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**Cartoons of the Variation of Financial Prices and of
Brownian Motions in Multifractal Time**

by

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Cartoons of the variation of financial prices and of Brownian motions in multifractal time

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♦ **Abstract.** This article describes a versatile family of functions increasingly roughened by successive interpolations. They provide models of the variation of financial prices. More importantly, they are helpful "cartoons" of Brownian motions in multifractal time, BMMT, which are better models described in the next article. Ordinary Brownian motion and two models the author proposed in the 1960s correspond to special cartoons. More general cartoons are richer in structure but (by choice) remain parsimonious and easily computed. Their outputs reproduce the main features of financial prices: continually varying volatility, discontinuity or concentration, and other events far outside the mildly behaving Brownian "norm." ♦

I. Introduction

A. The Challenge of Modeling the Financial Price Records.

Financial prices, such as those of securities, commodities, foreign exchange or interest rates, are largely unpredictable but one can evaluate the odds for or against some desired or feared outcomes, the most extreme being "ruin." Those odds are essential to the scientist who seeks to understand the financial markets and other aspects of the economy. They must also be used as inputs for decisions concerning economic policy or institutional arrangements. To handle all those issues, the first step - but far

from the last! – is to represent different prices' variation by suitable random processes.

This article and the next contribute in diverse ways to the “multifractal” approach to finance put forward in Mandelbrot (1997), especially in Chapter 6. The theme must be stated cautiously because not all prices have been investigated. This being granted, the variation in time of a variety of financial prices is well accounted for by choosing suitably within an altogether new random processes called “Brownian motions (Wiener or fractional) in multifractal time.” Those processes will be referred to as BMMT and the Brownian motions in the ordinary clock time will be WBM and FBM, respectively.

The key terms, “fractional” and “multifractal,” are nonclassical but do not belong to esoteric mathematics, and their practical consequences for finance and economic policy are numerous and important. Those terms, and “fractal” itself, will be explained. To bring this article and the next closer to being self-contained, both incorporate substantial background material, improved over the exposition in Mandelbrot (1997).

This paper's ambition is to present original material and results and help understanding while holding mathematics to a minimum. The underlying conceptual ideas will be motivated, explained in detail and illustrated graphically. An extensive mathematical basis already exists for multifractals, for example, in Mandelbrot (1999). However, for intrinsic reasons made more serious by raw novelty, BMMT is delicate and hard to grasp fully. But the fear that fractals/multifractals are far removed from clumsy and confused reality would be unwarranted because their mathematics strikes close to the main features of the underlying phenomena.

It is true that, without mastering the formulas and diagrams, the claims and contributions concerning multifractals cannot be fully understood and appreciated. It is also true that much of the underlying mathematics, without being at all difficult, is not widely familiar. However, in order to make the central point of this text, the best procedure is to draw pictures. To be sure, in many cases pictures lie as effectively as words, statistics and opaque formulas. But in the present case, the message is crystal clear, as the reader will see.

B. Legitimate a Priori Reservations Against Fractals/Multifractals in Finance.

Fractals (Mandelbrot 1982) are endlessly repeated geometric figures. They are best known for describing the shapes of coastlines and clouds and the distribution of galaxies, and on a more abstract level for having

led to the Mandelbrot set. As an extension of fractals, I introduced multifractals in papers on turbulence from the early 1970s. These are reproduced and discussed in Mandelbrot (1999). Because of those origins, the notion that related thoughts also describe financial prices may seem driven by mere imitation, therefore unpromising. In fact, the historical sequence went the other way, the build up towards fractal geometry having *begun* in my work in finance. But history does not matter here, only the fact that fractals and multifractals will not become accepted until several conditions have been met.

Good and parsimonious fit to the evidence is the ultimate condition. But diverse a priori reservations must be acknowledged. Therefore, this paper's first task will be to try and create a certain level of a priori goodwill or receptivity, or at least to preempt certain common objections. This will take some time and the "working part" of this paper will not begin until Section VI.

A preliminary obstacle to good will is that coin-tossing and some statistical distributions, such as the Gaussian and Poisson lognormal, are near-unanimously viewed as "natural" and "normal." Therefore, new uses are often "waved by" and used without fresh justifying argument. In sharp contrast, fractals/multifractals are largely unfamiliar to economic statisticians, consequently viewed as "artificial" or "contrived" and in need of a higher level of justification. Besides, they are inconvenient insofar as they fail to involve a large family of explicit analytic expressions that have many parameters and hence can be bent for many purposes. That is, they do not fall into the Carl Pearson pattern made familiar and accepted a century ago.

Precisely contrary views are defended in this paper. Firstly, it argues that in finance the Gaussian is, in fact, *not* "normal" at all. As recalled in Section II, departures from the Gaussian fill the newspaper headlines. The usual reaction was to call them "outliers," namely, "exceptions that confirm the rule" and do not deny the primacy of the Gaussian as the "norm." I challenged that reaction forty years ago and advanced substitute new "rules," now improved upon using multifractals.

Secondly, this paper argues that notwithstanding their acknowledged negatives, fractals/multifractals are absolutely compelling and near-unavoidable in finance among many other fields. The main fact is that fractal/multifractal techniques are by no means specific to the fields where they arose. To the contrary, they are the long-sought tool to tackle two universal concerns.

Roughness will be faced in Section III and Sections IV and V will be concerned with *hierarchies*. As to Sections VI to X, they hope to prepare the reader for multifractals, in particular to study BMMT in the next paper, by describing a rich family of easier-to-handle processes one can view as approximants or "cartoons" of BMMT.

C. *The Usefulness of Simple Cartoons.*

Given the cartoons' importance, it is useful to sketch their role without delay. The most widely used price variation model originated with Bachelier in 1900 and will be described in Section IIA. It is WBM, the "ordinary" Wiener Brownian motion in continuous clock time. That process is very familiar, yet exhibits very serious complications and remains best understood when studied in parallel with the discrete coin-tossing and random walk. These processes can be viewed as cartoons of the increments of WBM and WBM itself.

The random walk has no direct counterpart in the case of BMMT but splendid cartoons in a very different style were developed and sketched in Mandelbrot (1997). They are limits of discrete-parameter sequences of successive interpolations drawn on a continually refined temporal grid. This paper illustrates the power those interpolative cartoons preserve even when restricted to a very special family.

By design, this family is as simple as can reasonably be expected. As will be recalled, reality combines very long-tailed marginal distributions and long dependence, and one can expect the representation of each of those features to require at least one parameter. Indeed, the cartoons to be investigated involve only two parameters, which can be chosen to be the coordinates of a point in a square map called "phase diagram." Also by design, our family of cartoons is *not* simple to excess. Indeed we shall single out special behaviors associated with suitable special regions or "loci" in that square phase diagram and show that those loci yield cartoons of three existing models and thereby throw fresh light on those models' nature. Therefore, this article fulfills a third role, that of relating BMMT to a segment of the literature.

A single extremely special cartoon in this family, described as "Fickian," is a deep but non-destructive simplification of the "coin-tossing" model of financial prices, therefore of the Bachelier and Wiener form of Brownian motion.

Two less narrowly constrained special cartoons are, again, deep but non-destructive simplifications of my two early models of price variation.

One, first proposed in Mandelbrot (1963), used Lévy stable random processes to tackle long-tailedness; it is discussed in the second half of Mandelbrot (1997). The other, first sketched in Mandelbrot (1965), introduced fractional Brownian motions to tackle global memory, also called infinite memory or dependence; it is discussed in many papers collected in Mandelbrot (2000), which devotes a chapter to finance. Within the current wider conceptual framework, those early models are classified as “mesofractal” and “unifractal,” respectively. Once again, multifractality was introduced into finance in the first half of Mandelbrot (1997) and is being developed in this paper and the next. This article also hopes to make plain the relations between all those different old and new “flavors” of fractality.

II. Large Financial Price Movements and Their High Odds; “Normality” of Coin-Tossing, Compared to the Reality

A. The Brownian Motion as “Universal” Model or the Variation of Every Kind of Financial Price.

Reliance on illustrations has many advantages and in this instance presents none of the usual risks, since every statement can be buttressed by full mathematical developments, including those reported in the next paper. By design, Figures 1 and 2 mix in identical styles some actual and some model price series. The actual series may concern security, commodity, foreign exchange or interest rate prices. The model series are old or new. For the actual prices, the abscissa is time and the ordinate is either a price $P(t)$ – as in Figure 1, or its increment $\Delta P(t) = P(t) - P(t - 1)$ – as in Figure 2.

Present in both figures is Bachelier's WBM $B(t)$ and the sequence of its increments, called “white Gaussian noise” $B(t)$ is the continuous-time version of the “coin-tossing model” which assumes that all financial price continually moves either up a bit or down a bit following the toss of a coin.

It was soon observed that plain coin-tossing would eventually lead to negative prices. This is avoided and the overall logic is improved by assuming that, instead of the prices, the coin-tossing process rules the logarithm of price. (Observe that records of interest rates followed as function of time require a different transformation, since a daily interest rate is itself roughly a difference between two logarithms: that of the reimburse-

ment tomorrow and that of the loan today.) In this paper, transformations made to avoid negative values will be disregarded because the complications they create do not affect the basic points to be made.

WBM is the oldest and by far the easiest model of price variation. Part of its extreme simplicity follows from the fact that it includes a bold "assumption of universality." Rarely stressed as strongly as it should be, this assumption expresses that, except for one parameter – a scale factor that measures volatility, all prices follow a single universally valid process. Invoking roughness, a notion to which we shall return, the assumption is that all price charts are equally rough.

On the basis of Figure 1, the universality assumption is actually almost believable, since the intermixed real and forged records look alike. This impression is confirmed by analogous diagrams in the financial press and books devoted to the mathematics of finance. The optimist will rush to conclude that coin-tossing is perfectly acceptable.

B. The Irredeemable Lack of Fit of the Brownian Model.

Unfortunately, the mutual resemblance between the curves in Figure 1 is nothing but an artefact of plotting and completely vanishes in Figure 2. On that figure, the would-be "universal," namely white Gaussian noise, is plotted on line 1 from the top and actual price series mingle among the bottom five lines.

The white Gaussian noise line is extremely monotonous and reduces to a static background of small price changes, analogous to the static noise from a radio. Volatility stays uniform with no sudden jumps. In a historical record of this kind, daily chapters would vary from one another, but all the monthly chapters would read very much alike. A look at line 1 makes obvious that coin-tossing and financial reality are in sharp qualitative disagreement.

In other words, important differences are not enhanced on Figure 1, but instead are hidden. That is, plots of prices themselves are a very misleading way of presenting information. As known to students of the psychology of visual perception, position (Figure 1) is perceived less accurately than change (Figure 2).

In any event, qualitative impressions must be buttressed quantitatively. In the Gaussian distribution used to draw line 1 of Figure 2, it is true that 95% of all cases fall into the range between -2σ and 2σ . However, all the major events that matter fall into the remaining 5%. In actual data, to the contrary, large fluctuations often exceed 10σ . Under

the Gaussian assumption, the corresponding probability is of a few millionths of a millionth of a millionth of a millionth, that is, approximately the equivalent of one day out of ten million million million years. Such exquisitely tiny risks would indeed be truly negligible, not worth even a passing thought. Unfortunately, this tiny value grossly contradicts the evidence. In the real world of finance, "ten sigma" events are *not* freakish but occur on a regular basis. The actual value of their probability should be expected to be a few thousandths or even hundredths, a substantial part of the 5% not accounted for in coin-tossing.

Let us continue scanning Figure 2. Lines 2 and 3 also differ from reality. It will be seen in Sections VIII and IX that they correspond to my two early models, respectively, the M 1963 mesofractal model involving infinite variance, and the M 1965 unifractal model involving infinite dependence.

Lines 4 to 8 in Figure 2 intermix actual prices and models based on multifractals. The reason for intermixing is to emphasize that those lines are remarkably alike hence difficult to identify.

All exhibit a constant background of small up and down price movements. Invariably, however, a more striking aspect resides in a substantial number of sudden large changes. On the chart they appear as spikes that shoot up and down and stand out from the background of the moderate perturbations. In addition, their times of occurrence are not independent of one another but strikingly clustered. Moreover, defining volatility, generically as the rough order of magnitude of the smallish price movements, it is typical for it to remain roughly constant for a short or extended period and then - suddenly and unpredictably - change to assume a different value for another short or extended period.

All those "real-looking" model price changes were obtained using BMMT, that is, Brownian motion in *multifractal* time. In BMMT, the very long tails of the M 1963 model and the infinite dependence are of the M 1965 model harmoniously combined. This, and the next paper, proposes to explain how this goal is achieved.

C. Financial Hurricanes and the Difference - not in Degree but in Kind - Between "Mild" and "Wild" Randomness. The Need for Versatility Among Proposed Substitutes for the Brownian Motion.

An important distinction must now be mentioned, but only in passing. The coin-tossing model exemplifies a form of randomness (a "state of randomness") that Mandelbrot (1997, Chapter E5) proposed to call "mild." In physics, mildness characterizes a gas that reached equilibrium after

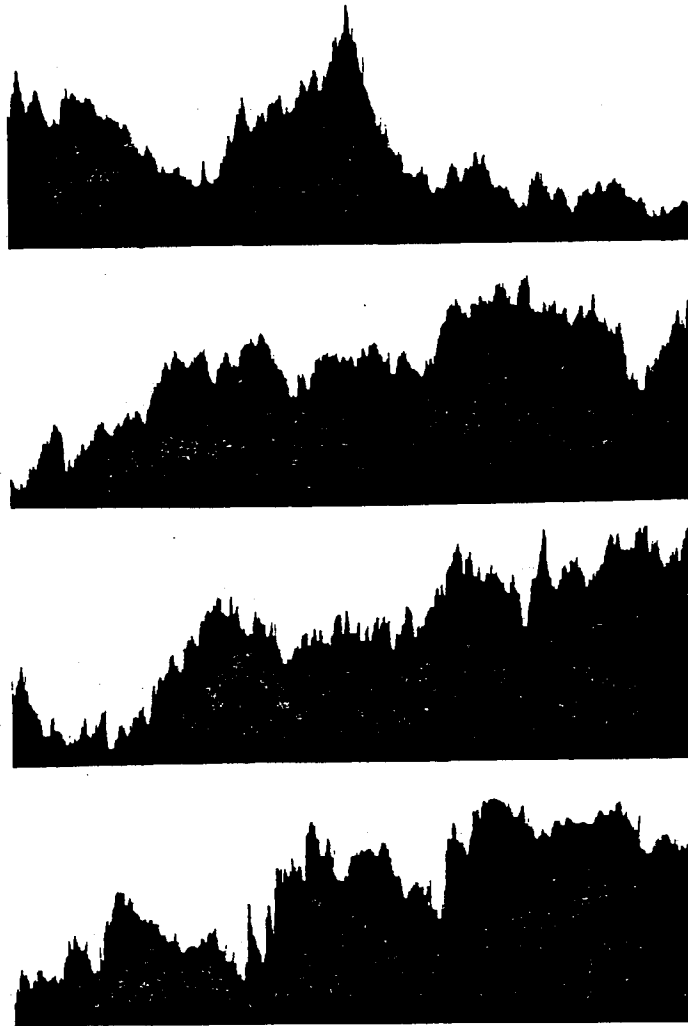


FIGURE H1-1. A collection of diagrams, illustrating - in no particular order - the behavior in time of at least one actual financial price and of at least one mathematical model of this behavior. It would be difficult to identify the models.

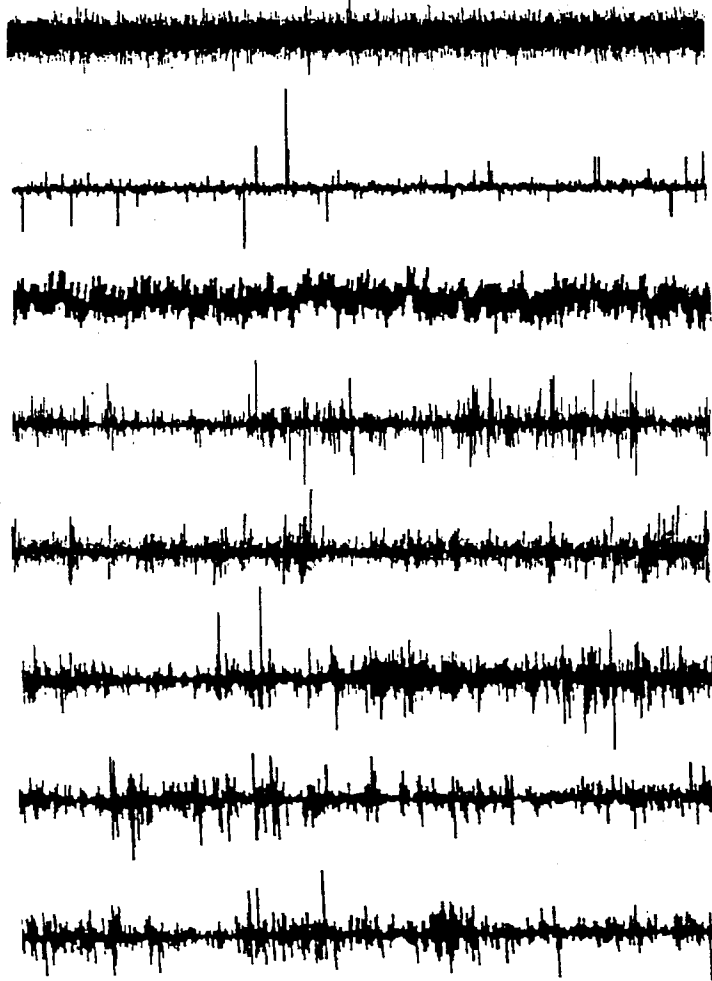


FIGURE H1-2. A stack of diagrams, illustrating the successive “daily” differences in at least one actual financial price and some mathematical models. Obviously, the top three lines do not report on data but on models; to the contrary, to identify the models among the lower five lines is difficult.

sitting for a long time at constant temperature and pressure. The behavior of turbulent gases is very different, not only in degree but in kind, and the behavior of the atmosphere is characterized by storms and hurricanes. I think financial prices bear no resemblance to gases in equilibrium. It is best to view the financial system as being violently turbulent and characterized by financial hurricanes.

Every "flavor" of fractal models, *mesofractal, unifractal and multifractal*, as well as the multifractal model of turbulence (see Mandelbrot 1999) are described in Mandelbrot (1997, Chapter E5) as not mildly, but "wildly" variable.

D. Alternatives to Brownian Motion. The Contrast Between two General Approaches to Science: Micro- versus Macromodeling.

Some scientific models are rightly viewed as "micromanaged" and, in the worst scenario, reduce to eclectic statistical summaries of the data. Other models are "macromanaged" and properly constructed from a small number of organizing principles. The terms micro and macro are familiar in economics but the nuance used in in this paper comes directly from physics. The study of financial prices appears to be dominated by the former, except for my work and its (by now, numerous) follow-ups.

A collection of moments is not in itself a model, only a summary of the data. For random variables, the same is true of a collection of best-fitted parameters relative to a prescribed combination of a statistical criterion of fit and a family of statistical distributions. A standard example is the already mentioned Pearson system and a new example is the so-called "generalized hyperbolic distributions." For random sequences the same comment applies to representations in terms of trigonometric functions or wavelets in terms of ARMA or even in terms of FARIMA, which are ARMAs artificially generalized to include global dependence.

Micromanaged approaches typically proceed through a series of "fixes." Each fix puts a "patch" on a perceived defect of coin-tossing, independently of other defects and the corresponding patches. The number of parameters rapidly becomes large and no property is present that was not knowingly incorporated in the construction.

In my mind at least, experience of successful modeling in other fields has fostered a priori doubts about the prospects of micromanaged modeling in finance. But personal prejudices would not matter if, a posteriori, the micromanaged modeling had proven effective. I think it has not. This

may be one reason why it keeps being challenged by a strong extremely bearish attitude, namely, the claim a priori that large market swings are individual "acts of God" that could not conceivably present any statistical regularity.

My own work over many years views a priori bearishness as unwarranted and micromanaged modeling as ineffective. Taking a decidedly bullish position, I have long been arguing that the variation of financial prices can be accounted for by a model one can call "macromanaged," because it is guided globally by a principle of fractal invariance to which we shall come soon.

III. Roughness, an Ill-Defined but Fundamental Issue in Many Sciences, First Faced and Quantified by Fractals

Many sciences arose directly from the desire to describe and understand some combination of basic messages the brain receives from the senses. Visual signals led to the notions of bulk and shape and of brightness and color, hence to geometry and optics. The sense of heavy versus light led to mechanics and the sense of hot versus cold led to the theory of heat. Other signals (for example, auditory) require no comment. Proper measures of mass and size go back to prehistory and temperature, a proper measure of hotness, dates back to Galileo.

Against this background, the sense of smooth versus rough suffered from a level of neglect that is noteworthy – the more so far being seldom, if ever, pointed out. Roughness is ubiquitous, always concretely relevant, and often essential. Yet, not only the theory of heat has no parallel in a theory of roughness, common, but temperature itself had no parallel concept until the advent of fractal geometry.

Even in the inanimate objective and non controversial context of metal fractures, roughness was generally measured by a borrowed expression: the root mean square, rms, deviation from an interpolating plane. In other words, the metallurgists and the economists, "volatility" were implicitly identified with a phenomenon already solved in the textbooks. But metallurgists viewed this measurement as suspect because different regions of a presumably homogeneous fracture emerged as being of different "r.m.s. volatility." The same was the case for different samples that were carefully prepared and later broken following precisely identical protocols.

To the contrary, as shown in Mandelbrot et al (1984) and confirmed by every later study, fractals have a characteristic called the fractal dimension D that provides, for a first time, an invariant measure of roughness. The quantity $3 - D$ is called "codimension" or "Hölder exponent" by mathematicians and has now come to be called "roughness exponent" by metallurgists.

The roles exponents play in fractal geometry will become clear in Sections V to X. But it is good to sketch it here. The surfaces' intersections by approximating orthogonal planes are formally identical to the price charts in Figure 1. Had these functions been differentiable, they could be studied through the derivative defined by $P'(t) = \lim_{\epsilon \rightarrow 0} (1/\epsilon)[P(t + \epsilon) - P(t)]$. For fractal functions, however, this limit does not exist. Instead, those functions' local behavior is studied through the parameters of a relation of the form $dP \sim F(t)(dt)^\alpha$. Here, $F(t)$ is called "prefactor" and the most important parameter is the exponent

$$\alpha = \lim_{\epsilon \rightarrow 0} \{ \log[P(t + \epsilon) - P(t)] / \log \epsilon \}.$$

This replacement of ratios of infinitesimals by ratios of logarithms of infinitesimals is an important innovation. It was not directed by trial and error but by theory. It is not a panacea but a tool that need not fear a shortage of applications. There is an adage that, when you only own a hammer, everything begins to look like a nail. This adage does not apply to roughness, which is an old problem in almost every field, not a new one marketed to find customers for a tool.

IV. A Paradoxical Contrast: Hierarchical Structure is Widespread in the Economy and Absent from Coin-Tossing

Ultimately, once again, the replacement of coin-tossing by fractals/multifractals must be judged by the quality of the fit a new model provides, as balanced against its parsimony. In parsimonious models, the parameters are few in numbers and enter intrinsically, not via textbook statistical procedures. Elaborating concretely on the comments in Section III, this section seeks to create good will towards the new approach. It seeks to introduce fractals as being a needed broadened implementation and new application of the well-known, artificial but useful, notion of hierarchical structure.

This way of introducing fractals responds to the paradoxical conflict that exists in current thinking in economics and finance.

On the one hand, part of the extreme simplicity of coin-tossing is traceable to its being the most *unstructured* process of probability theory. It is the paradigm of the mere *juxtaposition* of local phenomena, devoid of any structure. In the basic theorems of probability obeyed by coin-tossing, the limits are paradigms of absence of structure.

On the other hand, prices strongly illustrate the general observation that economic variability is non-periodic but cyclic. It is subjected to oscillations on which rides a hierarchy of faster oscillations and which themselves ride on slower oscillations. As a matter of fact, almost everything in society and economics (including industrial organization and business geography) and finance is filled with all kinds of rich and complex structures. One kind of structure consists in seasonals, either natural (daily and yearly) or man-made (weekly and monthly). To accommodate them, both the Brownian and the multifractal model require adjustments. Seasonals are very important issues but beyond this paper or the next.

The most extreme and conspicuous structures to be considered are ordered by the highly visible hands of armies, churches or totalitarian states. There, everything is governed by a hierarchy (or several juxtaposed ones) wherein small units interact strongly, increasingly larger units increasingly less so. A very geometric example that many geographers endorse is the "central place theory," in which the perpetuation of a hierarchy reflects a balance between the costs of purchase and access. Without the benefit of a central planning office, many small neighborhood stores coexist with successive layers of increasingly few but increasingly large stores that cater to increasingly large areas.

In more realistic cases, a hierarchy is only rough rather than extreme and obvious. Be that as it may, if modeling of price variation had been an unexplored field, it would have been very hard to credit the price-setting mechanism of the financial markets as being capable of the following. They should smother the observed hierarchical structure of society and economics into the absence of any structure.

Turning the table around, the gross failure of coin-tossing to represent the evidence suggests that, when seeking improved models of price variation, it is best to look among constructions that exhibit a strict or loose hierarchical structure. These last words are a characterization and almost a definition of fractals and multifractals, hence a strong reason to believe that fractals and multifractals enter inevitably in the study of finance. The remainder of this paper is devoted to elaborating on this belief.

V. Fractal Structures are Tightly Associated with Hierarchies, hence Unavoidable in Finance

A. Self-similarity and Fractal Dimension: a "Canonical" Example.

By design, the strongest and most obvious hierarchical structure is present in several constructions one can call "protofractal," because they were discovered before fractal geometry itself was organized. Many examples are described in Mandelbrot (1982). Their having a strict hierarchical structure unavoidably follows from the fact that they are constructed from the top down using recursive interpolation. Later on, recursive interpolation will be carried out in time instead of space and yield functions in time that can serve as "cartoons" of financial prices' variability.

The protofractals include the Cantor set and the Koch snowflake curve, but the best example pedagogically is the "Sierpinski gasket" drawn in Figure 3 and described in its caption. In the gasket, the "whole" being the (set-theoretical) "union" of $N=3$ "parts." Each part is identical to the whole except for a linear reduction in the ratio $r=1/2$ that leaves fixed one corner of the whole. If the parts are reduced linearly in a smaller ratio, the remainder, instead of being a connected curve, is a disconnected dust.

The gasket and its $r < 1/2$ variants exhibit a hierarchic geometrical structure described as "self-similar." To the contrary, the hierarchies of the armies and society are abstract. Therefore, the embedding of the hierarchy in the plane must not introduce new and irrelevant features. Consider the k th level of the hierarchy. If $r < 1/3$, any two points that lie in the same triangle at that level are closer to one another than any two points that lie within distinct triangles at the same level. Therefore, constructions with $r < 1/3$ avoid introducing artificial distance relationships.

Be that as it may, this subsection only seeks to provide a perspicuous illustration. It does not seek to model society, only prepare for the variability of prices, which was already described as being hierarchic in a one-dimensional geometric way along the time axis.

Returning to the Sierpinski gasket, everything about it follows from the quantities N and r . By analogy with physics, they can be described as "microscopic." More complicated recursively constructed fractals involve more numerous and complicated microscopic characteristics. And when the gasket and all other recursive structures are replaced by more realistic non-recursive structures, numerous alternative microscopic characteristics

enter into contention. Some or all may be needed in certain problems, both in physics and finance. But for many purposes, the details hardly matter. The usefulness of fractal geometry can be traced to the fact that it identified intrinsic characterizations that summarize the microscopic data. The most important one, and the only one needed in a first-approximation investigation, is a quantity called "fractal dimension," which is defined as

$$D = \frac{\log N}{\log(1/r)} = \frac{\log N^k}{\log(1/r^k)} = \frac{\log 3}{\log 2} \sim 1.5849\dots$$

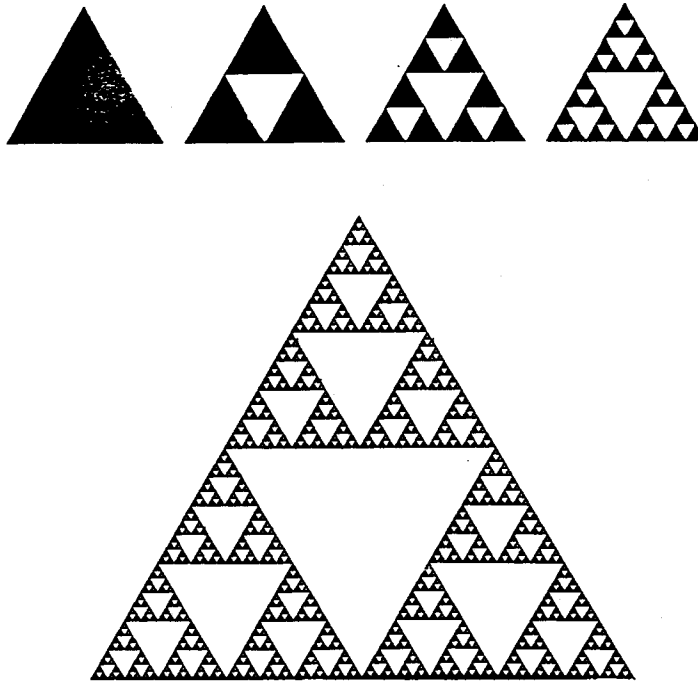


FIGURE H1-3. The Sierpinski gasket, an example of "a whole that is *more* than the sum of its parts." The construction is immediately obvious. It begins with a filled-in equilateral triangle, then divides it into four equal subtriangles and erases the middle one. The same construction then resumes within each sub-triangle. In the limit, one is left with a curve that includes an infinite collection of non-filled-in triangles plus their limit points. In terms that will serve in constructing price change cartoons, the original triangle is an "initiator," and the "generator" is made of the three triangles left after the middle one has been erased.

This quantity brings in a distinction that is closely related to that between micro and macromanaged models. Indeed, D is a fractal/multifractal measurement that is conceptually akin to a basic "macroscopic" notion in physics, namely temperature. Compared to a huge list of microscopic details, the macroscopic description is simplified drastically, but not to the point of becoming useless. In physics, such macroscopic quantities are well-known to exist and be useful for many purposes; the novelty is that, through fractals, the same turns out to be the case in finance. The strength and value of the fractal measurements will shine as we proceed to non-strict hierarchies.

B. Strict Hierarchies are Overspecified and Overly Complicated in One Way, and Grossly Oversimplified in Another. It is Typical of Suitably Chosen Random Fractals that Hierarchies are Perceived in the Samples but are Absent from the Generating Mechanism.

For the purpose of modeling empirical evidence of any kind, strict hierarchies are brutal approximations. They are overcomplicated because a fully specified construction algorithm may involve a large number of steps and of parameters. Some inputs are individually important, others become important in combination. As a result, a major unfilled need is created. The approximation error inevitable in scientific models cannot be assessed until the difference between truth and model can be measured quantitatively. For hierarchies, such measures are not part of standard statistics, but arise naturally in the theories of fractals and multifractals. In the case of structures in space, the best known is the (already mentioned) fractal dimension. For structures that illustrate variation in time, later notions will show that more than one measurement is needed. Furthermore, as already mentioned, strict hierarchies are grossly oversimplified because most physical or social hierarchies are at best not strict but loose and even elusive.

In any event, fractals go well beyond Sierpinski-like endlessly repeated geometrical figures. In fact, all random fractals of direct usefulness in modeling reality are defined by procedures other than recursive interpolation. For example, the BMMT process is constructed in the next paper without resorting to recursion. This feature follows upon the fact that fractional Brownian motion FBM is fully defined as being a Gaussian process that satisfies a few requirements that look very unspecific and bear absolutely no hint of strict hierarchy.

Naturally enough, no strict hierarchy is present in the outputs of FBM. However, Mandelbrot (1982) made the striking observation that the generating rule and the generated samples differ in a very significant and unexpected way. When presented with samples of non-hierarchical fractal processes, humans universally and spontaneously interpret them as involving a loose but widely agreed-upon hierarchy.

The prime example follows from the fact that Brownian coastlines are the source of many well-known fractal models, including the earliest fractal models of Earth's relief (again, see Mandelbrot 1982). Strikingly, Brownian coastlines are invariably perceived as filled with big bays and promontories on which "ride" smaller bays and promontories, and so on until scales become so small as to lack geographical significance.

A fact that is connected to the preceding observation (but in a rather complicated way) may be remembered by older students of economic statistics. Long ago, Slutsky observed that the curve obtained by adding gains and losses from coin-tossing seems to be decomposable into a hierarchy of cycles. The curve in question is essentially a Brownian motion and happens to be a fractal curve – an atypical one which no one has in mind when talking of fractal models! Those "Slutsky cycles" are total artefacts with no explanatory or predictive value whatsoever. I think it helpful to mention them here, but they must not be misconstrued: I agree with the general view that the cycles or economic hierarchies discussed in Section III are not artefacts but real.

C. Hierarchy versus Fractality: Last Comments.

It becomes clear now that, compared to simple hierarchy, fractality is a richer notion, enormously less constrained, hence amenable to far more flexible modeling.

The next important point is the following asset of fractal measurements like the dimension. They were introduced for the needs of strictly hierarchical structures, but their scope of validity is much broader and extends to all fractal structures. In particular, they make it possible to carry out numerical comparisons between real data, that are loosely but not strictly hierarchical, and fractal models whose samples are not really hierarchical but are perceived as such by all observers.

D. "Half-time" Summary.

The ambition of Sections III to V was to help sway the economists' opinion, from viewing fractals/multifractals as strange, to viewing them as plausible candidates for close examination and comparison with financial reality. In swaying this opinion from plausible to unavoidable, help is provided by the financial cartoons to which we now proceed.

VI. Example of a Cartoon Function of Time Constructed by Recursive Interpolation

A. From Hierarchies in Space to Hierarchies in Time and From Self-Similarity to Self-Affinity. A View of Scientific Research as a Search for Invariants.

Neglecting both seasonals and other constraints, trading must be viewed as proceeding in continuous time. To price options, if that is the goal, the probability distribution of the price change $P(t + \delta) - P(t)$ must be known for all values of δ . It does not help if a model is simple for some value of δ – for example, $\delta = 1$ day – but completely unwieldy for all other values. Such is often the case but a model cannot be called “simple” unless it is simple for all relevant values of δ . This is automatically the case for fractals.

Let us restate the characterization of fractals as geometric shapes that separate into parts, each of which is a reduced-scale version of the whole. As applied to finance, this concept is not a rootless abstraction but a theoretical reformulation of a down-to-earth bit of market folklore. Indeed, it is widely asserted that the charts of the price of a stock or currency all look alike when a market chart is enlarged or reduced so that it fits some prescribed time and price scales. This implies that an observer cannot tell which data concern prices change from week to week, day to day, or hour to hour. This property defines the charts as fractal curves and many powerful tools of mathematical and computer analysis become available.

The technical term for this form of close resemblance between the parts and the whole is *self-affinity*. This concept is related to the better-known property of *self-similarity*, which Section VA described as characteristic of the Sierpinski gasket. However, like all records of functions, financial market charts cannot be self-similar. If we simply focus on a detail of a graph, the features become increasingly higher than they are wide – as are the individual up-and-down price ticks of a stock. Hence, the shrinkage ratio from the whole to the parts must be larger along the time scale (the horizontal axis) than along the price scale (the vertical axis).

This is the kind of reduction performed by copiers using lasers. The geometric relation of the whole to its parts is said to be one of self-affinity.

Unchanging properties are not given much weight by most economists and statisticians. But they are beloved of physicists and mathematicians like myself, who call them invariances and are happiest with models that present an attractive invariance property. A good idea of what I mean is provided by a simple chart that uses recursion to insert (interpolate) price changes from time 0 to a later time 1 in successive steps. The intervals themselves can be interpreted at will; they may represent a second, an hour, a day or a year.

B. The Process of Recursion in an Increasingly Refined Grid.

We begin with price variation reduced to a "trend," that is, represented by a straight line called the "initiator," as shown in the top panel of Figure 4. Next, a broken line called the "generator" replaces the trend-initiator with a relatively slow up-down-and-up price oscillation. In the next stage, each of the generator's three pieces is interpolated by three shorter ones. One must squeeze the horizontal axis (time scale) and the vertical axis (price scale) to fit the horizontal and vertical boundaries of each piece of the generator. Repeating these steps reproduces the generator at increasingly compressed scales.

Only four construction stages are shown in Figure 4, but the same process continues. In theory, it has no end, but in practice, it makes no sense to interpolate down to time intervals shorter than those between trading transactions, which may be of the order of a minute. Each piece ends up with a shape like the whole, expressing scale invariance that is present simply because it was built in.

C. The Novelty and Surprising Creative Power of Simple Forms of Recursion.

The resulting very simply defined self-affine fractal curves can exhibit a wealth of structure. This finding exemplifies one of the most compelling features of fractal geometry. For this feature to be present and surprising, it is essential for the number and exact positions of the pieces of the generator to be completely specified and kept fixed. But those assumed microscopic properties do not amount to micromanaged representations in which everything of interest must be inputted separately. To the contrary, all that matters is some macromanaged features of the instructions.

Yet, as already mentioned, the construction's outcome, if plotted as in Figure 2, may be extremely sensitive to the exact shape of the generator. Indeed, Sections VII to X will show that generators that might seem close to one another may generate *qualitatively* distinct "price" behaviors. It will be necessary to construct a phase diagram in which different parts or "loci" lead to different behaviors. Being sensitive, the construction is also very versatile: it is general enough to range from the coin-tossing model's "mildness" to the "wild" and tumultuous real markets – and even beyond.

If, to the contrary, the generator fails to be exactly specified or if (worse!) one allows oneself the right to fiddle with the generator during the construction, no prediction could be made.

An analogous construction with a two-piece generator could serve diverse purposes but could not simulate a price that moves up and down. When the generator consists of many more than three pieces, it involves many parameters and creates the impression of micromanagement. Even if no accusation is made, the surprise provoked by the versatility of the procedure is psychologically dampened.

D. Randomly Shuffled Grid-bound Cartoons.

The recursion described in the preceding sections is called "grid-bound," because each recursion stage divides a time interval into three. This fixed pattern was chosen for its unbeatable simplicity and is clearly not part of the economic reality. Unfortunately, it remains visible even after many iterations, especially when the generator is symmetric, as in this paper. This artificiality is the main reason for referring to the resulting constructions as "cartoons." To achieve a higher level of realism, the next easiest step is to inject randomness. This is best done in two stages.

The first stage consists in randomizing the grids by shuffling the sequence of the generator's intervals before each use. Altogether, the fact that there are three intervals allow the following six permutations:

1,2,3; 1,3,2; 2,1,3; 2,3,1; 3,1,2 and 3,2,1.

Conveniently, a die has six sides; imagine that each bears the image of one of the six permutations. Before each interpolation, the die is thrown and the permutation that comes up is selected. When the generator is symmetric, there are only three distinct permutations and their effect is lesser.

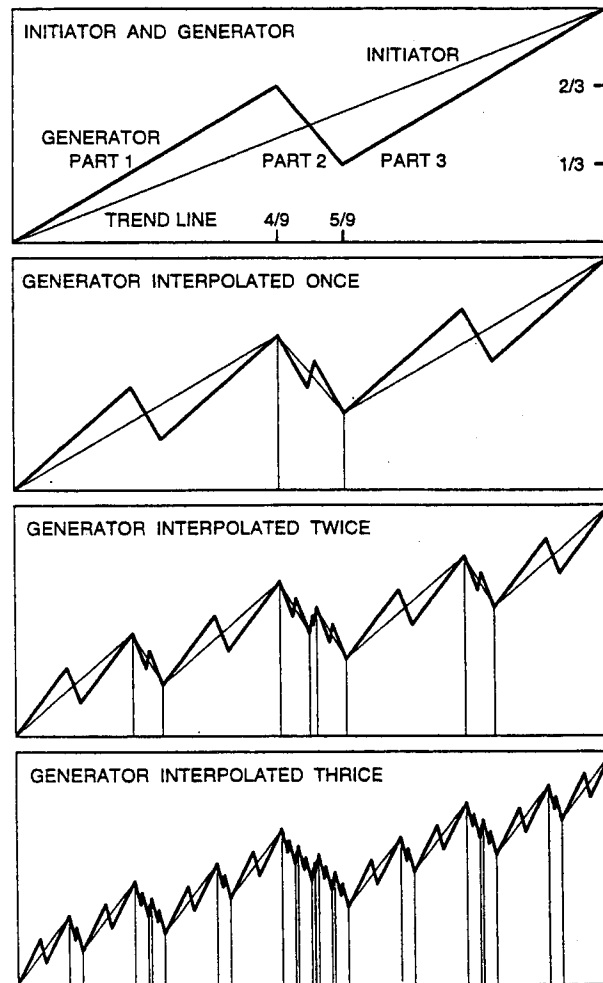


FIGURE H1-4. Constructing a "Fickian cartoon" of the idealized coin-tossing model that underlies modern portfolio theory. The construction starts with a linear trend ("the initiator") and breaks it repeatedly by following a prescribed "generator." The interpolated generator is inverted for each descending piece. In terms of Figure 1, the pattern that emerges increasingly resembles market price oscillation. However, this resemblance is misleading. The record of the increments of this pattern is thoroughly unrealistic because it is close to the top line of the more demanding Figure 2.

E. The Most Desirable Proper Randomization.

Shuffled cartoons have many virtues; however, the shuffled versions of all the cartoons we shall examine in sequence (Fickian, unifractal, mesofractal and multifractal) are grid-bound, therefore unrealistic. Fortunately, we shall see that each major category of cartoons was designed to fit a natural random and grid-free counterpart.

VII. Cartoons of Brownian Motion, Fickian and Beyond; Variance and Additional - Fuller and More Demanding - Measurements of "Volatility"

A. The "Fickian" Square-root Rule.

Moving from qualitative to quantitative examination, the non-shuffled Figure 4 uses a three-piece generator that is very special. Indeed, let the initiator-trend have one time unit as width and one price unit as height. This being granted, the generator intervals in Figure 4 are such that the heights of each - namely, $2/3$, $1/3$ or $2/3$ - are the square-roots of the corresponding width - namely, $4/9$, $1/9$ or $4/9$.

An integer-time form of this "square-root rule" is familiar in elementary statistics. Indeed, the standard deviation of the sum of N independent and identical random variables is the product of the standard deviation of each addend by \sqrt{N} . Therefore, the sum is said to "disperse" like \sqrt{N} .

In continuous grid-free time the square-root rule is called "Fickian" and characterizes the Wiener Brownian motion and "simple diffusion."

In our grid-bound interpolation, the square-root rule is strengthened by becoming non-random and weakened by holding only for the time intervals that belong to the recursive generating grid. The result is a behavior that is only pseudo-Brownian: it is, close, without being identical, to the continuous-time WBM.

Fickian diffusion is extraordinarily important in innumerable fields, but for financial prices it is not applicable. Fortunately, mildness is not a consequence of the recursive character of our construction, only of the special square-root rule imposed on the generator.

B. Symmetric 3-Interval Generators Beyond the Fickian Case. Their Basic "Phase Diagram."

Indeed, let us show how one can preserve the idea behind Figure 4 but modify it to allow for a wealth of behavior that differs greatly from the Brownian and from one another. We argued early in this paper that it is essential to keep those generalizations as simple as possible and capable of being followed on a simple two-dimensional diagram. It will suffice to preserve two features of Figure 4. Its generator includes 3 intervals and is symmetric with respect to the center of the original box.

Hence the coordinates of its first break determine those of the second break by taking complements to 1. It follows that a 3-interval symmetric generator is fully determined by the position of its first break. The resulting "phase diagram" is drawn as Figure 5. The point P will be called its "function address" in the "address space" defined as the left half of the unit square in Figure 5. Since we wish the construction to yield curves that oscillate up and down, all the possibilities will be covered by allowing the address to range over the top left quarter of the unit square. The bottom left quarter is also interesting but yields nondecreasing measures rather than oscillating functions. Many of those measures will turn up in a later section.

Active actual experimentation is very valuable at this stage and is accessible to the reader with a moderate knowledge of computer programming. Playing "hands-on," that reader will encounter a variety of behaviors that are extremely versatile, hence justify concentrating on 3-interval symmetric generators. Section VIIC lists rapidly the possibilities that will be discussed later in this paper.

C. Two Fundamental but Very Special Loci, Called "Unifractal" and "Mesofractal," and the "Multifractal" Remainder of the Phase Diagram.

The terms describing the simplest loci in Figure 5 are new but I have explored the underlying concepts in the 1960s.

Indeed, it will be seen in Section IX that the mesofractal cartoons correspond to my earliest partial improvement on Bachelier's work, namely the "M 1963" model built in Mandelbrot (1963) the stable random processes of Cauchy and Lévy. Price increments according to that model are illustrated by the rather simple second line of Figure 2. Compared to line 1 which reports on Bachelier, line 2 is less unrealistic, because it shows many spikes; however, these are isolated against an unchanging background in which the overall variability of prices remains constant.

As to the unifractal cartoons, they will be seen in Section VIII to correspond to my next earliest improvement on Bachelier, namely the "M 1965" model I built in Mandelbrot (1965) while introducing fractional Brownian motion. Price increments according to that model are illustrated by the third line of Figure 2. Compared to the M 1963 model, the strengths and failings were interchanged because it lacks any precipitous jumps.

The mesofractal and unifractal substitutes for coin-tossing deserved investigation but remain inadequate, except under certain special market conditions.

After examining those special regions, we shall proceed to the diagram's remainder. It consists of the multifractal cartoons which correspond to my current model of financial price variation, the "M1972/97 model" of fractional Brownian motion in multifractal trading time.

D. The "Special Root-mean-square" Definition of Volatility and Beyond.

The coin-tossing economics illustrated on the top line of Figure 2 has a single parameter, the root-mean square standard deviation σ and volatility is necessarily σ or perhaps an increasing function of σ . A strip of total width from -2σ to 2σ contains 95% of all price changes. If only implicitly, volatility is a relative concept: it concerns the comparison of the observed fluctuations to an ideal economy that achieved equilibrium and involves no fluctuation at all.

This implicit reference to equilibrium must be elaborated upon. Economics is clearly more complex than physics but the precise contrary would have been the case if the Brownian model were universally applicable. For example, the physical theory closest to coin-tossing finance is that of a gas in thermodynamical equilibrium, but such a system also depends on either volume or pressure. This is more complicated than coin-tossing which reduces to the parameter metaphorically closest to temperature.

The unifractal model illustrated on line 3 of Figure 2 and discussed in Section VIII is specified by σ and an exponent H . This H measures how far and how fast a constant-width "snake" oscillates along the time axis. H must be included in order to specify intuitive volatility quantitatively.

In the mesofractal model illustrated on line 2 of Figure 2 and discussed in Section IX, the standard deviation diverges. But the equally classical notion of percentiles remains meaningful. Hence volatility can be defined as including the two parameters that determine the process. One

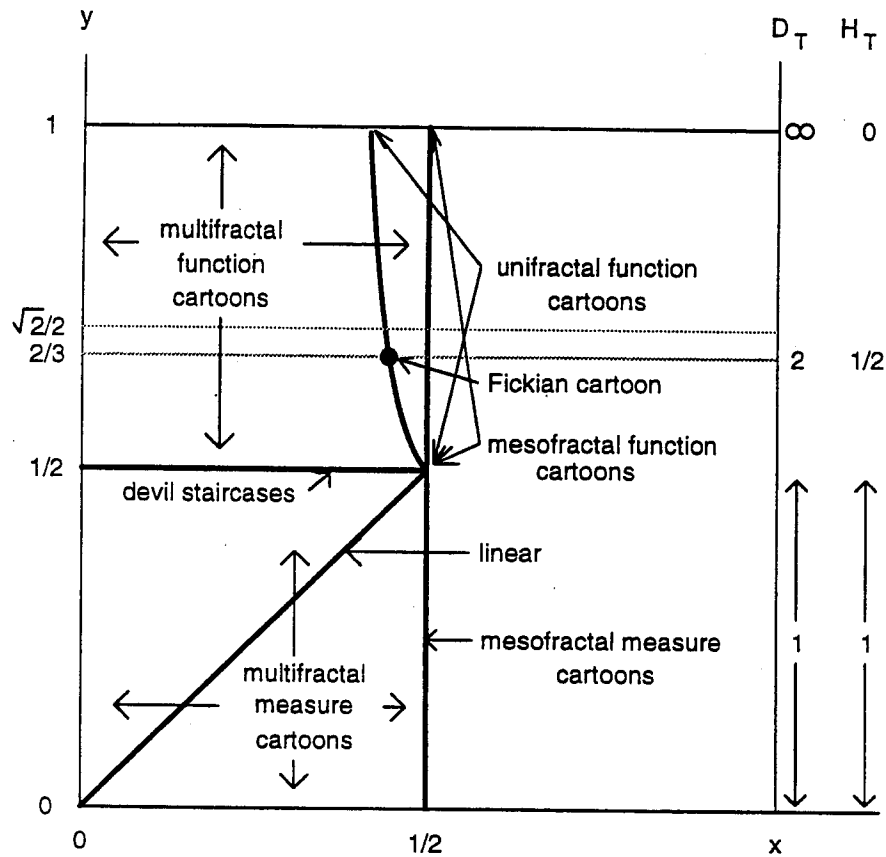


FIGURE H1-5. The “fundamental phase diagram” for the symmetric three-interval generator is drawn on the top left quarter of the unit square. Being restricted to 3 intervals, the generator is determined by the bottom left and top right corner of the square, plus two other points. Symmetry implies that those points are symmetric with respect to the center of the square. If the generated function is to be oscillating, the generator is determined by a point in the top left quarter, including its boundary to the right. This diagram is explored in four successive stages: first the “Fickian” dot, then the curved “unifractal locus” and the straight “mesofractal locus,” drawn in thicker lines starting at the center of the square. The final and most important stage of exploration tackles the remaining points in the upper left quarter; they form the “multifractal” locus, which is not a point or a curve but a domain. See also Figure 9.

is the width of the horizontal strip containing 95% of "price" changes. The second specifies the variability of the remaining 5% of large changes, which is ruled by an exponent α or its inverse, $H = 1/\alpha$.

VIII. Unifractals, Non-Periodic but Cyclic Behavior and Globality

A. The Exponent H and Equations that Characterize Unifractality.

Logically, if not quite so historically, cartoons that deserve to be called "unifractal" come immediately after the Fickian ones. Indeed, define the quantitative

$$\frac{\log(\text{ height of the } k\text{th generator interval })}{\log(\text{ width of the } k\text{th generator interval })} = H_k.$$

Given a single exponent that satisfies $0 < H < 1$, unifractality is defined by the condition that, $H_k = H$ for every k . The uniqueness of H is a major reason for the term "unifractal." The example of the Fickian "square-root" rule shows that the unifractality conditions can be implemented when $H = 1/2$. For other prescribed values of H , those conditions yield two "unifractality equations" $y = x^H$ and $2y - 1 = (1 - 2x)^H$. In particular, x is the root of the "generating equation" $2x^H - 1 = (1 - 2x)^H$. Solved numerically, as it must, this equation yields a single x , therefore a single $y = x^H$. That is, just as in the case $H = 1/2$, each allowable value of H is achieved by choosing for the function address P a single specified point in the address quarter square.

When lumped together, the points P form a "locus of unifractality" that takes the form of the only curve seen on Figure 5. This curve is of course far more restrictive than the whole allowable address space, which is a quarter square. The unifractality locus contains the unique Brownian address $(4/9, 2/3)$ but is far less restrictive.

We proceed to restate the unifractality condition in an alternative form and then in terms of a new quantity θ that will become essential in the multifractal case. The unifractality conditions can be rewritten as $(2y - 1)^{1/H} = 1 - 2x$ and $x = y^{1/H}$; eliminating x , these combine into $y^{1/H} + (2y - 1)^{1/H} + y^{1/H} = 1$.

This expresses that the sums of the intervals' absolute heights becomes 1 after each absolute height is raised to the same power. It follows that

one can define the quantities $\Delta_1\theta = y^{1/H}$, $\Delta_2\theta = (2y - 1)^{1/H}$, and $\Delta_3\theta = y^{1/H}$, which satisfy $\Delta_1\theta + \Delta_2\theta + \Delta_3\theta = 1$.

The auxiliary address point of coordinates x and $y = \Delta_1\theta$ will be called the generator's "time address." This unifractal case yields $\Delta_1\theta = x$, therefore the time address is located on the bisector of our diagram, between $(1/2, 1/2)$ and the $H=0$ point $(\sqrt{2} - 1, \sqrt{2} - 1)$ explained in the next paragraph. The time address of a generator fully determines its function address.

B. Persistence and Cyclic but non-Periodic Behavior.

Let us begin by extreme cases not included in the locus of unifractality.

The limit $H \rightarrow 0$. It corresponds to $y = 1 - \varepsilon \sim \exp(-\varepsilon)$, and $2y - 1 \sim \exp(-2\varepsilon) \sim y^2$. Hence the generating equation written in terms of $y^{1/H} = x$ becomes $x^2 + 2x - 1 = 0$, yielding $x = \sqrt{2} - 1$.

Combining this x and $y = 1$, the 3 intervals of the generator have heights $\Delta f = 1$, $\Delta f = -1$ and $\Delta f = 1$. In order to add to 1, the correlations between those three increments' are *negative* and as strong as can be. The limit is degenerate. But after an arbitrary number of recursions, each step in the approximation is equal in absolute value to 1, which is the increment of the function between any two points in the construction grid. This property is extreme but will be worth remembering when Section IXB discusses asymptotic negligibility.

The limit $H = 1$. It corresponds to a vanishing middle interval, therefore to a straight generator and a straight interpolated curve. In this disallowed limit case, price would be totally ruled by "inertia" and "persist" forever in its motion.

The Fickian $H = 1/2$. It represents a total absence of persistence.

In the $0 < H < 1/2$ part of the unifractal locus, there is a negative persistence or antipersistence.

In the more important $1/2 < H < 1$ part of the unifractal locus, persistence is positive and increases as H moves from $1/2$ to 1. It deserves close attention.

Let us now relate cyclic behavior and globality in graphs of a function $f(t)$ rather than of increments. The phenomenon of persistence manifests itself in patterns of change that are not periodic but perceived by everyone as "cyclic." As already mentioned, it was observed long ago by Slutsky

that the eye decomposes Brownian motion spontaneously into many cycles of periods ranging from very short to quite long. As the total duration of the sample is increased, new cycles appear without end. They correspond to nothing real, only the mere juxtaposition of random changes. To appreciate this fact, one should rethink the positive overall trend that is highly visible on Figure 4. Over a time space much shorter than the total space 1, the trend becomes negligible in comparison with local fluctuations. Hence, the up-down-up oscillation represented by the generator will be interpreted as a slow cycle.

As H increases above $1/2$, so does the relative intensity of this longest period cycle. It also ceases to be meaningless (à la Slutsky) and becomes increasingly real. While it does not promise the continuation of a periodic motion, it allows a certain degree of prediction. A nice illustration of what is happening is provided in a closely related context by Plates 264 and 265 of M 1982F(FGN). This is one aspect of the following property common to all values $H \neq 1/2$: the successive movement of $f(t)$ are not simply juxtaposed. In effect, they interact, their interdependence not being short, but long-range, or "global."

In any event, unifractal cartoons fail to generate either a variable volatility or the large spikes of variation that Figure 2 shows to be characteristic of finance. Therefore, the generalization of Fickian square-root must go beyond unifractality.

C. The Rule of "Three Cycles in Every Sample," and the Kondriateff Long Cycles of the Economy.

When H is about $3/4$, as is often the case, I found that the eye sees "about three cycles in a sample." This "three cycle" rule is a remarkable observation that cannot be elaborated here. It may perhaps help, or even suffice, to explain the celebrated but highly controversial slow cycles of the economy. Kondriateff, to whom they are credited, had a century worth of data and the slow thirty-odd years long cycles that he saw.

IX. Mesofractals and Price Discontinuity

A. The Locus of Discontinuous Behavior and the Distribution of the Discontinuities.

In the square that bounds the phase diagrams in Figure 5, discontinuous functions are associated with the unit length interval characterized by $x = 1/2$ and $0 < y < 1$. Aside from the Fickian point, this locus is the simplest and has the oldest roots in finance, insofar as the portion $1/2 < y < 1/\sqrt{2}$ will soon be linked with the M 1963 model of price variation (Mandelbrot 1963, 1997).

Recall the quantities

$$\frac{\log(\text{height of the } k\text{th generator interval})}{\log(\text{width of the } k\text{th generator interval})} = H_k.$$

For $x = 1/2$, the middle interval satisfies $H_k = 0$ and for the side intervals – by definition of \tilde{H} – satisfy $H_k = \tilde{H} = \log y / \log(1/2)$. The presence of two, not one, separate fractal exponents implies that the present construction no longer qualifies as unifractal. It is useful to define the intermediate category of mesofractality by the condition that $H_k = 0$ for at least one value of k and $H_k = \tilde{H}$ when $H_k \neq 0$. For reasons to be explained momentarily, one should denote $1/\tilde{H}$ as α . In the present very special generator, the exponents H and \tilde{H} are both functions of y , hence of each other, but this very peculiar feature disappears for more general cartoons.

To understand the role of our interval of overall length 1 and abscissa $x = 1/2$ as the locus of discontinuous behavior, let us continue the recursive repetition of the generator. The next iteration adds two smaller discontinuities of size $-y(2y - 1)$. Further iterations keep adding increasingly high numbers (4, 8, 16 and higher powers of 2^k to infinity) of increasingly smaller discontinuities of size $\lambda = -y^{k-1}(2y - 1)$. Altogether, the number of discontinuities of absolute size $> \lambda$ is easily seen to become, for small λ , proportional to $\lambda^{-\alpha}$. Depending on the exponent α , the discontinuity locus splits into three portions to be handled separately.

The portion from $0 < y < 1/2$, that is $0 < \alpha < 1$, corresponds to positive discontinuities hence to increasing functions. They generate a fractal trading time that is better discussed later, as a special case of the multifractal trading time.

The portion $1/2 < y < 1$, that is $\alpha > 1$, corresponds to negative discontinuities, hence to oscillating functions.

The notation α was adopted because when $\alpha < 2$, the above-written distribution of the discontinuities is the same as in the L-stable processes used in the M 1963 model. More precisely, all the jumps are negative here, while in the M 1963 model of price variation they can take either

sign. A distribution with two long tails can be achieved by using generators that include a positive and a negative discontinuity; this requires more than 3 intervals.

We digress on two technical questions that must be mentioned but can only be sketched. Firstly, why is the L-stable exponent α bounded by 2? The issue arises during attempted randomization of the mesofractal cartoons. Randomization involves the replacement of fixed numbers of discontinuities by random (Poisson) numbers. When $\alpha < 2$, this replacement introduces convergent integrals, but when $\alpha > 2$, it would introduce divergent integrals one cannot "renormalize."

Secondly, consider the quantity H defined as in the unifractal case by the previously used generating equation Σ (interval height) $^{1/H} = 1$, which will become essential in the multifractal case. When the vertical interval is excluded, the equation becomes $2y^{1/H} = 1$. Its solution is $\hat{H} = 1/\alpha$. When the vertical interval is not excluded, the solution is different from \hat{H} . The difference expresses that approximation the address point P with $x = 1/2$ from the left by the sequence of address points P_k is a singular process. This term means that the properties of the $f(t)$ corresponding to the point P are not the limits of the properties of the $f_k(t)$ corresponding to the point P_k .

B. Asymptotic non-Negligibility, the Mesofractal form of Concentration and an Additional Aspect of Globality.

The notion of concentration is most familiar in the context where it arose: firm sizes. Even in an industry that contains a large number of firms, concentration expresses that the largest firm's size is typically far larger than the average or median size. In highly concentrated industries, the largest firm's size may approach or exceed the size of all of the other firms taken together. The same is true of populations of cities.

Formulas simplify if one begins by ordering all firms by decreasing size within their industry, then reducing all firm sizes by division through their sums. Let S_r be the reduced size of the firm of rank r in the order by decreasing size. One has $\Sigma S_r = 1$. For the present purposes, let us say that if S_1 is small the industry is called non-concentrated. The higher S_1 the higher the concentration. High levels of concentration are well known for individuals' wealth or firm sizes. To represent those quantities, the Gaussian distributor is not inappropriate in degree, but in kind: it resides in a totally wrong "ballpark."

Fractals/multifractals clarify the notion of concentration and introduces it to the study of finance under two successive temporal versions, to be called meso- and multifractal. These versions help clarify the properties of the mesofractal and multifractal models and give fresh scope to a central issue of economics, namely, inequality.

While firm sizes are positive, price changes are not. To extend to them the notion of concentration, the simplest is to replace price changes by their squares.

For the sake of background and contrast, begin with coin-tossing. Every day's contribution could be ± 1 , and its square contribution would be 1. Over N days, each day's relative contribution to the sum is simply $1/N$, hence every contribution rapidly becomes negligible. Continue with the Brownian model. Its theoretical daily volatility is the expectation of the quantity $[P(t + \text{day}) - P(t)]^2$. Its empirical volatility is the sample average of the same quantity. This sample average rapidly converges to the expectation so that after N days the relative contribution of the wildest day is of the order of $1/N$ except for a negligible logarithmic factor.

The preceding property is called "asymptotic negligibility" of every individual contribution. It extends to the Fickian and other unifractal recursive cartoons. A heuristic argument proceeds as follows. Instead of pursuing the recursive contribution for the same number of steps throughout, prescribe $\epsilon > 0$ and stop the recursion as soon as the width of the intervals of the approximation becomes $< \epsilon$. The remaining intervals' widths Δx will range from $\epsilon(1 - 2x)$ to ϵ , where x as usual, is the abscissa of the function address P . Each of the remaining intervals contributes to $f(t)$ the amount $\pm(\Delta t)^H$; all those amounts become negligible as $\epsilon \rightarrow 0$.

Asymptotic negligibility is a wonderful notion in pure probability and in the study of many random phenomena. But it fails to account for important features in finance. As an example, consider a well diversified portfolio following the Standard & Poor 500 Index. Of the portfolio's positive returns over the 1980s, fully 40% was earned during *ten* days, about 0.5% of the number of trading days in a decade.

In the Brownian model, such a high level of concentration is not strictly impossible, but its probability is so minute that it should *never* happen. Unfortunately, by and large, it happens every decade.

Mesofractal cartoons behave in a totally different fashion: asymptotic negligibility is completely invalid and concentration prevails just as for firm sizes. Postponing the proof to the next paragraph, the conclusion can be stated in two approximations.

In a first approximation, mesofractality brings the variation of financial prices within a conceptual framework that is sufficiently broad to also accommodate the distributions of wealth and firm sizes. In a second approximation, however, we see that mesofractality goes too far and predicts a level of concentration that exceeds what is observed. This qualitative “mismatch” (which my earlier publication did not recognize sufficiently) is very important. It is one of many reasons why it is necessary to proceed beyond mesofractality to multifractality.

Proof of concentration for the above special mesofractal cartoon. Observe that after k iterations, the variation of $f_k(t)$ consists in $2 \cdot 2^k - 1$ intervals, alternatively inclined up and vertical down. Adding an arbitrarily small step down leaves 2^k “two-steps,” each defined as made of a step up increasingly short and steeply inclined, and a vertical step down.

The largest two-step's length is $-(2^k - 1)$, plus a quantity that is positive but small when k is large. Therefore (aside from its sign), the largest two-step is of the same order of magnitude as the *total* of all the two-steps, which is equal to one. The same – a fortiori – is true of the squares of the step.

I propose to call this property “mesofractal – or simply fractal – concentration,” to distinguish it from the more general and in a way less extreme “multifractal concentration” examined in Section XF.

X. Multifractals: Cartoons and Trading Time Functions

A. Variable Volatility, Revisited.

In the state diagram in Figure 5, the address points left to be examined belong to the top left quarter of the square but to neither the unifractal nor the mesofractal locus. For reasons that will transpire soon, those points will be called “multifractal.”

Back to Figure 2, focus on the five bottom lines. It was said that they intermix actual data with the best-fitting multifractal model. Asked to analyze any of those lines without being informed of “which is which,” a coin-tossing economist would begin by identifying short pieces here and there that vary sufficiently mildly to be approximated by suitable pieces of white Gaussian noise. These are pieces extracted from the first line, then widened or narrowed by being multiplied by a suitable r.m.s. volatility σ .

Those irregular records might have been increments of a non-stationary Brownian motion, a motion whose volatility varies in time. Furthermore, it is tempting to associate those changes in volatility to changes in market activity. Less mathematically-oriented observers describe the diverse lines at the bottom of Figure 2 (both the real data and forgeries) as corresponding to markets that proceed at different "speeds" at different times. This description remains purely qualitative until "speed" and the process that controls the variation of speed are quantified.

A similar situation occurring in physics should serve as warning. It concerns the notion of variable temperature. This is a forbiddingly messy and complicated problem and the best approaches are ad-hoc and not notable for being attractive or effective. The totally distinct approach that I took and to which we now proceed, consists in "leap-frogging" over nonuniform gases, all the way to turbulent fluids.

B. The Versatility of Multifractal Variation. In a Non-Gaussian Process, the Absence of Correlation is Compatible with a Great Amount of Structure; this Fact Reveals a Blind-spot of Correlation and Spectral Analysis.

Figure 6 illustrates a stack of multifractal cartoons that are shuffled at random before each use. In all cases, the ordinate of P is $2/3$, therefore $H = 1/2$. The column to the left is a stack of generators, the middle column, the stack of processes obtained as in Figure 4 but with shuffled generators, and the column to the right, the stack of the corresponding increments over identical time-increments Δt .

The line marked by a star (\star) is the shuffled form of Figure 4. The middle column is a cartoon of Brownian motion and its increments (right column) are a cartoon of white Gaussian noise and look like one, as expected. But what is quite unexpected is that the increments plotted on all the other lines in this stack are also uncorrelated to one another, that is, "spectrally white." As one moves up or down the stack, one encounters charts that diverge increasingly from the pseudo-Brownian model. Increasingly, they exhibit the combination of sharp, spiky price jumps and persistently large movements that characterize financial prices. The existence of such sharply non-Gaussian white noises proves that spectral whiteness, which is highly significant for Gaussian processes, is otherwise a rather weak constraint.

Figure 6 brings to this old-timer's mind an old episode that deserves to be revived because it carries a serious warning. After Fast Fourier Transform emerged, the newly-practical spectral analysis was promptly

applied to price change records. An approximately white spectrum emerged, and received varied interpretations. It was widely known that whiteness does not express statistical independence, only absence of correlation. But the temptation existed to view that distinction as mathematical nit-picking. Numerous scholars went on to list spectral whiteness as an experimental argument in favor of the Brownian motion or coin-tossing model. Other scholars, to the contrary, apparently recognized that data was in fact qualitatively incompatible with independence. Finding spectral whiteness to be incomprehensible, they dropped the spectral tool altogether as being unmanageable.

The hasty assimilation of spectral whiteness to independence was understandable but is clearly untenable. Figure 6 exhibits a variety of white noises whose high level of dependence is not a mathematical oddity but the inevitable result of self-affinity with the multifractal cartoons. By and large, points P close to the Fickian locus of Figure 5 will "tend" to produce wiggles that resemble those of financial markets. As one moves farther from the center, the resemblance tends to decrease then the chart becomes more extreme than any observed reality.

C. A Fundamental Representation Called "Baby Theorem."

Irresistibly, the question arises, can the overwhelming variety of white or non-white multifractal cartoons f be organized usefully? Most fortunately, it can, thanks to a remarkable representation that I discovered and facetiously called "baby theorem." It begins by classifying the generator by the values of H or equivalently of y .

On Figure 7, the small "window" near the top left shows the generators of two functions $f_u(t)$ and $f_m(t)$. One is unifractal with an address having the coordinates $x = x_u = 0.457$ and $y = 0.603$, hence $H = 0.646$. The other's address coordinates are the same y and H , but $x = x_m = 0.131$. This x_m is so small that the function $f_m(t)$ it generates is very unrealistic in the study of finance; but an extreme x_m was needed for Figure 7 to be legible. To transform a unifractal into a multifractal generator, the vertical axis is left untouched but the right and left intervals of the symmetric unifractal cartoons are shortened horizontally thus providing room for a horizontal lengthening of the middle piece.

Before we examine this transformation theoretically, it is useful to appreciate it intuitively. The body of Figure 7 illustrates the graphs of $f_u(t)$ and $f_m(t)$ obtained by interpolation using the above two generators. Examine them disregarding the bold portions, the dotted lines and the

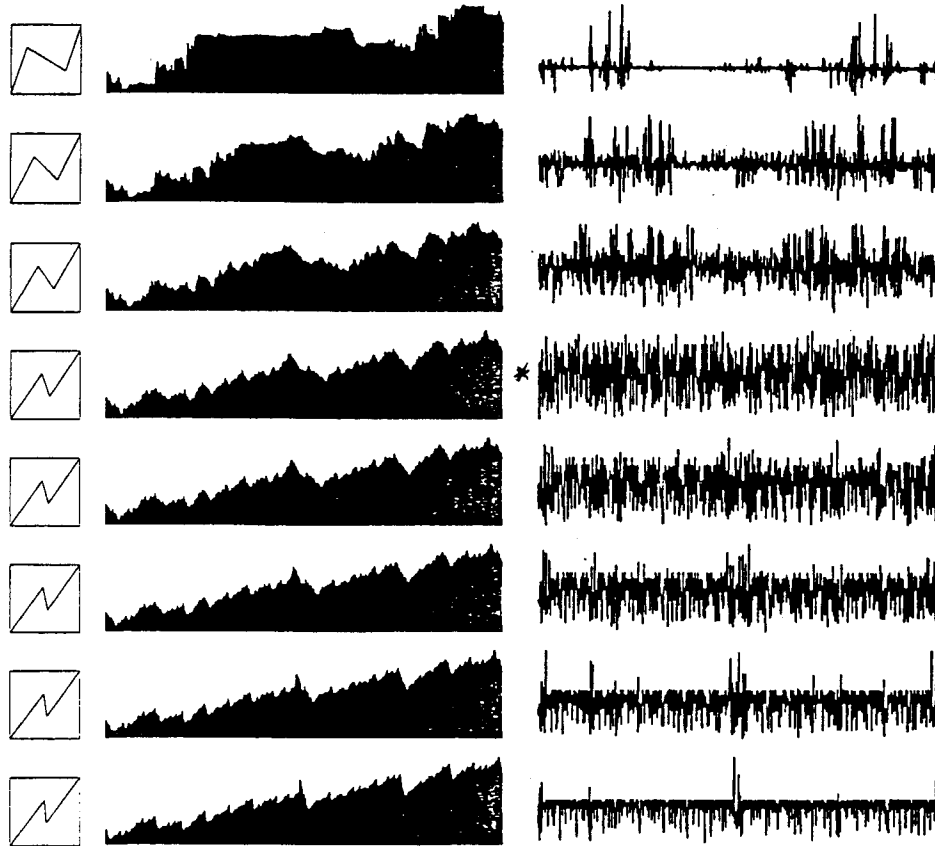


FIGURE H1-6. Stack of shuffled multifractal cartoons with $y=2/3$ therefore $H=1/2$ and – from the top down – the following values of x : 0.2222, 0.3333, 0.3889, 0.4444 (Fickian, starred), 0.4556, 0.4667, 0.4778, and 0.4889. Unconventional but true, all the increments plotted in the right column are spectrally white. But only one line in that column is near-Brownian; it is the starred Fickian line for $x=4/9$.

arrows. One observes that the unifractal curve $f_u(t)$ proceeds as already known, in measured up and down steps, while the multifractal curve $f_m(t)$ alternates periods of very fast and very slow change.

However, the fact that the two generators' addresses have the same y and H establishes a perfect one-to-one correspondence between "corresponding" pieces of two curves. This feature is emphasized by drawing three "matched" portions of each curve more boldly. Toward the right, between a local minimum and a local maximum, a gradual rise of the unifractal corresponds to a much faster rise of the multifractal. In the middle, between a local maximum and the center of the diagram, a gradual fall of the unifractal corresponds to a very slow fall of the multifractal - largely occurring between successive "plateaux" of very

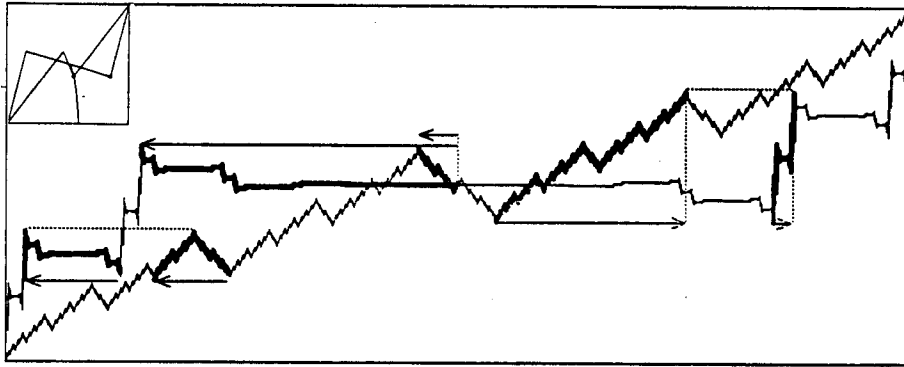


FIGURE H1-7. The small diagrams illustrate a unifractal and a multifractal generator corresponding to two address points situated on the same horizontal line in the phase space. The large diagram illustrates the resulting functions $f_u(t)$ and $f_m(t)$ and the one-to-one correspondence between them governed by the change from clock to trading time.

slow variation. Thirdly, between two local minima towards the left, a symmetric up and down unifractal configuration corresponds to a fast rise of the multifractal followed by a slow fall which, once again, proceeds by successive plateaux.

More generally, the choice of generators that share a common y insures that our two curves move up or down through the same values in the same sequence, but not at the same times. One would like to be more specific and say that they proceed at different "speeds," but the fractal context presents a major complication, already mentioned near the end of Section III. Differential calculus teaches us that when a function $f(t)$ increases by Δf when time increases by Δt , the ratio $\Delta f/\Delta t$ has a limit for $\Delta t \rightarrow 0$; this limit defines the derivative, which in turn measures the speed of variation.

Until recently, most sciences could take for granted that derivatives exist. But our cartoons have no positive and finite derivative. This fact is widely known to hold (almost surely, for almost all t) in the Brownian case. From the Fickian relation $\Delta f \sim \sqrt{\Delta t}$, it follows that, "as a rule," $\Delta f/\Delta t$ tends to ∞ as $\Delta t \rightarrow 0$.

But it was already mentioned at the end of Section III that not everything is lost. Indeed, there exists a non-traditional expression, $\log \Delta f / \log \Delta t$, that is not part of elementary differential calculus, but is well-behaved for the WBM $B(t)$. As $\Delta t \rightarrow 0$, it converges (for practical purposes) to a quantity called a Hölder exponent which coincides with $H = 1/2$.

More generally, in a unifractal cartoon all the increments in time Δt prove to be of the form $\Delta f_u(t) \sim (\Delta t)^H$, where the Hölder exponent H is identical to the constant that characterizes a unifractal.

Multifractal increments are altogether different. The theory shows that they also take the form $\Delta f_m(t) \sim (\Delta t)^{H(t)}$. However, $H(t)$ is no longer a constant but oscillates continually and can take any of a multitude of values. This is one of several alternative reasons for the prefix "multi-" in the term "multifractals."

In the present context of 3-interval symmetric generators, one has $0 \leq \min H(t) < 1$. The quantity $\min H(t)$ approaches 0 as one tends to the mesofractal locus corresponding to discontinuous variation.

As to $\max H(t)$, Section DXF will reveal two possibilities. For P_s located to the right of the unifractal locus, $\max H(t)$ cannot take a very large value; bounded values of $H(t)$ correspond to regions of near constant "volatility" of $f(t)$. But if P is located to the left of the unifractal locus,

$\max H(t)$ may be very large, corresponding to regions where $f(t)$ exhibits almost no volatility. This variety of possible behaviors is a major reason for the versatility of the multifractals. It is also, of course, a source of complexity, but this feature must be viewed as welcome, because the data themselves are indeed complex.

D. Compound Functions in Multifractal Trading Time.

Fortunately, this variety translates easily into the intuitive terms that were reported when discussing variable volatility. The key idea has already been announced: One can reasonably describe $f_u(t)$ as proceeding in a "clock time" that obeys the relentless regularity of physics. To the contrary, Section DXD implements the notion that $f_m(t)$ moves uniformly in its own subjective "trading" time, which – compared to clock time – flows slowly during some periods and fast during others.

The implementation generalizes the generating equation $y^{1/H} + (2y - 1)^{1/H} + y^{1/H} = 1$ already written down in the unifractal case, where it was of no special significance. Once this equation's root H has been determined, one defines (as before) the three quantities $y^{1/H} = \Delta_1\theta$; $(2y - 1)^{1/H} = \Delta_2\theta$ and $y^{1/H} = \Delta_3\theta$. Like in the unifractal case, these quantities satisfy $\Delta_1\theta + \Delta_2\theta + \Delta_3\theta = 1$. Moreover, $\Delta f_m = (\Delta\theta)^H$ as long as $\Delta\theta$ is an increment of θ that belongs to the hierarchy intrinsic to the generator.

The striking novelty brought by multifractality is that θ is a function of t that no longer reduces identically to t itself. That is, the time address $(x, y^{1/H})$ no longer lies on an interval of the main diagonal of the phase diagram. Instead, it lies within a horizontal rectangle that is defined by $0 < x < 1/2$ and $0 < y < \sqrt{2} - 1$. For given H , the rectangle reduces to a horizontal line. In Figure 7, the times taken to draw the generator's first interval are as follows: our unifractal $f_u(t)$ takes the time 0.457 and our multifractal $f_m(t)$ takes the extraordinarily compressed time 0.131. In the generator's middle interval, to the contrary, the multifractal is extraordinarily slowed down.

An extremely special case of compounding is familiar to econometricians. Called "subordination," it corresponds to the case where $\theta(t)$ is a random function with *independent* increments. Chapter E 21 of Mandelbrot (1997) reproduces a 1967 paper in which Taylor and I took for $\theta(t)$ a special process of independent positive increments, namely a process of L-stable increments of exponent $\alpha/2$. Brownian motion followed in this trading time reduces to the L-stable process postulated by the M 1963 model. More generally, independent increments in $\theta(t)$ lead to

a compound process that has independent increments and is called subordinated. The same Chapter E 21 also reproduces critical comments I published just after a 1973 paper by Clark. That paper selected a different subordinator $\theta(t)$, but preserves independent increments. Therefore, it also led to a price process with independent increments.

Many econometricians elaborated on Clark without questioning this independence. From their viewpoint, compounding that allows dependence may deserve to be called "generalized subordination." This usage (aside from being a bit silly) would be unfortunate because it would blur a major distinction. Preservation of the original association of subordination with independent price increments, clearly brands subordination as being unable to account for the obvious dependence in price records. Multifractal time shows one can account for dependence while preserving the reliance upon invariances I pioneered in 1963 and extending the path Taylor and I opened in 1967. Of course, the search for non-independent compounding could have proceeded in more traditional fashion, but it did not.

E. The Multifractal Behavior $\Delta f_m = (\Delta t)^{H(t)}$.

The theory of multifractals expresses the relation between θ and t as $\Delta\theta = (\Delta t)^{U(t)}$, therefore $\Delta f_m = (\Delta\theta)^H = (\Delta t)^{HU(t)} = (\Delta t)^{H(t)}$. This decomposition creates a "compound function," namely an oscillating unifractal function of exponent H , with the novelty that it proceeds in a trading time that is a non-oscillating multifractal function of clock time. Specifically, when $H = 1/2$, one has a Fickian (pseudo-Brownian) function of a multifractal time. When $H \neq 1/2$, one has a pseudo-fractional Brownian function of multifractal time.

In the next paper, the BMMT continuous time grid-free versions of the present cartoons will be seen to allow H and the multifractal time to be independent random variables. On this account the cartoons are more constrained and more complicated. In particular, the unifractal oscillation and the multifractal time cannot be chosen independently. Indeed, the address (x, y) of the unifractal function determines H and restricts the time address of the multifractal time to have the ordinate $y^{1/H}$ and an abscissa satisfying $x > 0$, $x \neq y^{1/H}$ and $x < 1/2$. However, those constraints are not fundamental but a peculiar feature of 3-interval symmetric generators. As the number of intervals in the generator increases, those constraints change; I expect them to become less demanding.

In any event, once again, the 3-interval symmetric generators do not pretend to exhaust all the possibilities offered by either theory or the facts. Their main virtue is to allow a wide range to illustrate the breadth of properties compatible with a very simple method of construction.

F. Fine-tuning of the Intermittence; the exponent $D(1)$ and its Role.

The next task is to confirm and quantify what Figure 7 tells us. A fully-developed theory exists and will now be sketched.

When analyzing a mesofractal cartoon in terms of "two-steps," the end of Section IXB noted that the largest two-step contributed a "price" change proportional to the sum of all the other two-steps, and independent of their number, N . That is, the largest two-step is of the order of N^0 ; the value of y does not affect the exponent, only a prefactor of proportionality. The largest two-step is also of the order of (sum)⁰. The same exponent also holds for the contribution the largest squared price changes make to the sum of the squares. The same extreme degree of mesofractal concentration occurs in the M 1963 model, which on this account is no more realistic than the mesofractal cartoon.

The intermittence exponent $D(1)$ for $H = 1/2$, that is, $y = 2/3$. In that case, consider a sum of N squared daily price changes, and denote by $M(N)$ the number of days that contributes the overwhelming bulk of that sum. The theory of multifractals tells us that $M(N) \sim N^{D(1)}$, where the exponent $D(1)$, a function of the address point P , is a new form of fractal dimension. This $D(1)$ originates in the fact that viewed in terms of the clock time t , the trading time is what mathematicians call a continuous singular function. It increases in every interval of clock time. However, most intervals contribute almost nothing. To the contrary, an arbitrarily high proportion of its variation occurs on a "support" that is a set of fractal dimension $D(1)$.

As expected, because of asymptotic negligibility and near-equality of the addends, $D(1) = 1$ in the unifractal special case, in which $M(N) \sim N$. At the other end $D(1) = 0$ in the mesofractal limit $x = 1/2$, in which $M(N) \sim N^0$, and also for $x = 0$. The properly multifractal cases yield $0 < D(1) < 1$. As one moves away from the unifractal locus marked on the phase diagram on Figure 8, the line $y = 2/3$ intersects the wavy curves at values of x that yield $D(1) = 0.9, 0.8, 0.7, 0.6, 0.5, 0.4$ and 0.3 . As x and therefore $D(1)$ decrease, the degree of intermittence seen on Figure 6 will increase. Therefore, a good definition of the degree of intermittence must include

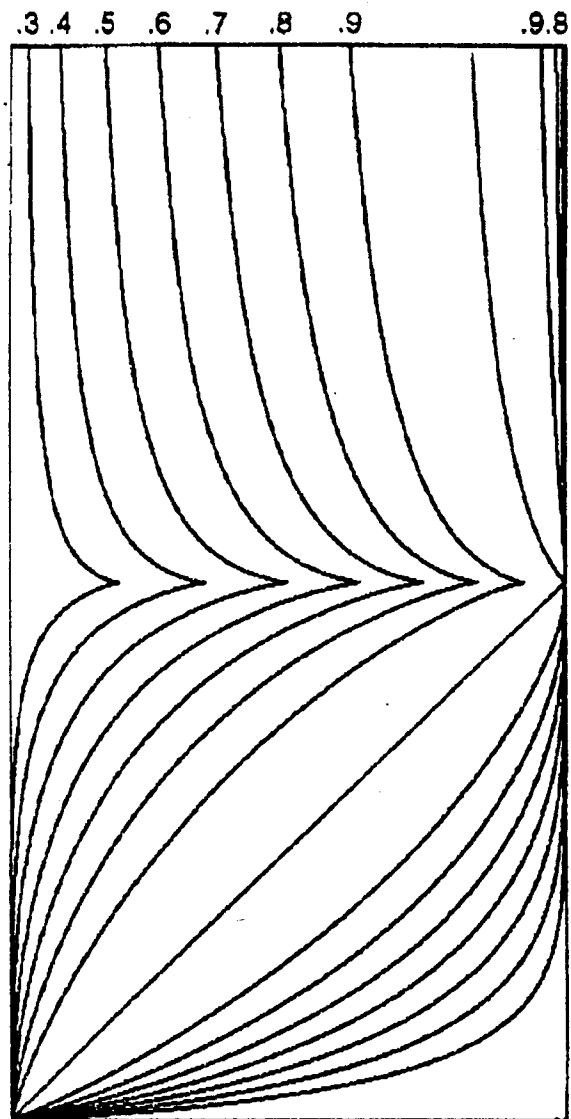


FIGURE H1-8. Iso-lines (lines of constant value) for the exponent of multifractal concentration, $C(1)$. It attains a maximum $D(1) = 1$ along the unifractal locus and the interval $0 < x = y < 1/2$; and decrease to 0 as y is fixed and x increases or decreases.

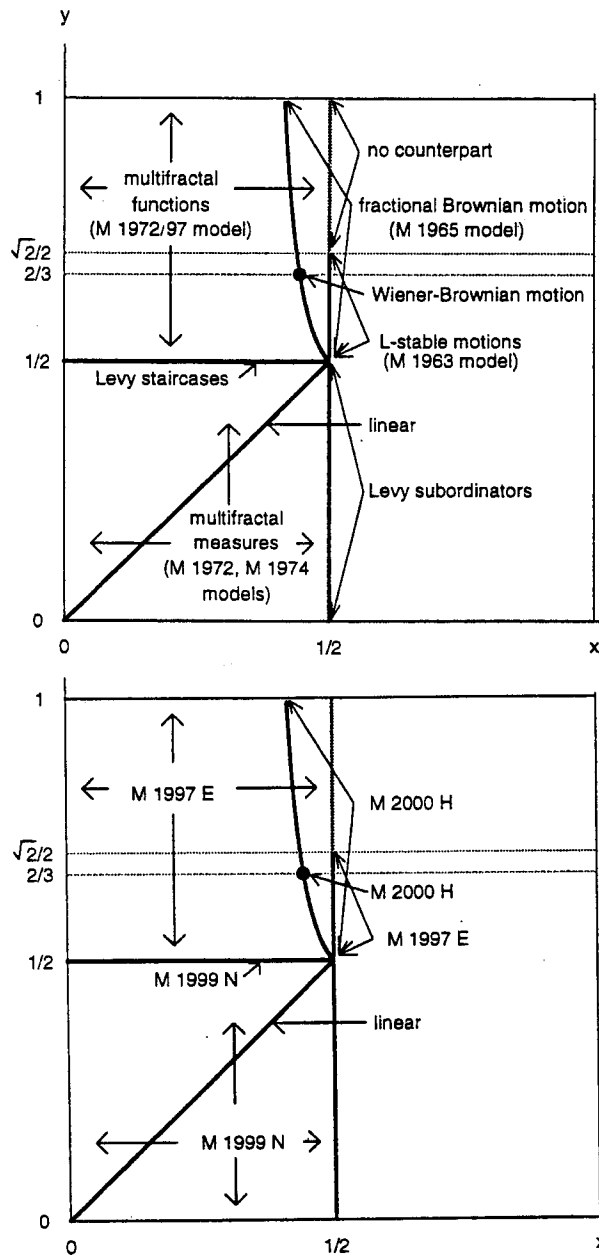


FIGURE H1-9. Two alternative versions of Figure 6, as explained in the text.

the quantity $1 - D(1)$. Section XH will show that one must also include other quantities.

The intermittence exponent $D(1)$ for H other than $1/2$. The interpretation of Figure 8 becomes a little different. The reason is subtle and can only be sketched here. It concerns the question of the best way to measure the deviations from the mean. The "normal" measure is once again the variance and its justification combines reasons of convenience and of principle. The old and universally valid reason of convenience is that variance is manageable with a slide-rule; before the computer, no alternative was present but the computer made this reason less compelling. An additional objective reason is often present in physics: a sum of squares is often an intrinsic quantity (for example, as energy) following basic laws of physics (for example, conservation). Another properly physical objective reason is restricted to the case of independent Gaussian variables: in that case, the first and second moments provide a "sufficient statistic."

Of these three reasons, only the last extends to the multifractal cartoons and it only holds for $H = 1/2$. When $H \neq 1/2$, the combination of multifractals and FBM puts forward a different intrinsic expression: the sum of absolute price increments raised to the power $1/H \neq 2$. Roughly speaking, it corresponds to the sum of increments of trading time over equal increments of clock time.

As to the expression $M(N) \sim N^{D(1)}$, its validity extends to $H \neq 1/2$, but only if, instead of being squared, the price increments are raised to the power $1/H$. We can now interpret the wavy lines beyond their intersections by the line $y = 2/3$. They are the loci where $D(1)$ takes the values 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, and 0.3.

G. A New, Multifractal Form of Concentration Based on $D(1)$

Let us now compare the data with the predictions of the mesofractal and the coin-tossing models that concern concentration.

No one denies that price variation is, in fact, concentrated, in contradiction with coin-tossing model of constant volatility. The usual argument is that the observed concentration simply reflects a variable volatility. The mesofractal prediction of the M 1963 model is that due to the extreme long-tailedness of the distribution, the largest of N daily price changes is of the order of magnitude of their sum. At first, this prediction is invariably perceived as completely shocking. After some thought, it is accepted as being on the right track but incomplete. It might be close to the mark on the short or middle run but is not reasonable on the long-run.

Providing totally new flexibility to the notion of concentration, multifractality allows a smooth transition between the preceding extremes, namely, the concentration $\sim (\text{size})^{-1}$ characteristic of unifractality, and the behavior $\sim (\text{size})^0$ characteristic of mesofractality.

When clock time is divided into very short increments Δt , the corresponding increments $\Delta\theta = (\Delta t)^{U(t)}$ vary enormously in size. In particular, the distribution of the exponents $U(t)$ is highly scattered. Both the casual glance and the lessons drawn from the well-known M 1963 model draw our attention to values that stand out as sharp spikes. They can indeed be extremely important, yet the multifractal case exhibits a "softer" form of concentration. Indeed, even the sharpest spike is asymptotically negligible compared to the whole. The fractal dimension $D(1)$ concerns smaller values of $U(t)$ within a range one can call "median." Taken separately, each is asymptotically negligible. But their number is $N^{D(1)}$: it is sufficiently large to insure that their total contribution is no longer asymptotically negligible, in fact is nearly equal to the whole increment of θ . Multifractal concentration consists in the fact that $D(1) < 1$.

H. Differences associated, for fixed y therefore H , with the value of $\min U(t)$, therefore the value of x being to the left or the right of unifractality.

The next simplest characteristics of a multifractal cartoon are $\min U(t)$ and $\max U(t)$. They are very important, because the former measures the degree of "peakedness" of the peaks of $\Delta\theta$, the latter, the duration and degree of flatness of the low lying parts of $\Delta\theta$.

The mathematical situation is as follows. To be concrete, take $H = 1/2$ and move x away from the unifractal value $x = 4/9$, either leftbound towards $x = 0$, or rightbound towards $x = 1/2 - \epsilon$. The value of $\min U(t)$ begins as 1 and tends to 0 in both cases. To the contrary, the behavior of $\max U(t)$ is very sensitive to the direction of motion. To the left, it increases without bound. To the contrary, one finds that to the right $\min U(t)$ only increases up to the limit $\log 3 / \log 2 \sim 1.5849$.

Concretely, this asymmetry creates a sharp and highly visible difference. For given $D(1)$, the probability of $U(t)$ being very small will be far greater for x to the left than to the right of the unifractal locus, that is, above or below the starred line on Figure 7. This prediction is clearly vindicated by Figure 7.

To stress the novelty of those predictions, they came after I drew Figure N1.4 of Mandelbrot (1999). That figure consisted, in effect, in moving always to the left of the unifractality and never to the right.

The above asymmetry between left and right can be expressed in terms of a theory that warrants a mention here, but only a very brief one: the variation of θ is "less lacunar" to the right of $x = 4/9$ than to the left.

XI. Concluding Remarks and Transition to the Next Paper

A. How do Simulations of the Multifractal Model Stand up Against Actual Records of Changes in Financial Prices?

To respond to the question raised in this subtitle, let us return to Figure 2, our key composite of several historical series of price changes with a few outputs of artificial models.

As already observed, the goal of modeling the real markets is certainly not fulfilled by the top three lines, which represent the Fickian, mesofractal and unifractal models. Considering the more important five lower lines at least one record is real and at least one is a computer-generated sample of the M 1972/97 model, the latest multifractal model and (once again) a proper random variant of cartoon multifractality.

I hope the forgeries will be perceived as surprisingly effective. In fact, only two are real graphs of market activity. Line 5 refers to the changes in price of IBM stock and Line 6 shows price fluctuations for the dollar-deutsche mark exchange rate. Lines 4, 7 and 8 strongly resemble their two real-world predecessors. But they are completely artificial.

B. Conceptual Issues. Spontaneous Resonances of the Financial Markets

The good fit of the multifractal model raises an endless string of hard questions. For example, price variation results in part from economic fundamentals. But it also results in part from structure of the financial institutions and the financial agents' responses to the fundamentals and other agents' actions. Which of these two causes is at the root of the very partial flavor of "order in chaos" that characterizes multifractality?

Multifractals are found throughout physics, but to assume that the regularities observed in price variations reflect regularities in the economic fundamentals would be extremely far-fetched and require hard evidence to be believed.

The alternative obvious thought seems far more likely, namely, that institutions and the complex interactions in financial markets end up by

creating some kind of order. Physics is skilled at studying the “spontaneous resonance of physical systems.” The behavior represented by the multifractal model may well be closest to “spontaneous resonances of the financial markets.”

If this last perspective proves fruitful, multifractality may provide a new handle on a perennial and very important practical issue. A better understanding might help improve society as well as some individual bank accounts.

Additional consequences of multifractality from the viewpoint of political economy are better considered elsewhere.

C. Transition to the Grid-Free Model Described in the Next Paper.

To recall and elaborate on history, the unifractal and mesofractal cartoons were constructed after the fact. While they turn out to be of intrinsic interest, they were designed to act as standbys/surrogates for two grid-free models, respectively, the M 1965 model based on fractional Brownian Motion and the M 1963 model based on Lévy stable processes.

As to the multifractal time, the cartoons that correspond to the lower left quarter of Figure 5 and a grid-free model arose in my mind near-simultaneously; the original papers issued in 1972 and 1974 are reprinted in Mandelbrot (1999).

Next, consider the Brownian motion (Wiener or fractional) in multifractal time. That process was conceived in the early 1970s, as described on p. 42 of Mandelbrot (1997). I introduced the multifractal cartoons much after the fact, as standbys/surrogates. That multifractal process was first investigated in Chapter E6, and other early chapters, of Mandelbrot (1997). In the next paper, this investigation, with many new facts, is described in free-standing fashion with no reference to the cartoons.

D. Relation between Grid-free Functions and their Cartoon Surrogates Explanation of Figure 9.

To a large extent, both parts of Figure 9 replicate Figure 6 except for the labels.

Figure 9 top relabels the loci of Figure 6 by the corresponding basic grid-free functions, when they exist, and indicates when they do not.

Figure 9 bottom refers to the recent books of mine, in which background material or all those grid-free models can be found. To explain the notation by an example, "M 1997E" stands for "Mandelbrot (1997), namely volume E in the author's *Selecta* series of books."

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