COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 1139

HYPERFINITE ASSET PRICING THEORY

M. Ali Khan and Yeneng Sun

November 1996

Hyperfinite Asset Pricing Theory*

M. Ali Khan† and Yeneng Sun‡

First version: July 1995; this version: October 1996

Abstract: We present a model of a financial market which unifies the capital-asset-pricing model (CAPM) of Sharpe-Lintner, and the arbitrage pricing theory (APT) of Ross. The model is based on a recent theory of hyperfinite processes, and it uncovers asset pricing phenomena which cannot be treated by classical methods, and whose asymptotic counterparts are not already, or even readily, apparent in the setting of a large but finite number of assets. In the model, an asset's unexpected return can be decomposed into a *systematic* and an *unsystematic* part, as in the APT, and the systematic part further decomposed into an *essential* and an *inessential* part, as in the CAPM. This tri-partite decomposition leads to a pricing formula expressed in terms of a beta that is based on a specific index portfolio identifying essential risk, and constructed from factors and factor loadings that are endogenously extracted from the process of asset returns. Furthermore, the valuation formulas of the two individual theories imply, and are implied by, the pervasive economic principle of no arbitrage. Explicit formulas for the characterization, as well as conditions for the existence, of important portfolios are furnished. The hyperfinite factor model possesses an optimality property which justifies the use of a relatively small number of factors to describe the relevant correlational structures. The asymptotic implementability of the idealized limit model is illustrated by an interpretation of selected results for the large but finite setting.

1 Introduction

The capital-asset-pricing model (CAPM) of Sharpe (1964) and Lintner (1965) yields the insight that the expected return to any asset is related solely to the non-diversifiable component of its total risk, such a risk is formalized in the model through the concept of a mean-variance efficient portfolio M, and is measured by the beta of the particular asset. If $\mu(t)$ denotes the expected return of a risky asset t chosen from a universe of assets T, x_t its random return, and $\mathrm{cov}\,(x_t, M)$ its covariance[1] with the portfolio M, then its beta β_t is given by $\mathrm{cov}(x_t, M)/V(M)$, and the model predicts the simple linear relationship

$$\mu(t) = \rho + \beta_t(E(M) - \rho),$$

where $E(M)$ and $V(M)$ are the expected return and the variance of the portfolio M, and ρ is the riskless interest rate. This formula can be phrased in terms of any mean-variance efficient portfolio, and under strong conditions on investor preferences and/or the process of asset returns, can be rationalized as a market portfolio.[2] What deserves emphasis is that the construction of the CAPM carrier portfolio M, and in particular its efficiency property, is relative to the particular finite universe of assets postulated in the model. The CAPM pricing formula has generated extensive empirical work, and has been one of the focal points for finance theory in the last three decades.[3]

Despite its success in both theoretical and empirical work, the CAPM has had to face criticism for the strong assumptions deriving its pricing formula,[4] and the arbitrage pricing theory (APT), developed in Ross (1976), has shifted the attention away from the microeconomics of mean-variance efficiency towards the gain from diversification based on portfolio size. Unlike the CAPM, in which a *single* variable (beta) determines the expected return of an asset, the APT allows for a *multiple* number of sources of market-wide risks to capture the intercorrelation among the various assets.[5] This ensemble of market-wide systematic risks is formalized by an *exogenously* given finite number of factors, with each of these factor risks affecting a non-negligible portion of the market. Then the purpose of the theory is to derive the result that the price or expected return of an asset depends only on factor risks and not on the residual unsystematic or idiosyncratic component. Put differently, it is desired that all risk that is not correlated with the factors can be completely diversified away and plays no role in equilibrium asset pricing. The systematic-unsystematic risk decomposition of an asset's unexpected return rests on the claim that even in a financial market with a large number of risk classes, there is only a relatively small number of factors relevant from macroscopic point of view, and that the residual part can be made arbitrarily small when large enough portfolios are taken. Ross' result then rests on three basic ideas: an *exogenously* given factor structure, a large number of assets to formalize the presence of unsystematic risks and to operationalize the notion of large portfolios, and the absence of asymptotic arbitrage in some specific sense. If systematic risks are captured by k exogenous factors $\varphi_n(\cdot)$ defined on some probability space $(\Omega, \mathcal{A}, P)$ and the random return of asset t is given by

$$x_t(\omega) = \mu(t) + \beta_{t1}\varphi_1(\omega) + \cdots + \beta_{tK}\varphi_K(\omega) + e_t(\omega),$$

ω a sample point in Ω, then a suitable version of the absence of arbitrage, and some assumption on the residual process $e_t(\cdot)$ formalizing unsystematic risks, yields the approximate linear equation

$$\mu(t) \simeq \rho + \tau_1\beta_{t1} + \cdots + \tau_K\beta_{tK}.$$

If the β_{tj} are called factor loadings, then the result simply states that an asset's expected return is approximately linearly related to its factor loadings.

If the ensemble of unsystematic risks present in the market is assumed to be constituted by mutually uncorrelated random variables, Ross' theorem follows as a consequence of the law of large numbers and a suitable version of the "no arbitrage" assumption. However, since the law of large numbers is conventionally presented in terms of an asymptotic approximate result for an infinite sequence of random variables, the APT is formulated in the context of an infinite sequence of assets with diversification necessarily incomplete, unsystematic risks only approximately removed, and a pricing formula that can only be expressed in an approximate form.[6] Indeed, given this emphasis on approximation, Chamberlain-Rothschild (1983) and Chamberlain (1983) present a version of Ross' theorem based only on an approximate factor structure.[7]

A natural question then arises as to the possibility of a mathematical framework which, in bypassing Ross' sequential formulation, can handle risks that are asset-specific, or affect at most a negligible subset of the financial market. In such a framework, unsystematic risks would be completely diversified away,[8] and at the same time, the concept of "no arbitrage" formulated in a more concrete and transparent way.[9] It is now well appreciated that the usual continuum – the unit interval endowed with Lebesgue measure, for example – does not offer the kind of mathematical framework that we seek. The reason is simply that there is no law of large numbers for such a continuum of random variables. The difficulties associated with a formulation of such a law are well understood.[10] Except for some trivial cases, the stochastic process $e_t(\omega)$, $t \in [0,1]$, $\omega \in \Omega$, regarded as a function of the two variables (t, ω) is not measurable, and even has no measurable standard modification with respect to the relevant product measure if the random variables $\{e_t(\cdot)\}_{t \in [0,1]}$ are assumed to be mutually independent. In fact, as noted in Judd (1985), one can construct independent and identically distributed random variables, indexed by the unit Lebesgue interval, for which it is not true that almost all sample functions[11] are measurable. Indeed, if one considers a natural extension of the probability measure on the sample space so that the measurability problem disappears, then the set of sample realizations satisfying the property that the expectation or distribution of the sample function is the theoretical one, has outer measure one and inner measure zero. This set is, therefore, not measurable. One can of course extend the measure on the sample space so that the law is satisfied by this very special process, but there is also another extension for which almost no sample realizations have the required properties.[12] Thus the law of large numbers for the standard continuum model can fail even for a simplest construction of independent and identically distributed random variables. In other words, instead of providing a better and more applicable version of the law than the discrete case, the continuum formulation messes up the basic intuition and complicates its scientific meaning.

In any case, one cannot simply graft onto a model of the financial market a particular continuous-parameter stochastic process for which the law of large numbers holds. It is not a question of assuming that the ensemble of unsystematic risks in the market follow one or another simple construction so that these risks can be canceled, since, as discussed above, there could be many other versions of unsystematic risks for which the law fails . In a viable model of the financial market, unsystematic risks have to be endogenously identified and extracted from the underlying process of asset returns under assumptions made on this basic primitive process, and the whole point is to obtain a framework in which such unsystematic risks derived in such a manner can be completely diversified away. And of course one can hardly have an arbitrage theory of asset pricing without asset specific risks in the market. Thus Chamberlain-Rothschild (1983; p.1282) flatly state, "Ross' heuristics cannot be made rigorous."[13]

In Sun (1994, 1996a), a particular class of measure spaces due to Loeb (1975) is used to model probabilistic phenomenon involving a large number of random variables in situations where there is no natural topology on the set T indexing these random variables.[14] We use the elements of this theory of hyperfinite processes, in particular the associated hyperfinite factor model,[15] to formalize the raw intuition underlying both the CAPM and the APT. Note that these two asset pricing theories deal with two different sets of risks, and this fact is reflected in the two different valuation formulas that they generate for the risk-premia associated with individual assets.[16] The CAPM distinguishes between ***non-diversifiable*** and ***diversifiable*** risks based on mean-variance efficiency, and thereby on the efficient diversification of a portfolio, while the APT distinguishes between ***systematic*** and ***unsystematic*** risks based on a more naive diversification depending simply on portfolio size.[17] The classical CAPM model, with its limitation to a universe with a finite number of assets, neglects unsystematic risks, while the APT, in its avoidance of any optimization, neglects non-diversifiable risks. Thus the two models seem to be disjoint, each addressing a different aspect of the premium-awarding scheme for taking risks in financial markets. It is, however, surprising that the framework of hyperfinite processes adopted here can nevertheless be used to unify their basic ingredients. In this framework, a "large number of assets" is formalized as a hyperfinite Loeb space and the random returns of the assets are described by a real valued hyperfinite process indexed by such a space. An associated hyperfinite factor model is used to derive endogenously all the different types of risks considered in CAPM and APT. Two results are crucial for the developments to follow: one is a law of large numbers for hyperfinite processes and the other is a biorthogonal representation theorem of the Karhunen-Loéve type.[18] In the remainder of this introduction, we outline the salient features of this application of hyperfinite processes.

The hyperfinite setting offers a mathematical framework that renders in a precise way the common sense behind unsystematic and systematic risks: the former is defined in terms of its affect on a "negligible" corner of the financial market, and the latter in terms of a "non-negligible" one, the measure-theoretic structure being used to formulate the intuitive notion of "negligibility".[19] One can now invoke the associated hyperfinite law of large numbers to show that unsystematic risks can be completely eliminated from a large portfolio.[20] Neither intelligence nor additional information is needed for naive diversification, and in the absence of profitable arbitrage opportunities, this yields the APT valuation formula. However, we can go beyond this to identify a portfolio I_0 based on the endogenous factors,[21] to define the beta β_t of any asset t to be the covariance of its random return with I_0, and finally to develop a CAPM type valuation formula in which the expected return of an asset t is linearly dependent on this recalculated β_t. We are then able to further decompose systematic risks into what we shall call here *essential* risks – scalar multiples of a common random variable obtained from I_0 – and *inessential* risks constituted by the relevant orthogonal complement. This additional terminology reflects the fact that, unlike the classical CAPM model where only two types of risks are considered, namely, diversifiable and non-diversifiable risks, we are now working with three types of risks. Unsystematic risk can be eliminated by naive diversification, while inessential risk, even though correlated with a nonnegligible portion of the market, by efficient diversification. The CAPM type formula based on I_0 shows that inspite of the market having many factor risks, it only rewards a risk which is essential and which can never be eliminated through either kind of diversification. Stated simply, the usual APT claim that the market only rewards systematic risks is simply not sharp enough; systematic risks can be reduced still further until a portfolio has only essential risk, and it is only this risk component of systematic risk that earns a premium. To repeat, though there could be *multiple* sources of industry-wide or market-wide factor risks, barring trivial cases, surprisingly there is only a *unique* source of risk, characterized by one random variable, that is rewarded by the market. Risk premium is paid to a factor risk only through the particular role of the factor in the definition of the essential risks. In brief, our framework clarifies three different types of risks in financial markets, and unifies them in a pricing formula that reflects the reward scheme for the holding of each type of risk. Previous work only focussed on two types of risks – each one at one time and in different settings.

Once the relevance and analytical viability of the hyperfinite framework is established, it is not surprising that in the absence of gains from arbitrage, an exact linear pricing equation should hold one way or another for the APT model. Hyperfinite models after all do capture the limiting behavior of large finite models. However, what is surprising and a further testimony to the fruitfulness of the approach, is that we can go beyond this to show that the absence of arbitrage is not only sufficient but also necessary for the validity of the APT pricing equation, and that this equation implies, and is implied by, the CAPM pricing equation. If APT is useful in the sense that by the empirical identification of factors, its valuation formula is testable, then by using the same factors to define the index portfolio I_0, one can obtain and test the associated valuation formula with I_0 as the CAPM carrier. Furthermore, if one is interested in a corner of the general market and APT is valid for that corner, then the valuation equation of the CAPM type can be obtained for this particular corner of the market as well. Since the risks which are essential for that corner are computed via the factors, there is no presumption that they reflect those of some market portfolio.

The advantages of being able to work in an ideal hyperfinite context do not stop here. Unlike the conventional APT literature where the asset returns are described by k *exogenously* given factors plus an *exogenously* given component of unsystematic risks, the hyperfinite factor model used here derives from the primitive assumption of square integrability of the process of asset returns both the collection of factors and the ensemble of unsystematic risks. The factors are the eigenfunctions of the autocorrelation function of the process of asset returns,[22] the latter being the infinite-dimensional analogue of what is the variance-covariance matrix of the returns to a finite number of assets. This responds to the criticism of the arbitrariness of the choice of factors in the conventional APT models.[23] In addition to the factors being orthonormal, the associated factor loadings in the hyperfinite factor model are also orthonormal. This biorthogonality property proves pivotal in the investigation of conditions for the existence of various important portfolios and in their characterization in terms of explicit formulas. It is reasonable to expect that these formulas will make the study of factors in financial markets relevant to the practical constructions of such portfolios in large asset markets. In any case, these characterizations give concreteness and depth to the theory.

The hyperfinite factor model has an additional optimality property: if one is limited to use only m sources to measure the ensemble of systematic risks of the market, m an arbitrarily given positive integer, then the best approximation is achieved by using the random variables derived from the m eigenfunctions corresponding to the largest eigenvalues of the autocorrelation function of the asset return process. We simply pick the first m factors specified in the hyperfinite factor model. This property the justifies the use of a relatively small number of factors, as defined in the framework, to understand the relevant correlational structures. The point can be put another way. There are of course many ways of choosing factors to describe the relevant correlational structures, but the optimality property singles out the procedure described above as the best way in a well-specified sense. If a different procedure of factor selection is adopted, then one has to ensure that factors with smaller contributions towards explaining the correlational structure, and thereby the systematic behavior, of the market are distinguished from those with larger ones.

Yet another payoff of the hyperfinite approach to asset pricing that is developed in this paper concerns the asymptotic formulation pertaining to a sequence of large but finite asset markets. As argued in Sun (1994), the asymptotic properties of stochastic processes on large finite probability spaces are equivalent to certain properties of internal processes on hyperfinite probability spaces, which in turn are usually equivalent to some properties of processes which are measurable with respect to the relevant Loeb product spaces. Thus, in order to understand the asymptotic nature of processes on large finite probability spaces, which correspond to the large finite phenomena being modelled, one only needs to consider processes which are measurable with respect to Loeb product spaces.[24] There is essentially no loss of scientific information if one models a probabilistic phenomenon by a hyperfinite process on a Loeb product space, and virtually no need in scientific modelling to consider processes beyond those which are Loeb product measurable.[25] This is the measurability assumption that we work with, but we nevertheless illustrate the validity of these general methodological remarks by presenting asymptotic versions of our unification and equivalence results.

In conclusion, we emphasize that the work reported here does not simply provide a framework in which epsilons can be rigorously equated to zero,[26] or somewhat less naively, rely on a particular stochastic process on a Loeb, or some other, measure space for which the law of large numbers holds.[27] The inapplicability of such constructions to the problem of asset-pricing hand has already been discussed above in the context of a Lebesgue measure space.[28] Our work belongs to the genre of ideal limit models that illustrate phenomena obscured in the discrete case, and not readily apparent in the large but finite case. One cannot discuss unsystematic risks in the context of a financial market with a finite number of assets, as in the classical CAPM, since each asset occupies a non-negligible portion of the market.[29] On the other hand, it is not clear how to introduce essential risks by simple explicit formulas based on factors into classical treatments of the APT where the well-diversified portfolios can only be defined as limits of finite portfolios. Moreover, in this asymptotic APT setting, the factor structures are not refined enough,[30] and the approximate results, in requiring the sum of an infinite series to be finite, do not give equal treatment to each asset, and are therefore not amenable for use in further constructions. These considerations might partially explain why the most important and relevant portfolios are never explicitly related in the APT literature to the associated factor structures.

The remainder of the paper is organized as follows. In Section 2, we present the mathematical preliminaries and the precise formalizations of common-sensical notions of systematic and unsystematic risks, along with their basic interrelationships. In Section 3, we turn to the equivalence of the valuation formulas of the CAPM and the APT, and develop a unified pricing formula based on the more refined tri-partite decomposition of the unexpected part of the rate of return to an individual asset. In Sections 4, 5 and 6, we turn to portfolios which are particularly important in finance – riskless and factor portfolios, mean and cost portfolios and the mean-variance efficient portfolios – and derive explicit formulas for them. In Section 7, we turn to the asymptotic interpretation of the model, and present approximate versions of the equivalence theorems and the pricing formula. After a concluding Section 8, we present two appendices. The first collects for the reader's convenience relevant results from the theory of hyperfinite processes; in particular, the hyperfinite factor model, as presented in Sun (1994, 1996a), and two of its refined asymptotic versions, as in Sun-Wang (1996). The second Appendix is devoted to the proofs of the results established in the paper; these involve routine Hilbert space manipulations once the basic framework is in place.

2 The Model

We work with Loeb spaces, standard measure spaces constituted by nonstandard entities, and with the Hilbert spaces of square integrable functions based on these Loeb spaces.[31] Our reliance on Hilbert space arguments is limited to projection maps.[32] The reader can make considerable headway towards understanding the framework simply by drawing both on a linear-algebraic and a measure-theoretic intuition to phrase the most basic concepts of asset pricing theories.

Let T be a hyperfinite set, $\mathcal{T}$ the internal power set of T, λ the internal counting probability measure on $(T, \mathcal{T})$, and $(T, L(\mathcal{T}), L(\lambda))$ the standardization[33] of the corresponding internal probability space, the Loeb space. Loeb measure spaces, even though constituted by nonstandard entities, are standard measure spaces in the specific sense that any result proved for an abstract measure space applies to them.[34] We shall use the space $(T, L(\mathcal{T}), L(\lambda))$ to index the assets in a market,[35] and another atomless Loeb space $(\Omega, L(\mathcal{A}), L(P))$ as the sample space. This sample space formalizes all possible uncertain social or natural states relevant to the asset market.

The internal product space $(T \times \Omega, \mathcal{T} \otimes \mathcal{A}, \lambda \otimes P)$ is also an internal probability space. Its standardization, which is to say its corresponding Loeb space, is denoted by $(T \times \Omega, L(\mathcal{T} \otimes \mathcal{A}), L(\lambda \otimes P))$ and referred to as the Loeb product space. As usual in the probabilistic literature,we shall refer to a measurable function of two variables as a process. Given a process g on the Loeb product space, for each $t \in T$, and $\omega \in \Omega$, g_t denotes the function $g(t, \cdot)$ on Ω and g_ω denotes the function $g(\cdot, \omega)$ on T. The functions g_t are usually called the random variables of the process g, while the g_ω form the sample functions of the process.[36]

Besides the Loeb product space, there is another type of product space corresponding to the standard spaces $(T, L(\mathcal{T}), L(\lambda))$ and $(\Omega, L(\mathcal{A}), L(P))$, and this is simply the standard product measure space[37] $(T \times \Omega, L(\mathcal{T}) \otimes L(\mathcal{A}), L(\lambda) \otimes L(P))$. It is an interesting fact that the completion of this standard product measure space, is strictly contained in the Loeb product space $(T \times \Omega, L(\mathcal{T} \otimes \mathcal{A}), L(\lambda \otimes P))$ except in the trivial case that one of the Loeb measures $L(\lambda)$ and $L(P)$ is purely atomic.[38] Since we assume $L(\lambda)$ to be the Loeb counting measure on T, and $L(P)$ to be atomless, the two types of product spaces are never identical. For simplicity, let $\mathcal{U}$ denote the product σ-algebra $L(\mathcal{T}) \otimes L(\mathcal{A})$. For an integrable real-valued process g on the Loeb product space, let $E(g|_{\mathcal{U}})$ denote the conditional expectation[39] of g with respect to $\mathcal{U}$. This conditional expectation is a key operation[40] introduced in Sun (1994) – it is used here to formalize the ensembles of systematic risks and unsystematic risks, and thereby model uncertainty from both macroscopic and microscopic point of view. The parallel is to all those attempts in the economic literature where a discrete or continuous parameter process with low intercorrelation is used to model individual uncertainty, and then the law of large numbers applied to remove this individual uncertainty.[41]

We shall model the financial market by a real-valued $L(\mathcal{T} \otimes \mathcal{A})$-measurable function x on $T \times \Omega$, and thus the real valued random variable x_t defined on $(\Omega, L(\mathcal{A}), L(P))$ is the one-period random return to an asset t in T. In order to use the notion of the variance of the return to any asset, we shall assume that the asset return process x has a finite second moment, and therefore belongs to the Hilbert space $\mathcal{L}^2(L(\lambda \otimes P))$ of real-valued square integrable functions on $L(\lambda \otimes P)$. Thus the square of the norm of x is given by the inner product

$$(x, x) = \int\int_{T \times \Omega} x^2(t, \omega) dL(\lambda \otimes P)(t, \omega) \ < \ \infty. \tag{1}$$

Let μ be the mean function of the random variables embodied in the process x of asset returns,[42] which is to say that $\mu(t) = \int_{\Omega} x(t, \omega) dL(P)(\omega)$ is the expected return of asset $t \in T$. By the Cauchy-Schwarz inequality, it is clear that

$$\int_T \mu^2(t) dL(\lambda) \leq \int\int_{T \times \Omega} x^2(t, \omega) dL(\lambda \otimes P)(t, \omega) < \infty, \tag{2}$$

and hence μ is $L(\lambda)$-square integrable and belongs to the Hilbert space $\mathcal{L}^2(L(\lambda))$. The centered process f, defined by $f(t, \omega) = x(t, \omega) - \mu(t)$, embodies the unexpected or the net random return of all the assets, and is also $L(\lambda \otimes P)$-square integrable.[43]

Since the universe of assets is specified by the Loeb measure space $(T, L(\mathcal{T}), L(\lambda))$, a portfolio is simply a function listing the amounts held of each asset. Since short sales are allowed, this function can take negative values. The cost of each asset is assumed to be unity, and hence the cost of a particular portfolio is simply its integral with respect to $L(\lambda)$. However, since we shall also be interested in the mean and variance of the return realized from a portfolio, we shall assume it to be a square integrable function. The random return from a particular portfolio then depends on the random return and the amounts held in the portfolio for each asset t.[44] Formally,

Definition 1 *A portfolio is a square integrable function p on $(T, L(\mathcal{T}), L(\lambda))$. The cost $C(p)$ of a portfolio p is given by $(p, 1) = \int_T p(t)dL(\lambda)(t)$. The random return of the portfolio p is given by $\mathcal{R}_p(\omega) = (p, x_\omega) = \int_T p(t)x(t,\omega)dL(\lambda)(t)$. The mean (or the expected return) $E(p)$ and the variance $V(p)$ of the portfolio p are the mean and the variance of the random return $\mathcal{R}_p$ respectively.*

Now by the hyperfinite factor model as specified in Theorem A in Appendix I, the process x of returns and the associated process f of net random returns have the following structure: for $L(\lambda \otimes P)$-almost all (t, ω) in $T \times \Omega$,

$$x(t,\omega) - \mu(t) = f(t,\omega) = \sum_{n=1}^{\infty} \lambda_n \psi_n(t)\varphi_n(\omega) + e(t,\omega), \tag{3}$$

where $E(e|\mathcal{U}) = 0$, $E(f|\mathcal{U})(t,\omega) = \sum_{n=1}^{\infty} \lambda_n\psi_n(t)\varphi_n(\omega)$,[45] the factors φ_n and the factor loadings ψ_n are orthonormal,[46] the scaling constants λ_n form a decreasing sequence of positive numbers, and e satisfies the law of large numbers in a strong sense. As remarked earlier, this is simply a hyperfinite version of the classical factor model with finite population, or alternatively, an idealization of the classical principal components model with nonnegligible components φ_1, φ_2, $\cdots$, together with random error terms with low intercorrelation. More to the point, we have a bivariate decomposition of the unexpected rate to any asset into the endogenously derived categories of unsystematic and systematic risks simply by the fact the underlying process of asset returns is square integrable. With the help of the properties as stated in Theorem A in Appendix I, we can substantiate this observation more fully.

Let us refer to a *risk* as any centered random variable defined on the sample space Ω and with a finite variance. As usual, we shall use its variance to measure its *level of risk*.[47] Thus the risk in asset t is simply the net random return f_t of asset t, and the process f is the ensemble of all the risks present in the financial market. By the intuitive definition of an unsystematic risk,[48] it is obvious to give the following formal definition.

Definition 2 *A centered random variable α on the sample space is said to be an unsystematic risk if α has finite variance and is uncorrelated to x_t for $L(\lambda)$-almost all $t \in T$.*

We now appeal to Theorem A in Appendix I to justify the use of the process e as expressing the ensemble of unsystematic risks present in the financial market. Theorem A (3) and (5) in Appendix I imply that for $L(\lambda)$-almost all $t \in T$, e_t is orthogonal to all the $\varphi_n, n \geq 1$, as well as to e_s for $L(\lambda)$-almost all $s \in T$. By ignoring a null set of assets, one can assume for convenience that this observation holds for *all* $t \in T$. With this assumption, it is obvious that for each $t \in T$, e_t is uncorrelated to x_s for $L(\lambda)$-almost all $s \in T$ and hence an unsystematic risk. By Theorem A (4) in Appendix I, the process e also satisfies the law of large numbers in a strong sense. This guarantees the possibility of complete diversification of unsystematic risks. Note that for an arbitrarily given $t \in T$, we cannot claim that e_t is orthogonal to e_s for all $s \in T$, since a continuum of null sets may accumulate to a set of positive measure. It may also happen that some unsystematic risk is uncorrelated with every asset, and thus has no presence at all in the financial market.

For a centered random variable α defined on the sample space Ω, Theorem A (5) in Appendix I implies that for $L(\lambda)$-almost all $t \in T$, $\int_\Omega \alpha f_t dL(P) = \sum_{n=1}^{\infty} \lambda_n \left(\int_\Omega \alpha \varphi_n dL(P)\right) \psi_n(t)$. Hence α is uncorrelated with x_t for $L(\lambda)$-almost all $t \in T$ if and only if α is orthogonal to all the factors φ_n. This means that a risk is unsystematic if and only if it is uncorrelated with all the factors. Thus, any nontrivial random variable α in the linear space $\mathcal{F}$ spanned by all the factors $\varphi_n, n \geq 1$ cannot be an unsystematic risk; and hence α must have correlation with a nonnegligible portion of the asset market. In fact, one simply notes that for each $n \geq 1$,

$$\operatorname{cov}(x_t, \varphi_n) = \int_\Omega f_t(\omega)\varphi_n(\omega) dL(P) = \lambda_n \psi_n(t) \neq 0 \tag{4}$$

holds on a nonnull subset of T. It is then natural to define systematic risks to be the random variables which belong to the space $\mathcal{F}$, and thus the conditional expectation $E(f|\mathcal{U})$ expresses the ensemble of the systematic risks for all assets. We include a formal definition below.

Definition 3 *A centered random variable β with finite variance on the sample space is said to be a systematic risk if β is in the linear space $\mathcal{F}$ spanned by all the factors $\varphi_n, n \geq 1$.*

It is common sense that one can in general divide risks into a systematic portion and an unsystematic portion – the above two exact definitions provide a sharper understanding of this fact. A given risk, which is to say, a centered random variable γ with finite variance, can be additively decomposed into an element β in the endogenously identified space $\mathcal{F}$, and an element α in its orthogonal complement.[49] Thus total risk f_t of asset t is the sum of its unsystematic portion e_t and systematic portion $\sum_{n=1}^{\infty} \lambda_n \psi_n(t) \varphi_n$.

As noted earlier, Theorem A (4) in Appendix I furnishes a strong version of the law of large numbers for the process e. It simply captures in an exact sense the common-sensical notion that unsystematic risks can be completely cancelled through diversification.[50] This kind of cancellation is very important in the usual APT models, and the viability of our approach also hinges crucially on it. For any portfolio p, Theorem A (4) in Appendix I yields

$$\begin{aligned} \mathcal{R}_p(\omega) &= \int_T p(t)\mu(t)dL(\lambda) + \sum_{n=1}^{\infty} \lambda_n \left(\int_T p(t)\psi_n(t)dL(\lambda)\right)\varphi_n(\omega) \\ &= (p,\mu) + \sum_{n=1}^{\infty} \lambda_n (p,\psi_n)\varphi_n(\omega), \end{aligned} \tag{5}$$

and hence by the fact that $\varphi_n,\ n \geq 1$ are orthonormal with means zero,

$$E(p) = (p,\mu) = \int_T p(t)\mu(t)dL(\lambda); V(p) = \sum_{n=1}^{\infty} \lambda_n^2 (p,\psi_n)^2 = \sum_{n=1}^{\infty} \lambda_n^2 \left(\int_T p(t)\psi_n(t)dL(\lambda)\right)^2 \tag{6}$$

In particular, if p is a riskless portfolio, which is to say that $V(p) = 0$, then p is orthogonal to all of the ψ_n.

We conclude the discussion of the model by presenting an optimality property concerning the factors. The result simply says that if one is allowed to use only m sources of risk to measure the systematic behavior of the market, then the best approximation is achieved by taking the first m factors as specified in the hyperfinite factor model.

Proposition 1 *For* $1 \leq i \leq m$, *let* $\mu_i \in \mathbb{R}, a_i \in \mathcal{L}^2(L(\lambda))$ *with* $\int_T a_i^2(t)dL(\lambda) = 1, b_i \in \mathcal{L}^2(L(P))$ *with* $\int_\Omega b_i^2(\omega)dL(P) = 1$. *Then*

$$\int\int_{T\times\Omega} [\sum_{i=1}^{m} \mu_i a_i(t) b_i(\omega) - f(t,\omega)]^2 dL(\lambda\otimes P) \geq \sum_{n=m+1}^{\infty} \lambda_n^2 + \int\int_{T\times\Omega} [f - E(f|\mathcal{U})]^2 dL(\lambda\otimes P),$$

or alternatively,

$$\Delta \equiv \int\int_{T\times\Omega} [\sum_{i=1}^{m} \mu_i a_i(t) b_i(\omega) - E(f|\mathcal{U})]^2 dL(\lambda\otimes P) \geq \sum_{n=m+1}^{\infty} \lambda_n^2.$$

The minimum is achieved at $\mu_i = \lambda_i,\ a_i = \psi_i,\ b_i = \varphi_i$ *for* $1 \leq i \leq m$. *If* λ_m *is an eigenvalue of unit multiplicity, and if the minimum is achieved by* $\sum_{i=1}^{m} \mu_i a_i(t) b_i(\omega) \equiv \beta(t,\omega)$, *then* $\beta(t,\omega) = \sum_{n=1}^{m} \lambda_n \varphi_n(\omega)\psi_n(t)$.

The result also indicates that even though one can use m sources of risk to approximate the ensemble of risks in the market in an optimal way, unsystematic risks remain irrespective of the size of m. On the other hand, since unsystematic risks have no significance from a macroscopic point of view, one can simply choose m so that the square sum of the remaining scaling constants is small enough.

3 The CAPM and the APT: A Unification

We begin with our rendition of Ross' theorem on asset pricing in large asset markets without arbitrage. Two points need re-emphasis in relation to the traditional treatment in the literature. First, we work with an endogenous factor structure. Second, the APT asset pricing formula is exact,[51] but more importantly, the assumption of no arbitrage is also shown to be necessary for its validity. We begin with a precise formulation of the common-sensical notion behind the absence of arbitrage opportunities; namely, a riskless and costless portfolio yields a zero rate of return.

Definition 4 *The market permits no arbitrage opportunities if for any portfolio p, $V(p) = C(p) = 0$ implies $E(p) = 0$.*

We are now ready to state the theorem on the equivalence of the validity of an APT type pricing formula with the economic principle of no arbitrage. The formula simply says that except for a null set of assets, the expected return of an asset is linearly dependent on its factor loadings in an exact way.

Theorem 1 *The market permits no arbitrage opportunities if and only if there is a sequence $\{\tau_n\}_{n=0}^{\infty}$ of real numbers such that for $L(\lambda)$-almost all $t \in T$, $\mu_t = \tau_0 + \sum_{n=1}^{\infty} \tau_n \psi_n(t)$.*

Let μ_s be the orthogonal complement of μ on the subspace spanned by the constant function 1 together with all the factor loadings ψ_n. The following corollary is clear from the above theorem.

Corollary 1 *The market permits no arbitrage opportunities if and only if $\mu_s = 0$.*

The following theorem proves the equivalence of the CAPM and APT pricing formulas without a concrete specification of a CAPM carrier portfolio.

Theorem 2 *The following two conditions are equivalent:*

(i) there is a portfolio M and a real number ρ such that for $L(\lambda)$-almost all $t \in T$, $\mu_t = \rho + cov(x_t, M)$;

(ii) there is a sequence $\{\tau_n\}_{n=0}^{\infty}$ of real numbers such that $\sum_{n=1}^{\infty}(\tau_n^2/\lambda_n^4) < \infty$, and for $L(\lambda)$-almost all $t \in T$, $\mu_t = \tau_0 + \sum_{n=1}^{\infty} \tau_n \psi_n(t)$.

If only finitely many factors are derived from the process x of asset returns, then the summability condition[52] in Theorem 2 (ii) is trivially satisfied. Otherwise, it means that the coefficient τ_n related to the risk premium of the n-th factor must be comparatively much smaller than the associated scaling constant. If one can earn relatively large amount of premium by holding small factor risks, then the risk premium awarding scheme in the market cannot be described by a CAPM type formula.

We develop some notation to translate the coefficients τ_n in Theorems 1 and 2 into directly observed market parameters. We first consider a portfolio h defined by projecting the constant function 1 on the closed subspace orthogonal to that spanned by $[\psi_1(\cdot), \psi_2(\cdot), \cdots]$. For each $n \geq 1$, let $s_n = \int_T \psi_n(t) dL(\lambda)(t) = (1, \psi_n)$, then h is defined by $h(t) = 1 - \sum_{n=1}^{\infty} s_n \psi_n(t)$. Given the orthogonality of h with the $\psi_n(\cdot)$, we obtain

$$(h, h) = \int_T h^2(t) dL(\lambda) = (h, 1) = \int_T h(t) dL(\lambda) = C(h) \equiv h_0. \tag{7}$$

We can now also define a parameter μ_0 by letting

$$\mu_0 = \begin{cases} \frac{\int_T \mu(t)h(t)dL(\lambda)}{\int_T h^2(t)dL(\lambda)} = \int_T \mu(t)h(t)dL(\lambda)(t)/h_0 & \text{if } h \not\equiv 0, \\ 0 & \text{if } h \equiv 0. \end{cases} \tag{8}$$

Finally, for each $n \geq 1$, let $\mu_n = (\mu, \psi_n) = \int_T \mu(t)\psi_n(t) dL(\lambda)$. Then

$$\mu = \mu_0 h + \sum_{n=1}^{\infty} \mu_n \psi_n + \mu_s. \tag{9}$$

Note that μ_n and s_n are also respectively the mean $E(\psi_n)$ and cost $C(\psi_n)$of the portfolio ψ_n. It is also clear that the spaces respectively spanned by $[h, \psi_1(\cdot), \psi_2(\cdot), \cdots]$ and $[1, \psi_1(\cdot), \psi_2(\cdot), \cdots]$ are the same. The following corollary relates the τ_n in Theorem 1 to the s_n and μ_n.

Corollary 2 *Assume that for $L(\lambda)$-almost all $t \in T$, $\mu_t = \tau_0 + \sum_{n=1}^{\infty} \tau_n \psi_n(t)$. Then we have the following:*

(i) if $h \not\equiv 0$, then $\tau_0 = \mu_0$ and $\tau_n = \mu_n - \mu_0 s_n$ for $n \geq 1$;

(ii) if $h \equiv 0$, then $\tau_n = \mu_n - \tau_0 s_n$, where τ_0 could be an arbitrary real number. In particular, one can take $\tau_0 = \mu_0 = 0$ and $\tau_n = \mu_n$ for $n \geq 1$.

The next corollary then characterizes the validity of CAPM linear pricing equation in terms of the market parameters s_n and μ_n.

Corollary 3 *Assume that the asset market permits no arbitrage opportunities. Then there is a portfolio M and a real number ρ such that for $L(\lambda)$-almost all $t \in T$, $\mu_t = \rho +$ cov (x_t, M) if and only if one of the following holds:*

(i) if $h \not\equiv 0$, then $\sum_{n=1}^{\infty}(\mu_n - \mu_0 s_n)^2/\lambda_n^4 < \infty$;

(ii) if $h \equiv 0$, then there exists a real number γ such that $\sum_{n=1}^{\infty}(\mu_n - \gamma s_n)^2/\lambda_n^4 < \infty$.

We now turn to the notion of essential risks and an associated unification theorem. We shall work with a fixed index portfolio I_0 explicitly constructed from the factor loadings in order to measure the risk most relevant to the risk premium of any asset. Towards this end, assume that $\sum_{n=1}^{\infty}(\mu_n - \mu_0 s_n)^2/\lambda_n^4 < \infty$, and let

$$I_0 = \sum_{n=1}^{\infty}((\mu_n - \mu_0 s_n)/\lambda_n^2)\psi_n, \text{ with net random return } X_0 = \sum_{n=1}^{\infty}((\mu_n - \mu_0 s_n)/\lambda_n)\varphi_n. \quad (10)$$

We can check that the cost, mean and variance of I_0 are respectively given by $C(I_0) = \sum_{n=1}^{\infty} s_n(\mu_n - \mu_0 s_n)/\lambda_n^2$, $E(I_0) = \sum_{n=1}^{\infty}\mu_n(\mu_n - \mu_0 s_n)/\lambda_n^2$ and $V(I_0) = \sum_{n=1}^{\infty}(\mu_n - \mu_0 s_n)^2/\lambda_n^2$. We can now use I_0 to present

Definition 5 *A centered random variable defined on the sample space is an essential risk if it is in the one dimensional linear space generated by the net random return X_0 of the index portfolio I_0, and is an inessential risk if it is in the orthogonal complement of X_0 in the space $\mathcal{F}$ of systematic risks.*

By the above definition, it is obvious that a systematic risk can be written as the sum of an essential risk and an inessential risk.

We shall now turn to the pricing formula based on portfolio I_0 and one that only rewards the holding of essential risks. If $V(I_0) \neq 0$, we define the beta β_t of asset t by

$$\beta_t = \frac{\text{cov}(x_t, I_0)}{V(I_0)} = \frac{\sum_{n=1}^{\infty}(\mu_n - \mu_0 s_n)\psi_n(t)}{\sum_{n=1}^{\infty}(\mu_n - \mu_0 s_n)^2/\lambda_n^2}.$$

For each asset t, let

$$Y(t,\omega) = \sum_{n=1}^{\infty}\left(\lambda_n\psi_n(t) - \frac{\beta_t}{\lambda_n}(\mu_n - \mu_0 s_n)\right)\varphi_n(\omega).$$

Then it is easy to check that Y_t is orthogonal to X_0 and the portion of systematic risk $\sum_{n=1}^{\infty}\lambda_n\psi_n(t)\varphi_n$ in the total risk f_t of asset t can be written as the sum of an essential risk $\beta_t X_0$ and an inessential risk Y_t, and hence

$$x(t,\omega) - \mu(t) = \beta_t X_0(\omega) + Y(t,\omega) + e(t,\omega).$$

That is, we can break down the total risk $x_t - \mu_t$ of an asset t into three components: the first involves the projection of the risk onto X_0, the component of essential risk; the second, the projection on the orthogonal complement of X_0 in the space of systematic risks, the component of inessential risk; and the third is the component of residual unsystematic risk. The following theorem shows that as long as there is no arbitrage, the risk premium of almost all assets t only depends on β_t, the level of essential risk held in the asset. More precisely, it is equal to the beta of the asset multiplied by the risk premium of the index portfolio. Note that the equivalence of (i) and (ii) below is simply a restatement of Theorem 1, which is included here to emphasize the unification of CAPM, APT and the no arbitrage condition.

Theorem 3 *Assume that $\sum_{n=1}^{\infty}(\mu_n - \mu_0 s_n)^2/\lambda_n^4 < \infty$ and the market is nontrivial in the sense that the expected return function μ is not the constant function μ_0 essentially. Then the following conditions are equivalent:*

(i) the market permits no arbitrage opportunities;

(ii) there is a sequence $\{\tau_n\}_{n=0}^{\infty}$ of real numbers such that for $L(\lambda)$-almost all $t \in T$, $\mu_t = \tau_0 + \sum_{n=1}^{\infty} \tau_n \psi_n(t)$;

(iii) for $L(\lambda)$-almost all $t \in T$, $\mu_t = \mu_0 + \beta_t (E(I_0) - \mu_0 C(I_0))$.

The importance of the theorem comes from the fact that the market only rewards a risk which is essential and which can never be eliminated through any sort of diversification. If two assets hold different systematic risks but still have the same essential risks, then they must earn the same risk premium. In other words, even though systematic risks are rewarded as indicated by the APT model, the relevant inessential risks among them do not earn a premium. Thus, the fact that APT allows *multiple* sources of industry-wide or market-wide factor risks does not mean the reward scheme for risk taking *must* involve a multiple number of risks; the market, in fact, only rewards the holding of one type of risk as described by X_0, and the risk premia for other types of risk depend on the correlation of these risks with the essential risk X_0.

4 Riskless Portfolios and Factor Portfolios

In this section we offer necessary and sufficient conditions for the existence of a riskless asset, as well as for all of the factors $\varphi_n(\cdot)$ to be portfolios. In the first case, we have to show the existence of a portfolio which is not costless, and whose non-zero return is independent of the state of nature $\omega \in \Omega$. In the second case, we have to find portfolios whose random returns are precisely φ_n. We begin with a formal specification of the notion of a riskless asset.

Definition 6 *A portfolio r is a riskless asset if $V(r) = 0$, $C(r) = 1$ and $E(r) \neq 0$.*

The following theorem characterizes the existence of a riskless asset. Note that this characterization is completely general. We do not even assume that the no arbitrage condition for the market.

Theorem 4 *There is a riskless asset if and only if one of the following holds: h is not null and either (i) $\mu_0 \neq 0$; or (ii) $\mu_0 = 0$, and the market permits arbitrage opportunities.*

In the following corollary, we do assume the no arbitrage condition. Since $h \equiv 0$ implies that $\mu_0 = 0$, the result singles out the importance of the parameter μ_0 for the existence of a riskless asset. It bears comparison with Chamberlain-Rothschild (1983; Proposition 2), but we can give an explicit formula for all the possible portfolios which make up the riskless asset.

Corollary 4 *Assume that there is no arbitrage in the market. Then there is a riskless asset if and only if $\mu_0 \neq 0$. In this case, r is a riskless asset if and only if $r = h/h_0 + r_s$, where r_s is a portfolio orthogonal to h and to all the ψ_n. Moreover, the return of a riskless asset must be μ_0.*

When there is no arbitrage in the market, let us call a portfolio q a dummy portfolio if q is orthogonal to h and to all the ψ_n. So the above corollary simply says that r is a riskless portfolio if and only if r is the sum of h/h_0 plus a dummy portfolio. The following simple lemma gives some conditions for a portfolio to be dummy.

Lemma 1 *Assume that there is no arbitrage in the market.. Let p be a portfolio. Then*

(i) p is a dummy portfolio if and only if $E(p) = C(p) = V(p) = 0$;

(ii) if either $\mu_0 \neq 0$ or $h \equiv 0$, then any portfolio p, with $\mathcal{R}_p \equiv 0$ must be a dummy portfolio.

Next, we characterize those markets which allow all factors to be portfolios. As in Theorem 4, even the no arbitrage condition is not assumed the following theorem. That condition is in fact used as part of the second equivalence condition.

Theorem 5 *All the factors φ_n are portfolios if and only if one of the following holds: (i) $\mu_0 \neq 0$; (ii) $\mu(t) = 0$ for $L(\lambda)$-almost all $t \in T$; or (iii) the market permits arbitrage opportunities.*

The following corollary is a simplification of Theorem 5 in the setting of no arbitrage.

Corollary 5 *Under the assumption of no arbitrage, all the factors are portfolios if and only if either $\mu_0 \neq 0$ or $\mu(t) = 0$ for $L(\lambda)$-almost all $t \in T$.*

If $\mu_0 \neq 0$ and the market permits no arbitrage, then there is a riskless asset and all factors are portfolios. In fact, for any random variable of the form $a_0 + \sum_{n=1}^{\infty} a_n \varphi_n(\omega)$ with $\sum_{n=1}^{\infty} a_n^2/\lambda_n^2 < \infty$, there is a portfolio with this random return.

5 Mean and Cost Portfolios

In the previous section, we looked for portfolios whose associated random returns in the space $\mathcal{L}^2(L(P))$ have a given form. In this section, we look for portfolios whose random returns furnish the mean and cost of the return x_t of a given asset $t \in T$, which is to say, the continuous linear functionals on $\mathcal{L}^2(L(P))$ defined by the random returns of these portfolios give the values μ_t and 1 to the elements x_t, for $L(\lambda)$-almost all $t \in T$. We develop necessary and sufficient conditions for the existence of such portfolios,[53] and then present a necessary and sufficient condition for the linear independence of the two portfolios. These conditions will be essential in our characterization of mean-variance efficient portfolios in the next section.

Definition 7 *A portfolio m is said to be a mean portfolio if the inner product[54] $(\mathcal{R}_m, x_t) = \mu_t$ for $L(\lambda)$-almost all $t \in T$. A portfolio c is said to be a cost portfolio if the inner product $(\mathcal{R}_c, x_t) = 1$ for $L(\lambda)$-almost all $t \in T$.*

The following easy lemma shows that the mean and cost portfolios defined above not only giving the right means and costs for individual assets but also for all the portfolios.[55]

Lemma 2 *If m is a mean portfolio, then for any portfolio p, the expected return $E(p)$ of the portfolio p is given by the inner product $(\mathcal{R}_m, \mathcal{R}_p) = \int_\Omega \mathcal{R}_m \mathcal{R}_p dL(P)$. Analogously, if c is a cost portfolio, then for any portfolio p, the cost $C(p)$ of the portfolio p is the inner product $(\mathcal{R}_c, \mathcal{R}_p) = \int_\Omega \mathcal{R}_c \mathcal{R}_p dL(P)$.*

We shall now characterize the existence of a mean portfolio. The explicit formula for such a portfolio is also given with the proof.

Theorem 6 *If either $\mu_0 \neq 0$ or μ_s is not null, then a mean portfolio exists. On the other hand, if $\mu_0 = 0$, $\mu_s \equiv 0$, and μ is not null, then there is a mean portfolio if and only if $\sum_{n=1}^{\infty} (\mu_n^2/\lambda_n^4) < \infty$.*

The next theorem characterizes the existence of a cost portfolio. As in the previous case, the relevant explicit formulas are also included in the proof.

Theorem 7 *(i) If $\mu_0 \neq 0$, then a cost portfolio exists if and only if the market permits no arbitrage opportunities, and $\sum_{n=1}^{\infty} (\mu_0 s_n - \mu_n)^2 / \lambda_n^4 < \infty$.*

(ii) If $\mu_0 = 0$ and h is not null, then there is no cost portfolio.

(iii) If $h \equiv 0$ and $\mu_s \neq 0$, then a cost portfolio exists if and only if $\sum_{n=1}^{\infty} s_n^2/\lambda_n^4 < \infty$.

(iv) If $h \equiv 0$, $\mu_s \equiv 0$, and a mean portfolio exists, then there is a cost portfolio if and only if $\sum_{n=1}^{\infty} s_n^2/\lambda_n^4 < \infty$.

Next, we turn to the question of uniqueness of the mean and cost portfolios. The following proposition says that if either $\mu_0 \neq 0$ or there is no arbitrage, then the mean and cost portfolios are essentially unique in the sense that different versions of the relevant portfolios must have the same random returns. Note that if there is no arbitrage, and either $\mu_0 \neq 0$ or $h \equiv 0$, then the following proposition and Lemma 1 imply that for $\mathcal{R}_{m_1} \equiv \mathcal{R}_{m_2}$, the difference portfolio $m_2 - m_1$ must be a dummy portfolio, and similar result holds for the cost portfolios.

Proposition 2 *The following statements are equivalent:*

(i) Either $\mu_0 \neq 0$ or $\mu_s \equiv 0$;

(ii) if p is any portfolio p such that the inner product $(\mathcal{R}_p, x_t) = 0$ for $L(\lambda)$-almost all $t \in T$, then $\mathcal{R}_p \equiv 0$;

(iii) $\mathcal{R}_{m_1} \equiv \mathcal{R}_{m_2}$ for all mean portfolios m_1, m_2;

(iv) $\mathcal{R}_{c_1} \equiv \mathcal{R}_{c_2}$ for all cost portfolios c_1, c_2.

In the literature on asset pricing of one time period, it is often assumed that the mean and cost portfolios span a two dimensional space. In the last proposition of this section, we characterize the case when the two are dependent, and hence the characterization for the independent case follows easily.

Proposition 3 *Let m be a mean portfolio and c a cost portfolio. Then the following are equivalent:*

(i) $\mathcal{R}_m$ and $\mathcal{R}_c$ are linearly dependent;

(ii) there is a real number α such that $\mu(t) = \alpha$ for $L(\lambda)$-almost all $t \in T$;

(iii) The following matrix is singular.

$$\begin{pmatrix} (\mathcal{R}_m, \mathcal{R}_m) & (\mathcal{R}_m, \mathcal{R}_c) \\ (\mathcal{R}_c, \mathcal{R}_m) & (\mathcal{R}_c, \mathcal{R}_c) \end{pmatrix} = \begin{pmatrix} E(m) & E(c) \\ E(c) & C(c) \end{pmatrix}.$$

6 Mean-Variance Efficient Portfolios

Throughout this section, we shall assume that there is no arbitrage in the asset market, namely $\mu_s \equiv 0$, and find all mean-variance efficient portfolios. We shall also identify all those efficient portfolios which can be used to measure the risk premium of any asset through a CAPM type equation. In addition, explicit formulas for the measurement of the trade-off between return and risk are furnished for all of the relevant situations.

Definition 8 *A portfolio p is said to be mean-variance efficient if, for some real numbers a and b, it is a solution of the following optimization problem*

$$\min V(p) \ \textit{ subject to } \ E(p) = a, \ C(p) = b.$$

The following simple lemma relates the market portfolio M used in the CAPM linear pricing equation to mean-variance efficient portfolios. Note that such a result is well known when mean and cost portfolios exist.[56] Our lemma covers the case even when such portfolios do not exist. However, we first need to specify the following simple concept.

Definition 9 *The asset market is said to be trivial if the expected return function $\mu(t)$ is a constant $L(\lambda)$-almost t in T.*

Lemma 3 *Let M be a portfolio, and ρ and α some real numbers such that a CAPM linear pricing equation holds, i.e., for $L(\lambda)$-almost all $t \in T$, $\mu_t = \rho + \alpha\ cov(x_t, M)$. If $\alpha \neq 0$, then M is mean-variance efficient. In particular, it is mean-variance efficient when the market is not trivial.*

By Proposition 3 in the last section, if mean and cost portfolios exist, then the market is trivial if and only if the two portfolios are linearly dependent. Here we also note that unlike Theorems 2 and 3 in Section 3, a coefficient α is inserted in the CAPM type equation in Lemma 3. One simply observes that the α can be moved together with the portfolio M to form a new portfolio. Thus for the existence problem of a portfolio with the relevant property, the α can be dropped without loss of generality. The reason to include α here is to have a sort of linearity among all the portfolios M which satisfies a CAPM type equation.

One can have nice interpretations for the coefficients ρ and α in the CAPM linear pricing equation. The interpretation in the following corollary presents the usual form of CAPM equation and returns to the discussion in the first paragraph of this paper, where the portfolio M is in fact assumed to be of cost 1. In the following corollary, if we choose M to have $C(M) = 1$, then we obtain exactly the same CAPM equation as in Section 1.

Corollary 6 *Let M be a portfolio, and ρ and α some real numbers such that for $L(\lambda)$-almost all $t \in T$, $\mu_t = \rho + \alpha\ cov(x_t, M)$. Assume $V(M) \neq 0$. Then $\alpha = (E(M) - \rho C(M))/V(M)$. Denote $\beta_t = cov(x_t, M)/V(M)$, the beta of the asset t. Then $\mu_t = \rho + \beta_t (E(M) - \rho C(M))$ for $L(\lambda)$-almost all $t \in T$. Moreover, for any portfolio p, we have $E(p) = \rho C(p) + \beta_p (E(M) - \rho C(M))$, where β_p is $cov(p, M)/V(M)$, the beta of the portfolio p. Furthermore the following statements hold.*

(i) If there is a riskless asset r, then ρ is the return μ_0 of the asset r, and thus $\mu_t - \mu_0 = \beta_t (E(M) - \mu_0 C(M))$ for $L(\lambda)$-almost all $t \in T$.

(ii) If there is no riskless asset, then ρ is the expected return of a zero beta portfolio with a unit cost.

As indicated in Lemma 3, the portfolio M in CAPM linear pricing equation is in general mean-variance efficient. To characterize portfolios satisfying the equation, we introduce the following definition.

Definition 10 *A mean-variance efficient portfolio p is said to be a CAPM carrier if there are real numbers ρ and α such that for $L(\lambda)$-almost all $t \in T$, $\mu_t = \rho + \alpha\ cov(x_t, p)$.*

The following simple lemma is useful for our explicit expressions of mean-variance efficient portfolios. Part (i) of the lemma is well known and part (ii) is also essentially known. We include the lemma for the sake of completeness.

Lemma 4 *Assume that mean and cost portfolios m and c exist and that they are linearly independent. Then the following statements are valid.*

(i) A portfolio q is mean-variance efficient if and only if $\mathcal{R}_q$ is in the linear span of $\mathcal{R}_m$ and $\mathcal{R}_c$.

(ii) The portfolio p where

$$p(t) = \frac{(aC(c) - bE(c))\, m(t) + (bE(m) - aE(c))\, c(t)}{E(m)C(c) - (E(c))^2},$$

is mean-variance efficient with mean a and cost b.

In order to exhibit the trade-off between mean and variance, we consider the situation when a trader tries to maximize the return for each additional unit of risk taken by him by formulating costless portfolios.

Definition 11 *Let* $\delta = \sup |E(p)|/V^{1/2}(p)$ *subject to* $C(p) = 0$ *and* $V(p) \neq 0$.

In the remainder of this section, we characterize mean-variance efficient portfolios, provide explicit formulas for the associated δ, and also present necessary and sufficient conditions for mean-variance efficient portfolios to be CAPM carriers. Throughout we work under the standing hypothesis of the *no arbitrage* condition, but consider various cases depending on whether μ_0 is, or is not, equal to zero. Since μ_0 can equal zero even when $h \not\equiv 0$, we focus separately on the cases when h is, or is not, the null function. The first three propositions relates to asset markets which are not trivial, but the last proposition also cover the other case.

We first consider the case $\mu_0 \neq 0$. The cost portfolio is also implicitly assumed to exist.[57] In this case, μ_0 is the riskless interest rate as noted in Corollary 3 and Theorem 6 also shows that a mean portfolio exists. In addition, Proposition 3 indicates that the mean and cost portfolios are independent since the market is not trivial.

Proposition 4 *Let a and b be some real numbers. Assume that* $\mu_0 \neq 0$, $\mu_s \equiv 0$, *and that the asset market is not trivial and cost portfolios exist. Then the portfolio p defined by*

$$p(t) = \frac{\left[a\left(1 + \sum_{n=1}^{\infty} \frac{(\mu_0 s_n - \mu_n)^2}{\lambda_n^2}\right) - b\mu_0\right] m(t) + \left(b\mu_0^2 - a\mu_0\right) c(t)}{\sum_{n=1}^{\infty} \frac{(\mu_0 s_n - \mu_n)^2}{\lambda_n^2}}$$

is mean-variance efficient with mean a and cost b, where $m(t) = h(t)/(\mu_0 h_0)$ *is a mean portfolio, and* $c(t) = c_0 h(t) + \sum_{n=1}^{\infty} c_n \psi_n(t)$ *is a cost portfolio with*

$$c_0 = \frac{(1/\mu_0) - \sum_{n=1}^{\infty} c_n \mu_n}{\mu_0 h_0}, \quad \text{and} \quad c_n = \frac{\mu_0 s_n - \mu_n}{\mu_0 \lambda_n^2} \quad \text{for all } n \geq 1.$$

Moreover, if q is any mean-variance efficient portfolio with mean a and cost b, then q is the sum of p and a dummy portfolio q_s.

By taking the portfolios described in the previous proposition, we can simply check when p is a CAPM carrier. The following corollary is then simple to establish. Here we note that by Corollary 2, the existence of a CAPM carrier is equivalent to $\sum_{n=1}^{\infty}(\mu_0 s_n - \mu_n)^2/\lambda_n^4 < \infty$, and hence the assumption on the existence of a cost portfolio is also necessary for the existence of a CAPM carrier in the case $\mu_0 \neq 0$.

Corollary 7 *Under the hypotheses of Proposition 4, a mean-variance efficient portfolio p is a CAPM carrier if and only if $E(p) \neq \mu_0 C(p)$, which is also equivalent to $V(p) \neq 0$.*

Note that if the trade-off parameter δ is achieved by some portfolio, then the portfolio must be mean-variance efficient. It is easy to obtain the following corollary by computing the variance of the portfolio p in Proposition 4 for the case $b = 0$.

Corollary 8 *Under the hypotheses of Proposition 4, we have*

$$\delta = \sqrt{\sum_{n=1}^{\infty}\left(\frac{\mu_0 s_n - \mu_n}{\lambda_n}\right)^2}.$$

Now we consider the case $h \equiv 0$. We also assume that both mean and cost portfolios exist.[58] The following proposition presents explicit formulas for all mean-variance efficient portfolios. Note that the relevant formulas are more complicated than those in Proposition 4. This is partially due to the fact that there is no riskless asset in the market and thus the mean portfolio given here is already quite complicated.

Proposition 5 *Let a and b be some real numbers. Assume that $h \equiv 0$, $\mu_s \equiv 0$, and that the asset market is not trivial and both mean and cost portfolios exist. Then the portfolio p defined by*

$$p(t) = \frac{\left(a\left[\left(1+\sum_{n=1}^{\infty}\frac{\mu_n^2}{\lambda_n^2}\right)\sum_{n=1}^{\infty}\frac{s_n^2}{\lambda_n^2} - \left(\sum_{n=1}^{\infty}\frac{\mu_n s_n}{\lambda_n^2}\right)^2\right] - b\sum_{n=1}^{\infty}\frac{\mu_n s_n}{\lambda_n^2}\right)m(t)}{\sum_{n=1}^{\infty}\frac{\mu_n^2}{\lambda_n^2}\sum_{n=1}^{\infty}\frac{s_n^2}{\lambda_n^2} - \left(\sum_{n=1}^{\infty}\frac{\mu_n s_n}{\lambda_n^2}\right)^2} + \frac{\left(b\sum_{n=1}^{\infty}\frac{\mu_n^2}{\lambda_n^2} - a\sum_{n=1}^{\infty}\frac{\mu_n s_n}{\lambda_n^2}\right)c(t)}{\sum_{n=1}^{\infty}\frac{\mu_n^2}{\lambda_n^2}\sum_{n=1}^{\infty}\frac{s_n^2}{\lambda_n^2} - \left(\sum_{n=1}^{\infty}\frac{\mu_n s_n}{\lambda_n^2}\right)^2}$$

is mean-variance efficient with mean a and cost b, where $m(t) = \sum_{n=1}^{\infty} m_n \psi_n(t)$ and $c(t) = \sum_{n=1}^{\infty} c_n \psi_n(t)$ are the mean and cost portfolios respectively with

$$m_n = \left(1 + \sum_{n=1}^{\infty}\frac{\mu_n^2}{\lambda_n^2}\right)^{-1}\frac{\mu_n}{\lambda_n^2},\ c_n = \frac{s_n - \gamma\mu_n}{\lambda_n^2} \text{ and } \gamma = \frac{\sum_{n=1}^{\infty}(\mu_n s_n / \lambda_n^2)}{1+\sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^2)}.$$

Moreover, if q is any mean-variance efficient portfolio with mean a and cost b, then q is the sum of p and a dummy portfolio q_s.

As in Corollaries 7 and 8, Corollary 9 characterizes the CAPM carrier portfolios and Corollary 10 provides an exact formula for the trade-off parameter δ. Both corollaries are straightforward to prove.

Corollary 9 *Under the hypotheses of Proposition 5, a mean variance efficient portfolio p is a CAPM carrier if and only if $E(p) \neq C(p)\frac{\tau}{\beta}$, where $\alpha = \sum_{n=1}^{\infty} \mu_n^2/\lambda_n^2$, $\beta = \sum_{n=1}^{\infty} s_n^2/\lambda_n^2$, and $\tau = \sum_{n=1}^{\infty} \mu_n s_n/\lambda_n^2$.*

Corollary 10 *Under the hypotheses of Proposition 5, we can obtain*

$$\delta = \sqrt{\frac{\sum_{n=1}^{\infty} \frac{\mu_n^2}{\lambda_n^2} \sum_{n=1}^{\infty} \frac{s_n^2}{\lambda_n^2} - \left(\sum_{n=1}^{\infty} \frac{\mu_n s_n}{\lambda_n^2}\right)^2}{\sum_{n=1}^{\infty} \frac{s_n^2}{\lambda_n^2}}}.$$

Next we consider a situation where cost portfolios do not exist. However, we can still find mean-variance efficient portfolios with any given means and costs. Note that we assume a mean portfolio exists. By Theorem 6, this existence is in fact equivalent to $\sum_{n=1}^{\infty} \mu_n^2/\lambda_n^4 < \infty$. We first prove a version of Lemma 4.

Lemma 5 *Assume that $\mu_0 = 0$, $h \not\equiv 0$, $\mu_s \equiv 0$, $\mu \not\equiv 0$, and there is a mean portfolio m. Then a portfolio q is mean-variance efficient if and only if $\mathcal{R}_q$ is a multiple of $\mathcal{R}_m$.*

The following proposition characterizes all the mean-variance portfolios in this setting.

Proposition 6 *Let a and b be some real numbers. Assume that $\mu_0 = 0$, $h \not\equiv 0$, $\mu_s \equiv 0$, $\mu \not\equiv 0$, and that a mean portfolio exists. Then the portfolio p defined by*

$$p(t) = \left[\frac{b}{h_0} - \frac{a \sum_{n=1}^{\infty} \frac{\mu_n s_n}{\lambda_n^2}}{h_0 \sum_{n=1}^{\infty} (\mu_n^2/\lambda_n^2)}\right] h(t) + a \frac{\sum_{n=1}^{\infty} \frac{\mu_n}{\lambda_n^2} \psi_n(t)}{\sum_{n=1}^{\infty} (\mu_n^2/\lambda_n^2)}$$

is mean-variance efficient with mean a and cost b. Moreover, if q is any mean-variance efficient portfolio with mean a and cost b, then q is the sum of p and a dummy portfolio q_s.

Let q be a mean variance efficient portfolio with mean a and cost b in the situation considered in Proposition 6. Then it is easy to check that

$$\mathrm{cov}(x_t, q) = \frac{a\mu(t)}{\sum_{n=1}^{\infty} \mu_n/\lambda_n^2}.$$

Hence the following corollary is clear.

Corollary 11 *Under the hypotheses of Proposition 6, a mean-variance efficient portfolio q is a CAPM carrier if and only if $E(q) \neq 0$, which is equivalent to $V(q) \neq 0$.*

Next, let p be a mean-variance efficient portfolio with mean a and any cost b as in Proposition 6. Then it is easy to check that $V(p) = a^2 / \sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^2)$, and hence we can obtain a formula for δ as follows.

Corollary 12 *Under the hypotheses of Proposition 6, we obtain $\delta = \sqrt{\sum_{n=1}^{\infty} \frac{\mu_n^2}{\lambda_n^2}}$.*

Finally, we move to the case that the market is trivial. By Proposition 3, the mean and cost portfolios are linearly dependent if they exist. In this case, there is no meaningful trade-off between mean and variance. Taking more risk without spending more will not lead to a higher return. The expected return is simply linearly dependent on the cost with a fixed coefficient. Thus, it is not true that for every pair of given amount of return and cost, there is a mean-variance efficient portfolio corresponding to it. The following proposition describes all the relevant mean-variance efficient portfolios. In part (ii) of the proposition, we explicitly assume convergence of the series $\sum_{n=1}^{\infty} \frac{s_n^2}{\lambda_n^4}$, which implies the existence of a cost portfolio by Theorem 7 (iv).

Proposition 7 *Let b be a real number. Assume that the asset market is trivial, with the expected return function μ being equal to a constant α.*

(i) If $h \not\equiv 0$, then the variance of the portfolio p defined by $p(t) = b\, h(t)/h_0$ is zero with mean αb and cost b, and hence also mean-variance efficient.

(ii) If $h \equiv 0$ and $\sum_{n=1}^{\infty} \frac{s_n^2}{\lambda_n^4} < \infty$, then the portfolio p defined by

$$p(t) = b \frac{\sum_{n=1}^{\infty} \frac{s_n}{\lambda_n^2} \psi_n(t)}{\sum_{n=1}^{\infty} \frac{s_n^2}{\lambda_n^2}}$$

is mean-variance efficient with mean αb and cost b.

Moreover, if q is any mean-variance efficient portfolio with cost b, then q is the sum of the portfolio p in (i) or (ii), and a dummy portfolio q_s.

Note that when the asset market is trivial, it is clear that every mean-variance efficient portfolio is a CAPM carrier. One can simply take the coefficient in front of the covariance in Definition 10 to be zero.[59] As noted earlier, there is no meaningful trade-off between mean and variance for a trivial market. However, we can define a trade-off between cost and variance.

Definition 12 *Let $\delta_c = \inf V^{1/2}(p)/C(p)$ subject to $C(p) \neq 0$.*

Note that δ_c measures the minimum risk for each additional cost. As in the previous cases, we can obtain the following simple corollary, whose proof will be omitted.

Corollary 13 *Assume the market is trivial. Then (i) if $h \not\equiv 0$, then $\delta_c = 0$; (ii) if $h \equiv 0$ and $\sum_{n=1}^{\infty} \frac{s_n^2}{\lambda_n^4} < \infty$, then $\delta = 1/\left(\sqrt{\sum_{n=1}^{\infty}(s_n^2/\lambda_n^2)}\right)$.*

7 CAPM, APT and No Arbitrage: Asymptotics

We have already referred in the introduction to the property of the nonstandard extension whereby a result for the idealized nonstandard model can be translated into a standard asymptotic one for a large but finite setting. This metatheorem notwithstanding, one does get insight into the idealized limit case by translating the results (and their proofs) into the asymptotic setting. The motivation behind such an exercise has by necessity to be illustrative – it would be tedious to translate each result, with each of its associated formulas, into an approximate discrete setting of the asset or sample space or both.[60] We limit ourselves here to the case where the sample space is an arbitrary probability space and kept fixed for an increasing sequence of finite asset markets. We present asymptotic versions of the results of Section 3. The formulation of the results presented here rely heavily on Sun-Wang (1996), much as those of Section 3 relied on Sun (1994, 1996a). The proofs make no reference to nonstandard analysis.[61]

We consider a sequence of markets $\mathcal{M}_n$, $n = 1, 2, \cdots$, where in each market $\mathcal{M}_n$, there are n assets indexed by the set $T_n = \{1, 2, \cdots, n\}$, and endowed with the uniform probability measure λ_n on T_n. Each asset t in the n-th market has unit cost and a one-period random return x_{nt}, a real valued random variable on a fixed common probability space $(\Omega, \mathcal{A}, P)$. We shall make the mild assumption of uniform boundedness of the average of the second moment of the random returns in the sequence of markets. Formally, we assume that there exists a positive number M such that for the sequence of random variables $x_{nt}, 1 \leq t \leq n$,

$$\frac{1}{n}\sum_{t=1}^{n}\int_{\Omega} x_{nt}^2 dP \leq M \text{ for all } n = 1, 2, \cdots. \tag{11}$$

For notational simplicity, we shall regard x_n as a process on the product space $(T_n \times \Omega, \mathcal{T}_n \otimes \mathcal{A}, \lambda_n \otimes P)$, where $\mathcal{T}_n$ is the power set on T_n. We shall often use the notation of integration on T_n instead of summation in order to emphasize that the results presented here are asymptotic interpretations of those presented above for the idealized limit model. Note that the collection $\{x_n\}_{n=1}^{\infty}$ is also called a triangular array of random variables.

A portfolio p_n in the n-th market is simply a vector in $\mathbb{R}^n$, but we can also regard it as a function on T_n. The cost $C(p_n)$ of p_n is simply $\int_{T_n} p_n(t)d\lambda_n$. The random return[62] $\mathcal{R}_{p_n}(\omega)$ of the portfolio p_n is $\int_{T_n} p_n(t)x_{nt}(\omega)d\lambda_n(t)$. Thus the return $E(p_n)$ and the variance $V(p_n)$ of the portfolio are respectively the mean and variance of the random return $\mathcal{R}_{p_n}(\omega)$. For a random variable α on Ω, we use $\|\alpha\|_2$ to denote the square root of its second moment, i.e., $\|\alpha\|_2 = \left(\int_\Omega \alpha^2 dP\right)^{1/2}$. For a portfolio p_n on T_n, we also use $\|p_n\|_2 = \left(\int_{T_n} p_n^2 d\lambda_n\right)^{1/2}$. For any sequence $\{p_n\}_{n=1}^{\infty}$, p_n defined on T_n, let $\|p\|_2$ denote $\sup\{\|p_n\|_2 : n \geq 1\}$. We shall only work with those sequences $\{p_n\}_{n=1}^{\infty}$ of portfolios such that $\|p\|_2$ is finite.[63] We can now present the first substantive concept of this section.

Definition 13 *We say the sequence of markets permits no asymptotic arbitrage if for any sequence $\{p_n\}_{n=1}^{\infty}$ of portfolios, $\lim_{n\to\infty} C(p_n) = \lim_{n\to\infty} V(p_n) = 0$ always implies that* $\lim_{n\to\infty} E(p_n) = 0$.

For the t-th asset in the n-th market $\mathcal{M}_n$, let $\mu_n(t)$ be its expected return $\mu_n(t) = \int_\Omega x_{nt}(\omega)dP$, and f_n be the centered process $x_n - \mu_n$ on $(T_n \times \Omega, \mathcal{T}_n \otimes \mathcal{A}, \lambda_n \otimes P)$ embodying the unexpected portion of these returns. In order to develop an asymptotic analogue of Theorems 1-3, an asymptotic analogue of the hyperfinite factor model as presented in Theorem A in Appendix I is needed. Theorem B in Appendix I presents a version of the asymptotic endogenous factor model from Sun-Wang (1996). It shows that the triangular array of random variables $\{f_n\}_{n=1}^{\infty}$ has the following structure: for any given $\varepsilon > 0$,

$$x_n(t,\omega) - \mu_n(t) = f_n(t,\omega) = \sum_{i=1}^{\ell_n(\varepsilon)} \lambda_{ni}\psi_{ni}(t)\varphi_{ni}(\omega) + e_n(\varepsilon)(t,\omega), \qquad (12)$$

where for each $n \geq 1$, the factors φ_{ni} and the factor loadings ψ_{ni} are orthonormal, the scaling constants λ_{ni} form a decreasing finite sequence of positive numbers, and $e_n(\varepsilon)$ satisfies the law of large numbers approximately. In addition, for any fixed $\varepsilon > 0$, the number $\ell_n(\varepsilon)$ of factors for each market has an upper bound independent of n. In analogy with the idealized case above, and as in the literature, the principal part formalizes systematic or factor risks, and the residual part $e_n(\varepsilon)$ unsystematic risks. Note that even though the residual term is more general than what is typically assumed, it has the desired property that it will be "almost cancelled" when general linear combinations are taken.[64] This is a version of an approximate law of large numbers, or alternatively, an asymptotic version of the informal statement that "aggregation removes individual uncertainty".

Before presenting the asymptotic analogue of Theorem 1, we introduce some notation. For $\varepsilon > 0$ and $n \geq 1$, let μ_{ns} be the orthogonal complement of μ_n on the subspace spanned by $\{1, \psi_{n1}, \cdots, \psi_{n\ell_n(\varepsilon)}\}$ and h_n be the orthogonal complement of the constant function 1 on the linear subspace of $L_2(\lambda_n)$ spanned by $\{\psi_{n1}, \cdots, \psi_{n\ell_n(\varepsilon)}\}$. Then μ_n can be written as

$$\mu_n(t) = \mu_{n0} h_n(t) + \sum_{i=1}^{\ell_n(\varepsilon)} \mu_{ni} \psi_{ni}(t) + \mu_{ns},$$

where μ_{n0} is taken to be zero if $h_n = 0$. The next proposition shows an equivalence, for large enough asset markets, of asymptotic APT linear pricing and the absence of asymptotic arbitrage.

Proposition 8 *The following two statements are equivalent:*
(1) The sequence of markets permits no asymptotic arbitrage.
(2) For any $\eta > 0$, there is $\varepsilon > 0$ such that for all $n > 1/\varepsilon$,

(i) $\|\mu_{ns}\|_2 < \eta$;

(ii) if $\|h_n\|_2 < \varepsilon$, *then* $\|\mu_{n0} h_n\|_2 < \eta$.

For any degree of approximation η, there exists a positive number ε such that in any market with the number of assets $n > 1/\varepsilon$, one can identify from the process of asset returns, a set of $\ell_n(\varepsilon)$ factors such that an APT pricing formula based on these factors is valid within the given degree of approximation. Note that the type of condition in Proposition 8 (2) (ii) does not arise in the ideal setting as in Theorem 1. In Theorem 1, if $h \equiv 0$, then one automatically has $\mu_0 h \equiv 0$. However, in the sequential setting, we have to make sure that when $\|h_n\|_2$ is small, $\|\mu_{n0} h_n\|_2$ must also be small.[65] Otherwise, one can construct a sequence of portfolios based on h_n to create arbitrage opportunities in the asymptotic sense. If the statement and proof of the asymptotic version of Theorem 1 in the above proposition are compared with that of Theorem 1 itself, it is clear that the asymptotic case is much more involved. This brings out how the messy approximations of the asymptotic case are effectively and elegantly handled in the idealized framework used in Theorem 1.

Proposition 8 depends on a given level of error η and the relevant ε. It will also be useful to consider an asymptotic context in which, rather than working with a prior given level of error, the effect of the error term tends to zero as the number of assets increase. It is important that in such a setting the number of factors be significantly smaller than the number of assets. Theorem C in Appendix I presents another version of the asymptotic endogenous factor model from Sun-Wang (1996) in which one can choose any unbounded nondecreasing sequence $\{m_n\}_{n=1}^{\infty}$ of positive integers as the upper bounds for the numbers of factors at appropriate stages. It shows that the sequence of random processes $\{x_n\}_{n=1}^{\infty}$ has the following structure:

$$x_n(t,\omega) = \mu_n(t) + f_n(t,\omega) = \mu_n(t) + \sum_{i=1}^{s_n} \lambda_{ni}\psi_{ni}(t)\varphi_{ni}(\omega) + e_n(t,\omega), \tag{13}$$

where, as before, for each $n \geq 1$, the factors φ_{ni} and the factor loadings ψ_{ni} are orthonormal, the scaling constants λ_{ni} form a decreasing finite sequence of positive numbers, and e_n satisfies the law of large numbers in a strong sense.[66] We can now present an asymptotic analogue of Theorem 2; namely, the equivalence of the asymptotic linear pricing formulas of CAPM and APT.

Proposition 9 *The following two conditions are equivalent:*

(i) there is a sequence of portfolios $M_n, n \geq 1$, and and a sequence of real numbers $\rho_n, n \geq 1$, such that $\lim_{n\to\infty} \|\mu_n(t) - (\rho_n + cov(x_n(t), M_n))\|_2 = 0$;

(ii) there is a sequence $\{\tau_{ni}\}_{i=0}^{s_n}$ of real numbers and a positive number B such that for all $n \geq 1$, $\sum_{i=1}^{s_n}(\tau_{ni}^2/\lambda_{ni}^4) \leq B$ and $\lim_{n\to\infty} \|\mu_n - (\tau_{n0} + \sum_{i=1}^{s_n} \tau_{ni}\psi_{ni})\|_2 = 0$.

All that remains is the identification of the index portfolios which capture essential risks for financial markets with large but finite number of assets. We have already referred to the riskless rate of return μ_{n0} for the epsilon-delta context of Proposition 8; we can also compute it and the parameters μ_{ni}, s_{ni} for the set-up of Proposition 9 by projecting the rate of return and unit functions on the space of the s_n factors extracted for market $\mathcal{M}_n$. In particular, for each $n \geq 1$, let h_n be the orthogonal complement of the constant function 1 on the linear subspace of $L_2(\lambda_n)$ spanned by $\{\psi_{n1}, \cdots, \psi_{ns_n}\}$ and $\mu_{n0}h_n$ the projection of μ_n on h_n (here, if $h_n = 0$, then μ_{n0} is also taken to be 0); in addition, for each $1 \leq i \leq s_n$, $\mu_{ni} = \int_{T_n} \mu_n \psi_{ni} d\lambda_n$ and $s_{ni} = \int_{T_n} \psi_{ni} d\lambda_n$.

Definition 14 *Assume that there is a positive number B such that for all $n \geq 1$, $\sum_{i=1}^{s_n}(\mu_{ni} - \mu_{n0}s_{ni})^2/\lambda_{ni}^4 < B$. We call the portfolios I_n index portfolios where for all $t \in T_n$,*

$$I_n(t) = \sum_{i=1}^{s_n} \frac{\mu_{ni} - \mu_{n0}s_{ni}}{\lambda_{ni}^2} \psi_{ni}(t).$$

We can now present the following asymptotic analogue of Theorem 3; namely, a pricing formula for which rewards only the essential risk embodied in the rate of return to a particular asset.

Proposition 10 *Assume that there is a positive number B such that for all $n \geq 1$, $\sum_{i=1}^{s_n}(\mu_{ni} - \mu_{n0}s_{ni})^2/\lambda_{ni}^4 < B$. Then the following conditions are equivalent:*

(i) the sequence of markets permits no asymptotic arbitrage;

(ii) $\lim_{n\to\infty} \|\mu_n(t) - (\mu_{n0} + \sum_{i=1}^{s_n}(\mu_{ni} - \mu_{n0}s_{ni})\psi_{ni})\|_2 = 0$*;*

(iii) $\lim_{n\to\infty} \|\mu_n(t) - (\mu_{n0} + \ cov(x_n(t), I_n))\|_2 = 0$*.*

Note that no asymptotic arbitrage, together with the assumption $\sum_{i=1}^{s_n}(\mu_{ni} - \mu_{n0}s_{ni})^2/\lambda_{ni}^4 < B$ for all $n \geq 1$, imply the boundedness of the sequence $\mu_{n0}, n \geq 1$. In fact, one can use the same proof to show the boundedness of the sequence $\rho_n, n \geq 1$ in Proposition 9. However, it is not at all clear whether the condition of no asymptotic arbitrage alone is sufficient to show such boundedness. This is the rationale for condition (ii) in Proposition 8(2).

8 Conclusion

We conclude the paper by outlining its marginal contribution. The principal point to note about the hyperfinite asset pricing theory is of course the unification of the CAPM and the APT that it elaborates. This unification spans both the conceptual and the technical dimensions: conceptual in that two sets of risks, systematic/unsystematic and essential/inessential, can be discussed within the same framework; and technical in that the pricing formula of one theory implies, and is implied by, that of the other, and that both are essentially equivalent to the absence of arbitrage opportunities. However, the important point to stress is that we can build on this unification to identify an index portfolio whose normalized covariance with an asset's expected return measures that component of its unexpected return that is rewarded in the market. Thus, as in the CAPM, investors need only know the particular beta of an asset, but one computed on the basis of the index portfolio constructed on the basis of multiple sources of market-wide risks embodied in the factors, as in the APT. However, in our case, the factors are themselves extracted from the process of asset returns, and they, along with their loadings and scaling coefficients, can be used not only to compute the index portfolio, but also a variety of other important portfolios, including, most importantly, mean-variance efficient portfolios. Thus, it appears to us highly plausible that the only obstacles to the concrete practical implementation of the hyperfinite asset pricing theory reported above may lie in informational and computational limitations.

9 Appendix I: Law of Large Numbers, Hyperfinite and Asymptotic Endogenous Factor Models

We now collect for the reader's convenience relevant results from Sun (1994, 1996a) and Sun-Wang (1996). The first theorem is a hyperfinite version of the classical factor models with finite population, and is referred to as a hyperfinite model (see Corollary 3.8 in Sun (1994)); it is part of Theorems 1-3 in Sun (1996a) and also Theorems 3.2, 3.5, and Corollary 3.3 in Sun (1994; Section 3.2).

Theorem A *Let f be a real valued square integrable centered process on the Loeb product space $(T \times \Omega, L(\mathcal{T} \otimes \mathcal{A}), L(\lambda \otimes P))$. Then f has the following expression*

$$f(t,\omega) = \sum_{n=1}^{\infty} \lambda_n \psi_n(t)\varphi_n(\omega) + e(t,\omega),$$

with properties:
(1) $\lambda_n, 1 \leq n < \infty$ is a decreasing sequence of positive numbers; the collection $\{\psi_n : 1 \leq n < \infty\}$ is orthonormal; and $\{\varphi_n : 1 \leq n < \infty\}$ is a collection of orthonormal and centered random variables.
(2) $E(f|_{\mathcal{U}})(t,\omega) = \sum_{n=1}^{\infty} \lambda_n \psi_n(t)\varphi_n(\omega)$ and $E(e|_{\mathcal{U}}) = 0$.
(3) The random variables e_t are almost surely orthogonal, which is to say that for $L(\lambda \otimes \lambda)$-almost all $(t_1, t_2) \in T \times T$, $\int_\Omega e_{t_1}(\omega)e_{t_2}(\omega)dL(P)(\omega) = 0$.
(4) If p is a square integrable real-valued function on $(T, L(\mathcal{T}), L(\lambda))$, then for $L(P)$-almost all $\omega \in \Omega$, $\int_T p(t)e_\omega(t)L(\lambda) = 0$, and

$$\int_T p(t)f(t,\omega)dL(\lambda)(t) = \sum_{n=1}^{\infty} \lambda_n \left(\int_T p(t)\psi_n(t)dL(\lambda)(t)\right) \varphi_n(\omega).$$

(5) If α is a square integrable random variable on $(\Omega, L(\mathcal{A}), L(P))$, then for $L(\lambda)$-almost all $t \in T$, it is orthogonal to e_t, and

$$\int_\Omega \alpha(\omega)f(t,\omega)dL(P)(\omega) = \sum_{n=1}^{\infty} \lambda_n \left(\int_\Omega \alpha(\omega)\varphi_n(\omega)dL(P)(\omega)\right) \psi_n(t).$$

Let $R(t_1,t_2) = \int_\Omega f(t_1,\omega)f(t_2,\omega)dL(P)$ be the associated autocorrelation function of the process f. By serving as a kernel, the autocorrelation function R defines an integral operator K on the space $\mathcal{L}^2(L(\lambda))$ of square integrable functions on $(T, L(\mathcal{T}), L(\lambda))$. That is, $K(h)(t_1) = \int_T R(t_1,t_2)h(t_2)dL(\lambda)(t_2)$ for $h \in \mathcal{L}^2(L(\lambda))$. Then it is easy to check that λ_n^2 is in fact the n-th positive eigenvalue of the operator K with eigenfunction ψ_n, with all the eigenvalues listed according to reverse order and repeated up to their corresponding multiplicities. Note that if there are only m positive eigenvalues, then the infinite sum in Theorem A should be read as a finite sum of m terms. It is also clear that $\varphi_n(\omega) = (1/\lambda_n)\int_T f_\omega(t)\psi_n(t)dL(\lambda)$. Theorem A (1) and (2) simply say that the conditional expectation $E(f|_{\mathcal{U}})$ has a sort of biorthogonal expansion in which both the random variables φ_n and the functions ψ_n are orthogonal among themselves. The corresponding continuous analogue for processes which are continuous in quadratic means on an interval is often called the Karhunen-Loéve expansion theorem and is also well-known. Theorem A (3) says that the residual process e has low intercorrelation, which is a requirement for the error term in the classical factor models. Thus it is natural to adopt the terminologies of Factor Analysis. The centered random variables φ_n are called factors, the corresponding functions ψ_n factor loadings, and the decreasing sequence of numbers λ_n scaling constants since the size of λ_n measures the role of the factor φ_n in understanding the correlational structure of f. The structural result in Theorem A is then naturally called a hyperfinite factor model. Since the factors are endogenously derived from the autocorrelation function, the model is also called a hyperfinite endogenous factor model. The first part of Theorem A (4) is a strong version of the law of large numbers for the process e. It says that as long as p is square integrable over T, almost all sample functions of the process $p(t)x(t,\omega)$ have zero means. As noted earlier, this law of large numbers, which allows complete diversification of unsystematic risks, together with the Karhunen-Loéve type biorthogonal expansion are crucial for the development of the results presented in this paper. The later part of Theorem A (4) and Theorem A (5) say that when integrals are concerned, one can simply ignore the error terms and focus on the factors.

Next, we move to an analogue of Theorem A in the asymptotic setting, i.e., an asymptotic endogenous factor model. For $n \geq 1$, consider the set $T_n = \{1, 2, \cdots, n\}$, endowed with the uniform probability measure λ_n on T_n. Let $(\Omega, \mathcal{A}, P)$ be a fixed common probability space. For each $n \geq 1$, let f_n be a process on $(T_n \times \Omega, \mathcal{T}_n \otimes \mathcal{A}, \lambda_n \times P)$ which describes some probabilistic model; then the collection of all f_n is called a triangular array of random variables describing a sequence of probabilistic models. Let $\|f_n\|_2$ denote the square root of the integral of f_n^2 on the product space; for a function p_n on T_n, and a random variable α on Ω, $\|p_n\|_2$ and $\|\alpha\|_2$ have similar meanings. The following result presents a version of the asymptotic endogenous factor model with $\varepsilon > 0$ denoting a given level of error. The result is taken from Sun-Wang (1996; Theorem 1), but note that some of the ε's here correspond to ε^3 there.

Theorem B *Let $f_n, n \geq 1$, be a triangular array of centered random variables on $(T_n \times \Omega, \mathcal{T}_n \otimes \mathcal{A}, \lambda_n \times P)$, $n \geq 1$. Assume that there is a positive real number M such that $\|f_n\|_2 \leq M$ for all n. Fix any $\varepsilon > 0$. Then for each $n \geq 1$, f_n has the following expression*

$$f_n(t,\omega) = \sum_{i=1}^{\ell_n(\varepsilon)} \lambda_{ni}\psi_{ni}(t)\varphi_{ni}(\omega) + e_n(\varepsilon)(t,\omega)$$

with properties:
(1) $\ell_n(\cdot)$ is a nonincreasing function from $(0,\infty)$ into the set of natural numbers with $\ell_n(\varepsilon) \leq M^4/\varepsilon^2$, and $\lambda_{n\ell_n(\varepsilon)} \geq \varepsilon/M$.
(2) The collection $\{\psi_{ni} : 1 \leq i \leq \ell_n(\varepsilon)\}$ is orthonormal, and $\{\varphi_{ni} : 1 \leq i \leq \ell_n(\varepsilon)\}$ is a collection of orthonormal and centered random variables.
(3) Let p_n be a function on T_n; then $\int_T p_n(t)e_n(\varepsilon)(t,\omega)d\lambda_n$ is centered, orthogonal to φ_{ni} for all $1 \leq i \leq \ell_n(\varepsilon)$, and its variance is dominated by $\varepsilon\|p_n\|_2^2$, i.e.,

$$\left\| \int_T p_n(t)e_n(\varepsilon)(t,\omega)d\lambda_n \right\|_2^2 \leq \varepsilon\|p_n\|_2^2.$$

As usual, for the n-th probabilistic model, φ_{ni} are called the factors, ψ_{ni} the factor loadings, λ_{ni} the scaling constants. When the prior error ε is given, the number of factors in the n-th probabilistic model has an upper bound depending on ε only. Thus, as the number n of random variables in the n-th probabilistic model goes to infinity, the number of factors is indeed comparatively small. This is consistent with the primary objective of factor analysis, i.e., to reduce the study of a large number of random variables through a relatively small number of latent random variables. Such kind of reduction is made possible by Theorem B (3) which guarantees that the contribution from error term $e_n(\varepsilon)$ will be small when general linear combinations are taken, and thus one can just focus on the factors.

Finally, we include another version of an asymptotic endogenous factor model from Sun-Wang (1996; Theorem 2). Unlike the previous case, no prior level of error is needed here. One simply choose any unbounded nondecreasing sequence $\{m_n\}_{n=1}^{\infty}$ of positive integers as the upper bounds for the number of factors at appropriate stages. As before, the φ_{ni}, ψ_{ni}, and λ_{ni} are respectively called the factors, factor loadings, scaling constants. In the setting of Theorem C, one can certainly choose the sequence $\{m_n\}_{n=1}^{\infty}$

such that the ratio between the number of factors and the number of observed random variables goes to zero, i.e., $\lim_{n\to\infty} s_n/n = 0$. Note that the inequality in Theorem B (3) becomes a limit in Theorem C (3).

Theorem C *Under the hypotheses of Theorem B, let $\{m_n\}_{n=1}^{\infty}$ be any unbounded nondecreasing sequence of positive integers. Then for each $n \geq 1$, there is a natural number $s_n \leq m_n$ such that*

$$f_n(t,\omega) = \sum_{i=1}^{s_n} \lambda_{ni}\psi_{ni}(t)\varphi_{ni}(\omega) + e_n(t,\omega)$$

with the following properties:
(1) $\lambda_{ni}, 1 \leq i \leq s_n$ is a decreasing sequence of positive numbers.
(2) The collection $\{\psi_{ni} : 1 \leq i \leq s_n\}$ is orthonormal, and $\{\varphi_{ni} : 1 \leq i \leq s_n\}$ is a collection of orthonormal and centered random variables.
(3) Let p_n be a function on T_n; then $\int_T p_n(t)e_n(t,\omega)d\lambda_n$ is centered and orthogonal to φ_{ni} for all $1 \leq i \leq s_n$; moreover, if for all $n \geq 1$, $\|p_n\|_2 \leq \|p\|_2 < \infty$, then the sequence of the variances of the linear combinations $\int_T p_n(t)e_n(t,\omega)d\lambda_n$ converges to zero, i.e.,

$$\lim_{n\to\infty} \left\| \int_T p_n(t)e_n(t,\omega)d\lambda_n \right\|_2 = 0.$$

To conclude Appendix I, we emphasize again that unlike classical factor models with finite population or their asymptotic generalizations in the literature, where factors and error terms are exogenously assumed, the asymptotic factor models considered in Theorems B and C derive factors and error terms endogenously from the uniform boundedness of the second moments of the f_n.

10 Appendix II: Proofs of Results

The proof of Proposition 1 hinges on the projections of the functions $a_i(\cdot)$ and $b_i(\cdot)$ on suitable subspaces of the corresponding Hilbert spaces.

Proof of Proposition 1: We begin the proof by deriving the second expression from the first. Towards this end, note from Loéve (1977 b; p. 16) that $E(g|_{\mathcal{U}}) = 0$ for a fixed $g \in \mathcal{L}^2(L(\lambda \otimes P))$ implies $E(gh|_{\mathcal{U}}) = 0$ for any $h \in \mathcal{L}^2(L(\lambda \otimes P))$, and therefore that $\int\int_{T\times\Omega} g.h dL(\lambda \otimes P) = 0$. Hence

$$\int\int_{T\times\Omega} (g+h)^2 dL(\lambda\otimes P) = \int\int_{T\times\Omega} g^2 dL(\lambda\otimes P) + \int\int_{T\times\Omega} h^2 dL(\lambda\otimes P).$$

¿From the equality $E[(f - f|_{\mathcal{U}})|_{\mathcal{U}}] = 0$, we obtain

$$\int\int_{T\times\Omega} [\sum_{i=1}^{m} \mu_i a_i(t) b_i(\omega) - f(t,\omega)]^2 dL(\lambda\otimes P) =$$
$$\int\int_{T\times\Omega} [\sum_{i=1}^{m} \mu_i a_i(t) b_i(\omega) - E(f|_{\mathcal{U}})]^2 dL(\lambda\otimes P) + \int\int_{T\times\Omega} [f - E(f|_{\mathcal{U}})]^2 dL(\lambda\otimes P).$$

Next, we show in three steps that $\Delta \geq \sum_{n=m+1}^{\infty} \lambda_n^2$.
Step 1: For each $1 \leq i \leq m$, let

$$b_i(\omega) = b_i'(\omega) + b_i''(\omega),$$

where b'_i is the projection of b_i on the space spanned by the φ_n. Thus, for each i, b''_i is orthogonal to all the φ_n. Hence

$$\begin{aligned}
\Delta &= \int\int_{T\times\Omega}\left[\sum_{i=1}^{m}\mu_i a_i(t)b'_i(\omega) - \sum_{n=1}^{\infty}\lambda_n\varphi_n(\omega)\psi_n(t) + \sum_{i=1}^{m}\mu_i a_i(t)b''_i(\omega)\right]^2 dL(\lambda\otimes P) \\
&= \int_T\left(\int_\Omega\left[\sum_{i=1}^{m}\mu_i a_i(t)b'_i(\omega) - \sum_{n=1}^{\infty}\lambda_n\varphi_n(\omega)\psi_n(t)\right]^2 dL(P)\right. \\
&\qquad \left. + \int_\Omega\left[\sum_{i=1}^{m}\mu_i a_i(t)b''_i(\omega)\right]^2 dL(P)\right) dL(\lambda) \\
&\geq \int_T\int_\Omega\left[\sum_{i=1}^{m}\mu_i a_i(t)b'_i(\omega) - E(f|\mu)\right]^2 dL(P)dL(\lambda).
\end{aligned}$$

Step 2: Take an orthonormal basis $\{d_1,\ldots,d_q\}$ for the space $\mathrm{sp}[b'_1,\cdots,b'_m]$. Note that the b'_i are not assumed to be linearly independent and hence $q\leq m$. Hence we can write

$$\sum_{i=1}^{m}\mu_i a_i(t)b'_i(\omega) = \sum_{i=1}^{q}\nu_i c_i(t)d_i(\omega),$$

with $\{d_1,\ldots,d_q\}$ orthonormal, $\int_T c_i^2(t)dL(\lambda) = 1$, $\nu_i\in\mathbb{R}$ for $1\leq i\leq q$. Since the d_i are in the space spanned by the φ_n, we have

$$d_i = \sum_{n=1}^{\infty}(d_i,\varphi_j)\varphi_j(\omega) \text{ where } (d_i,\varphi_j) = \int_\Omega d_i(\omega)\varphi_j(\omega)dL(\lambda).$$

Hence we obtain

$$\begin{aligned}
\Delta &\geq \int_T\int_\Omega\left[\sum_{i=1}^{q}\nu_i c_i(t)\sum_{j=1}^{\infty}(d_i,\varphi_j)\varphi_j(\omega) - \sum_{n=1}^{\infty}\lambda_j\varphi_j(\omega)\psi_j(t)\right]^2 dL(P)dL(\lambda) \\
&= \int_T\int_\Omega\left[\sum_{j=1}^{\infty}\left(\sum_{i=1}^{q}\nu_i(d_i,\varphi_j)c_i(t) - \lambda_j\psi_j(t)\right)\varphi_j(\omega)\right] dL(P)dL(\lambda) \\
&= \int_T\int_\Omega\sum_{j=1}^{\infty}\left(\sum_{i=1}^{q}\nu_i(d_i,\varphi_j)c_i(t) - \lambda_j\psi_j(t)\right)^2 dL(\lambda) \\
&= \sum_{j=1}^{\infty}\lambda_j^2\int_T\left(\psi_j(t) - \sum_{i=1}^{q}\frac{\nu_i(d_i,\varphi_j)}{\lambda_j}c_i(t)\right)^2 dL(\lambda).
\end{aligned}$$

Step 3: Take an orthonormal basis $v_1,\cdots,v_p$ for the space $\mathrm{sp}[c_1,\cdots,c_q]$. Then $p\leq q$, and we can write

$$\sum_{i=1}^{q}\frac{\nu_i(d_i,\varphi_j)}{\lambda_j}c_i(t) = \sum_{i=1}^{p}\alpha_i^j v_i(t).$$

Now, note that

$$\int_T\left(\psi_j(t) - \sum_{i=1}^{q}\alpha_i^j v_i(t)\right)^2 dL(\lambda) \;\geq\; \int_T\left(\psi_j(t) - \sum_{i=1}^{p}(\psi_j,v_i)v_i(t)\right)^2 dL(\lambda)$$

$$= \int_T \psi_j^2(t) - \sum_{i=1}^{p}(\psi_j, v_i)^2 \int_T v_i(t)^2 dL(\lambda)$$
$$= 1 - \sum_{i=1}^{p}(\psi_j, v_i)^2.$$

Hence we obtain

$$\Delta \geq \sum_{j=1}^{\infty}\lambda_j^2\left[1-\sum_{i=1}^{p}(\psi_j, v_i)^2\right]$$
$$= \sum_{j=1}^{p}\lambda_j^2\left[1-\sum_{i=1}^{p}(\psi_j, v_i)^2\right] + \sum_{j=p+1}^{\infty}\lambda_j^2\left[1-\sum_{i=1}^{p}(\psi_j, v_i)^2\right]. \quad (14)$$

Since $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \geq \lambda_{n+1} \geq \cdots$, we obtain

$$\Delta \geq \sum_{j=1}^{p}\lambda_p^2\left[1-\sum_{i=1}^{p}(\psi_j, v_i)^2\right] + \sum_{j=p+1}^{\infty}\lambda_j^2\left[1-\sum_{j=1}^{p}(\psi_j, v_i)^2\right] \quad (15)$$
$$= \sum_{j=p+1}^{\infty}\lambda_j^2 + \lambda_p^2\sum_{i=1}^{p}\left[1-\sum_{j=1}^{p}(\psi_j, v_i)^2\right] - \sum_{j=p+1}^{\infty}\lambda_j^2\sum_{i=1}^{p}(\psi_j, v_i)^2$$
$$\geq \sum_{j=p+1}^{\infty}\lambda_j^2 + \lambda_p^2\sum_{i=1}^{p}\left[1-\sum_{j=1}^{p}(\psi_j, v_i)^2\right] - \sum_{j=p+1}^{\infty}\lambda_p^2\sum_{i=1}^{p}(\psi_j, v_i)^2 \quad (16)$$
$$= \sum_{j=p+1}^{\infty}\lambda_j^2 + \lambda_p^2\sum_{i=1}^{p}\left[1-\sum_{j=1}^{\infty}(\psi_j, v_i)^2\right].$$

Note that $\sum_{j=1}^{\infty} < v_i, \psi_j > \psi_j$ is the projection of v_i on the space spanned by the ψ_n. Hence, $1 = \int_T v_i^2(t)dL(\lambda) \geq \sum_{j=1}^{\infty}(\psi_j, v_i)^2$. This implies that $1 - \sum_{j=1}^{\infty}(\psi_j, v_i)^2 \geq 0$ for $1 \leq i \leq p$. Since $p \leq m$, we obtain $\Delta \geq \sum_{j=p+1}^{\infty}\lambda_j^2 \geq \sum_{j=m+1}^{\infty}\lambda_j^2$. The proof of the assertion is complete, and we turn to the uniqueness claim.

Note the fact that the minimum is achieved implies that all of the inequalities involved must become equalities. In particular

$$\Delta = \sum_{j=m+1}^{\infty}\lambda_p^2 \text{ if and only if } p = m \text{ and hence } q = m.$$

Since $\lambda_{m-1} > \lambda_m > \lambda_{m+1}$, we obtain from (12) that

$$\sum_{i=1}^{m}(\psi_j, v_i)^2 = 0 \text{ for } j \geq m+1. \quad (17)$$

Note that $p = m$, and $\lambda_j < \lambda_m$, for $j \geq m+1$. Furthermore, we obtain

$$\int_T v_i^2(t)dL(\lambda) = \sum_{j=1}^{\infty} < \psi_j, v_i >^2 \text{ for } 1 \leq i \leq m. \quad (18)$$

(14) and (15) imply that each v_i, $1 \le i \le m$, is in the space $\mathrm{sp}[\psi_1, \cdots, \psi_m]$. From (12) and (13), we obtain

$$1 = \int_T \psi_j^2(t)dL(\lambda) = \sum_{i=1}^{m}(\psi_j, v_i)^2 \text{ for } 1 \le j \le m-1 \tag{19}$$

since for such a j, $\lambda_j > \lambda_m$. By (15) and (16), $1 = \sum_{j=1}^{m}(\psi_j, v_i)^2$ for $1 \le i \le m$, which implies that

$$\begin{aligned} m &= \sum_{i=1}^{m}\sum_{j=1}^{\infty}(\psi_j, v_i)^2 \\ &= \sum_{i=1}^{m}(\psi_m, v_i)^2 + \sum_{j=1}^{m-1}\sum_{i=1}^{m}(\psi_j, v_i)^2 \\ &= \sum_{i=1}^{m}(\psi_m, v_i)^2 + \sum_{j=1}^{m-1} 1 \qquad \text{by (11)} \\ &= \sum_{i=1}^{m}(\psi_m, v_i)^2 + m - 1. \end{aligned}$$

This implies that

$$1 = \sum_{i=1}^{m} < \psi_m, v_i >^2 = \int_T \psi_m^2(t)dL(\lambda). \tag{20}$$

(16) and (17) imply that each of $\psi_1, \cdots, \psi_m$ is in the space $\mathrm{sp}[v_1, \cdots, v_m]$. Hence $\psi_1, \cdots, \psi_m$, span the same space as $v_1, \cdots, v_m$, and hence $c_1, \cdots, c_m$, and hence $a_1, \cdots, a_m$. Note that the c_i are in the space spanned by $a_1, \cdots, a_m$. Hence there is a nonsingular $m \times m$ matrix A such that $(a_1, \cdots, a_m) = (\psi_1, \cdots, \psi_m)A$. The function

$$\beta(t, \omega) = (a_1, \cdots, a_m)\begin{pmatrix} \mu_1 b_1 \\ \vdots \\ \mu_m b_m \end{pmatrix} = (\psi_1, \cdots, \psi_m)A\begin{pmatrix} \mu_1 b_1 \\ \vdots \\ \mu_m b_m \end{pmatrix}.$$

Now let

$$A\begin{pmatrix} \mu_1 b_1 \\ \vdots \\ \mu_m b_m \end{pmatrix} = \begin{pmatrix} \tau_1 \eta_1(\omega) \\ \vdots \\ \tau_m \eta_m(\omega) \end{pmatrix},$$

where $\int_\Omega \eta_i^2(\omega)dL(P) = 1$. Then

$$\beta(t, \omega) = \sum_{i=1}^{m} \tau_i \psi_i(t)\eta_i(\omega).$$

Since the following integral is minimized,

$$\begin{aligned} \int\int_{T\times\Omega}\left(\sum_{n=1}^{\infty}\lambda_n\varphi_n(\omega)\psi_n(t) - \sum_{n=1}^{m}\tau_n\eta_n(\omega)\psi_n(t)\right)^2 dL(\lambda\otimes P) &= \sum_{n=m+1}^{\infty}\lambda_n^2 \\ = \int_T\int_\Omega\left(\sum_{n=1}^{m}(\lambda_n\varphi_n(\omega) - \tau_n\eta_n(\omega))\psi_n(t) + \sum_{n=m+1}^{\infty}\lambda_n\varphi_n(\omega)\psi_n(t)\right)^2 dL(P)dL(\lambda) \\ = \int_\Omega\left(\sum_{n=1}^{m}(\lambda_n\varphi_n(\omega) - \tau_n\eta_n(\omega))^2 + \sum_{n=m+1}^{\infty}\lambda_n^2\varphi_n^2(\omega)\right)dL(P) \end{aligned}$$

$$= \sum_{n=1}^{m} \int_{\Omega} (\lambda_n \varphi_n(\omega) - \tau_n \eta_n(\omega))^2 dL(P) + \sum_{n=m+1}^{\infty} \lambda_n^2. \tag{21}$$

(18) follows by virtue of the φ_n being orthonormal. Hence we obtain $\lambda_n \varphi_n(\omega) = \tau_n \eta_n(\omega)$ for $1 \leq n \leq m$. Therefore $\beta(t,\omega) = \sum_{n=1}^{m} \lambda_n \varphi_n(\omega) \psi_i(t)$, which is to say that the minimum is achieved at a unique function. ∎

Next, we turn to the proof of Theorem 1. To show that asymptotic no arbitrage implies an APT linear equation, Huberman (1982) uses the projection of the expected return function onto the closed subspace generated by the factor loadings ψ_n together with the constant function 1, where a relevant orthogonal vector to become arbitrary small as the number of assets becomes arbitrary large. Here we use a similar idea to show the necessity part of Theorem 1; however, the analogue of the relevant vector has to be identically zero in this setting. Such kind of exact equality leads to the discovery that the no arbitrage condition is also implied by the relevant APT linear equation. The details are as follows.

Proof of Theorem 1: We begin with necessity. For an arbitrary portfolio p, let p_r be the projection of p on the closed subspace spanned by the constant function 1 and all the ψ_n. Denote $p_s = p - p_r$. Since $\mu \in \mathcal{L}^2(L(\lambda))$ from (2) above, we can also project it on the same closed subspace, and define μ_r and μ_s accordingly. If p is costless and riskless, then it is clear from Definition 1 and (6) above that p is orthogonal to 1 and to all of the ψ_n. This implies that $p_r = 0$. In this case, we obtain that

$$E(p) = \int_T p_s(t)\mu(t) dL(\lambda)(t) = \int_T p_s(t)\mu_s(t) dL(\lambda)(t).$$

Thus, no arbitrage means that $\int_T p_s(t)\mu_s(t)dL(\lambda)(t) = 0$ for any p_s, and in particular, it is true when $p_s = \mu_s$. Hence, we obtain $\int_T \mu_s^2(t) dL(\lambda) = 0$, and thus $\mu_s(t) = 0$ for $L(\lambda)$-almost all $t \in T$. By the definition of μ_r, there are real numbers $\{\tau_n\}_{n=0}^{\infty}$ such that $\mu(t) = \mu_r(t) = \tau_0 + \sum_{n=1}^{\infty} \tau_n \psi_n(t)$ for $L(\lambda)$-almost all $t \in T$.

On the other hand, for the sufficiency part of the claim, the validity of the arbitrage pricing formula clearly implies $\mu_s = 0$, which also furnishes us the no arbitrage condition. ∎

Proof of Theorem 2: For $(i) \Longrightarrow (ii)$, let $M(t) = M_0 h(t) + \sum_{n=1}^{\infty} M_n \psi_n(t) + M_s(t)$ such that M_s is orthogonal to h and all the ψ_n. Then by (5), $\mathcal{R}_M - E(M) = \sum_{n=1}^{\infty} \lambda_n M_n \varphi_n$. Since M is square integrable, Bessel's inequality (Rudin (1974; p. 88)) guarantees that $\sum_{n=1}^{\infty} M_n^2 \leq (M, M) < \infty$. It is also clear that $\mathcal{R}_M - E(M)$ is square integrable, and thus we can then appeal to Theorem A (5) to assert that for $L(\lambda)$-almost all $t \in T$,

$$\text{cov}(x_t, M) = \int_{\Omega} (x_t - \mu_t)(\omega)\, (\mathcal{R}_M(\omega) - E(M))\, dL(P) = \sum_{n=1}^{\infty} \lambda_n^2 M_n \psi_n(t).$$

Hence it follows from the CAPM linear pricing equation in (i) that

$$\mu_t = \rho + \sum_{n=1}^{\infty} \left(\lambda_n^2 M_n\right) \psi_n(t).$$

By letting $\tau_0 = \rho$ and $\tau_n = \lambda_n^2 M_n$ for $n \geq 1$, we obtain the APT linear pricing equation. Since $\sum_{n=1}^{\infty} M_n^2 < \infty$, we have $\sum_{n=1}^{\infty} (\tau_n^2/\lambda_n^4) < \infty$.

Next, we consider $(ii) \Longrightarrow (i)$. Assume that the APT linear pricing equation in (ii) holds, and $\sum_{n=1}^{\infty} (\tau_n^2/\lambda_n^4) < \infty$. Define a portfolio M as follows:

$$M(t) = \sum_{n=1}^{\infty} \left(\frac{\tau_n}{\lambda_n^2}\right) \psi_n(t).$$

By $\sum_{n=1}^{\infty}(\tau_n^2/\lambda_n^4) < \infty$, we know that M is $L(\lambda)$-square integrable, and hence a well defined portfolio in our setting. By (5), it is easy to see that the difference of $\mathcal{R}_M$ with its mean is $\sum_{n=1}^{\infty}\left(\lambda_n \frac{\tau_n}{\lambda_n^2}\right)\varphi_n(\omega)$. Thus, by Theorem A (5), we obtain that for $L(\lambda)$-almost all $t \in T$,

$$\begin{aligned}\mathrm{cov}(x_t, M) &= \int_{\Omega}(x_t - \mu_t)(\omega)(\mathcal{R}_M(\omega) - E(M))dL(P) \\ &= \sum_{n=1}^{\infty}(\lambda_n\psi_n(t))\left(\frac{\tau_n}{\lambda_n}\right) = \sum_{n=1}^{\infty}\tau_n\psi_n(t).\end{aligned}$$

Let $\rho = \tau_0$. Hence, by the APT linear pricing equation in (ii), we obtain $\mu_t = \rho + \mathrm{cov}(x_t, M)$ for $L(\lambda)$-almost all $t \in T$, which is to say that (i) holds. ∎

Proof of Corollary 2: By the assumption, we have for $L(\lambda)$-almost all $t \in T$, $\mu_t = \tau_0 + \sum_{n=1}^{\infty}\tau_n\psi_n(t)$, and hence by substituting $1 = h + \sum_{n=1}^{\infty}s_n\psi_n$, we derive $\mu = \tau_0 h + \sum_{n=1}^{\infty}(\tau_n + \tau_0 s_n)\psi_n$. By Equation (9) and the fact that $\mu_s = 0$, we obtain that $\tau_0 h = \mu_0 h$, and $\tau_n + \tau_0 s_n = \mu_n$ for all $n \geq 1$. For (i), we simply note that $h \not\equiv 0$ implies that $\tau_0 = \mu_0$, and thus $\tau_n = \mu_n - \mu_0 s_n$ for all $n \geq 1$. For (ii), we only have to worry whether $\tau_n + \tau_0 s_n = \mu_n$ for all $n \geq 1$, since $\tau_0 h = \mu_0 h = 0$ is always satisfied in this case no matter what τ_0 is. Therefore, one can take $\tau_n = \mu_n - \tau_0 s_n$ for all $n \geq 1$ with τ_0 being a variable. ∎

Proof of Corollary 3: Since there is no arbitrage, Corollary 2 implies that $\mu = \mu_0 + \sum_{n=1}^{\infty}(\mu_n - \mu_0 s_n)\psi_n$. If $h \not\equiv 0$, then such a representation is unique as shown in Corollary 2 (i), and hence we can appeal to the equivalence of (i) and (ii) in Theorem 2, to obtain the desired equivalence. Thus (i) is shown.

For (ii), assume $h \equiv 0$. By Theorem 2, there is a portfolio M and a real number ρ such that for $L(\lambda)$-almost all $t \in T$, $\mu_t = \rho + \mathrm{cov}\,(x_t, M)$ if and only if there is a sequence $\{\tau_n\}_{n=0}^{\infty}$ of real numbers such that $\sum_{n=1}^{\infty}(\tau_n^2/\lambda_n^4) < \infty$, and for $L(\lambda)$-almost all $t \in T$, $\mu(t) = \tau_0 + \sum_{n=1}^{\infty}\tau_n\psi_n(t)$. On the other hand, if for $L(\lambda)$-almost all $t \in T$, $\mu(t) = \tau_0 + \sum_{n=1}^{\infty}\tau_n\psi_n(t)$, then Corollary 2 (ii) implies that $\tau_n = \mu_n - \tau_0 s_n$. Therefore, the equivalence follows by taking τ_0 to be some number γ. ∎

Proof of Theorem 3: The equivalence of (t) and (ii) is already shown in Theorem 1. To show the rest, note that the relevant formulas for the cost, return and variance of the index portfolio I_0 imply that $V(I_0) = E(I_0) - \mu_0 C(I_0)$, and hence by the definition of β_t, we obtain

$$\mu_0 + \beta_t\,(E(I_0) - \mu_0 C(I_0)) = \mu_0 + \sum_{n=1}^{\infty}(\mu_n - \mu_0 s_n)\psi_n(t).$$

If (ii) holds, then Corollary 2 implies that for $L(\lambda)$-almost all $t \in T$, $\mu(t) = \mu_0 + \sum_{n=1}^{\infty}(\mu_n - \mu_0 s_n)\psi_n(t)$, and hence $\mu(t) = \mu_0 + \beta_t\,(E(I_0) - \mu_0 C(I_0))$ for $L(\lambda)$-almost all $t \in T$, i.e., (iii) holds.

On the other hand, if we assume (iii), then the computation in the first paragraph implies that (ii) holds. Therefore all the three statements are equivalent. ∎

Proof of Theorem 4: If $\mu_0 \neq 0$, let $r(t) = h(t)/h_0$. Then it is easy to check using Equations (5), (7) and (9) that $C(r) = 1$ and the random return $\mathcal{R}_r \equiv \mu_0$. Thus $V(r) = 0$, $E(r) = \mu_0 \neq 0$, and hence r is a riskless asset. Next, we consider the second case when $\mu_0 = 0$, and both h and μ_s are not null functions, the latter through an appeal to Corollary 1. Let $r'(t) = h(t)/h_0 + \mu_s(t)$. Then by Equations (5), (7) and (9) again, we obtain $C(r') = 1$ and $\mathcal{R}_{r'} \equiv \int_T \mu_s^2(t)dL(\lambda)$. Since μ_s is not null, $E(r') > 0$, and hence r' is a riskless asset.

Now, we assume that the market has a riskless asset r''. On projecting $r''(t)$ to the subspace $\mathrm{sp}[h, \psi_1(t), \psi_2(t), \cdots]$, we obtain

$$r''(t) = r_0 h(t) + \sum_{n=1}^{\infty} r_n \psi_n(t) + r_s(t),$$

where $r_s(t)$ is orthogonal to $\mathrm{sp}[h, \psi_1(t), \psi_2(t), \cdots]$. Then by Equation 5,

$$\mathcal{R}_{r''} = \int_T r''(t)\mu(t)dL(\lambda) + \sum_{n=1}^{\infty} r_n \lambda_n \varphi_n(\omega).$$

Since $V(r'') = 0$, we obtain $r_n = 0$ for all $n \geq 1$. This fact together with Equation (9) imply that

$$\begin{aligned}
\mathcal{R}_{r''} &= \int_T r''(t)\mu(t)dL(\lambda) = r_0 \int_T h(t)\mu(t)dL(\lambda) + \int_T r_s(t)\mu(t)dL(\lambda) \\
&= r_0\mu_0 \int_T h^2(t)dL(\lambda) + \int_T r_s(t)\mu_s(t)dL(\lambda) \\
&= r_0\mu_0 h_0 + \int_T r_s(t)\mu_s(t)dL(\lambda)
\end{aligned}$$

Furthermore,

$$C(r'') = \int_T r''(t)dL(\lambda) = r_0 h_0.$$

Since we require that $C(r'') = 1$ we must have h nonnull. On the other hand, $E(r'')$ is assumed to be nonnull. Hence we cannot have both $\mu_0 = 0$ and $\mu_s \equiv 0$. On combining these conditions together, we obtain either condition (i) or condition (ii). ■

Proof of Corollary 4: The first result is clear from Theorem 4. Then we assume $\mu_0 \neq 0$. Let r be a portfolio with the expansion

$$r(t) = r_0 h(t) + \sum_{n=1}^{\infty} r_n \psi_n(t) + r_s(t),$$

as the portfolio r" in the proof of Theorem 4. If r is a riskless asset, then the computation in the proof of Theorem 4 suggests that $C(r) = r_0 h_0 = 1$ and $r_n = 0$ for all $n \geq 1$, and hence $r = h/h_0 + r_s$, where r_s is orthogonal to h and all the ψ_n. The rest is clear. ■

Proof of Lemma 1: The necessity part of (i) is clear. To prove the sufficiency part of (i), assume that $E(p) = C(p) = V(p) = 0$. By Equation (6), p is orthogonal to all the ψ_n. Since $C(p) = 0$, p is also orthogonal to the constant function 1, and hence p is dummy.

For (ii), project p to the closed subspace $[h, \psi_1, \psi_2, \cdots]$, to obtain

$$p = p_0 h + \sum_{n=1}^{\infty} p_n \psi_n + p_s.$$

By Equation (5), $\mathcal{R}_p = \int_T p(t)\mu(t)dL(\lambda) + \sum_{n=1}^{\infty} p_n \lambda_n \varphi_n$. Hence the assumption $\mathcal{R}_p \equiv 0$ implies that $\int_T p(t)\mu(t)dL(\lambda) = (p, \mu) = 0$, and $p_n = 0$ for all $n \geq 1$. Since $\mu = \mu_0 h + \sum_{n=1}^{\infty} \mu_n \psi_n$ in the absence of arbitrage, $(p, \mu) = \mu_0 p_0 h = 0$, and hence p is orthogonal to h under the hypothesis that either $\mu_0 \neq 0$ or $h \equiv 0$. ■

Now, we assume that the market has a riskless asset r''. On projecting $r''(t)$ to the subspace $\text{sp}[h, \psi_1(t), \psi_2(t), \cdots]$, we obtain

$$r''(t) = r_0 h(t) + \sum_{n=1}^{\infty} r_n \psi_n(t) + r_s(t),$$

where $r_s(t)$ is orthogonal to $\text{sp}[h, \psi_1(t), \psi_2(t), \cdots]$. Then by Equation 5,

$$\mathcal{R}_{r''} = \int_T r''(t)\mu(t)dL(\lambda) + \sum_{n=1}^{\infty} r_n \lambda_n \varphi_n(\omega).$$

Since $V(r'') = 0$, we obtain $r_n = 0$ for all $n \geq 1$. This fact together with Equation (9) imply that

$$\begin{aligned}
\mathcal{R}_{r''} &= \int_T r''(t)\mu(t)dL(\lambda) = r_0 \int_T h(t)\mu(t)dL(\lambda) + \int_T r_s(t)\mu(t)dL(\lambda) \\
&= r_0\mu_0 \int_T h^2(t)dL(\lambda) + \int_T r_s(t)\mu_s(t)dL(\lambda) \\
&= r_0\mu_0 h_0 + \int_T r_s(t)\mu_s(t)dL(\lambda)
\end{aligned}$$

Furthermore,

$$C(r'') = \int_T r''(t)dL(\lambda) = r_0 h_0.$$

Since we require that $C(r'') = 1$ we must have h nonnull. On the other hand, $E(r'')$ is assumed to be nonnull. Hence we cannot have both $\mu_0 = 0$ and $\mu_s \equiv 0$. On combining these conditions together, we obtain either condition (i) or condition (ii). ∎

Proof of Corollary 4: The first result is clear from Theorem 4. Then we assume $\mu_0 \neq 0$. Let r be a portfolio with the expansion

$$r(t) = r_0 h(t) + \sum_{n=1}^{\infty} r_n \psi_n(t) + r_s(t),$$

as the portfolio r'' in the proof of Theorem 4. If r is a riskless asset, then the computation in the proof of Theorem 4 suggests that $C(r) = r_0 h_0 = 1$ and $r_n = 0$ for all $n \geq 1$, and hence $r = h/h_0 + r_s$, where r_s is orthogonal to h and all the ψ_n. The rest is clear. ∎

Proof of Lemma 1: The necessity part of (i) is clear. To prove the sufficiency part of (i), assume that $E(p) = C(p) = V(p) = 0$. By Equation (6), p is orthogonal to all the ψ_n. Since $C(p) = 0$, p is also orthogonal to the constant function 1, and hence p is dummy.

For (ii), project p to the closed subspace $[h, \psi_1, \psi_2, \cdots]$, to obtain

$$p = p_0 h + \sum_{n=1}^{\infty} p_n \psi_n + p_s.$$

By Equation (5), $\mathcal{R}_p = \int_T p(t)\mu(t)dL(\lambda) + \sum_{n=1}^{\infty} p_n \lambda_n \varphi_n$. Hence the assumption $\mathcal{R}_p \equiv 0$ implies that $\int_T p(t)\mu(t)dL(\lambda) = (p, \mu) = 0$, and $p_n = 0$ for all $n \geq 1$. Since $\mu = \mu_0 h + \sum_{n=1}^{\infty} \mu_n \psi_n$ in the absence of arbitrage, $(p, \mu) = \mu_0 p_0 h = 0$, and hence p is orthogonal to h under the hypothesis that either $\mu_0 \neq 0$ or $h \equiv 0$. ∎

Hence, by the definition of a mean portfolio,

$$(\mathcal{R}_m, \mathcal{R}_p) = \int_T p(t) \int_\Omega \mathcal{R}_m(\omega) x(t,\omega) dL(P) dL(\lambda) = \int_T p(t)\mu(t) dL(\lambda) = E(p),$$

and we are done. The proof for the case of cost portfolios is the same. ∎

Proof of Theorem 6: As in the proof of Theorem 3, if $\mu_0 \neq 0$ or μ_s is not null, we can respectively define the mean portfolio by

$$m(t) = \frac{h(t)}{\mu_0 h_0} \text{ or by } m(t) = \frac{\mu_s(t)}{\int_T \mu_s^2(t) dL(\lambda)}.$$

It is clear that $\mathcal{R}_m \equiv 1$, and hence $E(p) = (\mathcal{R}_m, \mathcal{R}_p)$ for any portfolio p. Therefore m is a mean portfolio.

It remains to consider the case when $\mu_0 = 0$ and $\mu_s \equiv 0$. Let m be a mean portfolio, i.e., $(\mathcal{R}_m, x_t) = \mu_t$ for $L(\lambda)$-almost all $t \in T$. Let $m(t) = m_0 h(t) + \sum m_n \psi_n(t) + m_s(t)$, where m_s is orthogonal to h and all the ψ_n. Note that $\mathcal{R}_m(\omega) = \int_T m(t)\mu(t) dL(\lambda) + \sum_{n=1}^\infty m_n \lambda_n \varphi_n(\omega)$ and $E(m) = \int_T m(t)\mu(t) dL(\lambda)$. Then for $L(\lambda)$-almost all $t \in T$,

$$(\mathcal{R}_m, x_t) = \mu(t) E(m) + \sum_{n=1}^\infty m_n \lambda_n^2 \psi_n(t) = \mu(t).$$

By the fact that $\mu(t) = \sum_{n=1}^\infty \mu_n \psi_n(t)$, we obtain

$$\left(\sum_{n=1}^\infty \mu_n \psi_n(t)\right)(E(m) - 1) + \sum_{n=1}^\infty m_n \lambda_n^2 \psi_n(t) = 0.$$

Thus, for $L(\lambda)$-almost all $t \in T$,

$$\sum_{n=1}^\infty \left(\mu_n(E(m) - 1) + m_n \lambda_n^2\right) \psi_n(t) = 0.$$

Hence it follows from the orthogonality of the ψ_n that $\mu_n(E(m) - 1) + m_n \lambda_n^2 = 0$ for each $n \geq 1$. This means that $m_n = (\mu_n(1 - E(m)))/\lambda_n^2$. If $E(m) = 1$, then $m_n = 0$; since $E(m) = \sum_{n=1}^\infty m_n \mu_n$, we also obtain $E(m) = 0$. This is a contradiction. Therefore $E(m) \neq 1$. By the fact that $\sum_{n=1}^\infty m_n^2 < \infty$, we obtain that $\sum_{n=1}^\infty \mu_n^2/\lambda_n^4 < \infty$.

On the other hand, if $\sum_{n=1}^\infty \mu_n^2/\lambda_n^4 < \infty$, then we can simply define a portfolio m by letting

$$m(t) = \sum_{n=1}^\infty m_n \psi_n(t),\ m_n = \left(1 + \sum_{n=1}^\infty \frac{\mu_n^2}{\lambda_n^2}\right)^{-1} \frac{\mu_n}{\lambda_n^2},\ n \geq 1.$$

Note that the convergence of the series $\sum_{n=1}^\infty \mu_n^2/\lambda_n^4$ implies the convergence of $\sum_{n=1}^\infty \mu_n^2/\lambda_n^2$, since $\lim_{n\to\infty} \lambda_n = 0$. Thus the above portfolio m is well defined. By the previous computation, it is clear that m is a mean portfolio with $E(m) = (\sum_{n=1}^\infty \mu_n^2/\lambda_n^2)/(1 + \sum_{n=1}^\infty \mu_n^2/\lambda_n^2)$, and we are done. ∎

Proof of Theorem 7: Let $c(t) = c_0 h(t) + \sum c_n \psi_n(t) + c_s(t)$ be a cost portfolio, i.e., $(\mathcal{R}_c, x_t) = 1$ for $L(\lambda)$-almost all $t \in T$, where c_s is orthogonal to h and all the ψ_n. By Equation (5),

$$\mathcal{R}_c(\omega) = \int_T c(t)\mu(t) dL(\lambda) + \sum_{n=1}^\infty c_n \lambda_n \varphi_n(\omega)$$

and $E(c) = \int_T c(t)\mu(t)dL(\lambda)$. Then $L(\lambda)$-almost all $t \in T$,

$$\begin{aligned}
(\mathcal{R}_c, x_t) &= E(c)\mu(t) + \sum_{n=1}^{\infty} c_n \lambda_n^2 \psi_n(t) \\
&= E(c)\mu_0 h(t) + \sum_{n=1}^{\infty} \left(E(c)\mu_n + c_n\lambda_n^2\right)\psi_n(t) + E(c)\mu_s(t) \\
&= 1 = h(t) + \sum_{n=1}^{\infty} s_n\psi_n(t).
\end{aligned}$$

By the fact that μ_s, h, and all the ψ_n are mutually orthogonal, we can obtain from the above formula that $E(c)\mu_0 h = h$, $E(c)\mu_n + c_n\lambda_n^2 = s_n$, and $E(c)\mu_s \equiv 0$.

For the proof of (i), note that $\mu_0 \neq 0$ implies that h is not null. Consider a cost portfolio c as above. By the identity $E(c)\mu_0 h = h$, we have $E(c) = 1/\mu_0$. The identity $E(c)\mu_s \equiv 0$ implies $\mu_s = 0$. Moreover, for $n \geq 1$, $c_n = (\mu_0 s_n - \mu_n)/(\mu_0\lambda_n^2)$. Since c is a portfolio, we have $\sum_{n=1}^{\infty} c_n^2 < \infty$, and hence $\sum_{n=1}^{\infty}\left(\frac{\mu_0 s_n - \mu_n}{\lambda_n^2}\right)^2 < \infty$. On the other hand, if $\mu_0 \neq 0$, $\mu_s = 0$, and $\sum_{n=1}^{\infty}\left(\frac{\mu_0 s_n - \mu_n}{\lambda_n^2}\right)^2 < \infty$, then we can define a portfolio c' by letting

$$c'(t) = c_0' h(t) + \sum_{n=1}^{\infty} c_n' \psi_n(t),$$

and where

$$c_0' = \frac{(1/\mu_0) - \sum_{n=1}^{\infty} c_n' \mu_n}{\mu_0 h_0}, \text{ and } c_n' = \frac{\mu_0 s_n - \mu_n}{\mu_0 \lambda_n^2} \text{ for all } n \geq 1.$$

It is easy to check that $E(c') = 1/\mu_0$. By the computation in the previous paragraph, it is clear that c′ is a cost portfolio.

For the proof of (ii), consider $\mu_0 = 0$ and h is not null. Suppose c is a cost portfolio. Then as shown in the first paragraph of this proof, $E(c)\mu_0 h = h$. This implies that $h \equiv 0$, which contradicts the hypothesis. Therefore there is no cost portfolio in this case. In fact, in this case, the portfolio h has a positive cost $C(h) = \int_T h dL(\lambda) = \int_T h^2 dL(\lambda)$ but with random return $\mathcal{R}_h \equiv 0$. Thus $C(h)$ is not equal to the inner product of $\mathcal{R}_h$ with the random return of any other portfolio.

For (iii), we move to the case $h \equiv 0$ and $\mu_s \neq 0$. If there is a cost portfolio c, then it follows from the identity $E(c)\mu_s \equiv 0$ in the first paragraph that $E(c) = 0$. Hence $c_n = s_n/\lambda_n^2$. By the fact that $\sum_{n=1}^{\infty} c_n^2 < \infty$, we have $\sum_{n=1}^{\infty} s_n^2/\lambda_n^4 < \infty$.

On the other hand, if $\sum_{n=1}^{\infty} s_n^2/\lambda_n^4 < \infty$, then we can define a portfolio c' by letting

$$c'(t) = \sum_{n=1}^{\infty}\left(\frac{s_n}{\lambda_n^2}\right)\psi_n(t) - \sum_{n=1}^{\infty}\frac{\mu_n s_n}{\lambda_n^2 \int_T \mu_s^2 dL(\lambda)}\mu_s(t).$$

It is easy to check that $E(c') = 0$. By the computation in the first paragraph, it is clear that c' is a cost portfolio.

Finally, for (iv), we consider the case when $h \equiv 0$, $\mu_s \equiv 0$, and a mean portfolio exists. Let c be a cost portfolio as in the first paragraph. Then we obtain $c_n = (s_n - E(c)\mu_n)/\lambda_n^2$ for all $n \geq 1$. For any real numbers a, b, it is easy to check that $a^2 \leq 2(a-b)^2 + 2b^2$. Hence

$$\left(\frac{s_n}{\lambda_n^2}\right)^2 \leq 2c_n^2 + 2\left(\frac{E(c)\mu_n}{\lambda_n^2}\right)^2.$$

Since a mean portfolio exists, it follows from Theorem 6 that $\sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^4) < \infty$. By the fact that $\sum_{n=1}^{\infty} c_n^2 < \infty$, we obtain that $\sum_{n=1}^{\infty} s_n^2/\lambda_n^4 < \infty$.

On the other hand, if $\sum_{n=1}^{\infty} s_n^2/\lambda_n^4 < \infty$, then by the fact that $\sum_{n=1}^{\infty} \mu_n^2 < \infty$, we can claim the convergence of the series $\sum_{n=1}^{\infty} \mu_n s_n/\lambda_n^2$. Since $\sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^4) < \infty$, it is clear that $\sum_{n=1}^{\infty} \mu_n^2/\lambda_n^2 < \infty$. We define a real number a by

$$a = \frac{\sum_{n=1}^{\infty}(\mu_n s_n \ / \ \lambda_n^2)}{1 + \sum_{n=1}^{\infty}(\mu_n^2/\lambda_n^2)}.$$

Define a portfolio c' by letting

$$c'(t) = \sum_{n=1}^{\infty} c_n' \psi_n(t), \ c_n' = \frac{s_n - a\mu_n}{\lambda_n^2}, \ n \geq 1.$$

It is easy to check that $E(c') = a$. By the computation in the previous paragraph, it can be checked that c' is indeed a cost portfolio. ∎

Proof of Proposition 2: Let $p(t) = p_0 h(t) + \sum p_n \psi_n(t) + p_s(t)$ be a portfolio such that p_s is orthogonal to h and all the ψ_n. Then $\mathcal{R}_p(\omega) = \int_T p(t)\mu(t)dL(\lambda) + \sum_{n=1}^{\infty} p_n \lambda_n \varphi_n(\omega)$ and

$$E(p) = \int_T p(t)\mu(t)dL(\lambda) = p_0\mu_0 h_0 + \sum_{n=1}^{\infty} p_n \mu_n + \int_T p_s(t)\mu_s(t)dL(\lambda).$$

Thus $(\mathcal{R}_p, x_t) = E(p)\mu(t) + \sum_{n=1}^{\infty} p_n \lambda_n^2 \psi_n(t)$.

Now we consider $(ii) \Longrightarrow (i)$. Suppose both $\mu_0 = 0$ and $\mu_s \not\equiv 0$. Let

$$p_n = \frac{\mu_n \int_T \mu_s^2 dL(\lambda)}{\lambda_n^2} \text{ for } n \geq 1 \text{ and } p_s(t) = -\left(1 + \sum_{n=1}^{\infty} \mu_n^2/\lambda_n^2\right) \mu_s(t).$$

It is easy to check that $E(p) = -\int_T \mu_s^2 dL(\lambda)$. Thus $\mathcal{R}_p \not\equiv 0$. It is also straightforward to check that $(\mathcal{R}_p, x_t) = 0$ for $L(\lambda)$-almost all $t \in T$. This contradicts (ii). Hence (i) holds.

Next, for $(i) \Longrightarrow (ii)$, assume that $(\mathcal{R}_p, x_t) = E(p)\mu(t) + \sum_{n=1}^{\infty} p_n \lambda_n^2 \psi_n(t) = 0$ for $L(\lambda)$-almost all $t \in T$. Then we obtain $E(p)\mu_0 h \equiv 0$, $E(p)\mu_s \equiv 0$, and $E(p)\mu_n + \lambda_n^2 p_n = 0$ for all $n \geq 1$. If $\mu_0 \neq 0$, then $h \not\equiv 0$, and hence $E(p) = 0$. This implies $p_n = 0$ for all $n \geq 1$. Therefore $\mathcal{R}_p \equiv 0$.

On the other hand, if we have $\mu_0 = 0$ and $\mu_s \equiv 0$, we shall show that $E(p) = 0$. Suppose $E(p) \neq 0$. Then by the identity $p_n = -E(p)\mu_n/\lambda_n^2$ for each $n \geq 1$, we know that the series $\sum_{n=1}^{\infty} \mu_n^2/\lambda_n^2$ is convergent and $E(p) = \sum_{n=1}^{\infty} p_n \mu_n = -E(p) \sum_{n=1}^{\infty} \mu_n^2/\lambda_n^2$. Hence we obtain $E(p) = 0$, which contradicts the hypothesis. Thus $E(p)$ must be zero. This implies $\mathcal{R}_p \equiv 0$ as before.

To show the equivalence of (iii) and (iv) with (i), one simply observes that for $L(\lambda)$-almost all $t \in T$, the inner products $(\mathcal{R}_{m_1 - m_2}, x_t)$ and $(\mathcal{R}_{c_1 - c_2}, x_t)$ are both equal to zero, and then apply the equivalence of (i) and (ii) to the portfolios $m_1 - m_2$ and $c_1 - c_2$. ∎

Proof of Proposition 3: If $\mathcal{R}_m$ and $\mathcal{R}_c$ are linearly dependent, then there is a real number α such that $\mathcal{R}_m = \alpha\mathcal{R}_c$ since $\mathcal{R}_c$ is never zero. Then, by the definition of mean and cost portfolios, we have for $L(\lambda)$-almost all $t \in T$, $\mu_t = (\mathcal{R}_m, x_t) = \alpha(\mathcal{R}_c, x_t) = \alpha$. Hence $(i) \Longrightarrow (ii)$.

For $(ii) \Longrightarrow (i)$, we assume that $\mu(t) \equiv \alpha$ for some real number α. In this case, we certainly have $\mu_s = 0$. Note that for $L(\lambda)$-almost all $t \in T$, $\mu_t = (\mathcal{R}_m, x_t) = \alpha = \alpha(\mathcal{R}_c, x_t)$, and hence $(\mathcal{R}_m - \alpha\mathcal{R}_c, x_t) = 0$ for $L(\lambda)$-almost all $t \in T$. By Proposition 2, we have $\mathcal{R}_m - \alpha\mathcal{R}_c = 0$, and hence (i) holds.

Note that $(i) \Longrightarrow (iii)$ is obvious. It remains to see that $(iii) \Longrightarrow (i)$. Assume the matrix in (iii) is singular. Then there is a real number α such that

$$((\mathcal{R}_m, \mathcal{R}_m), (\mathcal{R}_m, \mathcal{R}_c)) = (\alpha(\mathcal{R}_c, \mathcal{R}_m), \alpha(\mathcal{R}_c, \mathcal{R}_c)).$$

Thus $(\mathcal{R}_m - \alpha\mathcal{R}_c, \mathcal{R}_m) = 0$ and $(\mathcal{R}_m - \alpha\mathcal{R}_c, \mathcal{R}_c) = 0$, which implies that $(\mathcal{R}_m - \alpha\mathcal{R}_c, \mathcal{R}_m - \alpha\mathcal{R}_c) = 0$. Hence $\mathcal{R}_m = \alpha\mathcal{R}_c$. ∎

Proof of Lemma 3: If $V(M)$ is zero, M is already mean-variance efficient. So we can assume $V(M) \neq 0$. For a portfolio p, since $E(p) = \int_T p(t)\mu(t)dL(\lambda)$, it is easy to obtain from the CAPM equation that $E(p) = \rho C(p) + \alpha \text{ cov}(\mathcal{R}_p, \mathcal{R}_M)$ by changing the relevant integrals. In particular, we have $E(M) = \rho C(M) + \alpha V(M)$, which implies that $V(M) = (E(M) - \rho C(M))/\alpha$. Now let p be an arbitrarily given portfolio with mean $E(M)$ and cost $C(M)$. Then the equality $E(p) = \rho C(p) + \alpha \text{ cov}(\mathcal{R}_p, \mathcal{R}_M)$ implies that $(E(M) - \rho C(M))/\alpha = \text{cov}(\mathcal{R}_p, \mathcal{R}_M)$. Hence $V(M) = \text{cov}(\mathcal{R}_p, \mathcal{R}_M) \leq \sqrt{V(p)}\sqrt{V(M)}$. Thus $V(M) \leq V(p)$, which means that M is mean-variance efficient. ∎

Proof of Corollary 6: As shown in the proof of Lemma 3, $E(M) = \rho C(M) + \alpha V(M)$, and hence $\alpha = (E(M) - \rho C(M))/V(M)$. It is also obvious that for any portfolio p, we have $E(p) = \alpha C(p) + (E(M) - \rho C(M))\beta_p$. The rest is clear. ∎

Proof of Lemma 4: We begin with the proof of (i). For a portfolio q, let $\alpha\mathcal{R}_m + \beta\mathcal{R}_c$ be the projection of $\mathcal{R}_q$ on the space spanned by $\mathcal{R}_m$ and $\mathcal{R}_c$. Define a portfolio u and w by letting $u(t) = \alpha m(t) + \beta c(t)$ and $w(t) = q(t) - u(t)$. Then $\mathcal{R}_w$ is orthogonal to both $\mathcal{R}_m$ and $\mathcal{R}_c$, and hence $E(w) = C(w) = 0$, $E(u) = E(q)$ and $C(u) = C(q)$. Since $\mathcal{R}_u$ and $\mathcal{R}_w$ are orthogonal, we have $V(q) = V(u) + V(w)$. Therefore, the portfolio q is mean-variance efficient if and only if $V(w) = 0$, and thus (i) follows.

For the proof of (ii), note that by (i), we only have to choose real numbers α and β so that the portfolio $p = \alpha m + \beta c$ has mean a and cost b. That is, $\alpha E(m) + \beta E(c) = a$ and $\alpha C(m) + \beta C(c) = b$. Since $C(m) = \int_\Omega \mathcal{R}_c\mathcal{R}_m dL(P) = E(c)$, it is equivalent to solve the following equations:

$$\begin{pmatrix} E(m) & E(c) \\ E(c) & C(c) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Since the relevant matrix is nonsingular by Proposition 3, we can obtain that

$$\alpha = \frac{(aC(c) - bE(c))}{E(m)C(c) - (E(c))^2} \text{ and } \beta = \frac{(bE(m) - aE(c))}{E(m)C(c) - (E(c))^2}.$$

We are done. ∎

Proof of Proposition 4: By Theorems 5 and 6, mean and cost portfolios have the form as given in the statement of the Proposition. It is clear that $E(m) = 1$, $E(c) = 1/\mu_0$, and

$$C(c) = \frac{1}{\mu_0^2} + \sum_{n=1}^{\infty} \lambda_n^2 c_n^2 = \frac{1}{\mu_0^2}\left(1 + \sum_{n=1}^{\infty} \frac{(\mu_0 s_n - \mu_n)^2}{\lambda_n^2}\right).$$

The rest is clear from Lemmas 1 and 4. ∎

Proof of Corollary 7: Let q be a mean variance efficient portfolio with mean a and cost b in the situation considered in Proposition 3. Then it is easy to check that

$$\begin{aligned}
\text{cov}(x_t, q) &= \frac{(b\mu_0^2 - a\mu_0)\sum_{n=1}^{\infty} \lambda_n^2 c_n \psi_n(t)}{\sum_{n=1}^{\infty}(\mu_0 s_n - \mu_n)^2/\lambda_n^2} \\
&= \frac{(b\mu_0 - a)\sum_{n=1}^{\infty}(\mu_0 s_n - \mu_n)\psi_n(t)}{\sum_{n=1}^{\infty}(\mu_0 s_n - \mu_n)^2/\lambda_n^2} \\
&= \frac{(a - b\mu_0)\left(\sum_{n=1}^{\infty}\mu_n\psi_n(t) - \mu_0(1 - h(t))\right)}{\sum_{n=1}^{\infty}(\mu_0 s_n - \mu_n)^2/\lambda_n^2} \\
&= \frac{(a - b\mu_0)(\mu(t) - \mu_0)}{\sum_{n=1}^{\infty}(\mu_0 s_n - \mu_n)^2/\lambda_n^2},
\end{aligned}$$

and the rest is clear. ■

Proof Corollary 8: For any portfolio q with $V(q) \not\equiv 0$, $E(q) = a$ and $C(q) = 0$, consider the portfolio p in Proposition 4 with $E(p) = a$ and $C(p) = 0$. Then p becomes

$$p(t) = \frac{a\left(1 + \sum_{n=1}^{\infty} \frac{(\mu_0 s_n - \mu_n)^2}{\lambda_n^2}\right) m(t) - a\mu_0 c(t)}{\sum_{n=1}^{\infty} (\mu_0 s_n - \mu_n)^2 / \lambda_n^2},$$

and hence

$$V(p) = \frac{a^2 \mu_0^2 \sum_{n=1}^{\infty} \lambda_n^2 c_n^2}{\left(\sum_{n=1}^{\infty} (\mu_0 s_n - \mu_n)^2 / \lambda_n^2\right)^2} = \frac{a^2}{\sum_{n=1}^{\infty} \left(\frac{\mu_0 s_n - \mu_n}{\lambda_n}\right)^2}.$$

But $|E(p)|/V^{1/2}(p) = \sqrt{\sum_{n=1}^{\infty} \left(\frac{\mu_0 s_n - \mu_n}{\lambda_n}\right)^2}$ for $a \neq 0$. Since p is mean-variance efficient, we know $V(p) \leq V(q)$, and hence, for $a \neq 0$,

$$|E(q)|/V^{1/2}(q) \leq |E(p)|/V^{1/2}(p) = \sqrt{\sum_{n=1}^{\infty} \left(\frac{\mu_0 s_n - \mu_n}{\lambda_n}\right)^2}.$$

If $a = 0$ and $V(q) \neq 0$, we have $|E(q)|/V^{1/2}(q) = 0$, and thus our formula for δ is validated. ■

Proof of Proposition 5: By Theorems 5 and 6, the given m and c are indeed mean and cost portfolios with $E(m) = (\sum_{n=1}^{\infty} \mu_n^2/\lambda_n^2)/(1 + \sum_{n=1}^{\infty} \mu_n^2/\lambda_n^2)$ and $E(c) = \gamma$. We also have

$$C(c) = \sum_{n=1}^{\infty} c_n s_n = \sum_{n=1}^{\infty} \frac{s_n^2}{\lambda_n^2} - \gamma \sum_{n=1}^{\infty} \frac{\mu_n s_n}{\lambda_n^2} = \frac{\left(1 + \sum_{n=1}^{\infty} \frac{\mu_n^2}{\lambda_n^2}\right) \sum_{n=1}^{\infty} \frac{s_n^2}{\lambda_n^2} - \left(\sum_{n=1}^{\infty} \frac{\mu_n s_n}{\lambda_n^2}\right)^2}{1 + \sum_{n=1}^{\infty} \frac{\mu_n^2}{\lambda_n^2}}.$$

Hence

$$E(m)C(c) - (E(c))^2 = \frac{\sum_{n=1}^{\infty} \frac{\mu_n^2}{\lambda_n^2} \sum_{n=1}^{\infty} \frac{s_n^2}{\lambda_n^2} - \left(\sum_{n=1}^{\infty} \frac{\mu_n s_n}{\lambda_n^2}\right)^2}{1 + \sum_{n=1}^{\infty} \frac{\mu_n^2}{\lambda_n^2}}.$$

Note that the fact that the market is not trivial implies that $E(m)C(c) - (E(c))^2$ is not zero. The rest is clear from Lemmas 1 and 4. ■

Proof of Corollary 9: By using the notation introduced in the statement of the corollary, the portfolio defined by

$$p(t) = \frac{(a[(1+\alpha)\beta - \tau^2] - b\tau)\, m(t) + (b\alpha - a\tau c(t)}{\alpha\beta - \tau^2}$$

has mean a and cost b. Then it is easy to obtain that

$$\mathcal{R}_p = \frac{(a[(1+\alpha)\beta - \tau^2] - b\tau)\, \mathcal{R}_m + (b\alpha - a\tau \mathcal{R}_c}{\alpha\beta - \tau^2}.$$

Note that $\text{cov}(x_t, p) = (x_t, \mathcal{R}_p) - E(x_t)E(p)$, and hence

$$\text{cov}(x_t, p) = \frac{(a[(1+\alpha)\beta - \tau^2] - b\tau)\, \mu(t) + b\alpha - a\tau}{\alpha\beta - \tau^2} - a\mu(t) = \frac{(a\beta - b\tau)\mu(t) + b\alpha - a\tau}{\alpha\beta - \tau^2},$$

where the fact that m and c are the mean and cost portfolios is used. Hence p is a CAPM carrier if and only if $E(p) \neq C(p)\frac{\tau}{\beta}$. ∎

Proof of Corollary 10: As in the proof of Corollary 8, our δ is equal to $|E(p)|/V^{1/2}(p)$ for p in Proposition 5 with $E(p) = 1$ and $C(p) = 0$. We use the notation of Corollary 9. Hence

$$p(t) = \frac{\left[(1+\alpha)\beta - \tau^2\right] m(t) - \tau\, c(t)}{\alpha\beta - \tau^2}.$$

Thus

$$\begin{aligned}
V(p) &= \frac{\sum_{n=1}^{\infty} \lambda_n^2 \left(\left[(1+\alpha)\beta - \tau^2\right] m_n - \tau c_n\right)^2}{\left(\alpha\beta - \tau^2\right)^2} \\
&= \frac{\sum_{n=1}^{\infty} \lambda_n^2 \left(\left[(1+\alpha)\beta - \tau^2\right] \frac{1}{(1+\alpha)} \frac{\mu_n}{\lambda_n^2} - \tau \frac{s_n}{\lambda_n^2} + \tau^2 \frac{1}{1+\alpha} \frac{\mu_n}{\lambda_n^2}\right)^2}{\left(\alpha\beta - \tau^2\right)^2} \\
&= \frac{\sum_{n=1}^{\infty} \left(\beta \frac{\mu_n}{\lambda_n} - \tau \frac{s_n}{\lambda_n}\right)^2}{\left(\alpha\beta - \tau^2\right)^2} \\
&= \frac{\beta^2 \sum_{n=1}^{\infty} (\frac{\mu_n}{\lambda_n})^2 - 2\beta\tau \sum_{n=1}^{\infty} \frac{\mu_n s_n}{\lambda_n^2} + \tau^2 \sum_{n=1}^{\infty} (\frac{s_n}{\lambda_n})^2}{\left(\alpha\beta - \tau^2\right)^2} \\
&= \frac{\beta^2\alpha - 2\beta\tau^2 + \tau^2\beta}{\left(\alpha\beta - \tau^2\right)^2} = \frac{\beta}{\alpha\beta - \tau^2}.
\end{aligned}$$

Therefore $\delta = \sqrt{\frac{\alpha\beta - \tau^2}{\beta}}$, and we are done. ∎

Proof of Lemma 5: For a portfolio q with mean a and cost b, define portfolios

$$u = \left(\frac{a}{E(m)}\right) m + \left(\frac{b}{h_0} - \frac{ac(m)}{h_0 E(m)}\right) h$$

and $w = q - u$. Note that, by the second part of Theorem 6, the assumptions of the lemma imply that $E(m) \neq 0$, and hence the portfolios are well defined. It is easy to check that $C(u) = b$ and $\mathcal{R}_u$ is $(a/E(m))\mathcal{R}_m$, the projection of $\mathcal{R}_q$ on the linear space spanned by $\mathcal{R}_m$, and hence $\mathcal{R}_w$ is orthogonal to $\mathcal{R}_m$. Thus $V(q) = V(u) + V(w)$. Therefore, the portfolio q is mean-variance efficient if and only if $V(w) = 0$, and thus the lemma follows. ∎

Proof of Proposition 6: As shown in Lemma 5, the portfolio

$$p = \left(\frac{a}{E(m)}\right) m + \left(\frac{b}{h_0} - \frac{ac(m)}{h_0 E(m)}\right) h$$

is mean-variance efficient. Here we choose m to be the mean portfolio defined in the last paragraph of the proof of Theorem 6. Then

$$E(m) = \frac{\sum_{n=1}^{\infty} \mu_n^2/\lambda_n^2}{1 + \sum_{n=1}^{\infty} \mu_n^2/\lambda_n^2} \text{ and } C(m) = \sum_{n=1}^{\infty} m_n s_n = \left(1 + \sum_{n=1}^{\infty} \frac{\mu_n^2}{\lambda_n^2}\right)^{-1} \sum_{n=1}^{\infty} \frac{\mu_n s_n}{\lambda_n^2},$$

and hence we obtain the formula for p as described.

Next, let $q = q_0h + \sum_{n=1}^{\infty} q_n\psi_n + q_s$ such that q is mean-variance efficient with mean a and cost b. Then we must have $\mathcal{R}_q = \mathcal{R}_p$, and hence $\mathcal{R}_{q-p} = 0$. This implies that $q_n - p_n = 0$ for all $n \geq 1$. Now

$$C(q) = q_0h_0 + \sum_{n=1}^{\infty} q_n s_n = q_0h_0 + \sum_{n=1}^{\infty} p_n s_n$$

and $C(p) = p_0h_0 + \sum_{n=1}^{\infty} p_n s_n$. By $C(p) = C(q)$ and $h_0 \neq 0$, we obtain $p_0 = q_0$. Therefore $q = p + q_s$. ∎

Proof of Proposition 7: We begin with the proof of (i). Since $\mu \equiv \alpha$, it is clear that $E(q) = \alpha C(q)$ for any portfolio q. We can only allow the value of cost to be a variable. The portfolio $p = bh/h_0$ has mean αb and cost b. Since its variance is zero, it must be mean-variance efficient. Next, let $q = q_0h + \sum_{n=1}^{\infty} q_n\psi_n + q_s$ be a mean-variance efficient portfolio with cost b. Then $\mathcal{R}_q$ must be constant, which implies that $q_n = 0$ for all $n \geq 1$. Since $C(q) = q_0h_0 = b$, we can obtain $q_0 = b/h_0$, and hence $q = p + q_s$.

For the proof of (ii), note that the cost portfolio exists in this case. Since we do not have to consider a constraint involving the mean, as in Lemma 5, we can obtain that a portfolio q is mean-variance efficient if and only if $\mathcal{R}_q$ is a multiple of $\mathcal{R}_c$. By Theorem 7 (iv) and the fact that $\mu = \alpha$, we obtain a cost portfolio c with

$$c(t) = \left(1 + \alpha^2 \sum_{n=1}^{\infty} s_n^2\right)^{-1} \sum_{n=1}^{\infty} \frac{s_n}{\lambda_n^2}\psi_n(t).$$

The portfolio p defined in (ii) is a multiple of c with cost b, and hence also mean-variance efficient.

Next, if q is a is mean-variance efficient portfolio with cost b, then $\mathcal{R}_{q-p}$ must be zero. By Lemma 1 (ii), $q - p$ must be a dummy portfolio. ∎

Proof of Proposition 8: We shall first assume that Part (2) holds. To prove Part (1), let $\{p_n\}_{n=1}^{\infty}$ be a sequence of portfolios such that

$$\lim_{n\to\infty} C(p_n) = \lim_{n\to\infty} V(p_n) = 0.$$

As remarked earlier, $\|p_n\|_2$ is assumed to be bounded with supremum $\|p\|_2$. We need to show that $\lim_{n\to\infty} E(p_n) = 0$. Fix η and $\varepsilon > 0$ as in the statement of the theorem. Let

$$p_n(t) = p_{n0}h_n(t) + \sum_{i=1}^{\ell_n(\varepsilon)} p_{ni}\psi_{ni}(t) + p_{ns},$$

where p_{ns} be the orthogonal complement of p_n on the space spanned by $\{1, \psi_{n1}, \cdots, \psi_{n\ell_n(\varepsilon)}\}$. By the fact that

$$x_n(t,\omega) = \mu_n(t) + \sum_{i=1}^{\ell_n(\varepsilon)} \lambda_{ni}\psi_{ni}(t)\varphi_{ni}(\omega) + e_n(\varepsilon)(t,\omega),$$

it is easy to obtain the following formula for the random return $\mathcal{R}_{p_n}$ of the portfolio p_n

$$\mathcal{R}_{p_n}(\omega) = \int_{T_n} p_n(t)\mu_n(t)d\lambda_n + \sum_{i=1}^{\ell_n(\varepsilon)} \lambda_{ni}p_{ni}\varphi_{ni}(\omega) + \int_{T_n} p_n(t)e_n(\varepsilon)(t,\omega)d\lambda_n.$$

By Theorem B (3) in Appendix I, we know that a linear combination of the residual part is orthogonal to all the factors, and hence the orthogonality of the factors implies that

$$V(p_n) = \sum_{i=1}^{\ell_n(\varepsilon)} \lambda_{ni}^2 p_{ni}^2 + \left(\int_{T_n} p_n(t)e_n(\varepsilon)(t,\omega)d\lambda_n\right)^2.$$

By Theorem B (1), $\lambda_{ni} \geq \varepsilon/M$ for all $1 \leq i \leq \ell_n(\varepsilon)$. Thus $\sum_{i=1}^{\ell_n(\varepsilon)} p_{ni}^2 \leq M^2/\varepsilon^2 V(p_n)$. Hence $\lim_{n\to\infty} \sum_{i=1}^{\ell_n(\varepsilon)} p_{ni}^2 = 0$. It is clear that $h_n = 1 - \sum_{i=1}^{\ell_n(\varepsilon)} (\int_{T_n} \psi_{ni} d\lambda_n)\psi_{ni}$ and $\int_{T_n} h_n d\lambda_n) = \|h_n\|_2^2$. Thus

$$p_{n0}\|h_n\|_2^2 = \int_{T_n} p_n(t)h_n(t)d\lambda_n = \int_{T_n} p_n(t)d\lambda_n - \sum_{i=1}^{\ell_n(\varepsilon)} p_{ni} \int_{T_n} \psi_{ni}(t)d\lambda_n.$$

Note that $C(p_n) = \int_{T_n} p_n(t)d\lambda_n$ and $\lim_{n\to\infty} C(p_n) = 0$. Since $\sum_{i=1}^{\ell_n(\varepsilon)} \left(\int_{T_n} \psi_{ni}(t)d\lambda_n\right)^2 \leq 1$, we have $\lim_{n\to\infty} p_{n0}\|h_n\|_2^2 = 0$. By $E(p_n) = \int_{T_n} p_n(t)\mu_n(t)d\lambda_n$, we can obtain

$$E(p_n) = p_{n0}\mu_{n0}\|h_n\|_2^2 + \sum_{i=1}^{\ell_n(\varepsilon)} p_{ni}\mu_{ni} + \int_{T_n} p_{ns}(t)\mu_{ns}(t)d\lambda_n.$$

By the Cauchy-Schwarz inequality,

$$|E(p_n)| \leq |p_{n0}\mu_{n0}|\|h_n\|_2^2 + \left(\sum_{i=1}^{\ell_n(\varepsilon)} p_{ni}^2 \sum_{i=1}^{\ell_n(\varepsilon)} \mu_{ni}^2\right)^{1/2} + \|p_{ns}\|_2\|\mu_{ns}\|_2.$$

It is clear that $\sum_{i=1}^{\ell_n(\varepsilon)} \mu_{ni}^2 \leq M^2$, $\|\mu_{n0}h_n\|_2 \leq M$ and $\|p_{n0}h_n\|_2 \leq \|p\|_2$.

Now we assume $n > 1/\varepsilon$. By Condition (ii) in (2), if $\|h_n\|_2 < \varepsilon$ then $\|\mu_{n0}h_n\|_2 < \eta$, and in this case

$$|p_{n0}\mu_{n0}|\|h_n\|_2^2 = \|p_{n0}h_n\|_2\|\mu_{n0}h_n\|_2 \leq \|p\|_2\eta.$$

On the other hand, if $\|h_n\|_2 \geq \varepsilon$, then

$$|p_{n0}\mu_{n0}|\|h_n\|_2^2 = \|p_{n0}h_n\|_2^2\|\mu_{n0}h_n\|_2/\|h_n\|_2 \leq \frac{M}{\varepsilon}|p_{n0}|\|h_n\|_2^2.$$

Thus, $|p_{n0}\mu_{n0}|\|h_n\|_2^2 \leq \|p\|_2\eta + \frac{M}{\varepsilon}|p_{n0}|\|h_n\|_2^2$ for all $n > 1/\varepsilon$. Since $\lim_{n\to\infty} p_{n0}\|h_n\|_2^2 = 0$, we can obtain $\overline{\lim}_{n\to\infty}|p_{n0}\mu_{n0}|\|h_n\|_2^2 \leq \eta\|p\|_2$ Hence, the fact that $\|\mu_{ns}\|_2 < \eta$ for $n > 1/\varepsilon$ together with $\|p_{ns}\|_2 \leq \|p\|_2$ imply that $\overline{\lim}_{n\to\infty}|E(p_n)| \leq 2\|p\|_2\eta$. By the arbitrary choice of η, we obtain that $\lim_{n\to\infty} E(p_n) = 0$. Therefore, the market permits no asymptotic arbitrage.

Next, we assume that the market permits no asymptotic arbitrage. Suppose Part (2) fails. Then there is an $\eta_0 > 0$ such that for any $\varepsilon > 0$, there is an integer $n > 1/\varepsilon$ such that either $\|\mu_{ns}\|_2 \geq \eta_0$, or $\|h_n\|_2 < \varepsilon$ but $\|\mu_{n0}h_n\|_2 \geq \eta_0$. Hence we can find a strictly increasing sequence $\{k_m\}_{m=0}^{\infty}$ of positive integers with $k_0 = 1$ such that either

$$d(\mu_{k_m}, span\{1, \psi_{k_m 1}, \cdots, \psi_{k_m \ell_{k_m}(1/k_{m-1})}\}) \geq \eta_0, \tag{22}$$

or

$$\|h_{k_m}\|_2 < 1/k_{m-1} \text{ but } \|\mu_{k_m 0}h_{k_m}\|_2 \geq \eta_0. \tag{23}$$

One of the previous equations must be true for infinitely many k_m's. First consider the case that Equation (22) is satisfied by infinitely many k_m's. Thus, we can find two subsequences $\{j_m\}_{m=1}^{\infty}$ and $\{j'_m\}_{m=1}^{\infty}$ of the sequence $\{k_m\}_{m=0}^{\infty}$ such that

$$d(\mu_{j_m}, span\{1, \psi_{j_m 1}, \cdots, \psi_{j_m \ell_{j_m}(1/j'_m)}\}) \geq \eta_0. \tag{24}$$

Define $p_n = 0$ if $n \neq j_m$ for any $m \geq 1$, and $p_n = \mu_{j_m s}$ if $n = j_m$ for some $m \geq 1$, where $\mu_{j_m s}$ is the orthogonal complement of μ_{j_m} on the space spanned by $\{1, \psi_{j_m 1}, \cdots, \psi_{j_m \ell_{j_m}(1/j'_m)}\}$. Then $E(p_{j_m}) = \|\mu_{j_m s}\|_2^2 \geq \eta_0^2$. By Theorem B (3) in Appendix I, we have

$$V(p_{j_m}) = \int_{\Omega} \left(\int_{T_{j_m}} \mu_{j_m s}(t)e_{j_m}(1/j'_m)(t,\omega)d\lambda_{j_m}\right)^2 dP \leq (1/j'_m)\|\mu_{j_m s}\|_2^2.$$

Note that $\|\mu_{j_m s}\|_2 \le M$ for all m. Thus, the sequence $\{p_n\}_{n=1}^{\infty}$ of portfolios is clearly bounded, and $\lim_{n\to\infty} V(p_n) = 0$. Since $C(p_n) = 0$ for all n, we know that $\lim_{n\to\infty} C(p_n) = \lim_{n\to\infty} V(p_n) = 0$, but $\lim_{n\to\infty} E(p_n) \neq 0$, which contradicts the hypothesis of no asymptotic arbitrage.

Now assume that Equation (23) is satisfied by infinitely many k_m's. As above, we can find two subsequences $\{r_m\}_{m=1}^{\infty}$ and $\{r'_m\}_{m=1}^{\infty}$ of the sequence $\{k_m\}_{m=0}^{\infty}$ such that

$$\|h_{r_m}\|_2 < 1/r'_m \text{ but } \|\mu_{r_m 0} h_{r_m}\|_2 \ge \eta_0, \tag{25}$$

where h_{r_m} is the orthogonal complement of the constant function 1 on the space spanned by $\{\psi_{r_m 1}, \cdots, \psi_{r_m \ell_{r_m}(1/r'_m)}\}$. Define $p_n = 0$ if $n \neq r_m$ for any $m \ge 1$, and $p_n = h_{r_m}/\|h_{r_m}\|_2$ if $n = r_m$ for some $m \ge 1$. It is obvious that $\|p_n\|_2 \le 1$ for all $n \ge 1$. It can be checked that $C(p_{r_m}) = \|h_{r_m}\|_2 < 1/r'_m$ and $E(p_{r_m}) = \mu_{r_m 0}\|h_{r_m}\|_2 \ge \eta_0$. By Theorem B (3) in Appendix I, we have

$$V(p_{r_m}) = \int_{\Omega} \left(\int_{T_{r_m}} p_{r_m s}(t) e_{r_m}(1/r'_m)(t,\omega) d\lambda_{r_m} \right)^2 dP \le (1/r'_m).$$

Then it is easy to see that $\lim_{n\to\infty} C(p_n) = \lim_{n\to\infty} V(p_n) = 0$, and $\lim_{n\to\infty} E(p_n) \neq 0$, which contradicts the hypothesis in (1) again. ∎

Proof of Proposition 9: Assume (i). We let

$$M_n(t) = M_{n0} h_n(t) + \sum_{i=1}^{s_n} M_{ni} \psi_{ni}(t) + M_{ns}(t),$$

where h_n and M_{ns} are the respective orthogonal complements of M_n on the space spanned by $\{1, \psi_{n1}, \cdots, \psi_{ns_n}\}$ and the constant function 1 on the space spanned by $\{\psi_{n1}, \cdots, \psi_{ns_n}\}$. Then

$$\mathcal{R}_{M_n}(\omega) = \int_{T_n} M_n(t)\mu_n(t) d\lambda_n + \sum_{i=1}^{s_n} \lambda_{ni} M_{ni} \varphi_{ni}(\omega) + \int_{T_n} M_n(t') e_n(t',\omega) d\lambda_n(t')$$

For each fixed $t \in T_n$, Theorem C (3) in Appendix I says that $\int_{T_n} M_n(t') e_n(t',\omega) d\lambda_n(t')$ and $e_n(t,\omega)$ are orthogonal to all the φ_{ni}, and hence,

$$\operatorname{cov}(x_n(t), \mathcal{R}_{M_n}) = \sum_{i=1}^{s_n} \lambda_{ni}^2 M_{ni} \psi_{ni}(t) + \int_{\Omega} e_n(t,\omega) \int_{T_n} M_n(t') e_n(t',\omega) d\lambda_n(t') dP(\omega).$$

It is also easy to see that for all $t \in T_n$,

$$\left| \int_{\Omega} e_n(t,\omega) \int_{T_n} M_n(t') e_n(t',\omega) d\lambda_n(t') dP(\omega) \right| \le \|e_{nt}\|_2 \left\| \int_{T_n} M_n(t') e_n(t',\omega) d\lambda_n(t') \right\|_2 .$$

It is clear that $\|e_n\|_2 \le M$. By (i) and Theorem C (3), we can obtain

$$\begin{aligned}
&\overline{\lim}_{n\to\infty} \left\| \mu_n(t) - \left(\rho_n + \sum_{n=1}^{s_n} \lambda_{ni}^2 M_{ni} \psi_{ni}(t) \right) \right\|_2 \\
\le\ & \lim_{n\to\infty} \|\mu_n(t) - (\rho_n + \operatorname{cov}(x_n(t), M_n))\|_2 + \overline{\lim}_{n\to\infty} \|e_n\|_2 \left\| \int_{T_n} M_n(t') e_n(t',\omega) d\lambda_n(t') \right\|_2 \\
\le\ & \overline{\lim}_{n\to\infty} M \left\| \int_{T_n} M_n(t') e_n(t',\omega) d\lambda_n(t') \right\|_2 = 0.
\end{aligned}$$

Note that the above derivation also uses the implicit assumption that $\|M_n\|_2^2$, $n \geq 1$, have an upper bound B for some positive number B. Let $\tau_{n0} = \rho_n$ and $\tau_{ni} = \lambda_{ni}^2 M_{ni}$ for $1 \leq i \leq s_n$. Since $\sum_{i=1}^{s_n} M_{ni}^2 \leq \|M_n\|_2^2 \leq B$, we have $\sum_{i=1}^{s_n}(\tau_{ni}^2/\lambda_{ni}^4) \leq B$ for all n.

Next, assume (ii) holds. Define $M_n(t) = \sum_{n=1}^{s_n} (\tau_{ni}/\lambda_{ni}^2)\, \psi_{ni}(t)$ and $\rho_n = \tau_{n0}$. Then by the assumed inequality, we know that $\|M_n\|_2 \leq B$ for all n, and hence the sequence satisfies our boundedness requirement. By the computation in the first paragraph as well as (ii) and Theorem C (3), we can obtain

$$\begin{aligned}
\overline{\lim}_{n\to\infty} \quad & \|\mu_n(t) - (\rho_n + \ \mathrm{cov}(x_n(t), M_n))\|_2 \\
\leq \quad & \lim_{n\to\infty} \left\| \mu_n(t) - \left(\rho_n + \sum_{n=1}^{s_n} \tau_{ni}\psi_{ni}(t) \right) \right\|_2 + \overline{\lim}_{n\to\infty} \|e_n\|_2 \left\| \int_{T_n} M_n(t') e_n(t', \omega) d\lambda_n(t') \right\|_2 \\
\leq \quad & \overline{\lim}_{n\to\infty} M \left\| \int_{T_n} M_n(t') e_n(t', \omega) d\lambda_n(t') \right\|_2 = 0,
\end{aligned}$$

and we are done. ∎

Proof of Proposition 10: The equivalence of (2) and (3) is already shown in Proposition 9.

We consider $(iii) \Longrightarrow (i)$. Let $\mu_n(t) = \mu_{n0} + \ \mathrm{cov}(x_n(t), I_n) + \alpha_n(t)$. Then (iii) implies that $\lim_{n\to\infty} \|\alpha_n\|_2 = 0$. By the uniform boundedness condition on the asset returns x_n and on the portfolios I_n, the following inequality

$$|\mu_{n0}| \leq \|\mu_n\|_2 + \|\alpha_n\|_2 + \|\mathrm{cov}(x_n(t), I_n)\|_2$$

implies the boundedness of the sequence $\{\mu_{n0}\}_{n=1}^{\infty}$. For a sequence $\{p_n\}_{n=1}^{\infty}$ of portfolios with finite supremum L_2 norm $\|p\|_2$,

$$\begin{aligned}
E(p_n) &= \int_{T_n} p_n(t)\mu_n(t) d\lambda_n \\
&= \mu_{n0} \int_{T_n} p_n(t) d\lambda_n + \mathrm{cov}\left(\left(\int_{T_n} p_n(t) x_n(t) d\lambda_n, I_n \right) + \int_{T_n} p_n(t)\alpha_n(t) d\lambda_n \right. \\
&= \mu_{n0} C(p_n) + \int_{\Omega} (\mathcal{R}_{p_n} - E(p_n))(\mathcal{R}_{I_n} - E(I_n)) dP + \int_{T_n} p_n(t)\alpha_n(t) d\lambda_n.
\end{aligned}$$

Given the boundedness of the sequence of portfolios $I_n, n \geq 1$, it is easy to check that $\|\mathcal{R}_{I_n} - E(I_n)\|_2$ is bounded. We can now find upper bounds for $|E(p_n)|$:

$$\begin{aligned}
|E(p_n)| &\leq |\mu_{n0} C(p_n)| + \|\mathcal{R}_{p_n} - E(p_n)\|_2 \|\mathcal{R}_{I_n} - E(I_n)\|_2 + \|p_n\|_2 \|\alpha_n\|_2 \\
&\leq |\mu_{n0} C(p_n)| + V(p_n)^{1/2} \|\mathcal{R}_{I_n} - E(I_n)\|_2 + \|p\|_2 \|\alpha_n\|_2.
\end{aligned}$$

If $\lim_{n\to\infty} C(p_n) = \lim_{n\to\infty} V(p_n) = 0$, then by the boundedness of the sequence $\{\mu_{n0}\}_{n=1}^{\infty}$, we can obtain that $\lim_{n\to\infty} E(p_n) = 0$.

Next, let $\mu_{ns} = \mu_n(t) - (\mu_{n0} + \sum_{i=1}^{s_n}(\mu_{ni} - \mu_{n0}s_{ni})\psi_{ni})$. Then it is easy to check that μ_{ns} is orthogonal to the constant function 1 and all the ψ_{ni}. Suppose (i) holds and (ii) fails. Then there is an $\eta > 0$ and a strictly increasing sequence $\{k_m\}_{m=1}^{\infty}$ of positive integers such that $\|\mu_{k_m s}\|_2 \geq \eta$. As in the proof of Proposition 8, define $p_n = 0$ if $n \neq k_m$ for any $m \geq 1$, and $p_n = \mu_{k_m s}$ if $n = k_m$ for some $m \geq 1$. Then $E(p_{k_m}) = \|\mu_{k_m s}\|_2^2 \geq \eta^2$. It is easy to see that

$$V(p_{k_m}) = \int_{\Omega} \left(\int_{T_{k_m}} \mu_{k_m s}(t) e_{k_m}(t, \omega) d\lambda_{k_m} \right)^2 dP.$$

Note that $\|\mu_{k_m s}\|_2 \leq M$ for all m. Thus, the sequence $\{p_n\}_{n=1}^{\infty}$ of portfolios is clearly bounded. By Theorem C (3) in Appendix I, we have $\lim_{n\to\infty} V(p_n) = 0$. Since $C(p_n) = 0$ for all n, we know that $\lim_{n\to\infty} C(p_n) = \lim_{n\to\infty} V(p_n) = 0$, but $\lim_{n\to\infty} E(p_n) \neq 0$, which contradicts the hypothesis of no asymptotic arbitrage. ∎

Footnotes

*This work was presented on April 22, 1996 at a Finance/Economics interdepartmental seminar at the University of Notre Dame. Part of the research was done when Yeneng Sun visited the Department of Economics at Johns Hopkins during June-July 1995; both authors thank Lou Maccini and the Department for support. Yeneng Sun is also grateful to the mathematical finance interest group in NUS for helpful conversations and stimulating seminars.

†Department of Economics, The Johns Hopkins University, Baltimore, MD 21218.

‡Cowles Foundation, Yale University, New Haven, CT 06520; Department of Mathematics, National University of Singapore, Singapore 119260.

[1]In the sequel, as in the literature, we shall use M to denote both the portfolio as well as its random return; no confusion will result.

[2]See Sharpe (1964; pp. 440-441 and Footnote 26 in particular), and also Markowitz (1952, 1959), Samuelson (1967a) and Tobin (1958).

[3]For an intuitive popular account, see Malkiel (1990; Part 3) and his references. For a more technical survey, see Rothschild (1986; Section 4) and his references. For a survey of empirical work, see Jenson (1972), Fischer et al. (1972), Brown (1989) and their references.

[4]See Borch (1969), Feldstein (1969) and Samuelson (1967b, 1970).

[5]Though the CAPM allows correlation among securities, it does not specify the underlying factors responsible for such a correlation.

[6]The no arbitrage assumption is, by necessity, also expressed in an asymptotic form; see Ross (1976) and Huberman (1982, 1987). There has been work on the exact version of the APT pricing formula but under additional assumptions; see Chamberlain (1983) and his references. This additional structure is not our concern here.

[7]The authors see the viability of this concept for asset pricing as one of the two goals of their research; see also Rothschild (1987).

[8]That is, is there a viable version of the law of large numbers in an exact form? Such a property is also informally referred to in the literature as *"aggregation removes individual uncertainty"*; see, for example, Anderson (1991), Feldman-Gilles (1985), Green (1994), Judd (1985), Sun (1994, 1996a), and Uhlig (1996) and their references.

[9]The absence of gains from arbitrage is suggested by the popular aphorism *"there are no profitable opportunities without cost or risk"* rather than the more elaborate *asymptotic* no arbitrage conditions in the literature.

[10]See Doob (1937; Theorem 2.2), Doob (1953; p. 67), Judd (1985), and Feldman and Gilles (1985).

[11]In the context of the random variables $e_t(\cdot)$, a sample function for a given value of ω is $e_\omega(t)$ viewed as a function of t.

[12]It means that even if one obtains the law as described, the *ad hoc* nature of the results based on such a law renders them unacceptable – one can also claim the almost sure failure of the relevant results if the measure was extended differently.

[13]However, see Uhlig (1996) and AL-Najjar (1995) for how far an investigation based on the usual continuum can be pushed.

[14]Sun (1996a) is an announcement and shortened discussion of some of the results in Sun (1994). These include some necessary and sufficient conditions for the law of large numbers and also the biorthogonal representation theorem. Also see Section 2 below and the references therein for definitions and discussion of a Loeb space. Except for some selected paragraphs, the reader may simply think of these as standard probability spaces with cardinality the same as that of the continuum.

[15]See Basilevsky (1994; Section 6.3). The model can also be seen as an analogue of the classical principal components model; see Basilevsky (1994; Section 3.2).

[16]It is often confused in the literature that the CAPM is the one factor version of the APT; see Rothschild (1986; Footnote 11).

[17]See Radcliffe (1994; pp. 173-197) for the discussion of naive and efficient diversification. We note here that there is no uniform terminology in the literature. For example, the terms *non-diversifiable* risk and *diversifiable* risk used here for the CAPM model are also called *systematic* risk and *unsystematic* risk in Sharpe (1964). On the other hand, *systematic* risk and *unsystematic* risk used here for the APT model are also referred to as *non-diversifiable* and *diversifiable* risk in, for example, Rothschild (1986; pp. 116-120), and Ross et al. (1996).

[18]The Karhunen-Loéve expansion theorem is itself the infinite dimensional analogue of the classical principal components model; see Loéve (1977b; Chapter XI), Ash (1965; Appendix), Basilevsky (1994; Chapter 7.8). The theorem has many applications in statistical factor analysis, pattern recognition, and other fields; see, for example, Basilevsky (1994), Fukunaga (1990) and Oja (1983) and their references.

[19]In their textbook, Ross et al. (1996; pp. 293-294) define an unsystematic risk to be one that "specifically affects a single asset or a small group of assets," and a systematic risk to be one that "affects a large number of assets, each to a greater or lesser degree".

[20]This formalization of unsystematic or idiosyncratic risks, though straightforward, may have more significance than one perceives. Indeed, after having presented the intuitive definition of an unsystematic risk, Ross et al. (1996; p. 294) note that "we may not be able to define a systematic risk and an unsystematic risk exactly, but we know them when we see them."

[21]This is a hyperfinite analogue of the classical principal components model or the classical factor model (see Basilevsky (1994; Sections 3.2 and 6.3)). It also has its refined asymptotic counterpart in Sun-Wang (1996).

[22]The procedure is very much in the spirit of the well known Karhunen-Loéve representation of continuous time processes; one essential difference is that the lack of a topology on the universe T of asset indecis prevents an appeal to any ideas of continuity. On the other hand, the Karhunen-Loéve expansion is itself the continuous analogue of the usual principal components model for a finite population. See Loéve (1977; p. 144), Basilevsky (1994; Section 7.5.1), and in particular, the paragraph above Theorem 7.2 there.

[23]This criticism has been most forcefully made by Gilles-LeRoy (1991; pp. 215, 217) who see as their principal result the claim that "any finite-dimensional subspace of [a Hilbert space] M can serve as the factor space F, [and that this] arbitrariness of factor spaces invalidates the APT."

[24]This is a technical assertion pertaining to the *nonstandard* extension. For the convenience of those readers with no background in nonstandard analysis, we will not exploit it even when we turn to the asymptotic versions of our results.

[25]See also the last section in Sun (1996a).

[26]For example, one can reinterpret the classical APT type results of Ross in a hyperfinite model and then obtain exact results by rounding off some infinitesimals. Nothing really new is obtained by this procedure.

[27]The existence of such a process is trivial. One can simply transfer an independent sequence of random variables to obtain a hyperfinite process by standardization. The validity of the law for this hyperfinite process follows from the classic law. See, for example, Keisler (1976), Anderson (1991), Nelson (1987) and Stroyan-Bayod (1986). Green (1994) contains another construction based on a different measure space.

[28]That is, one cannot simply assume that the ensemble of unsystematic risks in a market is described by a particularly constructed stochastic process.

[29]An obvious analogy is to treatments of the concept of perfect competition with only finitely many economic agents.

[30]Unlike the factor structures used in the usual APT models, the factor loadings in the hyperfinite factor model in Sun (1994) are, as noted earlier, also orthonormal (see Theorem A in Appendix I). This

additional orthogonality plays a pivotal role in the various constructions.

[31]For details on Loeb spaces, see Loeb (1975), Cutland (1983), Lindstrom (1988), Anderson (1991) and their references. For a mathematical introduction to nonstandard analysis, see, for example, Cutland (1983), Hurd-Loeb (1985), Lindstrom (1988) and Anderson (1991). For applications of nonstandard analysis to several issues in economic theory, see, for example, Rashid (1987) and Anderson (1991).

[32]As exposited, for example, in Rudin (1974; Chapter 4) and Kadison-Ringrose (1983; Chapter 2). These maps on Hilbert spaces have played a fundamental role in the work of Chamberlain (1983), Chamberlain-Rothschild (1983), Huberman (1987) and Gilles-LeRoy (1991) on APT models.

[33]Loeb spaces are constructed as a simple consequence of Carathéodory's extension theorem and the $\aleph_1$-saturation property of the nonstandard model. For a reference to Carathéodory's extension theorem, see for example Loéve (1977a; pp. 88-90).

[34]Loeb spaces are only used here as standard measure spaces with some desired special standard measure-theoretic properties as stated in Theorem A in Appendix I below. These properties fail for the traditional measure spaces. For readers with no background in nonstandard analysis, they can simply apply these stated special measure-theoretic properties without going into the details of the constructions of hyperfinite sets and Loeb spaces on them. The relevant analogy is to the use of a Lebesgue measure space without going into the details of the Dedekind set-theoretic construction of real numbers, or on the particular construction of Lebesgue measure. For other special standard properties of Loeb spaces and their applications to game theory, see Sun (1993, 1996b) and Khan-Sun (1995a). For applications of the hyperfinite law of large numbers in Sun (1994, 1996a) to other economic problems, see Khan-Sun (1995b).

[35]It is not necessary to use the counting measure on T; it is simply a natural one.

[36]Note that the measurability of g_t and g_ω is a simple consequence of a Fubini type result for Loeb product measures. It is also referred to as Keisler's Fubini theorem; see, Keisler (1984, 1988), and Loeb (1985).

[37]For product measures, and for integration on such measures, see, for example, Loéve (1977a; Chapter VIII) or Rudin (1974; Chapter 7).

[38]See Proposition 5.6 in Sun (1994). If either $L(\lambda)$ or $L(P)$ is purely atomic, then the two are in fact equal to each other. When T is a hyperfinite set, Ω its internal power set, and both endowed with the Loeb counting measure, the proper inclusion for this special case was first observed by Hoover; see Albeverio et al. (1986; Example 3.12.13).

[39]See Loéve (1977b; Chapter VIII) for details as to conditional expectations.

[40]Note that such an operation involves both a product σ-algebra and a natural but significant extension of it – $\mathcal{U}$ and the Loeb product algebra $L(\mathcal{T} \otimes \mathcal{A})$. As such, it has no natural counterpart in standard mathematical practice or in nonstandard mathematics using only internal entities.

[41]Given the discussion in the introduction, the continuous case does not allow a viable law of large numbers, and this removal in the discrete can only be approximate.

[42]It is clear that x is also $L(\lambda \otimes P)$-integrable. An appeal to the Fubini type theorem for Loeb measures as shown by Keisler, then guarantees that μ is a Loeb integrable function on $(T, L(\mathcal{T}), L(\lambda))$. One can understand a Fubini type result on iterated integrals in the hyperfinite Loeb measure setting as the simple observation that hyperfinite sums can be exchanged. Hereafter, when the need arises, we simply change the order of integrals without an explicit statement on the application of any Fubini type results. In the sequel, $\mu(t)$ will also be denoted by μ_t.

[43]As noted in Ross et al. (1996; p. 293), "the unanticipated part of the return, that portion resulting from surprises, is the true risk of any investment. After all, if we had already got what we had expected, there would be no risk and no uncertainty."

[44]Heuristically, if $dL(\lambda)(t)$ is interpreted as an infinitesimal part of an amount of asset t with cost unity (this amount is supposed to be taken to be some big number), then $p(t)dL(\lambda)(t)$ is the amount and cost of shares of asset t with a return $p(t)x(t,\omega)dL(\lambda)$ in the portfolio p. Since these terms are integrable as a function of t, the amount invested and the return pertaining to any asset is infinitesimal,

and therefore any portfolio is well-diversified automatically. Note that $dL(\lambda)(t)$ can be regarded as some small accounting unit in a certain sense.

[45]If there are only m nontrivial factors in the market, the infinite sum should be replaced by a sum with m terms, which will not be mentioned explicitly in the sequel.

[46]Note that it is a trivial matter to require the factors to be orthonormal. It is nontrivial to show that both factors and factor loadings can be orthogonal among themselves.

[47]In the literature, risk and the level of risk are often not distinguished.

[48]See Footnote 18

[49]According to the intuitive discussion in Ross et al. (1996; pp. 293-294), as long as $\alpha \neq 0$, one should call γ a systematic risk. On the other hand, our formal definition does not involve any unsystematic portion, i.e., it is purely systematic. For convenience, we will still refer the γ (α) as a risk (a systematic risk) rather than a systematic risk (a purely systematic risk).

[50]In the terminology of Chamberlain (1983; p.1306), all portfolios are "well-diversified" since they contain only factor variance and no idiosyncratic variance; also Chamberlain-Rothschild (1983; Footnote 3).

[51]As noted earlier, such an exact result is more or less expected within an idealized framework, since approximate results are already known in the large finite case. The proof of the result bypasses any sequential arguments and simply formalizes Ross' (1976) heuristics. Also see Huberman (1982) and Rothschild (1987; Theorem 2; p.118). The second is in the context of an approximate factor structure.

[52]The summability condition follows from the existence of the CAPM carrier portfolio M by Bessel's inequality (Rudin (1974; p. 88)).

[53]It should be noted that unlike Chamberlain-Rothschild (1983), we can go beyond an appeal to the Reisz representation theorem, and give explicit formulas for these portfolios.

[54]It is clear from (6) above that for any portfolio p, $\mathcal{R}_p$ belongs to $\mathcal{L}^2(L(P))$.

[55]This is consistent with the usual practice in the literature; see, for example, Rothschild (1986).

[56]See, for example, Rothschild (1986).

[57]By Theorem 7 (i), this is equivalent to $\sum_{n=1}^{\infty} (\mu_0 s_n - \mu_n)^2 / \lambda_n^4 < \infty$.

[58]By Theorem 6 and Theorem 7 (iv), this is equivalent to $\sum_{n=1}^{\infty} \mu_n^2/\lambda_n^4 < \infty$ and $\sum_{n=1}^{\infty} s_n^2/\lambda_n^4 < \infty$. From the convergence of these two series, it is clear that all the series in Proposition 5 are convergent.

[59]Note that here we already use α to be the constant expected return of all the assets.

[60]The analogue would be a mandatory illustration of every new result using differential calculus by its approximate counterpart in the calculus of finite differences, or one pertaining to differential equations by its difference equation counterpart. The ideal models typically provide a more analytic framework than their discrete counterparts, see Samuelson's Foreward to Merton (1990). Discrete analogues are usually obtained only when numerical computations are considered.

[61]In particular, we do not use transfer arguments for the convenience of readers without any background in nonstandard analysis – the translation of a result on hyperfinite Loeb space to a large finite result necessarily involves the language of nonstandard analysis. Moreover, the results in Sun-Wang (1996) only require uniform boundedness rather than uniform integrability, and thus yield sharper results than those obtained through direct transfer.

[62]Since we are considering a sequence of markets to analyze their asymptotic properties, our definitions of cost and random return are arithmetic averages rather than sums – this is the usual practice, for example, in general equilibrium theory. One can interpret the weight $1/n$ as a sort of small unit allowed to purchase an asset t. Since the cost of x_t is assumed to be one, $1/n$ will be a small number compared with 1 when n is large.

[63]Note that here the notation $||p||_2$ is simply introduced for convenience and the structure of sequences of portfolios will not be exploited.

[64]See Theorem B (3) below, and note that $||x_n||_2^2 = ||f_n||_2^2 + ||\mu_n||_2^2$, and hence $||\mu_n||_2 \leq M$ and $||f_n||_2 \leq M$.

[65]In general, we have no control over the choice of μ_{n0}. If $||h_n||_2$ goes to zero but $||\mu_{n0}||$ goes to infinity too fast, then $||\mu_{n0}h_n||_2$ may not go to zero.

[66]It says that as long as the asymptotic behavior of linear combinations of the observed random variables is concerned, one can ignore the error terms and focus on the factors.

References

ADMATI, A. R., AND P. PFLEIDERER (1985): "Interpreting the Factor Risk Premia in the Arbitrage Pricing Theory" *Journal of Economic Theory* **35**, 191-195.

ALBEVERIO, S., J. E. FENSTAD, R. HOUGH-KROHN, AND T. L. LINDSTROM (1986): *Nonstandard Methods in Stochastic Analysis and Mathematical Physics.* Orlando, Florida: Academic Press.

AL-NAJJAR, N. I. (1995): "Factor Structures and Arbitrage Pricing in Large Asset Markets" mimeo., Northwestern University.

ANDERSON, R. M. (1976): "A Nonstandard Representation for Brownian Motion and Ito Integration" *Israel Journal of Mathematics* **25**, 15-46.

ANDERSON, R. M. (1991): "Non-standard Analysis with Applications to Economics" in *Handbook of Mathematical Economics* ed. W. Hildenbrand and H. Sonnenschein. New York: North-Holland Publishing Company.

ASH, R. (1965): *Information Theory.* New York: Dover Publications, Inc.

BASILEVSKY, A. (1994): *Statistical Factor Analysis and Related Methods.* New York: John Wiley.

BLACK, F., M. C. JENSEN AND M. SCHOLES (1972): "The Capital Asset Pricing Model: Some Empirical Tests" in M. C. Jensen (ed.) *Studies in the Theory of Capital Markets.* New York: Praeger.

BROWN, S. J. (1989): "The Number of factors in Security Returns" *Journal of Finance* **44**, 1247-1262.

CHAMBERLAIN, G. (1983): "Funds, Factors and Diversification in Arbitrage Pricing Models" *Econometrica* **51**, 1305-1323.

CHAMBERLAIN, G., AND M. ROTHSCHILD (1983): "Arbitrage, Factor Structure and Mean-Variance Analysis on Large Asset Markets" *Econometrica* **51**, 1281-1304.

CUTLAND, N. (1983). "Nonstandard Measure Theory and its Applications" *Bull. London Math. Soc.* **15** 529-589.

DOOB, J. L. (1953): *Stochastic Processes.* New York: John Wiley and Sons.

DYBVIG, P., AND S. A. ROSS (1987): "Arbitrage" in J. Eatwell, P. K. Newman, M. Milgate (eds.) *The New Palgrave.* London: MacMillan Publishing Company.

FELDMAN, M., AND C. GILLES (1985): "An Expository Note on Individual Risk without Aggregate Uncertainty" *Journal of Economic Theory* **35**, 26-32.

FUKUNAGA, K. (1990): *Introduction to Statistical Pattern Recognition,* 2nd Edition. New York: Academic Press.

GILLES, C., AND S. F. LEROY (1991): "On the Arbitrage Pricing Theory" *Economic Theory* **1**, 213-230.

GREEN, E. J. (1994): "Individual-level Randomness in a Nonatomic Population" mimeo., University of Minnesota.

HARMAN, H. H. (1968): *Modern Factor Analysis,* 2nd edition. Chicago: University of Chicago Press.

HUBERMAN, G. (1982): "A Simple Approach to Arbitrage Pricing Theory" *Journal of Economic Theory* **28**, 183-191.

HUBERMAN, G. (1987): "Arbitrage Pricing Theory" in J. Eatwell, P. K. Newman, M. Milgate (eds.) *The New Palgrave.* London: MacMillan Publishing Company.

HURD, A. E., AND P. A. LOEB (1985). *An Introduction to Nonstandard Real Analysis.* Orlando: Academic Press.

INGERSOLL, J. (1984): "Some Results on the Theory of Arbitrage Pricing" *Journal of Finance* **39**, 1021-1039.

JENSEN, M. C. (1972): "Capital Markets, Theory and Evidence" *Bell Journal of Economics and management Science* **3** 357-398.

JUDD, K. L. (1985): "The Law of Large Numbers with a Continuum of iid Random Variables" *Journal of Economic Theory* **35**, 19-25.

KADISON, R. V., AND J. R. RINGROSE (1983): *Fundamentals of the Theory of Operator Algebras.* New York: Academic Press, Inc.

KEISLER, H. J. (1984): "An Infinitesimal Approach to Stochastic Analysis" *Memoirs of the American Mathematical Society* **48**, 1-184.

KEISLER, H. J. (1988): "Infinitesimals in Probability Theory" in N. Cutland (ed.) *Nonstandard Analysis and its Applications* (New York: Cambridge University Press), 106-139.

KHAN, M. ALI, AND Y. N. SUN (1995a): "Non-Cooperative Games on Hyperfinite Loeb Spaces" Johns Hopkins Working Paper No. 359.

KHAN, M. ALI, AND Y. N. SUN (1995b): "Characterizations of Individual Risks in Large Markets" Memeo.

LINDSTROM, T. (1988): "An Invitation to Nonstandard Analysis" in N. Cutland (ed.) *Nonstandard Analysis and its Applications* (New York: Cambridge University Press), 1-106.

LINTNER, J. (1965): "The Valuation of Risky Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets, *Review of Economics and Statistics* **47**, 13-37.

LOEB, P. A. (1975): "Conversion from Nonstandard to Standard Measure Spaces and Applications in Probability Theory" *Transactions of the American Mathematical Society* **211**, 113-122.

LOEB, P. A. (1985): "A Nonstandard Functional Approach to Fubini's Theorem" *Proceedings of the American Mathematical Society* **93**, 343-346.

LOÉVE, M. (1977a): *Probability Theory I,* 4th Edition. New York: Springer-Verlag.

LOÉVE, M. (1977b): *Probability Theory II,* 4th Edition. New York: Springer-Verlag.

MALKIEL, B. G. (1990): *A Random Walk down Wall Street* New York: W. W. Norton.

MARKOWITZ, H. (1952): "Portfolio selection" *Journal of Finance* **7**, 77-91.

MARKOWITZ, H. (1959): *Portfolio Selection: Efficient Diversification of Investment* New York: Wiley.

MERTON, R. C. (1990): *Continuous-Time Finance,* Oxford, Basil Blackwell.

MOSSIN, J. (1966): "Equilibrium in a Capital Asset Market," *Econometrica* **34**, 768-783.

NELSON, E. (1987): *Radically Elementary Probability Theory,* Princeton: Princeton University Press.

OJA, E. (1983): *Subspace Methods of Pattern Recognition,* New York: Wiley.

REISMAN, H. (1988): "A General Approach to the Arbitrage Pricing Theory (APT)" *Econometrica* **56**, 473-476.

ROSS, S. A. (1976): "The Arbitrage Theory of Capital Asset Pricing" *Journal of Economic Theory* **13**, 341-360.

ROSS, S. A., R. W. WESTERFIELD, AND J. JAFFE (1996): *Corporate Finance,* 4th Edition. Chicago: Irwin.

ROTHSCHILD, M. (1986): "Asset Pricing Theories" in *Uncertainty, Information and Communication: Essays in honour of K. J. Arrow,* ed. by W. P. Heller, R. M. Starr and D. Starett. Cambridge: Cambridge University Press.

RUDIN, W. (1974): *Real and Complex Analysis,* New York: McGraw-Hill Book Co.

SAMUELSON, P. A. (1967a): "General Proof that Diversification Pays" *Journal of Financial and Quantitative Analysis* **2**, 1-13.

SAMUELSON, P. A. (1967b): "Efficient Portfolio Selection for Pareto-Levy Investments" *Journal of Financial and Quantitative Analysis* **2**, 107-122.

SAMUELSON, P. A. (1970): "The Fundamental Approximation Theorem of Portfolio Analysis in Terms of Means, Variances and Higher Moments" *Review of Economic Studies* **37**, 537-542.

SHANKEN, J. (1992): "The Current State of the Arbitrage Pricing Theory" **47**, 1569-1574.

SHARPE, W. (1964): "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk" *Journal of Finance* **33**, 885-901.

SUN, Y. N. (1993): "Integration of Correspondences on Loeb Spaces" mimeo., National University of Singapore, forthcoming in *Transactions of the American Mathematical Society.*

SUN, Y. N. (1994): "A Theory of Hyperfinite Processes" mimeo., National University of Singapore.

SUN, Y. N. (1996a): "Hyperfinite Law of Large Numbers" *Bulletin of Symbolic Logic* **2**, 189-198.

SUN, Y. N. (1996b): "Distributional Properties of Correspondences on Loeb Spaces" *Journal of Functional Analysis* **139**, 68-93.

SUN, Y. N., AND L. WANG (1996): "Asymptotic Endogenous Factor Models" mimeo., National University of Singapore.

TOBIN, J. (1958): "Liquidity Preference as Behavior towards Risk" *Review of Economic Studies* **25**, 65-86.

UHLIG, H. (1996): "A Law of Large Numbers for Large Economies" *Economic Theory* **8**, 41-50.

VARIAN, H. (1993) "A Portfolio of Nobel Laureates: Markowitz, Miller and Sharpe" *Journal of Economic Perspectives* **7**, 169-169.