mu_0, a_0, \mu_1) + r(\mu_0, a_0, \mu_1) U(\mu_1, a_1; \mu_2, a_2, ...)$$

we get:

$$[r(\mu_0, a_0, \mu_1) - \widetilde{r}(\mu_0, a_0, \mu_1)] U(\mu_1, a_1; \mu_2, a_2, ...) =$$
$$= \frac{1}{\beta}[-\alpha + \widetilde{u}(\mu_0, a_0, \mu_1) + \alpha \widetilde{r}(\mu_0, a_0, \mu_1) - \beta u(\mu_0, a_0, \mu_1)].$$

Let $(\widetilde{\mu}_2, \widetilde{a}_2, \widetilde{\mu}_3, \widetilde{a}_3, ...) \in B$ satisfy $U(\mu_1, a_1; \mu_2, a_2, \mu_3, a_3, ...) \neq U(\mu_1, a_1; \widetilde{\mu}_2, \widetilde{a}_2, \widetilde{\mu}_3, \widetilde{a}_3, ...)$. The above equality also holds when changing the sequence $(\mu_2, a_2, \mu_3, a_3, ...)$ with $(\widetilde{\mu}_2, \widetilde{a}_2, \widetilde{\mu}_3, \widetilde{a}_3, ...)$. Subtracting, we get that

$$[r(\mu_0, a_0, \mu_1) - \widetilde{r}(\mu_0, a_0, \mu_1)][U(\mu_1, a_1; \mu_2, a_2, ...) - U(\mu_1, a_1; \widetilde{\mu}_2, \widetilde{a}_2, \widetilde{\mu}_3, \widetilde{a}_3, ...) = 0 \Rightarrow$$

$$\Rightarrow r(\mu_0, a_0, \mu_1) = \widetilde{r}(\mu_0, a_0, \mu_1) \Rightarrow$$

$$\Rightarrow \frac{1}{\beta}[-\alpha + \widetilde{u}(\mu_0, a_0, \mu_1) + \alpha\widetilde{r}(\mu_0, a_0, \mu_1) - \beta u(\mu_0, a_0, \mu_1)] = 0 \Rightarrow$$

$$\widetilde{u} = \alpha(1 - r) + \beta u = \alpha(1 - \exp(-\rho)) + \beta u. \blacksquare$$

Proof of Proposition 3: The proof traces the proof of Propositions 1 and 2.

From Axioms 2* and 3*, it follows that:

$$U(\omega, \mu_0, a_0; s, q, a, x) = A_\omega(\mu_0, a_0, \mu^B(\mu_0, s, q)) + B_\omega(\mu_0, a_0, \mu^B(\mu_0, s, q))U(\omega, \mu_-, a_0; s_-, q_-, a, x)$$

where μ_- is the null belief whose existence is guaranteed by Axiom 1*, $B_\omega(\mu_0, a_0, \mu^B(\mu_0, s, q)) > 0$, and s_-, q_- are such that $\mu^B(\mu_0, s, q) = \mu^B(\mu_-, s_-, q_-) \equiv \mu_1^B$. Note that such s_-, q_- exist by Lemma 1.

Similarly, from Axiom 1*,

$$U(\omega, \mu_-, a_0; s_-, q_-, a, x) = \widetilde{A}_\omega(a_0, \mu^B(\mu_-, s_-, q_-)) + \widetilde{B}_\omega(a_0, \mu^B(\mu_-, s_-, q_-))U(\omega, \mu_1^B, a; x)$$

where $\widetilde{B}_\omega(a_0, \mu^B(\mu_-, s_-, q_-)) > 0$.

Combining the last two equalities,

$$U(\omega, \mu_0, a_0; s, q, a, x) = u_\omega(\mu_0, a_0, \mu_1^B) + r_\omega(\mu_0, a_0, \mu_1^B)U(\omega, \mu_1^B, a; x) \quad (2)$$

and $r_\omega(\mu_0, a_0, \mu_1^B) > 0$.

Define

$$\rho_\omega(\mu_0, a_0, \mu_1) \equiv -\ln r_\omega(\mu_0, a_0, \mu_1) \Rightarrow r_\omega(\mu_0, a_0, \mu_1) = \exp(-\rho_\omega(\mu_0, a_0, \mu_1))$$

Applying equation (2) many times we get:

$$U(\omega, \mu_0, a_0; s_1, q_1, a_1, s_2, q_2, a_2, ...) = \sum_{i=0}^{\infty} u_\omega(\mu_i^B, a_i, \mu_{i+1}^B) \exp(-\sum_{j=0}^{i-1} \rho_\omega(\mu_j^B, a_j, \mu_{j+1}^B)).$$

In particular, assume that $\rho_\omega(\mu, a, \mu') \equiv \rho$ is a constant and define $\delta \equiv \exp(-\rho)$.

$$\begin{aligned} & U(\omega, \mu_0, a_0; s, \tfrac{1}{2}, a_0, s_1, q_1, a_1, s_1, \tfrac{1}{2}, a_1, s_1, \tfrac{1}{2}, a_1, ...) \\ = \ & u_\omega(\mu_0, a_0, \mu_0) + \delta u_\omega(\mu_0, a_0, \mu_1^B) + \sum_{i=0}^{\infty} \delta^2 u_\omega(\mu_1^B, a_1, \mu_1^B)\delta^i = \end{aligned}$$

$$= u_\omega(\mu_0, a_0, \mu_0) + \delta u_\omega(\mu_0, a_0, \mu_1^B) + \frac{\delta^2 u_\omega(\mu_1^B, a_1, \mu_1^B)}{1 - \delta}.$$

When Axiom 4* holds,

$$U(\omega, \mu_0, a_0; s, \frac{1}{2}, a_0, s_1, q_1, a_1, s_1, \frac{1}{2}, a_1, s_1, \frac{1}{2}, a_1, ...) =$$

$$\begin{aligned} &= u_\omega(\mu_0, a_0, \mu_0) + u_\omega(\mu_0, a_0, \mu_1^B) + \frac{\delta u_\omega(\mu_1^B, a_1, \mu_1^B)]}{1-\delta} = \\ &= \alpha_\omega(\mu_0, a_0, a_1) + \beta_\omega(\mu_0, a_0, a_1) v_{a_0}(\mu_1^B). \end{aligned}$$

In particular, when $q_1 = \frac{1}{2}$, $a_1 = a_0$,

$$\frac{u_\omega(\mu_0, a_0, \mu_0)}{1-\delta} = \alpha_\omega(\mu_0, a_0, a_0) + \beta_\omega(\mu_0, a_0, a_0) v_{a_0}(\mu_0).$$

Combining these two equalities we get:

$$u(\mu, a, \mu') = \alpha_1^*(\mu, a) + \alpha_1^*(\mu', a) + \beta^*(\mu, a) v_a(\mu'). \blacksquare$$