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PARETO IMPROVING PRICE REGULATION WHEN THE ASSET MARKET IS INCOMPLETE

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Pareto Improving Price Regulation When the Asset Market Is Incomplete *

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Abstract

When the asset market is incomplete, competitive equilibria are constrained suboptimal, which provides a scope for pareto improving interventions. Price regulation can be such a pareto improving policy, even when the welfare effects of rationing are taken into account. An appealing aspect of price regulation is that it that it operates anonymously on market variables.

Fix-price equilibria exist under weak assumptions. Such equilibria permit a competitive analysis of an economy with an incomplete asset market that is out of equilibrium. Arbitrage opportunities may arise: with three or more assets actively traded, an individual may hold an arbitrage portfolio at equilibrium.

The local existence of fix-price equilibrium for prices that are almost competitive may fail for robust examples. Under necessary and sufficient conditions for the local existence of fix-price equilibria, Pareto improving price regulation is generically possible.

Key words: incomplete asset market, fix-price equilibria, Pareto improvement.

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1 Introduction

Prices in competitive markets may fail to attain equilibrium; a variety of reasons, such as institutional constraints on price formation and lags in the adjustment of prices may underlie this failure. The theory of general competitive equilibrium does not account for the formation of prices; moreover, empirical evidence indicates the presence of persistent deviations from market clearing.

The failure of prices to attain market clearing is most plausible as a short-run phenomenon in an economy subject to stochastic shocks and, consequently, with an operative asset market; the extension of fix-price analysis to such a framework is, thus, pertinent.

Fix-price equilibria, following Bénassy (1975) and Drèze (1975), characterize the allocation of resources at arbitrary prices. The definition extends to economies with uncertainty and an incomplete asset market. Under weak assumptions fix-price equilibria exist.

With the prices of commodities fixed, the distinction between nominal assets, denominated in units of account, and real assets, denominated in one or multiple commodities, vanishes. The argument for the existence of fix-price equilibria is an adaptation of the argument for the existence of competitive equilibria for economies with a complete market in contingent commodities[1] or for economies with assets whose payoffs are denominated in a numeraire commodity — Geanakoplos and Polemarchakis (1986).

The prices of assets may allow for arbitrage: with three or more assets effectively traded, an individual may hold an arbitrage portfolio at equilibrium.

When the asset market is complete, competitive equilibrium allocations are Pareto optimal.[2] Moreover generically, they are regular: locally, they are unique, and they vary continuously with the parameters of the economy.[3]

When the asset market is incomplete, competitive equilibrium allocations generically fail to satisfy the criterion of constrained Pareto optimality that recognizes the incompleteness of the asset market: there exist reallocations of portfolios that yield Pareto improvements in welfare after spot commodity markets adjust to attain equilibrium — Geanakoplos and Polemarchakis (1986); this is the case, even if the payoffs of assets are denominated in a numeraire commodity and the set of states of the world is finite, which allow for the existence and regularity of competitive equilibrium allocations.

The informational requirements of improving interventions can be recovered from the market behavior of individuals — Geanakoplos and Polemarchakis (1990), Kübler and Polemarchakis (1999).

The failure of constrained optimality casts doubt on the desirability of non-intervention with competitive markets, such as the laissez faire policy in international

[1] Arrow and Debreu (1954), McKenzie (1954).
[2] Arrow (1951, 1953), Debreu (1951, 1960).
[3] Debreu(1970).

trade — Newbery and Stiglitz (1984). The heterogeneity of individuals and the requirement of anonymity may interfere with improving interventions — Citanna, Kajii and Villanacci (1998), Kajii (1994), Mas-Colell (1985 b). Nevertheless, the failure of constrained optimality raises the possibility of active policy.

Financial innovation, the introduction of new assets, may lead to a Pareto deterioration — of Hart (1975); conditions for Pareto improving financial innovation to be possible are rather restrictive — Hara (1997), Cass and Citanna (1998), Elul (1995).

An alternative to the reallocation of portfolios or financial innovation is the regulation of prices in spot commodity markets. When the asset market is incomplete, there exist variations in prices that lead to a Pareto improvement over a competitive allocation after rationing attains market clearing.

The deviation of prices from their competitive equilibrium values can be chosen independently of the state of the world;[4] thus, price regulation is comparable to the reallocation of portfolios carried out before the resolution of uncertainty. More importantly, it is anonymous. The volume of trade in the markets for assets as well as commodities is endogenously determined. The information required for the implementation of Pareto improving price regulation is null.

The incompleteness of the asset market makes competitive allocations targets for regulation; compared with the reallocation of portfolios, the mode of intervention here, price regulation, has the advantage that it involves only aggregate, market variables, the prices of commodities: regulation can be decentralized.

Direct antecedants of this result are the argument in Polemarchakis (1979), which showed that fixed wages that need not match shocks in productivity may yield higher expected utility in spite of the loss of output in an economy of overlapping generations; and the argument in Drèze and Gollier (1993), which employed the capital asset pricing model to determine optimal schedules of wages that differ from the marginal productivity of labor. An example of Pareto improving price regulation was developed in Kalmus (1997).

The desirability of price stability was evoked earlier in the literature of international trade — Waugh (1944), Howell (1945), Oi (1961, 1972) — where Samuelson (1972 a, b) raised the issue of feasibility and pointed out that price stabilization can be Pareto improving only if constraints prevent the Pareto optimality of competitive allocations. In a different context, Weitzman (1974, 1977) argued that quantities may dominate prices as planning instruments; but the argument does not distinguish efficiency from distribution; even with a complete asset market, a quantity based mechanism may indeed allocate resources more effectively to "those who need it most."

Minimum wages and price supports for agricultural products, often advocated and imposed on grounds of equity, even fixed exchange rates, may be called for on grounds of efficiency when the asset market fails to price all risks.

[4]John Geanakoplos insisted on this point. Hamid Sabourian suggested the alternative of state-independent quantity constraints.

2 The Economy

Individuals are $i \in \mathcal{I} = \{1, \ldots, I\}$. States of the world are $s \in \mathcal{S} = \{1, \ldots, S\}$. Commodities are $l \in \mathcal{L} = \{1, \ldots, L+1\}$; commodity l in state of the world s is (l, s). A bundle of commodities in state of the world s is[5] $x_s = (\ldots, x_{l,s}, \ldots)'$, and a bundle of commodities across states of the world is $x = (\ldots, x_s, \ldots)'$.

Individual i is described by his consumption set, $\mathcal{X}^i$, the set of consumption plans, bundles of commodities across states of the world, his utility function, u^i, with domain the consumption set, and by his endowment, e^i, a bundle of commodities across states of the world.

Assets are $a \in \mathcal{A} = \{1, \ldots, A+1\}$. A portfolio of assets is $y = (\ldots, y_a, \ldots)'$. The payoffs of assets are denominated in the numeraire commodity, $(L+1, s)$, in every state of the world. The payoff of asset a in state of the world s is $R_{s,a}$; the payoffs of the asset across states of the world are $R_{\cdot a} = (\ldots, R_{s,a}, \ldots)'$. The payoffs of assets in state of the world s are $R_{s\cdot} = (\ldots, R_{s,a}, \ldots)$; the payoffs of assets across states of the world are

$$R = (\ldots, R_{\cdot a}, \ldots) = \begin{pmatrix} \vdots \\ R_{s\cdot} \\ \vdots \end{pmatrix}.$$

The asset market is complete if all reallocations of revenue across states of the world are attainable: the matrix of payoffs of assets, R, has column span of dimension S; otherwise, it is incomplete.

An economy is

$$\mathcal{E} = ((\mathcal{X}^i, u^i, e^i) : i \in \mathcal{I}, R).$$

The aggregate endowment is $e^a = \sum_{i \in \mathcal{I}} e^i$.

An allocation of commodities is $x^{\mathcal{I}} = (\ldots, x^i, \ldots)$, such that $x^i \in \mathcal{X}^i$, for every individual; aggregate consumption is $x^a = \sum_{i \in \mathcal{I}} x^i$, and the allocation is feasible if aggregate consumption coincides with the aggregate endowment: $x^a = e^a$. An allocation of portfolios is $y^{\mathcal{I}} = (\ldots, y^i, \ldots)$; the aggregate portfolio is $y^a = \sum_{i \in \mathcal{I}} y^i$, and the allocation is feasible if the aggregate portfolio vanishes: $y^a = 0$. An allocation is $(x^{\mathcal{I}}, y^{\mathcal{I}})$, a pair of an allocation of commodities and an allocation of assets, and it is feasible if both the allocation of commodities and the allocation of assets are feasible.

Definition 1 *An allocation of commodities, $x^{\mathcal{I}}$, Pareto dominates another, $\widehat{x}^{\mathcal{I}}$, if $u^i(x^i) \geq u^i(\widehat{x}^i)$, for every individual, with strict inequality for some. A feasible allocation of commodities is Pareto optimal if no feasible allocation Pareto dominates it.*

[5] "$'$" denotes the transpose.

The price of commodity l in the spot market in state of the world s is $p_{l,s}$; the price of the numeraire commodity, $(L+1,s)$ in every state of the world, is $p_{L+1,s}=1$. Prices of commodities in state of the world s are $p_s=(\ldots,p_{l,s},\ldots)$, and prices of commodities across states of the world are $p=(\ldots,p_s,\ldots)$. The domain of prices of commodities is $\mathcal{P}=\{p : p_{L+1,s}=1, s\in\mathcal{S}\}$.

The price of asset a is q_a; the price of the numeraire asset, which, without loss of generality, can be chosen to be $A+1$, is $q_{A+1}=1$. Prices of assets are $q=(\ldots,q_a,\ldots)$. The domain of prices of assets is $\mathcal{Q}=\{q : q_{A+1}=1\}$.

Commodities other than the numeraire are $\check{\mathcal{L}}=\{1,\ldots,L\}$, and assets other than the numeraire are $\check{\mathcal{A}}=\{1,\ldots,A\}$. With prices of the numeraire deleted, prices of commodities in state of the world s are $\check{p}_s$, prices of commodities across states of the world are $\check{p}$, and prices of assets are $\check{q}$. The domain of prices of commodities other than the numeraire is $\check{\mathcal{P}}$, and the domain of prices of assets other than the numeraire is $\check{\mathcal{Q}}$.

Prices are a pair, (p,q), of prices of commodities and prices of assets; the domain of prices is $\mathcal{P}\times\mathcal{Q}$. With prices of the numeraires deleted, prices are $(\check{p},\check{q})$, and their domain is $\check{\mathcal{P}}\times\check{\mathcal{Q}}$.

At prices of commodities and assets (p,q), the set of non-numeraire commodities $\check{\mathcal{L}}$ is partitioned into the subsets of commodities with positive prices, $\mathcal{L}_+$, negative prices, $\mathcal{L}_-$, and free commodities, $\mathcal{L}_0$; the set of non-numeraire assets $\check{\mathcal{A}}$ is partitioned into the subsets of assets with positive prices, $\mathcal{A}_+$, negative prices, $\mathcal{A}_-$, and free assets, $\mathcal{A}_0$.

An economy with fixed prices, (p,q), is

$$\mathcal{E}(p,q)=((\mathcal{X}^i,u^i,e^i) : i\in\mathcal{I},R,(p,q)).$$

The economy satisfies the following assumptions

For every individual, the consumption set is the set of non-negative commodity bundles: $\mathcal{X}^i=\{x : x\geq 0\}$, the utility function, u^i, is continuous, quasi-concave and weakly monotonically increasing in the numeraire commodity in every state of the world:[6] $u^i(x+k\mathbf{1}_{(L+1)s,(L+1)S})\geq u^i(x)$, for all $k\geq 0$, and the endowment is an element of the consumption set: $e^i\in\mathcal{X}^i$.

These restrictions on the characteristics of individuals are weak. The matrix of payoffs of assets and the prices of commodities and assets other than the numeraires are unrestricted.

In commodities and assets other than the numeraire, uniform rationing across individuals serves to attain market clearing.

Rationing in the supply of commodity l in the spot market in state of the world s is $\underline{z}_{l,s}$, for $l\in\check{\mathcal{L}}$. Rationing in the supply of commodities, other than the numeraire in state of the world s is $\underline{z}_s=(\ldots,\underline{z}_{l,s},\ldots,\underline{z}_{L,s})'$, and rationing in the supply of commodities across states of the world is $\underline{z}=(\ldots,\underline{z}_s,\ldots)'$. The domain of rationing in the supply of commodities is $\underline{\mathcal{Z}}=\{\underline{z} : \underline{z}_s\leq 0,\ s\in\mathcal{S}\}$. Rationing in the supply of

[6] "$\mathbf{1}_{k,K}$"denotes the k - th unit vector of dimension K.

asset a, is $\underline{y}_a$, for $a \in \check{\mathcal{A}}$. Rationing in the supply of assets is $\underline{y} = (\ldots, \underline{y}_a, \ldots, \underline{y}_A)$. The domain of rationing in the supply of assets is $\underline{\mathcal{Y}} = \{\underline{y} : \underline{y} \leq 0\}$.

Rationing in the demand for commodity l in the spot market in state of the world s is $\overline{z}_{l,s}$, for $l \in \check{\mathcal{L}}$. Rationing in the demand for commodities, other than the numeraire, in state of the world s is $\overline{z}_s = (\ldots, \overline{z}_{l,s}, \ldots, \overline{z}_{L,s})'$, and rationing in the demand for commodities across states of the world is $\overline{z} = (\ldots, \overline{z}_s, \ldots)'$. The domain of rationing in the demand for commodities is[7] $\overline{\mathcal{Z}} = \{\overline{z} : \overline{z}_s \geq 0, s \in \mathcal{S}\}$. Rationing in the demand for asset a, is $\overline{y}_a$, for $a \in \check{\mathcal{A}}$. Rationing in the demand for assets is $\overline{y} = (\ldots, \overline{y}_a, \ldots, \overline{y}_A)$. The domain of rationing in the demand for assets is $\overline{\mathcal{Y}} = \{\overline{y} : \overline{y} \geq 0\}$.

A rationing scheme in commodities is a pair, $(\underline{z}, \overline{z})$, of rationing of supply and rationing of demand. A rationing scheme in assets is a pair, $(\underline{y}, \overline{y})$, of rationing of supply and rationing of demand. A rationing scheme is a pair, $((\underline{z}, \overline{z}), (\underline{y}, \overline{y}))$, of rationing in commodities and rationing in assets.

At prices and rationing scheme $(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y})$, the budget set of individual i is

$$\beta^i(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y}) = \left\{ (x, y) : \begin{array}{l} qy \leq 0, \\ p_s(x_s - e^i_s) \leq R_s.y, \quad s \in \mathcal{S}, \\ \underline{z}_{l,s} \leq (x_{l,s} - e^i_{l,s}) \leq \overline{z}_{l,s}, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S}, \\ \underline{y}_a \leq y \leq \overline{y}_a, \quad a \in \check{\mathcal{A}}, \\ x \in \mathcal{X}^i \end{array} \right\}.$$

The optimization problem of the individual is to

$$\begin{array}{ll} \max & u^i(x) \\ \text{s.t} & (x, y) \in \beta^i(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y}). \end{array}$$

The solution to the optimization problem, $\delta^i(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y})$, defines the demand correspondence, δ^i, of the individual.

Definition 2 (Effective rationing) *At prices and rationing scheme $(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y})$, individual i is effectively rationed in his supply (demand) in the market for commodity $(\bar{l}, \bar{s})$ if there exists $(x, y) \in \beta^i(p, q, \underline{z}', \overline{z}, \underline{y}, \overline{y})$ $((x, y) \in \beta^i(p, q, \underline{z}, \overline{z}', \underline{y}, \overline{y}))$, such that $u^i(x)$ exceeds the utility at $\delta^i(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y})$, where $\underline{z}'_{\bar{l},\bar{s}} = -\infty$, while $\underline{z}'_{l,s} = \underline{z}_{l,s}$ $(\overline{z}'_{\bar{l},\bar{s}} = +\infty$, while $\overline{z}'_{l,s} = \overline{z}_{l,s})$, for all $(l, s) \in \check{\mathcal{L}} \times \mathcal{S} \setminus \{(\bar{l}, \bar{s})\}$.*

There is effective supply (demand) rationing in the market for commodity $(\bar{l}, \bar{s})$ if at least one individual is effectively rationed on his supply (demand).

Individual i is effectively rationed in his supply (demand) in the market for asset $\overline{a}$ if there exists $(x, y) \in \beta^i(p, q, \underline{z}, \overline{z}, \underline{y}', \overline{y})$ $((x, y) \in \beta^i(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y}'))$, such that $u^i(x)$

[7] "$\gg$," "$>$"and "$\geq$"are vector inequalities; also "$\ll$," "$<$"and "$\leq$."

exceeds the utility at $\delta^i(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y})$, *where* $\underline{y}'_{\overline{a}} = -\infty$, *while* $\underline{y}'_a = \underline{y}_a$ ($\overline{y}'_{\overline{a}} = +\infty$, *while* $\overline{y}'_a = \overline{y}_a$), *for all* $a \in \check{\mathcal{A}} \setminus \{\overline{a}\}$.

There is effective supply (demand) rationing in the market for asset $\overline{a}$ *if at least one individual is effectively rationed on his supply (demand).*

The consumption sets and utility functions of individuals and the matrix of payoffs of assets are held fixed; an economy is fully described by the allocation of endowments of individuals: $\mathcal{E} = e^{\mathcal{I}} = (\ldots, e^i, \ldots)$, and an economy with fixed prices by the allocation of endowments of individuals and the prices of commodities and assets: $\mathcal{E}(p, q) = (e^{\mathcal{I}}, (p, q))$.

The set of economies, Ω, is a bounded, open subset of euclidean space of dimension $I(L+1)S$. An economy is $\mathcal{E}(\omega) = e^{\mathcal{I}}(\omega)$ or, simply, ω, and an economy with fixed prices is $\mathcal{E}(\omega)(p, q) = (e^{\mathcal{I}}(\omega), (p, q))$ or, simply, $(\omega, (p, q))$. A generic set of economies is an open subset of the set of economies of full lebesgue measure; a property holds generically if it holds for a generic set of economies.

Definition 3 (Fix-price equilibrium) *A fix-price equilibrium for the economy* $\mathcal{E} = ((\mathcal{X}^i, u^i, e^i) : i \in \mathcal{I}, R)$ *at prices* (p, q) *or, equivalently, a competitive equilibrium for the economy with fixed prices* $\mathcal{E}(p, q) = ((\mathcal{X}^i,\ u^i, e^i) : i \in \mathcal{I}, R, (p,\ q))$ *is a pair,* $((\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*),\ (x^{\mathcal{I}*}, y^{\mathcal{I}*}))$, *of a rationing scheme and an allocation, such that*

1. *for every individual,* $(x^{i*}, y^{i*}) \in \delta^i(p, q, \underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*)$,

2. $x^{a*} = e^a$, *and* $y^{a*} = 0$,

3. *for every commodity other than the numeraire, if, for some individual,* $x^{i'*}_{l,s} - e^{i'}_{l,s} = \underline{z}^*_{l,s}$, *then, for every individual,* $x^{i*}_{l,s} - e^i_{l,s} < \overline{z}^*_{l,s}$, *while, if, for some individual,* $x^{i'*}_{l,s} - e^{i'}_{l,s} = \overline{z}^*_{l,s}$ *then, for every individual,* $x^{i*}_{l,s} - e^i_{l,s} > \underline{z}^*_{l,s}$, *and*

4. *for every asset other than the numeraire, if, for some individual,* $y^{i'*}_a = \underline{y}^*_a$, *then, for every individual,* $y^{i*}_a < \overline{y}^*_a$, *while, if, for some individual,* $y^{i'*}_a = \overline{y}^*_a$ *then, for every individual,* $y^{i*}_a > \underline{y}^*_a$.

Conditions 1 and 2 are the usual optimization and market clearing conditions. Conditions 3 and 4, together with the convexity of the consumption sets and the quasi-concavity of the utility functions of individuals, imply that there is no effective rationing, simultaneously, on both sides of a market: markets are transparent.

Definition 4 (Competitive equilibrium) *A competitive equilibrium for the economy* $\mathcal{E} = ((\mathcal{X}^i, u^i, e^i) : i \in \mathcal{I}, R)$ *is a triple,* $((p^*, q^*), (\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*), (x^{\mathcal{I}*},\ y^{\mathcal{I}*}))$, *of prices, a rationing scheme and an allocation, such that*

1. *for every individual,* $(x^{i*}, y^{i*}) \in \delta^i(p^*, q^*, \underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*)$,

2. $x^{a*} = e^a$, *and* $y^{a*} = 0$, *and*

3. *for every individual, for every commodity other than the numeraire,* $\underline{z}^*_{l,s} < x^{i*}_{l,s} - e^i_{l,s} < \overline{z}^*_{l,s}$*, and, for every asset other than the numeraire,* $\underline{y}^*_a < y^{i*}_a < \overline{y}^*_a$.

In a competitive equilibrium there is no effective rationing in any market if preferences are convex.

If (p^*, q^*) are competitive equilibrium prices, a fix-price equilibrium at prices (p, q^*) with $p \neq p^*$ may require effective rationing in the markets for assets, as well as in the markets for commodities; this is the case when the prices of commodities are regulated away from their competitive equilibrium values in order to effect a Pareto improvement, while the prices of assets are held fixed at their competitive equilibrium values.

If $((p^*, q^*), (\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*), (x^{\mathcal{I}*}, y^{\mathcal{I}*}))$ is a competitive equilibrium for the economy $\mathcal{E}$, $((\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*), (x^{\mathcal{I}*}, y^{\mathcal{I}*}))$ is a fix-price competitive equilibrium for the economy $\mathcal{E}(p^*, q^*)$; nevertheless, it is possible that there are fix-price competitive equilibria $((\underline{z}^{**}, \overline{z}^{**}, \underline{y}^{**}, \overline{y}^{**}), (x^{\mathcal{I}**}, y^{\mathcal{I}**}))$ such that $x^{\mathcal{I}**}$ is not a competitive equilibrium allocation of commodities — Madden (1982), for an example in an economy with a complete asset market.

2.1 Arbitrage

An arbitrage portfolio, $\widehat{y}$, is such that $q\widehat{y} \leq 0$, while $R\widehat{y} > 0$. Prices of assets allow for arbitrage if an arbitrage portfolio exists. Fix-price equilibria exist when prices of assets allow for arbitrage — Proposition 3 — but the presence of arbitrage opportunities imposes restrictions on equilibrium rationing schemes.

Proposition 5 *If* $A \geq 1$ *and the utility function of every individual is monotonically increasing in the numeraire commodity in every state of the world:* $u^i(x + k\, \mathbf{1}_{(L+1)s,(L+1)S}) > u^i(x)$*, for all* $k \geq 0$*, then, at a fix-price equilibrium, if* $\widehat{y}$ *is an arbitrage portfolio, there exists, for every individual, an asset* a*, other than the numeraire, such that either* $\underline{y}^*_a = y^{i*}_a$ *and* $\widehat{y}_a < 0$ *or* $\overline{y}^*_a = y^{i*}_a$ *and* $\widehat{y}_a > 0$.

Proof If, for some individual, i, $\widehat{y}_a \geq 0$ whenever $\underline{y}^*_a = y^{i*}_a$, and $\widehat{y}_a \leq 0$ whenever $\overline{y}^*_a = y^{i*}_a$, then $\underline{y}^*_a = y^{i*}_a$ implies $y^{i*}_a < \overline{y}^*_a$, and $\overline{y}^*_a = y^{i*}_a$ implies $\underline{y}^*_a < y^{i*}_a$. It follows that, for some $\lambda > 0$, $\underline{y}^*_a \leq y^{i*}_a + \lambda\widehat{y}_a \leq \overline{y}^*_a$, for all $a \in \check{\mathcal{A}}$. But then, the pair of a consumption plan and a portfolio (x^i, y^i) defined by $y^i = y^{i*} + \lambda\widehat{y}$, $x^i_{l,s} = x^{i*}_{l,s}$, for all $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, and $x^i_{L+1,s} = x^{i*}_{L+1,s} + \lambda R_s.\widehat{y}$, for all $s \in \mathcal{S}$, is an element of the budget set $\beta^i(p, q, \underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*)$, while $u^i(x^i) > u^i(x^{i*})$, since the utility function is monotonically increasing in the numeraire commodity in every state of the world, a contradiction. □

At a fix-price equilibrium, $((\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*), (x^{\mathcal{I}*}, y^{\mathcal{I}*}))$, the market for asset a is closed if $\underline{y}^*_a = 0$ or $\overline{y}^*_a = 0$; if the market is not closed, then it is open — the market for asset $A + 1$ is always open. The set of all assets for which markets are open is $\mathcal{A}_\circ$; the associated effective prices of assets are $q_\circ$, an effective portfolio is $y_\circ$, and the matrix of effective payoffs of assets is $R_\circ$. An effective arbitrage portfolio, $\widehat{y}_\circ$, is such that $q_\circ \widehat{y}_\circ \leq 0$, while $R_\circ \widehat{y}_\circ > 0$.

Proposition 6 *If the utility function of every individual is monotonically increasing in the numeraire commodity in every state of the world, then, at a fix-price equilibrium for an economy with at most two assets for which markets are open,*[8] $|\mathcal{A}_\circ| \leq 2$, *there is no effective arbitrage portfolio.*

Proof If $|\mathcal{A}_o| = 1$, the argument is trivial.

If $|\mathcal{A}_o| = 2$, there exists a non-numeraire asset $\overline{a} \in \mathcal{A}_o$. If $\widehat{y}_o$ is an effective arbitrage portfolio, then either $\widehat{y}_{\overline{a}} = 0$ or $\widehat{y}_{\overline{a}} \neq 0$. If $\widehat{y}_{\overline{a}} = 0$, then $q_\circ \widehat{y}_\circ \leq 0$ and $R_\circ \widehat{y}_\circ > 0$ implies $R_{\cdot A+1} < 0$, and, since individuals do not face constraints in the supply of asset $A+1$, a fix-price equilibrium does not exist, a contradiction. If $\widehat{y}_{\overline{a}} > 0$, then by Proposition 1, $\overline{y}^*_{\overline{a}} = y^{i*}_a$, for all $i \in \mathcal{I}$, and thus, by market clearing, $\overline{y}^*_{\overline{a}} = 0 :$ the market for asset $\overline{a}$ is not open, a contradiction. If $\widehat{y}_{\overline{a}} < 0$, similarly, the market for asset $\overline{a}$ is not open, a contradiction. $\square$

The result does not extend to fix-price equilibria with more than two assets for which markets are open. With three assets and three individuals, it is even possible that, at a fix-price equilibrium, one individual holds an arbitrage portfolio that the other two individuals, together, supply; which is peculiar and obviously implies the existence of effective arbitrage portfolios.

An example

Individuals are $i \in \mathcal{I} = \{1, 2, 3\}$; there is only one commodity, the numeraire: $l \in \mathcal{L} = \{1\}$ — the subscript that indicates the commodity is not necessary; states of the world are $s \in \mathcal{S} = \{1, 2, 3\}$, and there are two assets other than the numeraire: $a \in \mathcal{A} = \{1, 2, 3\}$.

Individuals have utility functions $u^i = a^i x_1 + b^i x_2 + c^i x_3$, $x \geq 0$, and endowments $e^i = (e^i_1, e^i_2, e^i_3)'$. For individual 1, $(a^1, b^1, c^1) = (2, 1, 2)$, and $e^1 = (3, 9, 3)'$; for individual 2, $(a^2, b^2, c^2) = (1, 2, 2)$, and $e^2 = (9, 3, 3)'$; for individual 3, $(a^3, b^3, c^3) = (1, 1, 2)$ and $e^3 = (5, 5, 5)'$.

The matrix of payoffs of assets is

$$R = \begin{pmatrix} -4 & 2 & -2 \\ 2 & -4 & -2 \\ 2 & 2 & 6 \end{pmatrix}.$$

Prices of commodities and assets are

$$p = (1, 1, 1), \quad q = (1, 1, 2).$$

[8] "| |"denotes the cardinality of a set.

An arbitrage portfolio satisfies

$$-4y_1 + 2y_2 - 2y_3 \geq 0,$$

$$2y_1 - 4y_2 - 2y_3 \geq 0,$$

$$2y_1 + 2y_2 + 6y_3 \geq 0,$$

with at least one strict inequality, and

$$y_1 + y_2 + 2y_3 \leq 0.$$

For $\lambda > 0$, the portfolio $y_\lambda = (-\lambda, -\lambda, \lambda)$ is an arbitrage portfolio: $Ry_\lambda = (0, 0, 2\lambda)' > 0$, while $qy_\lambda = 0$.

The budget constraint can be shown to be holding with equality, so $y_3 = -(1/2)y_1 - (1/2)y_2$. Since

$$R \begin{pmatrix} y_1 \\ \\ y_2 \\ \\ -\frac{1}{2}y_1 - \frac{1}{2}y_2 \end{pmatrix} = \begin{pmatrix} -3y_1 + 3y_2 \\ \\ 3y_1 - 3y_2 \\ \\ -y_1 - y_2 \end{pmatrix},$$

it follows that an individual with a utility function $u^i(x) = a^i x_1 + b^i x_2 + c^i x_3$ solves the optimization problem

$$\begin{aligned} \max \quad & (-3a^i + 3b^i - c^i)y_1 + (3a^i - 3b^i - c^i)y_2, \\ \text{s.t.} \quad & y_1 - y_2 \leq \tfrac{1}{3}e_1^i, \\ & y_2 - y_1 \leq \tfrac{1}{3}e_2^i, \\ & y_1 + y_2 \leq e_3^i, \\ & \underline{y}_1 \leq y_1 \leq \overline{y}_1, \\ & \underline{y}_2 \leq y_2 \leq \overline{y}_2. \end{aligned}$$

If $((\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*), (x^{\mathcal{I}*}, y^{\mathcal{I}*}))$ is a fix-price equilibrium, since, for any $\lambda > 0$, y_λ is an arbitrage portfolio, it follows by Proposition 1 that all individuals are rationed on the supply of asset 1 or asset 2. If no individual is rationed in the supply of asset 2, then every individual is rationed in the supply of asset 1, and market clearing implies that $\underline{y}_1^* = 0$. Irrespective of rationing in the demand of asset 2, individual 2 supplies 2 units of asset 2 and individual 3 supply 4/3 units of asset 2, whereas individual 1 demands at most 2 units of this asset, which is a contradiction. Similarly, there is no fix-price equilibrium without rationing in the supply of asset market 1. Consequently, in every fix-price equilibrium, there is rationing in the supply of both assets, while there is no rationing in the demand of any asset. Therefore, without loss of generality, the demand for assets 1 and 2, and, hence, for asset 3 as well as for commodities, is a function of the rationing scheme on the supplies of the assets.

If $\underline{y}^* = (-1,-1)'$, $\overline{y}^* > (2,2)'$ (exact choice does not matter), then $x^{1*} = (12,0,2)'$, $x^{2*} = (0,12,2)'$, $x^{3*} = (5,5,7)'$, $y^{1*} = (-1,2,-1/2)'$, $y^{2*} = (2,-1,-1/2)'$, and $y^{3*} = (-1,-1,1)'$; this describes the unique fix-price equilibrium, where equilibria are equivalent if they differ only with respect to non-binding rationing schemes.

Indeed, the demands of individuals as functions of the rationing scheme on the supplies are

$$x^1(\underline{y}) = (12, 0, \min\{-2\underline{y}_1, 6 - 2\underline{y}_2\})',$$

$$y^1(\underline{y}) = (\max\{\underline{y}_1, \underline{y}_2 - 3\}, \max\{3 + \underline{y}_1, \underline{y}_2\}, \min\{-1\tfrac{1}{2} - \underline{y}_1, 1\tfrac{1}{2} - \underline{y}_2\})',$$

$$x^2(\underline{y}) = (0, 12, \min\{-2\underline{y}_2, 6 - 2\underline{y}_1\})',$$

$$y^2(\underline{y}) = (\max\{3 + \underline{y}_2, \underline{y}_1\}, \max\{\underline{y}_2, \underline{y}_1 - 3\}, \min\{-1\tfrac{1}{2} - \underline{y}_2, 1\tfrac{1}{2} - \underline{y}_1\})',$$

$$x^3(\underline{y}) = (5 - 3\underline{y}_1 + 3\underline{y}_2, 5 + 3\underline{y}_1 - 3\underline{y}_2, 5 - \underline{y}_1 - \underline{y}_2)',$$

$$y^3(\underline{y}) = (\underline{y}_1, \underline{y}_2, -\tfrac{1}{2}\underline{y}_1 - \tfrac{1}{2}\underline{y}_2)'.$$

The equality of supply and demand for assets 1 and 2, necessary and sufficient for equilibrium yields

$$\max\{\underline{y}_1, \underline{y}_2 - 3\} + \max\{3 + \underline{y}_2, \underline{y}_1\} + \underline{y}_1 = 0,$$

$$\max\{3 + \underline{y}_1, \underline{y}_2\} + \max\{\underline{y}_2, \underline{y}_1 - 3\} + \underline{y}_2 = 0.$$

The unique solution is $\underline{y} = (-1,-1)'$.

At the fix-price equilibrium, individuals 1 and 2, together, supply the arbitrage portfolio that individual 3 holds.

3 The Existence of Fix-price Equilibria

For the existence of fix-price equilibria, it is essential that budget constraints hold with equality. Either one imposes this condition directly on the budget set, or one makes the following assumption.

- The numeraire asset is weakly desirable, $R_{\cdot A+1} \geq 0$.

Since the utility functions of individuals are weakly monotonically increasing in the numeraire commodity while the numeraire asset is weakly desirable, with no loss of generality, the budget constraints of the individual in the market for assets as well as in the spot markets for commodities are satisfied with equality.

The effective consumption set of individual i is

$$\widehat{\mathcal{X}}^i = \{x \in \mathcal{X}^i : x_{l,s} \leq e^a_{l,s}, (l,s) \in \mathcal{L} \times \mathcal{S}\}.$$

If $x^{\mathcal{I}}$ is a feasible allocation of commodities, then $x^i \in \widehat{\mathcal{X}}^i$, for every individual.

A revenue plan is

$$w = (\ldots, w_s, \ldots)'.$$

Associated with a consumption plan of individual i, $x \in \mathcal{X}^i$, there is a revenue plan

$$w^i(x) = (\ldots, w^i_s(x_s), \ldots)',$$

where $w^i_s(x_s) = p_s(x_s - e^i_s)$.

The set of effective revenue plans of individual i is

$$\mathcal{W}^i = \{w : w = w^i(x), \text{ for some } x \in \widehat{\mathcal{X}}^i\}.$$

The set of effective portfolios of assets of individual i is

$$\mathcal{Y}^i = \{y : w = Ry, qy = 0, \text{ for some } w \in \mathcal{W}^i\}.$$

The sets $\widehat{\mathcal{X}}^i$ and $\mathcal{W}^i$ are compact; not necessarily so for the set of effective portfolios of assets of an individual, since the matrix of payoffs of assets need not have full column rank.

3.1 Minimal asset trades

The set of effective feasible allocations of assets for the economy is

$$\mathcal{Y}^{\mathcal{I}} = \{y^{\mathcal{I}} \in \times_{i \in \mathcal{I}} \mathcal{Y}^i : y^a = 0\}.$$

Equivalently, $y^{\mathcal{I}} \in \mathcal{Y}^{\mathcal{I}}$ if

$$My^{\mathcal{I}} = (\ldots, w^i, \ldots, 0, 0)',$$

for some $w^i \in \mathcal{W}^i$, $i \in \mathcal{I}$, where[9,10]

$$M = \begin{pmatrix} R & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & R \\ q' & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & q' \\ I_{A+1} & \cdots & I_{A+1} \end{pmatrix}.$$

[9] "I_{A+1}" denotes the unit matrix of dimension $A+1$.

[10] The matrix M is of dimension $(I(S+1)+A+1) \times I(A+1)$.

The set of minimal effective feasible allocations of assets is [11,12]

$$\widehat{\mathcal{Y}}^{\mathcal{I}} = \{\hat{y}^{\mathcal{I}} \in \mathcal{Y}^{\mathcal{I}} : \nexists y^{\mathcal{I}} \in \mathcal{Y}^{\mathcal{I}}, \text{ with } \operatorname{sign}(y^{\mathcal{I}}) = \operatorname{sign}(\hat{y}^{\mathcal{I}}), |y^{\mathcal{I}}| < |\hat{y}^{\mathcal{I}}|\}.$$

The set $\widehat{\mathcal{Y}}^{\mathcal{I}}$ contains the effective feasible allocations of assets that are minimal: there is no effective feasible allocation of assets such that at least one individual could attain the same revenue plan with less trade, in absolute value, in at least one of the assets.

Since M need not have full column rank, the left-inverse of M may not exist.

By the singular value decomposition, there exist orthogonal matrices, U, of dimension $(I(S+1)+A+1)\times(I(S+1)+A+1)$, and V, of dimension $I(A+1)\times I(A+1)$, such that[13] $U'MV = \operatorname{diag}(\sigma_1, \ldots, \sigma_{I(A+1)})$, and there is r such that the first r elements of $\operatorname{diag}(\sigma_1, \ldots, \sigma_{I(A+1)})$ are positive and the others are zero. The Moore–Penrose inverse of M is defined by

$$M^+ = V\Sigma^+ U',$$

where $\Sigma^+ = \operatorname{diag}(1/\sigma_1, \ldots, 1/\sigma_r, 0, \ldots, 0)$. If M has full column rank, then

$$M^+ = (M'M)^{-1}M'.$$

If $y_R^{\mathcal{I}}$ is such that $y_R^{\mathcal{I}} = M^+ z$, for some z, then $y_R^{\mathcal{I}}$ is an element in the row space of $M : z = My_R^{\mathcal{I}}$, and $y_R^{\mathcal{I}}$ is the unique element of the row space of M with this property.

Lemma 7 *The set, $\widehat{\mathcal{Y}}^{\mathcal{I}}$, of minimal effective feasible allocations of assets is bounded.*

Proof If not, there exists a sequence, $(y_n^{\mathcal{I}} \in \widehat{\mathcal{Y}}^{\mathcal{I}} : n = 1, \ldots)$, such that $\|y_n^{\mathcal{I}}\|_\infty \geq n$. For $n = 1, \ldots,$ $w_n^{\mathcal{I}} = (\ldots, Ry_n^i, \ldots)'$. Since $\mathcal{W}^i$ is compact, the sequence $(w_n^{\mathcal{I}} : n = 1, \ldots)$ has a convergent subsequence, denoted the same. The corresponding subsequence of $(y_n^{\mathcal{I}} \in \widehat{\mathcal{Y}}^{\mathcal{I}} : n = 1, \ldots)$ is also denoted the same. Moreover, without loss of generality, $\operatorname{sign}(y_n^{\mathcal{I}})$ is independent of n. For $n = 1, \ldots,$

$$y_{R,n}^{\mathcal{I}} = M^+(w_n^{\mathcal{I}}, 0, 0)'$$

and

$$y_{N,n}^{\mathcal{I}} = y_n^{\mathcal{I}} - y_{R,n}^{\mathcal{I}}.$$

The sequence $(y_{R,n}^{\mathcal{I}} : n = 1, \ldots)$ is convergent, and therefore bounded. Since $(y_n^{\mathcal{I}} \in \widehat{\mathcal{Y}}^{\mathcal{I}} : n = 1, \ldots)$ is unbounded, without loss of generality, the sequence

$$\left(\frac{1}{\|y_{N,n}^{\mathcal{I}}\|_\infty} y_{N,n}^{\mathcal{I}} : n = 1, \ldots\right)$$

[11] "$\operatorname{sign}(x)$" denotes the sign vector associated with the vector x; an element of $\operatorname{sign}(x)$ is $1, 0$ or -1 if the corresponding element of x is $> 0, 0$ or < 0, respectively.

[12] "$|x|$" denotes the absolute value vector associated with the vector x; an element of $|x|$ is the absolute value of the corresponding element of x.

[13] "$\operatorname{diag}(\ldots, a_k, \ldots)$" denotes the diagonal matrix of appropriate dimension with elements $\ldots, a_k, \ldots$ on the diagonal.

is well-defined and convergent, with limit $\overline{y}_N^{\mathcal{I}}$. Evidently, $M\overline{y}_N^{\mathcal{I}} = 0$, and there is i° such that $\overline{y}_N^{i^\circ} \neq 0$.

Moreover, $\overline{y}_{N,a}^i \neq 0$ implies $\lim_{n\to\infty} |y_{a,n}^i| = \infty$, $\text{sign}(y_{a,n}^i) > 0$ implies $\overline{y}_{N,a}^i \geq 0$, $\text{sign}(y_{a,n}^i) = 0$ implies $\overline{y}_{N,a}^i = 0$, and $\text{sign}(y_{a,n}^i) < 0$ implies $\overline{y}_{N,a}^i \leq 0$.

So, there exists n° such that for $n \geq n^\circ$, $\text{sign}(y_n^{\mathcal{I}} - \overline{y}_N^{\mathcal{I}}) = \text{sign}(y_n^{\mathcal{I}})$.

Furthermore, for $n \geq n^\circ$, $M(y_n^{\mathcal{I}} - \overline{y}_N^{\mathcal{I}}) = My_n^{\mathcal{I}}$, whereas $|y_{a,n}^i - \overline{y}_{N,a}^i| \leq |y_{a,n}^i|$ and there is a° such that $|y_{a^\circ,n}^{i^\circ} - \overline{y}_{N,a^\circ}^{i^\circ}| < |y_{a^\circ,n}^{i^\circ}|$.

Hence, for $n \geq n^\circ$, $y_n^{\mathcal{I}} \notin \widehat{\mathcal{Y}}^{\mathcal{I}}$, a contradiction. □

It is surprising that, even with arbitrage possibilities or with payoffs of assets that are linearly dependent, it is possible to restrict attention to a bounded set of possible sales and purchases of assets.

Since $\widehat{\mathcal{Y}}^{\mathcal{I}}$ is bounded, there exists $\alpha > 0$, such that $\|y^{\mathcal{I}}\|_\infty < \alpha$ for all $y^{\mathcal{I}} \in \widehat{\mathcal{Y}}^{\mathcal{I}}$.

At a rationing scheme $(\underline{z}, \overline{z}, \underline{y}, \overline{y})$, the exact budget set, $\widetilde{\beta}^i(\underline{z}, \overline{z}, \underline{y}, \overline{y})$, of individual i is the set of elements, $(x, y) \in \beta^i(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y})$, that satisfy the budget constraint in every state with equality: $qy = 0$ and $p_s(x_s - e_s^i) = R_s.y$. The exact demand set, $\widetilde{\delta}^i(\underline{z}, \overline{z}, \underline{y}, \overline{y})$ of the individual is the set of elements $(x, y) \in \widetilde{\beta}^i(\underline{z}, \overline{z}, \underline{y}, \overline{y})$ that maximize utility.

Non-emptiness of $\delta^i(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y})$ implies non-emptiness of $\widetilde{\delta}^i(\underline{z}, \overline{z}, \underline{y}, \overline{y})$, since the utility function is weakly monotonically increasing in the numeraire commodity in every state, where there are no rationing constraints. Nevertheless, $\widetilde{\delta}^i(\underline{z}, \overline{z}, \underline{y}, \overline{y})$ can be a proper subset of $\delta^i(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y})$, since the utility function is not strictly monotonically increasing.

Lemma 8 *The correspondence $\widetilde{\delta}^i$ is non-empty, compact and convex valued, and upper hemi-continuous.*

Proof For $(\underline{z}, \overline{z}, \underline{y}, \overline{y}) \in \underline{\mathcal{Z}} \times \overline{\mathcal{Z}} \times \underline{\mathcal{Y}} \times \overline{\mathcal{Y}}$, the set $\widetilde{\beta}^i(\underline{z}, \overline{z}, \underline{y}, \overline{y})$ is non-empty: $(e^i, 0) \in \widetilde{\beta}^i(\underline{z}, \overline{z}, \underline{y}, \overline{y})$, closed and convex. For $(x, y) \in \widetilde{\beta}^i(\underline{z}, \overline{z}, \underline{y}, \overline{y})$, $-\underline{y}_a \leq y_a \leq \overline{y}_a$, for $a \in \check{\mathcal{A}}$, and

$$y_{A+1} = -\textstyle\sum_{a\in\check{\mathcal{A}}} q_a y_a \geq -\sum_{a\in\mathcal{A}_-} q_a \underline{y}_a - \sum_{a\in\mathcal{A}_+} q_a \overline{y}_a,$$

$$y_{A+1} = -\textstyle\sum_{a\in\check{\mathcal{A}}} q_a y_a \leq -\sum_{a\in\mathcal{A}_-} q_a \overline{y}_a - \sum_{a\in\mathcal{A}_+} q_a \underline{y}_a,$$

and, thus, the asset demands are bounded. Moreover,

$$0 \leq x_{l,s} \leq e_{l,s}^i + \overline{z}_{l,s}, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$0 \leq x_{L+1,s} \leq e_{L+1,s}^i - \textstyle\sum_{(l,s)\in\mathcal{L}_-} p_{l,s}\overline{z}_{l,s} + \sum_{(l,s)\in\mathcal{L}_+} p_{l,s} e_{l,s}^i + R_s.y, \quad s \in \mathcal{S},$$

and it follows, from the boundedness of feasible asset demands, that the feasible spot market demands are bounded as well; $\widetilde{\beta}^i(\underline{z}, \overline{z}, \underline{y}, \overline{y})$ is compact. By the continuity and quasi-concavity of the utility function, $\widetilde{\delta}^i(\underline{z}, \overline{z}, \underline{y}, \overline{y})$ is compact and convex.

If $((\underline{z}_n, \overline{z}_n, \underline{y}_n, \overline{y}_n) \in \underline{\mathcal{Z}} \times \overline{\mathcal{Z}} \times \underline{\mathcal{Y}} \times \overline{\mathcal{Y}} : n = 1, \ldots)$ is a sequence that converges to $(\underline{z}, \overline{z}, \underline{y}, \overline{y})$, for any sequence $((x_n, y_n) \in \widetilde{\delta}^i(\underline{z}_n, \overline{z}_n, \underline{y}_n, \overline{y}_n) : n = 1, \ldots)$,

$$-\underline{y}_{a,n} \leq y_{a,n} \leq \overline{y}_{a,n}, \quad a \in \check{\mathcal{A}},$$

$$-\sum_{a\in\mathcal{A}_-} q_a \underline{y}_{a,n} - \sum_{a\in\mathcal{A}_+} q_a \overline{y}_{a,n} \leq y_{A+1,n} \leq -\sum_{a\in\mathcal{A}_-} q_a \overline{y}_{a,n} - \sum_{a\in\mathcal{A}_+} q_a \underline{y}_{a,n},$$

and $\lim_{n\to\infty}(\underline{y}_n, \overline{y}_n) = (\underline{y}, \overline{y})$; it follows that the sequence $(y_n : n = 1, \ldots)$ is bounded. Similarly, since

$$0 \leq x_{l,s,n} \leq e^i_{l,s} + \overline{z}_{l,s,n}, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$0 \leq x_{L+1,s,n} \leq e^i_{L+1,s} - \sum_{(l,s)\in\mathcal{L}_-} p_{l,s} \overline{z}_{l,s,n} + \sum_{(l,s)\in\mathcal{L}_+} p_{l,s} e^i_{l,s} + R_s.y^n,$$

$$s \in \mathcal{S},$$

the sequence $((\underline{z}_n, \overline{z}_n) : n = 1, \ldots)$ is convergent, as is, as a consequence, the sequence $(x_n : n = 1, \ldots)$. It follows that $((x_n, y_n) : n = 1, \ldots)$ has a convergent subsequence, also denoted $((x_n, y_n) : n = 1, \ldots)$, with limit $(\widehat{x}, \widehat{y}) \in \widetilde{\beta}^i(\underline{z}, \overline{z}, \underline{y}, \overline{y})$.

If there exists $(\widetilde{x}, \widetilde{y}) \in \widetilde{\delta}^i(\underline{z}, \overline{z}, \underline{y}, \overline{y})$, such that $u^i(\widetilde{x}) > u^i(\widehat{x})$, for $\widetilde{\mathcal{L}}_-$, $\widetilde{\mathcal{L}}_+$, $\widetilde{\mathcal{A}}_-$, and $\widetilde{\mathcal{A}}_+$, the sets of non-numeraire commodities and non-numeraire assets, respectively, for which $\widetilde{x}_{l,s} - e^i_{l,s}$ is negative, positive, $\widetilde{y}_a$ is negative, and positive, respectively, and for

$$\lambda^n =$$

$$\min\left\{1, \frac{\underline{z}^n_{l,s}}{\widetilde{x}_{l,s} - e^i_{l,s}}, (l, s) \in \widetilde{\mathcal{L}}_-, \frac{\overline{z}^n_{l,s}}{\widetilde{x}_{l,s} - e^i_{l,s}}, (l, s) \in \widetilde{\mathcal{L}}_+, \frac{\underline{y}^n_a}{\widetilde{y}_a}, a \in \widetilde{\mathcal{A}}_-, \frac{\overline{y}^n_a}{\widetilde{y}_a}, a \in \widetilde{\mathcal{A}}_+\right\},$$

$$n = 1, \ldots,$$

and

$$\widetilde{x}^n = e^i + \lambda^n(\widetilde{x} - e^i), \quad n = 1, \ldots$$

$$\widetilde{y}^n = \lambda^n \widetilde{y}, \quad n = 1, \ldots,$$

since

$$q\widetilde{y}^n = \lambda^n q\widetilde{y} = 0,$$

$$p_s(\widetilde{x}^n_s - e^i_s) = \lambda^n p_s(\widetilde{x}_s - e^i_s) = \lambda^n R_{s\cdot}\widetilde{y} = R_{s\cdot}\widetilde{y}^n,$$

$$\widetilde{x}^n_{l,s} - e^i_{l,s} = \lambda^n(\widetilde{x}_{l,s} - e^i_{l,s}) \geq \frac{\underline{z}^n_{l,s}}{\widetilde{x}_{l,s} - e^i_{l,s}}(\widetilde{x}_{l,s} - e^i_{l,s}) = \underline{z}^n_{l,s}, \quad (l,s) \in \widetilde{\mathcal{L}}_-,$$

$$\widetilde{x}^n_{l,s} - e^i_{l,s} = \lambda^n(\widetilde{x}_{l,s} - e^i_{l,s}) \geq 0 \geq \underline{z}^n_{l,s}, \quad (l,s) \in (\check{\mathcal{L}} \times \mathcal{S}) \setminus \widetilde{\mathcal{L}}_-,$$

$$\widetilde{x}^n_{l,s} - e^i_{l,s} = \lambda^n(\widetilde{x}_{l,s} - e^i_{l,s}) \leq \frac{\overline{z}^n_{l,s}}{\widetilde{x}_{l,s} - e^i_{l,s}}(\widetilde{x}_{l,s} - e^i_{l,s}) = \overline{z}^n_{l,s}, \quad (l,s) \in \widetilde{\mathcal{L}}_+,$$

$$\widetilde{x}^n_{l,s} - e^i_{l,s} = \lambda^n(\widetilde{x}_{l,s} - e^i_{l,s}) \leq 0 \leq \overline{z}^n_{l,s}, \quad (l,s) \in (\check{\mathcal{L}} \times \mathcal{S}) \setminus \widetilde{\mathcal{L}}_+,$$

$$\underline{y}^n_a = \frac{\underline{y}^n_a}{\widetilde{y}_a}\widetilde{y}_a \leq \lambda^n \widetilde{y}_a = \widetilde{y}^n_a \leq 0 \leq \overline{y}^n_a, \quad a \in \widetilde{\mathcal{A}}_-,$$

$$\underline{y}^n_a \leq 0 \leq \widetilde{y}^n_a = \lambda^n \widetilde{y}_a \leq \frac{\overline{y}^n_a}{\widetilde{y}_a}\widetilde{y}_a = \overline{y}^n_a, \quad a \in \widetilde{\mathcal{A}}_+,$$

it holds that $(\widetilde{x}^n, \widetilde{y}^n) \in \widetilde{\beta}^i(\underline{z}^n, \overline{z}^n, \underline{y}^n, \overline{y}^n)$. Moreover, $\widetilde{x}^n_{l,s} = (1-\lambda^n)e^i_{l,s} + \lambda^n \widetilde{x}_{l,s} \geq 0$, for $(l,s) \in \mathcal{L} \times \mathcal{S}$, and $\widetilde{x}^n \geq 0$. Evidently, $\lim_{n\to\infty} \lambda^n = 1$, and $\lim_{n\to\infty}(\widetilde{x}^n, \widetilde{y}^n) = (\widetilde{x}, \widetilde{y})$. By the continuity of the function u^i, $u^i(\widetilde{x}^n) > u^i(x^n)$ for n sufficiently large, which contradicts $(x^n, y^n) \in \widetilde{\delta}^i(\underline{z}^n, \overline{z}^n, \underline{y}^n, \overline{y}^n)$. Consequently, $\widetilde{\delta}^i$ is upper hemi-continuous. □

The demand of individuals depends in an upper hemi-continuous way on the constraints they face in the markets of the non-numeraire assets and commodities. It is not necessary to compactify consumption sets in order to get this result, even though there are no restrictions whatsoever in the markets of the numeraire assets and the numeraire commodities.

At a fix-price equilibrium $((\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*), (x^{\mathcal{I}*}, y^{\mathcal{I}*}))$, in the market for commodity $(l,s) \in \check{\mathcal{L}} \times \mathcal{S}$, if there is an individual, i°, such that $x^{i^\circ *}_{l,s} - e^{i^\circ}_{l,s} = \underline{z}^*_{l,s}$, so that the individual is constrained on his supply in market (l,s), then by the definition of a fix-price equilibrium, no individual is constrained on his demand in market (l,s) : $x^{i*}_{l,s} - e^i_{l,s} < \overline{z}^*_{l,s}$. For a fixed $\varepsilon > 0$, since, for every individual, $x^{i*}_{l,s} - e^i_{l,s} \leq x^{a*}_{l,s} = e^a_{l,s}$, if $\overline{z}_{l,s} = \varepsilon + e^a_{l,s}$, then $x^{i*}_{l,s} - e^i_{l,s} < \overline{z}_{l,s}$. If there is an individual, i°, such that $x^{i^\circ *}_{l,s} - e^{i^\circ}_{l,s} = \overline{z}^*_{l,s}$, so that the individual is constrained on his demand in market (l,s), then no individual is constrained on his supply in market (l,s) : $x^{i*}_{l,s} - e^i_{l,s} > \underline{z}^*_{l,s}$. Since, for every individual, $x^{i*}_{l,s} - e^i_{l,s} \geq -e^a_{l,s}$, if $\underline{z}_{l,s} = -\varepsilon - e^a_{l,s}$, then $x^{i*}_{l,s} - e^i_{l,s} > -\underline{z}_{l,s}$.

There is a minimal effective feasible allocation of assets $\widehat{y}^{\mathcal{I}} \in \widehat{\mathcal{Y}}$ satisfying $\sum_{i\in\mathcal{I}} \widehat{y}^i = 0$, and, for every individual, $R\widehat{y}^i = Ry^{i*}$, $q\widehat{y}^i = qy^{i*}$, $\text{sign}(\widehat{y}^i) = \text{sign}(y^{i*})$, and $|\widehat{y}^i_a| \leq |y^{i*}_a|$, for all $a \in \mathcal{A}$ — it is possible that $\widehat{y}^{\mathcal{I}} = y^{\mathcal{I}*}$. This implies that

$(x^{i*}, \widehat{y}^i) \in \widetilde{\delta}^i(\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*)$, for every individual, and that $((\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*), (x^{\mathcal{I}*}, \widehat{y}^{\mathcal{I}}))$ is a fix-price equilibrium.

In the market for some asset $a \in \check{\mathcal{A}}$, if there is an individual, i°, such that $\widehat{y}_a^{i^\circ} = \underline{y}_a^*$, so that the individual is constrained in his supply in the market of asset a, then no individual is constrained in his demand in the market of asset a : so $\widehat{y}_a^i < \overline{y}_a^*$. Since, for every individual, $\widehat{y}_a^i < \alpha$, if $\overline{y}_a^* = \alpha$, then $\widehat{y}_a^i < \overline{y}_a^*$, for every individual. If there is an individual, i°, such that $\widehat{y}_a^{i^\circ} = \overline{y}_a^*$, so that the individual is constrained in his demand in the market of asset a, then no individual is constrained in his supply in the market of asset a : $\widehat{y}_a^i > \underline{y}_a^*$. Since, for every individual, $\widehat{y}_a^i > -\alpha$, if $\underline{y}_a^* = -\alpha$, then $\widehat{y}_a^i > \underline{y}_a^*$, for every individual.

The state of the market of commodity $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$ is described by $r_{l,s} \in [0, 1]$. The rationing scheme in commodities is then fully determined by a function[14,15]

$$(\widehat{\underline{z}}, \widehat{\overline{z}}) : \mathcal{C}^{LS} \to -\mathcal{R}_+^{LS} \times \mathcal{R}_+^{LS}.$$

If $0 \leq r_{l,s} \leq 1/2$, then there may be supply rationing in the market of commodity (l, s), while demand rationing is excluded by putting $\widehat{\overline{z}}_{l,s}(r) = \varepsilon + e_{l,s}^a$; if $1/2 \leq r_{l,s} \leq 1$, then there may be demand rationing in the market of commodity (l, s), while supply rationing is excluded by putting $\widehat{\underline{z}}_{l,s}(r) = -\varepsilon - e_{l,s}^a$.

The state of the market of asset $a \in \check{\mathcal{A}}$ is described by $\rho_a \in [0, 1]$. The rationing scheme in assets is then fully determined by a function

$$(\widehat{\underline{y}}, \widehat{\overline{y}}) : \mathcal{C}^A \to -\mathcal{R}_+^A \times \mathcal{R}_+^A.$$

If $0 \leq \rho_a \leq 1/2$, then there may be supply rationing in the market of asset a, while demand rationing is excluded by putting $\widehat{\overline{y}}_a(\rho) = \alpha$; if $1/2 \leq \rho_a \leq 1$, then there may be demand rationing in the market of asset a, while supply rationing is excluded by putting $\widehat{\underline{y}}_a(\rho) = -\alpha$.

More precisely, the functions $(\widehat{\underline{z}}, \widehat{\overline{z}})$ and $(\widehat{\underline{y}}, \widehat{\overline{y}})$ are defined by

$$\widehat{\underline{z}}_{l,s}(r) = -\min\{2r_{l,s}(\varepsilon + e_{l,s}^a), \varepsilon + e_{l,s}^a\}, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S}, r \in \mathcal{C}^{LS},$$

$$\widehat{\overline{z}}_{l,s}(r) = \min\{(2 - 2r_{l,s})(\varepsilon + e_{l,s}^a), \varepsilon + e_{l,s}^a\}, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S}, r \in \mathcal{C}^{LS},$$

$$\widehat{\underline{y}}_a(\rho) = -\min\{2\rho_a\alpha, \alpha\}, \quad a \in \check{\mathcal{A}}, \rho \in \mathcal{R}^A,$$

$$\widehat{\overline{y}}_a(\rho) = \min\{(2 - 2\rho_a)\alpha, \alpha\}, \quad a \in \check{\mathcal{A}}, \rho \in \mathcal{R}^A,$$

for a fixed $\varepsilon > 0$.

The correspondences $\widehat{\delta}^i$, for all i, and $\widehat{\zeta}$, with domain $\mathcal{C}^{LS} \times \mathcal{C}^A$ are defined by

$$\widehat{\delta}^i(r, \rho) = \widetilde{\delta}^i(\widehat{\underline{z}}(r), \widehat{\overline{z}}(r), \widehat{\underline{y}}(\rho), \widehat{\overline{y}}(\rho)),$$

$$\widehat{\zeta}(r, \rho) = \textstyle\sum_{i \in \mathcal{I}} \widehat{\delta}^i(r, \rho) - \{e^a\}.$$

[14] "$\mathcal{R}^K$" denotes the euclidean space of dimension K; "$\mathcal{R}_+^K$" denotes the positive orthant and "$\mathcal{R}_{++}^K$" the strictly positive orthant; $\mathcal{R}^1 = \mathcal{R}$, $\mathcal{R}_+^1 = \mathcal{R}_+$, and $\mathcal{R}_{++}^1 = \mathcal{R}_{++}$.

[15] "$\mathcal{C}^K$" denotes the unit cube: $\mathcal{C}^K = \{r \in \mathcal{R}^K : 0 \leq r_k \leq 1\}$, of dimension K.

The correspondence $\widehat{\delta}^i$, can be seen as a restriction of the correspondence $\widetilde{\delta}^i$, with rationing schemes being parametrized by the sets $\mathcal{C}^{LS}$ and $\mathcal{C}^A$.

Lemma 9 *If* $0 \in \widehat{\zeta}(r^*, \rho^*)$, *then there exists* $(x^{i*}, y^{i*}) \in \widehat{\delta}^i(r^*, \rho^*)$, $i \in \mathcal{I}$, *such that* $((\widehat{\underline{z}}(r^*), \widehat{\overline{z}}(r^*), \widehat{\underline{y}}(\rho^*), \widehat{\overline{y}}(\rho^*)), (x^{\mathcal{I}*}, y^{\mathcal{I}*}))$ *is a fix-price equilibrium of the economy.*

Proof If $(r^*, \rho^*) \in \mathcal{C}^{LS} \times \mathcal{C}^A$ is such that $0 \in \widehat{\zeta}(r^*, \rho^*)$, then there exists $(x^{i*}, y^i) \in \widehat{\delta}^i(\widehat{\underline{z}}(r^*), \widehat{\overline{z}}(r^*), \widehat{\underline{y}}(\rho^*), \widehat{\overline{y}}(\rho^*))$, for all i, such that $x^{a*} = e^a$ and $y^a = 0$. There is a minimal effective feasible allocation of assets $y^{\mathcal{I}*} \in \widehat{\mathcal{Y}}$, such that $y^{a*} = 0$, and, for every individual, $Ry^{i*} = Ry^i$, $qy^{i*} = qy^i$, $\text{sign}(y^{i*}) = \text{sign}(y^i)$, and $|y^{i*}_a| \leq |y^i_a|$, for all a. This implies that $(x^{i*}, y^{i*}) \in \widetilde{\delta}^i(\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*)$, for every individual, and that (1) and (2) of the definition of a fix-price equilibrium are satisfied by $((\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*), (x^{\mathcal{I}*}, y^{\mathcal{I}*}))$.

If for $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, $x^{i^\circ *}_{l,s} - e^{i^\circ}_{l,s} = \widehat{\underline{z}}_{l,s}(r^*)$ for some $i^\circ \in \mathcal{I}$, then $\widehat{\underline{z}}_{l,s}(r^*) \geq -e^{i^\circ}_{l,s} > -\varepsilon - e^a_{l,s}$. So $r^*_{l,s} < \frac{1}{2}$, and $\widehat{\overline{z}}_{l,s}(r^*) = \varepsilon + e^a_{l,s}$. It follows that $x^{i*}_{l,s} - e^i_{l,s} < \widehat{\overline{z}}_{l,s}(r^*)$, for every individual.

If for $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, $x^{i^\circ *}_{l,s} - e^{i^\circ}_{l,s} = \widehat{\overline{z}}_{l,s}(r^*)$ for some $i^\circ \in \mathcal{I}$, then $\widehat{\overline{z}}_{l,s}(r^*) \leq x^{i^\circ *}_{l,s} < \varepsilon + e^a_{l,s}$. So $r^*_{l,s} > \frac{1}{2}$, and $\widehat{\underline{z}}_{l,s}(r^*) = -\varepsilon - e^a_{l,s}$. It follows that $x^{i*}_{l,s} - e^i_{l,s} > \widehat{\underline{z}}_{l,s}(r^*)$, for every individual.

If for $a \in \check{\mathcal{A}}$, $y^{i^\circ *}_a = \widehat{\underline{y}}_a(\rho^*)$ for some $i^\circ \in \mathcal{I}$, then $\widehat{\underline{y}}_a(\rho^*) > -\alpha$ since $y^{i^\circ *} \in \widehat{\mathcal{Y}}$. So $\rho^*_{l,s} < \frac{1}{2}$, and $\widehat{\overline{y}}_a(\rho^*) = \alpha$. It follows immediately that $y^{i*}_a(\rho^*) < \widehat{\overline{y}}_a$, for every individual.

If for $a \in \check{\mathcal{A}}$, $y^{i^\circ *}_a = \widehat{\overline{y}}_a(\rho^*)$ for some $i^\circ \in \mathcal{I}$, then $\widehat{\underline{y}}_a(\rho^*) < \alpha$ since $y^{i^\circ *} \in \widehat{\mathcal{Y}}$. So $\rho^*_{l,s} > \frac{1}{2}$, and $\widehat{\underline{y}}_a(\rho^*) = -\alpha$. Again, it follows immediately that $y^{i*}_a(\rho^*) > \widehat{\underline{y}}_a$, for every individual.

Hence, (3) is satisfied as well in the definition of a fix-price equilibrium. □

The preparatory work is complete; it remains to show that there exists a zero point of $\widehat{\zeta}$ and thereby, a fix-price equilibrium. Since there is no rationing in the market of the numeraire asset nor in the market of the numeraire commodities, existence of an equilibrium is not obvious.

Proposition 10 *A fix-price equilibrium exists.*

Proof Since the correspondence $\widetilde{\delta}^i$, for all i, is upper hemi-continuous and compact valued, and the functions $\widehat{\underline{z}}, \widehat{\overline{z}}, \widehat{\underline{y}}$, and $\widehat{\overline{y}}$ are continuous, it follows that $\widehat{\delta}^i = \widetilde{\delta}^i \circ (\widehat{\underline{z}}, \widehat{\overline{z}}, \widehat{\underline{y}}, \widehat{\overline{y}})$, with domain $\mathcal{C}^{LS} \times \mathcal{C}^A$, for all i, is a compact valued upper hemi-continuous correspondence, and so $\widehat{\zeta}$ is a compact valued upper hemi-continuous correspondence. It follows that the set $\widehat{\zeta}(\mathcal{C}^{LS} \times \mathcal{C}^A)$ is compact.

The set $\mathcal{Z}$ is a compact, convex set that contains the projection on the first $(L+1)S$ coordinates of the set $\widehat{\zeta}(\mathcal{C}^{LS} \times \mathcal{C}^A)$; $\mathcal{Y}$ is a compact, convex set that contains the projection on the last $A+1$ coordinates of the set $\widehat{\zeta}(\mathcal{C}^{LS} \times \mathcal{C}^A)$. The correspondence $\mu : \mathcal{Z} \to \mathcal{C}^{LS}$ is defined by

$$\mu(z) = \arg\max\{ \sum_{(l,s) \in \check{\mathcal{L}} \times \mathcal{S}} r_{l,s} z_{l,s} : r \in \mathcal{C}^{LS}, z \in \mathcal{Z}\};$$

the correspondence $\nu : \mathcal{Y} \to \mathcal{C}^A$ is defined by

$$\nu(y) = \arg\max\{\sum_{a \in \check{\mathcal{A}}} \rho_a y_a : \rho \in \mathcal{C}^A, y \in \mathcal{Y}\};$$

the correspondence $\varphi : \mathcal{Z} \times \mathcal{Y} \times \mathcal{C}^{LS} \times \mathcal{C}^A \to \mathcal{Z} \times \mathcal{Y} \times \mathcal{C}^{LS} \times \mathcal{C}^A$ is defined by

$$\varphi(z, y, r, \rho) = \widehat{\zeta}(r, \rho) \times \mu(z) \times \nu(y).$$

The correspondence φ is a non-empty, compact, convex valued, upper hemi-continuous correspondence defined on a non-empty, compact, convex set. By Kakutani's fixed point theorem, φ has a fixed point, (z^*, y^*, r^*, ρ^*).

If, for some $a \in \check{\mathcal{A}}$, $y_a^* < 0$, then, by the definition of ν, $\rho_a^* = 0$, and $y_a^* \geq 0$, a contradiction.

If, for some $a \in \check{\mathcal{A}}$, $y_a^* > 0$, then, by the definition of ν, $\rho_a^* = 1$, and $y_a^* \leq 0$, a contradiction.

Consequently, $y_a^* = 0$, for all $a \in \check{\mathcal{A}}$. Moreover, $y_{A+1}^* = -\sum_{a \in \check{\mathcal{A}}} q_a y_a^* = 0$.

If, for some $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, $z_{l,s}^* < 0$, then, by the definition of μ, $r_{l,s}^* = 0$, and $z_{l,s}^* \geq 0$, a contradiction.

If, for some $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, $z_{l,s}^* > 0$, then, by the definition of μ, $r_{l,s}^* = 1$, and $z_{l,s}^* \leq 0$, a contradiction.

Consequently, $z_{l,s}^* = 0$, for all $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$. Moreover, for every $s \in \mathcal{S}$, $z_{L+1,s}^* = -\sum_{(l,s) \in \check{\mathcal{L}} \times \mathcal{S}} p_{l,s} z_{l,s}^* + R_{s \cdot} y^* = 0$.

It follows that $0 \in \widehat{\zeta}(r^*, \rho^*)$, and, hence, a fix-price equilibrium exists. □

The conditions under which equilibrium existence can be shown are very weak. Of particular interest is that endowments can be on the boundaries of the consumption sets of individuals, and there is no restriction that the aggregate endowment of every commodity be positive; this is of particular interest in a world with uncertainty, since one can imagine states of nature in which certain commodities are fully unavailable.

4 Local Comparative Statics

The characterization of the local behavior requires that the economy be smooth:

1. For every individual, the consumption set is $\mathcal{X}^i = \{x : x \geq 0\}$; the utility function is continuous and quasi-concave; in the interior of the consumption set,[16] Int $\mathcal{X}^i$, it is twice continuously differentiable, $\partial u^i \gg 0$ and $\partial^2 u^i$ is negative definite on[17] $(\partial u^i)^\perp$; the endowment is strictly positive: $e^i \in$ Int $\mathcal{X}^i$, and $u^i(e^i) > u^i(x)$, for every $x \in$ Bd $\mathcal{X}^i$.

2. The matrix of payoffs of assets has full column rank. The numeraire asset, has positive payoff: $R_{\cdot A+1} > 0$.

[16] "Int" denotes the interior of a set and "Bd" the boundary.
[17] "$\perp$" denotes the orthogonal complement.

In a smooth economy, the solution to the individual decision problem, $d^i(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y})$, is unique, and the demand correspondence of an individual is a piece-wise continuously differentiable function, d^i.

With convex consumtion sets and quasi-concave utility functions, if an individual is effectively rationed in his supply in the market for commodity (l, s), then $d^i_{l,s}(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y}) - e^i_{l,s} = \underline{z}_{l,s}$, and, similarly, if the individual is effectively rationed in his demand in the market for commodity (l, s), then $d^i_{l,s}(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y}) - e^i_{l,s} = \overline{z}_{l,s}$. If an individual is effectively rationed on his supply in the market for asset a, then $d^i_a(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y}) = \underline{y}_a$, and, similarly, if the individual is effectively rationed on his demand in the market for asset a, then $d^i_a(p, q, \underline{z}, \overline{z}, \underline{y}, \overline{y}) = \overline{y}_a$.

A sign vector is a vector with components $-1, 0, 1$.

The state of markets at a fix-price equilibrium is described by a sign vector

$$r = (\ldots, r_{l,s}, \ldots, r_{L,s}, \ldots, r_a, \ldots, r_A).$$

If there is effective supply rationing in the market for a commodity or an asset, the associated component of the sign vector is –1, if there is effective demand rationing it is +1, and if there is no effective rationing it is 0.

For a sign vector r, the set $\mathcal{PQ}(r)$ is the set of prices $(p, q) \in \mathcal{P} \times \mathcal{Q}$, for which there exists a fix-price competitive equilibrium at prices (p, q) with state of the markets r.

For prices $(p, q) \in \mathcal{P} \times \mathcal{Q}$, the set of fix-price equilibrium allocations is $\mathcal{D}(p, q)$, and, for a sign vector r, the set of fix-price equilibrium allocations with state of the markets r is $\mathcal{D}(p, q, r)$.

Definition 11 (Local uniqueness) *The allocation* $(x^{\mathcal{I}*}, y^{\mathcal{I}*})$, *at a competitive equilibrium,* $((p^*, q^*), (\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*), (x^{\mathcal{I}*}, y^{\mathcal{I}*}))$, *is locally unique as a fix-price equilibrium allocation if there exists a neighborhood,* $\overline{\mathcal{N}}_{x^{\mathcal{I}*}, y^{\mathcal{I}*}}$, *of* $(x^{\mathcal{I}*}, y^{\mathcal{I}*})$, *such that, for every neighbourhood* $\mathcal{N}_{x^{\mathcal{I}*}, y^{\mathcal{I}*}}$ *of* $(x^{\mathcal{I}*}, y^{\mathcal{I}*})$ *that is contained in* $\overline{\mathcal{N}}_{x^{\mathcal{I}*}, y^{\mathcal{I}*}}$, *there exists a neighbourhood,* $\mathcal{N}_{p^*, q^*}$, *of* (p^*, q^*), *with the set* $\mathcal{D}(p, q) \cap \mathcal{N}_{x^{\mathcal{I}*}, y^{\mathcal{I}*}}$ *a singleton, for every* $(p, q) \in \mathcal{N}_{p^*, q^*}$.

If a competitive equilibrium allocation is locally unique as a fix-price equilibrium allocation, then, for prices close to competitive equilibrium prices, there is exactly one fix-price equilibrium allocation close to the competitive allocation. Even if a competitive equilibrium allocation is locally unique as a fix-price competitive equilibrium allocation, variations in non-binding rationing schemes yield inessentially distinct fix-price equilibria.

The prices of numeraire commodities and of the numeraire asset are equal to one, so neighbourhoods of prices are subsets of the domain $\mathcal{P} \times \mathcal{Q}$.

With a complete asset market, generically, the set of fix-price competitive equilibrium allocations can be represented by a finite number of continuously differentiable functions of prices and endowments — Laroque and Polemarchakis (1978). Nevertheless, under standard assumptions, competitive equilibria are not locally unique as fix-price equilibria — Laroque (1978), Madden (1982). Although fix-price equilibrium allocations exist for all prices even with an incomplete asset market — Proposition 3 — there may be robust local non-existence at competitive prices. The equilibrium manifold has a particularly complicated structure at competitive prices, which have lebesgue measure zero — as the generic regularity of fix-price equilibria requires; the incompleteness of the asset market does not alleviate the problem.

Local uniqueness of fix-price equilibrium allocations at competitive equilibria is not too strong a requirement; given the upper hemi-continuity of the equilibrium correspondence, it is less demanding than the requirement of uniqueness of fix-price equilibrium allocations, which, in turn, is weaker than the requirements for stability.

Comparative statics require a differentiable form of local uniqueness.

Definition 12 (Differentiable local uniqueness) *The allocation* $(x^{\mathcal{I}*}, y^{\mathcal{I}*})$*, at a competitive equilibrium* $((p^*, q^*), (\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*), (x^{\mathcal{I}*}, y^{\mathcal{I}*}))$*, is differentiably locally unique as a fix-price equilibrium allocation if it is locally unique and there is a neighbourhood,* $\mathcal{N}_{p^*,q^*}$*, of* (p^*, q^*)*, and a neighbourhood* $\mathcal{N}_{x^{\mathcal{I}*},y^{\mathcal{I}*}}$ *of* $(x^{\mathcal{I}*}, y^{\mathcal{I}*})$*, such that, for every sign vector r, the function* $(\widehat{x}^r, \widehat{y}^r) : \mathcal{N}_{p^*,q^*} \cap \mathcal{PQ}(r) \to \mathcal{R}^{I(L+1)S+I(A+1)}$*, obtained by associating the unique fix-price equilibrium allocation in* $\mathcal{N}_{x^{\mathcal{I}*},y^{\mathcal{I}*}} \cap \mathcal{D}(p,q,r)$ *to* $(p,q) \in \mathcal{N}_{p^*,q^*} \cap \mathcal{PQ}(r)$*, is differentiable* [18].

For a locally unique fix-price competitive equilibrium allocation, the requirement that it be differentiably locally unique is not very demanding; this is the case, since the requirement of differentiability applies separately to different states of the markets.

The function $(\widehat{x}, \widehat{y}) : \mathcal{N}_{p^*,q^*} \to \mathcal{R}^{I(L+1)S+I(A+1)}$ is obtained by associating the unique fix-price equilibrium allocation in $\mathcal{N}_{x^{\mathcal{I}*},y^{\mathcal{I}*}}$ to $(p,q) \in \mathcal{N}_{p^*,q^*}$. The indirect utility function of an individual at a locally unique fix-price equilibrium is defined by

$$v^i(p,q) = u^i(\widehat{x}^i(p,q)), \quad (p,q) \in \mathcal{N}_{p^*,q^*}.$$

Lemma 13 *At a differentiably locally unique competitive equilibrium allocation, for every individual, the indirect utility function* v^i*, with domain* $\mathcal{N}_{p^*,q^*}$*, is differentiable and*

$$\partial_{p_{l,s}} v^i(p^*, q^*) = -\partial_{x_{L+1,s}} u^i(x^{i*})(x^{i*}_{l,s} - e^i_{l,s}), \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S}.$$

Proof For every sign vector, r, the restriction of the function v^i to $\mathcal{N}_{p^*,q^*} \cap \mathcal{PQ}(r)$, denoted v^{i^r}, is differentiable. From the differentiation of the budget constraints,

$$q\widehat{y}^{i^r}(p,q) = 0,$$

and

$$p_s(\widehat{x}^{i^r}_s(p,q) - e^i_s) = R_s.\widehat{y}^{i^r}(p,q), \quad s \in \mathcal{S},$$

with respect to $p_{\overline{l},\overline{s}}$, and the first order conditions for individual optimization at a competitive equilibrium,

$$\partial_{x^i_{l,s}} u^i(x^{i*}) = \partial_{x^i_{L+1,s}} u^i(x^{i*}) p^*_{l,s}, \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S},$$

[18] A function with domain a subset of euclidean space which is not necessarily open is differentiable if it has a differentiable extension to an open neighborhood of its domain of definition.

and

$$\sum_{s\in\mathcal{S}} \partial_{x^i_{L+1,s}} u^i(x^{i*}) R_{s\cdot} = \mu^i q^*, \text{ for some } \mu^i > 0,$$

it follows that

$$\partial_{p_{\bar{l},\bar{s}}} v^{i^r}(p^*, q^*) = -\partial_{x^i_{L+1,\bar{s}}} u^i(x^{i*})(x^{i*}_{\bar{l},\bar{s}} - e^i_{\bar{l},\bar{s}}).$$

Since the derivative is independent of the sign vector, r, the result follows. □

The effect of a change in the spot market price of commodity $(l,s) \in \check{\mathcal{L}} \times \mathcal{S}$ is equal to the negative of the marginal utility of the numeraire commodity in state s multiplied by the excess demand of commodity (l,s) at the competitive equilibrium.

We analyse the local comparative statics of fix-price equilibria in the neighbourhood of a competitive price system. This analysis follows Laroque (1978, 1981) for economies with a complete asset market and leads to necessary and sufficient conditions for differentiable local uniqueness.

At a competitive equilibrium, $((p^*, q^*), (\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*), (x^{\mathcal{I}*}, y^{\mathcal{I}*}))$, $z^-_{l,s}$, $z^+_{l,s}$, y^-_a and y^+_a, defined by

$$z^-_{l,s} = \min_i x^{i*}_{l,s} - e^i_{l,s}, \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$z^+_{l,s} = \max_i x^{i*}_{l,s} - e^i_{l,s}, \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$y^-_a = \min_i y^{i*}_a, \quad a \in \check{\mathcal{A}},$$

$$y^+_a = \max_i y^{i*}_a, \quad a \in \check{\mathcal{A}},$$

determine the minimal and the maximal excess demands on both the spot and the asset markets. If

$$\underline{\mathcal{I}}_{l,s} = \{i : x^{i*}_{l,s} - e^i_{l,s} = z^-_{l,s}\}, \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$\overline{\mathcal{I}}_{l,s} = \{i : x^{i*}_{l,s} - e^i_{l,s} = z^+_{l,s}\}, \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$\underline{\mathcal{I}}_a = \{i : y^{i*}_a = y^-_a\}, \quad a \in \check{\mathcal{A}},$$

$$\overline{\mathcal{I}}_a = \{i : y^{i*}_a = y^+_a\}, \quad a \in \check{\mathcal{A}},$$

then in a neighbourhood of the competitive equilibrium, only individuals in $\underline{\mathcal{I}}_{l,s}$ may be rationed on supply in the spot market (l,s), only individuals in $\overline{\mathcal{I}}_{l,s}$ on demand in the spot market (l,s), only individuals in $\underline{\mathcal{I}}_a$ on supply in the asset market a, and individuals in $\overline{\mathcal{I}}_a$ on demand in asset market a.

Lemma 14 *Generically,*

$$|\underline{\mathcal{I}}_{l,s}| = |\overline{\mathcal{I}}_{l,s}| = 1, \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S},$$

and

$$|\underline{\mathcal{I}}_a| = |\overline{\mathcal{I}}_a| = 1, \quad a \in \check{\mathcal{A}}.$$

Proof It follows from a standard transversality argument. □

There is a generic set of economies, for which, there is exactly one individual in each market with the minimal excess demand and exactly one individual with the maximal excess demand; one restricts attention to this set, which does not include economies with Pareto optimal endowments.

At a competitive equilibrium, $((p^*, q^*), (\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*), (x^{\mathcal{I}*}, y^{\mathcal{I}*}))$, $\mathcal{N}^i_{x^{i*}, y^{i*}}$ is a neighbourhood of (x^{i*}, y^{i*}) with the property that, for every $(x^{\mathcal{I}}, y^{\mathcal{I}}) \in \mathcal{N}_{x^{\mathcal{I}*}, y^{\mathcal{I}*}} = \times_{i \in \mathcal{I}} \mathcal{N}^i_{x^{i*}, y^{i*}}$, for all $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$,

$$x^{i'}_{l,s} - e^{i'}_{l,s} < 0 \text{ and } x^{i'}_{l,s} - e^{i'}_{l,s} < x^i_{l,s} - e^i_{l,s}, \quad i \neq i', i' \in \underline{\mathcal{I}}_{l,s}$$

$$x^{i'}_{l,s} - e^{i'}_{l,s} > 0 \text{ and } x^{i'}_{l,s} - e^{i'}_{l,s} > x^i_{l,s} - e^i_{l,s}, \quad i \neq i', i' \in \overline{\mathcal{I}}_{l,s}$$

and, for all $a \in \check{\mathcal{A}}$,

$$y^{i'}_a < 0 \text{ and } y^{i'}_a < y^i_a, \quad i \neq i', i' \in \underline{\mathcal{I}}_a$$

$$y^{i'}_a > 0 \text{ and } y^{i'}_a > y^i_a, \quad i \neq i', i' \in \overline{\mathcal{I}}_a.$$

In the optimization problem an individual faces when determining his demand for commodities and assets, the lagrange multipliers corresponding to the rationing constraints in the markets for commodities are $\pi = (\ldots, \pi_s, \ldots)$, with $\pi_s = (\ldots, \pi_{l,s}, \ldots, \pi_{L,s})$, and the multipliers corresponding to the rationing constraints in the markets for assets are $\rho = (\ldots, \rho_a, \ldots, \rho_A)$.

At prices and lagrange multipliers (p, q, π, ρ) the modified demand correspondence $\widehat{d^i}$ of an individual is defined by the solution to the optimization problem

$$\begin{aligned} \max \quad & u^i(x) - \textstyle\sum_{(l,s) \in \check{\mathcal{L}} \times \mathcal{S}} \pi_{l,s} x_{l,s} - \sum_{a \in \check{\mathcal{A}}} \rho_a y_a, \\ \text{s.t.} \quad & qy \leq 0, \\ & p_s(x_s - e^i_s) \leq R_s.y, \quad s \in \mathcal{S}. \end{aligned}$$

Although the correspondence $\widehat{d^i}$ may have empty values, this is not the case in a neighbourhood of $(p^*, q^*, 0, 0)$; and it is single-valued, and, hence, a function, whenever it is non-empty valued.

Lemma 15 *Generically, at a competitive equilibrium,* $((p^*, q^*), (\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*), (x^{\mathcal{I}*}, y^{\mathcal{I}*}))$, *the function* $\widehat{d^i}$ *is continuously differentiable on an open neighbourhood* $\mathcal{N}_{p^*, q^*, 0, 0}$ *of* $(p^*, q^*, 0, 0)$.

Proof It follows from a standard transversality argument. □

For every individual, the function $c^i : \mathcal{R}^{LS} \times \mathcal{R}^A \to \mathcal{R}^{LS} \times \mathcal{R}^A$ is defined by

$$c^i_{l,s}(\pi, \rho) = \begin{cases} \pi_{l,s}, & \text{if } \pi_{l,s} \leq 0 \text{ and } i \in \underline{\mathcal{I}}_{l,s} \\ \\ & \text{if } \pi_{l,s} \geq 0 \text{ and } i \in \overline{\mathcal{I}}_{l,s}, \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S}, \\ \\ 0, & \text{otherwise,} \end{cases}$$

and by

$$c^i_a(\pi, \rho) = \begin{cases} \rho_a, & \text{if } \rho_a \leq 0 \text{ and } i \in \underline{\mathcal{I}}_a \\ \\ & \text{if } \rho_a \geq 0 \text{ and } i \in \overline{\mathcal{I}}_a, \quad a \in \check{\mathcal{A}}, \\ \\ 0, & \text{otherwise.} \end{cases}$$

It relates the lagrange multipliers, (π, ρ), to the fix-price equilibria in the neighborhood of a competitive equilibrium.

The aggregate, modified excess demand function for commodities and assets other than the numeraire, $\widehat{z}^a = (\ldots, \widehat{z}^a_{l,s}, \ldots, \widehat{z}^a_{L,s}, \ldots, \widehat{z}^a_a, \ldots, \widehat{z}^a_A)$, is defined, on the neighbourhood $\mathcal{N}_{p^*,q^*,0,0}$ of $(p^*, q^*, 0, 0)$, by

$$\widehat{z}^a_{l,s}(p, q, \pi, \rho) = \sum_{i \in \mathcal{I}} \widehat{d}^i_{l,s}(p, q, c^i(\pi, \rho)) - \sum_{i \in \mathcal{I}} e^i, \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S},$$

and by

$$\widehat{z}^a_a(p, q, \pi, \rho) = \sum_{i \in \mathcal{I}} \widehat{d}^i_a(p, q, c^i(\pi, \rho)), \quad a \in \check{\mathcal{A}}.$$

For fix-price equilibria in the neighbourhood of a competitive equilibrium, it is sufficient to restrict attention to the function $\widehat{z}^a$.

Lemma 16 *Generically, $(x^{\mathcal{I}}, y^{\mathcal{I}}) \in \mathcal{D}(p,q) \cap \mathcal{N}_{(x^{\mathcal{I}*}, y^{\mathcal{I}*})}$ if and only if there exists (π, ρ), such that $(p, q, \pi, \rho) \in \mathcal{N}_{p^*,q^*,0,0}$, $\widehat{d}^i(p, q, c^i(\pi, \rho)) = (x^i, y^i)$, for all $i \in \mathcal{I}$, and $\widehat{z}^a(p, q, \pi, \rho) = (0, 0)$.*

Proof It follows from the first order conditions. □

At a competitive equilibrium, the function $\widehat{z}^a$ vanishes: $\widehat{z}^a(p^*, q^*, 0, 0) = (0, 0)$. The function $\widehat{z}^a$ is Lipschitz continuous because of the differentiability of the functions $\widehat{d}^i$, and the Lipschitz continuity of the functions c^i, for every individual; it is differentiable at each point $(p, q, \pi, \rho) \in \mathcal{N}_{p^*,q^*,0,0}$ where all components of π and ρ are non-zero. Lemma 7 establishes that fix-price equilibria in the neighbourhood of a competitive equilibrium, are characterized by studying the zero points of $\widehat{z}^a$.

For a sign vector r,

$$\mathcal{N}^r_{p^*,q^*,0,0} =$$

$$\{(p, q, \pi, \rho) \in \mathcal{N}_{p^*,q^*,0,0} : \pi_{l,s} r_{l,s} > 0, (l,s) \in \check{\mathcal{L}} \times \mathcal{S}, \rho_a r_a > 0, a \in \check{\mathcal{A}}\}.$$

The function $\widehat{z}^a$ is differentiable on $\mathcal{N}^r_{p^*,q^*,0,0}$. If no component of r is equal to zero, then the limit of its jacobian, $\lim_{n\to\infty}\partial\widehat{z}^a(p^n,q^n,\pi^n,\rho^n)$, along a sequence $((p^n,q^n,\pi^n,\rho^n)\in\mathcal{N}^r_{p^*,q^*,0,0}:n=1,\ldots)$ that converges to $(p^*,q^*,0,\,0)$, exists and is denoted $\partial\widehat{z}^{a^r}(p^*,q^*,0,0)$. Since

$$\partial_{\check{p},\check{q}}\widehat{z}^{a^r}_{l,s}(p^*,q^*,0,0)=\sum_{i\in\mathcal{I}}\partial_{\check{p},\check{q}}\widehat{d}^i_{l,s}(p^*,q^*,0,0)=\partial_{\check{p},\check{q}}z^a_{l,s}(p^*,q^*),$$

and

$$\partial_{\check{p},\check{q}}\widehat{z}^{a^r}_a(p^*,q^*,0,0)=\sum_{i\in\mathcal{I}}\partial_{\check{p},\check{q}}\widehat{d}^i_a(p^*,q^*,0,0)=\partial z^a_a(p^*,q^*),$$

where $z^a=(\ldots,z^a_{l,s},\ldots,z^a_{L,s},\ldots,z^a_a,\ldots,z^a_A)$ denotes the unconstrained total excess demand function for commodities and assets other than the numeraires, at a competitive equilibrium, the jacobian with respect to $(\check{p},\check{q})$ is independent of r.

Proposition 17 *If $((p^*,q^*),(\underline{z}^*,\overline{z}^*,\underline{y}^*,\overline{y}^*),(x^{\mathcal{I}*},y^{\mathcal{I}*}))$ is a competitive equilibrium and r is a sign vector without zero, and if $\partial z(p^*,q^*)$ is of full rank, then the tangent cone at (p^*,q^*) to the set of price systems having a fix-price equilibrium with state of the markets r is*

$$\{(p,q)\in\mathcal{P}\times\mathcal{Q}:(\check{p},\check{q})=(\partial z(p^*,q^*))^{-1}\partial_{\pi,\rho}\widehat{z}^{a^r}(p^*,q^*,0,0)(\pi,\rho),$$

$$\pi_{l,s}r_{l,s}>0,(l,s)\in\check{\mathcal{L}}\times\mathcal{S},\rho_a r_a>0,a\in\check{\mathcal{A}}\}.$$

Proof The restriction of the function $\widehat{z}^a$ to $\mathcal{N}^r_{p^*,q^*,0,0}$ extends to a differentiable function $\widetilde{z}^a$ on $\mathcal{N}_{p^*,q^*,0,0}$ as follows: for $i\in\mathcal{I}$, the function $\widetilde{c}^i$ is defined by $\widetilde{c}^i_{l,s}(\pi,\rho)=\pi_{l,s}$ if $i\in\underline{\mathcal{I}}_{l,s}$ and $r_{l,s}=-1$ or $i\in\overline{\mathcal{I}}_{l,s}$ and $r_{l,s}=+1$, or $\widetilde{c}^i_{l,s}=0$ otherwise and $\widetilde{c}^i_a(\pi,\rho)=\rho_a$ if $i\in\underline{\mathcal{I}}_a$ and $r_a=-1$ or $i\in\overline{\mathcal{I}}_a$ and $r_a=+1$, and $\widetilde{c}^i_a(\pi,\rho)=0$, otherwise. The function $\widetilde{z}^a(\ldots,\widetilde{z}^a_{l,s},\ldots,\widetilde{z}^a_{L,s},\ldots,\widetilde{z}^a_a,\ldots,\widetilde{z}^a_A)$ is defined by $\widetilde{z}^a_{l,s}(p,q,\pi,\rho)=\sum_{i\in\mathcal{I}}\widehat{d}^i_{l,s}(p,q,\widetilde{c}^i(\pi,\rho))-\sum_{i\in\mathcal{I}}e^i$, and by $\widetilde{z}^a_a(p,q,\pi,\rho)=\sum_{i\in\mathcal{I}}\widehat{d}^i_a(p,q,\widetilde{c}^i(\pi,\rho))$. Since $\partial z(p^*,q^*)$ is of full rank, it follows by the implicit function theorem that the solution to $\widetilde{z}^a(p,q,\pi,\rho)=(0,0)$ determines p and q as a function of π and ρ in a neighbourhood of $(0,0)$. The derivative of this function at $(0,0)$ with respect to π and ρ is given by $(\partial z^a(p^*,q^*))^{-1}\partial_{\pi,\rho}\widetilde{z}^a(p^*,q^*,0,0)$. The expression in the proposition follows immediately if one takes into account that only π 's and ρ 's satisfying $\pi_{l,s}r_{l,s}>0$, for all $(l,s)\in\check{\mathcal{L}}\times\mathcal{S}$, and $\rho_a r_a>0$, for all $a\in\check{\mathcal{A}}$, should be considered. □

The assumption that $\partial z(p^*,q^*)$ has full rank at every competitive equilibrium holds generically — Geanakoplos and Polemarchakis (1986), Proposition 2. Proposition 1 characterizes the tangent cones to the regions in the price space having a fix-price equilibrium with state of the markets r in the neighbourhood of a competitive equilibrium. It guarantees neither that the closures of these tangent cones cover the price space nor that the tangent cones are full-dimensional nor that the tangent cones do not intersect. If this were the case, one would have local uniqueness of the fix-price competitive equilibrium; in fact, differentiable local uniqueness, since p and

q are differentiable functions of π and ρ, and the demand functions of individuals are differentiable as a function of p, q, π, and ρ on $\mathcal{N}_{p^*,q^*,0,0}$. Even in the case of a complete asset market, the local uniqueness of the fix-price equilibrium fails in robust examples — Madden (1982).

The function $\widehat{z}^a$ is Lipschitz continuous. The generalized jacobian of a lipschitz continuous function, f, at a point, x, is the convex hull of all matrices that are the limits of the sequence $(\partial f(x^n) : n = 1, \ldots)$, where $(x^n : n = 1, \ldots)$ is a convergent sequence with $\lim_{n\to\infty} x^n = x$ and f is differentiable at $x^n, n = 1, \ldots$.

If a function f is lipschitz continuous, $f(\widehat{x}, \widehat{y}) = 0$, and every matrix M in $\partial_x f(\widehat{x}, \widehat{y})$ has full rank, then there exist a neighbourhood, $\mathcal{N}_{\widehat{x},\widehat{y}}$ of $(\widehat{x}, \widehat{y})$, a neighbourhood $\mathcal{N}_{\widehat{y}}$ of $\widehat{y}$, and a lipschitz continuous function, g, on $\mathcal{N}_{\widehat{y}}$, such that $(x, y) \in \mathcal{N}_{\widehat{x},\widehat{y}}$ and $f(x, y) = 0$ if and only if $y \in \mathcal{N}_{\widehat{y}}$ and $x = g(y)$.[19]

Proposition 18 *If $((p^*, q^*), (\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*), (x^{\mathcal{I}*}, y^{\mathcal{I}*}))$ is a competitive equilib- rium, and if the determinants of the matrices $\partial_{\pi,\rho}\widehat{z}^r(p^*, q^*, 0, 0)$, with r sign vectors without zero components, are either all equal to -1 or all equal to $+1$, then the competitive equilibrium allocation is differentiably locally unique as a fix-price equilibrium allocation.*

Proof The generalized jacobian $\partial_{\pi,\rho}\widehat{z}^a(p^*, q^*, 0, 0)$ is equal to the convex hull of the matrices $M^r = \partial_{\pi,\rho}\widehat{z}^{a^r}(p^*, q^*, 0, 0)$, with r any sign vector without zero components. Moreover, column (l, s) of such a matrix only depends on $r_{l,s}$ and column $LS + a$ only on r_a. Therefore, any matrix M in $\partial_{\pi,\rho}\widehat{z}^a(p^*, q^*, 0, 0)$ can be written as

$$(\ldots, \lambda_{l,s}M^-_{l,s} + (1 - \lambda_{l,s})M^+_{l,s}, \ldots, \lambda_{L,s}M^-_{L,s} + (1 - \lambda_{L,s})M^+_{L,s},$$

$$\ldots, \lambda_a M^-_a + (1 - \lambda_a)M^+_a, \ldots, \lambda_A M^-_A + (1 - \lambda_A)M^+_A),$$

with $\lambda_{l,s} \in [0, 1]$, and $\lambda_a \in [0, 1]$, for all $a \in \check{\mathcal{A}}$, with $M^-_{l,s}$ corresponding to column (l, s) of a matrix M^r with $r_{l,s} = -1$, $M^+_{l,s}$ to column (l, s) of a matrix M^r with $r_{l,s} = +1$, M^-_a to column $LS + a$ of a matrix M^r with $r_a = -1$ and M^+_a to column $LS + a$ of a matrix M^r with $r_a = +1$. The determinant of M, is equal to the sum over all sign vectors without zero components of

$$\times_{(l,s)\in\check{\mathcal{L}}\times\mathcal{S}}(\tfrac{1}{2} + (\tfrac{1}{2} - \lambda_{l,s})r_{l,s}) \times_{a\in\check{\mathcal{A}}} (\tfrac{1}{2} + (\tfrac{1}{2} - \lambda_a)r_a) \det(M^r).$$

If, the sign of every $\det(M^r)$ is negative, then the sum is negative, whereas the sum is positive otherwise. So, M has full rank. By the extension of the implicit function proposition, π and ρ are described as a Lipschitz continuous function of p and q on a neighbourhood of (p^*, q^*) to guarantee that $\widehat{z}^a(p, q, \pi, \rho) = (0, 0)$. Since fix-price competitive equilibria are, locally, lipschitz continuous functions of (p, q), the competitive equilibrium is a locally unique fix-price equilibrium. The implicit function theorem applied to the function $\widetilde{z}^a$, as constructed in the proof of Proposition 1 for

[19]Laroque (1978, p. 121), following Clarke (1976).

any sign vector r without zero components, yields that the competitive equilibrium is differentiably locally unique as a fix-price competitive equilibrium. $\square$

There exist utility functions of individuals and matrices of payoffs of assets such that the set of economies satisfying Proposition 5 at all competitive equilibria is non-empty and open.

Assumption 1 *For every economy, $\omega \in \Omega$, every competitive equilibrium allocation is differentiably locally unique as a fix-price equilibrium allocation.*

As in Laroque (1981), whenever there are two sign vectors, r^1 and r^2, without zero components such that the determinants of $\partial_{\pi,\rho}\widehat{z}^{a^{r^1}}(p^*, q^*, 0, 0)$ and $\partial_{\pi,\rho}\widehat{z}^{a^{r^2}}(p^*, q^*, 0, 0)$ have opposite signs, and $\partial z^a(p^*, q^*)$ has full rank, then, for every neighbourhood, $\mathcal{N}_{x^{\mathcal{I}*}, y^{\mathcal{I}*}}$, of $(x^{\mathcal{I}*}, y^{\mathcal{I}*})$, there exists, for every neighbourhood, $\mathcal{N}_{p^*,q^*}$, of (p^*, q^*), a price system, $(p, q) \in \mathcal{N}_{p^*,q^*}$, with at least two fix-price competitive equilibrium allocations in $\mathcal{N}_{x^{\mathcal{I}*}, y^{\mathcal{I}*}}$: the conditions in Proposition 2 are "almost necessary."

It is an open question whether the interior of the set of allocations of endowments for which all competitive equilibrium allocations of the economy are differentiably locally unique as fix-price equilibrium allocations can be empty.

5 Pareto Improving Price Regulation

Price regulation can Pareto improve on a competitive equilibrium [20] $((p^*, q^*), (\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*), (x^*, y^*))$ if there exist prices of commodities, p, such that a fix-price equilibrium allocation of commodities, x, at prices of commodities and assets (p, q^*) Pareto dominates the allocation x^*. The ambiguity introduced by the possibility of multiple fix-price equilibrium allocations of commodities at prices (p, q^*) is circumvented by considering local variations at competitive equilibria allocations that are differentiably locally unique as fix-price equilibria.

Price regulation at competitive equilibrium prices p^* is uniform if the deviation of prices of commodities from their competitive equilibrium values, $p^*_s - p_s$, does not vary across states of the world.

Definition 19 (Pareto improving price regulation) *A competitive equilibrium,*

$$((p^*, q^*), (\underline{z}^*, \overline{z}^*, \underline{y}^*, \overline{y}^*), (x^*, y^*))$$

can be Pareto improved by price regulation if it is differentiably locally unique as fix-price equilibrium and there exists an infinitesimal variation in the prices of commodities,

$$d\check{p} = (\ldots, d\check{p}_s, \ldots),$$

[20]The superscript $\mathcal{I}$ that indicates allocations is omitted in this section.

such that

$$\sum_{(l,s)\in\check{\mathcal{L}}\times\mathcal{S}} \partial_{p_{l,s}} v^i(p^*, q^*) dp_{l,s} > 0, \quad i \in \mathcal{I}.$$

A competitive equilibrium, can be Pareto improved by uniform price regulation if it can be Pareto improved by price regulation with

$$d\check{p}_s = d\check{p}_{s'}, \quad s, s' \in \mathcal{S}.$$

Pareto improvement by price regulation is possible only if the competitive equilibrium allocation is not Pareto optimal, which is generically the case, and the asset market is incomplete.

Assumption 2 $A + 1 < S$.

Another necessary requirement is that the economy allows for heterogeneous individuals.

Assumption 3 $I > 1$.

The function φ is defined by

$$\varphi(x, \widetilde{\lambda}, \widetilde{p}, e) = \begin{pmatrix} \vdots \\ \partial u^i(x^i) - \widetilde{\lambda}^i \widetilde{p}, \quad i \in \mathcal{I} \\ \vdots \\ \sum_{s\in\mathcal{S}} \widetilde{p}_s (x^i_s - e^i_s), \quad i \in \mathcal{I} \\ \vdots \\ \sum_{i\in\mathcal{I}} (x^i_{l,s} - e^i_{l,s}), \quad (l,s) \in \mathcal{L}\times\mathcal{S} \setminus \{(L+1, S)\} \\ \vdots \\ \sum_{s\in\mathcal{S}} n_s \widetilde{p}_s (x^i_s - e^i_s), \quad i \in \mathcal{I}\setminus\{1\} \\ \vdots \end{pmatrix},$$

where $n \neq 0$ is a fixed vector, such that $nR = 0$; prices of commodities, $\widetilde{p}$, are discounted prices, with only the price of commodity $(L+1, S)$ normalized to 1, and the lagrangian multipliers , $\widetilde{\lambda}^i$, do not vary with the state of the world.[21]

For e fixed, the function φ_e is defined by

$$\varphi_e(x, \widetilde{\lambda}, \widetilde{p}) = \varphi(x, \widetilde{\lambda}, \widetilde{p}, e).$$

Lemma 20 *Generically, competitive equilibrium allocations are not Pareto optimal.*

[21] The dimension of the domain of the function φ is $I(L+1)S + I + (L+1)S - 1 + I(L+1)S$, while the dimension of the range is $I(L+1)S + I + (L+1)S - 1 + I - 1$,

Proof A necessary condition for x to be a Pareto optimal competitive equilibrium allocation for an economy, e, is that $\varphi_e(x, \widetilde{\lambda}, \widetilde{p}) = 0$.

Since the dimension of the domain of the function φ_e is lower than the dimension of the range,[22] whenever the function is transverse to 0, a solution to the equation $\varphi_e(x, \widetilde{\lambda}, \widetilde{p}) = 0$ does not exist.

By a standard argument, the function φ is transverse to 0. By the transversal density theorem,[23] the set of economies for which the function φ_e is transverse to 0 has full lebesgue measure; by Assumption 1 and a standard argument, this set is open and, hence, generic. □

The function ψ is defined by

$$\psi(\xi, e) = \left(\begin{array}{c} \vdots \\ \partial_{x_s^i} u^i(x^i) - \lambda_s^i p_s, \quad i \in \mathcal{I}, s \in \mathcal{S} \\ \vdots \\ p_s(x_s^i - e_s^i) - R_s . y^i, \quad i \in \mathcal{I}, s \in \mathcal{S} \\ \vdots \\ \lambda^i R - \mu^i q, \quad i \in \mathcal{I} \\ \vdots \\ \sum_{i \in \mathcal{I}} (x_{l,s}^i - e_{l,s}^i), \quad (l,s) \in \check{\mathcal{L}} \times \mathcal{S} \\ \vdots \\ \sum_{i \in \mathcal{I}} y_a^i, \quad a \in \check{\mathcal{A}} \\ \vdots \\ q y^i, \quad i \in \mathcal{I} \\ \vdots \end{array} \right),$$

where[24]

$$\xi = (x, \lambda, y, \mu, \check{p}, \check{q}).$$

The vector ξ is restricted to the set Ξ defined by

$$\Xi = \mathcal{R}_{++}^{I(L+1)S} \times \mathcal{R}_{++}^{IS} \times \mathcal{R}^{I(A+1)} \times \mathcal{R}^{I} \times \check{P} \times \check{Q}.$$

The dimension of Ξ is denoted by N.

For fixed e, the function ψ_e is defined by[25]

$$\psi_e(\xi) = \psi(\xi, e).$$

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[22]The dimension of the domain of the function φ_e, is $I(L+1)S + I + (L+1)S - 1$, while the dimension of the range is $I(L+1)S + I + (L+1)S - 1 + I - 1$.

[23]Mas-Colell (1985a), Proposition 8.3.1, p. 320.

[24]The dimension of the domain of the function ψ is $I(L+1)S+IS+I(A+1)+I+LS+A+I(L+1)S$, while the dimension of the range is $I(L+1)S + IS + I(A+1) + LS + A + I$.

[25]The dimension of the domain of the function ψ_e is $I(L+1)S + IS + I(A+1) + I + LS + A$, while the dimension of the range is $I(L+1)S + IS + I(A+1) + LS + A + I$.

A competitive equilibrium, $((p^*,q^*),(\underline{z}^*,\overline{z}^*,\underline{y}^*,\overline{y}^*),(x^*,y^*))$, is characterized by the necessary and sufficient first order conditions

$$\psi_e(\xi^*)=0,$$

with $\underline{z}^*_{l,s} < x^{i*}_{l,s} - e^i_{l,s} < \overline{z}^*_{l,s}$, for $(l,s) \in \check{\mathcal{L}} \times \mathcal{S}$, and $\underline{y}^{i*}_a < y^{i*}_a < \underline{y}^{i*}_a$, for $a \in \check{\mathcal{A}}$.

The function h is defined by

$$h(x,\lambda,\alpha,e)=\begin{pmatrix} \vdots \\ \sum_{i\in\mathcal{I}} \alpha^i \lambda^i_s (x^i_{l,s}-e^i_{l,s}), \quad (l,s)\in\check{\mathcal{L}}\times\mathcal{S} \\ \vdots \\ \sum_{i\in\mathcal{I}}(\alpha^i)^2-1 \end{pmatrix},$$

where α is a vector of dimension I.

A competitive equilibrium can be improved by means of price regulation if the matrix of partial derivatives of the indirect utility functions with respect to prices has full rank. By Lemma 4, this matrix is guaranteed to have full rank if there is no solution to the first order conditions in combination with the equation

$$h(x,\lambda,\alpha,e)=0.$$

The function $\widetilde{\psi}$ is defined by

$$\widetilde{\psi}(\xi,\alpha,e)=\begin{pmatrix} \psi(\xi,e) \\ h(x,\lambda,\alpha,e) \end{pmatrix}.$$

For fixed e, the function $\widetilde{\psi}_e$ is defined by[26]

$$\widetilde{\psi}_e(\xi,\alpha)=\widetilde{\psi}_e(\xi,\alpha,e).$$

If the function $\widetilde{\psi}$ is transverse to 0, then it follows from the transversal density theorem that, for a subset of economies of full Lebesgue measure, the function $\widetilde{\psi}_e$ is transverse to 0. Since the dimension of the range exceeds that of the domain, transversality of the function $\widetilde{\psi}_e$ implies that there are no solutions to the associated system of equations: it is possible to Pareto improve all competitive equilibria.

Proposition 21 *If $LS \geq I$, then, generically, all competitive equilibria can be Pareto improved by price regulation.*

[26] The dimension of the domain of the function $\widetilde{\psi}_e$, is $I(L+1)S+IS+I(A+1)+I+LS+A+I$, while the dimension of the range is $I(L+1)S+IS+I(A+1)+LS+A+I+LS+1$.

Proof One fixes $(\bar{l}, \bar{s}) \in \check{\mathcal{L}} \times \mathcal{S}$ and Ω^*, an open set of endowments of full measure, such that no competitive equilibria of the associated economy $\mathcal{E}$ are Pareto optimal.

The function g is defined by

$$g(x, \lambda, e) = \sum_{s \in \mathcal{S} \setminus \{\bar{s}\}} \sum_{i \in \mathcal{I}} \frac{\lambda_s^i}{\lambda_{\bar{s}}^i} (x_{\bar{l},s}^i - e_{\bar{l},s}^i).$$

The function $\widehat{\psi}$ is defined by

$$\widehat{\psi}(\xi, e) = \begin{pmatrix} \psi(\xi, e) \\ \\ g(x, \lambda, e) \end{pmatrix}.$$

For fixed e, the function $\widehat{\psi}_e$ is defined by

$$\widehat{\psi}_e(\xi) = \widehat{\psi}(\xi, e).$$

If $(\xi, e) \in \Xi \times \Omega^*$ is such that $\widehat{\psi}(\xi, e) = 0$, then the matrix, $\widehat{M}$, of partial derivatives of $\widehat{\psi}$ evaluated at (ξ, e) has full row rank: if $v'\widehat{M} = 0$, then $v = 0$.

The components of v are denoted $v_{1,i,l,s}$, $i \in \mathcal{I}$, $(l, s) \in \mathcal{L} \times \mathcal{S}$, $v_{2,i,s}$, $i \in \mathcal{I}$, $s \in \mathcal{S}$, $v_{3,i,a}$, $i \in \mathcal{I}$, $a \in \mathcal{A}$, $v_{4,l,s}$, $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, $v_{5,a}$, $a \in \check{\mathcal{A}}$, $v_{6,i}$, $i \in \mathcal{I}$, and v_9, according to the labelling of the equations defining $\widehat{\psi}$.

If v is such that $v'\widehat{M} = 0$, then $0 = v'\partial_{e_{L+1,s}^i} \widehat{\psi}(\xi, e) = -v_{2,i,s}$, $i \in \mathcal{I}$, $s \in \mathcal{S}$, and, thus,

$$v_{2,i,s} = 0, \quad i \in \mathcal{I}, s \in \mathcal{S}.$$

It follows that, for $i \in \mathcal{I}$,

$$0 = v'\partial_{e_{l,s}^i} \widehat{\psi}(\xi, e) = -v_{4,l,s}, \quad (l, s) \in (\check{\mathcal{L}} \setminus \{\bar{l}\}) \times \mathcal{S},$$

$$0 = v'\partial_{e_{\bar{l},s}^i} \widehat{\psi}(\xi, e) = -v_{4,\bar{l},s} - v_9 \frac{\lambda_s^i}{\lambda_{\bar{s}}^i} = 0, \quad s \in \mathcal{S} \setminus \{\bar{s}\},$$

$$0 = v'\partial_{e_{\bar{l}}^i(\bar{s})} \widehat{\psi}(\xi, e) = -v_{4,\bar{l},\bar{s}}.$$

Consequently, if $v_{4,\bar{l},\widehat{s}} = 0$ for some $\widehat{s} \in \mathcal{S} \setminus \{\bar{s}\}$, then $v_9 = 0$ and $v_{4,\bar{l},s} = 0$, for all $s \in \mathcal{S} \setminus \{\bar{s}\}$. If, on the contrary, $v_{4,\bar{l},s} \neq 0$, for all $s \in \mathcal{S} \setminus \{\widehat{s}\}$, then

$$\frac{\lambda_s^i}{\lambda_{\bar{s}}^i} = -\frac{v_{4,\bar{l},s}}{v_9} = \frac{\lambda_s^{i'}}{\lambda_{\bar{s}}^{i'}}, \quad i, i' \in \mathcal{I}, \ s \in \mathcal{S} \setminus \{\bar{s}\}.$$

Hence, for $i, i' \in \mathcal{I}$, for $s^1, s^2 \in \mathcal{S}$,

$$\frac{\lambda_{s^1}^i}{\lambda_{s^2}^i} = \frac{\lambda_{s^1}^i}{\lambda_{\bar{s}}^i} \frac{\lambda_{\bar{s}}^i}{\lambda_{s^2}^i} = \frac{\lambda_{s^1}^{i'}}{\lambda_{\bar{s}}^{i'}} \frac{\lambda_{\bar{s}}^{i'}}{\lambda_{s^2}^{i'}} = \frac{\lambda_{s^1}^{i'}}{\lambda_{s^2}^{i'}}.$$

Since $\widehat{\psi}(\xi, e) = 0$, the economy e has a Pareto optimal competitive equilibrium induced by ξ, contradicting $e \in \Omega^*$. Consequently, $v_{4,\bar{l},s} = 0$, $s \in \mathcal{S} \setminus \{\bar{s}\}$, and $v_9 = 0$.

Summarizing,

$$v_{4,l,s} = 0, \quad (l, s) \in \check{\mathcal{L}} \times \mathcal{S},$$

$$v_9 = 0.$$

For $i \in \mathcal{I}$, for $(l, s) \in \mathcal{L} \times \mathcal{S}$,

$$0 = v' \partial_{x^i_{l,s}} \widehat{\psi}(\xi, e) = v'_{1,i,\cdot,\cdot} \partial_{x^i_{l,s}} \partial u^i(x^i).$$

It is possible to represent a utility function satisfying Assumption 1 by one with $\partial^2 u^i(x^i)$ negative definite on a bounded subset of the consumption set.[27] Therefore, without loss of generality, $\partial^2 u^i(x^i)$ is assumed to be negative definite, and, thus,

$$v_{1,i,l,s} = 0, \quad i \in \mathcal{I}, \ (l, s) \in \mathcal{L} \times \mathcal{S}.$$

For $i \in \mathcal{I}$, $0 = v' \partial_{y^i_{A+1}} \widehat{\psi}(\xi, e) = v_{8,i}$, and, thus

$$v_{8,i} = 0, \quad i \in \mathcal{I}.$$

Also, for $a \in \check{\mathcal{A}}$, $0 = v' \partial_{y^i_a} \widehat{\psi}(\xi, e) = v_{5,a}$, and, thus,

$$v_{5,a} = 0, \quad a \in \check{\mathcal{A}}.$$

Finally, $0 = v' \partial_{\lambda^i_s} \widehat{\psi}(\xi, e) = v'_{3,i,\cdot} R'_{s\cdot}$, $i \in \mathcal{I}$, $s \in \mathcal{S}$. Since R has full column rank,

$$v_{3,i,a} = 0, \quad i \in \mathcal{I}, a \in \check{\mathcal{A}}.$$

Therefore, $v = 0$, $\widehat{M}$ has rank full row rank, $N + 1$, and $\widehat{\psi}$ is transverse to 0. Moreover, $\widehat{\psi}$ is continuously differentiable. If the set of endowments such that $\widehat{\psi}_e$ is transverse to zero is $\widehat{\Omega}_{\bar{l},\bar{s}}$, then, by the transversal density proposition, $\Omega^* \setminus \widehat{\Omega}_{\bar{l},\bar{s}}$ has lebesgue measure zero. For $e \in \widehat{\Omega}_{\bar{l},\bar{s}}$, $\widehat{\psi}_e$ is a function from an N-dimensional C^∞ manifold into an $(N+1)$-dimensional C^∞ manifold, $\widehat{\psi}_e \in C^1(\Xi, \mathcal{R}^{N+1})$, and $\widehat{\psi}_e$ is transverse to 0, so $(\widehat{\psi}_e)^{-1}(\{0\}) = \emptyset$. The same arguments can be repeated for every choice of $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$.

The set $\widehat{\Omega} = \cap_{(l,s) \in \check{\mathcal{L}} \times \mathcal{S}} \widehat{\Omega}_{l,s}$ is of full measure and, for $e \in \widehat{\Omega}$, (ξ, e) is a solution to first order conditions for a competitive equilibrium only if

$$\sum_{s \in \mathcal{S} \setminus \{\bar{s}\}} \sum_{i \in \mathcal{I}} \frac{\lambda^i_s}{\lambda^i_{\bar{s}}} (x^i_{\bar{l},s} - e^i_{l,s}) \neq 0, \quad (\bar{l}, \bar{s}) \in \check{\mathcal{L}} \times \mathcal{S}.$$

By Assumption 1 and a standard argument, $\widehat{\Omega}$ is open with no loss of generality.

One restricts attention to economies $e \in \widehat{\Omega}$.

[27] Mas-Colell (1985 a), Proposition 2.6.5, p. 81.

For (ξ, α, e), such that $\widetilde{\psi}(\xi, \alpha, e) = 0$, $\widetilde{M}$ is the matrix of partial derivatives of $\widetilde{\psi}$ evaluated at (ξ, α, e).

If v is such that $v'\widetilde{M} = 0$, and components of v are denoted by $v_{1,i,l,s}$, $v_{2,i,s}$, $v_{3,i,a}$, $v_{4,l,s}$, $v_{5,a}$, $v_{6,i}$, $v_{7,l,s}$, and v_8, then,

$$0 = v'\partial_{e^i_{L+1,s}}\widetilde{\psi}(\xi, \alpha, e) = -v_{2,i,s}, \quad i \in \mathcal{I},\ s \in \mathcal{S}.$$

Hence,

$$0 = v'\partial_{e^i_{l,s}}\widetilde{\psi}(\xi, \alpha, e) = -v_{4,l,s} - \alpha^i\lambda^i_s v_{7,l,s}, \quad i \in \mathcal{I},\ (l, s) \in \check{\mathcal{L}} \times \mathcal{S}.$$

Since $\sum_{i\in\mathcal{I}}(\alpha^i)^2 = 1$, there is i' such that $\alpha^{i'} \neq 0$. If there is $\overline{s} \in \mathcal{S}$ such that, for $i \in \mathcal{I} \setminus \{i'\}$, $\alpha^{i'}\lambda^{i'}_{\overline{s}} - \alpha^i\lambda^i_{\overline{s}} = 0$, then, for any $l \in \check{\mathcal{L}}$,

$$\begin{aligned} 0 &= \textstyle\sum_{s\in\mathcal{S}\setminus\{\overline{s}\}}\sum_{i\in\mathcal{I}}\alpha^i\lambda^i_s(x^i_{l,s} - e^i_{l,s}) \\ &= \textstyle\sum_{s\in\mathcal{S}\setminus\{\overline{s}\}}\sum_{i\in\mathcal{I}}\frac{\alpha^{i'}\lambda^{i'}_{\overline{s}}}{\lambda^i_{\overline{s}}}\lambda^i_s(x^i_{l,s} - e^i_{l,s}) \\ &= \alpha^{i'}\lambda^{i'}_{\overline{s}}\textstyle\sum_{s\in\mathcal{S}\setminus\{\overline{s}\}}\sum_{i\in\mathcal{I}}\frac{\lambda^i_s}{\lambda^i_{\overline{s}}}(x^i_{l,s} - e^i_{l,s}). \end{aligned}$$

Since $\alpha^{i'} \neq 0$,

$$\sum_{s\in\mathcal{S}\setminus\{\overline{s}\}}\sum_{i\in\mathcal{I}}\frac{\lambda^i_s}{\lambda^i_{\overline{s}}}(x^i_{l,s} - e^i_{l,s}) = 0,$$

a contradiction since $e \in \widehat{\Omega}$. Consequently, for every $s \in \mathcal{S}$, there is $i \in \mathcal{I} \setminus \{i'\}$ such that $\alpha^{i'}\lambda^{i'}_s - \alpha^i\lambda^i_s \neq 0$. For $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, $(\alpha^{i'}\lambda^{i'}_s - \alpha^i\lambda^i_s)v_{7,l,s} = 0$, so $v_{7,l,s} = 0$, and, thus $v_{4,l,s} = 0$.

Also, $0 = v'\partial_{\alpha^{i'}}\widetilde{\psi}(\xi, \alpha, e) = 2\alpha^{i'}v_8$, so, since $\alpha^{i'} \neq 0$, $v_8 = 0$. It follows as in the first part of the proof that $v_{1,i,l,s} = 0$, $i \in \mathcal{I}$, $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, that $v_{6,i} = 0$, $i \in \mathcal{I}$, that $v_{5,a} = 0$, $a \in \check{\mathcal{A}}$, and that $v_{3,i,a} = 0$, $i \in \mathcal{I}$, $a \in \mathcal{A}$.

Therefore, $\widetilde{M}$ has rank $N + LS + 1$ and $\widetilde{\psi}$ intersects 0 transversally; $\widetilde{\psi}$ is continuously differentiable.

If $\widetilde{\Omega}$ is the set of economies such that $\widetilde{\psi}$ is transverse to 0, $\widehat{\Omega} \setminus \widetilde{\Omega}$ has lebesgue measure zero. Without loss of generality, $\widetilde{\Omega}$, is an open set.

The set $\Omega^* \cap \widetilde{\Omega}$ is open and of full lebesgue measure. For every economy in $\Omega^* \cap \widetilde{\Omega}$, the matrix of partial derivatives of the indirect utility function, evaluated at any competitive equilibrium is invertible; which implies that there is a price regulation effecting a Pareto improvement. □

A competitive equilibrium can be Pareto improved by uniform price regulation if the matrix of partial derivatives of the indirect utility functions with respect to uniform price regulation has full rank.

The function k is defined by

$$k(\xi, \alpha, e) = \begin{pmatrix} \sum_{s\in\mathcal{S}} h_{l,s}(x, \lambda, \alpha, e), \quad l \in \check{\mathcal{L}} \\ \\ \sum_{i\in\mathcal{I}} (\alpha^i)^2 - 1 \end{pmatrix},$$

where α is a vector of dimension I.

By Lemma 4, this matrix is guaranteed to have full rank if there is no solution to the first order conditions for a competitive equilibrium augmented by the equations

$$k(\xi, \alpha, e) = 0.$$

Proposition 22 *If $L \geq I$, then, generically, all competitive equilibria can be Pareto improved by uniform price regulation.*

Proof The argument follows that in the proof of Proposition 6. The equations that characterize Pareto optimality are replaced by the equations that characterize Pareto improving regulation to define a function $\widetilde{\psi}$; the matrix $\widetilde{M}$ gives the partial derivatives of $\widetilde{\psi}$ evaluated at some (ξ, α, e) with $\widetilde{\psi}(\xi, \alpha, e) = 0$. If $v'\widetilde{M} = 0$, by considering the partial derivatives with respect to $e^i_{l,s}$, it follows that

$$v_{2,i,s} = 0, \quad i \in \mathcal{I}, s \in \mathcal{S},$$

$$v_{4,l,s} + \alpha^i \lambda^i_s v_{7,l} = 0, \quad i \in \mathcal{I},\ (l, s) \in \check{\mathcal{L}} \times \mathcal{S}.$$

If i' is such that $\alpha^{i'} \neq 0$, and if $\overline{s} \in \mathcal{S}$ such that, for $i \in \mathcal{I} \setminus \{i'\}$, $\alpha^{i'}\lambda^{i'}_{\overline{s}} - \alpha^i \lambda^i_{\overline{s}} = 0$, then, for any $l \in \check{\mathcal{L}}$,

$$0 = \sum_{i\in\mathcal{I}} \alpha^i \sum_{s\in\mathcal{S}} \lambda^i_s (x^i_{l,s} - e^i_{l,s}) = \alpha^{i'} \lambda^{i'}_{\overline{s}} \sum_{i\in\mathcal{I}} \sum_{s\in\mathcal{S}} \frac{\lambda^i_s}{\lambda^i_{\overline{s}}} (x^i_{l,s} - e^i_{l,s}).$$

Thus

$$0 = \sum_{i\in\mathcal{I}} \sum_{s\in\mathcal{S}} \frac{\lambda^i_s}{\lambda^i_{\overline{s}}} (x^i_{l,s} - e^i_{l,s}) = \sum_{i\in\mathcal{I}} \sum_{s\in\mathcal{S}\setminus\{\overline{s}\}} \frac{\lambda^i_s}{\lambda^i_{\overline{s}}} (x^i_{l,s} - e^i_{l,s}),$$

which contradicts $e \in \widehat{\Omega}$. It follows that $v_{4,l,s} = 0$, $(l, s) \in \check{\mathcal{L}} \times \mathcal{S}$, and $v_{7,l} = 0$, $l \in \check{\mathcal{L}}$. The remainder of the proof follows the argument in the proof of Proposition 6. □

Uniform price regulation is effective when $L \geq I$; it complements the constrained suboptimality result of Geanakoplos and Polemarchakis (1986), which applies when $I \geq 2L$.

6 An Example

There are two individuals: $\mathcal{I} = \{1, 2\}$, three states of the world : $\mathcal{S} = \{1, 2, 3\}$, two commodities: $\mathcal{L} = \{1, 2\}$, with commodity 2 the numeraire at every state of the world, and two assets: $\mathcal{A} = \{1, 2\}$, with asset 2 the numeraire in the market for assets. The utility function of individual i over strictly positive consumption of commodity 1 and positive consumption of commodity 2 across states of the world has an additively separable representation: $u^i = \sum_{s \in \mathcal{S}} \pi_s u^i_s$, with state dependent cardinal utility

$$u^i_s(x_s) = \alpha^i_s \ln x_{1,s} + \beta^i_s x_{2,s}, \quad \alpha^i_s > 0, \beta^i_s > 0,$$

and a strictly positive probability measure, $(\ldots, \pi_s, \ldots)$, over the states of the world; the endowment of the individual, e^i is strictly positive.

The payoffs of assets, denominated in the numeraire commodity, are $R_{\cdot 1} = (0, 1, 1)'$, and $R_{\cdot 2} = (1, 0, 0)'$, respectively.

The payoffs of assets allow for the following interpretation: consumption at state of the world 1 is concurrent with the trade in assets, while the only asset available, traded against consumption, is an indexed bond, with state-independent payoffs.

The price of the commodity other than the numeraire at each state of the world is p_s; across states of the world, $p = (\ldots, p_s, \ldots)$. The price of the asset other than the numeraire is q.

Rationing on the supply or demand of commodity 1 at state of the world s is $\underline{z}_s$, or $\overline{z}_s$, respectively, and rationing on the supply or demand of asset 1 is $\underline{y}$ or $\overline{y}$, respectively.

The parameters in the utility functions of individuals and their endowments are such that

$$\pi = \frac{\pi_1 \beta^1_1}{\pi_2 \beta^1_2 + \pi_3 \beta^1_3} = \frac{\pi_1 \beta^2_1}{\pi_2 \beta^2_2 + \pi_3 \beta^2_3},$$

and, for $\gamma^i_s = \alpha^i_s / \beta^i_s$,

$$\max \left\{ -e^1_{2,s} + \frac{\gamma^1_s e^2_{1,s} - \gamma^2_s e^1_{1,s}}{e^1_{1,s} + e^2_{1,s}} : s = 2, 3, -\pi e^2_{2,1} + \pi \frac{\gamma^2_1 e^1_{1,1} - \gamma^1_1 e^2_{1,1}}{e^1_{1,1} + e^2_{1,1}} \right\}$$

$$\leq \min \left\{ \pi e^1_{2,1} + \pi \frac{\gamma^2_1 e^1_{1,1} - \gamma^1_1 e^2_{1,1}}{e^1_{1,1} + e^2_{1,1}}, e^2_{2,s} + \frac{\gamma^1_s e^2_{1,s} - \gamma^2_s e^1_{1,s}}{e^1_{1,s} + e^2_{1,s}} : s = 2, 3 \right\},$$

which eliminates equilibria at the boundaries of their consumption sets.[28]

[28] A possible choice of parameters is, for instance,

$$\pi_1 = 1, \pi_2 = \pi_3 = \tfrac{1}{2},$$

$$\alpha^1_1 = \beta^1_1 = 1, \alpha^1_2 = \beta^1_2 = \tfrac{4}{3}, \alpha^1_3 = \beta^1_3 = \tfrac{2}{3},$$

$$\alpha^2_1 = \beta^2_1 = 1, \alpha^2_2 = \beta^2_2 = \tfrac{2}{3}, \alpha^2_3 = \beta^2_3 = \tfrac{4}{3},$$

$$e^1_1 = (1, 1)', e^1_2 = (1, 1)', e^1_3 = (2, 1)',$$

$$e^2_1 = (1, 1)', e^2_2 = (2, 1)', e^2_3 = (1, 1)'.$$

Competitive equilibrium prices and allocation are

$$
\begin{aligned}
p_s^* &= \tfrac{\gamma_s^1+\gamma_s^2}{e_{1,s}^1+e_{1,s}^2}, \quad s=1,2,3,\\
q^* &= \tfrac{1}{\pi},\\
x_{1,s}^{i*} &= \tfrac{\gamma_s^i}{\gamma_s^1+\gamma_s^2}(e_{1,s}^1+e_{1,s}^2), \quad i=1,2,\ s=1,2,3,\\
x_{2,1}^{i*} &= e_{2,1}^i + \tfrac{\gamma_1^{i'} e_{1,1}^i - \gamma_1^i e_{1,1}^{i'}}{e_{1,1}^1+e_{1,1}^2} - \tfrac{1}{\pi} y^{i*}, \quad i=1,2,\\
x_{2,s}^{i*} &= e_{2,s}^i + \tfrac{\gamma_s^{i'} e_{1,s}^i - \gamma_s^i e_{1,s}^{i'}}{e_{1,s}^1+e_{1,s}^2} + y^{i*}, \quad i=1,2,\ s=2,3,\\
y^{i*} &\le \pi e_{2,1}^i + \pi\tfrac{\gamma_1^{i'} e_{1,1}^i - \gamma_1^i e_{1,1}^{i'}}{e_{1,1}^1+e_{1,1}^2}, \quad i=1,2,\\
y^{i*} &\ge -e_{2,s}^i + \tfrac{\gamma_s^i e_{1,s}^{i'} - \gamma_s^{i'} e_{1,s}^i}{e_{1,s}^1+e_{1,s}^2}, \quad i=1,2,\ s=2,3,\\
y^{1*}+y^{2*} &= 0,\\
\underline{z}_s^* &< x_{1,s}^{i*} - e_{1,s}^i < \overline{z}_s^*, \quad i=1,2,\ s=1,2,3,\\
\underline{y}^* &< y^{i*} < \overline{y}^*, \quad i=1,2,
\end{aligned}
$$

where $i' \neq i$. After the choice of x^{1*}, x^{2*}, y^{1*} and y^{2*}, any choice of a non-binding rationing scheme yields an equilibrium. Owing to the linearity of utility in the amount consumed of the numeraire commodity in each state, the demand for the numeraire commodities is not uniquely determined in equilibrium. There is a trade-off between more consumption of the numeraire commodity in state 1 and an amount of consumption of the numeraire commodity in both states 2 and 3. The utility level of individuals is the same for all competitive equilibria.

Fix-price equilibrium exist for all prices of commodities, p, and prices of assets $q = \frac{1}{\pi}$. We assume i to be the individual such that

$$\frac{\gamma_s^i}{e_{1,s}^i} \le \frac{\gamma_s^{i'}}{e_{1,s}^{i'}},$$

and consider four different cases: (i) $0 < p_s \le \frac{\gamma_s^i}{e_{1,s}^i}$ (ii) $\frac{\gamma_s^i}{e_{1,s}^i} \le p_s \le \frac{\gamma_s^i+\gamma_s^{i'}}{e_{1,s}^i+e_{1,s}^{i'}}$ (iii) $\frac{\gamma_s^i+\gamma_s^{i'}}{e_{1,s}^i+e_{1s}^{i'}} \le p_s \le \frac{\gamma_s^{i'}}{e_{1,s}^{i'}}$ (iv) $\frac{\gamma_s^{i'}}{e_{1,s}^{i'}} \le p_s$.

In fact, α_s^i, β_s^i, and π_s, are chosen to coincide with π. There is, then a full dimensional set of parameters $e_{1,s}^i$ ans $e_{2,s}^i$ that satisfy the parameter restrictions: it suffices that $e_{2,s}^i$ be sufficiently large for every individual, at every state of the world.

(i) If $0 < p_s \leq \gamma_s^i / e_{1,s}^i$, both individuals have an excess demand for commodity 1. Equilibria obtain for $\overline{z}_s^* = 0$, $x_{1,s}^{i*} = e_{1,s}^i$, $x_{1,s}^{i'*} = e_{1,s}^{i'}$, and $y^{i'*} = -y^{i*}$. At $s = 1$, $x_{2,1}^i = e_{2,1}^i - (1/\pi) y^{i*}$, $x_{2,1}^{i'*} = e_{2,1}^{i'} + (1/\pi) y^{i*}$, $y^{i*} \leq \pi e_{2,1}^i$, and $y^{i*} \geq -\pi e_{2,1}^{i'}$. At $s = 2$ or $s = 3$, $x_{2,s}^{i*} = e_{2,s}^i + y^{i*}$, $x_{2,s}^{i'*} = e_{2,s}^{i'} - y^{i*}$, $y^{i*} \geq -e_{2,s}^i$, and $y^{i*} \leq e_{2,s}^{i'}$. The remaining parameters of the rationing scheme are set so as not to be binding.

(ii) If $\gamma_s^i / e_{1,s}^i \leq p_s \leq (\gamma_s^i + \gamma_s^{i'}) / (e_{1,s}^i + e_{1,s}^{i'})$, there is aggregate excess demand for commodity 1, but individual i supplies the commodity, and trade takes place, with individual i' rationed on his demand of the commodity. Equilibria obtain for $\overline{z}_{1,s}^* = e_{1,s}^i - \gamma_s^i / p_s$, $x_{1,s}^{i*} = \gamma_s^i / p_s$, $x_{1,s}^{i'*} = e_{1,s}^{i'} + e_{1,s}^i - \gamma_s^i / p_s$, and $y^{i'*} = -y^{i*}$. At $s = 1$, $x_{2,1}^{i*} = p_1 e_{1,1}^i + e_{2,1}^i - \gamma_1^i - (1/\pi) y^{i*}$, $x_{2,1}^{i'*} = e_{2,1}^{i'} - p_1 e_{1,1}^i + \gamma_1^i + (1/\pi) y^{i*}$, $y^{i*} \leq \pi (p_1 e_{1,1}^i + e_{2,1}^i - \gamma_1^i)$, and $y^{i*} \geq -\pi (e_{2,1}^{i'} - p_1 e_{1,1}^i + \gamma_1^i)$. At $s = 2$ or $s = 3$, $x_{2,s}^{i*} = p_s e_{1,s}^i + e_{2,s}^i - \gamma_s^i + y^{i*}$, $x_{2,s}^{i'*} = e_{2,s}^{i'} - p_s e_{1,s}^i + \gamma_s^i - y^{i*}$, $y^{i*} \geq -p_s e_{1,s}^i - e_{2,s}^i + \gamma_s^i$, and $y^{i*} \leq e_{2,s}^{i'} - p_s e_{1,s}^i + \gamma_s^i$. The remaining parameters of the rationing scheme are set so as not to be binding.

(iii) If $(\gamma_s^i + \gamma_s^{i'}) / (e_{1,s}^i + e_{1,s}^{i'}) \leq p_s \leq \gamma_s^{i'} / e_{1,s}^{i'}$, there is aggregate excess supply of commodity 1, and individual i supplies the commodity, rationed by the demand of individual i'. Equilibria obtain for $\underline{z}_{1,s}^* = e_{1,s}^{i'} - \gamma_s^{i'} / p_s$, $x_{1,s}^{i*} = e_{1,s}^i + e_{1,s}^{i'} - \gamma_s^{i'} / p_s$, $x_{1,s}^{i'*} = \gamma_s^{i'} / p_s$, and $y^{i'*} = -y^{i*}$. At $s = 1$, $x_{2,1}^{i*} = e_{2,1}^i - p_1 e_{1,1}^{i'} + \gamma_1^{i'} - (1/\pi) y^{i*}$, $x_{2,1}^{i'*} = p_1 e_{1,1}^{i'} + e_{2,1}^{i'} - \gamma_1^{i'} + (1/\pi) y^{i*}$, $y^{i*} \leq \pi (e_{2,1}^i - p_1 e_{1,1}^{i'} + \gamma_1^{i'})$, and $y^{i*} \geq -\pi (p_1 e_{1,1}^{i'} + e_{2,1}^{i'} - \gamma_1^{i'})$. At $s = 2$ or $s = 3$, $x_{2,s}^{i*} = e_{2,s}^i - p_s e_{1,s}^{i'} + \gamma_s^{i'} + y^{i*}$, $x_{2,s}^{i'*} = p_s e_{1,s}^{i'} + e_{2,s}^{i'} - \gamma_s^{i'} - y^{i*}$, $y^{i*} \geq -e_{2,s}^i + p_s e_{1,s}^{i'} - \gamma_s^{i'}$, and $y^{i*} \leq p_s e_{1,s}^{i'} + e_{2,s}^{i'} - \gamma_s^{i'}$. The remaining parameters of the rationing scheme are set so as not to be binding.

(iv) If $\gamma_s^{i'} / e_{1,s}^{i'} \leq p_s$, both individuals supply commodity 1, are fully rationed on their supply of the commodity and no trade takes place. Fix-price equilibria obtain for $\underline{z}_{1,s}^* = 0$, $x_{1,s}^{i*} = e_{1,s}^i$, $x_{1,s}^{i'*} = e_{1,s}^{i'}$, and $y^{i'*} = -y^{i*}$. At $s = 1$, $x_{2,1}^{i*} = e_{2,1}^i - (1/\pi) y^{i*}$, $x_{2,1}^{i'*} = e_{2,1}^{i'} + (1/\pi) y^{i*}$, $y^{i*} \leq \pi e_{2,1}^i$, and $y^{i*} \geq -\pi e_{2,1}^{i'}$. At $s = 2$ or $s = 3$, $x_{2,s}^{i*} = e_{2,s}^i + y^{i*}$, $x_{2,s}^{i'*} = e_{2,s}^{i'} - y^{i*}$, $y^{i*} \geq -e_{2,s}^i$, and $y^{i*} \leq e_{2,s}^{i'}$. The remaining parameters of the rationing scheme are set so as not to be binding.

The utility attained by each individual at a fix-price competitive equilibrium is unambiguously determined by the prices of commodities; if they coincide with the competitive equilibrium prices, the equilibrium allocations of commodities and the utility attained by each individual coincide as well. At competitive equilibrium prices of commodities, the utility of individual i is

$$v^i(p^*) = \sum_{s \in \mathcal{S}} \pi_s \left(\alpha_s^i \ln \left(\frac{\gamma_s^i}{\gamma_s^1 + \gamma_s^2} (e_{1,s}^1 + e_{1,s}^2) \right) + \beta_s^i \left(e_{2,s}^i + \frac{\gamma_s^{i'} e_{1,s}^i - \gamma_s^i e_{1,s}^{i'}}{e_{1,s}^1 + e_{1,s}^2} \right) \right).$$

At prices of commodities p, the utility of individual i at the fix-price competitive equilibrium is

$$v^i(p) = \sum_{s \in \mathcal{S}} \pi_s v_s^i(p_s),$$

where

(i) if $0 < p_s \leq \frac{\gamma_s^i}{e_{1,s}^i}$,

$$v_s^i(p_s) = \alpha_s^i \ln e_{1,s}^i + \beta_s^i e_{2,s}^i,$$

$$v_s^{i'}(p_s) = \alpha_s^{i'} \ln e_{1,s}^{i'} + \beta_s^{i'} e_{2,s}^{i'};$$

(ii) if $\frac{\gamma_s^i}{e_{1,s}^i} \leq p_s \leq \frac{\gamma_s^i + \gamma_s^{i'}}{e_{1,s}^i + e_{1,s}^{i'}}$,

$$v_s^i(p_s) = \alpha_s^i \ln(\tfrac{\gamma_s^i}{p_s}) + \beta_s^i \left(p_s e_{1,s}^i + e_{2,s}^i - \gamma_s^i\right),$$

$$v_s^{i'}(p_s) = \alpha_s^{i'} \ln\left(e_{1,s}^{i'} + e_{1,s}^i - \tfrac{\gamma_s^i}{p_s}\right) + \beta_s^{i'} \left(e_{2,s}^{i'} - p_s e_{1,s}^i + \gamma_s^i\right);$$

(iii) if $\frac{\gamma_s^i + \gamma_s^{i'}}{e_{1,s}^i + e_{1,s}^{i'}} \leq p_s \leq \frac{\gamma_s^{i'}}{e_{1,s}^{i'}}$,

$$v_s^i(p_s) = \alpha_s^i \ln\left(e_{1,s}^i + e_{1,s}^{i'} - \tfrac{\gamma_s^{i'}}{p_s}\right) + \beta_s^i \left(e_{2,s}^i - p_s e_{1,s}^{i'} + \gamma_s^{i'}\right),$$

$$v_s^{i'}(p_s) = \alpha_s^{i'} \ln\left(\tfrac{\gamma_s^{i'}}{p_s}\right) + \beta_s^{i'} \left(p_s e_{1,s}^{i'} + e_{2,s}^{i'} - \gamma_s^{i'}\right);$$

(iv) if $\frac{\gamma_s^{i'}}{e_{1,s}^{i'}} \leq p_s$,

$$v_s^i(p_s) = \alpha_s^i \ln e_{1,s}^i + \beta_s^i e_{2,s}^i,$$

$$v_s^i(p_s) = \alpha_s^{i'} \ln e_{1,s}^{i'} + \beta_s^{i'} e_{2,s}^{i'}.$$

Substitution of the competitive equilibrium prices in either case (ii) or case (iii) yields the utility levels at the competitive equilibrium. Moreover, the indirect utility function is differentiable at competitive prices which confirms Lemma 4; if p_s is as in case (ii), then

$$\partial_{p_s} v^i(p) = \pi_s \beta_s^i \left(\tfrac{-\gamma_s^i}{p_s} + e_{1,s}^i\right),$$

$$\partial_{p_s} v^{i'}(p) = \pi_s \beta_s^{i'} \left(\tfrac{\gamma_s^{i'} \gamma_s^i}{(p_s)^2 (e_{1,s}^{i'} + e_{1,s}^i) - p_s \gamma_s^i} - e_{1,s}^i\right),$$

and, if p_s is as in case (iii), then

$$\partial_{p_s} v^i(p) = \pi_s \beta_s^i \left(\tfrac{\gamma_s^i \gamma_s^{i'}}{(p_s)^2 (e_{1,s}^i + e_{1,s}^{i'}) - p_s \gamma_s^{i'}} - e_{1,s}^{i'}\right),$$

$$\partial_{p_s} v^{i'}(p) = \pi_s \beta_s^{i'} \left(\tfrac{-\gamma_s^{i'}}{p_s} + e_{1,s}^{i'}\right).$$

Indeed, substitution of the competitive price system in either (ii) or (iii) yields the same derivative for both households,

$$\partial_{p_s} v^i(p^*) = \pi_s \beta_s^i \frac{\gamma_s^{i'} e_{1,s}^i - \gamma_s^i e_{1,s}^{i'}}{\gamma_s^i + \gamma_s^{i'}} = -\pi_s \beta_s^i (x_s^{i*} - e_s^i),$$

$$\partial_{p_s} v^{i'}(p^*) = \pi_s \beta_s^{i'} \frac{\gamma_s^i e_{1,s}^{i'} - \gamma_s^{i'} e_{1,s}^i}{\gamma_s^{i'} + \gamma_s^i} = -\pi_s \beta_s^{i'} (x_s^{i'*} - e_s^{i'}).$$

If

$$v_s = \pi_s \frac{\gamma_s^2 e_{1,s}^1 - \gamma_s^1 e_{1,s}^2}{\gamma_s^1 + \gamma_s^2},$$

then

$$V = \begin{pmatrix} \partial v^1(p^*) \\ \\ \partial v^2(p^*) \end{pmatrix} = \begin{pmatrix} \beta_1^1 v_1 & \beta_2^1 v_2 & \beta_3^1 v_3 \\ \\ -\beta_1^2 v_1 & -\beta_2^2 v_2 & -\beta_3^2 v_3 \end{pmatrix}.$$

If the matrix V has full row rank, price regulation can Pareto improve the competitive equilibrium allocation.

If the ratios of the marginal utility of income are not the same across all states of the world,

$$\frac{\beta_1^1}{\beta_1^2} \neq \frac{\beta_2^1}{\beta_2^2} \quad \text{or} \quad \frac{\beta_3^1}{\beta_3^2} \neq \frac{\beta_2^1}{\beta_2^2},$$

for the matrix V to have full row rank it is sufficient that $v_s \neq 0$, for every state of the world. Since $v_s = 0$ if and only if $e_{1,s}^1/e_{1,s}^2 = \gamma_s^1/\gamma_s^2$, generically in the endowments of individuals it is possible to Pareto improve on the competitive allocation.[29] This is also the essence of Proposition 6; only here, because of linear utility in the numeraire commodity, variations in endowments do not affect the marginal utilities of income at equilibrium and an ad hoc argument is required.

Since $L < I$, it is not always possible to Pareto improve on the competitive equilibrium by a uniform price regulation. A Pareto improvement by a uniform price regulation may fail if $\beta_1^1 v_1 + \beta_2^1 v_2 + \beta_3^1 v_3$ and $-\beta_1^2 v_1 - \beta_2^2 v_2 - \beta_3^2 v_3$ have opposite

[29] For the specification of parameters given in footnote 12,

$$V = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & -\frac{1}{3} \end{pmatrix}.$$

Individual 1 demands commodity 1 in state 2 and therefore benefits from a decrease in the price of commodity 1 in state 2, and supplies commodity 1 in state 3 and therefore benefits from an increase in the price of commodity 1 in state 3. For individual 2 the utility effects are reversed. For individual 1 the marginal utility of income is higher in state 2, the state in which he has low endowment, than in state 3, where he has high endowments, and vice versa for individual 2. A decrease of the price of commodity 1 in state 2 leads to an increase of utility for individual 1 which exceeds the loss in utility of individual 2, and a decrease of the price of commodity 1 in state 3 leads to an increase of utility for individual 2 which exceeds the loss in utility of individual 1. Both individuals benefit if the price of commodity 1 in states 2 and 3 is fixed below its competitive equilibrium value. A Pareto improvement can even be achieved by a uniform price regulation, although this is not necessarily the case if $L < I$.

signs. This is by no means excluded. That uniform price regulation may fail in this example to achieve Pareto improvements is not surprising, since in general, it is not possible to attain I goals by only $L < I$ instruments.

7 Conclusion

Given any prices for commodities and assets, a competitive allocation of resources exists under weak assumptions, but does in general involve endogenously determined amounts of rationing. Under such circumstances it is possible for individuals to hold arbitrage portfolios in equilibrium, which is rather counterintuitive since markets are transparent and constraints on trade are endogenously determined.

Local comparative statics are complicated at competitive equilibrium prices. Arbitrarily small deviations from competitive prices may lead to discontinuous jumps in allocations and utilities. Necessary and sufficient conditions for local existence of fix-price equilibria in the neighborhood of competitive equilibria are derived. Provided those conditions hold, price regulation offers opportunities for efficiency gains when asset markets are incomplete and risk sharing is restricted. This conclusion does not change when uniform price regulation is considered only.

A serious concern are the informational requirements needed to determine, even compute, improving interventions. In the case of price regulation they involve knowledge of marginal utilities of income and excess demands for commodities across states. The characterization in Geanakoplos and Polemarchakis (1990) and in Kübler and Polemarchakis (1999) are only first steps towards an analysis of the informational requirements of active policy.

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