Inflationary Bias in a Simple Stochastic Economy

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Inflationary Bias in a Simple Stochastic Economy

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Abstract

We construct explicit equilibria for strategic market games used to model an economy with fiat money, one nondurable commodity, countably many time- periods, and a continuum of agents. The total production of the commodity is a random variable that fluctuates from period to period. In each period, the agents receive equal endowments of the commodity, and sell them for cash in a market; their spending determines, endogenously, the price of the commodity. All agents have a common utility function, and seek to maximize their expected total discounted utility from consumption.

Suppose an *outside bank* sets an interest rate ρ for loans and deposits. If $1+\rho$ is the reciprocal of the discount factor, and if agents must bid for consumption in each period before knowing their income, then there is no inflation. However, there is an inflationary trend if agents know their income before bidding.

We also consider a model with an *active central bank*, which is both accurately informed and flexible in its ability to change interest rates. This, however, may not be sufficient to control inflation.

Key Words: Inflation, strategic market game, control, interest rate, central bank, equilibrium.

JEL Classification: C7, C73, D81, E41, E58

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†John Geanakoplos joined this project after it was already underway.

1 Introduction

According to the famous Fisher equation, the rate of inflation in a stationary economy should be equal to the rate of interest less the time discount rate of the agents. In a nonstochastic, stationary economy, this is precisely the case: if the central bank sets the monetary rate of interest (the same for borrowers and depositors) equal to the rate of time preference of the agents, the equilibrium rate of inflation will be zero. But we show that when the central bank pursues the same strategy in a *stochastic* stationary economy with well-informed agents, who know their income before deciding on expenditures, the expected rate of inflation will be positive. The magnitude of the inflation depends on the shape of the agents' utility function (that is on their attitude toward risk) and on the volatility of output. On the other hand, if agents do not know their income before they are called upon to commit themselves to expenditures, then the Fisher equation is restored, [1] and setting the rate of interest equal to the discount rate will result in an expected rate of inflation equal to zero.

Finally, we show that it is impossible for a well-informed central bank to maintain a constant price level in an economy with stochastic output, even if the bank can instantaneously adjust the rate of inflation.

We construct equilibria for four variations of a simple economy with fiat money, one nondurable commodity, and a continuum of identical agents. The economy, and the lives of the agents, are assumed to stretch over an infinite number of time-periods $n = 1, 2, ...$ In each time-period n, the total production of commodity (output) is a random variable Y_n; agents are endowed with equal amounts of the commodity, which is sold for cash in a commodity market. Agents can choose how to allocate their money between saving and consumption expenditure in each period, and seek to maximize their expected total discounted utility from consumption.

Our model solves for a type-symmetric noncooperative equilibrium (TSNE), which in this instance is equivalent to the representative agent approach of Lucas (1978); we use dynamic programming methods in a microeconomic model of money, in the tradition of Shubik (1972), Shubik and Whitt (1973), Lucas (1980, 1990), Stokey and Lucas (1989) and Karatzas, Shubik and Sudderth (1994). The macroeconomic tradition of analyzing policy and money in a market-clearing model is vast; see, for example, Phelps (1967, 1970), Kydland and Prescott (1977), Barro (1990), Mankiw (1992), Sargent (1987, 1999), and Alvarez, Lucas and Weber (2001). To our knowledge, however, the questions addressed in this paper seem to be treated here for the first time. The model of Lucas (1990), for instance, is extremely close to ours, but analyzes the case where the central bank behavior is random and output is fixed; agents in this model may or may not know the interest rate when they make their savings consumption choice.

The four variations arise in our analysis, because: (1) we consider models with and without an outside bank, that lends and accepts deposits at a fixed rate of interest; and (2) we consider "low-information" models in which agents must bid (i.e., allocate cash out of their holdings) for consumption in each period n before learning the value of their endowment Y_n, as well as "high-information" models where agents do know Y_n when they bid. Agents always know the prevalent interest rate before they act. In any case, the price of the commodity fluctuates randomly from period to period.

In the absence of an outside bank, the money supply has a fixed value M and the prices $p_1, p_2, ...$ of the commodity in successive periods are independent, identically distributed

[1] In an actual economy, the control problem for a central bank is far more complicated than illustrated here or in the models of Lucas (1978, 1980, 1990). Only outside, or government (fiat), money is considered here. But in reality, the amount of credit in an economy is usually both larger and more volatile than the supply of government-money; thus, governmental control, of either the money supply or of the interest-rate, is far less effective than indicated in these models.

random variables. In the low-information model without an outside bank, these prices are given by $p_n = M/Y_n$; in particular, they do not depend on the agents' utility function. In the high-information model without an outside bank, prices depend critically on the utility function through a "no-arbitrage" condition.

In the low-information model *with an outside bank*, the money supply M_n in period n is equal to $[\beta(1+\rho)]^n M_0$, where $\beta \in (0,1)$ is the discount factor and $\rho \in (0,\infty)$ is the interest rate. There is a positive constant η such that the price in period n is $p_n = \eta M_{n-1}/Y_n$. Thus, the expected price inflates, deflates, or remains constant, according to whether $\beta(1+\rho)$ is larger than, smaller than, or equal to, one. In the high-information model with a bank, the successive levels of the money supply M_n and the price p_n are given by

$$M_n = M_0 \cdot \prod_{k=1}^{n} \tau(Y_k) \qquad \text{and} \qquad p_n \;=\; \frac{\eta(Y_n)}{Y_n}\, M_{n-1}\,,$$

respectively, for $n = 1, 2, \cdots$. Here $\eta(\cdot)$ is a suitable function with values in $(0, (1+\rho)/\rho)$, in terms of which we have $\tau(\cdot) = 1 + \rho - \rho\eta(\cdot)$, and the successive production (output) levels $Y_1, Y_2, \cdots$ are independent, identically distributed and positive random variables. The expected value of p_n is typically increasing when $\beta(1+\rho) = 1$; in other words, even with the "natural" rate of interest $\rho_* = 1/\beta - 1$ (equal to the "rate of time-preference for agents"), there is an "inflationary trend" in this economy.

The next section gives a precise formulation of the models and an appropriate definition of equilibrium. In Sections 3–6 equilibria are constructed for each of the four models in turn. Section 7 discusses the high-information model for an "active" bank, that has the ability to change the interest-rate $\rho > 0$ in each period; it is shown that it is possible for such a bank to control inflation perfectly, in the sense that it can achieve $\tau(Y_n) \equiv 1$ for all $n \in \mathbb{N}$ with probability one – but only under certain *very* stringent conditions on the discount factor β, the utility function $u(\cdot)$, and the distribution of the endowment variables Y_n.

2 The Models

There is a continuum of agents $\alpha \in I \triangleq [0,1]$, distributed according to a nonatomic probability measure φ on the Borel subsets $\mathcal{B}(I)$ of I. (We use the symbol "$\triangleq$" to mean "equal by definition.") On each day, or time-period, $n = 1, 2, ...$, every agent $\alpha \in I$ receives a random endowment $Y_n^\alpha(\omega)$ in units of a single perishable commodity. The random variables $\{Y_n^\alpha\}$ are defined on a given probability space $(\Omega, \mathcal{F}, P)$. In this paper, unlike our previous work in [KSS1], [KSS2] and [GKSS], we assume that, on any given period, all agents receive the same amount of commodity; this amount may vary from period to period.[2] Thus $Y_n^\alpha(\omega) = Y_n(\omega)$ is the same for all $\alpha \in I$, for each $n \in \mathbb{N}$. The total endowment (production, output) in the economy, namely

$$\int_I Y_n^\alpha(\omega)\varphi(d\alpha) = Y_n(\omega),$$

also has this same value. We assume that the random variables $Y, Y_1, Y_2, ...$ are independent, with common distribution λ that satisfies

$$\lambda((0,\infty)) = \mathbb{P}[Y > 0] = 1 \qquad \text{and} \qquad \int_0^\infty y\lambda(dy) = \mathbb{E}(Y) < \infty\,.$$

[2] Our assumptions here are close to those of the "representative agent" model. We stress, however, that we solve only for a type-symmetric equilibrium, and make no claim that every equilibrium that exists is of this form.

We shall denote throughout by $\mathcal{Y} \equiv \text{supp}(Y)$ the support of the distribution λ of the random variable Y.

In each time-period $n = 1, 2, ...$, every agent α begins with wealth $S_{n-1}^{\alpha}(\omega)$ in fiat money. At the beginning of period n, the *money-supply,* or total wealthof the agents, is

$$M_{n-1}(\omega) \triangleq \int_I S_{n-1}^{\alpha}(\omega)\varphi(d\alpha) \tag{2.1}$$

There is a commodity market, in which every agent α bids a nonnegative amount $b_n^{\alpha}(\omega)$ of fiat money, in period n. (The four specific models discussed below differ as to what bids are permitted.) The total bid for the period is

$$B_n(\omega) \triangleq \int_I b_n^{\alpha}(\omega)\varphi(d\alpha), \tag{2.2}$$

and the price of the commodity is formed as the ratio

$$p_n(\omega) \triangleq \frac{B_n(\omega)}{Y_n(\omega)} \tag{2.3}$$

of total bid over output. Agent α receives his bid's worth $x_n^{\alpha}(\omega) \triangleq b_n^{\alpha}(\omega)/p_n(\omega)$ of the commodity, consumes it immediately, and thereby receives $u(x_n^{\alpha}(\omega))$ in utility. The utility function $u(\cdot)$, common for all agents, is assumed to be defined, concave and increasing on $\mathbb{R}^+ = [0, \infty)$, and differentiable on $(0, \infty)$.

The total payoff for agent α is the infinite discounted sum

$$\sum_{n=1}^{\infty} \beta^{n-1} u(x_n^{\alpha}(\omega)), \tag{2.4}$$

where $\beta \in (0, 1)$ is a discount factor.

The law of motion for our dynamic game is given by the formula

$$S_n^{\alpha}(\omega) = (1+\rho)\left[S_{n-1}^{\alpha}(\omega) - b_n^{\alpha}(\omega)\right] + p_n(\omega)Y_n(\omega), \qquad n \in \mathbb{N}. \tag{2.5}$$

Here we set $\rho = 0$ for models without a central bank, and $0 < \rho < \infty$ denotes the interest rate for models with a central bank. When there is no bank, the formula says that agent α's wealth S_n^{α} at the end of period n is his wealth S_{n-1}^{α} at the beginning, less his bid b_n^{α}, plus his income $p_n Y_n$ from the sale of his endowment. When there is a bank, agent α either deposits the amount $S_{n-1}^{\alpha} - b_n^{\alpha}$ in the bank if $S_{n-1}^{\alpha} \geq b_n^{\alpha}$, or borrows the amount $b_n^{\alpha} - S_{n-1}^{\alpha}$ from the bank if $S_{n-1}^{\alpha} < b_n^{\alpha}$. Thus, the term $(1+\rho)(S_{n-1}^{\alpha} - b_n^{\alpha})$ represents what agent α is paid by the bank, or must pay back to the bank, depending on whether the term is positive or negative.

There is the possibility that an agent may not receive sufficient income to pay back his debt to the bank. This interesting possibility will not arise in the equilibria we construct here; see [GKSS] for models with active bankruptcy.

2.1 Strategies and Information

A strategy π^{α} for an agent $\alpha \in I$ specifies each bid b_n^{α} as a random variable that depends on the information available to the agent at the beginning of period n. We shall consider two distinct possibilities:

(i) **low information** — each agent α begins period n with information $\mathcal{L}_{n-1}^{\alpha}$, a σ-algebra that measures past wealth levels S_0^{α}, S_k^{α}, prices p_k, endowments Y_k, and levels of the money-supply M_k for $k = 1, 2, ..., n-1$.

(ii) **high information** — each agent α begins period n with information $\mathcal{H}_{n-1}^{\alpha}$, a σ -algebra that includes $\mathcal{L}_{n-1}^{\alpha}$ and also measures Y_n.

The crucial difference is that agents in high-information models know their endowments in any period n before bidding, whereas agents in low-information models do not.

We shall assume that the agents always play strategies $\{\pi^{\alpha},\ \alpha \in I\}$ so that, for every $n = 1, 2, ...$, the mapping $(\alpha, \omega) \mapsto b_n^{\alpha}(\omega)$ is $\mathcal{B}(I) \otimes \mathcal{L}_{n-1}^{\alpha}-$measurable (respectively, $\mathcal{B}(I) \otimes \mathcal{H}_{n-1}^{\alpha}-$measurable) for low- (respectively, high-) information models. We also assume that the initial wealths $\alpha \mapsto S_0^{\alpha}(\omega) = S_0^{\alpha}$ are non-random, $\mathcal{B}(I)-$measurable functions. Then the macroscopic quantities $M_0,\ M_n,\ B_n,\ p_n$ are well-defined for $n = 1, 2, ...$

2.2 The Stochastic Games

By an *initial distribution of wealth* we shall mean a probability measure ν_0 on $\mathcal{B}(I)$, such that

$$\nu_0(B) = \varphi(\{\alpha \in I : S_0^{\alpha} \in B\}).$$

Given such a ν_0 and a family of strategies $\{\pi^{\alpha},\ \alpha \in I\}$ as above, the expected value of the total payoff (2.4) to each agent α is determined.

Definition: An *equilibrium* consists of an initial distribution ν_0 and a family of strategies $\{\pi_{\alpha},\ \alpha \in I\}$ such that, for every $\alpha \in I$, π_{α} is optimal for agent α when all other agents $\beta \in I$, $\beta \neq \alpha$, play π_{β}.

Thus, given the initial distribution ν_0 of wealth, an equilibrium is, by this definition, a Nash equilibrium for the stochastic game.

In all of the models considered here, we shall see that there is a natural symmetric equilibrium, in which every agent plays the same strategy and has the same wealth at each stage of the game. However, the behavior of prices and the money supply will vary from model to model.

3 A Low-Information Model without a Bank

Suppose that agents must bid in each period without knowledge of their endowment in the period. Suppose also that there is no bank or other source of loans for the agents. An agent α with wealth S_{n-1}^{α} at the beginning of any period $n \in \mathbb{N}$ is then restricted to bid an amount $b_n^{\alpha} \in [0, S_{n-1}^{\alpha}]$. Let

$$M = M_0 = \int S_0^{\alpha} \varphi(d\alpha) > 0$$

be the initial money supply. The law of motion (2.5) takes the form

$$S_1^{\alpha}(\omega) = S_0^{\alpha} - b_1^{\alpha}(\omega) + p_1(\omega) Y_1(\omega)$$

in period $n = 1$. Integration over $\alpha \in I$ with respect to the probability measure φ, together with (2.1)–(2.3), gives

$$M_1(\omega) = M_0 - B_1(\omega) + \frac{B_1(\omega)}{Y_1(\omega)} \cdot Y_1(\omega) = M_0 = M.$$

Thus, by induction, the money supply stays fixed at $M_n = M$ for all $n \in \mathbb{N}$.

Theorem 3.1: *If the utility function $u(\cdot)$ is strictly concave and strictly increasing, then there is an equilibrium in which every agent has wealth M and bids M in every period.*

Proof: Suppose that every agent $\alpha \in I$ has wealth M and bids M at some stage of the game. If the value of the endowment variable is Y, then the price is $p(Y) = M/Y$. The next wealth level of such an agent will then be

$$M - M + Yp(Y) = M.$$

Thus the wealth of the agent stays fixed at M.

Now consider the one-person problem of a single agent playing against all the other agents who have wealth M and bid M every period. It suffices to show that M is the optimal bid of the single agent who has wealth M.

Consider more generally a single agent with initial wealth $s \geq 0$. The agent can bid any amount $b \in [0, s]$, receive $u(b/p(Y))$ in utility, and then move to $s - b + Yp(Y) = s - b + M$ at the next stage. Let $V(s)$ be the optimal reward for this agent. Then the function $V(\cdot)$ satisfies the Bellman equation

$$\begin{aligned} V(s) &= \sup_{0 \leq b \leq s} \left[Eu\left(\frac{b}{p(Y)}\right) + \beta V(s - b + M) \right] \\ &= \sup_{0 \leq b \leq s} [\tilde{u}(b) + \beta V(s - b + M)], \end{aligned} \tag{2.6}$$

where

$$\tilde{u}(b) \triangleq E\left[u\left(\frac{b}{p(Y)}\right)\right] = E\left[u\left(\frac{bY}{M}\right)\right], \qquad b \in [0, \infty)$$

can be regarded as another strictly concave utility function. Standard arguments show that $V(\cdot)$ is concave. Consequently,

$$\psi(b; s) \triangleq \tilde{u}(b) + \beta V(s - b + M), \quad 0 \leq b \leq s,$$

is strictly concave and has a unique point of maximum, namely

$$c(s) \triangleq \arg\max \psi(\cdot; s).$$

We need to show that $c(M) = M$. Of course, $c(M) \leq M$, by the rules of the game. Suppose, by way of contradiction, that $c(M) < M$.

Now

$$V(M) = \tilde{u}(c(M)) + \beta V(2M - c(M)),$$

and clearly,

$$V(c(M)) \geq \tilde{u}(c(M)) + \beta \cdot V(c(M) - c(M) + M) = \tilde{u}(c(M)) + \beta V(M),$$

from (2.6). Hence,

$$\begin{aligned} V(M) - V(c(M)) &\leq \beta[V(2M - c(M)) - V(M)] \\ &< V(2M - c(M)) - V(M), \end{aligned}$$

contradicting the concavity of $V(\cdot)$. (The final, strict inequality uses the easy-to-check fact that $V(\cdot)$ is strictly increasing, because $u(\cdot)$ is.) ■

Remark 3.1: A more refined argument gives $c(s) = s, \ \forall\, 0 \leq s \leq M$.

Remark 3.2: In the equilibrium of Theorem 3.1, the money supply stays fixed and the successive prices are $M/Y_1, M/Y_2, \ldots$. Although they fluctuate randomly, these prices have the same distribution. There is no inflation or deflation in this economy.

4 A Low-Information Model with a Bank

Again, agents must bid without knowledge of their endowment in each period. However, they are now permitted to borrow or make deposits in an outside bank. The bank charges borrowers and pays depositors at a fixed rate of interest $\rho \in (0, \infty)$.

To construct an equilibrium, suppose that, at some stage of the game, the money supply is M and every agent has wealth $\dot{M}$. Suppose further that every agent bids

$$b(M) \triangleq \eta M,$$

where $\eta > 0$ is a constant, to be determined below. With these bids, there is an associated random price

$$p_Y(M) = \frac{\eta M}{Y},$$

where $\eta M = \int b(M)\varphi(d\alpha)$ is the total bid and Y is the random endowment (output) in the period. However, the cash income of each agent from the sale of Y is

$$Y p_Y(M) = \eta M,$$

a constant. The wealth of each agent in the next period becomes

$$M' \; = \; (1+\rho)(M - \eta M) + \eta M \; = \; \tau M,$$

where $\tau \triangleq 1 + \rho - \rho\eta$ is the "rate of inflation". The quantity τM is also the new money-supply. Selecting $\eta = \eta(\rho, \beta) \triangleq (1+\rho)(1-\beta)/\rho$, we obtain the *Fisher equation*

$$\tau = \beta(1+\rho)\,, \tag{4.1}$$

which expresses the "rate of inflation" as a simple function of the prevalent interest rate $\rho > 0$ and the natural discount (time-preference) factor $\beta \in (0,1)$ in the economy

Let $\pi(\eta)$ be a strategy that bids the amount $b(M) = \eta M$, whenever an agent has wealth equal to the current money-supply. If all agents begin with wealth M_0 and follow $\pi(\eta)$, then the money-supply and wealth of every agent after n days will be

$$M_n = \tau^n M_0 = [\beta(1+\rho)]^n M_0.$$

The price of the commodity on the nth day will be

$$p_{Y_n}(M_{n-1}) = \frac{\eta M_{n-1}}{Y_n} = \frac{\eta[\beta(1+\rho)]^{n-1} M_0}{Y_n}.$$

Thus the money-supply will geometrically increase, decrease, or remain the same, depending on whether $\beta(1+\rho)$ is larger than, smaller than, or equal to, one. The prices, being random multiples of the money supply, will accordingly increase, decrease, or remain the same stochastically.

Theorem 4.1: *Suppose that the utility function $u(\cdot)$ is such that*

$$y u'(y) \leq \kappa\,, \quad \forall\; y \in \mathcal{Y} \tag{4.2}$$

holds for some constant $\kappa \in (0, \infty)$. Then there is an equilibrium, in which every agent has initial wealth equal to the initial money supply M_0 and uses the strategy $\pi(\eta)$, where $\eta = (1+\rho)(1-\beta)/\rho$.

Proof: Assuming that almost every agent α begins with wealth M_0 and plays $\pi(\eta)$, we need to show that the strategy $\pi(\eta)$ is optimal for a single agent with wealth M_0.

Thus, let us consider the one-person dynamic programming problem of an agent with wealth s, when the money-supply is M. The *Bellman equation* for the optimal reward function $V(s,M)$ takes the form

$$V(s,M) = \sup_{0\le b\le s+\eta M/(1+\rho)} \left[Eu\left(\frac{b}{p_Y(M)}\right) + \beta\cdot V((1+\rho)(s-b)+\eta M, \tau M)\right]. \quad (4.3)$$

It suffices to show that the optimal action at states of the form (M,M) is $c(M,M) = \eta M$, which attains then the supremum in (4.3) with $s=M$. We defer the proof of this fact to Section 6, where it will be seen to be a special case of Theorem 6.1; see Remark 6.1. ■

Remark 4.1: If the interest rate ρ satisfies $\beta(1+\rho)=1$, we get $\eta=\tau=1$. Then, in the equilibrium of Theorem 4.1, the money-supply stays fixed at $M=M_0$, and the successive prices $M/Y_1, M/Y_2, \ldots$ are independent and identically distributed, just as in the equilibrium of Theorem 3.1: there is no inflation or deflation in the economy. Indeed, with this choice of interest rate, agents neither borrow nor lend, the bank is inactive, and we recover the equilibrium of Theorem 3.1.

5 A High-Information Model without a Bank

We now assume that agents know the value of the endowment variable Y in each period before making their bids; this is the "high-information" model of subsection 2.1(ii). (Of course, agents do not know the values of their endowment variables for future periods.) For the model of this section, we also assume that there is no bank. Thus the money supply M remains constant, as it did for the low-information model without a bank. We also assume in this section that the utility function satisfies condition (4.2) and that the expectations $E[u(Y)]$ and $E[Yu'(Y)]$ are well-defined and finite.

Suppose that every agent begins with wealth M, knowing that the value of the first endowment variable Y_1 is y. Suppose further, that in this state (M,y), every agent bids the amount $b(y)\in(0,M]$. Then $b(y)$ is also the total bid, so the price of the commodity is

$$p(y) = \frac{b(y)}{y}. \quad (5.1)$$

Then each agent receives in utility

$$u\left(\frac{b(y)}{p(y)}\right) = u(y)\,,$$

and starts the next period with wealth

$$M - b(y) + yp(y) = M.$$

Thus, the marginal utility of an agent's first bid is

$$\frac{\partial}{\partial b}\left(u\left(\frac{b}{p(y)}\right)\right)\bigg|_{b=b(y)} = \frac{1}{p(y)}\,u'\left(\frac{b(y)}{p(y)}\right) = \frac{u'(y)}{p(y)} = \frac{yu'(y)}{b(y)},$$

and the expected, discounted marginal utility from bidding $b(Y)$ on the next day, is

$$\beta\cdot E\left[\frac{u'(Y)}{p(Y)}\right] = \beta\cdot E\left[\frac{Yu'(Y)}{b(Y)}\right].$$

To avoid the possibility of arbitrage, we need the following condition:

No-arbitrage condition: We have

$$\frac{u'(y)}{p(y)} \geq \beta \cdot E\left[\frac{u'(Y)}{p(Y)}\right], \quad \forall\, y \in \mathcal{Y}$$

with equality holding if $b(y) < M$ and strict inequality holding if $b(y) = M$. Equivalently,

$$\min\left[M - b(y)\ ,\ \frac{yu'(y)}{b(y)} - \beta \cdot E\left(\frac{Yu'(Y)}{b(Y)}\right)\right] = 0\,, \quad \forall\, y \in \mathcal{Y}. \tag{5.2}$$

Our construction of an equilibrium depends on the existence of prices that satisfy this condition.

Lemma 5.1: *There exist bids $b(y) \in (0, M]$ satisfying the no-arbitrage condition (5.2).*

Proof: For each $a \in (0, \infty)$, define bids

$$b_a(y) \triangleq \left\{\begin{array}{ll} \frac{yu'(y)}{a} & ;\text{ for } \quad yu'(y) \leq aM \\ M & ;\text{ for } \quad yu'(y) > aM \end{array}\right\} \;=\; M \wedge \left(\frac{yu'(y)}{a}\right) \tag{5.3}$$

and prices

$$p_a(y) \triangleq \frac{b_a(y)}{y} = \left\{\begin{array}{ll} \frac{u'(y)}{a} & ;\text{ for } \quad yu'(y) \leq aM \\ \frac{M}{y} & ;\text{ for } \quad yu'(y) > aM \end{array}\right\} \;=\; \left(\frac{u'(y)}{a}\right) \wedge \left(\frac{M}{y}\right).$$

Thus

$$\frac{u'(y)}{p_a(y)} \;=\; \left\{\begin{array}{ll} a & ;\text{ for } \quad yu'(y) \leq aM \\ \frac{yu'(y)}{M} & ;\text{ for } \quad yu'(y) > aM \end{array}\right\} \;=\; a \vee \left(\frac{yu'(y)}{M}\right). \tag{5.4}$$

Then there exists a number $a \in (0, \infty)$ such that $a = \beta \cdot E\left(\frac{u'(Y)}{p_a(Y)}\right)$. Indeed, by (5.4), this requirement is equivalent to

$$\frac{1}{\beta} = E\left(1 \vee \left(\frac{Yu'(Y)}{aM}\right)\right).$$

Now the right-hand side $f(a) \triangleq E[1 \vee (Yu'(Y)/aM)]$ defines a continuous, decreasing function $f : (0, \infty) \to [1, \infty)$ with $f(0+) = \infty$ and $f(\infty) = 1$. Since $0 < \beta < 1$, there obviously exists $a \in (0, \infty)$ with $f(a) = \beta^{-1}$.

For such a, we take $b(y) \equiv b_a(y)$ and $p(y) \equiv p_a(y)$. The bids $p(y)$ then satisfy the no-arbitrage condition. ■

From now on, $b(y)$ and $p(y)$ will denote the bids and no-arbitrage prices constructed in Lemma 5.1. We also set

$$a = \beta \cdot E\left(\frac{u'(Y)}{p(Y)}\right). \tag{5.5}$$

Theorem 5.1: *There is an equilibrium in which every agent has wealth equal to the money supply M and bids $b(y)$, whenever y is the value of the endowment variable.*

For the proof, consider a single agent $\alpha \in I$ and suppose that every other agent $\gamma \neq \alpha$ has wealth M and bids $b(y)$ when $Y = y$. Then the single agent $\alpha \in I$, with wealth $s \geq 0$ and knowing $Y = y$, faces a dynamic programming problem with optimal reward function $V(s, y)$ satisfying the Bellman equation

$$V(s,y) = \sup_{0 \leq b \leq s} \left[u\left(\frac{b}{p(y)} \right) + \beta \cdot E[V(s - b + b(y), Y)] \right] , \quad s \geq 0, \ y \in \mathcal{Y}.$$

Notice that this dynamic programming problem has state space $[0, \infty) \times \mathcal{Y}$, action sets $\mathcal{A}(s, y) = [0, s]$, law of motion

$$(s, y) \to (s - b + b(y), Y)$$

under action b, and daily reward $r((s, y), b) = u(b/p(y))$. It suffices to show that the optimal bid b at state (M, y) is $b(y)$, for every $y \in \mathcal{Y}$.

To prove that this is so, we introduce *another* dynamic programming problem with the same states (s, y), and law of motion, but with larger action sets $\tilde{\mathcal{A}}(s, y) = [-M, s]$ and with daily reward for taking action b at (s, y) equal to

$$\tilde{r}((s, y), b) = u_y(b) = A_y + \lambda_y b, \quad -M \leq b < \infty,$$

where

$$\lambda_y \triangleq \frac{u'(y)}{p(y)} = a \vee \left(\frac{y u'(y)}{M} \right) , \qquad A_y \triangleq u(y) - \lambda_y b(y). \tag{5.6}$$

Notice that

$$\begin{aligned} u_y(b(y)) &= u(y) = u\left(\frac{b(y)}{p(y)} \right), \\ u_y'(b(y)) &= \lambda_y = \frac{1}{p(y)} \ u'(y) = \frac{1}{p(y)} \ u'\left(\frac{b(y)}{p(y)} \right). \end{aligned}$$

Thus, the affine function $u_y(\cdot)$ is the tangent line to the graph of the concave function $b \mapsto u(b/p(y))$ at the point $b = b(y)$. In particular, $u_y(b) \geq u(b/p(y))$ for all $b \in [0, s]$. Consequently, the expected return from any strategy π, which is available in both problems, will be at least as large in the modified problem as it was in the original problem.

Let π^* be the strategy that, at each state (s, y), uses action

$$b^*(s, y) \triangleq \left\{ \begin{matrix} s & ; \text{if } \ \lambda_y > a \\ b(y) + s - M & ; \text{if } \ \lambda_y \leq a \end{matrix} \right\} \in \mathcal{A}(s, y) \, . \tag{5.7}$$

Notice that, for every $y \in \mathcal{Y}$, we have

$$b^*(M, y) \triangleq \left\{ \begin{matrix} M & ; \text{if } \ \lambda_y > a \\ b(y) & ; \text{if } \ \lambda_y \leq a \end{matrix} \right\} = b(y) \, ,$$

and that under the law of motion (2.5)

$$(M, y) \to (M - b^*(M, y) + b(y), Y) = (M, Y).$$

Thus, for an initial state (M, y), the return from π^* is the same in both problems; namely,

$$u(y) + \frac{\beta}{1 - \beta} \ E[u(Y)].$$

It now suffices to show that the strategy π^* is optimal in the modified problem, for it must then be optimal at states (M, y) in the original problem as well.

Let $W(s, y)$ be the optimal reward function in the modified problem. Then W satisfies the Bellman equation

$$W(s, y) = (TW)(s, y),$$

where T is the operator

$$(T\Phi)(s, y) \triangleq \sup_{-M \leq b \leq s} [u_y(b) + \beta \cdot E\Phi(s - b + b(y), Y)], \tag{5.8}$$

whose domain is the collection of functions $\Phi : [0, \infty) \times \mathcal{Y} \to \mathbb{R}$ for which the right-hand side of (5.8) is well-defined.

Define $Q(s, y)$ to be the expected return in the modified problem from the strategy π^* at the initial state (s, y), and let

$$v(y) \triangleq Q(M, y) = u(y) + \frac{\beta}{1 - \beta} E[u(Y)]. \tag{5.9}$$

Clearly $Q(s, y) \leq W(s, y)$, and $E[v(Y)] = (1 - \beta)^{-1} E[u(Y)]$, so

$$v(y) = u(y) + \beta \cdot E[v(Y)]. \tag{5.10}$$

Lemma 5.2: *For every initial state* (s, y)*, we have:* (i) $Q(s, y) = v(y) + \lambda_y(s - M)$*, and* (ii) $(TQ)(s, y) = Q(s, y)$.

Proof: By (5.7), we have

$$s - b^*(s, y) + b(y) = M. \tag{5.11}$$

Hence,

$$\begin{aligned} Q(s, y) &= u_y(b^*(s, y)) + \beta \cdot E[Q(M, Y)] \\ &= A_y + \lambda_y b^*(s, y) + \beta \cdot E[v(Y)] \\ &= A_y + \lambda_y b(y) + \lambda_y(s - M) + \beta \cdot E[v(Y)] \\ &= u(y) + \lambda_y(s - M) + \beta \cdot E[v(Y)] \\ &= v(y) + \lambda_y(s - M), \end{aligned} \tag{5.12}$$

thanks to (5.10), and (i) is verified. To verify (ii), let

$$\begin{aligned} \psi_y(b) &\triangleq u_y(b) + \beta \cdot E[Q(s - b + b(y), Y)] \\ &= A_y + \lambda_y b + \beta \cdot E[v(Y) + \lambda_Y(s - b + b(y) - M)]. \end{aligned} \tag{5.13}$$

The coefficient of b in this expression is

$$\lambda_y - \beta \cdot E[\lambda_Y] = \frac{u'(y)}{p(y)} - \beta \cdot E\left[\frac{u'(Y)}{p(Y)}\right].$$

By (5.2), this coefficient is positive for $\lambda_y > a$, and the maximum of $\psi_y(\cdot)$ on the interval $[-M, s]$ is then attained at $b^*(s, y) = s$; whereas for $\lambda_y = a$, the coefficient is zero and in this case every point of the interval, including $b^*(s, y)$, attains the maximum. In either case, we have: $(TQ)(s, y) = \max_{0 \leq b \leq s} \psi_y(b) = \psi_y(b^*(s, y)) = u_y(b^*(s, y)) + \beta \cdot E[Q(M, Y)] = Q(s, y)$, from (5.13) and (5.11), (5.12). ■

Proof of Theorem 5.1: We first consider the special case in which the utility function $u(\cdot)$ is bounded from below by a finite constant, say λ. Then, under condition (4.2), it can be checked that the function $(b, y) \mapsto u_y(b)$ is bounded from below on the set $[-M, s] \times \mathcal{Y}$; see the Appendix. Thus, by adding a positive constant to the daily reward, we obtain an equivalent problem with positive daily rewards. (Indeed, by adding a constant, say $d = -\lambda$, to the daily reward, we merely add $d/(1-\beta)$ to the total discounted reward.) In this case, a theorem of Blackwell (1966) states that the optimal reward function $W(\cdot)$ is the least nonnegative fixed point of the operator T. But $Q(\cdot)$ is such a fixed point, and $Q(\cdot) \leq W(\cdot)$ by definition, so $Q(\cdot) = W(\cdot)$ and π^* is optimal. Thus, for $u(\cdot)$ satisfying the assumptions of this section, we have

$$Q(M, y) = V(M, y) = u(y) + \frac{\beta}{1-\beta}\, E[u(Y)]. \tag{5.14}$$

Suppose now that $u(\cdot)$ is not bounded from below. For each $\theta > 0$, let $u^\theta(\cdot)$ be the concave truncation of $u(\cdot)$ defined by

$$u^\theta(\cdot) \triangleq \left\{ \begin{matrix} u(x) & ; \text{if } \ x \geq \theta \\ \ell^\theta(x) & ; \text{if } \ 0 \leq x < \theta \end{matrix} \right\},$$

where $\ell^\theta(\cdot)$ is the tangent line to $u(\cdot)$ at $x = \theta$. Then $u^\theta(\cdot) \geq u(\cdot)$ is bounded from below on $[0, \infty)$. Let $Q_\theta(\cdot)$ and $V_\theta(\cdot)$ denote the functions corresponding to $Q(\cdot)$ and $V(\cdot)$, respectively, for the problem with utility function $u^\theta(\cdot)$. Then we have

$$V(M, y) \leq V_\theta(M, y) = Q_\theta(M, y) = u^\theta(y) + \frac{\beta}{1-\beta} E[u^\theta(Y)].$$

Let $\theta \to 0$ and use the monotone convergence theorem to conclude that (5.14) remains true in the general case.

Remark 5.1 In the equilibrium of Theorem 5.1, the money-supply stays constant and the successive prices are $p(Y_1) = b(Y_1)/Y_1$, $p(Y_2) = b(Y_2)/Y_2, \ldots.$ As in Theorem 3.1, prices fluctuate as independent, identically distributed random variables. There is no inflation or deflation in this economy. Unlike Theorem 3.1, the prices here depend on the utility function through the no-arbitrage condition (5.2), or equivalently the form (5.3) of the optimal bid.

We conclude this section with a simple example with two types of agents who differ only in their information. As the example illustrates, differences in information may result in differences in wealth and consumption.

Example 5.1: Assume that there is no bank, every agent is risk neutral with utility function $u(x) \equiv x$, and the endowment variable Y takes on the values 1 and 5 with probability 1/2 each. Let the discount factor be $\beta = 1/2$ and let the supply of money held by the agents be $M = 1$. Finally suppose that half of the agents, called type 1, have low information in that they have no knowledge of the endowment variable Y before bidding in each period; and that the other half of the agents, called type 2, have high information in that they do know Y before bidding.

Then there is an equilibrium with two wealth states: In the first, type 1 agents have wealth $s = 1$ and type 2 agents have the same wealth $\tilde{s} = 1$; in the second type 1 agents have wealth $s = 3/5$ and type 2 agents have wealth $\tilde{s} = 7/5$. It can be shown that, in equilibrium, an optimal strategy for type 1 agents is always to bid their entire wealth, and an optimal strategy for type 2 agents is to bid all if $Y = 5$, but to bid 1/5 if $Y = 1$

and $\tilde{s} = 1$ and to bid 3/5 if $Y = 5$ and $\tilde{s} = 7/5$. The price depends on the value of Y. For example, if $s = \tilde{s} = 1$ and $Y = 1$, then the total bid is

$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{5} = \frac{3}{5}$$

and the price is

$$p_1 = \frac{3/5}{1} = 3/5.$$

The law of motion gives the new wealth values for the two types as

$$s_1 = 1 - 1 + \frac{3}{5} \cdot 1 = \frac{3}{5},\ \tilde{s}_1 = 1 - \frac{1}{5} + \frac{3}{5} \cdot 1 = \frac{7}{5}.$$

If $s = \tilde{s} = 1$ and $Y = 5$, then the price is

$$p_2 = \frac{\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1}{5} = \frac{1}{5},$$

and the new wealth values are

$$s_1 = \tilde{s}_1 = 1 - 1 + \frac{1}{5} \cdot 5 = 1.$$

Similar calculations show that for $s = 3/5$, $\tilde{s} = 7/5$, and $Y = 1$, the price is $p_1 = 3/5$ and the next wealth values are $s_1 = 3/5$, $\tilde{s}_1 = 7/5$; while for $s = 3/5$, $\tilde{s} = 7/5$, $Y = 5$, the price is $p_2 = 1/5$ and $s_1 = \tilde{s}_1 = 1$. If the economy is equally likely to start in either of the two wealth states (1,1) and (3/5,7/5), then another easy calculation shows that the average daily utility earned by type 1 agents is 8/3 and that earned by the better-informed type 2 agents is 10/3.

6 A High-Information Model with a Bank

In this section we assume that agents know the value of their endowment in each period before bidding, and may also borrow or deposit money in a bank at a fixed interest rate $\rho \in (0, \infty)$. We shall construct an equilibrium in which every agent has the same wealth, equal to the money-supply in every period, and every agent bids an amount proportional to his wealth but depending on the value y of the endowment variable. We assume in this section, just as we did in Section 5, that the utility function $u(\cdot)$ satisfies conditions (4.2), and that $E[u(Y)]$ and $E[Yu'(Y)]$ are both finite.

Suppose that at wealth M with observed Y-value y, every agent bids an amount

$$b_y(M) = \eta(y)M. \tag{6.1}$$

proportional to his welath. The price is formed as

$$p_y(M) = \frac{\eta(y)M}{y}\ , \tag{6.2}$$

and in the next period each agent's wealth, as well as the money supply, become

$$\begin{aligned} M_1 &= (1 + \rho)(M - b_y(M) + yp_y(M) \\ &= (1 + \rho - \rho\eta(y))M \ = \ \tau(y)M, \end{aligned} \tag{6.3}$$

where $\tau(y) \triangleq 1+\rho-\rho\eta(y)$ is the "inflation-rate" corresponding to the endowment value $Y=y$. Thus $M_1>0$ and $p_y(M)>0$, provided that

$$0<\eta(y)<(1+\rho)/\rho \tag{6.4}$$

for all $y \in \mathcal{Y}$. From the bid $b_y(M)$, each agent receives utility

$$u\left(\frac{b_y(M)}{p_y(M)}\right)=u(y),$$

and his marginal utility at the current price is

$$\frac{\partial}{\partial b}\left(u\left(\frac{b}{p_y(M)}\right)\right)_{b=b_y(M)}=\frac{1}{p_y(M)}u'\left(\frac{b_y(M)}{p_y(M)}\right)=\frac{u'(y)}{p_y(M)}=\frac{yu'(y)}{\eta(y)M}. \tag{6.5}$$

If the agent carries a unit of money over to the next period, the expected marginal utility to be earned, in view of interest and discounting, is

$$\beta(1+\rho)\cdot E\left[\frac{u'(Y)}{p_Y(\tau(y)M)}\right]=\frac{\beta(1+\rho)}{\tau(y)}\cdot E\left[\frac{Yu'(Y)}{\eta(Y)M}\right]. \tag{6.6}$$

Equating the expressions in (6.5) and (6.6), we obtain the following:

No-arbitrage condition: We have

$$\frac{yu'(y)}{\eta(y)}=\frac{\beta(1+\rho)}{\tau(y)}\cdot\ E\left[\frac{Yu'(Y)}{\eta(Y)}\right], \quad \forall\, y\in\mathcal{Y}. \tag{6.7}$$

Lemma 6.1: *There is a unique function* $\eta:(0,\infty)\to(0,(1+\rho)/\rho)$ *satisfying the no-arbitrage condition* (6.7), *namely,*

$$\eta(y)=\frac{1+\rho}{\rho}\left[\frac{(1-\beta)yu'(y)}{(1-\beta)yu'(y)+\beta E[Yu'(Y)]}\right], \quad \forall\, y\in\mathcal{Y}. \tag{6.8}$$

Proof: Rewrite (6.8) as

$$\frac{1}{\eta(y)}=\frac{\rho}{1+\rho}\left[1+\frac{\beta}{1-\beta}\cdot\frac{E[Yu'(Y)]}{yu'(y)}\right]. \tag{6.9}$$

Multiply through by the term $yu'(y)$ and take the expectation with respect to the distribution of Y, to obtain

$$\vartheta \triangleq E\left[\frac{Yu'(Y)}{\eta(Y)}\right]=\frac{\rho E[Yu'(Y)]}{(1+\rho)(1-\beta)}$$

or equivalently

$$E[Yu'(Y)]=\frac{(1+\rho)(1-\beta)\vartheta}{\rho}.$$

Substituting in (6.9), we have

$$\frac{1}{\eta(y)}=\frac{\rho}{1+\rho}+\frac{\beta\vartheta}{yu'(y)}$$

or, equivalently,

$$\frac{yu'(y)}{\eta(y)}=\frac{\beta(1+\rho)\vartheta}{1+\rho-\rho\eta(y)}.$$

But this is the no-arbitrage equation (6.7). The argument reverses, to prove uniqueness. ■

For the rest of this section, let $\eta(\cdot)$ be the function defined by (6.8). Also, let $\pi(\eta(\cdot))$ be a strategy that bids $b_y(M) = \eta(y)M$ in a time-period when the money-supply is M and the value of the endowment variable Y is y.

Theorem 6.1: *There is an equilibrium in which every agent has initial wealth equal to the money supply M_0 and uses the strategy $\pi(\eta(\cdot))$.*

The proof is similar to that of Theorem 5.1. We consider a single agent, and assume that every other agent begins with wealth $M = M_0$ and follows the strategy $\pi(\eta(\cdot))$. A single agent with wealth $s \geq 0$, knowing the value of Y is y, faces a dynamic programming problem with optimal reward function $V(s, y, M)$ which satisfies the Bellman equation:

$$V(s,y,M) = \sup_{0 \leq b \leq s + b_y(M)/(1+\rho)} \left[u\left(\frac{b}{p_y(M)}\right) + \beta \cdot E[V((1+\rho)(s-b) + b_y(M), Y, \tau(y)M)] \right]. \tag{6.10}$$

This dynamic programming problem has state space $[0,\infty) \times \mathcal{Y} \times [0,\infty)$, action sets $\mathcal{A}(s,y,M) = [0, s + b_y(M)/(1+\rho)]$, law of motion

$$(s,y,M) \to ((1+\rho)(s-b) + b_y(M), Y, \tau(y)M)$$

under action b, and daily reward $r((s,y,M),b) = u(b/p_y(M))$. It suffices to show that an optimal bid b at states of the form (M,y,M) is $b_y(M)$. For this will imply that $\pi(\eta(\cdot))$ is an optimal strategy for the single agent, when all other agents play $\pi(\eta(\cdot))$.

As in the proof of Theorem 5.1, we introduce a modified dynamic programming problem with the same states (s,y,M), and law of motion, but with larger action sets $\tilde{\mathcal{A}}(s,y,M) = [-M, s + b_y(M)/(1+\rho)]$, and with the daily reward

$$\tilde{r}((s,y,M),b) = u_{y,M}(b) = A_{y,M} + \lambda_{y,M} b, \tag{6.11}$$

for taking action b at state (s,y,M), where

$$\lambda_{y,M} \triangleq \frac{u'(y)}{p_y(M)} = \frac{yu'(y)}{M\eta(y)}, \qquad A_{y,M} = u(y) - \lambda_{y,M} b_y(M). \tag{6.12}$$

The affine function $u_{y,M}(\cdot)$ is the tangent line to the concave function $b \mapsto u(b/p_y(M))$ at the point $b = b_y(M)$. Thus $u_{y,M}(b) \geq u(b/p_y(M))$ for all b; so the return from any strategy available in both problems is at least as large in the modified problem as in the original problem.

Let π^* be the strategy that, at each state (s,y,M) uses the action

$$b^*(s,y,M) = b_y(M) + (s - M). \tag{6.13}$$

Since

$$b^*(M,y,M) = b_y(M),$$

and, under the law of motion (2.5),

$$(M,y,M) \ \to \ ((1+\rho)(M - b_y(M)) + b_y(M), Y, \tau(y)M) \ = \ (\tau(y)M, Y, \tau(y)M)\,,$$

we see that the strategies π^* and $\pi(\eta(\cdot))$ choose the same actions and have the same expected return for an initial state of the form (M,y,M). Thus, if π^* is optimal in the modified problem, then $\pi(\eta(\cdot))$ must be optimal in the original problem as well, for initial states (M, y, M).

To see that π^* is optimal in the modified problem, let $W(s, y, M)$ be the optimal reward function, and let $Q(s, y, M)$ be the expected return from π^*, for an initial state (s, y, M). The Bellman equation can be written as

$$W(s, y, M) = (TW)(s, y, M)$$

where

$$(T\Phi)(s, y, M) \triangleq \sup_{-M \le b \le s + b_y(M)/(1+\rho)} [u_{y,M}(b) + \beta \cdot E(\Phi((1+\rho)(s-b) + b_y(M), Y, \tau(y)M))] \tag{6.14}$$

is an operator acting on functions $\Phi : [0, \infty) \times \mathcal{Y} \times [0, \infty) \to \mathbb{R}$, for which the expectation on the right-hand side of the equation (6.14) above is well-defined. By analogy with (5.9), we also define

$$v(y) \triangleq Q(M, y, M) = u(y) + \frac{\beta}{1-\beta} \cdot E[u(Y)] \tag{6.15}$$

and observe that

$$v(y) = u(y) + \beta \cdot E[v(Y)]. \tag{6.16}$$

Lemma 6.2: *For every initial state* (s, y, M)*, we have* (i) $Q(s, y, M) = v(y) + \lambda_{y,M}(s - M)$*, and* (ii) $(TQ)(s, y, M) = Q(s, y, M)$.

Proof: (i) By (6.1), (6.3), and (6.13), we have

$$(1+\rho)(s - b^*(s, y, M)) + b_y(M) \;=\; (1+\rho)(M - b_y(M) + b_y(M) \;=\; \tau(y)M.$$

Hence, (6.11)–(6.13) and (6.15), (6.16) imply

$$\begin{aligned} Q(s, y, M) &= u_{y,M}(b^*(s, y, M)) + \beta \cdot E[Q(\tau(y)M, Y, \tau(y)M)] \\ &= A_{y,M} + \lambda_{y,M} b^*(s, y, M) + \beta \cdot E[v(Y)] \\ &= u(y) - \lambda_{y,M} b_y(M) + \lambda_{y,M}(b_y(M) + s - M) + \beta \cdot Ev(Y) \\ &= v(y) + \lambda_{y,M}(s - M). \end{aligned}$$

(ii) Define

$$\psi(b) \triangleq u_{y,M}(b) + \beta \cdot E[Q((1+\rho)(s-b) + b_y(M), Y, \tau(y)M)],$$

and observe that Part (i) implies

$$\psi(b) = A_{y,M} + \lambda_{y,M} b + \beta \cdot E[v(Y) + \lambda_{Y,\tau(y)M}((1+\rho)(s-b) + b_y(M) - \tau(y)M)].$$

The coefficient of b on the right-hand side of this expression is

$$\lambda_{y,M} - \beta(1+\rho) \cdot E[\lambda_{Y,\tau(y)M}] \;=\; \frac{u'(y)}{p_y(M)} - \frac{\beta(1+\rho)}{\tau(y)} \cdot E\left[\frac{u'(Y)}{p_Y(M)}\right] \;=\; 0\,,$$

by the no-arbitrage condition (6.7). Thus $\psi(\cdot)$ is a constant function of b, and (ii) follows trivially. ∎

Proof of Theorem 6.1: As in the proof of Theorem 5.1, we first assume that $u(\cdot)$ is bounded from below. It can then be checked that the function $(b, y, M) \mapsto u_{y,M}(b)$ as in (6.1) is bounded from below, uniformly for $b \in \tilde{\mathcal{A}}(s, y, M)$, $s > 0$, $y \in \mathcal{Y}$ and $M > 0$ (see the Appendix). Then a theorem of Blackwell (1966), and the fact that $Q(\cdot)$ is a fixed point of the operator T in (6.14), show that $Q(\cdot) = W(\cdot)$ and that π^* is optimal for the modified problem. Thus, the strategy $\pi(\eta(\cdot))$ is optimal in the original problem, for initial states (M, y, M). The case when $u(\cdot)$ is not bounded from below can be handled by a truncation argument, just as it was in Section 5. The proof of Theorem 6.1 is now complete. ■

We go on a brief digression, in order to complete the proof of Theorem 4.1.

Remark 6.1: Recall the one-person problem introduced in the proof of Theorem 4.1, with optimal reward function satisfying (4.1). This problem corresponds to a special case of the one-person problem in the proof of Theorem 6.1. To obtain the special case, we replace the utility function of Section 4 by

$$\tilde{u}(x) \triangleq E[u(xY)],$$

and then replace Y in Theorem 6.1 by the constant variable $\tilde{Y} \equiv 1$. The Bellman equation (4.1) then is equivalent to (6.10) with $p_y(M)$ replaced by $p_1(M) = \eta M$, $b_y(M)$ replaced by $b_1(M) = \eta M$, and $\tau(y)$ replaced by $\tau(1) = \tau = \beta(1 + \rho)$. ■

The rest of this section is devoted to an exploration of the properties of the successive values of the money-supply $\{M_n\}$ and of the prices $\{p_n\}$, under the assumption that all agents begin with wealth M_0 and follow the strategy $\pi(\eta(\cdot))$ as in the equilibrium of Theorem 6.1. First observe that, by repeated applications of (6.2) and (6.3), we have

$$M_n = M_0 \left(\prod_{k=1}^{n} \tau(Y_k) \right), \quad p_n = p_{Y_n}(M_{n-1}) = \frac{\eta(Y_n)}{Y_n} \cdot M_{n-1}\,, \tag{6.17}$$

for all $n = 1, 2, \ldots$. Thus, the key to an understanding of $\{M_n\}$ and, consequently $\{p_n\}$, is the random variable $\tau(Y)$. Indeed, we see from (6.17) that the money-supply and the prices will increase, decrease or stay the same in expectation, accordingly as $E[\tau(Y)]$ is greater than, less than, or equal to, one.

Recall first from (6.3) that, by definition, $\tau(y) = 1 + \rho - \rho\eta(y)$. Thus, by (6.8), we have the *generalized Fisher equation*

$$\tau(y) = \beta(1 + \rho) \cdot \frac{E[Yu'(Y)]}{(1 - \beta)yu'(y) + \beta \cdot E[Yu'(Y)]} \tag{6.18}$$

for the rate of inflation, as a function of the observed endowment value $y \in \mathcal{Y}$. Note also that, in contrast to the low-information case of Section 4, equation (4.1), the inflation rate depends now also on the shape of the utility function $u(\cdot)$.

By assumption, $yu'(y)$ is positive for every value y of Y; hence

$$0 < \tau(y) \leq 1 + \rho. \tag{6.19}$$

In the special case when $Yu'(Y)$ is a.s. constant (e.g., for degenerate income-variables Y, or for logarithmic utility functions), we have $\tau(y) = \beta(1 + \rho)$; in other words, we recover the Fisher equation of (4.1), valid for the low-information model of Section 4.

Theorem 6.2: (i) *If $Yu'(Y)$ is not constant, and $\beta(1+\rho) = 1$, then $E[\tau(Y)] > 1$. In other words, even for the "natural interest rate" $\rho_* = (1/\beta) - 1$, there is inflationary pressure in such an economy.*

(ii) *For $\rho > 0$ sufficiently small, we have $E[\tau(Y)] < 1$.*

(iii) *There is a unique interest rate $\rho = \rho^*$ such that $E[\tau(Y)] = 1$; indeed, we have*

$$1 + \rho^* = 1/\beta^* , \qquad \text{where} \qquad \beta^* \triangleq \beta \cdot E\left[\frac{E[Yu'(Y)]}{(1-\beta)Yu'(Y) + \beta \cdot E[Yu'(Y)]}\right]. \tag{6.20}$$

Proof: By (6.18), we have

$$E[\tau(Y)] = \beta(1+\rho) \cdot E\left[\frac{E[Yu'(Y)]}{(1-\beta)Yu'(Y) + \beta E[Yu'(Y)]}\right],$$

so (iii) is immediate. For (i), suppose that $\beta(1+\rho) = 1$. Then, by Jensen's inequality,

$$E[\tau(Y)] > \frac{E[Yu'(Y)]}{(1-\beta)E[Yu'(Y)] + \beta E[Yu'(Y)]} = 1.$$

For (ii), notice that, by (6.18), we have

$$\lim_{\rho \downarrow 0} \tau(y) = \frac{\beta \cdot E[Yu'(Y)]}{(1-\beta)yu'(y) + \beta \cdot E[Yu'(Y)]} < 1,$$

and apply Lebesgue's dominated convergence theorem. ■

By (6.17) and Theorem 6.2, we see that, for $1 + \rho = 1/\beta$, the expected value of M_n approaches infinity as $n \to \infty$, as does the expected value of p_n. Similarly, for small positive values of ρ, the expectation of M_n approaches zero and so does the expectation of p_n if $E(\eta(Y)/Y) < \infty$.

Suppose now that $\rho = \rho^*$, the critical value of Theorem 6.2(iii), for which $E[\tau(Y)] = 1$. Then $E(M_n) = M_0$ is constant. However, the variance of M_n tends to infinity if Y is not constant. To see this, observe that

$$\mathrm{Var}(M_n) = E(M_n^2) - M_0^2$$

and, by (6.17),

$$E(M_n^2) \;=\; \prod_{k=1}^{n} E(\tau(Y_k)^2) \cdot M_0^2 \;=\; (E[(\tau(Y_1))^2])^n \cdot M_0^2 \, ,$$

where

$$E[\tau(Y_1)^2] > [E(\tau(Y_1))]^2 = 1.$$

Furthermore, the sequence $\{M_n\}$ converges to zero almost surely, because

$$\log M_n = \log M_0 + \sum_{k=1}^{n} \log \tau(Y_k) \, , \quad n = 1, 2, \cdots$$

is a random walk with expected increment

$$E[\log \tau(Y)] < \log(E[\tau(Y)]) = \log 1 = 0.$$

Remark: As long as Y is not constant, and $\log \tau(Y)$ has positive, finite variance, then by the central limit theorem

$$\frac{\log M_n - \log M_0 - nE[\log \tau(Y)]}{\sqrt{n\mathrm{Var}[\log \tau(Y)]}}$$

converges in distribution to a standard normal random variable. Thus, for n large, M_n is approximately log-normal. Likewise, for n large, given that $Y_n = y$, the price p_n is approximately log-normal by (6.17).

Example 6.1: Utility function of power-type. Let us take $u(y) = 2\sqrt{y}$, so that $yu'(y) = \sqrt{y}$, and $\beta = 0.95$.

(i) For the distribution

$$P[Y = 4] = P[Y = 16] = 1/2 \tag{6.21}$$

with $E(Y) = 10$, we have $E(\sqrt{Y}) = 3$ and the constant of (6.20) is

$$\beta^* = E\left(\frac{1}{\frac{1-\beta}{\beta} \cdot \frac{\sqrt{Y}}{E(\sqrt{Y})} + 1}\right) = 0.9503. \tag{6.22}$$

For $1 + \rho = 1/\beta = 1.0526$, we get $E[\tau(Y)] = 1.0003$ (expected inflation rate of 0.03%).
For $1 + \rho = 1.05$, we get $E[\tau(Y)] = 0.9978$ (expected deflation rate of 0.22%).
For $1 + \rho = 1/\beta^* = 1.0523$, we get $E[\tau(Y)] = 1$.

(ii) Now consider the distribution

$$P[Y = 10] = P[Y = 12] = P[Y = 8] = 1/3, \tag{6.23}$$

which has the same mean $E(Y) = 10$ as that of (6.21) but much smaller variance, and $E(\sqrt{Y}) = 3.1516$. For this distribution, the constant of (6.20) is

$$\beta^* = 0.950018, \tag{6.24}$$

smaller than the number of (6.22) as should be expected (due to the stronger concentration of the distribution of Y in this case), and just slightly larger than β.
For $1 + \rho = 1/\beta = 1.0526$, we have $E[\tau(Y)] = 1.00002$.
For $1 + \rho = 1.05$, we get $E[\tau(Y)] = 0.950038$.
For $1 + \rho = 1/\beta^* = 1.052612$, we get $E[\tau(Y)] = 1$.

Example 6.2: Utility function of exponential type. Let us take now $u'(y) = 2^{-y}$, and $\beta = 0.95$ once again.

(i) For the distribution of (6.21), we have $E[Y2^{-Y}] = 0.125122$ and

$$\beta^* = E\left(\frac{1}{\frac{1-\beta}{\beta} \cdot \frac{Y2^{-Y}}{E(Y2^{-Y})} + 1}\right) = 0.952372,$$

a quantity significantly larger than that of (6.22).
For $1 + \rho = 1/\beta = 1.0526$, we get $E[\tau(Y)] = 1.00247$ (approximately 0.25% expected inflation).

For $1+\rho = 1.04$, we get $E[\tau(Y)] = 0.9905$ (0.5% expected deflation).
For $1+\rho = 1/\beta_* = 1.05$, we get $E[\tau(Y)] = 1$.

(ii) If Y has the distribution of (6.23), then we obtain $\beta^* = 0.95158$ for the quantity of (6.20), again significantly larger than the number of (6.24) (corresponding to a utility function of power-type).
For $1+\rho = 1/\beta = 1.0526$, we get $E[\tau(Y)] = 1.001633$ (approximately 0.16% inflation).
For $1+\rho = 1.05$, we get $E[\tau(Y)] = 0.99916$ (0.084% deflation).
For $1+\rho = 1/\beta^* = 1.0509$, we have $E[\tau(Y)] = 1$.

7 A High-Information Model with an Active Bank

Suppose now that the bank sets an interest rate $\rho(y) \in (0,\infty)$ in each period based on the observed value y of the endowment variable Y in the period. As in the previous section, we assume that every agent has wealth equal to the money-supply at each stage, and bids

$$b_y(M) = \eta(y)M\,,$$

where the observed $Y-$value is y and the money-supply is M. The price is formed as in (6.2), and the calculation of (6.3) yields the next value of the money supply as

$$M_1 = \tau(y)M,$$

where now the rate of inflation is

$$\tau(y) \triangleq 1+\rho(y)-\rho(y)\eta(y). \tag{7.1}$$

We shall construct an equilibrium for this model, which generalizes that of Theorem 6.1. Then we shall consider the question of whether the bank can select the interest rates $\rho(y) > 0$ in such a way that $\tau(y) \equiv 1$ in equilibrium, thereby *keeping the money-supply and expected prices constant.* At the end of this section, we consider the more difficult problem for the bank of holding prices, rather than expected prices, constant. We conclude that this is typically impossible for our models. We continue to assume (4.2) and that $E[u(Y)]$ and $E[Yu'(Y)]$ are finite.

The no-arbitrage condition (6.7) is replaced in this section by

$$\frac{yu'(y)}{\eta(y)} = \frac{\beta(1+\rho(y))}{\tau(y)} \cdot E\left[\frac{Yu'(Y)}{\eta(Y)}\right]\,, \quad \forall\, y \in \mathcal{Y}\,. \tag{7.2}$$

Lemma 7.1: *Given the interest rate function $\rho(\cdot)$, there is a unique function $\eta(\cdot)$ such that $0 < \eta(y) < (1+\rho(y))/\rho(y)$ and (7.2) holds, namely*

$$\frac{1}{\eta(y)} = \frac{\rho(y)}{1+\rho(y)} + \frac{\beta}{1-\beta} \cdot \frac{1}{yu'(y)} \cdot E\left(\frac{\rho(Y)}{1+\rho(Y)} \cdot Yu'(Y)\right). \tag{7.3}$$

Proof: Let $\vartheta \triangleq E[Yu'(Y)/\eta(Y)]$ and rewrite (7.2) as

$$\frac{yu'(y)}{\eta(y)} = \frac{\beta(1+\rho(y))\vartheta}{1+\rho(y)-\rho(y)\eta(y)},$$

or equivalently,

$$\frac{yu'(y)}{\eta(y)} = \frac{yu'(y)\rho(y)+\beta(1+\rho(y))\vartheta}{1+\rho(y)} = \beta\vartheta + yu'(y) \cdot \frac{\rho(y)}{1+\rho(y)}. \tag{7.4}$$

Integrating in this expression with respect to y according to the distribution λ of Y, we obtain

$$\vartheta = \beta\vartheta + E\left[\frac{\rho(Y)}{1+\rho(Y)} \cdot Yu'(Y)\right],$$

or equivalently,

$$\vartheta = \frac{1}{1-\beta} \cdot E\left[\frac{\rho(Y)}{1+\rho(Y)} \cdot Yu'(Y)\right]. \tag{7.5}$$

Substitute this expression for ϑ in (7.4) and divide by $yu'(y)$, to obtain (7.3).

We have shown that condition (7.2) implies (7.3). It is not difficult to reverse the argument, to see that (7.3) implies (7.2). ■

Theorem 7.1: *Let $0 < \rho(y) < \infty$, and let $\eta(y)$ be given by (7.3) for every y in the support of Y . Then there is an equilibrium, in which every agent has wealth equal to the money-supply at every stage, and bids $\eta(y)M$ when the money supply is M and the value of Y is y .*

We omit the proof of this result, as it is quite similar to the proof of Theorem 6.1.

Consider now the problem of selecting the function $\rho(\cdot)$ so that $\tau(y) = 1$ for all y. One possibility is to set $\rho(y) = 0$ for all y. However, there is then no natural limit on the amount an agent is allowed to borrow. For this reason, we shall continue to insist that $\rho(y) > 0$ for all y.

Theorem 7.2: *If* $yu'(y) > \beta \cdot E[Yu'(Y)]$ *for all* $y \in \mathcal{Y}$, then, with

$$\rho(y) \triangleq \frac{yu'(y)}{\beta \cdot E[Yu'(Y)]} - 1, \tag{7.6}$$

we have $\tau(y) \equiv 1$ in the equilibrium of Theorem 7.1.

Proof: Observe that (7.1), (7.3) give

$$\frac{1+\rho(y)}{\tau(y)} = 1 + \frac{1-\beta}{\beta} \cdot \frac{\frac{\rho(y)}{1+\rho(y)} \cdot yu'(y)}{E\left[\frac{\rho(Y)}{1+\rho(Y)} \cdot Yu'(Y)\right]},$$

and then that (7.6) leads to $\tau(y) \equiv 1$. ■

An obvious drawback to this result is that, in the absence of the strong conditions in Theorem 7.2, it can easily happen that some of the quantities on the right-hand side of (7.6) are negative. Furthermore, under these conditions, the result appears to be paradoxical in that the central bank selects the interest rate in each period in such a manner that the agents spend their entire wealth. Thus there is neither borrowing nor depositing. Despite all the efforts of the bank, it is superfluous and the equilibrium of Theorem 7.2 is equivalent to the high-information no-bank equilibrium of Theorem 5.1. Notice also that, if $Yu'(Y)$ is constant, we recover again $\rho(y) \equiv (1/\beta) - 1$, the non-inflationary interest rate of Section 4 for the low-information model with a bank.

Another goal for an active bank might be to hold prices exactly constant, rather than holding expected prices constant as in Theorem 7.2. However, if the endowment variable Y is not itself constant, *it is typically impossible for the bank to set interest rates so as* *to hold prices constant*. More precisely, there typically does not exist a type-symmetric equilibrium with constant prices, positive interest rates, and without the occurrence of bankruptcy. To avoid unenlightening technicalities, we shall give a proof only for the special case where the endowment variable Y takes two values a and c with respective probabilities

$$P[Y = a] = r\,, \qquad P[Y = c] = 1 - r\,,$$

where $0 < a < c$ and $0 < r < 1$. We assume this special stucture for the rest of the section.

Suppose that we want the price p_n in every period n to be the same, say $p_n \equiv 1$. Thus, for each value $y \in \mathcal{Y} \equiv \{a, c\}$, if $Y_n = y$, we require

$$p_n = B_n / y = 1.$$

For a type-symmetric equilibrium, we must have that the total bid B_n equals the bids $b(y)$ of the individual agents. Hence,

$$B_n = b(y) = y.$$

That is, every agent bids a when $Y = a$ and bids c when $Y = c$.

The no-arbitrage condition (7.2), which is necessary for the bids to be optimal, takes the form:

$$\frac{u'(y)}{1} = \beta(1 + \rho(y)) \cdot E\left[\frac{u'(Y)}{1}\right]\,, \qquad y \in \{a, c\}\,,$$

or equivalently

$$1 + \rho(a) = \frac{u'(a)}{\beta \cdot E[u'(Y)]}\,, \qquad 1 + \rho(c) = \frac{u'(c)}{\beta \cdot E[u'(Y)]}\,.$$

If $u'(c) < u'(a)$, then $u'(c) < E[u'(Y)]$. So, for β sufficiently close to 1, we have $1 + \rho(c) < 1$. However, for β sufficiently small, the interest rates $\rho(a)$, $\rho(c)$ are positive. We shall assume from now on that they are positive.

Next, we look at the behavior of the money supply, or equivalently, the cash holdings of an individual agent. The law of motion gives

$$M_{n+1} = (1 + \rho(Y_{n+1})) \cdot (M_n - Y_{n+1}) + Y_{n+1}.$$

An easy proof by induction shows that, if $Y_1 = Y_2 = \cdots = Y_n = y$, then,

$$M_n = (1 + \rho(y))^n \cdot (M_0 - y) + y.$$

Now consider possible values for the initial money-supply M_0. If $M_0 < c$ and $Y_1 = Y_2 = \cdots = Y_n = c$, then we have

$$M_n = (1 + \rho(c))^n \cdot (M_0 - c) + c \, < \, 0,$$

for n sufficiently large. In particular, bankruptcy occurs with positive probability. On the other hand, if $M_0 \geq c > a$, then it is easy to see from the law of motion that

$$M_{n+1} - M_n = \rho(Y_{n+1})(M_n - Y_{n+1}) \geq 0\,, \quad \forall\, n = 0, 1, \cdots\,,$$

and the inequality is strict when $Y_{n+1} = a$. Indeed, when $Y_{n+1} = a$, we have

$$M_{n+1} - M_n = \rho(a)(M_n - a) \geq \rho(a)(c - a) > 0.$$

Thus, with probability one, every agent eventually has wealth exceeding M_0. But then each agent can spend the excess over M_0, thereby earning additional utility, and continue with the original bidding strategy thereafter. Hence, the original strategy cannot be optimal, and we do not have an equilibrium.

8 Appendix

ON THE PROOF OF THEOREM 5.1: Under the assumption that $u(b) \geq \lambda > -\infty$ for all $b \geq 0$, we need to show that

$$u_y(b) \geq -k\,; \quad \forall \;\; b \in [-M, s]\,, \;\; y \in \mathcal{Y} \tag{8.1}$$

holds for some $k \in [0, \infty)$. Indeed,

$$\begin{aligned}
u_y(b) \;&=\; A_y + b\lambda_y \;\geq\; A_y - M\lambda_y \\
&=\; u(y) - \lambda_y\,[M + b(y)] \;=\; u(y) - [M + b(y)]\,\frac{u'(y)}{p(y)} \\
&\geq\; u(y) - 2M\,\frac{u'(y)}{p(y)} \;=\; u(y) - 2M\,\frac{yu'(y)}{b(y)} \\
&=\; \begin{Bmatrix} u(y) - 2aM & ;\text{ for } \;\; yu'(y) \leq aM \\ u(y) - 2yu'(y) & ;\text{ for } \;\; yu'(y) > aM \end{Bmatrix} \\
&\geq\; -\lambda - 2(\kappa \vee aM)
\end{aligned}$$

thanks to (5.1), (5.3), (5.6) and (4.2). This establishes (8.1) with $k = \lambda + 2(\kappa \vee aM)$.

ON THE PROOF OF THEOREM 6.1: Assuming that $u(b) \geq \lambda > -\infty$ for all $b \geq 0$, we need to show

$$u_{y,M}(b) \geq -k\,; \quad \forall \;\; b \in \left[-M, s + \frac{b_y(M)}{1+\rho}\right]\,, \;\; y \in \mathcal{Y}\,, M > 0 \tag{8.2}$$

for some $k \in [0, \infty)$. Indeed,

$$\begin{aligned}
u_{y,M}(b) \;&=\; A_{y,M} + b\lambda_{y,M} \;\geq\; A_{y,M} - M\lambda_{y,M} \\
&=\; u(y) - \lambda_{y,M}\,[M + b_y(M)] \;=\; u(y) - M[1 + \eta(y)]\,\frac{yu'(y)}{M\eta(y)} \\
&\geq\; u(y) - \frac{yu'(y)}{\eta(y)}\left[1 + \frac{1+\rho}{\rho}\right] \;=\; u(y) - \frac{\beta(1+\rho)\vartheta}{\tau(y)} \cdot \frac{1+2\rho}{\rho} \\
&=\; u(y) - \frac{\beta(1+2\rho)}{1-\beta} \cdot \frac{E[Yu'(Y)]}{\tau(y)} \\
&=\; u(y) - \frac{1+2\rho}{1+\rho}\left(yu'(y) + \frac{\beta}{1-\beta}\,E[Yu'(Y)]\right)
\end{aligned}$$

from (6.11), (6.12), (6.1), (6.4), the "no-arbitrage" condition (6.7), and (6.18). This shows that (8.2) is valid with

$$k = \lambda + \frac{1+2\rho}{1+\rho}\left(\kappa + \frac{\beta}{1-\beta}\,E[Yu'(Y)]\right),$$

thanks to condition (4.2).

9 Two Possible Extensions

It would be of considerable interest, to extend the results of this paper to a situation where the successive endowments

$$Y_1^\alpha, Y_2^\alpha, \ldots$$

of a particular agent $\alpha \in I$ are independent copies of the random variable Y, with common distribution λ, but can differ from agent to agent. (In a simpler context, such a situation was studied in our earlier work [KSS1], where there was no aggregate uncertainty, in the sense that $\int_I Y_n^\alpha(\omega)\varphi(d\alpha) = Q > 0$ was a positive constant for all $n \in \mathbb{N}$ and $\omega \in \Omega$.) In a context like this, one is unlikely to obtain such explicit results as in the present paper; nonetheless, some general, qualitative results about existence and characterization of equilibrium should be possible to achieve.

Another interesting extension of the models of this paper would be to generalize the information conditions. So far we have only allowed for the possibilities that agents and the bank either have no information or have perfect information about the value of the variable Y before making decisions in a period. A more general and more realistic assumption would be that they have partial information and that information may be different for different agents. Example 5.1 is a modest first step in this direction.

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