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ASSET MARKETS AND INVESTMENT DECISIONS

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Asset markets and investment decisions [1]

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Abstract

In an incomplete asset market, firms assign values to investment plans by projecting their payoffs on the span of the payoffs of marketed assets; equivalently, firms employ the Capital Asset Pricing Model. This is a criterion that does not require firms to possess information, such as the marginal valuation of revenue across date - events by shareholders, which is not observable; rather, it is based on information revealed by the prices and payoffs of marketed assets. Under standard assumptions, competitive equilibria exist. But, competitive equilibrium allocations need not satisfy a condition of constrained pareto optimality that recognizes the incompleteness of the asset market; and, even in the absence of nominal assets, competitive equilibrium allocations are generically indeterminate — they are determinate if firms consider the commodity payoffs of shares.

Key words: assets, profit, investment.

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1 Introduction

In a competitive, market economy, firms act on behalf of individual shareholders.

When all commodities are priced and exchanged in a universal, competitive market, where individuals optimize subject to one, overall budget constraint, the decision criterion of firms is unambiguous: firms maximize profit. The profit associated with an investment plan [1] is well defined, while individual shareholders, their heterogeneity in preferences notwithstanding, unanimously favor profit maximization. Importantly, the argument encompasses economies over time and under uncertainty, so long as there is a complete market in elementary securities — Arrow (1953) — or in contingent commodities — Debreu (1960); the prices of elementary securities, implicit or explicit, serve to define the profit associated with investment plans across date - events, the heterogeneity of individual shareholders now extended to allow for heterogeneity in time preferences and beliefs.

When the economy is subject to uncertainty, and the asset market is incomplete, individuals optimize under an irreducible multiplicity of budget constraints; equivalently, they optimize under two constraints: the overall budget constraint and the constraint that their expenditure across date - events be the payoff of a portfolio of marketed assets, the shares of firms among them — Radner (1972). As a consequence, marginal rates of substitution for revenue at different date - events typically differ across individuals; which poses a problem for firms in their attempt to value and choose among investment plans: the profit or market value of an investment plan not spanned by marketed assets is not defined.

Distinct, though related approaches to the problem have been put forward. One restricts attention to technologies of firms such that any investment plan is indeed spanned by portfolios of marketed assets — Diamond (1967), Ekern and Wilson (1974), Radner (1974); but this is excessively restrictive on the technologies of firms. Another supposes that firms value payoffs based on an average of the marginal valuations of individual shareholders — Drèze (1974), Grossman and Hart (1977); but this is excessively demanding on the information available to and used by firms, which is not revealed by the prices and payoffs of marketed assets. The attribution of objective functions to firms not linked to the preferences of shareholders — Drèze (1982), Leland (1974) — is arbitrary. More generally, firms employ any valuation of payoffs across date - events not incompatible with the prices of marketed assets and, hence, with the marginal valuations of individual shareholders — Duffie and Shafer (1987); this employs only information revealed by the prices of marketed assets, but it leads, typically, to a non - trivial continuum of distinct equilibrium allocations.

Here, firms compute the value of investment plans by approximating their

[1] The use of the term "investment" plan, as opposed to the more standard "production" plan serves to indicate that the problem addressed is of interest, mostly, in an intertemporal setting.

payoffs with the payoffs of portfolios of marketed assets; equivalently, firms employ the Capital Asset Pricing Model. This is a criterion that, by construction, does not require firms to possess information not revealed by the prices of marketed assets.

Under standard assumptions, competitive equilibria exist. But, competitive equilibrium allocations need not satisfy a condition of constrained pareto optimality that recognizes the incompleteness of the asset market; and, even in the absence of nominal assets, competitive equilibrium allocations are generically indeterminate — they are determinate if firms consider the commodity payoffs of shares.

The existence of competitive equilibria follows by a straightforward extension of the argument for economies with a complete asset market — Arrow and Debreu (1954), McKenzie (1984). With an incomplete asset market, variations in relative prices in commodity spot markets and the resulting discontinuities in the correspondence of attainable reallocations of revenue can cause competitive equilibria not to exist — Hart (1975), while the generic existence of competitive equilibria requires a different argument even for exchange economies — Duffie and Shafer (1985, 1986); the argument extends to economies with production — Duffie and Shafer (1987). Under the simplifying assumptions of one date and one, aggregate commodity, such complications are absent, as in the case of an exchange economy with asset payoffs denominated in a numeraire commodity — Geanakoplos and Polemarchakis (1986).

Constrained pareto optimality allows only transfers of revenue across date - events attainable through the available assets. This is the extension of the notion of pareto optimality satisfied by competitive allocations in economies with a complete asset market — Arrow (1951), and Debreu (1951) — that recognizes the incompleteness of the asset market. With multiple commodities, variations in relative spot commodity prices, generically, make for competitive equilibrium allocations that are not constrained pareto optimal; this is the case for the allocation of assets in exchange economies — Geanakoplos and Polemarchakis (1986); the argument extends to the choice of investment plans in economies with production — Drèze, Geanakoplos,Magill and Quinzii (1990), even if firms adopt the criterion of Drèze (1974): they adopt implcit prices for revenue across date - events that aggregate the marginal valuations of shareholders, with weights proportional to non - negative, equilibrium portfolio shares. In the absence of variations in relative spot prices for commodities, in particular with one date and one, aggregate commodity, competitive equilibrium allocations are constrained pareto optimal; this is the case for the allocation of assets in exchange economies; the argument extends to the choice of investment plans in economies with production if firms adopt the criterion of Drèze (1974), if the criterion of optimality is weakened further by not allowing for the reallocation of portfolios of shareholders along with the change in the production plans of firms.

When firms consider the projection of the payoff of shares on the span of the payoffs of marketed assets, they do not adopt implcit prices for revenue across date - events that aggregate the marginal valuations of shareholders; constrained pareto optimality may, thus, fail, even for economies with one, representative individual.

Competitive equilibrium allocations are determinate when they are, locally, functions of the exogenous, structural characteristics of the economy; determinacy allows for the prediction of equilibrium allocations as the parameters of the economy vary, and it implies that effective policy interventions modify structural characteristics, such as the distribution of income. For competitive economies with a complete asset market, determinacy holds generically — Debreu (1970). Competitive equilibrium allocations are indeterminate when the set of distinct equilibrium allocations contains an open set of non - zero dimension. The possible generic indeterminacy of equilibrium allocations is a hallmark of economies with an incomplete asset market; it obtains, most clearly, for economies with nominal assets, whose payoffs are denominated in revenue or abstract units of account — Balasko and Cass (1989), Cass (1985), Geanakoplos and Mas - Colell (1989), while, with real assets, whose payoffs are denominated in commodities, competitive equilibrium allocations are typically determinate — Geanakoplos and Polemarchakis (1986); alternatively, it obtains for economies with real assets but a set of states of the world of infinite cardinality — Mas - Colell (1991) — and for economies with real assets in which, alongside the incompleteness of the asset market, individuals face constraints on net trades that reflect the information available when net trades are decided — Polemarchakis and Siconolfi (1994).

The indeterminacy of equilibrium allocations when firms consider the projection of the payoff of shares on the span of the payoffs of marketed assets is a further instance of indeterminacy in an economy with an incomplete asset market that does not rely on nominal assets. Perhaps surprizingly, the set of equilibrium allocations coincides with the set of equilibrium allocations according to the definition of Duffie and Shafer (1987): variations in the price level yield the same set of equilibrium allocations as the adoption by firms of alternative implicit prices of revenue across date - events.

When equilibrium allocations are indeterminate, policy may be effective by selecting among equilibrium allocations; and desirable, when competitive equilibrium allocations fail to be constrined pareto optima. When firms compute the value of production plans by approximating the value of payoffs with the value of payoffs of portfolios of marketed assets, monetary policy, which determins the price level at alternative realizations of uncertainty, is effective and desirable.

2 The economy

State of the world are $s \in \mathcal{S} = \{1, \ldots, S\}$.

Investment decisions are made and assets are exchanged prior to the resolution of uncertainty. After the resolution of uncertainty, production occurs, assets pay off, and consumption occurs. There is one commodity at each state of the world.

A quantity of the commodity at a state of the world is x_s; across states of the world, a bundle of commodities is $x = (\ldots, x_s, \ldots)'$[2]. The price of the commodity at a state of the world is p_s; across states of the world, prices of commodities are $p = (\ldots, p_s, \ldots)$. With one commodity at each spot market, no exchange occurs; the price of the commodity corresponds to the overall price level in a multi - commodity world. The value, across states of the world, associated with the bundle x at prices of commodities p is $x(p) = Px$[3].

Agents in the economy are individuals, consumer - investors, $i \in \mathcal{I} = \{1, \ldots, I\}$, and firms, $j \in \mathcal{J} = \{1, \ldots, J\}$.

A firm is described by $(\mathcal{Y}^j, f^j)$: a production set, a set of investment plans, bundles of commodities; and an endowment, a bundle of commodities. The investment plan $y^j \in \mathcal{Y}^j$ is the commodity payoff of the shares of the firm.

Across firms, the allocation of endowments is $f^{\mathcal{J}} = (\ldots, f^j, \ldots)$, and the aggregate endowment of commodities of firms is $f^a = \sum_{j \in \mathcal{J}} f^j$. An allocation of investment plans is $y^{\mathcal{J}} = (\ldots, y^j, \ldots)'$, and the associated matrix, $Y = (\ldots, y^j, \ldots)$, is the matrix of commodity payoffs of shares of firms. The aggregate investment plan is $y^a = \sum_{j \in \mathcal{J}} y^j$.

The payoff of the shares of a firm is $y^j(p) = Py^j$, and the matrix of payoffs of shares is $Y(p) = PY$.

A portfolio of shares of firms is $z = (\ldots, z_j, \ldots)'$. The price of shares of a firm in the market for shares prior to the resolution of uncertainty is q_j; across firms, prices of shares are $q = (\ldots, q_j, \ldots)$. The value of the portfolio z at prices of shares of firms q is qz.

Shares of firms are the only marketed assets, available to investors for the transfer of revenue across states of the world. A transfer of revenue is attainable if and only if it is the payoff of a portfolio of shares. At prices of commodities p, if the allocation of investment plans is $y^{\mathcal{J}}$ and the matrix of payoffs of shares is $Y(p)$, the subspace of attainable transfers of revenue is $[Y(p)]$[4].

Prices are a pair, (p, q), of prices of commodities and prices of shares.

An individual is described by $(\mathcal{X}^i, \mathcal{R}^i, e^i, d^i)$: a consumption set, a set of consumption plans, bundles of commodities; a preference relation over consumption plans; an endowment of commodities, a bundle of commodities; and an endowment of shares, a portfolio of shares. Associated with the preference relation, there is $\mathcal{P}^i$, the strong preference relation, and $\mathcal{I}^i$, the indifference relation.

Across individuals, the allocation of endowments of commodities is $e^{\mathcal{I}} = (\ldots, e^i, \ldots)$, and the aggregate endowment of commodities of individuals is $e^a =$

[2] " $'$ " denotes the transpose

[3] For a vector $b = (\ldots, b_s, \ldots)$, $B = \text{diag}(b)$, where " diag(b) " denotes the diagonal matrix with diagonal elements $\text{diag}(b)_{s,s} = b_s$.

[4] " [ ] " denotes the span of the columns of a matrix or of a collection of vectors

$\sum_{i \in \mathcal{I}} e^i$. An allocation of portfolios of shares is $z^{\mathcal{I}} = (\ldots, z^i, \ldots)'$. The aggregate portfolio of shares is $z^a = \sum_{i \in \mathcal{I}} z^i$. The aggregate endowment of portfolios of shares is $\sum_{i \in \mathcal{I}} d^i = 1'_J$[5].

An allocation is $(z^{\mathcal{I}}, y^{\mathcal{J}})$, a pair of an allocation of portfolios of shares and an allocation of investment plans [6].

An allocation is feasible if and only if, for every individual, $x^i = e^i + Yz^i \in \mathcal{X}^i$, while $z^a = 1_J$. Associated with a feasible allocation, there is an allocation of consumption plans, $x^{\mathcal{I}}$.

At prices of shares q the budget constraint of an individual is

$$qz \leq qd^i;$$

at an allocation of investment plans $y^{\mathcal{J}}$, the consumption plan associated with a portfolio of shares is $x^i = e^i + Yz$; the demand for shares by the individual is $z^i(q, Y)$, and the demand correspondence is z^i.

With one commodity in each state of the world, prices of commodities do not enter the optimization problems of individuals.

At prices (p, q), if the allocation of investment plans is $y^{\mathcal{J}}$, and the matrix of payoffs of shares is $Y(p)$, the value of the projection of the payoff of a investment plan, y, on the span of the matrix of payoffs of shares is

$$v(y, p, q, Y) = q\alpha(y, p, Y),$$

where $\alpha(y, p, Y) = \arg\min\{\|(y(p) - Y(p)\alpha\|^7 : \alpha = (\ldots, \alpha_j, \ldots)'\}$ is the portfolio of shares whose payoff best approximates the payoff $y(p)$; equivalently, $Y(p)\alpha(y, p, Y)$ is the projection of the payoff on the subspace, $[Y(p)]$, of attainable transfers of revenue

A firm selects investment plans by maximizing the value of the projection of the payoff on the span of the matrix of payoffs of shares; the investment plan or the supply of the firm is

$$y^j(p, q, Y) = \arg\max\{v(y, p, q, Y) : y \in \mathcal{Y}^j\};$$

the supply correspondence is y^j, and the value of the firm is $\bar{v}^j(p, q, Y)$.

If the matrix $Y(p)$ has full column rank,

$$\alpha(y, p, Y) = (Y(p)'Y(p))^{-1}Y(p)'y(p),$$

and

$$v(y, p, q, Y) = q(Y(p)'Y(p))^{-1}Y(p)'y(p).$$

[5] " 1_K " denotes the vector of 1's of dimension K.

[6] Associated with a production set, there is a net production set, $\hat{\mathcal{Y}}^j = \mathcal{Y}^j - \{f^j\}$; with an investment plan, a net investment plan, $\hat{y}^j = y^j - f^j$; with a consumption plan, an excess consumption plan, $\hat{x}^i = x^i - e^i$; with a portfolio of shares, an excess portfolio of shares, $\hat{z}^i = z^i - d^i$; and similarly for payoffs of shares, aggregates and allocations.

[7] "$\| \|$" denotes the eucleidean norm

When a firm maximizes the value of the projection of the payoff on the span of the matrix of payoffs of shares, it maximizes the value of the payoff at implicit prices of revenue $\pi^f(y, p, Y) = q(Y(p)'Y(p))^{-1}Y(p)'$: prices and the allocation of production plans determine implicit prices of revenue for firms.

If a firm selects investment plans by maximizing the value of the projection of the commodity payoff on the span of the commodity payoffs of marketed shares, the supply correspondence is defined by $y^j(q, Y) = \arg\max\{v(y, q, Y) : y \in \mathcal{Y}^j\}$, where $v(y, q, Y) = q\alpha(y, Y)$, and $\alpha(y, Y) = \arg\min\{\|(y - Y\alpha)\| : \alpha = (\ldots, \alpha_j, \ldots)'\}$. With one commodity in each state of the world, commodity prices do not enter the optimization problems of firms either; equivalently, prices of commodities prices are $p = \mathbf{1}_S$.

Projection valuation and the CAPM

When a firm values the projection of the payoff of a production plan on the span of the matrix of payoffs of shares, it applies the Capital Asset Pricing Model (CAPM).

Implicit prices of revenue across states of the world are $\pi = (\ldots, \pi_s, \ldots)$.

Prices of shares, q, do not allow for arbitrage if and only if

$$Y(p)z > 0 \Rightarrow qz > 0^8.$$

If the matrix of payoffs of shares has full column rank, prices of shares do not allow for arbitrage if and only if there exist implicit prices of revenue, such that

$$q = \pi Y(p), \quad \pi \gg 0.$$

Alternatively, prices of shares, q, do not allow for arbitrage weakly if and only if

$$Y(p)z \gg 0 \Rightarrow qz > 0.$$

If the matrix of payoffs of shares has full column rank, and if there exists a portfolio of shares, $\bar{z}$, with strictly positive payoffs: $Y(p)\bar{z} \gg 0$, prices of shares do not allow for arbitrage weakly if and only if there exist implicit prices of revenue, such that

$$q = \pi Y(p), \quad \pi > 0.$$

This elucidates the link between an economy with a complete market in elementary securities and an economy with a general asset market, in particular a market for shares — Ross (1978).

The set of implicit prices of revenue compatible with prices of shares is $\Pi(p, q, Y) = \{\pi : q = \pi Y(p)\}$. When the market for shares is complete, the matrix of payoffs of shares, $Y(p)$, has full row rank, all transfers of revenue across states of the world are attainable, and the set $\Pi(p, q, Y) = \{\pi : q = \pi Y(p)\}$ is a

[8] Vector inequalities are "$\gg$," "$>$," and "$\geq$."

singleton; when the market is incomplete, $\dim[Y(p)] < S$, and the set $\Pi(p,q,Y)$ is an affine subspace of dimension $(S - \dim[Y(p)]) > 0$.

Under standard assumptions on the characteristics of individuals, there exist, for every individual, implicit prices of revenue, $\pi^i \in \Pi(p,q,Y)$, $\pi^i > 0$, such that the solution to individual optimization under the constraints $x = Yz + e^i, qz \leq qd^i$ and the solution under the constraint $p^i x \leq p^i(e^i + Yd^i)$ coincide, where $p^i = \pi^i P$; if the preferences of the individual are strictly monotonic, $\pi^i \gg 0$. Importantly, the implicit prices of revenue, π^i, such that the individual effectively optimizes under a single, overall budget constrain at prices $\pi^i P$, differ across individuals; the market for shares is effectively complete for $\hat{\pi}$ if every individual effectively optimizes under a single, overall budget constrain at prices $\hat{\pi} P$.

Shares of an investment plan y, with revenue payoff $y(p)$, are priced by arbitrage at $\tilde{q}$ if and only if, for every individual, if z^i is a solution to the individual optimization problem, $(z^i, 0)$ is a solution to the individual optimization problem over the modified budget set $\{qz + \tilde{q}\tilde{z} \leq qd^i\}$. This is of interest for investment plans whose shares are not marketed: $y \notin \{y^j : j \in \mathcal{J}\}$.

Shares of an investment plan. y, are redundant if and only if their payoff is spanned by the payoffs of marketed shares: $y(p) \in [Y(p)]$; equivalently, $y(p)$ is an attainable transfer of revenue.

Shares of a investment plan, y, that are redundant are priced by arbitrage at $\tilde{q} = \pi y(p), \pi \in \Pi(p,q,Y)$, which is well defined — it is independent of the choice of $\pi \in \Pi(p,q,Y)$.

When the market for shares is complete, shares of any production plan are redundant and priced by arbitrage.

If the market for shares is effectively complete for $\hat{\pi}$, shares of any investment plan can be priced at $\tilde{q} = \hat{\pi} y(p)$.

The rate of return of shares of an investment plan with price $\tilde{q} \neq 0$ and payoff $y(p) = (\ldots, p_s y_s, \ldots)$ is $\rho = (\ldots, \rho_s, \ldots)$, where $\rho_s = (p_s y_s / \tilde{q}) - 1$.

Implicit prices of revenue that are positive: $\pi > 0$, and sum up to 1: $\sum_{s \in S} \pi_s = 1$, can be interpreted as a probability measure on the set of states of the world.

With probability measure on the set of states of the world $\pi \in \Pi(p,q,Y)$, the expected rate of return of shares of any investment plan that are marketed vanishes: $E_\pi \tilde{\rho} = 0$; also, of any shares that are redundant. This is the martingale property satisfied by the price of any shares of investment plans that are marketed or redundant.

For π, a probability measure on the set of states of the world, there exist unique implicit prices of revenue. $\tilde{\pi} \in \Pi(p,q,Y)$, such that $\tilde{\pi}' \in [\Pi Y(p)]$; they are given by $\tilde{\pi} = q(Y(p)'\Pi Y(p))^{-1} Y(p)'\Pi$. Alternatively, for $\overline{\pi} \in \Pi(p,q,Y)$, implicit prices of revenue, there exists a probability measure on the set of states of the world, π, not necessarilly unique, such that $\bar{\pi}' \in [\Pi Y(p)]$; it is given, up to normalization, as the solution to the equation $\overline{\pi} = \Pi Y(p)\overline{z}$, where $\overline{z}$ is a portfolio of shares with strictly positive payoffs: $Y(p)\overline{z} \gg 0$, which is assumed to exist.

For $\pi \gg 0$, a strictly positive probability measure on the set of states of the world, the "market asset," m, is the asset with payoffs $y_m(p) = b\Pi^{-1}\bar{\pi}'$, for $b \neq 0$. Since $\bar{\pi}' \in [\Pi Y(p)], y_m(p) \in [Y(p)]$: shares of the market asset are redundant. The price of the market asset is $q_m = b\bar{\pi}\Pi^{-1}\bar{\pi}' \neq 0$, and its rate of return is $\rho_m = b\Pi^{-1}\bar{\pi}'(\text{diag}(1_S q_m))^{-1} - 1_S$.

For shares of investment plans that are marketed or redundant,

$$\mathrm{E}_\pi \rho = \frac{\mathrm{Cov}_\pi(\rho, \rho_m)}{\mathrm{Var}_\pi \rho_m} \mathrm{E}_\pi \rho_m{}^{9}.$$

This is the capital asset pricing model (CAPM) pricing equation

By direct substitution, $\bar{\pi}y(p) = \bar{\pi}y(p)^{\pi,Y(p)}$, where $y(p)^{\pi,Y(p)} = Y(p)(Y(p)'\Pi Y(p))^{-1}Y(p)'\Pi y(p)$, [10] is the $\pi-$ projection of payoffs $y(p)$ on $[Y(p)]$, the subspace of attainable transfers of revenue. The implicit prices of revenue $\bar{\pi}$ price approximately a non - redundant asset by pricing the associated redundant shares of the $\pi-$ projection of the payoffs of the asset on the subspace of attainable transfers of revenue. This is the generalized CAPM — Dutta and Polemarchakis (1991), Geanakoplos and Shubik (1990). The approximation is exact when the market for shares is effectively complete for $\bar{\pi}$.

If (i) the preferences of every individual have a quadratic von Neumann - Morgenstern representation $u^i(x) = E_\pi(x_s - \alpha^i x_s^2)$, $\alpha^i > 0$, with common probability measure, π, (ii) the endowment of commodities of each individual lies in the span of the matrix of revenue payoffs of shares: $e^i \in [Y(p)]$, (iii) there exists a "risk - free" firm, $f \in \mathcal{J}$, with investment plan $y^f = 1_S$, then, as long as the solution to the optimization problem of every individual satisfies $0 \leq x_s^i \leq \alpha^i/2$, the asset market is effectively complete for the probability measure on the set of states of the world common to the von Neumann - Morgenstern representations of the preferences of individuals, and the payoffs of the market asset coincide with the aggregate consumption. It suffices to observe that the gradient of the utility function of every individual lies in the span of the matrix of payoffs of assets: $Du^i(x^i) \in [\Pi Y]$. This is the classical CAPM — Lintner (1965), Sharp (1964) and Traynor (1961).

Definition 1 *A competitive equilibrium is $((p^*, q^*), (z^{\mathcal{I}*}, y^{\mathcal{J}*}))$, a pair of prices and a feasible allocation, such that, for every firm, $y^{j*} \in y^j(p^*, q^*, Y^*)$, and, for every individual, $z^{i*} \in z^i(q^*, Y^*)$.*

[9] "E_π" denoes the expectation with respect to the probability measure π,, and similarly for "Var_π" and "Cov_π."

[10] $y(p)^{\pi,Y(p)} = Y(p)\gamma$, where $\gamma = \arg\min\{(y(p) - Y(p)\gamma)'\Pi(y(p) - Y(p)\gamma) : \gamma = (\ldots, \gamma_s, \ldots)\}$.

At prices of commodities $\bar{p}$, a competitive equilibrium is $(q^*, (z^{\mathcal{I}*}, y^{\mathcal{J}*}))$, a pair of prices of shares and a feasible allocation, such that $((\bar{p}, q^*), (z^{\mathcal{I}*}, y^{\mathcal{J}*}))$ is a competitive equilibrium.

At a competitive equilibrium, the value of the objective function that a firm maximizes coincides with the price of its shares: $\tilde{v}^j(p^*, q^*, Y^*) = q_j^*$. Implicit prices of revenue for firms are $\pi^{j*} = q^*(Y^*(p^*)'Y^*(p^*))^{-1}Y^*(p^*)' \in \Pi(p^*, q*)$.

If firms consider the commodity payoffs of shares, a competitive equilibrium is a pair, $(q^*, (z^{\mathcal{I}*}, y^{\mathcal{J}*}))$, a competitive equilibrium at prices of commodities $\bar{p} = 1_S$.

Alternatively, as in Duffie and Shafer (1987), a firm selects production plans by maximizing the value of payoff of shares at some implicit prices of revenue $\pi \in \Pi(1_S, q)$ — without loss of generality, prices of commodities prices are $p = 1_S$; the supply correspondence is defined by $y^j(\pi) = \arg\max\{\pi y : y \in \mathcal{Y}^j\}$.

A competitive equilibrium according to Duffie and Shafer (1987) is $((\pi^*, q^*), (z^{\mathcal{I}*}, y^{\mathcal{J}*}))$, a pair of implicit prices of revenue and prices of shares and a feasible allocation, such that $\pi^* \in \Pi(1_S, q^*)$, for every firm, $y^{j*} \in y^j(\pi^*)$, and, for every individual, $z^{i*} \in z^i(q^*, Y^*)$.

An allocation $(z^{\mathcal{I}}, y^{\mathcal{J}})$, pareto dominates another, $(\tilde{z}^{\mathcal{I}}, \tilde{y}^{\mathcal{J}})$, if and only if $x^i \mathcal{R}^i \tilde{x}^i$, for every individual, with strict preference, $x^i \mathcal{P}^i \tilde{x}^i$, for some.

An allocation $(z^{\mathcal{I}}, y^{\mathcal{J}})$, pareto dominates another, $(\tilde{z}^{\mathcal{I}}, \tilde{y}^{\mathcal{J}})$, in production if and only if the allocations of shares coincide: $z^{\mathcal{I}} = \tilde{z}^{\mathcal{I}}$, and $x^i \mathcal{R}^i \tilde{x}^i$, for every individual, with strict preference, $x^i \mathcal{P}^i \tilde{x}^i$, for some.

An allocation $(z^{\mathcal{I}}, y^{\mathcal{J}})$, pareto dominates another, $(\tilde{z}^{\mathcal{I}}, \tilde{y}^{\mathcal{J}})$, in exchange if and only if the allocations of production plans coincide: $y^{\mathcal{J}} = \tilde{y}^{\mathcal{J}}$, and $x^i \mathcal{R}^i \tilde{x}^i$, for every individual, with strict preference, $x^i \mathcal{P}^i \tilde{x}^i$, for some.

Definition 2 *A feasible allocation is constrained pareto optimal if and only if no feasible allocation pareto dominates it.*

The criterion of constrained pareto optimality does not allow for allocations of commodities not attainable through the allocation of shares, as is appropriate.

Less demanding criteria are constrained pareto optimality in production: a feasible allocation is constrained pareto optimal in production if and only if no feasible allocation pareto dominates it in production; or in exchange: a feasible allocation is constrained pareto optimal in exchange if and only if no feasible allocation pareto dominates it in exchange; either is of questionable interest.

Definition 3 *Competitive equilibrium allocations are indeterminate of degree k if and only if the set of distinct competitive equilibrium allocations of commodities contains an open set of dimension k.*

Competitive equilibrium allocations are determinate if and only if the set of competitive equilibrium allocations of commodities is, locally, the range of a finite set of continuous functions.

An example

States of the world are $s = 1,2,3$. Bundles of commodities are $x = (x_1, x_2, x_3)$, and prices of commodities are $p = (1, p_2, p_3)$.

Firms are $j = 1,2$.

Firm 1 has endowment $f^1 = (1,0,0)$ and technology $\mathcal{Y}^1 = \{(1,0,0)\}$; firm 2 has endowment $f^2 = (0,1,0,)$ and technology $\mathcal{Y}^2 = \{(y_1, y_2, y_3) : y_1 = -k, y_2 = 1, y_3 = k, k \geq 0\}$.

The matrix of payoffs of shares is

$$Y(p) = \begin{pmatrix} 1 & -k \\ 0 & p_2 \\ 0 & p_3 k \end{pmatrix}.$$

The prices of shares are $q = (1, q_2)$. They do not allow for arbitrage if and only if $q_2 = (-1+\pi_3 p_3)k + \pi_2 p_2$, for implicit prices of revenue $\pi = (1, \pi_2, \pi_3) \gg 0$; in particular, if prices of shares do not allow for arbitrage,

$$q_2 + k > 0.$$

By direct computation,

$$(Y(p)'Y(p))^{-1}Y(p)' = (p_2^2 + p_3^2 k^2)^{-1} \begin{pmatrix} p_2^2 + p_3^2 k^2 & p_2 k & p_3 k^2 \\ 0 & p_2 & p_3 k \end{pmatrix}.$$

At (p, q) and allocation of investment plans (y^1, y^2), with $y^2 = (-\bar{k}, 1, \bar{k})$, implicit prices of revenue for firms are

$$\pi^j(p, q, Y) = (1, q_2)(Y(p)'Y(p))^{-1}Y(p)')$$

or

$$(p_2^2 + p_3^2 \bar{k}^2)^{-1}((p_2^2 + p_3^2 \bar{k}^2), p_2(q_2 + \bar{k}), p_3 \bar{k}(q_2 + \bar{k})).$$

If prices of shares do not allow for arbitrage, implicit prices of revenue for firms are positive: $\pi^j(p, q, Y) > 0$, and, for $k > 0$, strictly positive: $\pi^j(p, q, Y) \gg 0$; this follows from the structure of the matrix of payoffs of shares: there exists an invertible matrix,

$$T = \begin{pmatrix} 1 & \frac{k}{p_2} \\ 0 & \frac{1}{p_2} \end{pmatrix},$$

such that

$$Y(p)T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \frac{p_3 k}{p_2} \end{pmatrix},$$

is orthogonal and positive, and, for $k > 0$, $Y(p)T1_2 \gg 0$.

The objective function of firm 2 is to maximize

$$\pi^f(p, q, Y)(-k, p_2, p_3 k)$$

or, after simplification,

$$-p_2^2 k + q_2 p_3^2 \bar{k} k.$$

For $k > 0$, equilibrium in production requires that

$$\bar{k} = \frac{p_2^2}{q_2 p_3^2}.$$

The optimization problem of firm 1 is trivial.

There is one individual, with preferences represented by the utility function $u = \ln x_1 + \ln x_2 + \ln x_3$, endowment of commodities $e = (0, 0, 0)$ and endowment of shares $d = (1, 1)$.

The budget constraint of the individual is

$$z_1 + q_2 z_2 = 1 + q_2,$$

and his indirect utility function over shares of firm 2 is

$$\tilde{u} = \ln(1 + q_2(1 - z_2) - z_2 k) + \ln z_2 + \ln(z_2 k).$$

The individual's demand for shares of firm 2 is

$$z_2 = \frac{2(1 + q_2)}{3(q_2 + k)}.$$

Equilibrium in the market for shares requires that $z_2 = 1$ or

$$k = \frac{2 - q_2}{3}.$$

The family of equilibria is described by

$$0 < k^* < \frac{2}{3}.$$

Competitive equilibrium prices are

$$q_2^* = 2 - 3k^*,$$

and

$$\frac{p_2^*}{p_3^*} = \sqrt{k^*(2 - 3k^*)}.$$

The family of equilibria coincides with the family of equilibria following the alternative formulation, in Duffie and Shafer (1987), according to which a firm selects production plans by maximizing the value of payoff of shares at some implicit prices of revenue $\pi \in \Pi(1_S, q)$.

Commodity prices are normalized to $p = (1, 1, 1)$.

At prices of shares q and allocation of investment plans $y^{\mathcal{J}}$, implicit prices of revenue are $\pi = (\pi_1, \pi_2, \pi_3)$, such that $q = \pi Y$. Importantly, the incompleteness of the asset market prevents the prices of shares from determining, up to normalization, the implicit prices of revenue.

The optimization of individuals remains unchanged.

Equilibrium in the market for shares requires, as before, that

$$k = \frac{2 - q_2}{3}.$$

A firm selects investment plans by maximizing the value of the payoff, πy.

The objective function of firm 2 is to maximize

$$-\pi_1 k + \pi_2 + \pi_3 k,$$

for some $\pi = (\pi_1, \pi_2, \pi_3)$, such that $\pi Y = q$ or

$$\pi_1 = 1, \quad -\pi_1 k + \pi_2 + \pi_3 k = q_2.$$

The optimization of firm 1 is, as before, trivial.

For $k > 0$, equilibrium in production requires that

$$\pi_1 = \pi_3 = 1, \quad \pi_2 = q_2.$$

As before, the family of equilibria is described by

$$0 < k^* < \frac{2}{3}.$$

Competitive equilibrium prices are

$$q_2^* = 2 - 3k^*,$$

and

$$(\pi_1^*, \pi_2^*, \pi_3^*) = (1, q_2^*, 1).$$

In addition, there exists an essentially autarkic competitive equilibium, with $k = 0$; it obtains, for either definition of the optimization problem of firms, for an implicit price of revenue $\pi_3^* = 0$.

The sets of competitive equilibrium allocations when firms approximate the value of investment plans with the payoffs of marketed assets and spot commodity prices vary coincides with the set of competitive equilibria when firms adopt alternative implicit prices of revenue across realizations of uncertainty compatible with the observed prices of marketed assets.

Among equilibrium allocations, one is pareto optimal: it maximizes the utility of the individual over production allocations or

$$\hat{u} = \ln(1 - y) + \ln 1 + \ln y;$$

it involves

$$y^{**} = \frac{1}{2}.$$

Interestingly, it obtains for

$$q = \frac{1}{2}.$$

The price level that yields the pareto optimal allocation involves different price levels across realizations of uncertainty, which can be interpreted as active monetary policy.

3 Competitive equilibria

The economy satisfies standard assumptions if and only if

- for every individual, the consumption set, $\mathcal{X}^i$, coincides with the set of bundles of commodities that are non - negative; the preference relation, $\mathcal{R}^i$, is complete, transitive, continuous, convex, and strictly monotonically incresing: $x > \tilde{x} \Rightarrow x\mathcal{P}^i\tilde{x}$;
- for every individual, the endowments of commodities is strictly positive: $e^i \gg 0$, and the endowment of shares non - negative: $d^i \geq 0$;
- for every firm, the net production set, $\mathcal{Y}^j$, is closed and convex;
- for every firm, $0 \in \hat{\mathcal{Y}}^j$, and the endowment is non - negative: $f^j \geq 0$;
- across firms, the aggregate net production set, $\hat{\mathcal{Y}}^a$, is such that $\hat{\mathcal{Y}}^a \cap -\hat{\mathcal{Y}}^a = \{0\}$, while, whenever $y \in \hat{\mathcal{Y}}^a$ and $y \geq 0, y^a = 0$;
- there exist a partition of the set of states of the world, $\{\mathcal{S}^1, \ldots, \mathcal{S}^j, \ldots, \mathcal{S}^J\}$, into non - empty subsets such that, for every allocation of investment plans, $y^{\mathcal{J}}$, there exists an invertible matrix, T, such that

$$YT = (r^1, \ldots, r^j, \ldots, r^J),$$

where

$$r^j_s = \begin{cases} 1, & \text{if } s \in \mathcal{S}^j \\ 0, & \text{if } s \in \mathcal{S} \setminus \mathcal{S}^j. \end{cases}$$

The condition that, for every allocation of investment plans, there exists an invertible matrix, T, such that the matrix YT is orthogonal and positive and $YT1_S \gg 0$ guarantees that, for prices of shares of firms that do not allow for arbitrage: $q = \pi Y(p)$, for some $\pi \gg 0$, implicit prices of revenue for firms are strictly positive: $\pi^J = qY(p)(Y(p)'Y(p))^{-1}Y(p)' > 0$; it implies, in particular, that the matrix of commodity payoffs of shares, Y, has full column rank, and, also, that there exists a portfolio of shares with strictly positive payoffs: $Y(p)\overline{y} \gg 0$, for $\overline{y} = T1_J$; and it is inherited by the matrix $Y(p)$, for strictly positive prices of commodities. The condition is one of decomposability of the aggregate technology across states of the world; it is satisfied whenever the asset market is complete; for an incomplete asset market, it is restrictive, but it is not a condition of decomposability of the aggregate techology across firms or, equivalently, of the matrix of payoffs across firms or assets.

The condition that, for each individual, the consumption set coincides with the set of non - negative commodity bundles, while the preference relation is strictly monotonically increasing guarantees that commodities can be freely disposed of; in conjunction with the condition that there exists a portfolio of shares with strictly positive payoffs, it guarantees that the preference relations display local non - satiation. For the free disposal of commodities, a weaker condition, weak monotonicity: $x \geq \tilde{x} \Rightarrow x\mathcal{R}^i\tilde{x}$, suffices. For local non - satiation, a weaker condition, a portfolio, $\overline{y}$, with positive payoffs: $Y(p)\overline{y} > 0$, suffices. Strict monotonicity of the preference relation is not particularly restrictive in the present context with only one commodity at each state of the world.

Proposition 1 *If the economy satisfies standard assumptions, at prices of commodities $\overline{p} \gg 0$, competitive equilibria exist.*

Proof Without loss of generality, $\overline{p} = 1_S$; the payoffs and the commodity payoffs of shares coincide.

The domain of prices of shares, allocations of net investment plans of firms and aggregate excess portfolios of shares is

$$\mathcal{D} = \left\{ (q, \hat{y}^{\mathcal{J}}, \hat{z}^a) : \begin{array}{c} q = \pi Y, \pi \gg 0, \|q\| = 1, \\ \hat{y}^j \in \hat{\mathcal{Y}}^j : j \in \mathcal{J} \end{array} \right\}.$$

For $k > 0$, the closed, truncated domain of prices of shares, allocations of investment plans of firms and aggregate portfolios of shares is

$$\mathcal{D}_k = \left\{ (q, \hat{y}^{\mathcal{J}}, \hat{z}^a) : \begin{array}{c} q = \pi Y, \pi > 0, \|q\| = 1, \\ \hat{y}^j \in \hat{\mathcal{Y}}^j, \|\hat{y}^j\| \leq k : j \in \mathcal{J}, \\ \|\hat{z}^a\| \leq kI \end{array} \right\};$$

it is non - empty, compact and homeomorphic to a convex set.

On the domain $\mathcal{D}_k$, the truncated adjustment correspondence,

$$(q, \hat{y}^{\mathcal{J}}, \hat{z}^a) \rightarrow (q', \hat{y}^{\mathcal{J}\prime}, \hat{z}^{a\prime}),$$

is defined by

$$q' \in \arg\max\{qz^a : q = \pi Y, \pi > 0, \|q\| = 1\},$$

$$\hat{y}^{j\prime} \in \arg\max\{q(Y'Y)^{-1}Yy : y \in \mathcal{Y}^j, \|\hat{y}\| \leq k\} - \{f^j\}\},$$

$$\hat{z}^{a\prime} \in \sum_{i \in \mathcal{I}}(\arg\max\{\mathcal{R}^i : \sum_{j \in \mathcal{J}} \tilde{v}^j(1S, q, Y)\hat{z}^i_j, \|\hat{z}\| \leq k\} - \{d^i\});$$

the values of the correspondence are non - empty, compact and homeomorphic to convex sets, and the correspondence is upper - hemi continuous. There exists a fixed point, $(q^*_k, \hat{y}^{\mathcal{J}*}_k, \hat{z}^{a*}_k)$; for every firm, $\tilde{v}^j(1S, q^*_k, Y^*) = q^*_{j,k}$.

As $k \to \infty$, the sequence of fixed points has a limit point, $(q^*, \hat{y}^{\mathcal{J}*}, \hat{z}^{a*})$: prices of assets lie in a compact set; allocations of net investment plans, such that the aggregate net production plan is bounded below: $\hat{y}^a \geq -(e^a + f^a)$, lie in a compact set; if, along a subsequence, $\|\hat{z}^{a*}_k\| \to \infty$, the normalized excess portfolios $\hat{\hat{z}}^{a*}_k = (\|\hat{z}^{a*}_k\|)^{-1}\hat{z}^{a*}_k$ have a limit point, $\hat{\hat{z}}^{a*} \neq 0$, with $Y^*\hat{\hat{z}}^{a*} \geq 0$, and $q^*\hat{\hat{z}}^{a*} \leq 0$; by the optimization of individuals with strictly monotonically increasing preferences, $Y^*\hat{\hat{z}}^{a*} = 0$, and by the full column rank of the matrix of payoffs of assets, $\hat{\hat{z}}^{a*} = 0$, a contradiction.

At a limit point, $\hat{z}^{a*} = 0$: by the local non - satiation of the preference relations of individuals, at a limit point, $q^*\hat{z}^{a*} = 0$, and, by the definition of the adjustment correspondence for prices of shares, $Y^*\hat{z}^{a*} \leq 0$; if $Y^*\hat{z}^{a*} < 0$, the portfolio of shares $-\hat{z}^{a*}$ has positive payoffs, while it is valued at 0, which contradicts the optimization of individuals with strictly monotonically increasing preferences; hence, $Y^*\hat{z}^{a*} = 0$, and, since the matrix of payoffs of shares has full column rank, $z^{a*} = 0$.

Associated with the limit point, there is an allocation, $(z^{\mathcal{I}*}, y^{\mathcal{J}*}))$, such that, for every firm, $y^*_j \in \arg\max\{q^*(Y^{*\prime}Y^*)^{-1}Y^{*\prime}y : y \in \mathcal{Y}^j\}$. for every individual, $z^{i*} \in \arg\max\{\mathcal{R}^i : q^*z^{i*} \leq q^*d^{i*}\}$, and $z^{a*} = \sum_{i \in \mathcal{I}} z^{i*}, = 0$: the allocation is feasible, and $(q^*, z^{\mathcal{I}*}, y^{\mathcal{J}*}))$, is a competitive equilibrium at prices of commodities $\overline{p} = 1_S$. □

The assumption that, for every individual, the initial endowment of commodities is strictly positive: $e^i \gg 0$, and of shares of firms non - negative: $d^i \geq 0$, while $\pi Y \gg 0$, guarantees that, with income $\pi Y d^i \geq 0$, the individual is not a minimum wealth point, which, in turn, guarantees the continuity of the reaction correspondence; it could be replaced by the weaker assumption of resource relatedness, originally introduced by McKenzie (1959, 1961) for an economy without uncertainty or, equivalently, with a complete market in contingent commodities or elementary securities and subsequently adapted by Gottardi and Hens (1996) to take account of market incompleteness.

Corollary 1 *At a competitive equilibrium, the implicit prices of revenue for firms are strictly positive:* $\pi^{j*} \gg 0$.

The argument for the existence of competitive equilibria only requires that there exist a portfolio of shares with positive payoffs or that, for every allocation of investment plans, the matrix of payoffs of shares of firms, Y, has full column rank, while, for an invertible matrix, T, the matrix YT is orthogonal and positive. For the corollary, it is necessary that $YT\mathbf{1}_S \gg 0$. In the example, the latter fails for $k = 0$, and, indeed, the implicit prices of revenue at the autarkic equilibrium are not strictly positive.

Proposition 2 *A pair, $((p^*, q^*), (z^{\mathcal{I}*}, y^{\mathcal{J}*}))$, of prices of and a feasible allocation is a competitive equilibrium if and only if there exists a pair, (π^{**}, q^{**}), of strictly positive implicit prices of revenue: $\pi^{**} \gg 0$, and prices of shares, such that $\pi^{**} \in \Pi(\mathbf{1}_S, q^{**})$, for every firm, $y^{j*} \in y^j(\pi^{**})$, and, for every individual, $z^{i*} \in z^i(q^{**}, Y^*)$.*

Proof If

$$\pi^{**} = q^*(Y(p^*)'Y(p^*))^{-1}Y(p^*)'P^*, \quad \text{and} \quad q^{**} = \pi^{**}Y^*,$$

then

$$q^{**} = q^*, \quad \text{and} \quad \pi^{**}y = q^*(Y(p^*)'Y(p^*))^{-1}Y(p^*)'y(p^*).$$

By construction, for every firm, $y^j(\pi^{**}) = y^j(p^*, q^*, Y^*)$, and, hence, $y^{j*} \in y^j(\pi^{**})$; since the aggregate net production set allows for the free disposal of commodities, $\pi^{**} > 0$; for every individual, $z^i(q^{**}, Y^*) = z^i(q^*, Y^*)$, and, hence, $z^{i*} \in z^i(q^{**}, Y^*)$.

Since $Y^*\overline{z} \gg 0$, while $\pi^{**} \gg 0$, there exists $p^* \gg 0$, such that

$$\pi^{**} = \overline{z}'Y^{*\prime}P^{*\prime}P^*.$$

If

$$\pi^* = \overline{z}'Y^{*\prime}P^{*\prime} \quad \text{and} \quad q^* = \pi^*Y^*(p^*),$$

then

$$q^* = q^{**} \quad \text{and} \quad q^*(Y(p^*)'Y(p^*))^{-1}Y(p^*)'y(p^*) = \pi^{**}y.$$

By construction, for every firm, $y^j(p^*, q^*, Y^*)y^j(\pi^{**}) =$ and, hence, $y^{j*} \in y^j(p^*, q^*, Y^*)$; for every individual, $z^i(q^*, Y^*) = z^i(q^{**}, Y^*)$, and, hence, $z^{i*} \in z^i(q^{**}, Y^*)$. □

Competitive equilibrium allocations when firms compute the value of investment plans by approximating the value of payoffs with the value of portfolios of marketed shares essentially coincide with competitive equilibrium allocations when, according to Duffie and Shafer (1987), firms adopt implicit prices of revenue compatible with the prices of marketed shares. The argument that yields competitive equilibrium allocations when, according to Duffie and Shafer (1987), firms adopt implicit prices of revenue compatible with the prices of marketed shares as competitive equilibrium allocations when firms compute the value of investment plans by approximating the value of payoffs with the value of portfolios of marketed shares fails if the implicit prices of prices of revenue adopted by firms fail to be strictly positive at equilibrium.

A transfer of revenue is $\tau = (\dots, \tau^i, \dots)$, such that $\sum_{i \in \mathcal{I}} \tau^i = 0$.

With transfer of revenue τ^i, the budget constraint of an individual is $qz \leq qd^i + \tau^i$, and the demand for shares by the individual is $z^i(q, Y, \tau^i)$.

A competitive equilibrium with a transfer of revenue is $((p^*, q^*), \tau^*, (z^{\mathcal{I}*}, y^{\mathcal{J}*}))$, a triple of prices, a transfer of revenue and a feasible allocation, such that, for every firm, $y^{j*} \in y^j(p^*, q^*, Y^*)$, and, for every individual, $z^{i*} \in z^i(q^*, \tau^{i*}i, Y^*)$.

Proposition 3 *If the economy satisfies standard assumptions, a constrained pareto optimal allocation,* $(z^{\mathcal{I}*}, y^{\mathcal{J}*})$*, such that, for every individual,* $(e^i + Y^* z^{i*}) \gg 0$*, is a competitive equilibrium allocation with a transfer of revenue.*

Proof The sets

$$\mathcal{C}^1 = \{(y^a : y^a = \sum_{j \in \mathcal{J}} y^j, (e^i + Yz^i)\mathcal{R}^i(e^i + Y^* z^{i*}), i \in \mathcal{I}\} - \{y^{a*}\}$$

and

$$\mathcal{C}^2 = \{(y^a : \hat{y}^a \in \mathcal{Y}^a\} - \{y^{a*}\}$$

are convex. Since the allocation $(z^{\mathcal{I}*}, y^{\mathcal{J}*})$ is constrained pareto optimal, while tha preferences of individuals are continuous and do not display satiation, Rel Int $\mathcal{C}^1 \cap$ Rel Int $\mathcal{C}^2 = \emptyset$. Since the preferences of individuals are monotonic and, hence, locally non - satiated, $0 \in$ Bd $\mathcal{C}^1 \cap$ Bd $\mathcal{C}^2$. There exists a separating hyperplane: $\pi^* \neq 0$, such that $\inf\{\pi^* y^a : y^a \in \mathcal{C}^1\} < \sup\{\pi^* y^a : y^a \in \mathcal{C}^1\} \leq \inf\{\pi^* y^a : y^a \in \mathcal{C}^2\} < \sup\{\pi^* y^a : y^a \in \mathcal{C}^2\}$. Since the consumption seta and preference relations of individuals allow for the free disposal of commodities, $\pi^* > 0$. If $y \in \mathcal{Y}^i$, substitution of $y + \sum_{i' \in \mathcal{I} \setminus \{i\}} y^{i*}$ for y^a yields that $\pi^* y \leq \pi^* y^{i*}$. Similarly, for every individual, $\pi^*(e^i + Yz)\pi^* \geq \pi^*(e^i + Y^* z^{i*})$ if $(e^i + Yz)\mathcal{R}^i(e^i + Y^* z^{i*})$, and, since $(e^i + Y^* z^{i*}) \gg 0, \pi^*(e^i + Yz) > \pi^*(e^i + Y^* z^{i*})$ if $(e^i + Yz)\mathcal{P}^i(e^i + Y^* z^{i*})$. □

The assumption that consumption plans are not interior to the consumption sets of individuals guarantees that they are not minimum wealth points, in which case a constrained pareto optimal allocation need not be a competitive equilibrium allocation with a transfer of revenue; this is the case even for a complete asset market.

A competitive equilibrium allocation need not be constrained pareto optimal, as the example illustrates.

By a standard argument, with one commodity at each state of the world, a competitive equilibrium allocation is constrained pareto optimal in exchange. Also in production, as long as firms adopt implicit prices of revenue across states of the world defined by $\pi^{j*} = \sum_{i \in \mathcal{I}} z^i_j \pi^{i*}$ for π^{i*} implicit prices of revenue that support the consumption plans of individuals at equilibrium: $\pi^{i*} Y^* z > \pi^{i*} Y^* z^{i*}$ whenever $(e^i + Y^* z) \mathcal{P}^i (e^i + Y^* z^{i*})$ — Drèze (1974); the implicit prices of revenue adopted need not coincide across firms. A variant of the aggregation of the support prices of individual shareholders that excludes those with short positions in the shares of the firm — $z^{i*}_j < 0$ — does not affect the argument.

When firms compute the value of investment plans by approximating the value of payoffs with the value of portfolios of marketed shares, a competitive equilibrium allocation need not be constrained pareto optimal in production, as the example illustrates as well: with one individual, the notion coincides with constrained pareto optimality.

The economy satisfies standard smoothness assumptions if and only if it satisfies standard assumptions and

- for every individual, for strictly positive consumption plans, the preference relation is represented by a utility function, $u^i : x \mathcal{R}^i \tilde{x} \Leftrightarrow u^i(x) \geq u^i(\tilde{x})$; the utility function, with domain the interior of the consumption set, is twice continuously differentiable, differentiably strictly monotonically increasing: $Du^i \gg 0$, and differentiably strictly quasi - concave: $\xi \neq 0$ and $Du^i \xi = 0 \Rightarrow \xi' D^2 u^i \xi < 0$; for consumption plans, $x \gg 0$ and $\bar{x} \not\gg 0, x \mathcal{P}^i \bar{x}$;
- for every firm, the strict efficient boundary of the net production set, $\mathrm{Bd}_+ \hat{\mathcal{Y}}^j = \{\hat{y} : \hat{y} = \arg\max \pi^f \hat{y}, \pi^f \gg 0\}$, is described by a production function, $g^j : \hat{y} \in \mathrm{Bd}_+ \Leftrightarrow g^j(\hat{y}) = 0$; the production function, with domain an open set of investment plans, is twice continuously differentiable, differentiably strictly monotonically increasing: $Dg^j \gg 0$, and differentiably strictly quasi - concave: $\xi \neq 0$ and $Dg^j \xi = 0 \Rightarrow \xi' D^2 g^j \xi < 0$;
- for every firm, the endowment of commodities is strictly positive: $f^j \gg 0$.

With the preferences and endowments of shares of individuals and the technologies of firms held fixed, an economy is described by the endowments of individuals and firms, $\omega = (e^{\mathcal{I}}, f^{\mathcal{J}})$; the set of economies, Ω, is a bounded subset of the strictly positive orthant in euclidean space of dimension $IS + JS$.

The set of prices of commodities is $\mathcal{P}$, the strictly positive orthant in euclidean space of dimension S.

The set of economies and prices of commodities, $(\omega, p) = ((e^{\mathcal{I}}, f^{\mathcal{J}}), p)$, is $\Omega \times \mathcal{P}$, an open subset of the strictly positive orthant in euclidean space of dimension $IS + JS + S$.

A property holds generically if and only if it holds for an open subset of full lebesgue measure.

Proposition 4 *If the economy satisfies standard smoothness assumptions, competitive equilibrium allocations are, generically, indeterminate of degree $(S-A)$; competitive equilibria at fixed prices of commodities are, generically, determinate.*

Proof[11] If the function F of $(e^{\mathcal{I}}, f^{\mathcal{J}}, p, z^{\mathcal{I}}, \lambda^{\mathcal{I}}, y^{\mathcal{J}}, \mu^{\mathcal{J}})$ is defined by

$$F = \begin{pmatrix} (Du^i(e^i + Yz^i) - \lambda^i q & \\ & i \in \mathcal{I} \\ q(z - d^i) & \\ \sum_{i \in \mathcal{I}} \tilde{z}^i - \mathbf{1}_{(A-1)} & \\ q(Y(p)'Y(p))^{-1}Y(p)' - \mu^j Dg^j(\hat{y}^j) & \\ & j \in \mathcal{J} \\ g^j(\hat{y}^j) & \end{pmatrix},$$

where $\lambda^{\mathcal{I}} = (\ldots, \lambda^i, \ldots) \gg 0$, $\mu^{\mathcal{J}} = (\ldots, \mu^j, \ldots) \gg 0$, $\tilde{z} = (z_2, \ldots, z_A)$, and $q = (1, q_2, \ldots, q_A)$ — without loss of generality, $y^1 \gg 0$. Competitive equilibria are identified with $F^{-1}(0)$.

By a standard argument, the function F is transverse to 0. For a subset of economies and prices of commodities, $(\Omega \times \mathcal{P})^*$, the function $F_{(e^{\mathcal{I}}, f^{\mathcal{J}}, p)}$ is transverse to 0. The set of competitive equilibria for the economy $(\ldots, e^i, \ldots, f^j, \ldots)$ at prices of commodities p is finite and varies continuously with the parameters $(\ldots, e^i, \ldots, f^j, \ldots, p)$: at fixed prices of commodities, competitive equilibrium allocations are, generically, determinate

The subset, Ω^*, of economies, ω, such that $(\omega, p) \in (\Omega \times \mathcal{P})^*$, for some prices of commodities, p, is open and of full lebesgue measure. For a fixed economy, $\omega \in \Omega^*$, there exists an open subset of prices of commodities, $\mathcal{P}(\omega)$, such that $\omega \times \mathcal{P}(\omega) \in (\Omega \times \mathcal{P})^*$. It remains to show that, as prices of commodities vary in $\mathcal{P}(\omega)$ the associated set of competitive equilibriun allocations containd an open set of dimension $(S-A)$.

The subset of states of the world $\bar{\mathcal{S}} = \{s^1, \ldots, s^j, \ldots, s^J\}$ is chosen such that $s^j \in \mathcal{S}^j$, where $\{\mathcal{S}^1, \ldots, \mathcal{S}^j, \ldots, \mathcal{S}^J\}$ is the partition associated with the decomposition of the matrix of payoffs of shares.

Eucleidean space, of dimension S is the direct sum of subspaces $\mathcal{E}^A$ and $\mathcal{E}^{(S-A)}$, of dimension A and $(S-A)$, respectively, where $\mathcal{E}^A = \{k : k_s = k_{s'}, \text{ for } s, s' \in \mathcal{S}^j\}$ and $\mathcal{E}^{(S-A)} = \{k : k_s = 0, \text{ for } s \in \bar{\mathcal{S}}\}$.

[11] The argument is that of Duffie and Shafer (1987); we give a simple version, which is possible with one commodity per state of the world and constant rank of the matrix of payoffs of shares.

For prices of commodities p and p', such that $(p - p') \in \mathcal{E}^A$, the associated competitive equilibrium allocations coincide. If $Y(p) = PY$ is the matrix of payoffs of shares at a competitive equilibrium at prices of commodities p, and if the matrix $Y(p)T = (r^1, \ldots, r^j, \ldots, r^J)$ is such that $r^j_s = 1$, if $s \in \mathcal{S}^j$, and if $s \in \mathcal{S} \setminus \mathcal{S}^j$, the matrix of payoffs of shares at prices of commodities p', for the same allocation of investment plans is $Y(p') = P'Y$, and $Y(p')T = Y(p)T\Delta$, where $\delta = (\delta_{s1}, \ldots, \delta_{sj}, \ldots, \delta_{sJ})$.

For prices of commodities p and p', such that $(p - p') \in \mathcal{E}^{(S-A)}$, the associated competitive equilibrium allocations are distinct: If not, there exist prices of commodities p and p', such that $(p - p') \in \mathcal{E}^{(S-A)}$, and an allocation of investment plans with matrix of commodity payoffs of shares, Y, prices of shares, q, and implicit prices of revenue for firms, π^J, associated with competititive equiilbria at both p and p'. In particular, $\pi^J = qT(T'Y(p)'Y(p)T)^{-1}T'Y(p)' = qT(T'Y(p')'Y(p')T)^{-1}T'Y(p')'$. Since $qT = \pi Y(p)T = \pi' Y(p')T$, for some $\pi \gg 0$, $\pi' \gg 0$. By the structure of the matrix YT, since $p_s = p'_s$, for $s \in \overline{\mathcal{S}}$, $(T'Y(p)'Y(p)T)^{-1} = (T'Y(p')'Y(p')T)^{-1}$. But then, for $s \in \mathcal{S} \setminus \overline{\mathcal{S}}$, such that $p_s \neq p'_s$, it is not possible that the implicit price of revenue for firms, π^J_s, at p_s and p'_s coincide.

Since, for prices of commodities p and p', such that $(p - p') \in \mathcal{E}^{(S-A)}$, the associated competitive equilibrium allocations are distinct, while, for prices of commodities p and p', such that $(p - p') \in \mathcal{E}^A$, the associated competitive equilibrium allocations coincide, competitive equilibrium allocations are, generically, indeterminate of degree $(S - A)$. □

4 Conclusion

Economies with multiple commodities or, more interestingly, economies that extend over multiple dates require explicit treatment of the discontinuous correspondence that maps investment plans of firms and prices in commodity spot markets to attainable reallocations of revenue or equivalently, to the span of the matrix of payoffs of shares of firms and other assets. The argument is demanding, and Duffie and Shafer (1987) have handled a typical case; it is an extension of the argument for exchange economies. The simplifying assumptions here allow the argument to focus on the investment criteria for firms and the ensueing properties of competitive equilibrium allocations.

It is an open question whether operational investment criteria for firms lead to constrained pareto optimal allocations; whether they are robust to the voting or other power of shareholders — DeMarzo (1993), Gevers (1974); and whether they require information that can be recovered from the market behavior of shareholders — Dybvig and Polemarchakis (1981), Geanakoplos and Polemarchakis (1990), Green, Lau and Polemarchakis (1979).

Most interesting is the possibility that, investment criteria of firms may allow for effective and desirable policy, in particular monetary policy.

References

[1] Arrow, K. J. (1951), "A reformulation of the theorems of classical welfare economics," in J. Neyman (ed.), *Second Berkeley Symposium on Mathematical Statistics and Probability*, 488 - 500.

[2] Arrow, K. J. (1953), "Le rôle des valeurs boursières pour l'allocation la mellieure des risques," *Econometrie, Colloqus Internationaux du CNRS*, 11, 41 - 47.

[3] Arrow, K. J. and G. Debreu (1954), "Existence of equilibrium for a competitive economy," *Econometrica*, 22, 265 - 290.

[4] Balasko, Y. and D. Cass (1989), "The structure of financial equilibrium with exogenous yields," *Econometrica*, 57, 135 - 162.

[5] Cass, D. (1985), "On the 'number' of equilibrium allocations with incomplete financial markets," Working Paper No. 85 - 00, CARESS, University of Pennsylvania.

[6] Debreu, G. (1951), "The coefficient of resource utilization," *Econometrica*, 19, 273 - 292.

[7] Debreu, G. (1960), "Une économie de l'incertain," *Economie Appliqueé*, 13, 111 - 116.

[8] Debreu, G. (1970),"Economies with a finite set of equilibria," *Econometrica*, 38, 387 - 392.

[9] DeMarzo, P. M. (1993), "Majority voting and corporate control: the rule of the dominant shareholder," *Review of Economic Studies*, 60, 713 - 734.

[10] Diamond, P. (1967), "The role of the stock market in a general equilibrium model with technological uncertainty," *American Economic Review*, 57, 759 - 776.

[11] Drèze, J. H. (1974), "Investment under private ownership: optimality, equilibrium and stability," in J. H. Drèze (ed.), *Allocation under Uncertainty: Equilibrium and Optimality*, Macmillan Press, 129 - 166.

[12] Drèze, J. H. (1982), "Decision criteria for business firms," Discussion Paper No. 8211, CORE, Université Catholique de Louvain.

[13] Drèze, J. H. (1989), *Labour Management, Contracts and Capital Markets*, Yrjö Jahnsson Lectures, Basil Blackwell.

[14] Duffie, D. and W. Shafer (1985; 1986), "Equilibrium in incomplete markets, I: a basic model of generic existence; II: generic existence in stochastic economies, " *Journal of Mathematical Economics,* 14, 285 - 300; 15, 199 - 216.

[15] Duffie, D. and W. Shafer (1987), "Equilibrium and the role of the firm in incomplete markets," Working Paper No. 87/279, Department of Economics, European University Institute.

[16] Dutta, J. and H. M. Polemarchakis (1991), "Asset pricing and observability," in W. A. Barnett et al. (eds.), *Equilibrium Theory and Applications: Proceedings of the Sixth International Symposium in Economic Theory and Econometrics,* Cambridge University Press, 9 - 30.

[17] Dybvig, P. L. and H. M. Polemarchakis (1981), "Recovering cardinal utility," *Review of Economic Studies,* 48, 159 - 166.

[18] Ekern, S. and R. Wilson (1974), "On the theory of the firm in an economy with incomplete markets," *Bell Journal of Economics and Management Science,* 5, 171 - 180.

[19] Geanakoplos, J. D., M. J. P. Magill, M. Quinzii and J. H. Drèze (1990), "Generic inefficiency of stock market equilibrium when markets are incomplete," *Journal of Mathematical Economics,* 19, 113 - 152.

[20] Geanakoplos, J. D. and A. Mas - Colell (1989), "Real indeterminacy with financial assets," *Journal of Economic Theory,* 47, 22 - 38.

[21] Geanakoplos, J. D. and H. M. Polemarchakis (1986), "Existence, regularity and constrained suboptimality of competitive portfolio allocations when the asset market is incomplete," in W. P. Heller et al. (eds.), *Uncertainty, Information and Communication: Essays in honour of K. J. Arrow, Vol. III,* Cambridge University Press, 65 - 96.

[22] Geanakoplos, J. D. and H. M. Polemarchakis (1990), "Observability and optimality," *Journal of Mathematical Economics,* 19, 153 - 166.

[23] Geanakoplos, J. D. and M. Shubik (1990), "The capital asset pricing model with incomplete markets," *The Geneva Papers of Risk and Insurance Theory,* 15, 55 - 72.

[24] Gottardi, P. and T. Hens (1996), "The survival assumption and competitive equilibria when the asset market is incomplete, " *Journal of Economic Theory,* 71, 313 - 323.

[25] Gevers, L. (1974), "—," in J. H. Drèze (ed.), *Allocation under Uncertainty: Equilibrium and Optimality,* Macmillan Press, 000 - 000.

[26] Grossman, S. J. and O. D. Hart (1979), "A theory of competitive equilibrium in stock market economies," *Econometrica*, 47, 293 - 330.

[27] Green, J. R., L. J. Lau and H. M. Polemarchakis (1979), "On the recoverability of the von neumann - morgenstern utility function from asset demands," in J. R. Green and J. A. Scheinkman (eds.), *Equilibrium, Growth and Trade: Essays in Honor of L. McKenzie*, Academic Press, 151 - 161.

[28] Hart, O. J. (1975), "On the Optimality of equilibrium when the market structure is incomplete," *Journal of Economic Theory*, 21, 293 - 311.

[29] Leland, H. (1974), "Production theory and the stock market," *Bell Journal of Economics and Management Science*, 5, 125 - 144.

[30] Lintner, J. (1965), "The valuation of risky assets and the selection of risky investments in stock portfolios and capital budgets," *Review of Economics and Statistics*, 47, 13 - 37.

[31] Mas - Coloell (1991), "Indeterminacy in incomplete market economies," *Economic Theory*, 1, 45 - 61.

[32] McKenzie, L. (1954), "Equilibrium in Graham's model of world trade and other competitive systems," *Econometrica*, 22, 147 - 161.

[33] McKenzie, L. (1959), " On the existence of general equilibrium for a competitive market," *Econometrica*, 27, 54 - 71.

[34] McKenzie, L. (1961), "On the existence of general equilibrium: some corrections," *Econometrica*, 00, 000 - 000.

[35] Polemarchakis, H. M. and P. Siconolfi (1994), "Prices, asset markets and indeterminacy," Discussion Paper No. 9429, CORE, Université Catholique de Louvain.

[36] Radner, R. (1972), "Existence of equilibrium of plans, prices and price expectations in a sequence of markets," *Econometrica*, 40, 289 - 303.

[37] Radner, R. (1974), "A note on unanimity of stockholders' preferences among alternative production plans: a reformulation of the Ekern - Wilson model," *The Bell Journal of Economics and Management Science*, 5, 181 - 186.

[38] Ross, S. A. (1978), "A simple approach to the valuation of risky streams," *Journal of Business*, 5, 453 - 475.

[39] Sharp, W. (1964), "Capital asset prices: a theory of market equilibrium under conditions of risk," *Journal of Finance*, 19, 425 - 442.

[40] Traynor, J. (1961), "Towards a theory of market value of risky assets," mimeo.