Compromises between Cardinality and Ordinality

in Preference Theory and Social Choice

by

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Abstract:

By taking sets of utility functions as a primitive description of agents, we define an ordering over assumptions on utility functions that gauges their implicit measurement requirements. Cardinal and ordinal assumptions constitute two types of measurement requirements, but several standard assumptions in economics lie between these extremes. We first apply the ordering to different theories for why consumer preferences should be convex and show that diminishing marginal utility, which for complete preferences implies convexity, is an example of a compromise between cardinality and ordinality. In contrast, the Arrow-Koopmans theory of convexity, although proposed as an ordinal theory, relies on utility functions that lie in the cardinal measurement class. In a second application, we show that diminishing marginal utility, rather than the standard stronger assumption of cardinality, also justifies utilitarian recommendations on redistribution and axiomatizes the Pigou-Dalton principle. We also show that transitivity and order-density (but not completeness) characterize the ordinal preferences that can be induced from sets of utility functions, present a general cardinality theorem for additively separable preferences, and provide sufficient conditions for orderings of assumptions on utility functions to be acyclic and transitive.

1. Introduction

Ordinal utility theory asserts that only those assumptions on utility functions that are preserved under monotonically increasing transformations are proper primitives. The rationale is that any property P that is not preserved under increasing transformations cannot be verified through observations of choice behavior: if a utility u satisfies P, there will exist another utility u' that does not satisfy P but that represents the same preferences as u. Nonordinal properties are therefore needlessly restrictive: given a nonordinal assumption, one may always make a weaker assumption with the same implications for choice behavior. It is common, therefore, to contend that the only role for cardinal utility functions in economics is the normative one of representing interpersonal utility comparisons.

Ordinalism's first targets were diminishing marginal utility and concavity, which had long been used as arguments for why preferences are normally convex. Neither DMU nor concavity is preserved by increasing transformations and hence neither is an ordinal assumption. Rather, when a utility analogue for the convexity of preferences is necessary, the ordinal procedure is to assume that utility functions are quasiconcave. Many pioneer ordinalists, e.g., Arrow (1951), claimed in addition that diminishing marginal utility is tantamount to assuming that utility is cardinal. Arrow's position in the 1950's was typical and remains predominant: *either* an assumption on utility is ordinal *or* it is cardinal.

By taking sets of utility functions as primitive, I define a finer gradation of properties of utility that allows for intermediate standards of measurement. Ordinal preference theory, which takes the functions generated by all increasing transformations of a given utility function as primitive, lies at one end of the spectrum. Cardinal theory, which takes the functions generated by all increasing affine transformations of a given utility function as primitive, uses a much smaller set of utility representations and therefore leads to a stronger (more restrictive) theory. Outside of economics, ratio scales, which take still smaller sets of functions as primitive (the functions generated by all increasing linear transformations), are also common. But in addition to these well-known measurement scales, there is an infinity of intermediate cases. Diminishing marginal utility and concavity lie precisely in the middle ground between cardinality and ordinality. The set of all concave representations of a given preference relation is larger than any cardinal set of representations but smaller than the ordinal set. Concavity thus presupposes an intermediate standard of measurement and does not, therefore, rely on cardinalist foundations. Moreover, concavity is far more natural as a primitive in economics than is

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quasiconcavity: concavity exactly captures the core presumptions of economic psychology.

Given that only ordinal properties of utility can be tested with choice behavior, what advantage could there be in taking nonordinal properties of utility as primitive? One benefit is that nonordinal properties can provide rationales for assumptions on preference relations. Diminishing marginal utility, for instance, gives a psychological explanation of why preferences should be convex. To declare by fiat that preferences relations are convex (or that utility is quasiconcave), in contrast, offers no psychological rationale. This paper thus gives utility functions a purpose in positive preference theory, whereas for ordinalists they serve only as a convenient shorthand for preference relations. A second advantage is that we can order the strength of assumptions and thereby gauge just how nonordinal an assumption is. This ordering proves particularly valuable in social choice theory, where the need for nonordinal assumptions is well-recognized but the ordinal/cardinal dichotomy prevails.

We illustrate our ordering of properties of utility by considering another celebrated rationale for the convexity of preferences, Arrow's (1951) argument (following Koopmans) that an agent's leeway to determine the precise timing of consumption implies that preferences must be convex. Arrow reasoned that this rationale for convexity, unlike diminishing marginal utility, was free from any taint of cardinality. We show, however, that the utility theory that lies behind the Arrow/Koopmans position is cardinal. Bringing these results together, we see that the old neoclassical explanation, diminishing marginal utility or concavity, rests on less a demanding standard of measurement.

We also apply concavity as a measurement scale to social decision-making, and take issue with another common belief about measurement in economics, that utilitarian interpersonal comparisons rely on cardinal utility scales. By taking a set of concave interpersonal utility functions as a primitive we derive utilitarian conclusions about the benefits of redistributing utility from rich to poor. Indeed, the ordering that arises from a set of concave utilities provide a better approximation of classical utilitarian writings than the cardinalist approach of taking a single interpersonal utility function and its affine transformations as primitive. For instance, concave utilities characterize the famous Pigou-Dalton principle, which adds egalitarian constraints to standard utilitarian orderings.

Our ordering of properties of utility draws on two literatures. The first is measurement theory (see, e.g., Krantz et al. (1971) and Roberts (1979), and originally Stevens (1946)), which identifies measurement classes with sets of transformations. Thus, ratio scales are defined by the set of increasing

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linear transformations, interval or cardinal scales by the set of increasing affine transformations, and ordinal scales by the set of all increasing transformations. Measurement theory implicitly orders measurement classes by set inclusion; notice that in the cases mentioned the sets of transformations are nested. A set inclusion ordering of transformations has similarities to some of the orderings of properties we consider (see section 6), and under some circumstances, our approach and the measurement theory approach coincide. But traditional measurement theory considers only a few prominent cases and fails to define a sufficiently rich array of measurement classes.

The second literature consists of social choice models that vary the set of admissible transformations of utility functions according to the desired degree of interpersonal comparability (Sen (1970), d'Aspremont and Gevers (1977), Roberts (1980b)). These models, which employ multiple-agent profiles of utility functions, place restrictions on what transformations can be applied to any individual utility function and on whether transformations are permitted to vary across individuals. Applying a smaller set of transformations imposes a tighter interpersonal comparability requirement.

The drawback of both literatures is that they identify a standard of measurement or interpersonal comparability with a set of transformations applied to utility functions. At first glance, this appears to be an advantage: any utility function can then be a member of any of the standard measurement classes. But taking arbitrary sets of utility functions as primitive admits a greater variety of measurement standards and is more flexible. For instance, the set of continuous utility functions defines a measurement standard that cannot be characterized by the set of continuous transformations (since a continuous transformation applied to a noncontinuous utility function will not generate a continuous utility). Just as importantly, taking sets of utility functions as primitive allows us to identify the implicit measurement requirements of assumptions on utility functions and hence to compare the measurement requirements of different assumptions. It might seem that taking sets of utilities as primitive would not allow assumptions to be ordered except for trivial cases where assumptions α and β are such that α is satisfied whenever β is. We will see, however, that debilitating incompleteness can be avoided and measurement classes can be calibrated finely. Indeed, claims that at first appear nonsensical – e.g., "concavity" is a weaker assumption than "additive separability" – can be given a precise meaning.

The ordinalist approach to cardinal utility has always been a puzzle. Considerable work (e.g., Debreu (1960), Krantz et al. (1971)) has gone into specifying axioms on binary preference relations that

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ensure that preferences can be represented by a utility unique up to an increasing affine transformation. But from the ordinal point of view the significance of such representation results remains limited. If binary relations are the primitives of preference theory, the cardinal utility whose existence is established has no significance beyond the notational. But if utility is primitive, cardinality has an immediate purpose: that an assumption on utility admits a cardinal set of representations indicates the extent of the assumption's nonordinality. Other nonordinal properties of utility (e.g., concavity and continuity, which also have plausible psychological foundations) admit larger sets of utility representations and are thus weaker assumptions. Our ordering of properties can thus compare how demanding are the measurement requirements of different assumptions.

Basu (1979) has also explored room for compromise between ordinal and cardinal utility theory. In a spirit similar to the present paper, Basu contends that DMU resides in this middle ground and remarks on the advantages of taking nonordinal assumptions as primitive. But Basu sticks to the method of characterizing measurement classes via utility transformations. Furthermore, as Basu (1982) shows, the middle ground that Basu (1979) linked to DMU ends up being equivalent to full-scale cardinality in classical commodity spaces. Basu concludes that utility theory prior to the ordinal revolution used assumptions that were tantamount to cardinality (even when, as in Lange's case, they were attempting to rid themselves of cardinal assumptions). Using sets of utility functions to compare measurement standards, in contrast, permits compromises between cardinality and ordinality that are robust to the specification of the commodity space.

In sections 2 and 3, we consider sets of utility functions, which we call *psychologies*, our orderings of psychologies and properties of utility, and cardinal and ordinal properties. We also provide a sufficient condition that ensures that ordinal preferences can be induced by psychologies: outside of a technicality, any transitive preference relation can be induced by some psychology. Section 4 establishes that concavity is weaker than any cardinal property of utility and stronger than any ordinal property and shows that the Arrow/Koopmans utility theory is cardinal. Section 5 links sets of concave interpersonal utility functions to standard utilitarian results on the optimality of redistributing utility. The Pigou-Dalton principle then appears as a theorem rather than as an assumption. Section 6 specifies conditions under which our orderings of properties are acyclic and transitive.

2. Psychologies and orderings of psychologies

Let X be a nonempty set of consumption options and, for any nonempty $A \subset X$, let $\mathscr{F}_A$ be the set of functions from A to $\mathbb{R}$. An agent is characterized by a nonempty set $U \subset \mathscr{F}_X$, which lists the utility functions that accurately depict the agent's psychological reactions to the options in X. We say that U is a *psychology* and that X is the *domain of* U.

Preference relations on X emerge straightforwardly from psychologies. Call $A \subset X$ *decisive* for U if and only if, for all $u, v \in U$ and $x, y \in A$, $u(x) \geq u(y) \Leftrightarrow v(x) \geq v(y)$. Define the binary relation $\succeq_U$ $\subset X \times X$, *the induced preference relation* of U, by:

$$x \succeq_U y \Leftrightarrow \{x, y\} \text{ is decisive and there exists a } u \in U \text{ such that } u(x) \geq u(y).$$

Induced preference relations can satisfy most of the standard assumptions imposed on ordinal preferences. For instance, $\succeq_U$ is complete if X is decisive for U. We will speak interchangeably below of $\succeq_U$ satisfying an assumption on binary relations and the underlying psychology U satisfying the same assumption. Psychologies as we have defined them plainly cannot be intransitive. But, the theorem below (which applies a classical representation result in Richter (1966) to sets of utility functions) shows that, outside of transitivity and a standard technicality, any binary relation can be induced by a psychology. At the end of this section, we briefly show how to extend the definition of psychologies to handle the intransitive and other cases not covered here.

A binary relation $\succeq$ on X is *countably order-dense* if there exists a countable $Y \subset X$ such that for all $x, z \in X$ with $x \succeq z$ and not $z \succeq x$, there exists a $y \in Y$ such that $x \succeq y \succeq z$.

Theorem 2.1 If the binary relation $\succeq$ on X is transitive and countably order-dense, there exists a psychology U with domain X whose induced preference relation is $\succeq$.

Proof: If $\succeq$ is complete in addition to transitive and order-dense, the proof that there exists a U with $\succeq_U$ $= \succeq$ is the standard existence theorem for utility functions. So assume that $\succeq$ is not complete. Let $X/\sim$ denote the indifference classes of $\succeq$ and define $\succ$ on $X/\sim$ by $I \succ J$ if and only if $I \neq J$ and $x \succeq y$ for some $x \in I$ and $y \in J$. For any $z \in X$, let $I(z)$ denote the indifference class that z belongs to. For any $x, y \in X$ such that neither $x \succeq y$ nor $y \succeq x$ holds, define two strict partial orders $\succ_x$ and $\succ_y$ on $X/\sim$ by $\succ \cup (I(x), I(y))$ and $\succ \cup (I(y), I(x))$ respectively. Let $\succ_x^t$ and $\succ_y^t$ denote the transitive closures of $\succ_x$ and $\succ_y$

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respectively.

By assumption, there is a countable set of indifference classes, say Y, that is order-dense with respect to $\succ$. Let $Y' = Y \cup \{I(x), I(y)\}$. To see that Y' is order-dense with respect to $\succ_x^t$ and $\succ_y^t$, suppose not. Then, to take the case of $\succ_x^t$, there exist $I, J \in X/\sim \setminus Y'$ such that $I \succ_x^t J$ and such that for all $K \in Y'$, not $I \succ_x^t K \succ_x^t J$. That is, there are two indifference classes not in Y' that are unranked according to $\succ_x$ but that are ranked according to $\succ_x^t$. By the definition of a transitive closure, there must exist a finite set of indifference classes, say $I_1, ..., I_n$ such that $I \succ_x I_1 \succ_x ... \succ_x I_n \succ_x J$. But since $\succ_x$ is transitive, at least one of the elements I_1 to I_n has to be $I(x)$ or $I(y)$, which contradicts the assumption that Y' is not order-dense. By Theorem 3.2 of Fishburn (1979) (which generalizes Richter (1966)), there exists a utility function $u_{x,y}$ on $X/\sim$ such that $L \succ_x^t M$ implies $u(L) > u(M)$. Similarly, there exists a $u_{y,x}$ on $X/\sim$ such that $L \succ_y^t M$ implies $u(L) > u(M)$. Define the utility functions $v_{x,y}$ and $v_{y,x}$ by letting each element of any indifference class inherit the utility number of its indifference class given by $u_{x,y}$ and $u_{y,x}$ respectively. Let U be defined by $v \in U$ if and only if $v \in \{v_{x,y}, v_{y,x}\}$ for some $x, y \in X$ such that not $x \succeq y$ and not $y \succeq x$. It is immediate that $\succeq_U = \succeq$. ∎

Theorem 2.1 differs from standard utility representation results only in that no completeness assumption is present. Transitivity and countable order-density are retained without change. Ok (1999) has also recently analyzed when an incomplete preference relation $\succeq$ can be represented by a set of utility functions. Ok's definition of representation is the same as ours, except that he concentrates on the case where the set of utility functions is finite. Ok shows that transitivity and order-density by themselves do not imply that a preference relation can be represented by a finite set of utilities and also provides sufficient conditions that guarantee representation by an infinite set of utilities. Theorem 2.1 makes do with weaker conditions and indeed shows that infinite-dimensional representability requires only the standard assumptions. See Dubra, Maccheroni, and Ok (2001) for infinite-dimensional representability of possibly incomplete preferences on lotteries.

We now define the key ordering of psychologies. In this definition and in several to follow, we distinguish between complete and possibly incomplete psychologies. While the incomplete case is more general, the reader is invited for concreteness to focus on complete psychologies. To cover the incomplete case, let $U|A$ (the restriction of U to A), where A is a subset of the domain of U, denote the

set $\{w \in \mathscr{F}_A : w = u \mid A \text{ for some } u \in U\}$. A pair of psychologies U and V *have the same decisive sets* if and only if, for all $A \subset X$, A is decisive for $U \Leftrightarrow A$ is decisive for V.

Definition 2.1:

(Complete case) Suppose U and V are complete. Then U is *no stronger than V* if and only if $U \supset V$.

(Incomplete case) U is *no stronger than V* if and only if U and V have the same decisive sets and, for each decisive A, $U \mid A \supset V \mid A$.

Other natural orderings of psychologies may be defined. For example, we could say instead: U is no stronger than $V \Leftrightarrow$ for each $A \subset X$ that is decisive for U, $U \mid A \supset V \mid A$. This ordering ranks more psychologies, but the additional discrimination is unnecessary for our applications and so we use Definition 2.1.

It is immediate that the "no stronger than" relation on psychologies is transitive and, when $|X| > 1$, incomplete. We define a "weaker than" relation on psychologies as the asymmetric part of the "no stronger than" relation: U is *weaker* than V if and only if U is no stronger than V and it is not the case that V is no stronger than U.

In closing this section, we point briefly out how to extend the above model of psychologies so that psychologies can generate arbitrary binary preference relations (even intransitive ones). Instead of each utility function in a psychology being defined on the same domain X, we instead allow utilities to be defined on arbitrary subsets of some universal domain X. Specifically, an *extended psychology U* on X is a family of functions such that $u \in U$ if and only if $u \in \mathscr{F}_A$ for some $A \subset X$. Recall that for a simple psychology U and a $A \subset X$, $U \mid A$ (the restriction of U to A) is the set $\{w \in \mathscr{F}_A : w = u \mid A \text{ for some } u \in U\}$. The same definition holds unchanged for extended psychologies. For any $A \subset X$, $U \mid A$ is interpreted as the set of utility functions on A that accurately depict the agent's psychological reactions to the consumption possibilities in A.

For any extended psychology U on X, let the extended domain of U denote $\{A \subset X : A =$ Domain u for some $u \in U\}$. If the extended domain of U is $\{X\}$, then the extended psychology satisfies our previous model and we say it is *simple*. We redefine $A \subset X$ to be *decisive* for U if and only if, for all $x, y \in A$ and all $u, v \in U$ such that $\{x, y\} \subset$ Domain $u \cap$ Domain v, $u(x) \geq u(y) \Leftrightarrow v(x) \geq v(y)$.

Preference relations are induced by extended psychologies in the same way they are induced by simple psychologies: given an extended psychology U, define R_U by $x\, R_U\, y$ if and only if $\{x, y\}$ is decisive and there exists a $u \in U$ such that $u(x) \geq u(y)$. It is clear that for any binary relation $\succeq$ on X, there exists an extended psychology U on X such that $R_U = \succeq$. A suitable U can be assembled in many ways: for instance, for each $(x, y) \in \succeq$, let $u_{x,y} : \{x, y\} \to \mathbb{R}$ be a function that satisfies $u(x) \geq u(y)$ if and only if $x \succeq y$, and let $U = \bigcup_{(x, y) \in \succeq} \{u_{x,y}\}$. Extended psychologies therefore constitute a more general model of agents than ordinal preferences.

Our ordering of psychologies extends straightforwardly: we can define the extended psychology U to be weaker than the extended psychology V if and only if (1) U and V have the same decisive sets, and (2) for any decisive A, $U|A \supset V|A$. Much of what follows, e.g., the theory of ordinal and cardinal psychologies and ordinal and cardinal properties of utility (see Definitions 3.3 - 3.5 below), can be recast in terms of extended psychologies. To keep notation simple, we stick to simple psychologies.

3. Properties of utility and orderings of properties

We now use the ordering over psychologies to generate orderings over properties of utility functions. Formally, a property P is simply a set of functions into the real line. The domains of the functions in P may differ. A utility function $u: A \to \mathbb{R}$ *satisfies property P* if and only if $u \in P$.

Definition 3.1:
(Complete case) A complete U *maximally satisfies* property P if and only if (1) for each $u \in U$, u satisfies P, and (2) $\nexists$ a complete psychology $V \supsetneq U$ such that each $v \in V$ satisfies P.
(Incomplete case) U *maximally satisfies* property P if and only if (1) for each $u \in U$ and each A that is decisive for U, there exists a $B \supset A$ that is decisive for U such that $u|B$ satisfies property P, and
(2) $\nexists V \supsetneq U$ with the same decisive sets as U that meets condition (1).

In words, U maximally satisfies P if it is largest among psychologies that share the same family of decisive sets and that, for each u in U and each decisive A, own a decisive B containing A such that u satisfies P on B.

The "containing" sets B in Definition 3.1 are unavoidable: since some properties (e.g.,

quasiconcavity or concavity) can only be satisfied on certain domains (convex sets), we cannot speak of those properties as satisfied on arbitrary decisive sets. Also, note that the domain of a psychology may determine whether it maximally satisfies some properties. For instance, if $|X|$ is finite, any psychology maximally satisfies continuity, but not when, for example, $X = \mathbb{R}^n$.

Our earlier ordering of psychologies suggests a natural ordering of properties of utility. (Formally speaking, the complete case below defines a distinct ordering, which we include only as an illustration; all explicit references to $\succeq_{NS}$ refer to the incomplete case.)

Definition 3.2:

(Complete case) Property P is *no stronger than* property Q, or $P \succeq_{NS} Q$, if and only if for all U that maximally satisfy P and all V that maximally satisfy Q, $U \cap V \neq \emptyset$ implies $U \supset V$.

(Incomplete case) Property P is *no stronger than* property Q, or $P \succeq_{NS} Q$, if and only if whenever U maximally satisfies P, V maximally satisfies Q, U and V have the same decisive sets, and $U|A \cap V|A \neq \emptyset$ for all decisive A, then $U|A \supset V|A$ for all decisive A.

(Both cases) P is *weaker than* Q, or $P \succeq_W Q$, if and only if $P \succeq_{NS} Q$ and not $Q \succeq_{NS} P$.

In the incomplete case, the "weaker than" part of Definition 3.2 can be rephrased: $P \succeq_W Q$ if and only if $P \succeq_{NS} Q$ and there exists a U that maximally satisfies P and a V that maximally satisfies Q such that U and V have the same decisive sets, $U|A \cap V|A \neq \emptyset$ for all decisive A, and $U|A \supsetneq V|A$ for some decisive A.

The relations $\succeq_{NS}$ and $\succeq_W$ need not be transitive or complete. Since psychologies do not have to be nested, incompleteness is unremarkable. The incompleteness of $\succeq_W$ can be particularly extensive since P and Q are unranked by $\succeq_W$ if P and Q are inconsistent, i.e., if $P \cap Q = \emptyset$. The intransitivity of $\succeq_{NS}$ and $\succeq_W$ may come as more of a surprise. We defer this subject until section 6, where we discuss domains on which $\succeq_W$ and other related orderings are transitive or at least acyclic.

Definition 3.3 (Ordinality) The functions u and v *agree* on A if and only if, for all $x, y \in A$, $u(x) \geq u(y)$ $\Leftrightarrow v(x) \geq v(y)$. A psychology U with domain X is *ordinal* if and only if $u \in U$ implies that if $v \in \mathscr{F}_X$ and u and v agree on each A that is decisive for U, then $v \in U$.

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Equivalently, a psychology U with domain X is ordinal if and only if $u \in U \Rightarrow (v \in U \Leftrightarrow v \in \mathscr{F}_X$ and, for each decisive A, there exists an increasing transformation g such that $v|A = g \circ u|A$).

Definition 3.4 (Cardinality) A function $g: E \to \mathbb{R}$, where $E \subset \mathbb{R}$, is an *increasing affine transformation* if and only if there exist $a > 0$ and b such that, for all $x \in E$, $g(x) = ax + b$. A psychology U with domain X is *cardinal* if and only if $u \in U \Rightarrow (v \in U \Leftrightarrow v \in \mathscr{F}_X$ and, for each decisive A, there exists an increasing affine transformation g such that $v|A = g \circ u|A$).

We can now define properties of utility as ordinal or cardinal.

Definition 3.5 A property P is *ordinal* (resp. *cardinal*) if and only if any U that maximally satisfies P is ordinal (resp. cardinal).

Most of the standard assumptions used nowadays in utility theory are ordinal properties. As an example, consider quasiconcavity. A function $u: Z \to \mathbb{R}$ is *quasiconcave* if Z is a convex set and, for all $x, y \in Z$ and $\lambda \in [0, 1]$, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$. To confirm that quasiconcavity is an ordinal property of utility, let U maximally satisfy quasiconcavity, let u be an arbitrary element of U, and suppose, for all decisive A, that $u|A$ and $v|A$ agree. For any decisive A, there exists a decisive $B \supset A$ such that $u|B$ satisfies quasiconcavity (given Definition 3.1 (1)). Since B is decisive, $u|B$ and $v|B$ agree. Since $u|B$ and $v|B$ agree, there is an increasing transformation f: Range $u|B \to \mathbb{R}$ such that $f \circ u|B = v|B$; since f is increasing, for all $x, y \in B$ and all $\lambda \in [0, 1]$, $v(\lambda x + (1-\lambda)y) \geq \min\{v(x), v(y)\}$. Since $u|B$ satisfies quasiconcavity, B is convex; hence $v|B$ satisfies quasiconcavity. So, by Definition 3.1 (2), $v \in U$.

As an example of a cardinal property, we consider additive separability, which will later be important in our examination of the convexity of preferences.

Definition 3.6 A function $u: A \to \mathbb{R}$ satisfies *additive separability* if and only if, for some integer $n \geq 2$, there exist component spaces A_i, $i = 1, ..., n$, such that $A = A_1 \times ... \times A_n$ and functions $u_i: A_i \to \mathbb{R}$, $i = 1, ..., n$, such that (1) for each $x \in A$, $u(x) = \sum_{i=1}^{n} u_i(x_i)$, (2) for each component i, cl Range u_i is an

interval, and (3) two of these intervals have nonempty interior.

The *property* of additive separability consists of the set of all functions that satisfy additive separability.

Theorem 3.1 Additive separability is a cardinal property.

The proof, and all other proofs missing from the text, are in the appendix.

Two features distinguish Theorem 3.1 from the existing literature on additive separability (see Debreu (1960) and Krantz et al. (1971)). First, since psychologies contain sets of functions – and those functions need not be ordinally identically – Theorem 3.1 extends classical cardinality results; specifically, incomplete preferences are covered. Second, the standard approach to additively separable functions proves cardinality as a by-product of existence theorems that specify conditions on ordinal preferences that imply the existence of an additively separable utility representation. The existence question is difficult, however, and so this technique ends up imposing overly strong restrictions. By separating cardinality from existence, Theorem 3.1 makes do with much weaker conditions relative to the literature, which usually supposes that utility functions are continuous.

4. Convexity of preferences

4.1 Concavity as a primitive

This section presents explanatory rationales for when one should expect preferences to be convex. For complete preferences defined on a convex domain X, we define the convexity of a preference relation $\succeq$ in the standard way: the binary relation $\succeq \subset X \times X$ satisfies *convexity* if and only if, for all $y \in X$, the set $\{x \in X: x \succeq y\}$ is convex. To cover preference relations that can be incomplete, we need an expanded definition. A binary relation $\succeq \subset X \times X$ is *complete on* $A \subset X$ if and only if, for all $\{x, y\} \subset A$, either $x \succeq y$ or $y \succeq x$. We say that $\succeq \subset X \times X$ satisfies *generalized convexity* if and only if, for any convex set $A \subset X$ such that $\succeq$ is complete on A and any $y \in X$, $\{x \in A: x \succeq y\}$ is convex. If $\succeq \subset X \times X$ is complete and satisfies generalized convexity and X is convex, then $\succeq$ obviously satisfies convexity. Without completeness, however, a $\succeq$ that satisfies generalized convexity may fail to be convex, e.g., the relation $\succeq \subset \mathbb{R}^2 \times \mathbb{R}^2$ defined by $x \succeq y$ if and only if ($x_1 \geq y_1$ and $x_2 = y_2$) or ($x_1 = y_1$ and $x_2 \geq y_2$).

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A function $u: Z \to \mathbb{R}$ satisfies *concavity* if and only if Z is a convex set and, for all $x, y \in Z$ and all $\lambda \in [0, 1]$, $u(\lambda x + (1 - \lambda) y) \geq \lambda u(x) + (1 - \lambda) u(y)$.

Theorem 4.1 If psychology U maximally satisfies concavity, then $\succeq_U$ satisfies generalized convexity.

Given the above example of a $\succeq$ that satisfies generalized convexity but not convexity, it should also be clear that psychology can maximally satisfy concavity even though $\succeq_U$ is not convex. But while concavity of utility by itself does not imply that preferences are convex, Theorem 4.1 indicates that the concavity and completeness of a psychology do.

Proof of Theorem 4.1: Let U be a psychology that maximally satisfies concavity, let X denote the domain of U, let u be an arbitrary element of U, and let $A \subset X$ be an arbitrary convex set that is decisive for U. Since A is decisive, there exists a decisive $B \supset A$ such that $u | B$ satisfies concavity. Since A is convex, $u | A$ also satisfies concavity. For any $y \in X$, define $y_p = \{x \in A:$ for all $v \in U$, $v(x) \geq v(y)\}$. We need to show that y_p is convex. If not, there would exist $z, z' \in y_p$ and $\lambda \in [0, 1]$ such that $\hat{z} = \lambda z + (1 - \lambda) z' \notin y_p$. Since A is convex, however, $A \supset \text{co } y_p$ (where "co" is the convex hull) and therefore $u | \text{co } y_p$ satisfies concavity. Hence, for all $v \in U$, $v(\hat{z}) \geq \lambda v(z) + (1 - \lambda) v(z') \geq v(y)$. Therefore $\hat{z} \in y_p$. ∎

Definition 4.1 For n a positive integer, property P has *range* $> n$ if and only if, for each $u \in P$, $|\text{Range } u| > n$.

Definition 4.2 Property P *intersects* property Q if and only if there exists some U maximally satisfying P and some V maximally satisfying Q such that U and V have the same decisive sets and, for each decisive A, $U | A \cap V | A \neq \varnothing$.

Theorem 4.2:
(1) Any ordinal property is no stronger than concavity and concavity is no stronger than any cardinal property.

(2) Concavity is weaker than any cardinal property with range > 2 that intersects concavity.

(3) Any ordinal property with range > 1 that intersects the property of being concave and continuous is weaker than the property of being concave and continuous.

Theorem 4.2 confirms a simple intuition about concavity. Along a line, concavity as a psychology assumes that an agent experiences each successive unit of consumption as delivering a smaller increment of utility. But concavity does *not* require that each utility increment is a specific fraction of the previous increment. Agents experience diminishing marginal utility but no additional extra-ordinal precision. Cardinality, in contrast, requires that agents experience any pair of utility increments as a precise ratio. Cardinality thus imposes considerably more – implausibly more – psychological structure.

Remark on Theorem 4.2 (3): due to the fact that a concave function need not be continuous on the boundaries of its domain, there are nontrivial preferences – preferences that exhibit strict preference between arbitrarily many pairs of consumption bundles – whose only utility representations are concave. It follows that there are ordinal properties with arbitrarily large range that are not weaker than concavity.

By strengthening our ordering of properties, we can tighten Theorem 4.2 (2) and (3).

Definition 4.3 Property P is *strictly weaker* than property Q, or $P \geq_{SW} Q$, if and only if P intersects Q and whenever U maximally satisfies P, V maximally satisfies Q, U and V have the same decisive sets, and $U|A \cap V|A \neq \varnothing$ for all decisive A, then U is weaker than V.

In contrast, property P is weaker than Q if it is merely the case that P is no stronger than Q and there is *some* U that maximally satisfies P and *some* V that maximally satisfies Q, where U and V have the same decisive sets and $U|A \cap V|A \neq \varnothing$ for all decisive A, such that U is weaker than V.

One may show (by adapting the proof of Theorem 4.2 in the appendix):

(2′) concavity is strictly weaker than any cardinal property with range > 2 that intersects concavity,

(3′) any ordinal property with range > 1 that intersects the property of being concave and continuous is strictly weaker than the property of being concave and continuous.

Concavity can be ranked relative to some other classical assumptions in utility theory. If X is a nonempty open subset of $\mathbb{R}^n$, the property of continuity on X is no stronger than concavity on X and any ordinal property is no stronger than continuity. Also, the property of being continuous and nonconstant on X is weaker than concavity. These assertions follow from the fact that any concave function on an open set is continuous, but not vice versa and the fact that any continuous increasing transformation preserves continuity, but noncontinuous increasing transformations do not preserve continuity. We omit the details, which vary only slightly from the proof of Theorem 4.2.

The property being concave and continuous and the property of being continuous are each associated with a set of utility transformations, namely increasing concave and increasing continuous functions from $\mathbb{R}$ to $\mathbb{R}$. This feature is by no means shared by all properties of utility (but see section 6). Moreover, even for the cases at hand, the properties should not be confused with their associated transformations: the transformation must be applied to a function that satisfies the property in question. For instance, an increasing concave transformation of a nonconcave utility need not be concave.

4.2 The Arrow/Koopmans theory

Arrow (1951), following unpublished remarks by Koopmans, argued that if an agent holds a consumption bundle for a period of time, say $[0, T]$, and can decide on the timing of how that bundle is consumed, then the agent's preferences must be convex. Arrow's reasoning implicitly supposed that the agent's total utility is the integral of the utility achieved at each moment from 0 to T. An agent holding consumption vector $z \in \mathbb{R}^n_+$ chooses a function from $[0, T]$ to $\mathbb{R}^n_+$ that maximizes this integral. The informal argument for convexity is that an agent holding the vector $\lambda x + (1 - \lambda) y$, $\lambda \in [0, 1]$, could consume λx in λT units of time and $(1 - \lambda) y$ during the remaining $(1 - \lambda) T$ time units. If the agent's utility at each instant is independent of the consumption at other instants, it seems plausible that λx consumed in λT time units will deliver total utility that is λ times the utility of x consumed in T time units, and similarly for $(1 - \lambda) y$. So, if x and y are indifferent, $\lambda x + (1 - \lambda) y$ will leave the agent at least as well off as x or y, that is, indifference curves are convex. Arrow argued that this explanation of convexity, unlike the supposedly cardinalist stories that rely on diminishing marginal utility, is free from cardinal influences.

We follow Grodal's (1974) formalization of the Arrow/Koopmans theory. Assume that the

binary relation $\succeq_{AK}$ on $\mathbb{R}^n_+$ can be represented by a utility function $U: \mathbb{R}^n_+ \to \mathbb{R}$ that takes the form

$$U(z) = \sup_x \int_0^T u(x(t), t)\, d\mu(t) \quad \text{s.t.} \quad \int_0^T x_i(t)\, d\mu(t) \leq z_i, \ i = 1, ..., n,$$

where $x: [0, T] \to \mathbb{R}^n$, $u: \mathbb{R}^n_+ \times [0, T] \to \mathbb{R}$, μ is Lebesgue measure, $t \to u(x(t), t)$ is Lebesgue integrable, $z \in \mathbb{R}^n_+$, and the supremum is taken over all integrable x such that $x_i(t) \geq 0$ for all $t \in [0, T]$, $i = 1, ..., n$.

Theorem 4.3 U is concave and therefore $\succeq_{AK}$ is convex.

We turn to the measurement requirements of the utility function, $\int_0^T u(x(t), t)\, d\mu(t)$, that underlies the above maximization problem. We generalize somewhat.

Definition 4.4 Let Λ denote the set of Lebesgue measurable functions from $[0, T]$ to $\mathbb{R}^n$. A set $Y \subset \Lambda$ is *closed under a.e. replacement* if and only if whenever $z \in \Lambda$ and, for a.e. $t \in [0, T]$, there exists a $x \in Y$ such that $z(t) = x(t)$, then $z \in Y$.

The obvious consumption sets for a consumer with the utility U defined above, e.g., $\{x \in \Lambda: 0 \leq x_i(t) \leq k_i$ for each $i = 1, ..., n$, and each $t \in [0, T]\}$, where each k_i is a nonnegative real number, are closed under a.e. replacement.

Definition 4.5 A function $U: Y \to \mathbb{R}$ satisfies *utility integrability* if and only if, for some positive integer n, (i) Y is closed under a.e. replacement, (ii) there exists a $u: \mathbb{R}^n_+ \times [0, T] \to \mathbb{R}$ with $t \to u(x(t), t)$ integrable such that $U(x) = \int_0^T u(x(t), t)\, d\mu(t)$ for all $x \in Y$, (iii) $|\text{Range } U| > 1$.

For a *psychology* to maximally satisfy utility integrability, it must meet the conditions of Definition 3.1. In the complete case, therefore, $\mathbf{U}$ maximally satisfies utility integrability if it is a largest complete psychology such that each function $U \in \mathbf{U}$ satisfies utility integrability. For the *property* of utility integrability to qualify as cardinal it must be that every psychology $\mathbf{U}$ that maximally satisfies utility integrability consists precisely of the functions that, when restricted to any decisive set A of $\mathbf{U}$, are the

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affine transformations of any function in $U|A$.

Theorem 4.4 Utility integrability is a cardinal property.

Theorems 4.4 and 4.2 together imply that the Arrow/Koopmans theory imposes stricter measurement requirements on agents than does concavity. Utility integrability has range > 2 and intersects concavity; so Theorem 4.2 implies that concavity is weaker than utility integrability. Thus, concavity, despite its preeminent place in preordinal utility theory, is *nearer* to ordinalist standards of measurability.

5. Concave utilitarianism: between cardinal and ordinal interpersonal comparisons

It is common to think that although preference theory should be based solely on ordinal assumptions, utilitarianism is necessarily a cardinal enterprise. More generally, formal social choice theory concentrates on the standard set of measurement classes: ordinal scales, interval or cardinal scales, and, occasionally, ratio scales. But as in the case of individual preference theory, a rich terrain lies between ordinal and cardinal measurement. Specifically, the portions of utilitarianism that have earned the widest consensus rely on a measurement scale that is weaker than cardinality: concave psychologies reproduce the key utilitarian recommendation that income be redistributed from high-utility agents with low marginal utilities of income to low-utility agents with low marginal utilities of income. Not every utilitarian conclusion can be derived from concavity, but the missing results are the anti-egalitarian recommendations that social choice theorists, and even many utilitarians, have often regarded as ethically suspect: e.g., redistributions that harm the lowest-utility agents.

Let $\mathbb{I} = \{1, ..., I\}$ be a finite set of agents and X a set of social choices. As an example, we will later consider the important case where $X \subset \mathbb{R}_{+}^{I}$ is a set of income profiles of the I agents. Our primitive is a psychology U, where each $u \in U$ is a utility function from $\mathbb{I} \times X$ to $\mathbb{R}$. To keep the distinction between social and individual choice in mind, call the current U's *social psychologies*. Following d'Aspremont and Gevers (1977) (see also Suppes (1966) and Hammond (1976)), each $u \in U$ expresses a set of interpersonal welfare comparisons: $u(i, x) \geq u(j, y)$ means that agent i with social choice x is at least as well off as agent j with social choice y. When combined with an aggregation rule, e.g.,

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$\Sigma_{i \in \mathbb{I}} \, u(i, x)$ or $\min_{i \in \mathbb{I}} u(i, x)$, a u generates a binary relation on X. We also suppose, as is standard, that, for each $i \in \mathbb{I}$, $u(i, \cdot)$ represents agent i's individual preferences.

One may view the interpersonal judgments represented by u as the preferences of an individual contemplating what it would be like to be various agents under various social outcomes. A social psychology is then interpreted in the same way as our earlier "individual" psychologies: it lists the functions that accurately depicts preference judgments over a pertinent domain. For social psychologies, the domain is $\mathbb{I} \times X$ rather than an arbitrary set.

We consider only social psychologies U that are complete: U contains only, though not necessarily all, increasing transformations of some $u \in U$. In the language of social choice theory, completeness implies that the minimum measurement standard that any resulting social welfare functional satisfies is *coordinality* (in Hammond (1976)'s terminology) or *ordinal level comparability* (in Roberts (1980b)'s terminology). Since, however, we will not require U to contain all ordinally equivalent utilities, our social psychologies can obey a stronger measurement standard.[1]

To specify social choice rankings for more than a single psychology, we define three collections of psychologies. Let $\mathcal{U}$ denote the set of functions from $\mathbb{I} \times X$ to $\mathbb{R}$, and let $\wp(\mathcal{U})$ denote the set of all subsets of $\mathcal{U}$.

(1) The *cardinal* collection of social psychologies:

$\mathbb{C}_C = \{U \in \wp(\mathcal{U}): u \in U \Rightarrow (v \in U \leftrightarrow$ there exists an increasing affine g such that $v = g \circ u)\}$,

with generic element $U_C \in \mathbb{C}_C$.

(2) The *ordinal* collection of social psychologies:

$\mathbb{C}_O = \{U \in \wp(\mathcal{U}): u \in U \Rightarrow (v \in U \leftrightarrow$ there exists an increasing g such that $v = g \circ u)$, with

generic element $U_O \in \mathbb{C}_O$.

To define the third collection, let $u: \mathbb{I} \times X \to \mathbb{R}$ be *coordinately concave* if and only if X is convex and, for each $i \in \mathbb{I}$, $u(i, \cdot)$ is concave and continuous and let $\mathcal{U}_{CC}$ denote the set of coordinately concave functions.

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[1] Some theories of social choice consider utility transformations that vary as a function of the agent i, and which in our framework translate into social psychologies that are not complete. For example, invariance with respects to individual origins of utility, used by d'Aspremont and Gevers (1977) to axiomatize utilitarianism, admits affine transformations of the form $a \, u(i, \cdot) + b_i$.

(3) The *concave* collection of social psychologies:

$\mathbb{C}_{CC} = \{ \boldsymbol{U} \in \wp(\mathcal{U}): \boldsymbol{U}$ is complete, $\boldsymbol{U} \subset \mathcal{U}_{CC}$, and $\nexists$ a complete $\boldsymbol{V} \supsetneq \boldsymbol{U}$ such that $\boldsymbol{V} \subset \mathcal{U}_{CC}\}$, with

generic element $\boldsymbol{U}_{CC} \in \mathbb{C}_{CC}$.

We consider utilitarian social welfare rankings, which order according to the sum of the utilities $\sum_{i \in \mathbb{I}} u(i, x)$. Given a social psychology $\boldsymbol{U}$, the utilitarian ordering $\succeq_{\boldsymbol{U}} \subset X \times X$ is defined by $x \succeq_{\boldsymbol{U}} y \Leftrightarrow \sum_{i \in \mathbb{I}} v(i, x) \geq \sum_{i \in \mathbb{I}} v(i, y)$ for all $v \in \boldsymbol{U}$. Different social psychologies evidently generate different utilitarian rankings. Call $\succeq_{\boldsymbol{U}_C}$ and $\succeq_{\boldsymbol{U}_{CC}}$ *cardinal* and *concave utilitarianism*, respectively. To avoid an oxymoron, we simply call $\succeq_{\boldsymbol{U}_O}$ *ordinalism*.

Cardinal utilitarianism is the standard utilitarianism: if we rank social choices according to the sum of utilities, the ranking will be unchanged if we apply the same increasing affine transformation to all individual utility functions. But cardinal utilitarianism has the drawback that it rests on a demanding standard of psychological measurement. Ordinalism, at the other extreme, relies on the weakest possible measurement standard. Ordinalism's drawback is that, since it requires any ranking of x over y to pass a larger set of sum-of-utilities tests, it ranks fewer social choices than cardinal utilitarianism. Indeed, ordinalism can often make only a trivial set of rankings. Since a concave social psychology $\boldsymbol{U}_{CC}$ contains the utilities generated by the affine transformations of any $u \in \boldsymbol{U}_{CC}$, and since the set of all ordinal representations of u contains the concave representations, concave utilitarianism produces more rankings than ordinalism but fewer rankings than cardinal utilitarianism. We record this as a theorem.

Theorem 5.1 If $\boldsymbol{U}_C \cap \boldsymbol{U}_{CC} \neq \varnothing$, then $\boldsymbol{U}_C \subset \boldsymbol{U}_{CC}$ and $\succeq_{\boldsymbol{U}_C} \supset \succeq_{\boldsymbol{U}_{CC}}$. If $\boldsymbol{U}_{CC} \cap \boldsymbol{U}_O \neq \varnothing$, then $\boldsymbol{U}_{CC} \subset \boldsymbol{U}_O$ and $\succeq_{\boldsymbol{U}_{CC}} \supset \succeq_{\boldsymbol{U}_O}$.

Concave utilitarianism relaxes the stringent measurement requirements of cardinal utilitarianism, and, as in the case of preference theory, concavity as a psychology precisely models the characteristic neoclassical intuitions about economic psychology. But it remains to see whether concave utilitarianism retains the decisiveness and egalitarianism of cardinal utilitarianism. Specifically, does concave utilitarianism recommend redistributions from rich to poor and does it rank sufficiently many social choices?

The ordering $\succeq_U$ and a $u \in U$ induce an ordering over utility vectors in $\mathbb{R}^I$ which will help characterize $\succeq_U$. Given a social psychology U and $u \in U$, define $R_{u \in U} \subset \mathbb{R}^I \times \mathbb{R}^I$ by

$\mathbf{v} \, R_{u \in U} \, \mathbf{w} \Leftrightarrow \exists \, x, y \in X$ such that $\mathbf{v} = (u(1, x), ..., u(I, x))$, $\mathbf{w} = (u(1, y), ..., u(I, y))$ and $x \succeq_U y$.

We write that $\mathbf{u}$ is in the coordinate range of u, or $\mathbf{u} \in \text{Range}\{u_i\}$ if and only if $\exists \, x \in X$ such that $\mathbf{u} = (u(1, x), ..., u(I, x))$.

Not surprisingly, $\mathbf{v}$ is ranked higher than $\mathbf{w}$ according to any $R_{u \in U_C}$ if and only if the sum of the coordinates in $\mathbf{v}$ is at least as large as the sum in $\mathbf{w}$. Ordinalism is also easily characterized. For any $\mathbf{v} \in \mathbb{R}^I$, let $\mathbf{v}^*$ denote $\mathbf{v}$ with the coordinates placed in increasing order.

Theorem 5.2:

(1) Suppose $u \in U_C$ and $\mathbf{v}, \mathbf{w} \in \text{Range}\{u_i\}$. Then $\mathbf{v} \, R_{u \in U_C} \, \mathbf{w} \Leftrightarrow \sum_{i \in \mathbb{I}} \mathbf{v}_i \geq \sum_{i \in \mathbb{I}} \mathbf{w}_i$.

(2) Suppose $u \in U_O$ and $\mathbf{v}, \mathbf{w} \in \text{Range}\{u_i\}$. Then $\mathbf{v} \, R_{u \in U_O} \, \mathbf{w} \Leftrightarrow \mathbf{v}^* \geq \mathbf{w}^*$.

Result 5.2 (2) says that ordinalism judges $x \succeq_{U_O} y$ if and only if, according to any $u \in U_O$, the ith best off agent under x is at least as well off as the ith best off under y for all $i \in \mathbb{I}$. Thus, modulo utilitarianism's anonymity requirement (which implies that if $\mathbf{u}$ and $\mathbf{v}$ merely rearrange indices without changing the utility level of the ith best off agent for any i then $R_{u \in U_O}$ ranks $\mathbf{u}$ and $\mathbf{v}$ as indifferent), ordinalism recommends only Pareto improvements. So let us say that $\mathbf{u}$ is a (weak) *anonymous Pareto improvement* over $\mathbf{v}$ if $\mathbf{u}^* \geq \mathbf{v}^*$ (sometimes this is called a *Suppes-Sen* improvement). In particular, the redistributive conclusions of cardinal utilitarianism are lost: ordinalism will never recommend a transfer of wealth from a high-utility agent to a low-utility agent. Cardinal utilitarianism of course recommends such transfers if the low-utility agent has a higher marginal utility of wealth (according to any $u \in U_C$) than the high-utility agent.

As for concave utilitarianism, first recall the definition of a least concave function. Let X be convex and let V denote a nonempty (nonsocial) psychology consisting of all the continuous and concave functions $v: X \to \mathbb{R}$ that represent a fixed preference relation $\succeq$ on X. The function $\bar{v} \in V$ is *least concave* if and only if for every $v \in V$ there exists an increasing concave transformation $g: \text{Range } \bar{v} \to \mathbb{R}$ such that $v = g \circ \bar{v}$. Debreu (1976) proved that if V meets the above assumptions then a least concave $\bar{v}$ exists. The definition of least concavity does not apply to a concave social

psychology U_{CC} since the domain of each $u \in U_{CC}$ is not convex. The definition can be generalized in a couple of ways. Let co S denote the convex hull of a set S.

Definition 5.1 Let U_{CC} be a concave social psychology. Then,

(1) $\bar{u} \in U_{CC}$ is *least coordinately concave* or *lcc* if and only if for all $u \in U_{CC}$ there exists an increasing transformation g: Range $\bar{u} \to \mathbb{R}$ such that $u = g \circ \bar{u}$ and, for all $i \in \mathbb{I}$, $g \mid \text{Range } \bar{u}(i, \cdot)$ is concave.

(2) $\hat{u}$ is *strongly lcc* if and only if for all $u \in U_{CC}$ there exists an increasing transformation $g : \text{co}(\text{Range } \hat{u}) \to \mathbb{R}$ such that $u = g \circ \hat{u}$ and g is concave.

Debreu's existence theorem does not itself imply that an lcc element in U_{CC} exists, but his proof extends easily. Strongly lcc utilities are in some respects the more powerful tool, but Example 5.2, at the end of this section, shows that a concave social psychology can fail to have a strongly lcc element. If $\bar{u} \in U_{CC}$ is lcc and any $u \in U_{CC}$ satisfies the range condition that $\bigcup_{i \in \mathbb{I}} \text{Int } u(i, X)$ is convex, then $\bar{u}$ is strongly lcc. The range condition says that the range of each agent's utility either directly or indirectly overlaps the range of any other agent's utility. We discuss what this means and implies when we come to Example 5.2.

 For our purposes, the following sufficient condition will adequately distinguish concave utilitarianism from cardinal utilitarianism and ordinalism. A simple characterization of concave utilitarianism is possible when strongly lcc utilities exist, as we also explain later.

Theorem 5.3 Suppose $\bar{u} \in U_{CC}$ is lcc and $\mathbf{v}, \mathbf{w} \in \text{Range}\{\bar{u}_i\}$. Then $\mathbf{v} \in \text{co}\{\mathbf{u} \in \text{Range}\{\bar{u}_i\}: \mathbf{u}^* \geq \mathbf{w}^*\}$ implies $\mathbf{v} \, R_{\bar{u} \in U_{CC}} \, \mathbf{w}$.

Hence, if one examines $(\bar{u}(1, x), ..., \bar{u}(I, x))^*$ and $(\bar{u}(1, y), ..., \bar{u}(I, y))^*$ and finds that the first vector is a convex combination of Pareto improvements of the second, then $x \geq_{U_{CC}} y$. The proof to follow needs only notational changes to show that if $\hat{u}$ is strongly lcc, then the weaker condition $\mathbf{v} \in \text{co}\{\mathbf{u} \in \mathbb{R}^I: \mathbf{u}^* \geq \mathbf{w}^*\}$ will imply $\mathbf{v} \, R_{\hat{u} \in U_{CC}} \, \mathbf{w}$.

Proof of Theorem 5.3: Given that $\mathbf{v} \in \text{co}\{\mathbf{u} \in \text{Range}\{\bar{u}_i\}: \mathbf{u}^* \geq \mathbf{w}^*\}$, there exist $\lambda^1, ..., \lambda^m \in [0, 1]$ and

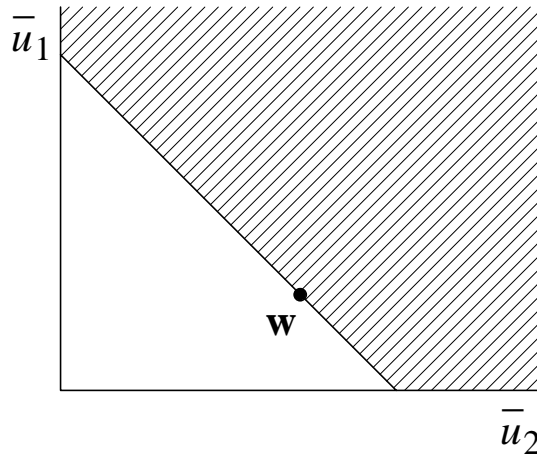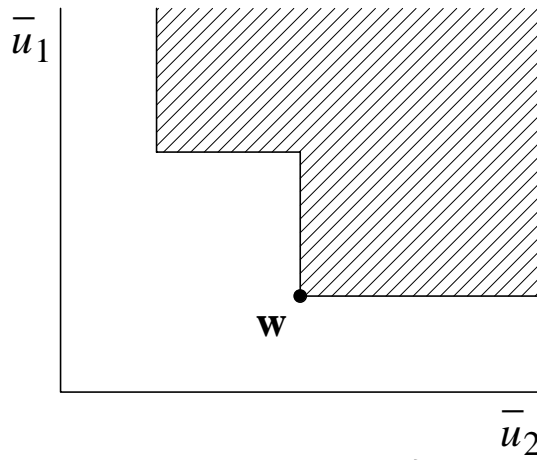Figure 1: $\{\mathbf{v}: \mathbf{v} \, R_{\overline{u} \in U_C} \, \mathbf{w}\}$

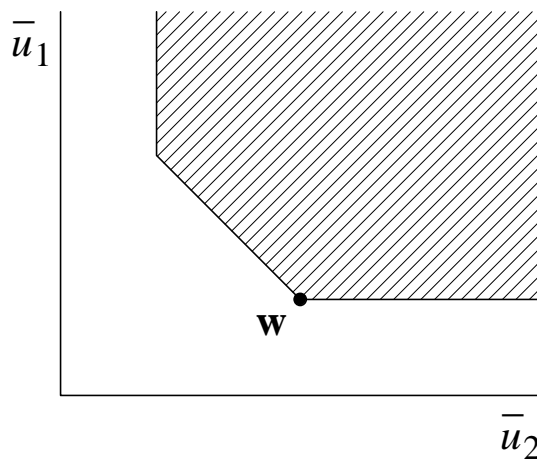Figure 2: $\{\mathbf{v}: \mathbf{v} \, R_{\overline{u} \in U_O} \, \mathbf{w}\}$

Figure 3: $\{\mathbf{v}: \mathbf{v} \, R_{\overline{u} \in U_{CV}} \, \mathbf{w}\}$

$\mathbf{z}^1, ..., \mathbf{z}^m \in \text{Range}\{\bar{u}_i\}$ such that $\sum_{k=1}^m \lambda^k = 1$, $\mathbf{v} = \sum_{k=1}^m \lambda^k \mathbf{z}^k$, and each $\mathbf{z}^{k*} \geq \mathbf{w}^*$. For any $g: \text{Range } \bar{u} \rightarrow \mathbb{R}$ such that $g \mid \text{Range } \bar{u}(i, \cdot)$ is concave and increasing for each $i \in \mathbb{I}$,

$$\sum_{i \in \mathbb{I}} g(v_i) = \sum_{i \in \mathbb{I}} g(\sum_{k=1}^m \lambda_i^k \mathbf{z}_i^k) \geq \sum_{i \in \mathbb{I}} [\sum_{k=1}^m \lambda_i^k g(\mathbf{z}_i^k)] \geq \sum_{i \in \mathbb{I}} [\sum_{k=1}^m \lambda^k g(w_i)] = \sum_{i \in \mathbb{I}} g(w_i).$$

The first inequality follows from the concavity of $g \mid \text{Range } \bar{u}(i, \cdot)$ and the second from the fact that $\mathbf{z}^{k*} \geq \mathbf{w}^*$. Since $\mathbf{v}, \mathbf{w} \in \text{Range}\{\bar{u}_i\}$, there exist $x, y \in X$ such that $\mathbf{v} = (\bar{u}(1, x), ..., \bar{u}(I, x))$ and $\mathbf{w} = (\bar{u}(1, y), ..., \bar{u}(I, y))$. Since $\bar{u}$ is lcc, for each $u \in \boldsymbol{U}_{CC}$ there is a g meeting the above assumptions such that $u = g \circ \bar{u}$. Hence $\sum_{i \in \mathbb{I}} u(i, x) \geq \sum_{i \in \mathbb{I}} u(i, y)$, $x \succeq_{\boldsymbol{U}_{CC}} y$, and $\mathbf{v} \, R_{\bar{u} \in \boldsymbol{U}_{CC}} \, \mathbf{w}$. ∎

To see the relations $R_{u \in \boldsymbol{U}_C}$, $R_{u \in \boldsymbol{U}_O}$, and $R_{\bar{u} \in \boldsymbol{U}_{CC}}$ graphically, let $I = 2$, and suppose $\boldsymbol{U}_C, \boldsymbol{U}_O, \boldsymbol{U}_{CC}$ have a common element $\bar{u}$, which is lcc for $\boldsymbol{U}_{CC}$. Fix some $\mathbf{w} \in \text{Range}\{\bar{u}_i\}$ and assume that if $\mathbf{v}^* \geq \mathbf{w}^*$ then $\mathbf{v} \in \text{Range}\{\bar{u}_i\}$. The upper contour sets $\{\mathbf{v}: \mathbf{v} \, R \, \mathbf{w}\}$ for $R \in \{R_{u \in \boldsymbol{U}_C}, R_{u \in \boldsymbol{U}_O}, R_{\bar{u} \in \boldsymbol{U}_{CC}}\}$ are pictured in Figures 1 through 3. In this two-dimensional case, $\{\mathbf{v}: \mathbf{v} \, R_{\bar{u} \in \boldsymbol{U}_{CC}} \, \mathbf{w}\}$ exactly coincides with the convex hull given in Theorem 5.3 (see Theorem 5.4 below).

Figures 2 and 3 indicate that concave utilitarianism ranks a richer set of utility vectors than does ordinalism. Not only are the anonymous Pareto improvements ranked superior to $\mathbf{w}$, but any redistribution of utility (with no net loss) from the utility-rich agent to the utility-poor agent is also superior. On the other hand, in contrast to cardinal utilitarianism, concave utilitarianism does not declare any utility vector that lowers the welfare of the worse-off agent to be superior to $\mathbf{w}$. Indeed, changes from $\mathbf{w}$ that harm the worse-off agent are the only orderings made by $R_{\bar{u} \in \boldsymbol{U}_C}$ but not by $R_{\bar{u} \in \boldsymbol{U}_{CC}}$. Concave utilitarianism thus stakes out an egalitarian compromise between standard (cardinal) utilitarianism and the narrow Paretian judgments made by ordinalism.

Cardinal utilitarianism has long been criticized for ignoring welfare levels and in particular for recommending that low-utility agents should undergo arbitrarily large utility losses whenever those losses lead to greater utility gains for high-utility agents. Concave utilitarianism does not suffer from this defect. Moreover, concave utilitarianism does not arrive at its prohibition on harming the least well-off by invoking an equity axiom (as in Hammond (1976)). The egalitarianism flows directly from the psychological content of concavity.

To characterize this egalitarianism more precisely, consider the important *Pigou-Dalton principle* (see, e.g., Moulin (1988)). Given a utility $u: \mathbb{I} \times X \rightarrow \mathbb{R}$, an ordering R over utility vectors in

Range $\{u_i\}$ satisfies the Pigou-Dalton principle if, for any $j, k \in \mathbb{I}$ and any $\mathbf{v}, \mathbf{w} \in$ Range $\{u_i\}$ such that $\mathbf{v}_m = \mathbf{w}_m$ for $m \notin \{j, k\}$:

(5.1) $\qquad\qquad \mathbf{v}_j + \mathbf{v}_k \geq \mathbf{w}_j + \mathbf{w}_k$ and $\min\{\mathbf{v}_j, \mathbf{v}_k\} \geq \min\{\mathbf{w}_j, \mathbf{w}_k\} \Rightarrow \mathbf{v} \, R \, \mathbf{w}$.

In words, R satisfies Pigou-Dalton if it recommends a change in utility levels affecting a pair of individuals that both increases the sum of utilities and does not leave the lower-utility person after the change worse off than the lower-utility person before the change.

Figure 3 suggests that concave utilitarianism should satisfy the Pigou-Dalton principle. The implication in (5.1) goes both ways, in fact. In this sense, concave utilitarianism characterizes Pigou-Dalton.

Theorem 5.4 Suppose $\hat{u}$ is a strongly lcc element of $\boldsymbol{U}_{CC}$, and let $\mathbf{v}, \mathbf{w} \in$ Range $\{\hat{u}_i\}$ be identical in all but two coordinates j and k. Then,

$$\mathbf{v}_j + \mathbf{v}_k \geq \mathbf{w}_j + \mathbf{w}_k \text{ and } \min\{\mathbf{v}_j, \mathbf{v}_k\} \geq \min\{\mathbf{w}_j, \mathbf{w}_k\} \Leftrightarrow \mathbf{v} \ R_{\hat{u} \in U_{CC}} \ \mathbf{w}.$$

We illustrate cardinal and concave utilitarianism's common ground, and the paucity of ordinalist rankings, with the classic problem of constructing a welfare ranking of income distributions.

Example 5.1. The set of social choices X on which social psychologies are defined is then $\mathbb{R}_+^I$, and a policymaker with aggregate income $\omega > 0$ to distribute will choose from $\Omega = \{x \in \mathbb{R}_+^I : \sum_{i \in \mathbb{I}} x_i = \omega\}$. Consider social psychologies $\boldsymbol{U}_C$, $\boldsymbol{U}_O$, and $\boldsymbol{U}_{CC}$ that have a common element u such that each $u(i, \cdot)$ is increasing, differentiable, and strictly concave in the ith coordinate of $\mathbb{R}_+^I$ and constant in the remaining $I - 1$ coordinates. (We suppress notation of all but the ith coordinate of $\mathbb{R}_+^I$ when writing $u(i, \cdot)$.) Let $\boldsymbol{U}_{CC}$ have the strongly lcc element $\hat{u}$.

First, suppose all individuals are identical: $u(i, \cdot) = u(j, \cdot)$ for any $i, j \in \mathbb{I}$. Although the cardinal and concave utilitarian orderings $\succeq_{U_C}$ and $\succeq_{U_{CC}}$ are not identical, they both rank $e = ((1/n)\omega, ..., (1/n)\omega)$ above any other distribution in Ω. For $\succeq_{U_O}$, in contrast, any unequal $x \in \Omega$ is unranked relative to e.

Next, allow $u(i, \cdot) \neq u(j, \cdot)$. As in the identical-agent case, $\succeq_{U_C}$ ranks the x such that $D_x u(i, x_i) = D_x u(j, x_j)$ to be superior to any other point in Ω. While $\succeq_{U_{CC}}$ need not make the same ordering in this

case, Theorem 5.4 implies, given some base distribution x, that some transfer of income from agent j to agent i will be superior according to $\succeq_{U_{CC}}$ if

$$D_x \hat{u}(i, x_i) > D_x \hat{u}(j, x_j) \text{ and } \hat{u}(i, x_i) < \hat{u}(j, x_j).$$

Rather than the equal derivative condition, any x such that $D\,\hat{u}(i, x)^*$ is increasing in i is undominated according to $\succeq_{U_{CC}}$ (where $\hat{u}(\cdot, x)^*$ denotes $\hat{u}(\cdot, x)$ with the coordinates arranged in increasing order.) Ordinalism again makes very few orderings: given some $x \in \Omega$, any change affecting only two agents i and j that raises i's welfare and hence lowers j's is not ranked relative to x. For instance, any point in Ω that leads to equal utility levels is always unranked relative to all other points in Ω. ∎

Although the cardinal and concave utilitarian orderings are not identical, it is concave rather than cardinal utilitarianism that provides the better rationale for classic utilitarian policy recommendations. The connection to the Pigou-Dalton principle is revealing. Pigou, the primary architect of neoclassical welfare economics, considered himself both a cardinalist and a utilitarian: at least in theory, each individual in society has a cardinal and interpersonally comparable utility function, and policy choices should be evaluated by summing the utility numbers that these utilities assign to policies. But Pigou (1932) recognized that analysts have no easy access to cardinal utility information, and argued therefore that welfare economics should recommend only policy changes that would be validated independently of that information. For instance, a policy that lowers the income of low-utility individual by a $1 and raises the income of a high-utility individual by x, where $x > 1$, could not be unambiguously recommended, no matter how large x is, since the marginal utility of income might be very small at high utility levels. On the other hand, if the low-utility agent gains by x, $x > 1$, and the high-utility agent loses by $1, then any sum-of-utilities test will recommend the policy change if agents are represented by concave utilities (and assuming we always hold to the same ordinal classification of agents into high-utility and low-utility individuals). Concave utilitarianism is in substance identical to Pigou's position, but it elevates the theoretical status of not having cardinal information about individuals. According to concavity as a social psychology, there is no unknown but nevertheless real cardinal utility function lurking out there – the most detailed information that theory might in principle provide about interpersonally comparable utilities is that they are concave.

In closing, we return to the characterization of $\succeq_{U_{CC}}$. Roberts (1980a) in effect brings up this

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issue and suggests that least concave utilities are the appropriate tool. (We should mention that Roberts does not take sets of concave utility functions as primitive, but argues, like Pigou, that such a set is useful only insofar as it is assumed to contain the one "true" cardinal welfare function.) Roberts slightly misstates his characterization result and does not formally define least concavity, but the proposition that Roberts has in mind, relying on a theorem of Hanoch and Levy (1969), is, in our notation:

(5.2) $$\mathbf{v} \; R_{\bar{u} \in U_{CC}} \; \mathbf{w} \leftrightarrow \sum_{i=1}^{n} \mathbf{v}_i^* \geq \sum_{i=1}^{n} \mathbf{w}_i^* \text{ for all } n \in \mathbb{I}.$$

where $\mathbf{v}, \mathbf{w} \in \text{Range} \{ \bar{u}_i \}$. Example 5.2 below shows that if we take $\bar{u}$ to be lcc, the equivalence (5.2) does not hold. If $\bar{u}$ is strongly lcc, on the other hand, then one may readily establish (5.2). But Example 5.2 also shows that concave social psychologies do not always have strongly lcc elements, in which case, of course, for $u \in U_{CC}$, $\bigcup_i \text{Int } u(i, X)$ cannot be convex.

Example 5.2. Let $X = [0, 1]$ and $I = 2$. Let U_{CC} contain the lcc utility $\bar{u}$ defined by

$$\bar{u}(1, x) = 2x, \quad \bar{u}(2, x) = -x + 4.$$

We then have $(\bar{u}(1, 1), \bar{u}(2, 1)) = (2, 3)$ and $(\bar{u}(1, 0), \bar{u}(2, 0)) = (0, 4)$. So $(2, 3) \; R_{U_{CC}}(\bar{u}) \; (0, 4)$ according to (5.2) and therefore $1 \succeq_{U_{CC}} 0$. But the $u \in U_{CC}$ defined by

$$u(1, x) = 2x, \quad u(2, x) = -3x + 6$$

yields $\sum_{i=1}^{2} u(i, 0) > \sum_{i=1}^{2} u(i, 1)$, and so it cannot be that $(2, 3) \; R_{\bar{u} \in U_{CC}} \; (0, 4)$.

To see that U_{CC} does not have a strongly lcc element, suppose to the contrary that $\hat{u}$ is strongly lcc. Both $\hat{u}(1, \cdot)$ and $\hat{u}(2, \cdot)$ must then be affine, i.e., $\hat{u}(1, x) = ax + b$ and $\hat{u}(2, x) = cx + d$, for some $a > 0$ and $c < 0$, where $\hat{u}(2, x) > \hat{u}(1, x)$ for all x. The u given by

$$u(1, x) = ax + b, \quad u(2, x) = \tilde{c}x + d,$$

is then also an element of U_{CC} for all $\tilde{c}$ sufficiently close to c. But if $\tilde{c} < c$ and g satisfies $u = g \circ \hat{u}$, then g cannot be concave, since g must have slope $= 1$ on $u(1, X)$ and slope > 1 on $u(2, X)$.

Observe that $\succeq_{U_{CC}} = \succeq_{U_o}$: no pair of points in X is ranked by $\succeq_{U_{CC}}$. The reason is that since the interiors of $u(1, X)$ and $u(2, X)$ are disjoint, we may, given some $u \in U_{CC}$, multiply $u(1, \cdot)$ and $u(2, \cdot)$ by arbitrary positive weights (and choose constant terms so that $u(2, x) > u(1, x)$ is preserved at each x) and arrive at another element of U_{CC}. ∎

The fact that the ranges of the two utility functions in the above example have disjoint interiors means that the concavity of the utilities in U_{CC} has no bite. In general, whenever $\bigcup_i \operatorname{Int} u(i, X)$ is not convex, a u in U_{CC} can be assembled in which the weight on some subset of agent utility functions is arbitrary relative to the weight on some other subset. In the extreme case where, for each i and j, $\operatorname{Int} u(i, X) \cap \operatorname{Int} u(j, X) = \varnothing$ (as above), then $\succeq_{U_{CC}}$ and $\succeq_{U_o}$ coincide.

6. Acyclic and transitive domains for properties

An example of a set of properties on which the "strictly weaker than" relation $\succeq_{SW}$ (see Definition 4.3) cycles will illustrate the intransitivity problem.

Example 6.1 Let X be a nonempty open convex subset of $\mathbb{R}^n$ and let $\succeq_1$, $\succeq_2$, and $\succeq_3$ be distinct complete binary relations on X. Suppose the relations $\succeq_1$ and $\succeq_2$ each have concave and nonconstant utility representations, u_1 for $\succeq_1$ and u_2 for $\succeq_2$, and suppose $\succeq_3$ has the utility representation u_3. Define the properties α, β, and γ as follows:

$$\alpha = \{u \in \mathscr{F}_X : u \text{ is a concave representation of } \succeq_1 \text{ or a concave representation of } \succeq_2\},$$

$$\beta = \{u \in \mathscr{F}_X : u \text{ is a continuous representation of } \succeq_2 \text{ or } u = u_3\},$$

$$\gamma = \{u \in \mathscr{F}_X : u \text{ is a positive linear transformation of } u_1 \text{ or } u_3\}.$$

It is immediate that β is strictly weaker than α, γ is strictly weaker than β, and α is strictly weaker than γ. ■

From the vantage point of trying to specify a well-behaved compromise between ordinality and cardinality, Example 6.1 depicts a worst-case intransitivity. Property α is strictly weaker than any cardinal property δ such that $\alpha \cap \delta \neq \varnothing$, and any cardinal property δ such that $\gamma \cap \delta \neq \varnothing$ is strictly weaker than γ (note that "linear" rather than "affine" appears in the definition of γ). Yet, one can move via $\succeq_{SW}$ from α to γ.

Still, the cycle here hinges on the fact that, since a ranking of properties P and Q depends only on the utility functions in $P \cap Q$, P can be weaker than Q even though Q may contain a comparatively large set of utilities for preferences not represented by any of the utilities in $P \cap Q$. To construct an acyclic domain of properties, therefore, properties must include only sets of utility representations that somehow treat different preference relations symmetrically. One way to proceed is to employ sets of

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utility transformations, similarly but not identical to the way they are used in measurement theory.

For ease of presentation, we restrict ourselves to complete psychologies. Henceforth, when we say that the psychology U maximally satisfies property P, we mean that (1) U is complete, (2) for all $u \in U$, u satisfies P, and (3) there does not exist a complete psychology $V \supsetneq U$ such that each $v \in V$ satisfies P. Given this restriction, P is no stronger than Q if and only if, for all U that maximally satisfy P and all V that maximally satisfy Q, $U \cap V \neq \varnothing$ implies $U \supset V$.

Definition 6.1 The psychology U has a *generator with respect to a set of transformations $F \subset \mathscr{F}_{\mathbb{R}}$* if and only if there is a $u \in U$ such that: $v \in U \Leftrightarrow$ there is a $f \in F$ such that $v = f \circ u$.

Definition 6.2 A property P is *transformational* if and only if there exists a set of transformations F_P such that, for all psychologies U that maximally satisfy P, U has a generator with respect to F_P. The set F_P is called a set of *P transformations*. $\mathscr{P}_T$ will denote the set of transformational properties.

By associating sets of utility transformations with properties, we are taking a step towards the traditional model of measurement classes. But a transformational property P differs in that the transformations in F_P must be applied to the generator of a psychology that maximally satisfies P rather than an arbitrary utility function; otherwise the utility functions generated need not satisfy P or one might not generate all of the functions that satisfy P. As an example, consider the property P_{CVC} consisting of the concave and continuous functions on some convex set X. The set $F_{CV} \subset \mathscr{F}_{\mathbb{R}}$ of all increasing concave transformations is a set of P_{CVC} transformations. Given a U maximally satisfying P_{CVC}, any of the "least concave" utility representations of $\succsim_U$ (see Debreu (1976) and section 5) may serve as a generator with respect to F_{CV}. If we apply any $f \in F_{CV}$ to a function u satisfying P_{CVC} we generate another function satisfying P_{CVC}. But in order to generate all of the functions in P_{CVC} that agree with u, we must apply the $f \in F_{CV}$ to a least concave utility. (Of course, if we apply F_{CV} to a nonconcave utility then some of the functions generated will not satisfy P_{CVC}.) This example illustrates that transformations by themselves do not define properties; properties therefore must still be defined as sets of utility functions.

Although the "strictly weaker than" relation can cycle on some sets of transformational

properties, transformational properties that are "comparable" to some cardinal property cannot circle around cardinal properties in the manner of Example 6.1.

Definition 6.3 A set of properties $\mathcal{P}$ is *acyclic with respect to cardinality* if there does not exist a finite set of properties $\{P_1, ..., P_n\} \subset \mathcal{P}$ such that P_1 is weaker than some cardinal property, some cardinal property is weaker than P_n, and, for $1 < i \le n$, P_i is weaker than P_{i-1}.

Definition 6.4 Property P is *comparable* to property Q if there exists some U maximally satisfying P and some V maximally satisfying Q such that $U \cap V \ne \varnothing$ and either $U \subset V$ or $U \supset V$.

Comparability is relatively weak: P and Q can be comparable even if it is neither the case that P is no stronger than Q nor the case that Q is no stronger than P.

Theorem 6.1 Any set of properties $\mathcal{P}_C \subset \mathcal{P}_T$ such that each $P \in \mathcal{P}_C$ is comparable to some cardinal property is acyclic with respect to cardinality.

Every property we discuss in this paper (except β in Example 6.1) is comparable to some cardinal property.

Proof of Theorem 6.1: Suppose there is a $\{P_1, ..., P_n\} \subset \mathcal{P}_C$ such that P_1 is weaker than some cardinal property, some cardinal property is weaker than P_n, and, for $1 < i \le n$, P_i is weaker than P_{i-1}. Let P_k be the element of $\{P_2, ..., P_n\}$ with the smallest index such that there exist U_{Q_k} and U_{P_k} meeting the conditions (1) U_{Q_k} maximally satisfies a cardinal property Q_k, (2) U_{P_k} maximally satisfies P_k, and (3) $U_{Q_k} \supset U_{P_k}$. Given the comparability assumption, there exist $U_{Q_{k-1}}$ and $U_{P_{k-1}}$ such that (a) $U_{Q_{k-1}}$ maximally satisfies a cardinal property Q_{k-1}, (b) $U_{P_{k-1}}$ maximally satisfies P_{k-1}, and (c) $U_{Q_{k-1}} \subset U_{P_{k-1}}$. (If $P_k = P_2$, this conclusion follows from our supposition on P_2 and in the other cases from the fact that P_k has the smallest index.) Let F_{P_k} be a set of P_k transformations and let $u_{P_k} \in U_{P_k}$ be a generator for U_{P_k} with respect to F_{P_k}. Since Q_k is cardinal, the set of Q_k transformations is $F_{IA} = \{f \in \mathscr{F}_{\mathbb{R}} : f$ is an increasing affine transformation$\}$ and each $u \in U_{Q_k}$ is a generator for U_{Q_k} with respect to F_{IA}. Hence

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u_{P_k} is a generator for $\boldsymbol{U}_{Q_k}$ with respect to F_{IA}. Since for each $f \in F_{P_k}, f \circ u_{P_k} \in \boldsymbol{U}_{P_k}$ and hence $f \circ u_{P_k}$ $\in \boldsymbol{U}_{Q_k}$, $F_{P_k} \subset F_{IA}$. Using the same reasoning, we may also infer that $F_{P_{k-1}} \supset F_{IA}$.

On the other hand, since P_k is weaker than P_{k-1}, there exists a $\hat{\boldsymbol{U}}_{P_k}$ that maximally satisfies P_k and a $\hat{\boldsymbol{U}}_{P_{k-1}}$ that maximally satisfies P_{k-1} such that $\hat{\boldsymbol{U}}_{P_k} \supsetneq \hat{\boldsymbol{U}}_{P_{k-1}}$. Let $\hat{u}_{P_k}$ be a generator for $\hat{\boldsymbol{U}}_{P_k}$ with respect to F_{P_k} and let $\hat{u}_{P_{k-1}}$ be a generator for $\hat{\boldsymbol{U}}_{P_{k-1}}$ with respect to $F_{P_{k-1}}$. For each $u_{P_k} \in \hat{\boldsymbol{U}}_{P_k}$, there exists $f_{u_{P_k}} \in F_{P_k}$ such that $f_{u_{P_k}} \circ \hat{u}_{P_k} = u_{P_k}$. Since $F_{IA} \supset F_{P_k}$, each $f_{u_{P_k}}$ is increasing and affine and hence so is $f_{u_{P_k}}^{-1}$. Since $\hat{\boldsymbol{U}}_{P_k} \supsetneq \hat{\boldsymbol{U}}_{P_{k-1}}$, $\hat{u}_{P_{k-1}} \in \hat{\boldsymbol{U}}_{P_k}$ and so there is a $f_{\hat{u}_{P_{k-1}}} \in F_{P_k}$ such that $f_{\hat{u}_{P_{k-1}}} \circ \hat{u}_{P_k} =$ $\hat{u}_{P_{k-1}}$. Hence $\hat{u}_{P_k} = f_{\hat{u}_{P_{k-1}}}^{-1} \circ \hat{u}_{P_{k-1}}$. We therefore have, for any $u_{P_k} \in \hat{\boldsymbol{U}}_{P_k}$, $u_{P_k} = f_{u_{P_k}} \circ f_{\hat{u}_{P_k}}^{-1} \circ \hat{u}_{P-1}$. Since $f_{u_{P_k}} \circ f_{\hat{u}_{P_k}}^{-1}$ is increasing and affine and $F_{P_{k-1}} \supset F_{IA}$, each u_{P_k} is an element of $\boldsymbol{U}_{P_{k-1}}$, which contradicts $\hat{\boldsymbol{U}}_{P_k} \supsetneq \hat{\boldsymbol{U}}_{P_{k-1}}$. ∎

The assumptions of Theorem 6.1 eliminate the most disturbing cases where the "weaker than" or the "strictly weaker than" relations cycle. Still, it is illuminating to investigate conditions under which some intuitive definition of "weaker than" will be transitive. A few hurdles stand in the way. First, a property P will vacuously be no stronger than property Q if P and Q do not intersect. Hence, one cannot expect $\succeq_{NS}$ to be transitive even on a well-behaved set of properties such as $\mathcal{P}_T$. To illustrate, let $X = \mathbb{R}$ and suppose P is the transformational property of mapping a set $A \subset \mathbb{R}$ onto the interval $[0, 1]$ and Q is the transformational property of mapping A onto $[2, 3]$. Vacuously, P is no stronger than Q (and Q is no stronger than P). To generate an intransitivity, let S be the property of mapping $B \subset \mathbb{R}$ onto $[2, 3]$, where $A \cap B = \varnothing$. Once again, vacuously, Q is no stronger than S, but obviously it is not the case that P is no stronger than S. No interesting domain restriction can eliminate such intransitivities.

Second, although the relations $\succeq_W$ or $\succeq_{SW}$ do not suffer from exactly the same vacuity that afflicts $\succeq_{NS}$, similar problems appear. For example, when $P \succeq_W Q$ and $Q \succeq_W S$ hold, in which case P is comparable to Q and Q is comparable to S, P can nevertheless not be comparable to S, implying that $P \succeq_W S$ cannot hold. One might at least hope for the acyclicity of $\succeq_W$ on a well-behaved domain. The following example shows, however, that $\succeq_W$ or $\succeq_{SW}$ can cycle on $\mathcal{P}_T$.

Example 6.2 For some nonempty open set $X \subset \mathbb{R}_+^n$, let $\succeq_1$, $\succeq_2$, and $\succeq_3$ be distinct complete binary relations on X, each of which has a concave utility representation. For $i = 1, 2, 3$, let $\underline{u}_i$ denote one such

representation. Let $F_{CV} \subset \mathscr{F}_\mathbb{R}$ be the set of increasing concave transformations. Given some $f' \in F_{CV}$ that is strictly concave on the range of each $\underline{u}_i$, define $\bar{u}_i = f' \circ \underline{u}_i$. Define the properties $\alpha, \beta,$ and γ as follows:

$$\alpha = \{u \in \mathscr{F}_X : u = f \circ \bar{u}_1 \text{ or } u = f \circ \underline{u}_3 \text{ for some } f \in F_{P_{CC}}\},$$

$$\beta = \{u \in \mathscr{F}_X : u = f \circ \bar{u}_2 \text{ or } u = f \circ \underline{u}_1 \text{ for some } f \in F_{P_{CC}}\},$$

$$\gamma = \{u \in \mathscr{F}_X : u = f \circ \bar{u}_3 \text{ or } u = f \circ \underline{u}_2 \text{ for some } f \in F_{P_{CC}}\}.$$

Each of these properties is transformational: for all three, F_{CV} may serve as the set of transformations, $\bar{u}_1$ and $\underline{u}_3$ are generators for α, $\bar{u}_2$ and $\underline{u}_1$ for β, and $\bar{u}_3$ and $\underline{u}_2$ for γ. Yet we have $\beta \succeq_{SW} \alpha$, $\gamma \succeq_{SW} \beta$, and $\alpha \succeq_{SW} \gamma$. ∎

The key to Example 6.2 is that while each property is ranked relative to the other two, no pair of psychologies that maximally satisfy distinct properties have a generator in common. Thus one property may be weaker than another even though they share the same set of transformations. One way to proceed, therefore, is to declare that when a pair of properties *never* have a generator in common they are not ranked.

Definition 6.5 A property P is *uniquely transformational* if and only if there exists one and only one set of transformations F_P, called the *unique P-transformations*, such that, for all psychologies U that maximally satisfy P, U has a generator with respect to F_P. Let $\mathscr{P}_{UT} \subset \mathscr{P}_T$ denote the set of uniquely transformational properties.

Definition 6.6 The relation $\succeq_{NS}^* \subset \mathscr{P}_{UT} \times \mathscr{P}_{UT}$ is defined by $P \succeq_{NS}^* Q \Leftrightarrow$ whenever U maximally satisfies P, V maximally satisfies Q, and there exists a $w \in U \cap V$ that is a generator for U with respect to the unique P-transformations and a generator for V with respect to the unique Q-transformations, then $U \supset V$. Let $\succeq_W^* \subset \mathscr{P}_{UT} \times \mathscr{P}_{UT}$ be defined by $P \succeq_W^* Q \Leftrightarrow P \succeq_{NS}^* Q$ and not $Q \succeq_{NS}^* P$.

Theorem 6.2 The relation $\succeq_W^*$ is acyclic.

Proof: Suppose to the contrary that there exists a finite set $\{P_1, ..., P_n\} \subset \mathscr{P}_{UT}$ such that $P_1 \succeq_W^* P_n$ and,

for $1 < i \le n$, $P_i \succeq_W^* P_{i-1}$. For $i \in \{2, ..., n\}$, there is a U_{P_i} maximally satisfying P_i and a $U_{P_{i-1}}$ maximally satisfying P_{i-1} such that (i) $U_{P_i} \supsetneq U_{P_{i-1}}$ and (ii) there exists a $w \in U_{P_i} \cap U_{P_{i-1}}$ that is a generator for U_{P_i} with respect to the set of unique P_i-transformations, say F_{P_i}, and a generator for $U_{P_{i-1}}$ with respect to the set of unique P_{i-1}-transformations, say $F_{P_{i-1}}$. For each $f \in F_{P_{i-1}}$, there exists $u_{P_{i-1}} \in U_{P_{i-1}}$ such that $f \circ w = u_{P_{i-1}}$. Since $U_{P_i} \supset U_{P_{i-1}}$, $f \in F_{p_i}$. Since $U_{P_i} \supsetneq U_{P_{i-1}}$, there exists a $f' \in F_{P_i}$ such that $f' \circ w \notin U_{P_{i-1}}$. So $f' \notin F_{P_{i-1}}$ and therefore $F_{P_i} \supsetneq F_{P_{i-1}}$. Repeating this argument for U_{P_1} and U_{P_n}, we have $F_{P_1} \supsetneq F_{P_n}$. So $F_{P_n} \supsetneq F_{P_n}$, a contradiction. ∎

Given Theorem 6.2, it is trivial to construct a transitive ordering, namely the transitive closure of $\succeq_W^*$, say $\overline{\succeq}_W^*$. That is, let $\succeq_{W\text{-}cl}^* \subset \mathcal{P}_{UT} \times \mathcal{P}_{UT}$ be defined by $P \overline{\succeq}_W^* Q$ if and only if $P \succeq_W^* Q$ or there exists a finite set of transformational properties $\{S_1, ..., S_n\}$ such that $P \succeq_W^* S_1 \succeq_W^* \cdots \succeq_W^* S_n \succeq_W^* Q$. The acyclicity of $\succeq_W^*$ ensures that $\overline{\succeq}_W^*$ is asymmetric, and so $\overline{\succeq}_W^*$ does not reverse any of the orderings in $\succeq_W^*$.

7. Conclusion

The orderings of properties analyzed in this paper gauge the strength of assumptions in utility analysis. If property or assumption P is weaker than Q then P imposes less demanding measurement requirements than Q. We illustrated the usefulness of this gauge by examining different arguments for the convexity of preferences.

I argued in section 4 that the Arrow/Koopmans explanation of convexity relies on utility functions that are cardinal. This position is open to objection. Although the Arrow-Koopmans theory presupposes that an agent's ordinal preferences have a utility representation taking the integral form, one might argue that the integral utility functions have no special significance; other utility representations for the same preferences do not take the integral form. Furthermore, one can impose assumptions on ordinal preferences that imply the existence of an integral utility representation (Grodal and Mertens (1968), Vind (1969) – just as there are axioms on preferences over finite numbers of goods that imply that additively separable utility representations exist (Debreu (1960)). One might conclude, therefore, that the Arrow-Koopmans theory gives genuinely ordinalist foundations for the convexity of preferences. But the key axiom needed for the existence of integral utility functions is an independence postulate, just as an independence assumption underlies the existence of additively separability utilities.

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Such assumptions – which assert that preferences over a subset of the time-dated goods are independent of the consumption level of goods with different dates – are ordinal but rely on a cardinal psychology. Independence assumptions, even when posed as a restriction on ordinal preferences, are motivated by the idea that consumption at one date does not affect the satisfaction derived from consumption at different dates. Thus, it will only be the additively separable utility functions (or in the infinite case, the integral functions) that are fully psychologically accurate; all other ordinally equivalent functions do not express the psychological presuppositions that make independence plausible.

Our orderings of properties of utility can shed light on the measurement requirements of rationales for other common assumptions on ordinal preferences. Consider continuity for example. An obvious justification for the continuity of preferences is to argue that satisfaction or happiness is a continuous psychological quantity. Although not quite ordinal, continuity (viewed as an assumption on utility functions) is weaker than several other assumptions we have considered (e.g., additive separability, concavity on open sets). Thus, as intuition no doubt suggests, the continuity of preferences can be justified using only a mild measurement assumption. Once again, an ordinalist might object that there is no need to assume that utility functions are continuous in order to give a rationale for preference continuity; the continuity of preference relations already stands as an ordinal axiom. But the psychology that motivates an assumption that upper and lower contour sets are open surely is nonordinal; it turns on a claim that satisfaction is a continuous quantity.

If any doubt lingers that nonordinal content can lie behind an axiom on preference relations, consider the following trivial ordinalization of assumptions on utility functions. For any property P, define the property P_O consisting of all functions that are ordinally equivalent to some u that satisfies P. Although P_O must be ordinal, the psychological theory that motivates P clearly need not be ordinal. For the same reason, one may conclude that the psychologies that motivate the independence and continuity assumptions also rest on nonordinal foundations.

Appendix.

Proof of Theorem 3.1: Let U with domain X maximally satisfy additive separability and let $u \in \mathscr{F}_X$ be an arbitrary element of U. For any decisive A, let $B_A \supset A$ denote a decisive set such that $u|B_A$ satisfies additive separability. Given $v \in \mathscr{F}_X$, suppose that for each decisive A there exists an increasing affine

transformation g: Range $u|A \to \mathbb{R}$ such that $g \circ u|A = v|A$. In particular, for the decisive set B_A, this supposition implies there is an increasing affine transformation g such that $g \circ u|B_A = v|B_A$. Since $v|B_A$ satisfies additive separability, $v \in U$.

In the other direction, we must show, for any $v \in U$ and any decisive A, that there exists an increasing affine transformation g: Range $u|A \to \mathbb{R}$ such that $v|A = g \circ u|A$. The remainder of the proof considers a fixed A and the associated $B \supset A$ such that $u|B$ satisfies additive separability. It is sufficient to show that if g is increasing and $g \circ u|B$ satisfies additive separability (i.e., $g \circ u|B \in U|B$), then g is affine. To simplify, we henceforth drop the notation "$|B$" indicating the restriction of v, u, etc., to B.

Note that since each cl Range u_i is an interval there exists some x' such that, for all i with u_i nonconstant, $u_i(x_i') \in$ Int cl Range u_i. By adding constants to the u_i, there exists an increasing affine transformation that, when applied to u, yields a $w: B \to \mathbb{R}$ that is additively separable and that satisfies $w_i(x_i') = 0$ for all i. Clearly, there is also an increasing affine transformation that, when applied to w, yields u.

Consider an increasing transformation g that, when applied to u, yields an additively separable $h: B \to \mathbb{R}$. For each i, define $k_i: B_i \to \mathbb{R}$ by $k_i(x_i) = h(x_i) - h(x_i')$ and define $k: B \to \mathbb{R}$ by $k(x) = \sum_{i=1}^{n} k_i(x_i)$. Since u is an increasing affine transformation of w, h is an increasing transformation of u, and k is an increasing affine transformation of h, (1) there is an increasing transformation f: Range $w \to \mathbb{R}$ such that $f \circ w = k$, and (2) if f is linear, g is affine.

We first show that f is continuous. If not, let $\bar{x} \in B$ be a point such that f is not continuous at $w(\bar{x})$. Since there are at least two components such that cl Range u_i has nonempty interior, there is a component i such that $w_i(\bar{x}_i) \in$ Int cl Range w.

For any l, by setting $x_j = x_j'$ for $j \neq l$, we have $f(w_l(x_l)) = k_l(x_l)$ for all x_l. Hence,

(A1) $\qquad f(\sum_{l=1}^{n} w_l(x_l)) = \sum_{i=1}^{n} k_l(x_l) = \sum_{l=1}^{n} f(w_l(x_l))$ for all $x \in B$.

In particular, for any given set of $\hat{x}_l$, $l \neq i$,

(A2) $\qquad f(w_i(x_i)) + \sum_{l \neq i} f(w_l(\hat{x}_l)) = f(w_i(x_i) + \sum_{l \neq i} w_l(\hat{x}_l))$ for all $x_i \in B_i$.

Hence, using $\hat{x}_l = \bar{x}_l$, $\ell \neq i$, and given that cl Range w_i is a nondegenerate interval, the discontinuity of f at $w(\bar{x})$ implies that f is also not continuous at $w_i(\bar{x}_i)$.

Since f is increasing, $f_+(z)$ and $f_-(z)$, the right and left hand limits of f at z, exist at any $z \in$ Int cl Range w. Define J: Int cl Range $w \to \mathbb{R}$ by $J(z) = f_+(z) - f_-(z)$. The discontinuity of f at $w_i(\bar{x}_i)$

implies $J(w_i(\bar{x}_i)) > 0$. Since there exists some component $j \neq i$ such that cl Range w_j is a nondegenerate interval containing 0 and given A1, we can, by varying only $w_j(x_j)$ and setting $w_i(x_i) = w_i(\bar{x}_i)$ and $w_l(x_l) = 0$ for $l \neq i, j$, assemble a set $Q \subset$ Range w that is dense on a bounded nondegenerate interval and such that $w_i(\bar{x}_i) \in Q$. Given A2, $J(w_i(x_i)) = J(w_i(x_i) + \sum_{l \neq i} w_l(\hat{x}_l))$ for any $w_i(x_i)$ and any set of $\hat{x}_l$, $l \neq i$. Hence, for any $z \in Q$, by setting $w_i(x_i) = w_i(\bar{x}_i)$ and $w_j(\hat{x}_j) = z - w_i(\bar{x}_i)$ (and the remaining $w_l(x_l) = 0$), we have $J(w_i(\bar{x}_i)) = J(z) > 0$. Given that Q has an infinite number of elements and f is increasing, $J(z) > 0$ for all $z \in Q$ contradicts the fact that Q is bounded. Hence f is continuous.

Since f is continuous, f has a unique continuous extension to cl Range w, say f_e. Given A2,

(A3) $$f_e(\Sigma_{l=1}^n a_l) = \Sigma_{l=1}^n f_e(a_l)$$

for all $a \in$ cl Range $w_1 \times ... \times$ cl Range w_n.

We turn to the linearity of f. Fix some $d > 0$ that satisfies $d \in$ cl Range w_i for all i such that w_i is not a constant function. (Such a d exists since, for all i, $w_i(x_i') = 0$, cl Range w_i is an interval, and, when cl Range w_i is not a singleton, $0 \in$ Int cl Range w_i.) Consider any $e' > 0$ that is an element of Range w_i for some i. For all $\epsilon > 0$, there exists a $e \in$ cl Range w_i and rational r such that $dr = e$ and $|e - e'| < \epsilon$. Let s and t be positive integers such that $r = s/t$. We have $d/t \in$ cl Range w_i for any i such that w_i is nonconstant. Let l and j be coordinates of nonconstant w_i. Choosing $a \in \mathbb{R}^n$ such that $a_l = a_j = d/t$ and the remaining $a_i = 0$, A3 implies $f_e((d/t) + (d/t)) = 2f_e(d/t)$. Iterating this argument t times, we have $f_e(d) = tf_e(d/t)$. (Since, for any positive integer $m \leq t$, $(md)/t \leq d$, $(md)/t \in$ cl Range w_l and $(md)/t \in$ cl Range w_j, which permits each stage of the iteration.) Changing the coordinate l if necessary so that $e \in$ cl Range w_l, apply the same iteration argument to conclude $f_e((sd)/t) = sf_e(d/t)$. (We now have, for any positive integer $m \leq s$, that $(md)/t \leq e$ and hence $(md)/t \in$ cl Range w_l.) So $f_e(e) = sf_e(d/t) = f_e(d)(s/t) = f_e(d)r = (f_e(d)/d)e$. The continuity of f_e implies that f, when restricted to $e \in \bigcup_{i=1}^n$ Range w_i such that $e > 0$, is linear. Now consider any $e \in$ Range w such that $e > 0$. For any such e, there exists a $a \in \mathbb{R}^n_+$ such that each $a_i \in$ Range w_i, $e = \Sigma_{i=1}^n a_i$, and $f(e) = \Sigma_{i=1}^n f(a_i)$. Hence, $f(e) = \Sigma_{i=1}^n a_i(f(d)/d) = e(f(d)/d)$. So f is linear on positive points of its domain.

Since we can repeat the argument of the previous paragraph for $d < 0$ and $e' < 0$, f can be locally nonlinear only at 0 (and g therefore is locally nonaffine only at $u(x')$). By repeating our

construction with some $\hat{x}$ such that $u(\hat{x}) \neq u(x')$ we can define new functions $\hat{k}$, $\hat{w}$, and $\hat{f}$ (where $\hat{k}$ is additively separable, $\hat{w}$ is an increasing affine transformation of u, and $\hat{f}$ is increasing) that satisfy $\hat{f} \circ \hat{w} = \hat{k}$. Just as with f, $\hat{f}$ can be locally nonlinear only at 0 and g can be locally nonaffine only at $u(\hat{x})$. The transformation g is therefore locally affine at $u(x')$ and so f is locally linear at 0. ∎

Proof of Theorem 4.2: We first show that concavity is no stronger than any cardinal property. Let U_C be cardinal, let U_{CV} maximally satisfy concavity, suppose U_C and U_{CV} have the same decisive sets, and assume for all decisive A that there exist $u \in U_C$ and $v \in U_{CV}$ such that $u|A = v|A$. For any decisive A, let $B_A \supset A$ denote a convex and decisive set such that $v|B_A$ satisfies concavity. Since B_A is itself decisive, by assumption there exist $\hat{u} \in U_C$ and $\hat{v} \in U_{CV}$ such that $\hat{u}|B_A = \hat{v}|B_A$. Since $\hat{v}|B_A \in U_C|B_A$, for any $u' \in U_C$ there exists an increasing affine transformation g such that $g \circ \hat{v}|B_A = u'|B_A$. Since an increasing affine transformation of a concave function is concave, $u'|B_A$ satisfies concavity. Moreover, since there exists such an increasing affine g for the B_A corresponding to any decisive A, each $u'|B_A$ satisfies concavity. Hence, $u' \in U_{CV}$. Concavity is therefore no stronger than any cardinal property. For result (2), assume in addition the cardinal property has range > 2. Let A'' be a decisive set for U_C such that $u'' \in U_C$ has $|\text{Range} (u''|A'')| > 2$. Let $f: \mathbb{R} \to \mathbb{R}$ be increasing and strictly concave and let v be any element of U_{CV}. For any decisive A there exists a decisive $B \supset A$ such that $v|B$ and hence $f \circ v|B$ satisfy concavity. So $f \circ v \in U_{CV}$. Since by assumption $u'' \in U_{CV}, f \circ u'' \in U_{CV}$. Since $|\text{Range} (u''|B_{A''})| > 2$, there exist $u_1, u_2, u_3 \in \text{Range } u''|B_{A''}$ and a $\lambda \in (0, 1)$ such that $u_1 < u_2 < u_3$ and $u_2 = \lambda u_1 + (1 - \lambda)u_3$. Since f is strictly concave, $f(u_2) > \lambda f(u_1) + (1 - \lambda)f(u_3)$. So $f|(\text{Range } u''|B_{A''})$ is not affine and therefore $f \circ u'' \notin U_C$. Hence, concavity is weaker than any cardinal property with range > 2 that intersects concavity.

As for concavity and ordinality, it is straightforward to show that each ordinal property is no stronger than concavity (or indeed no stronger than any property). To show result (3), that any ordinal property with range > 1 that intersects the property of being concave and continuous is weaker than the property of being concave and continuous, let U_O maximally satisfy an ordinal property with range > 1, let U_{CVC} maximally satisfy concavity and continuity and have the same decisive sets as U_O, assume for each A that is decisive for U_{CVC} that $U_O|A \cap U_{CVC}|A \neq \emptyset$, and let $v \in U_{CVC}$. Let A' be decisive for U_{CVC} and U_O. Since U_O maximally satisfies an ordinal property with range > 1, there is a

decisive $B' \supset A'$ such that $|\text{Range } v|B'| > 1$. For some convex and decisive $B \supset B'$, $v|B$ satisfies

concavity. Given continuity, Range $v|B$ is a nontrivial interval. Let $x_1, x_2 \in B$ satisfy $v(x_1) < v(x_2)$

and define $D = \{w \in B: w = \lambda x_1 + (1-\lambda)x_2 \text{ for some } \lambda \in [0, 1]\}$. Let $C \subset D$ be a connected set such

that v is monotone on C and let $z_1, z_2, z_3 \in C$ satisfy $z_3 = (1/2)z_1 + (1/2)z_3$ and $v(z_1) < v(z_2) < v(z_3)$.

Hence there exists an increasing transformation $g: \mathbb{R} \to \mathbb{R}$ such that $g(v(z_2)) < (1/2)g(v(z_1)) +$

$(1/2)g(v(z_3))$. We then have $g \circ v \notin U_{CVC}$, but since g is increasing, $g \circ v \in U_O$. $\blacksquare$

Proof of Theorem 4.3: See Grodal (1974). $\blacksquare$

Proof of Theorem 4.4: Let U with domain X maximally satisfy additive separability and let U be an

arbitrary element of U. We must show (1) if $V: X \to \mathbb{R}$ is such that for all A that are decisive for U

there exists an increasing affine transformation g that satisfies $g \circ U|A = V|A$, then V satisfies

integrability, and (2) for any $V \in U$ and any decisive A there exists an increasing affine transformation

g such that $V|A = g \circ U|A$. The proof of (1) is identical to the beginning of the proof of Theorem 3.1.

As for (2), we consider henceforth a fixed A and a fixed $B \supset A$ such that $U|B$ satisfies utility

integrability. It is sufficient to show that if g is an increasing transformation and $g \circ U|B = V|B$

satisfies utility integrability (i.e., $V|B \in U|B$), then g is affine.

Observe that since $|\text{Range } U| > 1$, there exist $x, x' \in B$ such that $U(x) > U(x')$, and hence there

also exists, for any $\varepsilon > 0$, a measurable $C_1 \subset [0, T]$ such that

$$0 < \int_{C_1} (u(x(t), t) - u(x'(t), t)) \, d\mu(t) < \varepsilon,$$

where $u: \mathbb{R}_+^n \times [0, T] \to \mathbb{R}$ is a function satisfying Definition 4.5 (ii). By setting ε sufficiently small,

we can partition $[0, T]$ into sets C_1 and C_2 such that

$$\int_{C_2} (u(x(t), t) - u(x'(t), t)) \, d\mu(t) > 0.$$

For $i = 1, 2$, let B_i be the restriction of B to C_i – that is, the set of functions from C_i to $\mathbb{R}^n$ defined by x_i

$\in B_i$ if and only if there exists $x \in B$ such that $x_i(t) = x(t)$ for all $t \in C_i$. Since B is closed under a.e.

replacement, so are B_1 and B_2. Let $U_i: B_i \to \mathbb{R}$ be defined by $U_i(x_i) = \int_{C_i} u(x_i(t), t) \, d\mu(t)$. We have B

$= B_1 \times B_2$, again due to the fact that B is closed under a.e. replacement. Using (x_1, x_2) to denote the x

$\in B$ such that $x(t) = x_i(t)$ for all $t \in C_i$, we have $U(x_1, x_2) = U_1(x_1) + U_2(x_2)$ for all $x \in B$. Since, for i

$\in \{1, 2\}$, $U_i(x_i) > U_i(x_i')$, $|\text{Range } U_i| > 1$.

We use the following result from the theory of integration of correspondences, sometimes called Lyapunov's theorem.

Lyapunov's theorem. Given an atomless measure space $(\Omega, \mathscr{F}, \lambda)$ and correspondence $P: \Omega \rightrightarrows \mathbb{R}$, $\int_\Omega P \, d\lambda \equiv \{ \int_\Omega p \, d\lambda : p$ is integrable and $p(\omega) \in P(\omega)$ for a.e. $\omega \in \Omega \}$ is a convex set.

By Lyapunov's theorem, Range U, Range U_1, and Range U_2 are convex sets and therefore intervals. For example, for the case of Range U_1, the measure space is Lebesgue measure on C_1 and the correspondence P would be defined by $P(t) = \{u(x_1(t), t) \in \mathbb{R}: x_1 \in B_1\}$ for each $t \in C_1$. Since B_1 is closed under a.e. replacement, Range $U_1 = \int_{C_1} P \, d\mu$, and so we may conclude that Range U_1 is convex. Note that since $|\text{Range } U_1| > 1$ and $|\text{Range } U_2| > 1$, Range U_1 and Range U_2 have nonempty interiors.

Since $V \in \mathbf{U}$, there exists a $v: \mathbb{R}_+^n \times [0, T] \to \mathbb{R}$, where $t \to v(x(t), t)$ is integrable and $V(x) = g(U(x)) = \int_0^T v(x(t), t) \, d\mu(t)$ for all $x \in B$. Just as we argued for the case of U, U_1, and U_2, we have $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$ for all $(x_1, x_2) \in B$, where $V_i(x_i) = \int_{C_i} v(x_i(t), t) \, d\mu(t)$, $i \in \{1, 2\}$. Hence $g(U_1(x_1) + U_2(x_2)) = V_1(x_1) + V_2(x_2)$ for all $(x_1, x_2) \in B$. Apply Theorem 3.1 to conclude that g is affine. ∎

Proof of Theorem 5.1: In text. See also the proof of Theorem 4.2. ∎

Proof of Theorem 5.2: We omit the details: 5.2 (1) is obvious and 5.2 (2) is a slight variant on common arguments in the social choice literature (see, e.g., Moulin (1988)). ∎

Proof of Theorem 5.4: For $\mathbf{y} \in \mathbb{R}^I$, $\mathbf{y}_{\max}$ will denote $\max\{\mathbf{y}_j, \mathbf{y}_k\}$ and $\mathbf{y}_{\min}$ will denote $\min\{\mathbf{y}_j, \mathbf{y}_k\}$.

Assume $\mathbf{v}_j + \mathbf{v}_k \geq \mathbf{w}_j + \mathbf{w}_k$ and $\mathbf{v}_{\min} \geq \mathbf{w}_{\min}$. If both $\mathbf{v}_j + \mathbf{v}_k = \mathbf{w}_j + \mathbf{w}_k$ and $\mathbf{v}_{\min} = \mathbf{w}_{\min}$, it follows that $\mathbf{v}^* = \mathbf{w}^*$ and the conclusion $\mathbf{v} \, R_{\hat{u} \in U_{CC}} \, \mathbf{w}$ is immediate. So assume that at least one inequality is strict. Define $\mathbf{x}$ by $\mathbf{x}_j = \mathbf{w}_{\min}$, $\mathbf{x}_k = \mathbf{v}_k + \mathbf{v}_j - \mathbf{w}_{\min}$, and $\mathbf{x}_i = \mathbf{v}_i$ for any $i \notin \{j, k\}$. Let $\mathbf{z}$ equal $\mathbf{x}$ but interchange the j and k coordinates. Given $\mathbf{v}_j + \mathbf{v}_k \geq \mathbf{w}_j + \mathbf{w}_k$, we conclude that $\mathbf{x}_k \geq \mathbf{w}_{\max}$ and therefore $\mathbf{x}^* \geq \mathbf{w}^*$ and $\mathbf{z}^* \geq \mathbf{w}^*$. Set

$$\lambda = \frac{\mathbf{v}_{\min} - \mathbf{w}_{\min}}{\mathbf{v}_j + \mathbf{v}_k - 2\,\mathbf{w}_{\min}},$$

which is well-defined given that either $\mathbf{v}_j + \mathbf{v}_k > \mathbf{w}_j + \mathbf{w}_k$ or $\mathbf{v}_{\min} > \mathbf{w}_{\min}$. Evidently $\lambda \in [0, 1]$, and one may confirm that $\mathbf{v} = \lambda\mathbf{x} + (1 - \lambda)\mathbf{z}$ (if $\mathbf{v}_{\min} = \mathbf{v}_k$) or $\mathbf{v} = \lambda\mathbf{z} + (1 - \lambda)\mathbf{x}$ (if $\mathbf{v}_{\min} = \mathbf{v}_j$). Hence Theorem 5.3 implies $\mathbf{v}\, R_{\hat{u} \in U_{CC}}\, \mathbf{w}$ (see the comment following the statement of Theorem 5.3).

Now assume $\mathbf{v}\, R_{\hat{u} \in U_{CC}}\, \mathbf{w}$. If $\mathbf{v}_j + \mathbf{v}_k < \mathbf{w}_j + \mathbf{w}_k$, then $\sum_{i \in \mathbb{I}} \mathbf{v}_i < \sum_{i \in \mathbb{I}} \mathbf{w}_i$, contradicting $\mathbf{v}\, R_{\hat{u} \in U_{CC}}\, \mathbf{w}$. Hence $\mathbf{v}_j + \mathbf{v}_k \geq \mathbf{w}_j + \mathbf{w}_k$. Suppose, contrary to the theorem, that $\mathbf{v}_{\min} < \mathbf{w}_{\min}$. Consider transformations $g : \mathrm{co}\,\mathrm{Range}\,\hat{u} \to \mathbb{R}$ that are piecewise-linear, increasing, concave, and whose only nondifferentiability occurs at $\mathbf{w}_{\min}$. As long as $Dg(u)\big|_{u < \mathbf{w}_{\min}} > Dg(u)\big|_{u > \mathbf{w}_{\min}} > 0$, g will be concave and increasing, but otherwise these two derivatives are arbitrary. So fix some $Dg(u)\big|_{u < \mathbf{w}_{\min}}$ and choose $Dg(u)\big|_{u > \mathbf{w}_{\min}}$ small enough that $g(\mathbf{v}_{\max}) \approx g(\mathbf{w}_{\max})$. (Note that $\mathbf{v}_{\max} > \mathbf{w}_{\min}$ since $\mathbf{v}_{\min} < \mathbf{w}_{\min}$ and $\mathbf{v}_j + \mathbf{v}_k \geq \mathbf{w}_j + \mathbf{w}_k$.) Hence $g(\mathbf{v}_j) + g(\mathbf{v}_k) < g(\mathbf{w}_j) + g(\mathbf{w}_k)$ and so $\sum_{i \in \mathbb{I}} g(\mathbf{v}_i) < \sum_{i \in \mathbb{I}} g(\mathbf{w}_i)$, again contradicting $\mathbf{v}\, R_{\hat{u} \in U_{CC}}\, \mathbf{w}$. Observe that here we use only the assumption that $\hat{u}$ is lcc. ∎

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