Bootstrapping Spurious Regression

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Bootstrapping Spurious Regression*

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Abstract

The bootstrap is shown to be inconsistent in spurious regression. The failure of the bootstrap is spectacular in that the bootstrap effectively turns a spurious regression into a cointegrating regression. In particular, the serial correlation coefficient of the residuals in the bootstrap regression does not converge to unity, so the bootstrap is not even first order consistent. The block bootstrap serial correlation coefficient does converge to unity and is therefore first order consistent, but has a slower rate of convergence and a different limit distribution from that of the sample data serial correlation coefficient. The analysis covers spurious regressions involving both deterministic trends and stochastic trends. The results reinforce earlier warnings about routine use of the bootstrap with dependent data.

JEL Classification: C22
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1. Introduction

The bootstrap is now a popular and practical tool of inference in econometric work. One aspect of its appeal is its wide applicability, enabling its use in many different econometric models and allowing for both cross section and time dependent data. In time series situations it is known to be important that the bootstrap capture accurately the temporal dependence properties of the original time series if it is to be a useful aid to inference. Two approaches are now in common use to deal with temporal dependence: the sieve bootstrap (Kreiss, 1992, and Buhlmann, 1997, 1998, among others), where a sequence of finite dimensional parametric models (like autoregressions) is used to remove temporal dependence; and block bootstrap methods (Carlstein, 1986, and Künsch, 1989) where blocking techniques are used to deal with dependence.

*Some of the results reported here were mentioned by the author at the York Econometrics Meeting in May, 2000. A draft version of the paper was written in February, 2001. The paper was typed in SW2.5. Computing and graphics were done in GAUSS.

If temporal dependence is poorly captured or ignored, then the bootstrap can not be expected to perform well even asymptotically and may well produce inconsistent estimates. Horowitz (1999) warns of the difficulties of using the bootstrap with dependent data, making uninformed applications of the bootstrap with dependent data one his bootstrap *Don'ts*. The present paper highlights these problems in a context of substantial econometric interest. We show that in spurious regressions use of the crude bootstrap (where dependence is ignored) produces a spectacular asymptotic failure. The crude bootstrap is shown to convert spurious regressions of full rank integrated time series into cointegrating regressions and spurious regressions of integrated times series on deterministic trends into trend stationary time series. The paper proves these results using almost sure invariance principles and the Loève Karhunen (LK) representation of Brownian motion used recently in Phillips (1998) for studying spurious regressions. The methods are likely to be useful in other applications involving nonstationary time series.

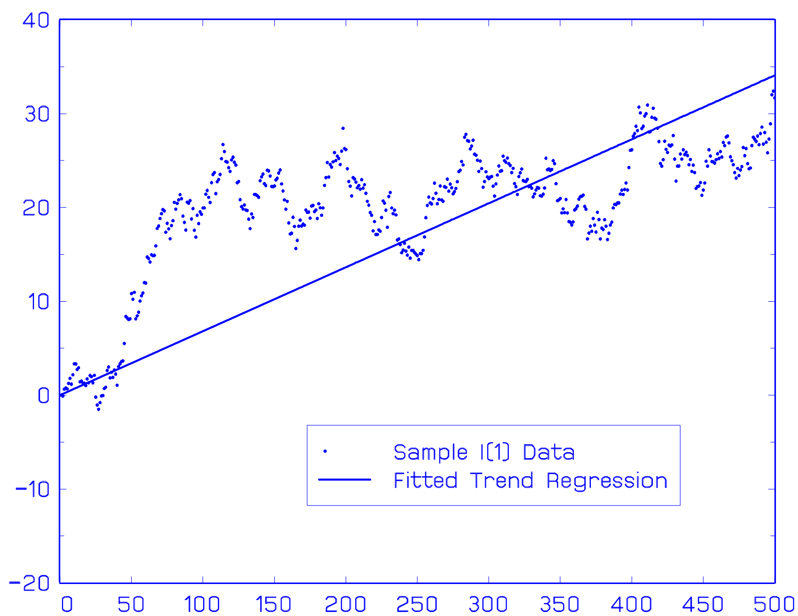

Fig. 1: Spurious Regression of I(1) Data on a Linear Trend

Figs. 1 and 2 illustrate these effects with some sample data and bootstrap sample data. In Fig. 1 the sample data $\{X_t : t = 1, ...n\}$ comprise $n = 500$ observations of a standard Gaussian random walk which is shown against the fitted least squares regression line $\widehat{X}_t = \widehat{b}t$. The random wandering characteristics about the regression line are apparent. From traditional limit theory (Durlauf and Phillips, 1988) it is known that the regression coefficient $\widehat{b} \rightarrow_p 0$ but is statistically significant with probability one as $n \rightarrow \infty$. The bootstrap sample $\{X_t^* : t = 1, ...n\}$ is generated from $X_t^* = \widehat{b}t + u_t^*$, where $\{u_t^* : t = 1, ..., n\}$ are random draws with replacement from the empirical distribution of centred versions of the residuals $\{\widehat{u}_t = X_t - \widehat{b}t : t = 1, ..., n\}$, that is from a crude application of the bootstrap. Fig. 2 shows a typical set of bootstrap sample data generated in this way for the given data $\{X_t : t = 1, ...n\}$. Included in Fig. 2 is the least squares regression line $\widehat{X}_t^* = \widehat{b}^* t$ for the bootstrap sample. The result is strikingly apparent - the bootstrap data now appear to be stationary, albeit widely varying, about the trend function $\widehat{b}^*$.

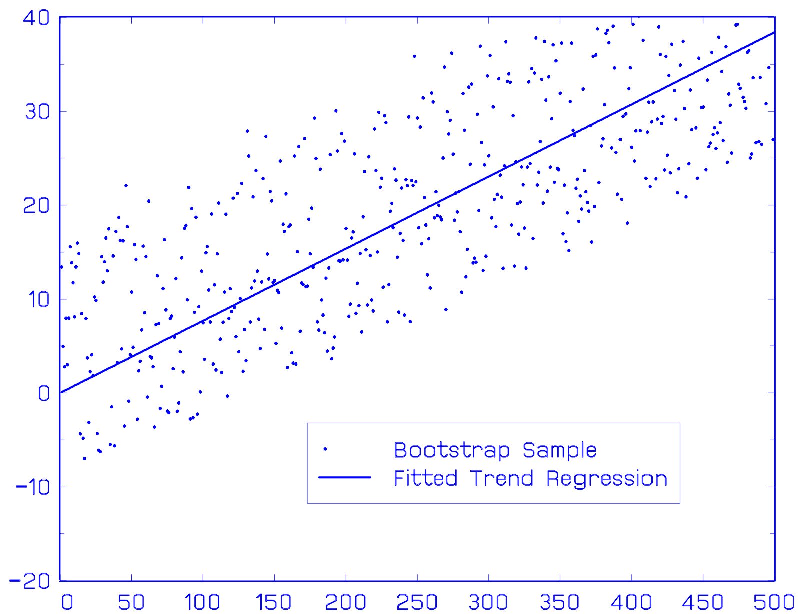

Fig. 2: Spurious Regression of Bootstrapped Sample on a Linear Trend

In a similar way, Fig. 3 shows sample data crossplots for two independent standard Gaussian random walks $\{Y_t, X_t : t = 1, ...n\}$ together with the (spurious) regression line $\widehat{Y}_t = \widehat{b}X_t$. The random wandering behavior of the sample data around the line is clearly evident. The bootstrap sample $\{Y_t^* : t = 1, ...n\}$ is generated from $Y_t^* = \widehat{b}X_t + u_t^*$, where $\{u_t^* : t = 1, ..., n\}$ are random draws with replacement from the empirical distribution of cntred versions of the residuals $\{\widehat{u}_t = Y_t - \widehat{b}X_t : t = 1, ..., n\}$, again a crude application of the bootstrap. Fig. 4 shows bootstrap sample data generated in this way, together with the least squares regression line $\widehat{Y}_t^* = \widehat{b}^* X_t$ for this bootstrap sample. The result is again striking. The bootstrap data $\{Y_t^*, X_t : t = 1, ...n\}$ now appear to cointegrate about the fitted regression line, albeit with wide variation.

This paper extends previous research primarily by developing methods for analyzing the asymptotic properties of the bootstrap and block bootstrap on integrated time series. There has been frequent speculation about the properties of such uses of the bootstrap (e.g., Li and Maddala, 1996 & 1997, and Hinkley, 1997) but no prior analysis. Using these methods the paper shows that direct application of the bootstrap converts integrated time series into asymptotically stationary series, thereby fundamentally changing the nature of the series. The failure is a first order inconsistency in that the serial correlation coefficients of a bootstrapped integrated process do not tend to unity. The impact of the inconsistency is that the bootstrap converts spurious regressions into cointegrating or trend stationary regressions. Unlike the bootstrap, the block bootstrap is shown to be first order consistent, maintaining the feature of an integrated process that its serial correlation coefficients tend to unity. However, the limit distribution of the bootstrapped serial correlation coefficient differs from that of the sample serial correlation coefficient, is no longer a unit root distribution and converges at a different rate.

Fig. 3: Spurious Regression of I(1) Data on Independent I(1) Data

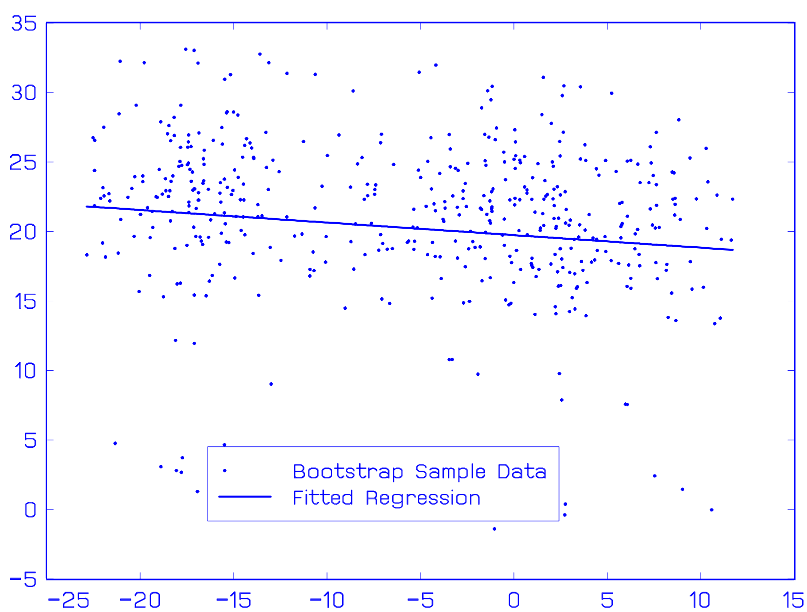

Fig. 4: Bootstrapped Spurious Regression of independent I(1) Data

The paper is organised as follows. Section 2 gives some preliminary theory and then develops bootstrap asymptotics for integrated time series, including direct bootstrap and block bootstrap approaches. Section 3 provides results for the bootstrap and block bootstrap in the prototypical spurious regression of an integrated process on a deterministic trend and Section 4 does the same for spurious regressions among full rank integrated time series. Implications of the results are considered in Section 5 and proofs are given in Section 6. A notational table is provided at the end of the paper.

2. Bootstrap Limit Theory for Integrated Time Series

(a) Preliminaries

The $\ell-$ vector time series x_t is a full rank unit root process generated by

$$x_t = x_{t-1} + u_t, \tag{1}$$

with $x_0 = O_{a.s}(1)$ and with u_t a linear process that satisfies the following condition, where $||a|| = \max_{ij} |a_{ij}|$ for matrix a.

Assumption L *For all $t > 0$, u_t has Wold representation*

$$u_t = C(L)\varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}, \quad \sum_{j=0}^{\infty} j^s ||c_j|| < \infty, \quad s \geq 1, C(1) \neq 0, \tag{2}$$

with $\varepsilon_t = iid(0, \Sigma_\varepsilon)$ and with $E(||\varepsilon_t||^q) < \infty$, for some $q > 4$.

The summability condition in (2) is satisfied by a wide class of parametric and nonparametric models for u_t and, in conjunction with the moment condition, enables the use of the asymptotic techniques in Phillips and Solo (1992) and an almost sure invariance principles (IP) for the partial sums of u_t. To perform the latter, which are especially useful here, we expand the probability space as needed so that the partial sum process $S_k = \sum_{t=1}^k u_t$ of u_t can be represented up to a negligible error in terms of a Brownian motion defined on the same space. An IP of this type for partial sums of scalar linear processes was proved in Phillips (1999) and is straightforwardly extended to the case $\ell \geq 1$.

2.1 Lemma *Let $S_k = \sum_{j=1}^k u_j$ for $k \geq 1$, and $S_0 = 0$, for $k = 0$, where u_j satisfies Assumption L. Then, the probability space on which the u_j and S_k are defined can be expanded in such a way that there is a process distributionally equivalent to S_k and a vector Brownian motion $B(\cdot)$ with variance matrix $\Omega = C(1)\Sigma_\varepsilon C(1)'$ on the new space for which*

$$\sup_{0 \leq k \leq n} \left|\left| \frac{S_k}{\sqrt{n}} - B(\frac{k}{n}) \right|\right| = o_{a.s}(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}). \tag{3}$$

provided $E||u_t||^q < \infty$ for some $q > 2p > 4$.

Since $x_t = S_t + O_{a.s.}(1)$, it follows from (3) that, after changing the probability space as required, we have

$$\sup_{0 \leq t \leq n} \left|\left| \frac{x_t}{\sqrt{n}} - B(\frac{t}{n}) \right|\right| = o_{a.s}(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}). \tag{4}$$

We shall often proceed as if this change has been made without continually adding the qualification and noting that in the original space this convergence translates into weak convergence of measures.

We shall also make extensive use of the LK representation of the stochastic process $B(r)$. Let $W(r)$ be $\ell-$ vector standard Brownian motion. Over the interval $r \in [0,1]$, the $\ell-$ vector process $W(r)$ has LK representation (equation (4) of Phillips, 1998)

$$W(r) = \sqrt{2}\sum_{k=1}^{\infty} \frac{\sin\left[\left(k-\frac{1}{2}\right)\pi r\right]}{\left(k-\frac{1}{2}\right)\pi}\xi_k, \quad \xi_k \equiv iidN(0, I_\ell),$$

which holds $a.s.$ and uniformly in r. For $B(r) = \Omega^{\frac{1}{2}}W(r)$ we have the corresponding representation

$$B(r) = \sqrt{2}\sum_{k=1}^{\infty} \frac{\sin\left[\left(k-\frac{1}{2}\right)\pi r\right]}{\left(k-\frac{1}{2}\right)\pi}\xi_k, \quad \xi_k \equiv iidN(0, \Omega). \tag{5}$$

With this approximation in hand, we can develop some preliminary limit theory for applications of the bootstrap to integrated time series.

(b) Direct Bootstrap Limits for Integrated Processes

We use the following scheme and notation for bootstrap samples. Suppose we are given data $(e_t)_1^n$ or estimates of this data (e.g. from regression residuals) denoted $(\widehat{e}_t)_1^n$ defined on a probability space $(\Omega, \mathcal{F}, P)$. Direct random resampling from the empirical distribution of $(\widehat{e}_t)_1^n$ produces the crude bootstrap sample $(e_t^*)_1^n$. Here, each observation e_t^* is drawn from the population $(\widehat{e}_t)_1^n$ and each point has equal probability, $\frac{1}{n}$, of being drawn. If we centre the original observations and draw from the population $(\widehat{e}_t - \frac{1}{n}\sum_{s=1}^n \widehat{e}_s)_1^n$, then the bootstrap sample is denoted $(\widetilde{e}_t^*)_1^n$. Following convention, we let P^* denote probability conditional on the realized trajectory of the orginal data $(\widehat{e}_t)_1^\infty$. Expectation with respect to P^* is denoted E^*, convergence in distribution (respectively, in probability) with respect to P^* is denoted $\rightarrow_{d^*}$ (respectively, $\rightarrow_{p^*}$) and when convergence occurs with P probability unity (i.e. for almost all realizations of $(e_t)_1^\infty$), we write $\rightarrow_{d^*}$ $a.s.$ and $\rightarrow_{p^*}$ $a.s.$

Suppose we have data $(x_t)_1^n$ generated as in (1) and let $(x_t^*)_1^n$ and $(\widetilde{x}_t^*)_1^n$ be crude Bootstrap samples constructed as above. We can write $x_t^* = x_j$ where j is a random index drawn from $\{1, 2, ..., n\}$. That is, j is uniform over the integers $\{1, 2, ..., n\}$ with mass probability $\frac{1}{n}$ on each integer. Then, in view of Lemma 2.1 and (4), there exists a probability space in which we can write

$$\frac{x_t^*}{\sqrt{n}} = \frac{x_j}{\sqrt{n}} = B\left(\frac{j}{n}\right) + o_{a.s}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right) = B(R_{nj}) + o_{a.s}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right), \tag{6}$$

where $R_{nj} = \frac{j}{n}$ is uniformly distributed over $\{\frac{1}{n}, \frac{2}{n}, .., 1\}$ for each j. Now, in a slight abuse of notation, we can write

$$R_{nj} \rightarrow_{d^*} R_j, \tag{7}$$

where R_j has an independent continuous uniform distribution over $[0,1]$ for each j, written as $iidU[0,1]$. The convergence (7) evidently also holds almost surely (P). It follows by the continuous mapping theorem that

$$\sin\left[\left(k-\frac{1}{2}\right)\pi R_{nj}\right] \rightarrow_{d^*} \sin\left[\left(k-\frac{1}{2}\right)\pi R_j\right] \quad a.s.(P). \tag{8}$$

Moreover, for $j \neq k$, R_{nj} and R_{nk} are statistically independent and so the limit variates R_j and R_k are also independent. It is therefore apparent from (5)-(8) that the standardized bootstrap process $n^{-\frac{1}{2}}x_t^*$ has an asymptotic representation in terms of a vector Brownian motion evaluated at a random argument $R_j \in [0,1]$, viz.,

$$\frac{x_t^*}{\sqrt{n}} = \sqrt{2}\sum_{k=1}^{\infty}\frac{\sin\left[\left(k-\frac{1}{2}\right)\pi R_{nj}\right]}{\left(k-\frac{1}{2}\right)\pi}\xi_k + o_{a.s}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right) \sim_{d^*} \sqrt{2}\sum_{k=1}^{\infty}\frac{\sin\left[\left(k-\frac{1}{2}\right)\pi R_j\right]}{\left(k-\frac{1}{2}\right)\pi}\xi_k. \quad (9)$$

Thus, just as x_t^* is a random draw from the empirical distribution of $(x_t)_1^n$, the large sample behavior of the standardized process $n^{-\frac{1}{2}}x_t^*$ is analogous to that of a random draw from the trajectory of the Brownian motion limit of $n^{-\frac{1}{2}}x_t$. This is formalized in the following lemma.

2.2 Lemma *Let $(x_t^*)_1^n$ and $(\widetilde{x}_t^*)_1^n$ be bootstrap samples from a unit root process $(x_t)_1^n$ generated as in (1) with u_j satisfying Assumption* L.

(a) $\frac{x_{[nr]}^*}{\sqrt{n}} \to_{d^*} B(R_r) \quad a.s.(P),$

(b) $\frac{\widetilde{x}_{[nr]}^*}{\sqrt{n}} \to_{d^*} B(R_r) - \int_0^1 B(s)\,ds \quad a.s.(P),$

where $\{R_r : r \in [0,1]\}$ is family of independent uniform variates on $[0,1]$.

The bootstrapped process $n^{-\frac{1}{2}}x_{[nr]}^*$ converges weakly to a randomly sampled version of the Brownian motion B to which $n^{-\frac{1}{2}}x_{[nr]}$ converges, and $n^{-\frac{1}{2}}\widetilde{x}_{[nr]}^*$ converges to a randomly sampled version of the corresponding demeaned Brownian motion $\underline{B}(r) = B(r) - \int_0^1 B(s)\,ds$. Unlike the Brownian motion process from which these draws are made, $B(R_r)$ is not pathwise continuous and is not a separable process.

We proceed to find distributional limits of bootstrapped statistics in regressions involving $I(1)$ variables like x_t. These limits are useful in the regression theory of the following section. In view of the different limit behavior of x_t^* and $\widetilde{x}_t^*$, we continue to report results for both versions of the bootstrap in what follows. Lemma 2.3 below gives some limit theory for sample moments of the bootstrapped data and original sample data.

2.3 Lemma *Under the same conditions as Lemma 2.2:*

(a) $n^{-\frac{3}{2}}\sum_{t=1}^n x_t^* \to_{d^*} \int_0^1 B(s)\,ds \quad a.s.(P),$

(b) $n^{-\frac{3}{2}}\sum_{t=1}^n \widetilde{x}_t^* \to_{p^*} \int_0^1 \underline{B}(s)\,ds = 0 \quad a.s.(P),$

(c) $n^{-2}\sum_{t=1}^n x_t^* x_t^{*\prime} \to_{d^*} \int_0^1 B(s)B(s)'\,ds \quad a.s.(P),$

(d) $n^{-2}\sum_{t=1}^n \widetilde{x}_t^* \widetilde{x}_t^{*\prime} \to_{d^*} \int_0^1 \underline{B}(s)\underline{B}(s)'\,ds \quad a.s.(P),$

(e) $n^{-2}\sum_{t=1}^n x_t^* x_t' \to_{d^*} \left(\int_0^1 B(s)\,ds\right)\left(\int_0^1 B(s)'\,ds\right) \quad a.s.(P),$

(f) $n^{-2}\sum_{t=1}^n \widetilde{x}_t^* x_t' \to_{d^*} \left(\int_0^1 \underline{B}(s)\,ds\right)\left(\int_0^1 B(s)'\,ds\right) = 0 \quad a.s.(P),$

(g) $n^{-5/2}\sum_1^n x_t^* t \rightarrow_{d^*} \left(\int_0^1 B(r)dr\right)\left(\int_0^1 rdr\right) \quad a.s.(P),$

(h) $n^{-5/2}\sum_1^n \widetilde{x}_t^* t \rightarrow_{d^*} \left(\int_0^1 \underline{B}(s)\,dr\right)\left(\int_0^1 rdr\right) = 0 \quad a.s.(P).$

Evidently, the bootstrap sample moments $n^{-\frac{3}{2}}\sum_{t=1}^n x_t^*$ and $n^{-2}\sum_{t=1}^n x_t^* x_t^{*\prime}$ have the same limits as the corresponding sample averages $n^{-\frac{3}{2}}\sum_{t=1}^n x_t$ and $n^{-2}\sum_{t=1}^n x_t x_t^{'}$. Correspondingly, sample moments of the centred bootstrap sample $(\widetilde{x}_t^*)_1^n$ have limits that correspond to those of the demeaned sample data. However, sample covariances between the bootstrap data and the original sample, $n^{-2}\sum_{t=1}^n x_t^* x_t'$ and $n^{-2}\sum_{t=1}^n \widetilde{x}_t^* x_t'$, differ from the limit of $n^{-2}\sum_{t=1}^n x_t x_t^{'}$; and the bootstrap covariances $n^{-5/2}\sum_1^n x_t^* t$ and $n^{-5/2}\sum_1^n \widetilde{x}_t^* t$ do not have the same limits as $n^{-5/2}\sum_1^n x_t t$ and $n^{-5/2}\sum_1^n (x_t - \overline{x})\,t$, which are $\int_0^1 B(r)rdr$ and $\int_0^1 \underline{B}(r)rdr$.

Next, consider the scalar case and set $\ell = 1$ in (1). In this case we shall use ω^2 to denote the variance of the Brownian motion B. Define the h'th order serial correlation of the data $\widehat{\rho}_h = \sum_{t=h}^n x_t x_{t-h} / \sum_{t=h}^n x_{t-h}^2$, and the bootstrap serial correlations $\widehat{\rho}_h^* = \sum_{t=h}^n x_t^* x_{t-h}^* / \sum_{t=h}^n x_{t-h}^{*2}$, $\widetilde{\rho}_h^* = \sum_{t=h}^n \widetilde{x}_t^* \widetilde{x}_{t-h}^* / \sum_{t=h}^n \widetilde{x}_{t-h}^{*2}$, and $\widehat{\rho}_h^e = \sum_{t=h}^n e_t^* e_{t-h}^* / \sum_{t=h}^n e_{t-h}^{*2}$, where $e_t = x_t^* - \overline{x}^*$.

2.4 Theorem *Serial correlations of scalar bootstrap samples $(x_t^*)_1^n$ and $(\widetilde{x}_t^*)_1^n$ from a unit root process satisfying the conditions of Lemma 2.2 have the following limits for $h \neq 0$ as $n \rightarrow \infty$:*

$$\widehat{\rho}_h^* \rightarrow_{d^*} \xi_\rho = \frac{\left[\int_0^1 (B(r))dr\right]^2}{\int_0^1 B(r)^2 dr} \quad a.s.\,(P)\,, \tag{10}$$

$$\widetilde{\rho}_h^*, \widehat{\rho}_h^e \rightarrow_{p^*} 0 \quad a.s.\,(P)\,. \tag{11}$$

The limit variate ξ_ρ in (10) evidently satisfies $0 < \xi_\rho < 1$, $a.s.(P)$. Thus, the theorem shows that the first order serial correlation coefficient of a crude bootstrap sample of an $I(1)$ process no longer has a probability limit of unity. Instead, $\widehat{\rho}_1^*$ converges weakly to a limit random variable that is strictly less than unity, but also strictly positive. Since $\widehat{\rho}_h \rightarrow_p 1$ for all h, the crude bootstrap $\widehat{\rho}_h^*$ is not even first order consistent. It follows that a unit root test on the bootstrap sample $(x_t^*)_1^n$ that is based on $\widehat{\rho}^*$ will reject the null hypothesis almost surely as $n \rightarrow \infty$. Note that this is the case even though $x_{[nr]}^*$ is of order $\sqrt{n}$ and $n^{-\frac{1}{2}} x_{[nr]}^*$ tends to a limiting stochastic process that is a randomized version of Brownian motion, as shown in Lemma 2.3 (a). Further, all serial correlations $\widehat{\rho}_h^*$ have the same limit ξ_ρ as $n \rightarrow \infty$ for any $h \neq 0$. Thus, the bootstrap process x_t^* is strongly dependent with temporal dependence characteristics analogous to those of a time series with a random fixed effect.

The serial correlation coefficients $\widetilde{\rho}_h^*$ for the centred bootstrap data are all zero in the limit. Thus, differences between centred and uncentred bootstrap sampling persist in the limit behavior of the serial correlations and the bootstrap sample $(\widetilde{x}_t^*)_1^n$ behaves asymptotically like an uncorrelated sequence. Again, unit root tests will reject the null hypothesis with probability one as $n \rightarrow \infty$. When the serial correlation coefficient $\widehat{\rho}_h^*$ is redefined in terms of deviations from sample means, i.e. as $\widehat{\rho}_h^e$, then (11) holds rather than (10). In all these cases, crude bootstrap resampling turns an $I(1)$ series into one that is asymptotically stationary.

(c) Block Bootstrap Limits for Integrated Processes

A similar analysis can be performed for the block bootstrap. We will use Carlstein (1986) blocking. Here, the data are subdivided into M successive blocks $\{A_j : j = 1, .., M\}$ of equal size m, with $n = mM$. Assume $\frac{1}{m} + \frac{m}{n} \to 0$, so that $\frac{1}{M} + \frac{M}{n} \to 0$ and then the size of each block as well as the number of blocks tend to infinity at a rate slower than n. With this schematic, we can write observations as $x_t = x_{(s-1)m+k}$, so that the t'th observation in the sample appears as the k'th observation in the s'th block A_s.

Block bootstrap samples $(x_t^{b*})_1^n$ are constructed by randomly sampling the M blocks (with replacement) and then arranging them end-to-end in the order in which they are sampled. Call the sampled blocks $\{A_j^* : j = 1, .., M\}$. A typical block bootstrap observation can then be written as $x_t^{b*} = x_{(j-1)m+k}$ where j is a random index drawn from $\{1, 2, ..., M\}$. That is, j is uniform over the integers $\{1, 2, ..., M\}$ with mass probability $\frac{1}{M}$ on each integer. We can construct centred block bootstrap samples $(\widetilde{x}_t^{b*})_1^n$ in the same way. With some minor changes in the proofs, all the results given below carry over to the moving block bootstrap (Künsch, 1989). In this blocking scheme, we let $N_1, ..., N_M$ be *iid* uniform random draws from $\{0, 1, 2, ..., n-m\}$. Then a typical moving block bootstrap observation is $x_{(j-1)m+k}^{mb*} = x_{N_j+k}$ for $1 \leq j \leq M$ and $1 \leq k \leq m$.

As in the previous section, it follows from Lemma 2.1 that there exists a probability space in which we can write

$$\frac{x_t^{b*}}{\sqrt{n}} = \frac{x_{(j-1)m+k}}{\sqrt{n}} = B\left(\frac{(j-1)m+k}{n}\right) + o_{a.s}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right) = B\left(R_{Mj} + \frac{k}{n}\right) + o_{a.s}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right), \quad (12)$$

where $R_{Mj} = \frac{j-1}{M}$ is uniformly distributed over $\{0, \frac{1}{M}, .., \frac{M-1}{M}\}$ for each j. As $n, M \to \infty$, with a slight abuse of notation, we have

$$R_{Mj} \to_{d^*} R_j, \quad (13)$$

where R_j is $iidU[0,1]$. Similarly, for any $r \in [0,1]$ there exist integers s_r and k_r for which $[nr] = (s_r - 1)m + k_r$ and we can write

$$\frac{x_{[nr]}^{b*}}{\sqrt{n}} = \frac{x_{(J_r-1)m+k_r}}{\sqrt{n}} = B\left(\frac{(J_r-1)m+k_r}{n}\right) + o_{a.s}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right) = B\left(R_{M,r} + \frac{k_r}{n}\right) + o_{a.s}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right), \quad (14)$$

where J_r is uniformly distributed over $\{1, 2, ..., M\}$, $R_{M,r} = \frac{J_r-1}{M}$ is uniformly distributed over $\{0, \frac{1}{M}, .., \frac{M-1}{M}\}$ and $R_{M,r} \to_{d^*} R_r$, where R_r is uniform over $[0,1]$ for each r.

Block bootstrap versions of Lemmas 2.2 and 2.3 follow in a straightforward way and we do not repeat them here. Instead, we give the main result, a block bootstrap version of Theorem 2.4. The serial correlation coefficients $\widehat{\rho}_h^{b*}, \widetilde{\rho}_h^{b*}$, and $\widehat{\rho}_h^{be}$ are defined as in Section 2 (b) above using the block bootstrap data.

2.5 Theorem *Serial correlations of scalar block bootstrap samples* $(x_t^{b*})_1^n$ *and* $(\widetilde{x}_t^{b*})_1^n$ *from a unit root process satisfying the conditions of Lemma 2.2 and with* $\frac{1}{m} + \frac{m}{n} \to 0$ *have the following limits for all fixed* $h \neq 0$ *as* $n \to \infty$ *:*

$$\widehat{\rho}_h^{b*}, \widetilde{\rho}_h^{b*}, \widehat{\rho}_h^{be} \to_{d^*} 1 \quad a.s.(P). \quad (15)$$

Thus, the serial correlation coefficients of the block bootstrapped time series x_t^{b*} converge to unity, even though the standardized process $n^{-\frac{1}{2}}x_{[nr]}^{b*}$ converges weakly to a randomly sampled version of the Brownian motion B just as in the crude bootstrap (Lemma 2.2). Hence, in contrast to the crude bootstrap, the block bootstrap serial correlation coefficients are first order consistent.

Some further analysis reveals that the distribution of the block bootstrap serial correlation coefficient $\widehat{\rho}_1^{b*}$ is inconsistent for the unit root limit distribution of the first order serial correlation coefficient of an integrated process. Its convergence rate $O(n/M)$ is also slower than the conventional order n rate for unit root distributions.

2.6 Theorem *The first order serial correlation coefficient of a scalar block bootstrap sample $(x_t^{b*})_1^n$ from a unit root process satisfying the conditions of Lemma 2.2 and with $\frac{1}{m}+\frac{m}{n}\to 0$ has the following limit distribution as $n\to\infty$:*

$$\frac{n}{M}\left(\widehat{\rho}_1^{b*}-1\right)\to_{d^*}-\frac{\left\{\int_0^1 B(r)^2dr-\left(\int_0^1 B(r)dr\right)^2\right\}}{\int_0^1 B\left(r\right)^2dr}<0 \quad a.s.\,(P)\,. \tag{16}$$

If centred bootstrap samples $(\widetilde{x}_t^{b*})_1^n$ are used then the corresponding limit is

$$\frac{n}{M}(\widehat{\rho}_1^{b*}-1)\to_{p^*}-1 \quad a.s.\,(P)\,. \tag{17}$$

The limit results (16) and (17) show that if a conventional unit root root test were applied to block bootstrapped sample data then the presence of a unit root would be rejected with a probability approaching unity as $n\to\infty$. Conversely, if the bootstrap distribution of $\widehat{\rho}_1^{b*}$ were used to construct critical values in inference based on $\widehat{\rho}_1$ about the presence of a unit autoregressive coefficient in (1), then there would be a zero rejection rate asymptotically for the unit root hypothesis.

The convergence rate of $\widehat{\rho}_1^{b*}$ is $O\left(m\right)$, revealing that in the block bootstrap it is the number of consecutive observations m in each block that determines the rate of convergence of the serial correlation coefficient not the number of blocks. In effect, each block is a small-infinity-sized trajectory of a unit root process. However, since the expression for $\widehat{\rho}_1^{b*}$ involves averages across blocks as well as within blocks, the limit of $m(\widehat{\rho}_1^{b*}-1)$ is not a conventional unit root distribution. Instead, as the proof of theorem 2.6 reveals, a bias term involving averages of differences of Brownian motion across randomized blocks dominates the sample covariance $\frac{1}{nM}\sum_{t=1}^n x_{t-1}^*\Delta x_t^*$ and gives rise to the limits in (16) and (17).

Fig. 5 graphs kernel estimates of the densities of the unit root distribution of $\widehat{\rho}_1$ against those of the crude boostrap estimate $\widehat{\rho}_1^*$, the block bootstrap estimate $\widehat{\rho}_1^{b*}$ and the centred bootstrap estimate $\widetilde{\rho}_1^*$. The simulation estimates used 10,000 replications, a sample size of 200 and block settings $m=10$, $M=20$. Apparently, the block bootstrap sampling distribution is closer to that of the unit root distribution than the other estimates but is substantially downward biased, consonant with the results in Theorem 2.6.

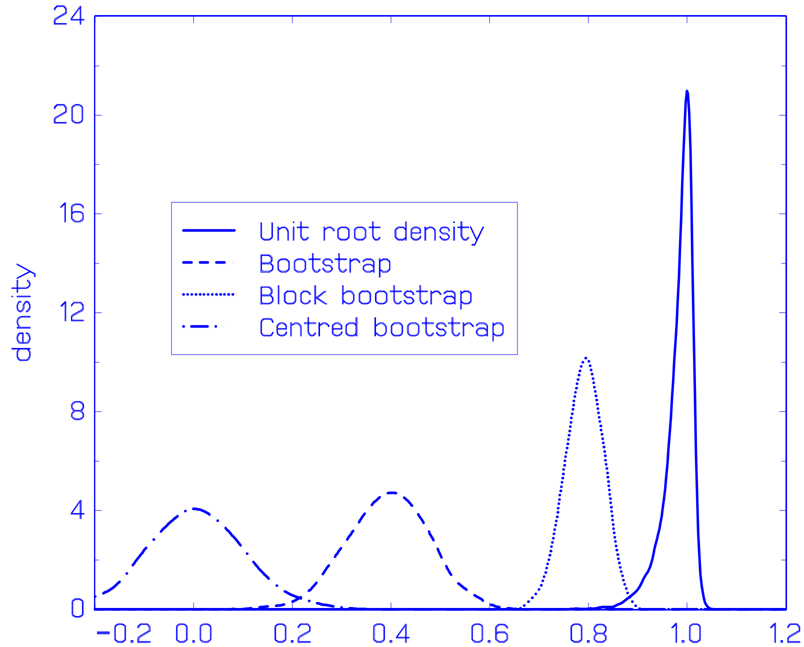

Fig. 5: Kernel Estimates of Densities of $\widehat{\rho}$, $\widehat{\rho}_1^*$, $\widehat{\rho}_1^{b*}$, $\widetilde{\rho}_1^*$.

3. Regression on a Deterministic Trend

We consider the prototypical spurious regresion of a scalar time series x_t generated as in (1) on a linear trend

$$x_t = \widehat{b}_n t + \widehat{u}_t, \tag{18}$$

Regressions on higher order polynomials and other deterministic functions produce analogous results and the simple regression (18) is chosen simply for convenience. Let $s_{\widehat{b}_n}$ be the standard error of $\widehat{b}_n$, $t(\hat{b}_n)$ be the corresponding $t-$statistic, and set $\widehat{\rho} = \sum_2^n \widehat{u}_t \widehat{u}_{t-1} / \sum_2^n \widehat{u}_{t-1}^2$ and $DW = \sum_2^n (\widehat{u}_t - \widehat{u}_{t-1})^2 / \sum_2^n \widehat{u}_{t-1}^2$. As in Durlauf and Phillips (1988),

$$\sqrt{n}\hat{b}_n = \frac{n^{-\frac{5}{2}} \sum_{t=1}^n x_t t}{n^{-3} \sum_{t=1}^n t^2} \rightarrow_d \frac{\int_0^1 rB(r)}{\int_0^1 r^2} \equiv \xi, \quad \text{say}, \tag{19}$$

$$\frac{1}{\sqrt{n}} t\left(\hat{b}_n\right) = \frac{\hat{b}_n}{s_{\widehat{b}_n}} \rightarrow_d \frac{\xi \left(\int_0^1 r^2\right)^{\frac{1}{2}}}{\left(\int_0^1 \left(B\left(r\right) - \xi r\right)^2\right)^{\frac{1}{2}}}, \tag{20}$$

and

$$\widehat{\rho} \rightarrow_p 1, \quad nDW \rightarrow_d \frac{\sigma_u^2 + \xi^2}{\int_0^1 \left(B\left(r\right) - \xi r\right)^2}, \tag{21}$$

where B is scalar Brownian motion with variance ω^2. By changing the probability space as discussed in the paragraph following Lemma 2.1, each of these convergences can be replaced by $a.s.$ convergence, and we shall commonly proceed below as if these changes have been made.

(a) Crude Bootstrapping of Residuals

Here bootstrap samples $(u_t^*)_1^n$ and $(\widetilde{u}_t^*)_1^n$ are formed by random sampling from $(\widehat{u}_t)_1^n$, bootstrap data $(x_t^{**})_1^n$ are generated according to

$$x_t^{**} = \widehat{b}_n t + u_t^*, \tag{22}$$

and least squares produces the fitted regression

$$x_t^{**} = \widehat{b}_n^* t + u_t^{**}, \tag{23}$$

which defines the residual process u_t^{**}. Let $s^{**2} = \frac{1}{n}\sum_{t=1}^n u_t^{**2}$, let $s_{\widehat{b}_n^*}$ and $t(\widehat{b}_n^*)$ be the standard error and $t-$ ratio of $\widehat{b}_n^*$ in (23), and let DW^{**} be the DW ratio in (23). Denote serial correlations by the notation $\widehat{\rho}_h^{**} = \sum_{t=h}^n u_t^{**}u_{t-h}^{**}/\sum_{t=1}^n u_{t-h}^{**2}$. The following results give the limit behavior of these bootstrap statistics.

3.1 Theorem *Let $(x_t^{**})_1^n$ be bootstrap data generated from (22) using $(u_t^*)_1^n$ where the original data $(x_t)_1^n$ in the trend regression (18) is a unit root process (1) with u_t satisfying Assumption* L.

(a) $\sqrt{n}\hat{b}_n^* \rightarrow_{d^*} \xi^* = \frac{\frac{1}{2}\int_0^1 B(r)dr + \frac{1}{12}\xi}{\int_0^1 r^2}$ $\quad a.s.(P)$,

(b) $\frac{1}{n}s^{**2} \rightarrow_{d^*} \sigma_\xi^2 = \int_0^1 \{B_r(s) + (\xi - \xi^*)\, s\}^2 dr \quad a.s.(P)$,

(c) $\frac{1}{\sqrt{n}}t(\widehat{b}_n^*) \rightarrow_{d^*} \frac{\xi^*\left(\int_0^1 r^2\right)^{\frac{1}{2}}}{\sigma_\xi} \quad a.s.(P)$,

(d) $\widehat{\rho}_h^{**} \rightarrow_{d^*} \xi_\rho^{breg}, \quad \left|\xi_\rho^{breg}\right| < 1 \quad a.s.(P)$, for all $h \neq 0$,

(e) $DW^{**} \rightarrow_{d^*} \xi_{DW}^{breg} > 0 \quad a.s.(P)$,

(f) $R^{**2} \rightarrow_{d^*} \xi_{R^2}^{breg} < 1, \quad a.s.(P)$,

where $B_r(s) = B(s) - \xi s$ *is detrended Brownian motion,* $B_r^*(s) = B_r(s) + (\xi - \xi^*)\, s = B(s) - \xi^* s$,

$$\xi_\rho^{breg} = \frac{\left(\int_0^1 B_r^*(s)ds\right)^2}{\int_0^1 B_r^*(s)^2 ds} \in (0,1) \quad a.s.(P)$$

$$\xi_{DW}^{breg} = \frac{2\int_0^1 B_r(s)^2 ds}{\sigma_\xi^2}\left[1 - \xi_\rho^b\right] > 0 \quad a.s.(P)$$

$$\xi_\rho^b = \frac{\left(\int_0^1 B_r(s)ds\right)^2}{\int_0^1 B_r(s)^2 ds} \in (0,1) \quad a.s.(P)$$

and

$$\xi_{R^2}^{breg} = 1 - \frac{\sigma_\xi^2}{\int_0^1 B\,(r)^2\, dr}.$$

3.2 Remarks

(i) From (a), $\hat{b}_n^* \to_{p^*} 0$ and so the bootstrap estimate $\hat{b}_n^*$ is first-order consistent. By a straightforward but lengthy calculation $\xi^* = N(0, v_b^*)$ with $v_b^* = \omega^2 \frac{207}{160}$, whereas $\xi = N(0, v_b)$ with $v_b = \frac{6}{5}\omega^2$. Evidently, the bootstrap statistic $\sqrt{n}\hat{b}_n^*$ and $\sqrt{n}\hat{b}_n$ are not asymptotically equivalent although the limit distribution $N(0, v_b^*)$ is close to the correct limit ($\frac{207}{160} \simeq 1.29$ versus $\frac{6}{5} = 1.20$).

(ii) The reason for the inconsistency of the bootstrap statistic $\sqrt{n}\hat{b}_n^*$ is that in the numerator of this statistic $n^{-5/2}\sum_1^n x_t^* t \not\sim_a n^{-5/2}\sum_1^n x_t t$. In fact, from Lemma 2.3 (g) we have $n^{-5/2}\sum_1^n x_t^* t \to_{d^*} (\int_0^1 B(r)dr)(\int_0^1 rdr)$ $a.s.(P)$, whereas by standard methods $n^{-5/2}\sum_1^n x_t t \to_{d^*} (\int_0^1 B(r)rdr)$ as in (19).

(iii) The bootstrap $t-$ statistic $t(\widehat{b}_n^*)$ diverges at the rate $\sqrt{n}$, just as the regression $t-$ statistic $t(\widehat{b}_n)$, but $t(\hat{b}_n^*)/\sqrt{n}$ has a different limit distribution from $t(\hat{b}_n)/\sqrt{n}$, which by standard methods is $\xi(\int_0^1 r^2 dr)^{\frac{1}{2}}/(\int_0^1 B_r(s)ds)$.

(iv) Most importantly, the bootstrap serial correlation $\widehat{\rho}_1^{**}$ has a random limit ξ_ρ^{breg} which is strictly less than unity with probability one. It follows that unit root tests on the residuals of (23) that are based on $\widehat{\rho}_1^{**}$ will reject the null hypothesis of a unit root and hence the null of a spurious regression. Thus, the original regression (18) and the bootstrap regression (23) have fundamentally different characteristics – (18) is a fitted trend with a significant coefficient and diagnostics that reveal the significance is spurious, whereas (23) is a trend regression with a similarly significant coefficient and diagnostics that do not invalidate the relationship asymptotically.

3.3 Theorem *Let $(x_t^{**})_1^n$ be bootstrap data generated from (22) using centred bootstrap residuals $(\widetilde{u}_t^*)_1^n$ where the original data $(x_t)_1^n$ in the trend regression (18) is a unit root process (1) with u_t satisfying Assumption* L.

(a) $\sqrt{n}\hat{b}_n^* \to_{d^*} \xi \quad a.s.(P)$,

(b) $\frac{1}{n}s^{**2} \to_{d^*} \int_0^1 \underline{B}_r(s)^2 dr \quad a.s.(P)$,

(c) $\frac{1}{\sqrt{n}}t(\widehat{b}_n^*) \to_{d^*} \frac{\xi\left(\int_0^1 r^2\right)^{\frac{1}{2}}}{\left(\int_0^1 \underline{B}_r(s)^2 ds\right)^{\frac{1}{2}}} \quad a.s.(P)$,

(d) $\widehat{\rho}_h^{**} \to_{p^*} 0, \quad a.s.(P)$, for all $h \neq 0$,

(e) $DW^{**} \to_{p^*} 2 \quad a.s.(P)$,

(f) $R^{**2} \to_{d^*} \frac{\xi^2 \int_0^1 r^2 dr}{\int_0^1 \underline{B}_r(s)^2 ds + \xi^2 \int_0^1 r^2 dr} < 1, \quad a.s.(P)$,

where $\underline{B}_r(s) = \underline{B}(s) - \xi\underline{s} = B_r(s) - \int_0^1 B_r(s)ds$.

3.4 Remarks

(i) The bootstrap estimate $\hat{b}_n^*$ is consistent and $\sqrt{n}\hat{b}_n^*$ has the same limit distribution as $\sqrt{n}\hat{b}_n$. Similarly, the bootstrap $t-$ statistic $t(\hat{b}_n^*)$ diverges at the rate $\sqrt{n}$, just as the regression $t-$ statistic $t(\widehat{b}_n)$,but the limit distribution of $t(\hat{b}_n^*)/\sqrt{n}$ given in part (c) is different from that of $t(\hat{b}_n)/\sqrt{n}$ (given in Remark 3.2(iii)).

(ii) According to parts (d) and (e), the residuals in the regression (23) are asymptotically uncorrelated. This means that bootstrap observations $(x_t^{**})_1^n$ generated from

$$x_t^{**} = \widehat{b}_n t + \widetilde{u}_t^* \tag{24}$$

behave in the regression (23) as trend stationary data with white noise residuals, whereas the orginal data $(x_t)_1^n$ is $I(1)$. Again, the original regression (18) and the bootstrap regression (23) have fundamentally different characteristics.

(b) Block Bootstrapping of Residuals

Block bootstrap samples of residuals $(u_t^{b*})_1^n$ are constructed by randomly sampling M blocks (with replacement) of the centred residuals $(\widehat{u}_t - n^{-1}\sum_{s=1}^n \widehat{u}_s)_1^n$ from (18). A typical block bootstrap residual can then be written as $\widetilde{u}_t^{b*} = \widehat{u}_{(j-1)m+k} - n^{-1}\sum_{s=1}^n \widehat{u}_s$, where j is a random index drawn from $\{1, 2, ..., M\}$. Block bootstrap data $(x_t^{b*})_1^n$ are generated according to

$$x_t^{b**} = \widehat{b}_n t + \widetilde{u}_t^{b*} \tag{25}$$

and least squares produces the fitted regression

$$x_t^{b**} = \widehat{b}_n^{b*} t + u_t^{b**}, \tag{26}$$

which defines the residual process u_t^{b**}. Let $s^{b**2} = \frac{1}{n}\sum_{t=1}^n u_t^{b**2}$, let s^{b*} and t^{b*} be the standard error and $t-$ ratio of $\widehat{b}_n^{b*}$ in (26), and let DW^{b**} be the DW ratio in (26). Again, we denote serial correlations by the notation $\widehat{\rho}_h^{b**}$. The following results give the limit behavior of these bootstrap statistics.

As in Section 2(c) and setting $t = (s-1)m + k$, there exists a probability space in which we can write

$$\frac{x_t^{b**}}{\sqrt{n}} = \sqrt{n}\widehat{b}_n \frac{t}{n} + \frac{\widehat{u}_{(j_s-1)m+k} - n^{-1}\sum_{s=1}^n \widehat{u}_s}{\sqrt{n}} \tag{27}$$

$$= [\xi + o_{a.s}(1)]\frac{(s-1)m+k}{n} + \underline{B}_r\left(\tfrac{(j_s-1)m+k}{n}\right) + o_{a.s}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right)$$

$$= \xi\left[\frac{(s-1)}{M} + \frac{k}{n}\right] + \underline{B}_r\left(R_{Mj_s} + \frac{k}{n}\right) + o_{a.s}(1), \tag{28}$$

where $R_{Mj_s} = \frac{j_s-1}{M}$ is uniformly distributed over $\{0, \frac{1}{M}, .., \frac{M-1}{M}\}$ for each s. As $n, M \to \infty$, with a slight abuse of notation, we have

$$R_{Mj_s} \to_{d^*} R_j,$$

where R_j is $iidU\,[0,1]$. Similarly, for any $r \in [0,1]$ there exist integers s_r and k_r for which $[nr] = (s_r - 1)m + k_r$ and we can write

$$\frac{x_{[nr]}^{b**}}{\sqrt{n}} = \xi r + \underline{B}_r\left(R_{M,r}\right) + o_{a.s}\left(1\right), \tag{29}$$

where $R_{M,r} = \frac{J_r - 1}{M}$ is uniformly distributed over $\{0, \frac{1}{M}, .., \frac{M-1}{M}\}$ and $R_{M,r} \to_{d^*} R_r$, where R_r is uniform over $[0,1]$ for each r.

We give a block bootstrap version of Theorem 3.3.

3.5 Theorem *Let $(x_t^{b**})_1^n$ be bootstrap data generated from (25) with $\frac{1}{m} + \frac{m}{n} \to 0$ and using centred block bootstrap residuals $(\widetilde{u}_t^{b*})_1^n$ where the original data $(x_t)_1^n$ in the trend regression (18) is a unit root process (1) with u_t satisfying Assumption* L. *Then:*

(a) $\sqrt{n}\hat{b}_n^{b*} \to_{d^*} \xi \quad a.s.(P)$,

(b) $\frac{1}{n}s^{b**2} \to_{d^*} \int_0^1 \underline{B}_r(s)^2 dr \quad a.s.(P)$,

(c) $\frac{1}{\sqrt{n}}t^{b*} \to_{d^*} \frac{\xi\left(\int_0^1 r^2\right)^{\frac{1}{2}}}{\left(\int_0^1 \underline{B}_r(s)^2 ds\right)^{\frac{1}{2}}} \quad a.s.(P)$,

(d) $\widehat{\rho}_h^{b**} \to_{p^*} 1, \quad a.s.(P)$, for all $h \neq 0$,

(e) $DW^{b**} \to_{p^*} 0 \quad a.s.(P)$,

(f) $R^{b**2} \to_{d^*} \frac{\xi^2 \int_0^1 r^2 dr}{\int_0^1 \underline{B}_r(s)^2 ds + \xi^2 \int_0^1 r^2 dr} < 1, \quad a.s.(P)$.

3.6 Remarks

(i) The block bootstrap results for the regression coefficient $\hat{b}_n^{b*}$, $t-$ ratio t^{b*} and R^2 are the same as those for the centred bootstrap given in Theorem 3.4.

(ii) The serial correlation properties of the residuals in the block bootstrap regression are different from those in Theorem 3.4. Just as in block bootstrapping $I(1)$ data (Theorem 2.5), the residual serial correlation coefficients $\widehat{\rho}^{b**}, \widehat{\rho}_h^{b**} \to_{p^*} 1$ and the Durbin Watson statistic $DW^{b**} \to_{p^*} 0$. Thus, these diagnostics are now first order consistent in the bootstrap regression (26). This regression is now spurious in the same sense as the original trend regression (23).

(iii) However, the limit distributions of neither $\widehat{\rho}_h^{b**}$ nor DW^{b**} are consistent. As in Theorem 2.6 we find that these bootstrap diagnostic statistics have different rates of convergence and different limits from those of the original sample data. Theorem 3.7 below gives the result for $\widehat{\rho}_1^{b**}$. Thus, bootstrapping these diagnostic statistics (or versions of them that are nuisance parameter free) will not produce an asymptotically valid inference procedure.

3.7 Theorem *Under the same conditions as Theorem 3.5*, $\widehat{\rho}_1^{b**}$ *has the following limit as* $n \to \infty$:

$$\frac{n}{M}\left(\widehat{\rho}_1^{b**} - 1\right) \to_{d^*} -1 \quad a.s. \, (P) \, .$$

4. Spurious Regression on Stochastic Trends

Let z_t be an $(\ell{+}1){-}$ vector integrated time series generated as in (1) with $\Delta z_t = u_{zt}$ satisfying L and z_t satisfying Lemma 2.1 with limiting Brownian motion B_z whose variance matrix is $\Omega_{zz} > 0$. Partition $z_t = (y_t, x_t')'$ into the scalar y_t and $\ell{-}$ vector x_t and let $B_z = (B_y, B_x')'$ be a conformable partition of B_z. Run the (spurious) regression

$$y_t = \widehat{b}_n' x_t + \widehat{u}_t. \tag{30}$$

Let s_i be the standard error of $\widehat{b}_{ni}$, t_i be the corresponding $t{-}$statistic, and set $\widehat{\rho} = \sum_2^n \widehat{u}_t \widehat{u}_{t-1} / \sum_2^n \widehat{u}_{t-1}^2$ and $DW = \sum_2^n (\widehat{u}_t - \widehat{u}_{t-1})^2 / \sum_2^n \widehat{u}_{t-1}^2$. As in Phillips (1986),

$$\hat{b}_n \to_d A_{xx}^{-1} a_{xy} = \xi_{yx}, \quad \text{say,} \tag{31}$$

$$\frac{1}{\sqrt{n}} t_i = \frac{\hat{b}_{ni}}{s_i} \to_d \frac{\xi_{yxi}}{\left(a_{yy.x}\left[A_{xx}^{-1}\right]_{ii}\right)^{\frac{1}{2}}} = \xi_{ti},$$

and

$$nDW \to_d \frac{\eta' \Sigma_{zz} \eta}{a_{yy.x}},$$

where, in conformable partitions, we define

$$\int_0^1 B_z(r) B_z'(r)\, dr = \begin{bmatrix} a_{yy} & a_{xy}' \\ a_{xy} & A_{xx} \end{bmatrix}, \quad a_{yy.x} = a_{yy} - a_{xy}' A_{xx}^{-1} a_{xy}, \quad \eta = \begin{bmatrix} 1 \\ -A_{xx}^{-1} a_{xy} \end{bmatrix},$$

and $\Sigma_{zz} = E\left(u_{zt} u_{zt}'\right)$. These convergences can be replaced by $a.s.$ convergence after appropriate changes in the probability space as we have discussed earlier.

Application of bootstrap methods to the residuals in (30) leads to results that are similar to those of the previous section. We therefore provide the main results here in brief. Bootstrap samples $(\widetilde{u}_t^*)_1^n$ are formed by random sampling from centred versions of the residuals $(\widehat{u}_t)_1^n$ in (30), bootstrap data $(y_t^{**})_1^n$ are generated according to

$$y_t^{**} = \widehat{b}_n' x_t + \widetilde{u}_t^* \tag{32}$$

and least squares produces the fitted regression

$$y_t^{**} = \widehat{b}_n^{*\prime} x_t + u_t^{**}, \tag{33}$$

which defines the residual process u_t^{**}. Let $s^{**2} = \frac{1}{n}\sum_{t=1}^n u_t^{**2}$, let s_i^* and t_i^* be the standard error and $t{-}$ ratio of $\widehat{b}_n^*$, and let DW^{**} be the serial correlation coefficient and DW ratio. As earlier, denote serial correlations by the notation $\widehat{\rho}_h^{**} = \sum_{t=h}^n u_t^{**} u_{t-h}^{**} / \sum_{t=1}^n u_{t-h}^{**2}$. The following results give the limit behavior of these bootstrap statistics. We confine the discussion to centred bootstrap samples.

4.1 Theorem *Let $(y_t^{**})_1^n$ be bootstrap data generated from (32) using $(\widetilde{u}_t^*)_1^n$ where the data $(y_t, x_t)_1^n$ in the sample regression (30) is generated as in (1) with increments $\Delta z_t = u_{zt}$ satisfying L.*

(a) $\hat{b}_n^* \rightarrow_{d^*} \xi_{yx} \quad a.s.(P)$,

(b) $\frac{1}{n}s^{**2} \rightarrow_{d^*} \eta' \left(\int_0^1 \underline{B}_z \underline{B}_z' \right) \eta \quad a.s.(P)$,

(c) $\frac{1}{\sqrt{n}} t_i^* \rightarrow_{d^*} \frac{\xi_{yxi}}{\left\{ \eta' \left(\int_0^1 \underline{B}_z \underline{B}_z' \right) \eta \left[A_{xx}^{-1} \right]_{ii} \right\}^{\frac{1}{2}}} = \xi_{ti}^* \quad a.s.(P)$,

(d) $\widehat{\rho}_h^{**} \rightarrow_{p^*} 0, \quad a.s.(P)$, for all $h \neq 0$,

(e) $DW^{**} \rightarrow_{p^*} 2 \quad a.s.(P)$,

(f) $R^{**2} \rightarrow_{d^*} \frac{\xi_{yx}' \left(\int_0^1 B_x B_x' \right) \xi_{yx}}{\eta' \left(\int_0^1 \underline{B}_z \underline{B}_z' \right) \eta + \xi_{yx}' \left(\int_0^1 B_x B_x' \right) \xi_{yx}} < 1, \quad a.s.(P)$.

The bootstrap regression coefficient in (33) is consistent for ξ_{yx} and the $t-$ ratio t_i^* diverges at the rate $\sqrt{n}$ just as in the spurious regresion (30), although the limit variate $\xi_{ti}^* \neq \xi_{ti}$. More importantly, however, the residuals in the regression (33) are asymptotically uncorrelated and so the bootstrap regression is asymptotically a cointegrating regression.

Different results apply when we use the block bootstrap. Using the same notation as in Section 3(c) with the affix b to signify use of the block bootstrap, we find the following limit theory.

4.2 Theorem *Let $(y_t^{b**})_1^n$ be bootstrap data generated from (32) using centred block bootstrap residuals $(\widetilde{u}_t^{b*})_1^n$ with $\frac{1}{m} + \frac{m}{n} \rightarrow 0$ and where the original data the data $(y_t, x_t)_1^n$ in the sample regression (30) is generated as in (1) with increments $\Delta z_t = u_{zt}$ satisfying L. Then:*

(a) $\hat{b}_n^{b*} \rightarrow_{d^*} \xi_{yx} \quad a.s.(P)$,

(b) $\frac{1}{n}s^{b**2} \rightarrow_{d^*} \eta' \left(\int_0^1 \underline{B}_z \underline{B}_z' \right) \eta \quad a.s.(P)$,

(c) $\frac{1}{\sqrt{n}} t_i^{b*} \rightarrow_{d^*} \xi_{ti}^* \quad a.s.(P)$,

(d) $\widehat{\rho}_h^{b**} \rightarrow_{p^*} 1, \quad a.s.(P)$, for all $h \neq 0$,

(e) $DW^{b**} \rightarrow_{p^*} 0 \quad a.s.(P)$,

(f) $R^{b**2} \rightarrow_{d^*} \frac{\xi_{yx}' \left(\int_0^1 B_x B_x' \right) \xi_{yx}}{\eta' \left(\int_0^1 \underline{B}_z \underline{B}_z' \right) \eta + \xi_{yx}' \left(\int_0^1 B_x B_x' \right) \xi_{yx}} < 1, \quad a.s.(P)$.

4.3 Theorem *Under the same conditions as Theorem 4.2, $\widehat{\rho}_1^{b**}$ has the following limit as* $n \rightarrow \infty$:

$$\frac{n}{M} \left(\widehat{\rho}_1^{b**} - 1 \right) \rightarrow_{p^*} -1 \quad a.s. (P) .$$

5. Conclusions and Implications

This paper shows that mechanical application of both the bootstrap and block bootstrap will give perverse results if used in residual based testing for unit roots to distinguish spurious and cointegrating regressions or trend stationary from integrated data. In particular, bootstrapping the residuals of such regressions turns spurious regressions into cointegrating regressions and thereby changes the character of the regressions in a fundamental way. While the block bootstrap retains the $I(1)$ feature of the regression residuals and has serial correlation coefficients that tend to unity, their rate of convergence is slower than the $O(n)$ rate of the sample residual serial correlations. Serial correlations of the block bootstrap converge at the rate $O(m)$, where m is the block size and the limit is no longer of the unit root type. Thus, the bootstrap and the block bootstrap both fail seriously in reproducing the properties of spurious regressions and are unsuitable for residual based testing in this context.

These results reinforce the warnings given in earlier research about the difficulties encountered by the bootstrap with dependent data (Horowitz, 1999, among many others) and with correlation coefficients (Hall, 1992, p.152). It seems that bootstrapping integrated data exacerbates known problems of bootstrap inconsistency in unit root inference (e.g. Basawa et. al. 1991), producing differences not only in the limit distributions between the bootstrap and the original statistic but first order differences in the limits of the statistics in some cases. The block bootstrap shows some robustness to nonstationarity in this regard, but as Hinkley (1997) remarks, it too is unable to withstand the integrated model.

These results have implications for the use of the bootstrap in residual based cointegration testing, where the null hypothesis is that the residuals in the regression are integrated. The results here show that routine applications of the bootstrap in such situations should be avoided. An alternative approach under the null is to difference the residuals and apply sieve bootstrap methods (Kreiss, 1992, Buhlmann, 1997, 1998) on the differenced residuals and then bootstrap invariance principles (Bickel and Buhlmann, 1999, and Park, 2001) to bootstrap the distribution of residual based cointegration tests. Pretesting for the presence of a unit root in the residuals prior to the use of the bootstrap by consistent methods of model selection (e.g. Phillips and Ploberger, 1996) is another alternative. Other approaches like hybrid resampling (Chuang and Lai, 2000) and subsampling (Politis, Romano and Wolf, 1999) have been found to offer improvements in some cases where the bootstrap performs poorly and these also seem worthy of study in this context.

6. Appendix

6.1 Proof of Lemma 2.1 This is a straightforward extension of lemma 3.1 of Phillips(1999). Under L, the BN decomposition (Phillips and Solo, 1992) $C(L) = C(1) + \widetilde{C}(L)(L-1)$ is valid, where $\widetilde{C}(L) = \sum_{j=0}^{\infty} \widetilde{c}_j L^j$ with $\widetilde{c}_j = \sum_{s=j+1}^{\infty} c_s$ and $\sum_{j=0}^{\infty} ||\widetilde{c}_j|| < \infty$. Then,

$$u_t = C(1)\varepsilon_t + \widetilde{\varepsilon}_{t-1} - \widetilde{\varepsilon}_t = C(1)\Sigma^{\frac{1}{2}}\eta_t + \widetilde{\varepsilon}_{t-1} - \widetilde{\varepsilon}_t,$$

and

$$S_t = C(1)\Sigma^{\frac{1}{2}} S_{\eta t} + \widetilde{\varepsilon}_0 - \widetilde{\varepsilon}_t,$$

where η_t is $iid(0, I)$, $\widetilde{\varepsilon}_t = \widetilde{C}(L)\varepsilon_t$ is stationary, and $S_{\eta t} = \sum_{j=1}^{t} \eta_j$. A strong approximation to the partial sum process $S_{\eta t}$ of η_j may be constructed componentwise as in lemma 3.1 of

Phillips (1999) leading to

$$\sup_{0\leq k\leq n}\left|S_{\eta_i k}-W_i(k)\right|=o_{a.s}(n^{\frac{1}{q}}),\quad i=1,..,m \tag{34}$$

giving a uniform approximation to $S_{\eta k}$ over $0\leq k\leq n$ in terms of the vector standard Brownian motion $W=(W_i)$. Then, setting $B(r)=C(1)\Sigma^{\frac{1}{2}}W(r)$, we have

$$\begin{aligned}\sup_{0\leq k\leq n}\left|\left|\frac{S_k}{\sqrt{n}}-B(\frac{k}{n})\right|\right| &\leq \left|\left|C(1)\Sigma^{\frac{1}{2}}\right|\right|\max_i\sup_{0\leq k\leq n}\left|\frac{S_{\eta_i k}}{\sqrt{n}}-W_i(\frac{k}{n})\right|+2\sup_{0\leq k\leq n}\frac{||\widetilde{\varepsilon}_k||}{\sqrt{n}}\\ &= o_{a.s.}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right),\end{aligned}$$

as in lemma 3.1 of Phillips (1999).

6.2 Proof of Lemma 2.2 For $r\in[0,1]$, we can write $x^*_{[nr]}=x_{J_r}$ where J_r is uniform over $\{1,2,...,n\}$ for each r and then as in (6) we have the embedding

$$\frac{x^*_{[nr]}}{\sqrt{n}}=\frac{x_{J_r}}{\sqrt{n}}=B\left(\frac{J_r}{n}\right)+o_{a.s}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right)=B(R_{n,r})+o_{a.s}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right), \tag{35}$$

where $R_{n,r}=\frac{J_r}{n}$ is uniformly distributed over $\{\frac{1}{n},\frac{2}{n},..,1\}$ for each r. Moreover, for $r\neq s$ and n large enough so that $n|r-s|>1$, J_r and J_s are independent draws. Hence, $R_{n,r}$ and $R_{n,s}$ are statistically independent for large n for all $r\neq s$.

Substituting the LK representation (5) for B in (35) we get

$$\frac{x^*_{[nr]}}{\sqrt{n}}=\sqrt{2}\sum_{k=1}^{\infty}\frac{\sin\left[\left(k-\frac{1}{2}\right)\pi R_{n,r}\right]}{\left(k-\frac{1}{2}\right)\pi}\xi_k+o_{a.s}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right). \tag{36}$$

Now $R_{n,r}\rightarrow_{d^*}R_r$, $a.s\ (P)$, where R_r has a continuous uniform distribution over $[0,1]$ for each r. It follows by the continuous mapping theorem that

$$\sin\left[\left(k-\frac{1}{2}\right)\pi R_{n,r}\right]\rightarrow_{d^*}\sin\left[\left(k-\frac{1}{2}\right)\pi R_r\right],\quad a.s.(P). \tag{37}$$

Hence,

$$\frac{x^*_{[nr]}}{\sqrt{n}}\rightarrow_{d^*}B(R_r)\quad a.s.(P). \tag{38}$$

Since $R_{n,r}$ and $R_{n,s}$ are asymptotically independent for all $r\neq s$, it follows that R_r and R_s are independent for $r\neq s$. Thus, $\{R_r:r\in[0,1]\}$ is family of independent uniform variates on $[0,1]$.

In a similar way we write the centred bootstrap as $\widetilde{x}^*_t=x_j-\overline{x}$, and then in the same notation as (35)

$$\begin{aligned}\frac{x^*_{[nr]}}{\sqrt{n}} &= \frac{x_{J_r}}{\sqrt{n}}=B\left(\frac{J_r}{n}\right)-\frac{1}{n}\sum_{t=1}^{n}B\left(\frac{t}{n}\right)+o_{a.s}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right)\\ &= B(R_{n,r})-\int_0^1 B(s)\,ds+o_{a.s}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right),\end{aligned} \tag{39}$$

and the stated result follows from (38) and (39).

6.3 Proof of Lemma 2.3

Part (a) Using (9) we have

$$\frac{1}{n}\sum_{t=1}^{n}\frac{x_t^*}{\sqrt{n}}=\sqrt{2}\omega\sum_{k=1}^{\infty}\frac{\frac{1}{n}\sum_{j=1}^{n}\sin\left[\left(k-\frac{1}{2}\right)\pi R_{nj}\right]}{\left(k-\frac{1}{2}\right)\pi}\xi_k+o_{a.s}\left(\frac{1}{n^{\frac{1}{2}-\frac{1}{p}}}\right), \tag{40}$$

and, in view of (8),

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\sin\left[\left(k-\frac{1}{2}\right)\pi R_{nj}\right] &= \frac{1}{n}\sum_{j=1}^{n}\sin\left[\left(k-\frac{1}{2}\right)\pi R_j\right]+o_{p^*}(1) &(41)\\ &\to_{p^*} E\left\{\sin\left[\left(k-\frac{1}{2}\right)\pi R_j\right]\right\} \quad a.s.\,(P)\\ &= \int_0^1\sin\left[\left(k-\frac{1}{2}\right)\pi r\right]dr. &(42)\end{aligned}$$

It follows from (40) - (42) that

$$\begin{aligned}\frac{1}{n^{\frac{3}{2}}}\sum_{t=1}^{n}x_t^* &\to_{d^*}\sqrt{2}\sum_{k=1}^{\infty}\frac{\int_0^1\sin\left[\left(k-\frac{1}{2}\right)\pi r\right]dr}{\left(k-\frac{1}{2}\right)\pi}\xi_k \quad a.s.\,(P)\\ &= \int_0^1 B(r)\,dr,\end{aligned}$$

the last line following by virtue of the uniform integrability of the LK series in this case.

Part (b) In a similar fashion, we find

$$\frac{1}{n^{\frac{3}{2}}}\sum_{t=1}^{n}\widetilde{x}_t^*\to_{d^*}\int_0^1\underline{B}(r)\,dr=0, \quad a.s.\,(P).$$

Part (c) Higher order sample moments are dealt with in the same way. Thus, for part (c) we have

$$\frac{1}{n^2}\sum_{t=1}^{n}x_t^*x_t^{*\prime}=\frac{1}{n}\sum_{t=1}^{n}\frac{x_t^*}{\sqrt{n}}\frac{x_t^*}{\sqrt{n}}^{\prime}=2\sum_{k,m=1}^{\infty}\frac{\frac{1}{n}\sum_{j=1}^{n}\sin\left[\left(k-\frac{1}{2}\right)\pi R_{nj}\right]\sin\left[\left(m-\frac{1}{2}\right)\pi R_{nj}\right]}{\left(k-\frac{1}{2}\right)\left(m-\frac{1}{2}\right)\pi^2}\xi_k\xi_m^{\prime},$$

and

$$\begin{aligned}&\frac{1}{n}\sum_{j=1}^{n}\sin\left[\left(k-\frac{1}{2}\right)\pi R_{nj}\right]\sin\left[\left(m-\frac{1}{2}\right)\pi R_{nj}\right]\\ \to\ &_{p^*}E\left\{\sin\left[\left(k-\frac{1}{2}\right)\pi R_j\right]\sin\left[\left(m-\frac{1}{2}\right)\pi R_j\right]\right\} \quad a.s.\,(P)\\ =\ &\int_0^1\sin\left[\left(k-\frac{1}{2}\right)\pi r\right]\sin\left[\left(m-\frac{1}{2}\right)\pi r\right]dr.\end{aligned}$$

Again, uniform convergence of the LK series implies that we can integrate term by term, leading to the result

$$\frac{1}{n^2}\sum_{t=1}^{n} x_t^* x_t^{*\prime} \rightarrow_{d^*} 2\sum_{k,m=1}^{\infty} \frac{\int_0^1 \sin\left[\left(k-\frac{1}{2}\right)\pi r\right]\sin\left[\left(m-\frac{1}{2}\right)\pi r\right]dr}{\left(k-\frac{1}{2}\right)\left(m-\frac{1}{2}\right)\pi^2}\xi_k\xi_m' = \int_0^1 B(r)\,B(r)'\,dr \quad a.s.(P)$$

Alternatively, by direct calculation we have

$$\begin{aligned}\int_0^1 \sin\left[\left(k-\frac{1}{2}\right)\pi r\right]\sin\left[\left(m-\frac{1}{2}\right)\pi r\right]dr &= \frac{1}{2}\int_0^1 \left\{\cos\left[(k-m)\pi r\right]-\cos\left[(k+m-1)\pi r\right]\right\}dr \\ &= \begin{cases} 0 & k\neq m \\ \frac{1}{2} & k=m \end{cases}\end{aligned}$$

and then

$$\frac{1}{n^2}\sum_{t=1}^{n} x_t^* x_t^{*\prime} \rightarrow_{d^*} \sum_{k=1}^{\infty}\frac{1}{\left[\left(k-\frac{1}{2}\right)\pi\right]^2}\xi_k\xi_m',$$

which is a series representation of $\int_0^1 B(r)\,B(r)'\,dr$. Part (d) follows by a similar calculation.

Part (e) Using (4) and (6) we can write

$$\begin{aligned}&\frac{1}{n}\textstyle\sum_1^n\left(\frac{x_t^*}{\sqrt{n}}\right)\left(\frac{x_t}{\sqrt{n}}\right)' \\ =\ &\frac{1}{n}\textstyle\sum_{j=1}^n\left[B(R_{nj})+o_{\text{a.s.}}\left(\frac{1}{n^{1/2-1/p}}\right)\right]\left[B\left(\frac{j}{n}\right)+o_{\text{a.s.}}\left(\frac{1}{n^{1/2-1/p}}\right)\right]' \\ =\ &\frac{1}{n}\sum_{j=1}^{n} B(R_{nj})B\left(\frac{j}{n}\right)'+o_{\text{a.s.}}(1) \\ =\ &2\frac{1}{n}\textstyle\sum_1^n\left(\displaystyle\sum_{k=1}^{\infty}\frac{\sin\left[\left(k-\frac{1}{2}\right)\pi R_{nj}\right]}{\left(k-\frac{1}{2}\right)\pi}\xi_k\right)\left(\displaystyle\sum_{\ell=1}^{\infty}\frac{\sin\left[\left(\ell-\frac{1}{2}\right)\pi\frac{j}{n}\right]}{\left(\ell-\frac{1}{2}\right)\pi}\xi_\ell'\right)+o_{\text{a.s.}}(1) \\ =\ &2\sum_{k,\ell=1}^{\infty}\frac{\xi_k\xi_\ell'}{\left(k-\frac{1}{2}\right)\left(\ell-\frac{1}{2}\right)\pi^2}\frac{1}{n}\sum_{j=1}^{n}\sin\left[\left(k-\frac{1}{2}\right)\pi R_{nj}\right]\sin\left[\left(\ell-\frac{1}{2}\right)\pi\frac{j}{n}\right]+o_{\text{a.s.}}(1)\end{aligned} \tag{43}$$

As in (41),

$$\begin{aligned}&\frac{1}{n}\textstyle\sum_1^n\sin\left[\left(k-\frac{1}{2}\right)\pi R_{nj}\right]\sin\left[\left(\ell-\frac{1}{2}\right)\pi\frac{j}{n}\right] \\ =\ &\frac{1}{n}\textstyle\sum_1^n E\left\{\sin\left[\left(k-\frac{1}{2}\right)\pi R_j\right]\right\}\sin\left[\left(\ell-\frac{1}{2}\right)\pi\frac{j}{n}\right] \\ &+\frac{1}{n}\textstyle\sum_1^n\left[\sin\left[\left(k-\frac{1}{2}\right)\pi R_{nj}\right]-E\sin\left\{\left[\left(k-\frac{1}{2}\right)\pi R_j\right]\right\}\right]\sin\left[\left(\ell-\frac{1}{2}\right)\pi\frac{j}{n}\right] \\ =\ &E\left\{\sin\left[\left(k-\frac{1}{2}\right)\pi R\right]\right\}\frac{1}{n}\textstyle\sum_1^n\sin\left[\left(\ell-\frac{1}{2}\right)\pi\frac{j}{n}\right]\end{aligned} \tag{44}$$

$$
\begin{aligned}
& + \frac{1}{n} \textstyle\sum_1^n \left[\sin \left[\left(k - \frac{1}{2} \right) \pi R_j \right] - E \sin \left\{ \left[\left(k - \frac{1}{2} \right) \pi R_j \right] \right\} \right] \sin \left[\left(\ell - \frac{1}{2} \right) \pi \frac{j}{n} \right] + o_{p^*}(1) \\
= \; & E \left\{ \sin \left[\left(k - \frac{1}{2} \right) \pi R \right] \right\} \frac{1}{n} \textstyle\sum_1^n \sin \left[\left(\ell - \frac{1}{2} \right) \pi \frac{j}{n} \right] + \frac{1}{n} \textstyle\sum_1^t \eta_j \sin \left[\left(\ell - \frac{1}{2} \right) \pi \frac{j}{n} \right] \quad (45) \\
= \; & E \left(\sin \left(k - \frac{1}{2} \right) \pi R \right) \left[\int_0^1 \sin \left(\ell - \frac{1}{2} \right) \pi s \, ds + o(1) \right] + o_{p^*}(1) \quad (46) \\
= \; & \left(\int_0^1 \sin \left(k - \frac{1}{2} \right) \pi r \, dr \right) \left(\int_0^1 \sin \left(\ell - \frac{1}{2} \right) \pi s \, ds \right) + o_{p^*}(1) \\
= \; & \frac{1}{\left(k - \frac{1}{2} \right) \pi} \frac{1}{\left(\ell - \frac{1}{2} \right) \pi} + o_{p^*}(1). \quad (47)
\end{aligned}
$$

In line (44) above, R is uniformly distributed on $[0,1]$, in line (45) $\eta_j = \sin\left[\left(k - \frac{1}{2}\right)\pi R_j\right] - E\sin\left\{\left[\left(k - \frac{1}{2}\right)\pi R_j\right]\right\} \equiv \text{iid}(0, \sigma_\eta^2)$, with $\sigma_\eta^2 = E(\eta_j^2)$, and line (47) follows since $\int_0^1 \sin\left(k - \frac{1}{2}\right)\pi r\, dr = 1/(k - \frac{1}{2})\pi$.

From (43) and (47) we deduce that

$$
\begin{aligned}
\frac{1}{n} \textstyle\sum_1^t \left(\frac{x_t^*}{\sqrt{n}} \right) \left(\frac{x_t}{\sqrt{n}} \right) \; = \; & 2 \sum_{k,\ell=1}^{\infty} \frac{\xi_k \xi_\ell'}{\left[\left(k - \frac{1}{2} \right) \pi \left(\ell - \frac{1}{2} \right) \pi \right]^2} + o_{p^*}(1) \\
\to_{d^*} \; & 2 \left\{ \sum_{k=1}^{\infty} \frac{\xi_k}{\left[\left(k - \frac{1}{2} \right) \pi \right]^2} \right\} \left\{ \sum_{k=1}^{\infty} \frac{\xi_k}{\left[\left(k - \frac{1}{2} \right) \pi \right]^2} \right\}' \quad a.s.(P) \\
= \; & \left\{ \int_0^1 B(r) dr \right\} \left\{ \int_0^1 B(r) dr \right\}',
\end{aligned}
$$

giving result (e). Part (f) follows in a similar way.

Part (g)

$$
\begin{aligned}
\frac{1}{n} \textstyle\sum_1^n \frac{x_t^*}{\sqrt{n}} \frac{t}{n} \; = \; & \frac{1}{n} \sum_{j-1}^{n} \left[B(R_{nj}) + o_{\text{a.s.}} \left(\frac{1}{n^{1/2-1/p}} \right) \right] \frac{j}{n} + o_{\text{a.s.}}(1) \\
= \; & \frac{1}{n} \textstyle\sum_1^n B(R_{nj}) \frac{j}{n} + o_{\text{a.s.}}(1).
\end{aligned}
$$

Set $a_k = E\left[\sin\left[\left(k - \frac{1}{2}\right)\pi R\right]\right] = 1/(k - \frac{1}{2})\pi$, where R is uniformly distributed on $[0,1]$. Then, we have

$$
\begin{aligned}
\frac{1}{n} \textstyle\sum_1^n B(R_j) \frac{j}{n} \; = \; & \sqrt{2} \sum_{k=1}^{\infty} \frac{\xi_k}{\left(k - \frac{1}{2} \right) \pi} \frac{1}{n} \sum_{j=1}^{n} \sin \left[\left(k - \frac{1}{2} \right) \pi R_{nj} \right] \frac{j}{n} \\
= \; & \sqrt{2} \sum_{k=1}^{\infty} \frac{\xi_k}{\left(k - \frac{1}{2} \right) \pi} \left\{ \frac{1}{n} \textstyle\sum_1^n a_k \frac{j}{n} + \frac{1}{n} \textstyle\sum_1^n \left[\sin \left(k - \frac{1}{2} \right) \pi R_{nj} - a_k \right] \frac{j}{n} \right\} \\
= \; & \sqrt{2} \sum_{k=1}^{\infty} \frac{\xi_k}{\left(k - \frac{1}{2} \right) \pi} \left\{ \frac{1}{n} \textstyle\sum_1^n a_k \frac{j}{n} + \frac{1}{n} \textstyle\sum_1^n \left[\sin \left(k - \frac{1}{2} \right) \pi R_j - a_k \right] \frac{j}{n} \right\} + o_{p^*}(1)
\end{aligned}
$$

$$
\begin{aligned}
&= \sqrt{2}\sum_{k=1}^{\infty}\frac{\xi_k}{\left[\left(k-\frac{1}{2}\right)\pi\right]^2}\int_0^1 rdr + o_{p^*}(1) \\
&= \frac{1}{2}\int_0^1 B(r)dr + o_{p^*}(1), \qquad (48)
\end{aligned}
$$

giving the stated result.

6.4 Proof of Theorem 2.4

Part (a) First consider $\widehat{\rho}^* = n^{-2}\sum_{t=1}^n x_t^* x_{t-1}^* / n^{-2}\sum_{t=1}^n x_{t-1}^{*2}$. From Lemma 2.3 (c), the denominator of $\widehat{\rho}^*$ has the following limit

$$
\frac{1}{n^2}\sum_1^n x_{t-1}^{*2} \rightarrow_{d^*} \int_0^1 B(r)^2 dr \quad a.s.\,(P). \qquad (49)
$$

Using (6), the numerator can be written as

$$
\begin{aligned}
\frac{1}{n^2}\sum_1^n x_t^* x_{t-1}^* &= \frac{1}{n}\sum_1^n \left(\frac{x_t^*}{\sqrt{n}}\right)\left(\frac{x_{t-1}^*}{\sqrt{n}}\right) \\
&= \frac{1}{n}\sum_{j=1}^n \left[B(R_{nj}) + o_{\text{a.s.}}(1)\right]\left[B(R_{nj-1}) + o_{\text{a.s.}}(1)\right]. \qquad (50)
\end{aligned}
$$

Here, R_{nj} and R_{nj-1} are independent draws from $1/n, ..., 1$ and, in view of (7),

$$
R_{nj} \rightarrow_{d^*} R_j, \quad R_{nj-1} \rightarrow_{d^*} R_{j-1} \qquad a.s.\,(P),
$$

where R_j and R_{j-1} are independent draws from $U[0,1]$. Thus, (50) is asymptotically equivalent to

$$
\frac{1}{n}\sum_{j=1}^n B(R_j)B(S_j), \qquad (51)
$$

where R_j, S_j are independent $U[0,1]$. Using the LK representation of B we therefore find

$$
\begin{aligned}
\frac{1}{n}\sum_{j=1}^n B(R_j)B(S_j) &= 2\omega^2\sum_{k,\ell=1}^{\infty}\frac{\xi_k}{\left(k-\frac{1}{2}\right)\pi}\frac{\xi_\ell}{\left(\ell-\frac{1}{2}\right)\pi}\frac{1}{n}\sum_1^n \sin\left[\left(k-\tfrac{1}{2}\right)\pi R_j\right]\sin\left[\left(\ell-\tfrac{1}{2}\right)\pi S_j\right] \\
&\rightarrow_{p^*} 2\omega^2\sum_{k,\ell=1}^{\infty}\frac{\xi_k\xi_\ell}{\left(k-\frac{1}{2}\right)\pi\left(\ell-\frac{1}{2}\right)\pi}E\left\{\sin\left[\left(k-\tfrac{1}{2}\right)\pi R_j\right]\right\}E\left\{\sin\left[\left(\ell-\tfrac{1}{2}\right)\pi S_j\right]\right\} \\
&= 2\omega^2\sum_{k,\ell=1}^{\infty}\frac{\xi_k\xi_\ell}{\left(k-\frac{1}{2}\right)\pi\left(\ell-\frac{1}{2}\right)\pi}\int_0^1 \sin\left[\left(k-\tfrac{1}{2}\right)\pi r\right]dr\int_0^1 \sin\left[\left(\ell-\tfrac{1}{2}\right)\pi s\right]ds \\
&= \left[\int_0^1 (B(r))dr\right]^2. \qquad (52)
\end{aligned}
$$

It follows from (49)-(52) that

$$
\widehat{\rho}^* \rightarrow_{d^*} \xi_\rho = \frac{\left[\int_0^1 (B(r))dr\right]^2}{\int_0^1 B(r)^2 dr} \quad a.s.\,(P),
$$

as stated.

Next consider $\widehat{\rho}_h^* = n^{-2}\sum_{t=h}^n x_t^* x_{t-h}^* / n^{-2}\sum_{t=1}^n x_{t-h}^{*2}$. The denominator has the same limit as (49). The numerator, following (50), can be written as

$$\begin{aligned}\frac{1}{n^2}\sum_h^n x_t^* x_{t-h}^* &= \frac{1}{n}\sum_h^n \left(\frac{x_t^*}{\sqrt{n}}\right)\left(\frac{x_{t-h}^*}{\sqrt{n}}\right) \\ &= \frac{1}{n}\sum_{j=h}^n \left[B(R_{nj}) + o_{\text{a.s.}}(1)\right]\left[B(R_{nj-h}) + o_{\text{a.s.}}(1)\right] \\ &= \frac{1}{n}\sum_{j=h}^n B(R_{nj})B(R_{nj-h}) + o_{\text{a.s.}}(1).\end{aligned}$$

Again, R_{nj} and R_{nj-h} are independent draws from $1/n, ..., 1$ and $R_{nj} \to_{d^*} R_j, \quad R_{nj-h} \to_{d^*} R_{j-h} \quad a.s.(P)$, where R_j and R_{j-h} are independent $U[0,1]$. Thus, (50) is asymptotically equivalent to (51), where R_j, S_j are independent $U[0,1]$ and independent over j. Thus, the limit (52) applies and we deduce the stated result.

Part (b) First consider $\widetilde{\rho}^* = n^{-2}\sum_{t=1}^n \widetilde{x}_t^* \widetilde{x}_{t-1}^* / n^{-2}\sum_{t=1}^n \widetilde{x}_{t-1}^{*2}$. From Lemma 2.3 (c), the denominator of $\widetilde{\rho}^*$ has the following limit

$$\frac{1}{n^2}\sum_1^n \widetilde{x}_{t-1}^{*2} \to_{d^*} \int_0^1 \underline{B}(r)^2 dr \quad a.s.(P). \tag{53}$$

Proceeding as in (50) and using (39), the numerator can be written as

$$\begin{aligned}&\frac{1}{n}\sum_1^n \left(\frac{\widetilde{x}_t^*}{\sqrt{n}}\right)\left(\frac{\widetilde{x}_{t-1}^*}{\sqrt{n}}\right) \\ =\ &\frac{1}{n}\sum_{j=1}^n \left[B(R_{nj}) - \int_0^1 B(r)dr + o_{\text{a.s.}}(1)\right]\left[B(R_{nj-1}) - \int_0^1 B(r)dr + o_{\text{a.s.}}(1)\right],\end{aligned} \tag{54}$$

which is asymptotically equivalent to

$$\frac{1}{n}\sum_{j=1}^n \left[B(R_j) - \int_0^1 B(r)dr\right]\left[B(S_j) - \int_0^1 B(r)dr\right] = \frac{1}{n}\sum_{j=1}^n \underline{B}(R_j)\underline{B}(S_j).$$

where R_j, S_j are independent $U[0,1]$. Calculations analogous to those leading to (52) show that

$$\frac{1}{n}\sum_{j=1}^n B(R_j) \to_{p^*} \int_0^1 B(r)dr \quad a.s.(P),$$

which, in combination with (52), reveal that

$$\frac{1}{n}\sum_{j=1}^n \underline{B}(R_j)\underline{B}(S_j) \to_{p^*} 0 \quad a.s.(P). \tag{55}$$

We deduce that $\widetilde{\rho}^* \to_{p^*} 0 \quad a.s.(P)$ and the result for $\widetilde{\rho}_h^*$ follows in the same way.

Finally, if the serial correlation coefficient is redefined in terms of deviations from means as $\widehat{\rho}_h^e = n^{-2}\sum_{t=h}^n e_t^* e_{t-h}^* / n^{-2}\sum_{t=1}^n e_t^{*2}$, then the numerator $n^{-2}\sum_h^n e_t^* e_{t-1}^*$ evidently has the same form as (54) above. Then (55) applies and we deduce that $\widehat{\rho}_h^e \to_{p^*} 0 \quad a.s.(P)$.

6.5 Proof of Theorem 2.5 Take the serial correlation $\widehat{\rho}^{b*} = n^{-2}\sum_{t=1}^{n} x_t^{b*}x_{t-1}^{b*}/n^{-2}\sum_{t=1}^{n} x_{t-1}^{b*2}$. In a manner analogous to the proof of Lemma 2.3 (c), the denominator of $\widehat{\rho}^{*}$ can be shown to have the following limit

$$\frac{1}{n^2}\sum_1^n x_{t-1}^{b*2} \to_{d^*} \int_0^1 B(r)^2 dr \quad a.s.\,(P)\,. \tag{56}$$

Using (12), the numerator can be written as

$$\begin{aligned} & \frac{1}{n}\sum_1^n \left(\frac{x_t^{b*}}{\sqrt{n}}\right)\left(\frac{x_{t-1}^{b*}}{\sqrt{n}}\right) \\ = & \frac{1}{mM}\sum_{j=1}^{M}\sum_{k=2}^{m}\left[B(R_{Mj}+\frac{k}{n})+o_{\text{a.s.}}(1)\right]\left[B(R_{Mj}+\frac{k-1}{n})+o_{\text{a.s.}}(1)\right] \\ & +\frac{1}{mM}\sum_{j=1}^{M}\left[B\left(R_{Mj}+\frac{1}{n}\right)+o_{\text{a.s.}}(1)\right]\left[B\left(R_{Mj-1}+\frac{m}{n}\right)+o_{\text{a.s.}}(1)\right] \end{aligned} \tag{57}$$

In (57), R_{Mj} and R_{Mj-1} are independent draws from $0, \frac{1}{M}, ..., \frac{M-1}{M}$ and, in view of (13),

$$R_{Mj} \to_{d^*} R_j, \quad R_{Mj-1} \to_{d^*} R_{j-1} \quad a.s.\,(P)\,,$$

where R_j and R_{j-1} are independent draws from $U[0,1]$. Thus, (57) is asymptotically equivalent to

$$\begin{aligned} & \frac{1}{M}\sum_{j=1}^{M}\frac{1}{m}\sum_{k=2}^{m}\left[B(R_{Mj}+\frac{k}{n})+o_{\text{a.s.}}(1)\right]\left[B(R_{Mj}+\frac{k-1}{n})+o_{\text{a.s.}}(1)\right]+o_{\text{a.s.}}\left(\frac{1}{m}\right) \\ = & \sum_{j=1}^{M}\int_{R_{Mj}+\frac{1}{n}}^{R_{Mj}+\frac{m}{n}} B\left(r\right)^2 dr + o_{\text{a.s.}}\left(1\right). \end{aligned} \tag{58}$$

By virtue of the continuity of Brownian motion and the fact that $\frac{m}{n} = \frac{1}{M} \to 0$, (58) is asymptotically equivalent to

$$\frac{1}{M}\sum_{j=1}^{M} B\left(R_{Mj}\right)^2$$

From (13), we have

$$\sin\left[\left(k-\frac{1}{2}\right)\pi R_{Mj}\right] \to_{d^*} \sin\left[\left(k-\frac{1}{2}\right)\pi R_j\right] \quad a.s.(P),$$

as in (8), and

$$\frac{1}{M}\sum_{j=1}^{M}\sin\left[\left(k-\frac{1}{2}\right)\pi R_j\right]\sin\left[\left(\ell-\frac{1}{2}\right)\pi R_j\right] \to_{p^*} E\left\{\sin\left[\left(k-\frac{1}{2}\right)\pi R_j\right]\sin\left[\left(\ell-\frac{1}{2}\right)\pi R_j\right]\right\}.$$

Then, as in the proof of Lemma 2.3 (c) we find that

$$\begin{aligned}\frac{1}{M}\sum_{j=1}^{M} B\left(R_{Mj}\right)^2 &\rightarrow_{d^*} 2\omega^2 \sum_{k,\ell=1}^{\infty} \frac{\xi_k \xi_\ell}{\left(k-\frac{1}{2}\right)\pi\left(\ell-\frac{1}{2}\right)\pi} \int_0^1 \sin\left[\left(k-\tfrac{1}{2}\right)\pi r\right] \sin\left[\left(\ell-\tfrac{1}{2}\right)\pi r\right] dr \\ &= \int_0^1 B(r)^2 dr. \end{aligned} \tag{59}$$

Combining (56)-(59) we deduce that

$$\widehat{\rho}^{b*} \rightarrow_{d^*} \frac{\int_0^1 B(r)^2 dr}{\int_0^1 B(r)^2 dr} = 1 \quad a.s.\,(P),$$

as stated. Proofs for $\widetilde{\rho}_h^{b*}$ and $\widehat{\rho}_h^{be}$ follow in a similar fashion.

6.6 Proof of Theorem 2.6 Write

$$\frac{n}{M}(\widehat{\rho}_1^* - 1) = \frac{1}{nM}\sum_{t=1}^{n} \Delta x_t^* x_{t-1}^* / \frac{1}{n^2}\sum_{t=1}^{n} x_{t-1}^{*2}. \tag{60}$$

The limit of the denominator is given in (56). For the numerator, the identity

$$\Delta \sum_{t=1}^{n} x_t^{*2} = \sum_{t=1}^{n} \Delta x_t^* x_t^* + \sum_{t=1}^{n} x_{t-1}^* \Delta x_t^* = 2\sum_{t=1}^{n} x_{t-1}^* \Delta x_t^* + \sum_{t=1}^{n} \left(\Delta x_t^*\right)^2$$

leads to

$$\begin{aligned} &\frac{1}{nM}\sum_{t=1}^{n} x_{t-1}^* \Delta x_t^* \\ =\ &\frac{1}{2M}\left\{\left(\frac{x_n^*}{\sqrt{n}}\right)^2 - \frac{1}{n}\sum_{t=1}^{n}\left(\Delta x_t^*\right)^2\right\} \\ =\ &\frac{1}{2M}\left\{\left(\frac{x_n^*}{\sqrt{n}}\right)^2 - \frac{1}{mM}\sum_{s=1}^{M}\sum_{k=2}^{m}\left(x_{(j_s-1)m+k} - x_{(j_s-1)m+k-1}\right)^2 - \frac{1}{n}\sum_{s=1}^{M}\left(x_{(j_s-1)m+1} - x_{(j_{s-1}-1)m+m}\right)^2\right\} \\ =\ &\frac{1}{2M}\left\{\left(\frac{x_{(j_M-1)m+m}}{\sqrt{n}}\right)^2 - \frac{1}{mM}\sum_{s=1}^{M}\sum_{k=2}^{m} u_{(j_s-1)m+k}^2 - \sum_{s=1}^{M}\left(\frac{x_{(j_s-1)m+1} - x_{(j_{s-1}-1)m+m}}{\sqrt{n}}\right)^2\right\} \end{aligned} \tag{61}$$

where $\{j_s : s = 1, ..., M\}$ are independent uniform draws from $1, ..., M$, and $j_0 = 0$. From (14) and a block bootstrap version of Lemma 2.2, we find that

$$\frac{x_{(j_M-1)m+m}}{\sqrt{n}} \rightarrow_{d^*} B\left(R\right) \quad a.s.\,(P), \tag{62}$$

where R is uniform over $[0, 1]$. Next, for all s

$$\frac{1}{m}\sum_{k=2}^{m} u_{(j_s-1)m+k}^2 \rightarrow_{p^*} \sigma_u^2 \quad a.s.\,(P) \tag{63}$$

where $\sigma_u^2 = E\left(u_t^2\right)$. Finally

$$\begin{aligned}
\frac{1}{M}\sum_{s=1}^{M}\left(\frac{x_{(j_s-1)m+1}-x_{j_{s-1}m}}{\sqrt{n}}\right)^2 &= \frac{1}{M}\sum_{s=1}^{M}\left[B(R_{Mj_s}+\frac{1}{n})-B(R_{Mj_{s-1}}+\frac{m}{n})+o_{\text{a.s.}}(1)\right]^2 \\
&= \frac{1}{M}\sum_{s=1}^{M}\left[B(R_{Mj_s})-B(R_{Mj_{s-1}})\right]^2+o_{\text{a.s.}}(1) \\
&\to_{d^*} \int_0^1\int_0^1 \left[B(r)-B(s)\right]^2 drds \quad a.s.(P) \qquad (64)
\end{aligned}$$

where R and S are independent uniform variates on $[0,1]$. Combining (61) - (64), we obtain the following limit for the numerator of (60)

$$\begin{aligned}
\frac{1}{nM}\sum_{t=1}^{n}x_{t-1}^{*}\Delta x_t^{*} &\to_{d^*} -\frac{1}{2}\int_0^1\int_0^1 \left[B(r)-B(s)\right]^2 drds \quad a.s.(P) \\
&= -\left\{\int_0^1 B(r)^2dr-\left(\int_0^1 B(r)dr\right)^2\right\}
\end{aligned}$$

and the stated result follows. If centred bootstrap resampling is used, then small modifications to the above argument reveal that

$$\begin{aligned}
\frac{1}{nM}\sum_{t=1}^{n}\widetilde{x}_{t-1}^{*}\Delta \widetilde{x}_t^{*} &\to_{d^*} -\frac{1}{2}\int_0^1\int_0^1 \left[\underline{B}(r)-\underline{B}(s)\right]^2 drds \quad a.s.(P) \\
&= -\int_0^1 \underline{B}(r)^2dr,
\end{aligned}$$

and then $\frac{n}{M}(\widehat{\rho}_1^{*}-1)\to_{p^*}-1$.

6.7 Proof of Theorem 3.1

Part (a) Since $\widehat{b}_n^{*}=\widehat{b}_n+\sum_{t=1}^{n}tu_t^{*}/\sum_{t=1}^{n}t^2$,

$$\sqrt{n}\left(\widehat{b}_n^{*}-\widehat{b}_n\right)=\frac{n^{-\frac{5}{2}}\sum_{t=1}^{n}u_t^{*}t}{n^{-3}\sum_{t=1}^{n}t^2}$$

The bootstrap sample $(u_t^{*})_1^n$ is drawn randomly from the residuals $(\widehat{u}_t=x_t-\widehat{b}_nt)_1^n$. From (4) and (19) we can write

$$\frac{\hat{u}_t}{\sqrt{n}}=\frac{x_t}{\sqrt{n}}-\sqrt{n}\hat{b}_n\frac{t}{n}=B\left(\frac{t}{n}\right)-(\xi+o_{\text{a.s.}}(1))\frac{t}{n}+o_{\text{a.s.}}(1),$$

which leads to the following representation for the bootstrap process u_t^{*}

$$\frac{\hat{u}_t^{*}}{\sqrt{n}}=B(R_{nj})-\xi R_{nj}+o_{\text{a.s.}}(1), \qquad (65)$$

as in (6), where R_{nj} is uniformly distributed over $\{\frac{1}{n}, \frac{2}{n}, .., 1\}$ for each j and satisfies (7). In (65)

$$B(R_{nj}) - \xi R_{nj} = B(R_{nj}) - \left(\int_0^1 rB(r)\right)\left(\int_0^1 r^2\right)^{-1} R_{nj} = B_r(R_{nj}), \tag{66}$$

where $B_r(s) = B(s) - (\int_0^1 rB(r)dr)(\int_0^1 r^2)^{-1}s$ is detrended Brownian motion. Then

$$\begin{aligned}
\sqrt{n}\left(\widehat{b}_n^* - \widehat{b}_n\right) &= \frac{n^{-\frac{5}{2}}\sum_{t=1}^n u_t^* t}{n^{-3}\sum_{t=1}^n t^2} = \frac{\frac{1}{n}\sum_{j=1}^n [B(R_{nj}) - \xi R_{nj} + o_{\text{a.s.}}(1)]\left(\frac{j}{n}\right)}{n^{-3}\sum_{t=1}^n t^2} \\
&= \frac{\frac{1}{n}\sum_{j=1}^n [B(R_{nj}) - \xi R_{nj} + o_{\text{a.s.}}(1)]\left(\frac{j}{n}\right)}{n^{-3}\sum_{t=1}^n t^2}.
\end{aligned}$$

As in (48)

$$\frac{1}{n}\sum_{j=1}^n B(R_{nj})\left(\frac{j}{n}\right) \to_{d^*} \int_0^1 B(r)dr \int_0^1 rdr = \frac{1}{2}\int_0^1 B(r)dr \quad a.s.(P),$$

and

$$\frac{1}{n}\sum_{j=1}^n R_{nj}\left(\frac{j}{n}\right) \to_{p^*} E(R_j)\int_0^1 rdr = \left(\int_0^1 rdr\right)^2 = \frac{1}{4}.$$

It follows that

$$\sqrt{n}\left(\widehat{b}_n^* - \widehat{b}_n\right) \to_{d^*} \frac{\int_0^1 B_r(s)ds \int_0^1 sds}{\int_0^1 r^2 dr} = \frac{\frac{1}{2}\int_0^1 B(r)dr - \frac{1}{4}\xi}{\frac{1}{3}} \quad a.s.(P),$$

and so

$$\begin{aligned}
\sqrt{n}\widehat{b}_n^* &\to_{d^*} \xi^* = \xi + \frac{\int_0^1 B_r(s)ds \int_0^1 sds}{\int_0^1 r^2 dr} \quad a.s.(P) \\
&= \frac{3}{2}\int_0^1 B(r)dr + \frac{1}{4}\xi,
\end{aligned} \tag{67}$$

as required.

Part (b) The standardized residual from the bootstrap regression (23) is

$$\begin{aligned}
\frac{u_t^{**}}{\sqrt{n}} &= \frac{x_t^{**}}{\sqrt{n}} - \sqrt{n}\widehat{b}_n^*\frac{t}{n} = \frac{\widehat{b}_n t + u_t^*}{\sqrt{n}} - \sqrt{n}\widehat{b}_n^*\frac{t}{n} \\
&= \frac{u_t^*}{\sqrt{n}} - \sqrt{n}\left(\widehat{b}_n^* - \widehat{b}_n\right)\frac{t}{n}.
\end{aligned} \tag{68}$$

From (65), $\frac{u_t^*}{\sqrt{n}} = B(R_{nj}) - \xi R_{nj} + o_{\text{a.s.}}(1)$, and from part (a) and (67) $\sqrt{n}\left(\widehat{b}_n^* - \widehat{b}_n\right) \to_{d^*} \xi^* - \xi, \quad a.s.(P)$. It follows from (65)-(68) that the limit distribution of $n^{-1}s^{**2} = n^{-2}\sum_{t=1}^n u_t^{**2}$

is the same as that of

$$\frac{1}{n}\sum_{j=1}^{n}\left\{B_r\left(R_{nj}\right)+\left(\xi-\xi^*\right)\frac{j}{n}\right\}^2=\frac{1}{n}\sum_{j=1}^{n}B_r\left(R_{nj}\right)^2+2\left(\xi-\xi^*\right)\frac{1}{n}\sum_{j=1}^{n}B_r\left(R_{nj}\right)\frac{j}{n}+\left(\xi^*-\xi\right)^2\frac{1}{n}\sum_{j=1}^{n}\left(\frac{j}{n}\right)^2. \tag{69}$$

As in Lemma 2.3 (c) and (g) we have

$$\frac{1}{n}\sum_{j=1}^{n}B_r\left(R_{nj}\right)^2 \quad \rightarrow_{d^*} \quad \int_0^1 B_r(s)^2 ds \quad a.s.(P), \tag{70}$$

$$\frac{1}{n}\sum_{j=1}^{n}B_r\left(R_{nj}\right)\frac{j}{n} \quad \rightarrow_{d^*} \quad \frac{1}{2}\int_0^1 B_r(r)dr=\int_0^1 B_r(s)ds\int_0^1 sds \quad a.s.(P). \tag{71}$$

It follows that

$$\begin{aligned}\frac{1}{n}s^{**2} \quad &\rightarrow_{d^*} \quad \int_0^1 B_r(s)^2 ds+\left(\xi-\xi^*\right)\int_0^1 B_r(s)ds\int_0^1 sds+\left(\xi-\xi^*\right)^2\int_0^1 sds \quad a.s.(P)\\ &= \int_0^1\left\{B_r(s)+\left(\xi-\xi^*\right)s\right\}^2 dr=\sigma_\xi^2, \quad \text{say,}\end{aligned} \tag{72}$$

giving the stated result.

Part (c) The bootstrap regression t-ratio is

$$t(\hat{b}_n^*)=\frac{\hat{b}_n^*}{s_{\hat{b}^*}}=\frac{\hat{b}_n^*}{[s^{**2}/\sum_1^n t^2]^{1/2}}=\sqrt{n}\frac{\sqrt{n}\hat{b}_n^*}{[\frac{1}{n}s^{**2}/\frac{1}{n^3}\sum_1^n t^2]^{1/2}}$$

and then

$$\frac{1}{\sqrt{n}}t(\hat{b}_n^*)\rightarrow_{d^*}\frac{\xi^*\left(\int_0^1 r^2\right)^{\frac{1}{2}}}{\sigma_\xi}=\frac{\xi^*}{\sqrt{3}\sigma_\xi} \quad a.s.(P).$$

Part (d) Write $\widehat{\rho}^{**}=n^{-2}\sum_{t=1}^n u_t^{**}u_{t-1}^{**}/n^{-2}\sum_{t=1}^n u_{t-1}^{**2}$. As in (68)-(69) above, $n^{-\frac{1}{2}}u_t^{**}$ behaves asymptotically like $B_r\left(R_{nj}\right)+\left(\xi-\xi^*\right)\frac{j}{n}$. It follows that $n^{-2}\sum_{t=1}^n u_t^{**}u_{t-1}^{**}$ is asymptotically equivalent to

$$\begin{aligned}&\frac{1}{n}\sum_{j=1}^{n}\left\{B_r\left(R_{nj}\right)+\left(\xi-\xi^*\right)\frac{j}{n}\right\}\left\{B_r\left(R_{nj-1}\right)+\left(\xi-\xi^*\right)\frac{j-1}{n}\right\}\\ =\ &\frac{1}{n}\sum_{j=1}^{n}B_r\left(R_{nj}\right)B_r\left(R_{nj-1}\right)+\left(\xi-\xi^*\right)\frac{1}{n}\sum_{j=1}^{n}\frac{j}{n}B_r\left(R_{nj-1}\right)+\left(\xi-\xi^*\right)\frac{1}{n}\sum_{j=1}^{n}\frac{j-1}{n}B_r\left(R_{nj}\right)\\ &+\left(\xi-\xi^*\right)^2\frac{1}{n}\sum_{j=1}^{n}\frac{j}{n}\frac{j-1}{n}.\end{aligned}$$

Proceeding as in (50)-(52) we find

$$\frac{1}{n}\sum_{j=1}^{n} B_r\left(R_{nj}\right) B_r\left(R_{nj-1}\right) \rightarrow_{d^*} \left(\int_0^1 B_r(s)ds\right)^2 \quad a.s.(P), \tag{73}$$

$$\frac{1}{n}\sum_{j=1}^{n} \frac{j}{n} B_r\left(R_{nj-1}\right) \rightarrow_{d^*} \left(\int_0^1 sds\right)\left(\int_0^1 B_r(s)ds\right), \tag{74}$$

and then

$$n^{-2}\sum_{t=1}^{n} u_t^{**}u_{t-1}^{**} \rightarrow_{d^*} \left(\int_0^1 B_r(s)ds + (\xi - \xi^*)\int_0^1 sds\right)^2 \quad a.s.(P).$$

As in (72)

$$n^{-2}\sum_{t=1}^{n} u_{t-1}^{**2} \rightarrow_{d^*} \int_0^1 \left\{B_r(s) + (\xi - \xi^*)\, s\right\}^2 ds = \sigma_\xi^2 \quad a.s.(P). \tag{75}$$

It follows that

$$\widehat{\rho}^{**} \rightarrow_{d^*} \xi_\rho^{breg} = \frac{\left(\int_0^1 \left[B_r(s) + (\xi - \xi^*)\, s\right] ds\right)^2}{\int_0^1 \left\{B_r(s) + (\xi - \xi^*)\, s\right\}^2 ds},$$

as given in the theorem. The result for $\widehat{\rho}_h^{**}$ follows in precisely the same manner (c.f. the proof of part (a) of Theorem 2.4).

Part (e) Write $DW^{**} = n^{-2}\sum_2^n (\Delta\hat{u}_t^{**})^2 / n^{-2}\sum_1^n \hat{u}_t^{**2}$ where $\Delta\hat{u}_t^{**} = \Delta x_t^{**} - \hat{b}_n^* = \hat{b}_n - \hat{b}_n^* + \Delta\hat{u}_t^*$ from (22) and (23). Hence

$$\begin{aligned}
\frac{1}{n^2}\sum_{t=2}^{n} (\Delta u_t^{**})^2 &= \frac{1}{n}\sum \left[\frac{u_t^*}{\sqrt{n}} - \frac{u_{t-1}^*}{\sqrt{n}} + o_{p^*}(1)\right]^2 \\
&= \frac{1}{n}\sum_{j=1}^{n} [(B(R_{nj}) - \xi R_{nj}) - (B(R_{nj-1}) - \xi(R_{nj-1}))]^2 + o_{p^*}(1),
\end{aligned}$$

whose asymptotic behavior is the same as

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n} [\{B(R_j) - \xi(R_j)\} - \{B(S_j) - \xi S_j\}]^2 &= \frac{1}{n}\sum_{j=1}^{n} [B_r(R_j) - B_r(S_j)]^2 \\
&= \frac{1}{n}\sum_{j=1}^{n} B_r(R_j)^2 + \frac{1}{n}\sum_{j=1}^{n} B_r(S_j)^2 - \frac{2}{n}\sum_{j=1}^{n} B_r(R_j)B_r(S_j),
\end{aligned}$$

where R_j, S_j are independent $iidU\,[0,1]$. As in (70) and (73) above

$$\frac{1}{n}\sum_{j=1}^{n} B_r(R_j)^2, \quad \frac{1}{n}\sum_{j=1}^{n} B_r(S_j)^2 \rightarrow_{d^*} \int_0^1 B_r(s)^2 ds \quad a.s.(P),$$

$$\frac{1}{n}\sum_{j=1}^{n} B_r(R_j)B_r(S_j) \rightarrow_{d^*} \left(\int_0^1 B_r(s)ds\right)^2 \quad a.s.(P),$$

so that

$$\frac{1}{n^2}\sum_{t=2}^{n}(\Delta u_t^{**})^2 \to_{d^*} 2\int_0^1 B_r(s)^2 ds - 2\left(\int_0^1 B_r(s)ds\right)^2 \quad a.s.(P),$$

which can be written in the form

$$2\int_0^1 B_r(s)^2 ds\left[1-\frac{\left(\int_0^1 B_r(s)ds\right)^2}{\int_0^1 B_r(s)^2 ds}\right] = 2\int_0^1 B_r(s)^2 ds\left[1-\xi_\rho^b\right],$$

where

$$\xi_\rho^b = \frac{\left(\int_0^1 B_r(s)ds\right)^2}{\int_0^1 B_r(s)^2 ds}.$$

From (75), the limit of the denominator of DW^{**} is

$$n^{-2}\sum_{t=1}^{n} u_t^{**2} \to_{d^*} \int_0^1 \{B_r(s) + (\xi - \xi^*)\, s\}^2 \, ds = \sigma_\xi^2 \quad a.s.(P),$$

and so

$$DW^{**} \to_{d^*} \frac{2\int_0^1 B_r(s)^2 ds}{\sigma_\xi^2}\left[1-\xi_\rho^b\right] \quad a.s.(P).$$

Part (f) Write $R^{**2} = 1 - n^{-2}\sum_1^n \hat{u}_t^{**2}/n^{-2}\sum_1^n x_t^{**2}$. We have $x_t^{**} = \widehat{b}_n t + u_t^*$ and so

$$\frac{1}{n}\sum_{t=1}^{n}\left(\frac{x_t^{**}}{\sqrt{n}}\right)^2 = \frac{1}{n}\sum_{t=1}^{n}\left(\sqrt{n}\widehat{b}_n\frac{t}{n} + \frac{u_t^*}{\sqrt{n}}\right)^2,$$

whose asymptotic behavior is the same as

$$\begin{aligned}
& \frac{1}{n}\sum_{j=1}^{n}\left(\xi\frac{j}{n} + B(R_j) - \xi R_j\right)^2 \\
= \;& \xi^2\frac{1}{n}\sum_{j=1}^{n}\left(\frac{j}{n}\right)^2 + \frac{1}{n}\sum_{j=1}^{n} B(R_j)^2 + \xi^2\frac{1}{n}\sum_{j=1}^{n} R_j^2 \\
& +2\xi\frac{1}{n}\sum_{j=1}^{n} B(R_j)\frac{j}{n} - 2\xi^2\frac{1}{n}\sum_{j=1}^{n} R_j\frac{j}{n} - 2\xi\frac{1}{n}\sum_{j=1}^{n} B(R_j)R_j \\
\to_{d^*} \;& \xi^2\int_0^1 r^2 dr + \int_0^1 B(r)^2 dr + \xi^2\int_0^1 r^2 dr \quad a.s.(P) \\
& +2\xi\int_0^1 B(r)\,dr\int_0^1 r dr - 2\xi^2\int_0^1 r^2 dr - 2\xi\int_0^1 B(r)\,dr\int_0^1 r dr \quad a.s.(P) \\
= \;& \int_0^1 B(r)^2 dr.
\end{aligned}$$

It follows that

$$R^2 \to_{d^*} 1 - \frac{\sigma_\xi^2}{\int_0^1 B(r)^2 dr} = 1 - \frac{\int_0^1 \{B(s) - \xi^* s\}^2 ds}{\int_0^1 B(r)^2 dr} \quad a.s.(P).$$

6.8 Proof of Theorem 3.3 The proofs here and in later arguments are similar to what has come before and so derivations henceforth are simply sketched.

Part (a) Since $\widehat{b}_n^* = \widehat{b}_n + \sum_{t=1}^n t\widetilde{u}_t^* / \sum_{t=1}^n t^2$,

$$\sqrt{n}\left(\widehat{b}_n^* - \widehat{b}_n\right) = \frac{n^{-\frac{5}{2}} \sum_{t=1}^n \widetilde{u}_t^* t}{n^{-3} \sum_{t=1}^n t^2}.$$

The bootstrap sample $(\widetilde{u}_t^*)_1^n$ is drawn randomly from the centred residuals $(\widehat{u}_t - n^{-1} \sum_{t=1}^n \widehat{u}_t = x_t - \overline{x} - \widehat{b}_n(t - \overline{t}))_1^n$. From (4) and (19) we can write

$$\frac{\hat{u}_t}{\sqrt{n}} - \frac{1}{n^{\frac{3}{2}}} \sum_{t=1}^n \widehat{u}_t = B\left(\frac{t}{n}\right) - \int_0^1 B(r)\,dr - (\xi + o_{\text{a.s.}}(1))\left(\frac{t}{n} - \int_0^1 rdr\right) + o_{\text{a.s.}}(1),$$

which leads to the following representation for the bootstrap process $\widetilde{u}_t^*$

$$\frac{\widetilde{u}_t^*}{\sqrt{n}} = \underline{B}(R_{nj}) - \xi \underline{R}_{nj} + o_{\text{a.s.}}(1) = \underline{B}_r(R_{nj}) + o_{\text{a.s.}}(1), \tag{76}$$

where $\underline{r} = r - \int_0^1 rdr$. Then, just as in part (a) of Theorem 3.1 we find that

$$\sqrt{n}\left(\widehat{b}_n^* - \widehat{b}_n\right) \to_{d^*} \frac{\int_0^1 \underline{B}_r(s)ds \int_0^1 sds}{\int_0^1 r^2 dr} = 0 \quad a.s.(P), \tag{77}$$

and so

$$\sqrt{n}\widehat{b}_n^* \to_{d^*} \xi \quad a.s.(P).$$

Parts (b) & (c) The standardized residual from the bootstrap regression (23) is

$$\begin{aligned} \frac{u_t^{**}}{\sqrt{n}} &= \frac{x_t^{**}}{\sqrt{n}} - \sqrt{n}\widehat{b}_n^* \frac{t}{n} = \frac{\widehat{b}_n t + \widetilde{u}_t^*}{\sqrt{n}} - \sqrt{n}\widehat{b}_n^* \frac{t}{n} \\ &= \frac{\widetilde{u}_t^*}{\sqrt{n}} - \sqrt{n}\left(\widehat{b}_n^* - \widehat{b}_n\right)\frac{t}{n}. \end{aligned}$$

It follows from (76) and (77) that the limit distribution of $n^{-1}s^{**2} = n^{-2}\sum_{t=1}^n u_t^{**2}$ is the same as that of

$$\frac{1}{n}\sum_{j=1}^n \underline{B}_r(R_{nj})^2 \to_{d^*} \int_0^1 \underline{B}_r(s)^2 ds \quad a.s.(P),$$

using the same arguments as those in Lemma 2.3 (c). The bootstrap regression t-ratio is

$$t(\hat{b}_n^*) = \frac{\hat{b}_n^*}{s_{\hat{b}^*}} = \frac{\hat{b}_n^*}{[s^{**2}/\sum_1^n t^2]^{1/2}} = \sqrt{n}\frac{\sqrt{n}\hat{b}_n^*}{[\frac{1}{n}s^{**2}/\frac{1}{n^3}\sum_1^n t^2]^{1/2}}$$

and then

$$\frac{1}{\sqrt{n}}t(\hat{b}_n^*) \to_{d^*} \frac{\xi\left(\int_0^1 r^2\right)^{\frac{1}{2}}}{\left(\int_0^1 \underline{B}_r(s)^2 ds\right)^{\frac{1}{2}}} \quad a.s.(P).$$

Part (d) Write $\widehat{\rho}^{**} = n^{-2}\sum_{t=1}^{n} u_t^{**}u_{t-1}^{**}/n^{-2}\sum_{t=1}^{n} u_{t-1}^{**2}$. As in (68)-(69) above, $n^{-\frac{1}{2}}u_t^{**}$ behaves asymptotically like $\underline{B}_r\left(R_{nj}\right) = B_r(R_{nj}) - \int_0^1 B_r(s)ds$. It follows that $n^{-2}\sum_{t=1}^{n} u_t^{**}u_{t-1}^{**}$ is asymptotically equivalent to

$$\begin{aligned} & \frac{1}{n}\sum_{j=1}^{n}\underline{B}_r\left(R_{nj}\right)\underline{B}_r\left(R_{nj-1}\right) \\ = & \frac{1}{n}\sum_{j=1}^{n}B_r\left(R_{nj}\right)B_r\left(R_{nj-1}\right) - \int_0^1 B_r(s)ds\frac{1}{n}\sum_{j=1}^{n}B_r\left(R_{nj-1}\right) \\ & - \int_0^1 B_r(s)ds\frac{1}{n}\sum_{j=1}^{n}B_r\left(R_{nj}\right) + \left(\int_0^1 B_r(s)ds\right)^2. \end{aligned}$$

As in (73) we find

$$\begin{aligned} \frac{1}{n}\sum_{j=1}^{n}B_r\left(R_{nj}\right)B_r\left(R_{nj-1}\right) & \rightarrow_{d^*} \left(\int_0^1 B_r(s)ds\right)^2 \quad a.s.(P), \\ \frac{1}{n}\sum_{j=1}^{n}B_r\left(R_{nj-1}\right) & \rightarrow_{d^*} \int_0^1 B_r(s)ds, \end{aligned}$$

and then

$$n^{-2}\sum_{t=1}^{n}u_t^{**}u_{t-1}^{**}\rightarrow_{p^*} 0 \quad a.s.(P),$$

from which the result for $\widehat{\rho}^{**}$ follows. The result for $\widehat{\rho}_h^{**}$ follows in the same way.

Part (e) Write $DW^{**} = n^{-2}\sum_2^n(\Delta\hat{u}_t^{**})^2/n^{-2}\sum_1^n \hat{u}_t^{**2}$ where $\Delta\hat{u}_t^{**} = \Delta x_t^{**} - \hat{b}_n^* = \hat{b}_n - \hat{b}_n^* + \Delta\widetilde{u}_t^*$ from (22) and (23). Hence

$$\begin{aligned} \frac{1}{n^2}\sum_{t=2}^{n}(\Delta u_t^{**})^2 & = \frac{1}{n}\sum\left[\frac{\widetilde{u}_t^*}{\sqrt{n}} - \frac{\widetilde{u}_{t-1}^*}{\sqrt{n}} + o_{p^*}(1)\right]^2 \\ & = \frac{1}{n}\sum_{j=1}^{n}[\underline{B}_r(R_{nj}) - \underline{B}_r(R_{nj-1})]^2 + o_{p^*}(1), \end{aligned}$$

whose asymptotic behavior is the same as

$$\frac{1}{n}\sum_{j=1}^{n}[\underline{B}_r(R_j) - \underline{B}_r(S_j)]^2 = \frac{1}{n}\sum_{j=1}^{n}\underline{B}_r(R_j)^2 + \frac{1}{n}\sum_{j=1}^{n}\underline{B}_r(S_j)^2 - \frac{2}{n}\sum_{j=1}^{n}\underline{B}_r(R_j)\underline{B}_r(S_j),$$

where R_j, S_j are independent $iidU\left[0,1\right]$. As in (70) and (73) above

$$\begin{aligned} \frac{1}{n}\sum_{j=1}^{n}\underline{B}_r(R_j)^2, \quad \frac{1}{n}\sum_{j=1}^{n}\underline{B}_r(S_j)^2 & \rightarrow_{d^*} \int_0^1 \underline{B}_r(s)^2 ds \quad a.s.(P), \\ \frac{1}{n}\sum_{j=1}^{n}B_r(R_j)B_r(S_j) & \rightarrow_{d^*} \left(\int_0^1 \underline{B}_r(s)ds\right)^2 = 0 \quad a.s.(P), \end{aligned}$$

so that

$$\frac{1}{n^2}\sum_{t=2}^{n}(\Delta u_t^{**})^2 \rightarrow_{d^*} 2\int_0^1 \underline{B}_r(s)^2 ds \quad a.s.(P),$$

From this result and part (b) we deduce that $DW^{**} \rightarrow_{p^*} 2 \quad a.s.(P)$, as stated.

Part (f) Write $R^{**2} = 1 - n^{-2}\sum_1^n \hat{u}_t^{**2}/n^{-2}\sum_1^n x_t^{**2}$. We have $x_t^{**} = \widehat{b}_n t + \widetilde{u}_t^*$ and so

$$\frac{1}{n}\sum_{t=1}^{n}\left(\frac{x_t^{**}}{\sqrt{n}}\right)^2 = \frac{1}{n}\sum_{t=1}^{n}\left(\sqrt{n}\widehat{b}_n\frac{t}{n} + \frac{\widetilde{u}_t^*}{\sqrt{n}}\right)^2,$$

whose asymptotic behavior is the same as

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\left(\xi\frac{j}{n} + \underline{B}_r(R_j)\right)^2 &= \xi^2\frac{1}{n}\sum_{j=1}^{n}\left(\frac{j}{n}\right)^2 + \frac{1}{n}\sum_{j=1}^{n}\underline{B}_r(R_j)^2 + 2\xi\frac{1}{n}\sum_{j=1}^{n}\underline{B}_r(R_j)\frac{j}{n} \\ &\rightarrow_{d^*} \xi^2\int_0^1 r^2 dr + \int_0^1 \underline{B}_r(s)^2 ds + 2\xi\left(\int_0^1 \underline{B}_r(r)\, dr\right)\left(\int_0^1 r dr\right) \quad a.s.(P) \\ &= \int_0^1 \underline{B}_r(s)^2 ds + \xi^2\int_0^1 r^2 dr\end{aligned}$$

It follows from this result and part (b) that

$$R^2 \rightarrow_{d^*} 1 - \frac{\int_0^1 \underline{B}_r(s)^2 ds}{\int_0^1 \underline{B}_r(s)^2 ds + \xi^2\int_0^1 r^2 dr} = \frac{\xi^2\int_0^1 r^2 dr}{\int_0^1 \underline{B}_r(s)^2 ds + \xi^2\int_0^1 r^2 dr} \quad a.s.(P).$$

6.8 Proof of Theorem 3.5

Part (a) Start with the formula

$$\sqrt{n}\left(\widehat{b}_n^{b*} - \widehat{b}_n\right) = \frac{n^{-\frac{5}{2}}\sum_{t=1}^n \widetilde{u}_t^{b*} t}{n^{-3}\sum_{t=1}^n t^2},$$

and, as in (27)-27, for $t = (s-1)m + k$ we can write

$$\frac{\widetilde{u}_t^{b*}}{\sqrt{n}} = \frac{\widehat{u}_{(j_s-1)m+k} - n^{-1}\sum_{s=1}^n \widehat{u}_s}{\sqrt{n}} = \underline{B}_r\left(R_{Mj_s} + \frac{k}{n}\right) + o_{a.s}(1) \tag{78}$$

Then, as in part (a) of Theorems 3.1 and 3.3 we find that

$$\sqrt{n}\left(\widehat{b}_n^{b*} - \widehat{b}_n\right) \rightarrow_{d^*} \frac{\int_0^1 \underline{B}_r(s)ds\int_0^1 sds}{\int_0^1 r^2 dr} = 0 \quad a.s.(P), \tag{79}$$

and so $\sqrt{n}\widehat{b}_n^* \rightarrow_{d^*} \xi \quad a.s.(P)$.

Parts (b) & (c) The standardized residual from the regression (26) is

$$\frac{u_t^{b**}}{\sqrt{n}} = \frac{x_t^{b**}}{\sqrt{n}} - \sqrt{n}\widehat{b}_n^{b*}\frac{t}{n} = \frac{\widehat{b}_n t + \widetilde{u}_t^{b*}}{\sqrt{n}} - \sqrt{n}\widehat{b}_n^{b*}\frac{t}{n} = \frac{\widetilde{u}_t^{b*}}{\sqrt{n}} - \sqrt{n}\left(\widehat{b}_n^{b*} - \widehat{b}_n\right)\frac{t}{n}. \tag{80}$$

It follows from (79) that $n^{-\frac{1}{2}}u_t^{b**}$ is asymptotically equivalent to $n^{-\frac{1}{2}}\widetilde{u}_t^{b*}$. Then, from (78) and the continuity of $\underline{B}_r(s)$, the limit distribution of $n^{-1}s^{b**2} = n^{-2}\sum_{t=1}^n u_t^{b**2}$ is the same as that of

$$\frac{1}{M}\sum_{s=1}^M \underline{B}_r(R_{Mj_s})^2 \to_{d^*} \int_0^1 \underline{B}_r(s)^2 ds \quad a.s.(P), \tag{81}$$

- compare the argument leading to (84) below. Part (c) follows directly.

Part (d) Take the serial correlation $\widehat{\rho}_1^{b**} = n^{-2}\sum_{t=1}^n u_t^{b**}u_{t-1}^{b**}/n^{-2}\sum_{t=1}^n u_{t-1}^{b**2}$. The denominator of $\widehat{\rho}^{b**}$ is covered in part (b). In view of (80) and (79), the numerator is asymptotically equivalent to

$$\begin{aligned} & \frac{1}{n}\sum\nolimits_1^n \left(\frac{\widetilde{u}_t^{b*}}{\sqrt{n}}\right)\left(\frac{\widetilde{u}_{t-1}^{b*}}{\sqrt{n}}\right) \\ = \; & \frac{1}{mM}\sum_{s=1}^M\sum_{k=2}^m \left[\underline{B}_r(R_{Mj_s} + \frac{k}{n}) + o_{\text{a.s.}}(1)\right]\left[\underline{B}_r(R_{Mj_s} + \frac{k-1}{n}) + o_{\text{a.s.}}(1)\right] \\ & + \frac{1}{mM}\sum_{s=1}^M \left[\underline{B}_r\left(R_{Mj_s} + \frac{1}{n}\right) + o_{\text{a.s.}}(1)\right]\left[\underline{B}_r\left(R_{Mj_s-1} + \frac{m}{n}\right) + o_{\text{a.s.}}(1)\right] \end{aligned} \tag{82}$$

In (82), R_{Mj_s} and R_{Mj_s-1} are independent draws from $0, \frac{1}{M}, ..., \frac{M-1}{M}$ and, in view of (13),

$$R_{Mj_s} \to_{d^*} R_j, \quad R_{Mj_s-1} \to_{d^*} R_{j-1} \quad a.s.\,(P)\,,$$

where R_j and R_{j-1} are independent draws from $U[0,1]$. Thus, (82) is asymptotically equivalent to

$$\begin{aligned} & \frac{1}{M}\sum_{s=1}^M \frac{1}{m}\sum_{k=2}^m \left[\underline{B}_r(R_{Mj_s} + \frac{k}{n}) + o_{\text{a.s.}}(1)\right]\left[\underline{B}_r(R_{Mj_s} + \frac{k-1}{n}) + o_{\text{a.s.}}(1)\right] + o_{\text{a.s.}}\left(\frac{1}{m}\right) \\ = \; & \sum_{s=1}^M \int_{R_{Mj_s}+\frac{2}{n}}^{R_{Mj_s}+\frac{m}{n}} \underline{B}_r(r)^2\, dr + o_{\text{a.s.}}(1) = \sum_{s=1}^M \int_{R_{Mj_s}+\frac{1}{n}}^{R_{Mj_s}+\frac{m}{n}} \underline{B}_r(r)^2\, dr + o_{\text{a.s.}}(1) \end{aligned} \tag{83}$$

By virtue of the continuity of $\underline{B}_r(s)$ and the fact that $\frac{m}{n} = \frac{1}{M} \to 0$, (83) is asymptotically equivalent to $M^{-1}\sum_{s=1}^M \underline{B}_r(R_{Mj_s})^2$. Then, just as in the proof of Theorem 2.3, we find that

$$\frac{1}{M}\sum_{s=1}^M \underline{B}_r(R_{Mj_s})^2 \to_{d^*} \int_0^1 \underline{B}_r(r)^2 dr. \tag{84}$$

Combining these results for the numerator and denominator, we deduce that

$$\widehat{\rho}_1^{b**} \to_{d^*} \frac{\int_0^1 \underline{B}_r(r)^2 dr}{\int_0^1 \underline{B}_r(r)^2 dr} = 1 \quad a.s.\,(P)\,,$$

as stated. The proof for $\widehat{\rho}_h^{b**}$ follows in a similar fashion.

Part (e) Write $DW^{b**} = n^{-2}\sum_2^n(\Delta u_t^{b**})^2/n^{-2}\sum_1^n u_t^{b**2}$ where $\Delta u_t^{b**} = \Delta x_t^{b**} - \hat{b}_n^{b*} = \hat{b}_n - \hat{b}_n^{b*} + \Delta\widetilde{u}_t^{b*}$. Using part (a) we have

$$\begin{aligned}
&\frac{1}{n^2}\sum_{t=2}^{n}(\Delta u_t^{b**})^2 \\
= \;&\frac{1}{n}\sum_{t=2}^{n}\left[\frac{\widetilde{u}_t^{b*}}{\sqrt{n}} - \frac{\widetilde{u}_{t-1}^{b*}}{\sqrt{n}} + o_{p^*}(1)\right]^2 \\
= \;&\frac{1}{n}\sum_{s=1}^{M}\sum_{k=2}^{m}[\underline{B}_r(R_{Mj_s} + \frac{k}{n}) - \underline{B}_r(R_{Mj_s} + \frac{k-1}{n}) + o_{\text{a.s.}}(1)]^2 \\
&+\frac{1}{mM}\sum_{s=1}^{M}\left[\underline{B}_r\left(R_{Mj_s} + \frac{1}{n}\right) - \underline{B}_r\left(R_{Mj_s-1} + \frac{m}{n}\right) + o_{\text{a.s.}}(1)\right]^2 + o_{p^*}(1). \qquad (85)
\end{aligned}$$

Now

$$\frac{1}{n}\sum_{s=1}^{M}\sum_{k=2}^{m}[\underline{B}_r(R_{Mj_s} + \frac{k}{n}) - \underline{B}_r(R_{Mj_s} + \frac{k-1}{n})]^2 = o_{p^*}(\frac{1}{n^{1-\delta}}), \quad \delta > 0, \qquad (86)$$

by virtue of the Holder continuity of $\underline{B}_r$ and, as in (64) above,

$$\frac{1}{M}\sum_{s=1}^{M}\left[\underline{B}_r\left(R_{Mj_s} + \frac{1}{n}\right) - \underline{B}_r\left(R_{Mj_s-1} + \frac{m}{n}\right)\right]^2 \to_{d^*} \int_0^1\int_0^1 [\underline{B}_r(r) - \underline{B}_r(s)]^2\, drds \quad a.s.(P) \qquad (87)$$

Thus,

$$\frac{1}{n^2}\sum_{t=2}^{n}(\Delta u_t^{**})^2 \to_{p^*} 0 \quad a.s.(P),$$

It follows directly that $DW^{b**} \to_{p^*} 0 \quad a.s.(P)$, as stated.

Part (f) Write $R^{b**2} = 1 - n^{-2}\sum_1^n u_t^{b**2}/n^{-2}\sum_1^n x_t^{b**2}$. We have $x_t^{b**} = \widehat{b}_n t + \widetilde{u}_t^{b*}$ and from (29), $n^{-\frac{1}{2}}x_{[nr]}^{b**} = \xi r + \underline{B}_r\left(R_{M,r}\right) + o_{a.s}\left(1\right)$. Then,

$$\frac{1}{n}\sum_{t=1}^{n}\left(\frac{x_t^{b**}}{\sqrt{n}}\right)^2 = \frac{1}{n}\sum_{t=1}^{n}\left(\sqrt{n}\widehat{b}_n\frac{t}{n} + \frac{\widetilde{u}_t^{b*}}{\sqrt{n}}\right)^2,$$

whose asymptotic behavior is the same as

$$\begin{aligned}
&\frac{1}{n}\sum_{s=1}^{M}\sum_{k=1}^{m}\left(\xi\frac{(s-1)\,m+k}{n} + \underline{B}_r(R_{Mj_s} + \frac{k}{n})\right)^2 \\
\to_{d^*} \;&\xi^2\int_0^1 r^2dr + \int_0^1 \underline{B}_r\left(s\right)^2 ds + 2\xi\left(\int_0^1 \underline{B}_r\left(r\right)dr\right)\left(\int_0^1 rdr\right) \quad a.s.(P) \\
= \;&\int_0^1 \underline{B}_r\left(s\right)^2 ds + \xi^2\int_0^1 r^2dr.
\end{aligned}$$

It follows from this result and part (b) that

$$R^2 \to_{d^*} 1 - \frac{\int_0^1 \underline{B}_r(s)^2 ds}{\int_0^1 \underline{B}_r\left(s\right)^2 ds + \xi^2\int_0^1 r^2dr} = \frac{\xi^2\int_0^1 r^2dr}{\int_0^1 \underline{B}_r\left(s\right)^2 ds + \xi^2\int_0^1 r^2dr} \quad a.s.(P).$$

6.9 Proof of Theorem 3.7 Write

$$\frac{n}{M}(\widehat{\rho}_1^{b**}-1)=\frac{1}{nM}\sum_{t=1}^{n}\Delta u_t^{b**}u_{t-1}^{b**}/\frac{1}{n^2}\sum_{t=1}^{n}u_{t-1}^{b**2}.$$

The limit of the denominator is given in (81). Next, in view of (79) and (80), for $t=(s-1)m+k$, we can write

$$n^{-\frac{1}{2}}u_t^{b**}=n^{-\frac{1}{2}}\widetilde{u}_t^{b*}+o_{p^*}(1)=n^{-\frac{1}{2}}\hat{u}_{(j_s-1)m+k}+o_{p^*}(1),$$

and, as in (85),

$$\frac{\Delta\widetilde{u}_t^{b*}}{\sqrt{n}}=\underline{B}_r(R_{Mj_s}+\frac{k}{n})-\underline{B}_r(R_{Mj_s}+\frac{k-1}{n})+o_{\text{a.s.}}(1).$$

where $\{j_s:s=1,...,M\}$ are independent uniform draws from $1,...,M$, and $j_0=0$. The identity

$$\Delta\sum_{t=1}^{n}u_t^{b**2}=\sum_{t=1}^{n}\Delta u_t^{b**}u_t^{b**}+\sum_{t=1}^{n}u_{t-1}^{b**}\Delta u_t^{b**}=2\sum_{t=1}^{n}u_{t-1}^{b**}\Delta u_t^{b**}+\sum_{t=1}^{n}\left(\Delta u_t^{b**}\right)^2$$

then leads to the representation

$$\frac{1}{nM}\sum_{t=1}^{n}u_{t-1}^{b**}\Delta u_t^{b**}=\frac{1}{2M}\left\{\left(\frac{u_n^{b**}}{\sqrt{n}}\right)^2-\sum_{t=1}^{n}\left(\frac{\Delta u_t^{b**}}{\sqrt{n}}\right)^2\right\}.$$

From (85)-(86) we have

$$\begin{aligned}\frac{1}{nM}\sum_{t=1}^{n}(\Delta u_t^{b**})^2 &= \frac{1}{M}\sum_{t=1}^{n}\left[\frac{\widetilde{u}_t^{b*}}{\sqrt{n}}-\frac{\widetilde{u}_{t-1}^{b*}}{\sqrt{n}}+o_{p^*}(1)\right]^2\\ &= \frac{1}{M}\sum_{s=1}^{M}\sum_{k=2}^{m}[\underline{B}_r(R_{Mj_s}+\frac{k}{n})-\underline{B}_r(R_{Mj_s}+\frac{k-1}{n})+o_{\text{a.s.}}(1)]^2\\ &\quad+\frac{1}{M}\sum_{s=1}^{M}\left[\underline{B}_r\left(R_{Mj_s}+\frac{1}{n}\right)-\underline{B}_r\left(R_{Mj_{s-1}}+\frac{m}{n}\right)+o_{\text{a.s.}}(1)\right]^2+o_{p^*}(1),\end{aligned}$$

and

$$\frac{1}{M}\sum_{s=1}^{M}\sum_{k=2}^{m}[\underline{B}_r(R_{Mj_s}+\frac{k}{n})-\underline{B}_r(R_{Mj_s}+\frac{k-1}{n})]^2=o_{p^*}(\frac{m}{n^{1-\delta}}),\quad \delta>0,$$

As in (87),

$$\frac{1}{M}\sum_{s=1}^{M}\left[\underline{B}_r\left(R_{Mj_s}+\frac{1}{n}\right)-\underline{B}_r\left(R_{Mj_{s-1}}+\frac{m}{n}\right)\right]^2\rightarrow_{d^*}\int_0^1\int_0^1[\underline{B}_r(r)-\underline{B}_r(s)]^2\,drds\quad a.s.\,(P),$$

from which we deduce that

$$\frac{1}{nM}\sum_{t=1}^{n} u_{t-1}^{b**}\Delta u_t^{b**} \quad \to_{d^*} \quad -\frac{1}{2}\int_0^1\int_0^1 \left[\underline{B}_r(r)-\underline{B}_r(s)\right]^2 drds \quad a.s.(P)$$
$$= \quad -\int_0^1 \underline{B}_r(r)^2 dr.$$

It follows that

$$\frac{n}{M}(\widehat{\rho}_1^{b**}-1) \to_{d^*} \quad -\frac{\int_0^1 \underline{B}_r(r)^2dr}{\int_0^1 \underline{B}_r(r)^2dr} = -1 \quad a.s.(P)$$

giving the stated result.

6.10 Proof of Theorem 4.1

Part (a) The estimation error is

$$\widehat{b}_n^* - \widehat{b}_n = \left(\frac{1}{n^2}\sum_{t=1}^n x_t x_t'\right)^{-1}\left(\frac{1}{n^2}\sum_{t=1}^n x_t \widetilde{u}_t^*\right).$$

The bootstrap sample $(\widetilde{u}_t^*)_1^n$ is drawn randomly from the centred residuals $(\widehat{u}_t - n^{-1}\sum_{t=1}^n \widehat{u}_t = y_t - \overline{y} - \widehat{b}_n(x_t-\overline{x}))_1^n$ and from (4) and (31) we can write (after appropriate redefinition of the probability space)

$$\frac{\hat{u}_t}{\sqrt{n}} - \frac{1}{n^{\frac{3}{2}}}\sum_{t=1}^n \widehat{u}_t = B_y\left(\frac{t}{n}\right) - \int_0^1 B_y(r)\,dr - \left(\xi_{yx}+o_{\text{a.s.}}(1)\right)'\left(B_x\left(\frac{t}{n}\right)-\int_0^1 B_x(r)\,dr\right) + o_{\text{a.s.}}(1)$$
$$= \eta' \underline{B}_z\left(\frac{t}{n}\right) + o_{\text{a.s.}}(1), \tag{88}$$

which leads to the following representation for the bootstrap process $\widetilde{u}_t^*$

$$\frac{\widetilde{u}_t^*}{\sqrt{n}} = \eta'\underline{B}_z\left(R_{nj_t}\right) + o_{\text{a.s.}}(1), \tag{89}$$

where $\underline{B}_z = B_z - \int_0^1 B_z$ and R_{nj_t} is uniform over $\frac{1}{n},\frac{2}{n},...,1$. Then, as in earlier derivations we find

$$\frac{1}{n}\sum_{t=1}^n \sin\left[\left(k-\frac{1}{2}\right)\pi\frac{t}{n}\right]\sin\left[\left(m-\frac{1}{2}\right)\pi R_{nj_t}\right]$$
$$\to_{d^*} \int_0^1 \sin\left[\left(k-\frac{1}{2}\right)\pi s\right]ds\int_0^1 \sin\left[\left(m-\frac{1}{2}\right)\pi r\right]dr \quad a.s.(P),$$

which in conjunction with the LK representation of B_z leads to the following.

$$\widehat{b}_n^* - \widehat{b}_n \to_{d^*} \left(\int_0^1 B_x B_x'\right)^{-1}\left(\int_0^1 B_x\right)\left(\int_0^1 \underline{B}_z'\right)\eta = 0 \quad a.s.(P). \tag{90}$$

Thus, $\widehat{b}_n^* \to_{d^*} \xi_{yx} \quad a.s.(P)$, as stated.

Part (b) &(c) The standardized residual from the bootstrap regression (33) is $y_t^{**} = \widehat{b}_n^{*\prime} x_t + u_t^{**}$

$$\frac{u_t^{**}}{\sqrt{n}} = \frac{y_t^{**}}{\sqrt{n}} - \widehat{b}_n^{*\prime} \frac{x_t}{\sqrt{n}} = \frac{\widetilde{u}_t^{*}}{\sqrt{n}} - \left(\widehat{b}_n^{*} - \widehat{b}_n \right) \frac{x_t}{\sqrt{n}}$$

It follows from (89) and (90) that

$$\frac{u_t^{**}}{\sqrt{n}} = \eta' \underline{B}_z \left(R_{nj_t} \right) + o_{\text{a.s.}}(1), \tag{91}$$

and then the limit distribution of $n^{-1} s^{**2} = n^{-2} \sum_{t=1}^{n} u_t^{**2}$ is the same as that of

$$\frac{1}{n} \sum_{j=1}^{n} \eta' \underline{B}_z(R_{nj}) \underline{B}_z(R_{nj})' \eta \to_{d^*} \eta' \left(\int_0^1 \underline{B}_z \underline{B}_z' \right) \eta \quad a.s.(P),$$

using arguments like those in Lemma 2.3 (c). Similarly, we find that the bootstrap regression t-ratio is

$$t_i^* = \frac{\hat{b}_{ni}^*}{s_i^*} = \frac{\hat{b}_n^*}{\left\{ s^{**2} \left[\left(\sum_1^n x_t x_t' \right)^{-1} \right]_{ii} \right\}^{1/2}} = \frac{\sqrt{n} \hat{b}_n^*}{\left\{ \frac{1}{n} s^{**2} \left[\left(\frac{1}{n^2} \sum_1^n x_t x_t' \right)^{-1} \right]_{ii} \right\}^{1/2}}$$

and then

$$\frac{1}{\sqrt{n}} t_i^* \to_{d^*} \frac{\xi_{yxi}}{\left\{ \eta' \left(\int_0^1 \underline{B}_z \underline{B}_z' \right) \eta \left[\left(\int_0^1 B_x B_x' \right)^{-1} \right]_{ii} \right\}^{\frac{1}{2}}} \quad a.s.(P).$$

Part (d) & (e) Write $\widehat{\rho}^{**} = n^{-2} \sum_{t=1}^{n} u_t^{**} u_{t-1}^{**} / n^{-2} \sum_{t=1}^{n} u_{t-1}^{**2}$. Using (91) $n^{-2} \sum_{t=1}^{n} u_t^{**} u_{t-1}^{**}$ is asymptotically equivalent to

$$\frac{1}{n} \sum_{j=1}^{n} \eta' \underline{B}_z \left(R_{nj} \right) \underline{B}_z \left(R_{nj-1} \right)' \eta \to_{d^*} \left(\int_0^1 \underline{B}_z(s) ds \right) \left(\int_0^1 \underline{B}_z(s)' ds \right) = 0 \quad a.s.(P).$$

and then $n^{-2} \sum_{t=1}^{n} u_t^{**} u_{t-1}^{**} \to_{p^*} 0 \quad a.s.(P)$, from which the result for $\widehat{\rho}^{**}$ follows. The result for $\widehat{\rho}_h^{**}$ follows in the same way. The result for $DW^{**} = n^{-2} \sum_2^n (\Delta \hat{u}_t^{**})^2 / n^{-2} \sum_1^n \hat{u}_t^{**2}$ follows as in Theorem 3.3 (e).

Part (f) Write $R^2 = 1 - n^{-2} \sum_1^n \hat{u}_t^{**2} / n^{-2} \sum_1^n y_t^{**2}$. We have $y_t^{**} = \widehat{b}_n' x_t + \widetilde{u}_t^*$ and so

$$\frac{1}{n} \sum_{t=1}^{n} \left(\frac{y_t^{**}}{\sqrt{n}} \right)^2 = \frac{1}{n} \sum_{t=1}^{n} \left(\widehat{b}_n' \frac{x_t}{\sqrt{n}} + \frac{\widetilde{u}_t^*}{\sqrt{n}} \right)^2,$$

whose asymptotic behavior is the same as

$$\frac{1}{n} \sum_{j=1}^{n} \left(\xi_{yx}' B_x \left(\frac{j}{n} \right) + \eta' \underline{B}_z \left(R_{nj} \right) \right)^2 \to_{d^*} \eta' \left(\int_0^1 \underline{B}_z \underline{B}_z' \right) \eta + \xi_{yx}' \left(\int_0^1 B_x B_x' \right) \xi_{yx} \quad a.s.(P)$$

It follows from this result and part (b) that

$$R^2 \to_{d^*} \frac{\xi_{yx}' \left(\int_0^1 B_x B_x' \right) \xi_{yx}}{\eta' \left(\int_0^1 \underline{B}_z \underline{B}_z' \right) \eta + \xi_{yx}' \left(\int_0^1 B_x B_x' \right) \xi_{yx}} \quad a.s.(P).$$

6.11 Proof of Theorem 4.2

Parts (a), (b) & (c) Start with the formula

$$\widehat{b}_n^{b*} - \widehat{b}_n = \left(\frac{1}{n^2}\sum_{t=1}^{n} x_t x_t'\right)^{-1}\left(\frac{1}{n^2}\sum_{t=1}^{n} x_t \widetilde{u}_t^{b*}\right).$$

The block bootstrap sample $(\widetilde{u}_t^{b*})_1^n$ is drawn by randomly selecting blocks from the centred residuals $(\widehat{u}_t - n^{-1}\sum_{t=1}^{n}\widehat{u}_t = y_t - \overline{y} - \widehat{b}_n(x_t - \overline{x}))_1^n$. Setting $t = (s-1)m + k$ and using the representation (88) above, we can write

$$\frac{\widetilde{u}_t^{b*}}{\sqrt{n}} = \frac{\widehat{u}_{(j_s-1)m+k} - n^{-1}\sum_{s=1}^{n}\widehat{u}_s}{\sqrt{n}} = \eta'\underline{B}_z\left(R_{Mj_s} + \frac{k}{n}\right) + o_{a.s}(1) \qquad (92)$$

where R_{Mj_s} is uniform over $0, \frac{1}{M}, ..., \frac{M-1}{M}$. Then, as in earlier derivations, we find

$$\frac{1}{n}\sum_{t=1}^{n}\sin\left[\left(k-\frac{1}{2}\right)\pi\frac{t}{n}\right]\sin\left[\left(m-\frac{1}{2}\right)\pi R_{Mj_s}\right]$$
$$\to_{d^*} \int_0^1 \sin\left[\left(k-\frac{1}{2}\right)\pi s\right]ds\int_0^1 \sin\left[\left(m-\frac{1}{2}\right)\pi r\right]dr \quad a.s.(P),$$

which leads to $\widehat{b}_n^* \to_{d^*} \xi_{yx} \quad a.s.(P)$, just as in part (a) of Theorem 4.1. Next,

$$\frac{u_t^{b**}}{\sqrt{n}} = \frac{y_t^{b**}}{\sqrt{n}} - \widehat{b}_n^{b*\prime}\frac{x_t}{\sqrt{n}} = \frac{\widetilde{u}_t^{b*}}{\sqrt{n}} - \left(\widehat{b}_n^{b*} - \widehat{b}_n\right)\frac{x_t}{\sqrt{n}} = \frac{\widetilde{u}_t^{b*}}{\sqrt{n}} + o_{p^*}(1), \qquad (93)$$

so the limit distribution of $n^{-1}s^{b**2} = n^{-2}\sum_{t=1}^{n} u_t^{b**2}$ is the same as that of

$$\frac{1}{M}\sum_{s=1}^{M}\eta'\underline{B}_z(R_{Mj_s})^2 \to_{d^*} \eta'\left(\int_0^1 \underline{B}_z\underline{B}_z'\right)\eta \quad a.s.(P). \qquad (94)$$

Parts (b) and (c) follow in a straightforward way.

Parts (d) and (e The denominator of $\widehat{\rho}^{b**} = n^{-2}\sum_{t=1}^{n} u_t^{b**}u_{t-1}^{b**}/n^{-2}\sum_{t=1}^{n} u_{t-1}^{b**2}$ is covered by (93) and (94). The numerator is asymptotically equivalent to

$$\begin{aligned}
&\frac{1}{n}\textstyle\sum_1^n \left(\frac{\widetilde{u}_t^{b*}}{\sqrt{n}}\right)\left(\frac{\widetilde{u}_{t-1}^{b*}}{\sqrt{n}}\right) \\
&= \frac{1}{mM}\sum_{s=1}^{M}\sum_{k=2}^{m}\left[\eta'\underline{B}_z(R_{Mj_s} + \frac{k}{n}) + o_{\text{a.s.}}(1)\right]\left[\underline{B}_z(R_{Mj_s} + \frac{k-1}{n})'\eta + o_{\text{a.s.}}(1)\right] \\
&\quad + \frac{1}{mM}\sum_{s=1}^{M}\left[\eta'\underline{B}_z\left(R_{Mj_s} + \frac{1}{n}\right) + o_{\text{a.s.}}(1)\right]\left[\underline{B}_z\left(R_{Mj_s-1} + \frac{m}{n}\right)'\eta + o_{\text{a.s.}}(1)\right] \\
&\to_{d^*} \eta'\left(\int_0^1 \underline{B}_z\underline{B}_z'\right)\eta \quad a.s.(P),
\end{aligned} \qquad (95)$$

using the same arguments as those in the proof of Theorem 3.5. Combining (94) and (95) we get

$$\widehat{\rho}^{b**} \to_{d^*} \frac{\eta' \left(\int_0^1 \underline{B}_z \underline{B}_z' \right) \eta}{\eta' \left(\int_0^1 \underline{B}_z \underline{B}_z' \right) \eta} = 1 \quad a.s. (P) .$$

as stated. The proofs for $\widehat{\rho}_h^{b**}$, DW^{b**} and R^{b**2} follow in a related fashion.

6.12 Proof of Theorem 4.3. Write

$$\frac{n}{M}(\widehat{\rho}_1^{b**} - 1) = \frac{1}{nM} \sum_{t=1}^{n} \Delta u_t^{b**} u_{t-1}^{b**} / \frac{1}{n^2} \sum_{t=1}^{n} u_{t-1}^{b**2}.$$

The limit of the denominator is given in (94). Next, using (92) and (88) and setting $t = (s-1)m + k$, we can write

$$n^{-\frac{1}{2}} u_t^{b**} = n^{-\frac{1}{2}} \widetilde{u}_t^{b*} + o_{p^*}(1) = n^{-\frac{1}{2}} \hat{u}_{(j_s-1)m+k} + o_{p^*}(1) = \eta' \underline{B}_z \left(R_{Mj_s} + \frac{k}{n} \right) + o_{p^*}(1),$$

and then

$$\frac{\Delta \widetilde{u}_t^{b*}}{\sqrt{n}} = \eta' \left\{ \underline{B}_z (R_{Mj_s} + \frac{k}{n}) - \underline{B}_z (R_{Mj_s} + \frac{k-1}{n}) \right\} + o_{p^*}(1),$$

where $\{j_s : s = 1, ..., M\}$ are independent uniform draws from $1, ..., M$, and $j_0 = 0$. Proceeding as in the proof of Theorem 3.7, we find that

$$\frac{1}{nM} \sum_{t=1}^{n} u_{t-1}^{b**} \Delta u_t^{b**} = \frac{1}{2M} \left\{ \left(\frac{u_n^{b**}}{\sqrt{n}} \right)^2 - \sum_{t=1}^{n} \left(\frac{\Delta u_t^{b**}}{\sqrt{n}} \right)^2 \right\}.$$

and

$$\begin{aligned} \frac{1}{M} \sum_{t=1}^{n} (\frac{\Delta u_t^{b**}}{\sqrt{n}})^2 \quad &\to_{d^*} \eta' \left(\int_0^1 \int_0^1 \left[\underline{B}_z(r) - \underline{B}_z(s) \right] \left[\underline{B}_z(r) - \underline{B}_z(s) \right]' drds \right) \eta \quad a.s. (P) \\ &= 2\eta' \left(\int_0^1 \underline{B}_z(r) \underline{B}_z(r)' dr \right) \eta, \end{aligned}$$

from which we deduce that

$$\frac{1}{nM} \sum_{t=1}^{n} u_{t-1}^{b**} \Delta u_t^{b**} \to_{d^*} -\eta' \left(\int_0^1 \underline{B}_z(r) \underline{B}_z(r)' dr \right) \eta \quad a.s. (P) .$$

It follows that

$$\frac{n}{M}(\widehat{\rho}_1^{b**} - 1) \to_{d^*} -\frac{\eta' \left(\int_0^1 \underline{B}_z(r) \underline{B}_z(r)' dr \right) \eta}{\eta' \left(\int_0^1 \underline{B}_z(r) \underline{B}_z(r)' dr \right) \eta} = -1 \quad a.s.(P),$$

giving the stated result.

7. Notation

$o_{a.s.}(1)$	tends to zero almost surely (P)	$\nsim_a$	not asymptotically equivalent to
$O_{a.s}(1)$	bounded almost surely (P)	$(e_t^*)_1^n$	direct bootstrap sample
$\rightarrow_{p^*}$	convergence in P^* probability	$(\tilde{e}_t^*)_1^n$	centred bootstrap sample
$\rightarrow_{d^*}$	weak convergence (P^*)	$(\tilde{e}_t^{b*})_1^n$	centred block bootstrap sample
$\rightarrow_{d^*}$ $a.s.$	weak convergence (P^*), almost surely (P)	$(e_t^{b*})_1^n$	block bootstrap sample
$\rightarrow_{p^*}$ $a.s.$	convergence in P^* probability, almost surely (P)	$[\cdot]$	integer part
$\sim_{d^*}$	asymptotically distributed as		

8. References

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