

**Nash and Walras Equilibrium via Brouwer**

**By**

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# Nash and Walras Equilibrium via Brouwer

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## Abstract

The existence of Nash and Walras equilibrium is proved via Brouwer's Fixed Point Theorem, without recourse to Kakutani's Fixed Point Theorem for correspondences. The domain of the Walras fixed point map is confined to the price simplex, even when there is production and weakly quasi-convex preferences. The key idea is to replace optimization with "satisficing improvement," i.e., to replace the Maximum Principle with the "Satisficing Principle."

Keywords: Equilibrium, Nash, Walras, Brouwer, Kakutani

JEL Classification: C6, C62

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\*I wish to thank Ken Arrow, Don Brown, and Andreu Mas-Colell for helpful comments. I first thought about using Brouwer's theorem without Kakutani's extension when I heard Herb Scarf's lectures on mathematical economics as an undergraduate in 1974, and then again when I read Tim Kehoe's 1980 Ph.D dissertation under Herb Scarf, but I did not resolve my confusion until I had to discuss Kehoe's presentation at the celebration for Herb Scarf's 65th birthday in September, 1995.

Mordecai Kurz has been an inspiration for a whole generation of economists. I vividly remember many blissful summers at the IMSSS in Stanford, listening to the programs Mordecai masterfully put together. Those summer sessions defined economic theory for their time, and defined the standards of excellence we all tried to live up to. In retrospect, the late 70s and early 80s appear clearly as a golden era in the history of economic theory, and it is hard to believe things would have turned out so well if it weren't for IMSSS, and for Mordecai's energy, enthusiasm, and tenacity as its director.

## 1 Introduction

The standard proofs of the existence of Nash and Walras equilibrium (including the original proofs by Nash [18], Arrow and Debreu [2], and McKenzie [16]) rely on Kakutani's Fixed Point Theorem for correspondences. I show that a slight perturbation of the standard arguments enables one to work entirely with Brouwer's Fixed Point Theorem for continuous functions.<sup>1</sup>

Nash himself [19] gave a Brouwer fixed point proof of Nash equilibrium for the special case of matrix games. McKenzie [17] derived the existence of Walras equilibrium with production from Brouwer's Fixed Point Theorem. The only advantage of the maps I propose is that some readers may think they are simpler. For example, in my Walras existence proof the domain of the fixed point map is the price simplex. There is no need to enlarge the domain to include excess demands, as done by Gale [9] and Debreu [6], [7], or the demands of each consumer, as done in the generalized game proofs of Debreu [5] and Arrow and Debreu [2], or to add the auxiliary commodities introduced by McKenzie [17].<sup>2</sup>

In Section 2, the existence of Nash equilibrium in concave games is proved. Let a game  $G = (u_n, \Sigma_n)_{n \in N}$  be described by its payoffs  $u_n$  and compact, convex strategy spaces  $\Sigma_n$ , for agents  $n \in N$ . The original proof by Nash relied on the best response correspondence  $B_n(\bar{\sigma}_n, \bar{\sigma}_{-n}) = \operatorname{argmax}_{\sigma_n \in \Sigma_n} u_n(\sigma_n, \bar{\sigma}_{-n})$ . My proof simply replaces  $B_n$  with a satisficing improvement *function*

$$\beta_n(\bar{\sigma}_n, \bar{\sigma}_{-n}) = \operatorname{argmax}_{\sigma_n \in \Sigma_n} [u_n(\sigma_n, \bar{\sigma}_{-n}) - \|\sigma_n - \bar{\sigma}_n\|^2].$$

If  $u_n$  is concave in  $\sigma_n$ , it can easily be shown that  $\beta_n$  always moves agent  $n$  part of the way to his optimal response against  $\bar{\sigma}_{-n}$ . Moving all the way to a best response

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<sup>1</sup>Of course Kakutani's FPT can be derived from Brouwer's FPT, so in a sense all these standard proofs are derivable from Brouwer. But I mean there is a single continuous function, not involving any approximations and selection, whose fixed points are Walras equilibria.

<sup>2</sup>Thus in the proofs (9), (6), (7) the dimension of the domain of the fixed point map is  $(L - 1) + (L - 1)$ , where  $L$  is the number of commodities. In the proofs (5), (2), the dimension of the domain is  $(L - 1) + (H + F)(L - 1)$ , where  $H$  is the number of households and  $F$  the number of firms. In the proof (16) the dimension is  $(L - 1) + F$ . All of the proofs (9), (6), (7), (5), (2) are based on Kakutani's fixed point theorem. My proof uses Brouwer's fixed point theorem on a domain of dimension  $(L - 1)$ .

is irrelevant to demonstrating that a fixed point is an equilibrium. Section 1 also includes a discussion of earlier demonstrations of Nash equilibrium based on Brouwer's FPT for *matrix* games.

In Section 3 the existence of Walras equilibrium is proved for economies  $E = ((u^h, e^h)_{h \in H}, (Y_f)_{f \in F}, (\theta_f^h)_{f \in F}^{h \in H})$  with quasi-concave utilities  $u^h$  and convex technologies  $Y_f$ . Let  $M^h(p, \bar{p})$  be the minimum net expenditure household  $h$  must make at prices  $p$  beyond its Walrasian income  $I^h(p)$  in order to achieve the same utility it would obtain if it faced prices  $\bar{p}$  and income  $I^h(\bar{p})$ .<sup>3</sup> It is well-known that  $M^h$  is continuous in  $(p, \bar{p})$  and concave in  $p$  for any fixed  $\bar{p}$ . Let  $M(p, \bar{p})$  be the sum of the  $M^h(p, \bar{p})$  over all households  $h$ . Let  $S$  be the price simplex. In Section 3 it is shown that the function  $\varphi : S \rightarrow S$  defined for each  $\bar{p}$  in  $S$  by

$$\varphi(\bar{p}) = \arg \max_{p \in S} [M(p, \bar{p}) - \|p - \bar{p}\|^2]$$

is continuous and has Walras equilibria as its fixed points.

The minimum expenditure function and its properties have been very closely studied since Hicks showed that the so-called Hicksian demand is more regular than the Marshallian demand. Intermediate textbooks often emphasize the duality between utility maximization and expenditure minimization. Precisely this duality guarantees (through the Maxmin theorem) that a fixed point of the function  $\varphi$  must be a Walras equilibrium. Nevertheless, though there are many closely related ideas to be found in the literature, to the best of my knowledge nobody has used the function  $M$  to demonstrate the existence of equilibrium.

To understand the genesis of the function  $M$ , let us temporarily suppose that the Walrasian demand correspondence  $D^h(\bar{p})$ , and the Walrasian supply correspondence  $Y_f(\bar{p}) = \arg \max_{y_f \in Y_f} \bar{p} \cdot y_f$ , and therefore also the Walrasian aggregate excess demand correspondence  $Z(\bar{p}) = \sum_{h \in H} (D^h(\bar{p}) - e^h) - \sum_{f \in F} Y_f(\bar{p})$ , are all single valued functions, which we denote by  $d^h(\bar{p})$ ,  $y_f(\bar{p})$ ,  $z(\bar{p})$ . (If utilities are strictly concave, and production sets strictly convex, this will be the case, assuming we enclose the economy in a compact space.) In that case we can define a continuous function  $\psi : S \rightarrow S$

$$\psi(\bar{p}) = \arg \max_{p \in S} [p \cdot z(\bar{p}) - \|p - \bar{p}\|^2]$$

whose fixed points are Walrasian equilibrium prices, as we show in Section 4.

When  $Z(\bar{p})$  is multivalued, there does not, at first glance, seem to be an analogue for  $\psi$ . However, define  $D_+^h(\bar{p})$  as the set of all consumption bundles (budget feasible and not) that make agent  $h$  at least as well off as his Walrasian demands  $D^h(\bar{p})$ . Define the "better than excess demand correspondence"  $Z_+$  by  $Z_+(\bar{p}) = \sum_{h \in H} (D_+^h(\bar{p}) - e^h) - \sum_{f \in F} Y_f$ , where firms choose anything feasible. A crucial advantage of  $Z_+$  over  $Z$  is that it is lower semicontinuous as well as upper semicontinuous. We show in Section 3 that

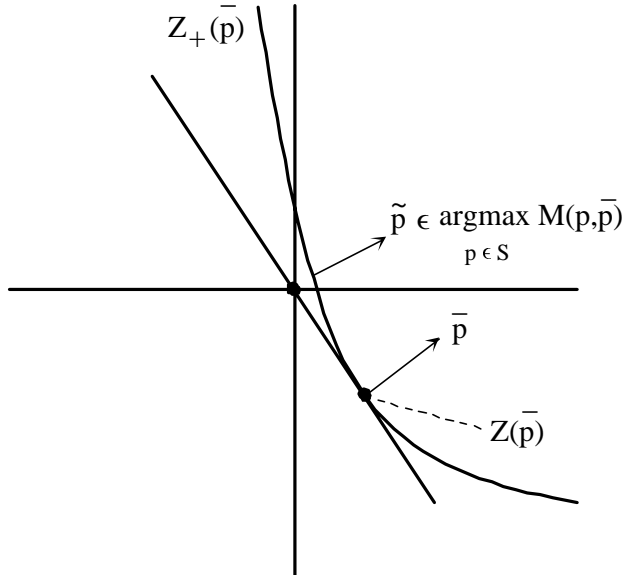
$$\varphi(\bar{p}) = \arg \max_{p \in S} \left[ \min_{z \in Z_+(\bar{p})} p \cdot z - \|p - \bar{p}\|^2 \right]$$

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<sup>3</sup>Income is defined by  $I^h(p) = p \cdot e^h + \sum_{f \in F} \theta_f^h \max_{y_f \in Y_f} p \cdot y_f$ .

defines a continuous *function* from the simplex to itself whose fixed points are Walrasian equilibria. In fact this is the same  $\varphi$  given earlier, since

$$M(p, \bar{p}) = \min_{z \in Z_+(\bar{p})} p \cdot z.$$



**Figure 1**

In the standard Kakutani existence proof pioneered by Debreu (see Arrow–Debreu [2]), the price player chooses  $p$  to maximize the value of a given excess demand  $z$ . The vector  $z$  is an independent argument in the fixed point map. In my proof the price player chooses  $p$  to maximize the cost of achieving a given social welfare  $(v^h)_{h \in H}$ , where  $v^h$  is a utility level for agent  $h$ . The  $(v^h)_{h \in H}$  are in turn derived from prices  $\bar{p}$ ,  $v^h = v^h(\bar{p})$ , the indirect utilities at Walrasian prices  $\bar{p}$ , so that prices are the lone independent variables.

The mapping  $\varphi$  naturally suggests a potential Lyapunov function  $L : S \rightarrow \mathbb{R}$  defined by

$$L(\bar{p}) \equiv \max_{p \in S} [M(p, \bar{p}) - \|p - \bar{p}\|^2].$$

It might be interesting to establish conditions for the underlying economy guaranteeing that  $L(\varphi(\bar{p})) < L(\bar{p})$  for all  $\bar{p} \in S$ , but this line of inquiry is not pursued here.

In Section 4 I examine several special economies with strictly quasi-concave utilities  $u^h$ , for which there are already standard proofs of Walras equilibrium based on Brouwer’s FPT. In the first special case we also take the  $Y_f$  strictly convex, so excess demand  $Z(\bar{p})$  is a *function*  $z(\bar{p})$ , as mentioned earlier. By replacing  $M$  with  $N \equiv \min_{z \in Z(\bar{p})} p \cdot z$ , obtaining

$$N(p, \bar{p}) \equiv \min_{z \in Z(\bar{p})} p \cdot z = p \cdot z(\bar{p})$$

we obtain the function  $\psi$  defined earlier.<sup>4</sup> The map  $\psi$  is quite different from the standard Brouwer map (deriving from Nash’s matrix game map) that is expounded in most textbooks, but it turns out that  $\psi(p)$  reduces to another one of the standard Brouwer maps, namely  $h(p) = \text{Proj}_S(p + \frac{1}{2}z(p))$ . But whereas it requires the Kuhn–Tucker theorem to verify that a fixed point of  $h$  is a Walras equilibrium, it is immediate that a fixed point of  $\psi$  is an equilibrium. Thus our perturbation  $-||p - \bar{p}||^2$  still simplifies matters, even when dealing with excess demand *functions*. We apply similar maps in other special cases, e.g., with constant-returns-to-scale technologies (CRS).<sup>5</sup> In this case  $\psi$  turns out to be closely related to the maps used by Todd [24] and Kehoe [12] to compute equilibria of economies with fixed coefficient technologies.

The only technical point in this paper occurs in showing that the function  $M(p, \bar{p})$  is continuous, which is tantamount to showing that the “better than” correspondence  $Z_+(\bar{p})$  is upper semi-continuous (USC) and lower semi-continuous (LSC). This in fact is trivial, but I prove it after introducing a new lemma called the Satisficing Principle, which could perhaps stand just behind the Maximum Principle as a useful tool in the theory of choice, because it guarantees LSC and USC. The impression the student is sometimes left holding is that LSC is less central than USC, but we should not forget that the Maximum Principle cannot be applied unless the budget correspondence of each agent is USC and LSC.

The Satisficing Principle supposes that an agent is maximizing a continuous utility  $u_\alpha(x)$  subject to a constraint  $x \in \beta(\alpha)$  over which he is locally nonsatiated. Suppose he is satisfied with a payoff  $w(\alpha) < v(\alpha)$ , where  $v(\alpha)$  is the maximum achievable utility given the exogenous parameters  $\alpha$ , and  $w$  is any continuous function. Then the correspondence  $W(\alpha)$  of all choices achieving payoff at least  $w(\alpha)$  is lower semi-continuous (LSC) as well as upper semi-continuous (USC) in  $\alpha$ , provided that  $\beta(\alpha)$  is. The Satisficing Principle complements the Maximum Principle, which guarantees that  $v(\alpha)$  is continuous and that the set of choices achieving  $v(\alpha)$  is USC but not necessarily LSC. One immediate application of the Satisficing Principle is that the Walrasian budget correspondence is LSC and USC when the endowment is strictly positive. More importantly, since the Walrasian indirect utility function  $w^h(p)$  is continuous, and by nonsatiation, strictly less than the maximal utility  $v^h(p) = v$  achievable without a budget constraint, the Satisficing Principle guarantees the LSC and USC of  $D_+^h(p)$ , and hence of  $Z_+(p)$ .

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<sup>4</sup>Note that for any pair  $(p, \bar{p})$ ,  $M(p, \bar{p}) \leq N(p, \bar{p})$ ; usually  $M(p, \bar{p}) < N(p, \bar{p})$ . Indeed when excess demand  $Z$  is a correspondence, as will typically be the case without further assumptions,  $N(p, \bar{p})$  is not continuous. Even when  $Z(\bar{p})$  is a function, and  $N$  is continuous,  $M(p, \bar{p}) \neq N(p, \bar{p})$ . The function  $N$  has nevertheless often been used to prove the existence of equilibrium. In one such approach the prices  $p$  are called “better” than the prices  $\bar{p}$  if  $N(p, \bar{p}) > 0$ . Walras equilibrium then exists if it can be shown that this partial ordering on prices has a maximal element. The problem is thus reduced to one of maximizing a (nontransitive) binary relation, for which see Nikaido [21], Fan [8], Sonnenschein [23], and Aliprantis and Brown [1]. For a lucid exposition of these ideas, see Border [3]. Along these lines, see also the proof of the K–K–M–S theorem via Brouwer in Krasa–Yannelis [14].

<sup>5</sup>An interesting feature of each successive Walras existence proof is that Brouwer’s fixed point theorem must be augmented by Farkas’ Lemma (when technology is given by a finite number of activities), the separating hyperplane theorem (when technology is given more generally by a cone), and the MinMax theorem (when technological possibilities are given by arbitrary convex sets).

The Satisficing Principle is stated and proved in Section 5, where it is also used to give a Brouwer FPT proof that quasi-concave games have Nash equilibria. In some sense the whole idea of this paper comes down to replacing optimization with satisficing improvement; first for the game players and the auctioneer, by subtracting  $\|\sigma_n - \bar{\sigma}_n\|^2$  or  $\|p - \bar{p}\|^2$ , and second for the households, in substituting  $Z_+(\bar{p})$  for  $Z(\bar{p})$ .

## 2 Games and Nash Equilibrium

### 2.1 Concave Perturbation Lemma

My proofs rely on the following concave perturbation lemma:

**Concave Perturbation Lemma** *Let  $X \subset \mathbb{R}^n$  be convex, and let  $\bar{x} \in X$ . Let  $u : X \rightarrow \mathbb{R}$  be concave. Then  $\arg \max_{x \in X} [u(x) - \|x - \bar{x}\|^2]$  is at most a single point, and if  $\bar{x} = \arg \max_{x \in X} [u(x) - \|x - \bar{x}\|^2]$ , then  $\bar{x} \in \arg \max_{x \in X} u(x)$ .*

**Proof** Since  $u$  is concave in  $x$ , and  $-\|x - \bar{x}\|^2$  is strictly concave in  $x$ ,  $[u(x) - \|x - \bar{x}\|^2]$  is strictly concave, and  $\arg \max_{x \in X} [u(x) - \|x - \bar{x}\|^2]$  cannot contain two distinct points. Suppose  $\bar{x} = \arg \max_{x \in X} [u(x) - \|x - \bar{x}\|^2]$ . Take any  $x \in X$ . By hypothesis, and by the convexity of  $X$  and the concavity of  $u$ , for any  $0 < \varepsilon < 1$ ,

$$\begin{aligned} 0 &\geq \{u([(1 - \varepsilon)\bar{x} + \varepsilon x]) - \|[(1 - \varepsilon)\bar{x} + \varepsilon x] - \bar{x}\|^2\} - \{u(\bar{x}) - \|\bar{x} - \bar{x}\|^2\} \\ &= u([(1 - \varepsilon)\bar{x} + \varepsilon x]) - \varepsilon^2\|x - \bar{x}\|^2 - u(\bar{x}) \\ &\geq (1 - \varepsilon)u(\bar{x}) + \varepsilon u(x) - u(\bar{x}) - \varepsilon^2\|x - \bar{x}\|^2 \\ &= \varepsilon(u(x) - u(\bar{x})) - \varepsilon^2\|x - \bar{x}\|^2 \end{aligned}$$

So

$$\begin{aligned} u(x) - u(\bar{x}) &\leq \varepsilon\|x - \bar{x}\|^2 \text{ for all } \varepsilon > 0, \text{ so} \\ u(x) - u(\bar{x}) &\leq 0 \end{aligned} \quad \blacksquare$$

### 2.2 Concave Games

Let a game  $G$  among  $N$  players be defined by compact and convex strategy spaces  $\Sigma_1, \dots, \Sigma_N$  in finite-dimensional Euclidean spaces, and by continuous payoff functions  $u_1, \dots, u_N$ , where for each  $n \in N$ ,  $u_n : \Sigma \equiv \Sigma_1 \times \dots \times \Sigma_N \rightarrow \mathbb{R}$ . We call  $G$  a concave game if for any fixed  $\bar{\sigma}_{-n} \equiv (\bar{\sigma}_1, \dots, \bar{\sigma}_{n-1}, \bar{\sigma}_{n+1}, \dots, \bar{\sigma}_N) \in \Sigma_{-n} \equiv \Sigma_1 \times \dots \times \Sigma_{n-1} \times \Sigma_{n+1} \times \dots \times \Sigma_N$ ,  $u_n(\sigma_n, \bar{\sigma}_{-n})$  is concave in  $\sigma_n$ .

Given a game  $G = (\Sigma_1, \dots, \Sigma_N; u_1, \dots, u_N)$ , a Nash equilibrium is a choice  $\bar{\sigma} = (\bar{\sigma}_1, \dots, \bar{\sigma}_N) \in \Sigma$  such that for all  $n \in N$  and all  $\sigma_n \in \Sigma_n$ ,

$$u_n(\bar{\sigma}) \geq u_n(\sigma_n, \bar{\sigma}_{-n}).$$

**Theorem** *Every concave game has a Nash equilibrium.*

**Proof** Define the function

$$\varphi_n : \Sigma \rightarrow \Sigma_n \text{ by}$$

$$\varphi_n(\bar{\sigma}_1, \dots, \bar{\sigma}_n, \dots, \bar{\sigma}_N) = \arg \max_{\sigma_n \in \Sigma_n} [u_n(\sigma_n, \bar{\sigma}_{-n}) - \|\sigma_n - \bar{\sigma}_n\|^2].$$

Observe that the maximand is the sum of a continuous, concave function in  $\sigma_n$ , and a negative quadratic function in  $\sigma_n$ , and hence is continuous and strictly concave. Since  $\Sigma_n$  is compact and convex,  $\varphi_n$  is a well-defined function. Furthermore, the maximand is continuous in the parameter  $\bar{\sigma} = (\bar{\sigma}_1, \dots, \bar{\sigma}_n)$ , hence by the maximum principle,  $\varphi_n$  is a continuous function.

Now define  $\varphi : \Sigma \rightarrow \Sigma$  by  $\varphi = (\varphi_1, \dots, \varphi_N)$ . Clearly  $\varphi$  is continuous, and so by Brouwer's theorem it has a fixed point  $\varphi(\bar{\sigma}) = \bar{\sigma}$ .

By the concave perturbation lemma, for all  $\sigma_n \in \Sigma_n$ ,  $u_n(\sigma_n, \bar{\sigma}_{-n}) \leq u_n(\bar{\sigma})$ . Hence  $\bar{\sigma}$  is a Nash equilibrium.  $\blacksquare$

Nash [18] suggested the correspondence  $\psi_n : \Sigma \rightrightarrows \Sigma_n$  defined by  $\psi_n(\bar{\sigma}) = \arg \max_{\sigma_n \in \Sigma_n} u_n(\sigma_n, \bar{\sigma}_{-n})$ . Since  $u_n$  is not necessarily strictly concave,  $\psi_n(\bar{\sigma})$  may contain multiple elements.

The maximand above is simply a perturbation of the Nash maximand. It guarantees that a player will always make some improvement when there is an opportunity to improve, but he will not necessarily move all the way to his best response. Another difference is that the Nash correspondence  $\psi_n$  throws away some information, since  $\psi_n$  actually is defined on  $\Sigma_{-n}$ . The map  $\varphi_n$  depends on all the coordinates, including  $\Sigma_n$ .

### 2.3 Matrix Games

Two player matrix games are defined by  $r \times s$  matrices  $A$  and  $B$ . Player  $\alpha$  has strategy space  $\Sigma_\alpha \equiv \{p \in \mathbb{R}_+^r : \sum_{i=1}^r p_i = 1\}$  and player  $\beta$  has strategy space  $\Sigma_\beta = \{q \in \mathbb{R}_+^s : \sum_{j=1}^s q_j = 1\}$ . The payoffs are defined by  $u_\alpha(p, q) \equiv p'Aq$  and  $u_\beta(p, q) \equiv p'Bq$ . Since  $u_n$  is linear on  $\Sigma_n$  for  $n = \alpha$  and  $\beta$ , these matrix games are indeed concave games.

Nash [19] showed that for matrix games, Brouwer's Fixed Point Theorem sufficed. He suggested using the excess return functions  $z_\alpha(\bar{p}, \bar{q}) = A\bar{q} - (\bar{p}'A\bar{q})1$  and  $z_\beta(\bar{p}, \bar{q}) = \bar{p}'B - (\bar{p}B\bar{q})1$ , which specify the surplus each agent can get by playing each pure strategy instead of his designated mixed strategy. He then defined the map

$$f(\bar{p}, \bar{q}) = \left( \frac{\bar{p} + [A\bar{q} - (\bar{p}'A\bar{q})1]^+}{1 + [A\bar{q} - (\bar{p}'A\bar{q})1]^+ \cdot 1}, \frac{\bar{q} + [\bar{p}'B - (\bar{p}B\bar{q})1]^+}{1 + [\bar{p}'B - (\bar{p}B\bar{q})1]^+ \cdot 1} \right),$$

where for any vector  $y$ ,  $[y]^+$  is the vector with  $i$ th coordinate  $\max(0, y_i)$ , and  $1$  is the vector of all 1's, or just the scalar 1, depending on the context. A fixed point of the Nash map can be shown to be a Nash equilibrium by observing that  $\bar{p}'[A\bar{q} - (\bar{p}'A\bar{q})1] = 0$ . Indeed this same trick is copied in the now standard existence proof for Walrasian equilibrium, where it crops up as Walras law. The Nash map  $f$  exploits the special form of matrix games.



The map  $\varphi$  can be used for any concave game, not just matrix games. In the special case of matrix games, a short computation shows that it reduces to

$$\varphi(\bar{p}, \bar{q}) = h(\bar{p}, \bar{q}) \equiv \left( \Pi_{\Sigma_\alpha} \left( \bar{p} + \frac{1}{2} A \bar{q} \right), \Pi_{\Sigma_\beta} \left( \bar{q} + \frac{1}{2} \bar{p}' B \right) \right),$$

where  $\Pi_K(x)$  is the closest point in  $K$  to  $x$ . The map  $h$  has already been used to prove the existence of Nash equilibrium in matrix games by Lemke–Howson [15], and to study the index of matrix game Nash equilibrium by Gul–Pearce–Stacchetti [11]. To see that  $\varphi$  reduces to  $h$  for matrix games, one needs to use the Kuhn–Tucker theorem. Indeed, one needs the Kuhn–Tucker theorem to verify that a fixed point of  $h$  is a Nash equilibrium.<sup>6</sup> But as we saw in the proof of our first theorem, using  $\varphi$  avoids the need for the Kuhn–Tucker theorem.

### 3 Walrasian Economies

#### 3.1 The Walrasian Economy

Let us represent an economy by

$$E = \left\{ H, (X^h, e^h, u^h)_{h \in H}, F, (Y_f)_{f \in F}, (\theta_f^h)_{f \in F}^{h \in H} \right\},$$

where  $H$  is a finite set of households,  $X^h \subset \mathbb{R}^L$  is the consumption set of household  $h$ ,  $e^h$  is the endowment, and  $u^h$  is the utility function of agent  $h \in H$ ,  $F$  is a finite set of firms,  $Y_f$  is the technology of firm  $f \in F$ , and  $\theta_f^h \in \mathbb{R}_+$  is the ownership share of firm  $f$  by agent  $h$ ,  $\sum_{h \in H} \theta_f^h = 1$  for all  $f \in F$ . Following Arrow–Debreu [2], we assume in addition that  $\forall h \in H$ ,

- (1)  $X^h$  is closed, convex, and bounded from below:  $\exists \underline{d}^h$  such that  $\underline{d}^h \leq x$  for all  $x \in X^h$
- (2)  $e^h \in X^h$  and  $\exists d^h \in X^h$  with  $d^h \ll e^h$
- (3a)  $u^h : X^h \rightarrow \mathbb{R}$  is continuous
- (3b)  $u^h$  is quasi-concave, i.e.,  $[u^h(x) > u^h(y) \text{ and } 0 < \lambda < 1] \Rightarrow [u^h(\lambda x + (1 - \lambda)y) > u^h(y)]$ , for all  $x, y \in X^h$
- (3c)  $u^h$  is nonsatiated, i.e.,  $\forall y \in X^h, \exists x \in X^h$  with  $u^h(x) > u^h(y)$   
and for all  $f \in F$ ,
- (4)  $Y_f$  is a closed convex subset of  $\mathbb{R}^L$ , and  $0 \in Y_f$   
and furthermore,
- (5) If  $Y \equiv \sum_{f \in F} Y_f$ , then  $Y \cap \mathbb{R}_+^L = \{0\}$
- (6) Irreversibility:  $Y \cap -Y = \{0\}$ .

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<sup>6</sup>By the Kuhn–Tucker theorem,  $\varphi(\bar{p}, \bar{q}) = (\varphi_\alpha(\bar{p}, \bar{q}), \varphi_\beta(\bar{p}, \bar{q}))$  satisfies  $A\bar{q} - 2(\varphi_\alpha(\bar{p}, \bar{q}) - \bar{p}) - \lambda e + \Lambda = 0$ , where  $\Lambda \geq 0$  is a diagonal matrix with  $\Lambda_{jj} > 0$  only if  $\varphi_{\alpha_j}(\bar{p}, \bar{q}) = 0$ . By the Kuhn–Tucker theorem, the map  $h(\bar{p}, \bar{q}) = (h_\alpha(\bar{p}, \bar{q}), h_\beta(\bar{p}, \bar{q}))$  satisfies  $-2(h_\alpha(\bar{p}, \bar{q}) - \frac{1}{2} A \bar{q} - \bar{p}) + \mu e + \Omega = 0$ , where  $\Omega \geq 0$  is a diagonal matrix with  $\Omega_{jj} > 0$  only if  $h_{\alpha_j}(\bar{p}, \bar{q}) = 0$ .

### 3.2 Walras Equilibrium

A Walras equilibrium (WE) for the economy  $E$  is a tuple  $(\bar{p}, (\bar{x}^h)_{h \in H}, (\bar{y}_f)_{f \in F}) \in \mathbb{R}_+^L \times \mathbf{X}_{h \in H} X^h \times \mathbf{X}_{f \in F} Y_f$  satisfying

$$(a) \quad \sum_{h \in H} \bar{x}^h \leq \sum_{h \in H} e^h + \sum_{f \in F} \bar{y}_f$$

$$(b) \quad \bar{y}_f \in \arg \max_{y_f \in Y_f} \bar{p} \cdot y_f, \forall f \in F$$

$$(c) \quad \bar{x}^h \in B^h(\bar{p}) = \{x \in X^h : \bar{p} \cdot x \leq \bar{p} \cdot e^h + \sum_{f \in F} \theta_f^h \max_{y_f \in Y_f} \bar{p} y_f \equiv I^h(\bar{p})\}, \forall h \in H$$

$$(d) \quad \bar{x}^h \in \arg \max_{x \in B^h(\bar{p})} u^h(x).$$

By nonsatiation and quasi-concavity, we know that at a WE each agent spends all his income, so the budget inequality in (c) reduces to equality, and we therefore conclude that in a WE,

$$\sum_{h \in H} x_i^h < \sum_{h \in Y} e_i^h + \sum_{f \in F} \bar{y}_{f,i} \Rightarrow \bar{p}_i = 0. \quad (1.1)$$

### 3.3 Easy Consequences of the Assumptions

It follows from (1.1) that we obtain an equivalent definition of equilibrium by strengthening the definition of equilibrium to require equality of supply and demand in condition (a), provided that we augment production by allowing free disposal, replacing  $Y$  with  $\hat{Y} = Y - \mathbb{R}_+^L$ . So without loss of generality we require equality in (a) but also assume

$$(7) \quad \text{Free disposal: } Y - \mathbb{R}_+^L = Y.$$

As shown in Arrow–Debreu [2], assumptions (1)–(6) have the consequence that  $\mathcal{A} \equiv \{(x^1, \dots, x^H, y_1, \dots, y_F) \in \mathbf{X}_{h \in H} X^h \times \mathbf{X}_{f \in F} Y_f : \sum_{h \in H} (x^h - e^h) - \sum_{f \in F} y_f \leq 0\}$  is compact. In view of the quasi-concavity of the utilities, restricting the consumption sets from  $X^h$  to  $X^h \cap \hat{X}^h$  and restricting the technologies from  $Y_f$  to  $Y_f \cap \hat{Y}_f$  where  $\hat{X}^h$  and  $\hat{Y}_f$  are compact and convex and such that  $\mathcal{A}$  is contained in the interior of  $\mathbf{X}_{h \in H} \hat{X}^h \times \mathbf{X}_{f \in F} \hat{Y}_f$  gives rise to an economy  $\hat{E}$  with exactly the same Walras equilibria as  $E$ . Thus without loss of generality, we may add assumption (8) and weaken assumption (3c):

$$(8) \quad X^h \text{ and } Y^f \text{ are compact for all } h \in H \text{ and } f \in F,$$

which requires weakening (3c) to

$$(3c) \quad [(x^1, \dots, x^H, y_1, \dots, y_F) \in \mathcal{A}] \Rightarrow [\forall h \in H, \exists \hat{x}^h \in X^h, u^h(\hat{x}^h) > u^h(x)].$$

An implication of the convexity of  $X^h$  from (1), and the quasi-concavity of  $u^h$  from (3b), is that

(3d)  $u^h$  is locally nonsatiated in  $X^h$  :  $\forall y \in X^h$ , if  $\exists x \in X^h$  with  $u^h(x) > u^h(y)$ , then  $\exists \{x(n)\}_{n=1}^\infty \subset X^h$ ,  $x(n) \rightarrow y$  with  $u^h(x(n)) > u^h(y)$  for all  $n$ .

We list six more simple observations. All lemmas rely on assumptions (1)-(8). Lemmas 1 and 2 rely on the definitions of USC and LSC, and on the Satisficing Principle, all of which are deferred to Section 5.

**Lemma 1** *The budget correspondence  $B^h(p)$  is USC, LSC, nonempty valued, and compact-valued on  $S = \{p \in \mathbb{R}_+^L : \sum p_\ell = 1\}$ .*

**Proof** This is a standard and trivial result. Instead of proving it directly, we note that it is a corollary of the satisficing principle proved in Section 5.

$$B^h(p) = \{x \in X^h : p \cdot x \leq I^h(p)\} = \{x \in X^h : -p \cdot x \geq -I^h(p)\}.$$

Let  $w(p) \equiv -I^h(p)$  be the satisficing threshold. Let  $v(p) \equiv \max_{x \in X^h} -p \cdot x$  be the maximal threshold. Since  $e^h \gg d^h$ , for all  $p \in S$ ,  $w(p) = -I^h(p) \leq -p \cdot e^h < -p \cdot d^h \leq \max_{x \in X^h} -p \cdot x = v(p)$ , so the lemma follows from the compactness of  $X^h$ , the continuity of  $I^h(p)$ , and the Satisficing Principle. ■

Let  $v^h(p) \equiv \max_{x \in B^h(p)} u^h(x)$  be the so-called indirect utility function of agent  $h$ . Since  $B^h(p)$  is USC and LSC, nonempty valued and compact-valued, by the Maximum Principle,  $v^h(p)$  must be continuous on  $S$ . Furthermore, let

$$D^h(p) \equiv \arg \max_{x \in B^h(p)} u^h(x)$$

be the demand correspondence of agent  $h$ . Again by the Maximum Principle,  $D^h(p)$  is USC. Unfortunately,  $D^h(p)$  may not be LSC, as is well known.

A central element of the existence proof given in Section 3.4 is the replacement of the demand correspondence  $D^h(p)$ , which may fail to be LSC, with the “demand or better” correspondence  $D_+^h(p)$ , which is always LSC. McKenzie [17] used a similar correspondence.

**Lemma 2**  *$D_+^h(p) = \{x \in X^h : u^h(x) \geq v^h(p)\}$  is USC, LSC, and nonempty-valued for  $p \in S$ . Hence so is the better than excess demand  $Z_+(p) = \sum_{h \in H} D_+^h(p) - \sum_{h \in H} e^h - \sum_{f \in F} Y_f$ .*

**Proof** The USC and nonemptiness of  $D_+^h$  follow immediately from the continuity of  $u^h$ . As for LSC, let  $p(n) \rightarrow p$  and let  $x \in D_+^h(p)$ . Let  $y(0) \in \arg \max\{u^h(y) : y \in X^h\}$ . If  $u^h(x) = u^h(y(0))$ , then  $u^h(x) \geq v^h(p(n)) \forall n$  and so letting  $x(n) = x$  for  $n \geq 1$  shows the LSC of  $D_+^h$  at  $p$ . If  $u^h(x) < u^h(y(0))$ , then by local nonsatiation (3d),  $\exists y(m) \rightarrow x$  with  $u^h(y(m)) > u^h(x)$  for all  $m \geq 1$ . Since the indirect utility  $v^h$  is continuous,  $v^h(p(n)) \rightarrow v^h(p)$ . Hence for  $n \geq 1$  we can define  $x(n) = y(m(n))$ , where  $m(n) \equiv \max_{0 \leq m \leq n} \{u^h(y(m)) \geq v^h(p(n))\}$ . Then  $x(n) \in D_+^h(p(n))$  and  $x(n) \rightarrow x$ , showing the LSC of  $D_+^h$ . The sum of USC (LSC) correspondences whose range is compact is also USC (LSC).

Lemma 2 can also be derived from the satisficing principle. Let  $w(p) \equiv v^h(p)$  be the satisficing threshold, and let  $v^*(p) = v^* = \max_{x \in X^h} u^h(x)$  be the maximum threshold. Apply the Satisficing Principle, noting that  $X^h$  and  $u^h$  are independent of  $p$ , and that  $v^h(p)$  is continuous. ■

**Lemma 3** *The minimum expenditure function*

$$M(p, \bar{p}) \equiv \min_{z \in Z_+(\bar{p})} p \cdot z$$

*is continuous in  $(p, \bar{p}) \in S \times S$ , and concave in  $p$  for any fixed  $\bar{p} \in S$ .*

**Proof** Lemma 2 and the Maximum Principle guarantee the continuity of  $M(p, \bar{p})$ . For any fixed  $\bar{p}$ ,  $M(p, \bar{p})$  is the minimum of a family of linear functions in  $p$ , hence it must be concave. ■

**Lemma 4** *For all  $\bar{p} \in S$ ,  $Z(\bar{p}) \subset Z_+(\bar{p})$ . Hence  $M(\bar{p}, \bar{p}) \leq 0$ .*

**Proof** Obvious. ■

The following Lemmas 5 and 6 show the role of the so-called “duality principle” that utility maximization and expenditure minimization are the same at points where nonsatiation holds. Lemma 5 also uses the linearity of unconstrained expenditure minimization.

**Lemma 5** *If for some  $\bar{p} \in S$ , there is  $\bar{z} \in Z_+(\bar{p})$  with  $\bar{z} \leq 0$ , then  $\exists \bar{x}^h \in X^h \forall h$  and  $\bar{y}_f \in Y_f \forall f$  such that  $(\bar{p}, (\bar{x}^h)_{h \in H}, (\bar{y}_f)_{f \in F})$  is a Walrasian equilibrium.*

**Proof** If  $\bar{z} \in Z_+(\bar{p})$ , then by definition there is  $\bar{x}^h \in D_+^h(\bar{p}) \forall h \in H$ , and  $\bar{y}_f \in Y_f \forall f \in F$  with  $\bar{z} = \sum_{h \in H} \bar{x}^h - \sum_{h \in H} e^h - \sum_{f \in F} \bar{y}_f$ . Since  $\bar{z} \leq 0$ , nonsatiation obtains from (3c) and we deduce from local nonsatiation (3d) that  $\bar{p} \cdot \bar{x}^h \geq I^h(\bar{p}) \forall h \in H$ . But  $\bar{z} \leq 0$  and  $\bar{p} \in S$  implies that  $0 \geq \bar{p} \cdot \bar{z} = \bar{p} \sum_{h \in H} \bar{x}^h - \bar{p} [\sum_{h \in H} e^h + \sum_{f \in F} \bar{y}_f] \geq \sum_{h \in H} I^h(\bar{p}) - \sum_{h \in H} I^h(\bar{p}) = 0$ . Hence  $\bar{p} \cdot \bar{x}^h = I^h(\bar{p}) \forall h \in H$  and  $\bar{y}_f \in \arg \max_{y_f \in Y_f} \bar{p} \cdot y_f, \forall f \in F$ . ■

It is worth noting that (assuming local nonsatiation), neither the quasi-concavity of the  $u^h$  nor the convexity of the  $Y_f$  played any role in proving Lemmas 1–5.

**Lemma 6** *If for some  $\bar{p} \in S$ ,  $\max_{p \in S} M(p, \bar{p}) = M(\bar{p}, \bar{p})$ , then  $\exists \bar{x}^h \in X^h \forall h$  and  $\bar{y}_f \in Y_f \forall f$  such that  $(\bar{p}, (\bar{x}^h)_{h \in H}, (\bar{y}_f)_{f \in F})$  is a Walrasian equilibrium.*

**Proof** We now invoke the convexity of the  $X^h$  and  $Y_f$ , and the quasi-concavity of  $u^h$ , to assert the convexity of  $Z_+(\bar{p})$ . The minmax theorem then guarantees that  $\exists \bar{z} \in Z_+(\bar{p})$  with  $M(\bar{p}, \bar{p}) = \max_{p \in S} p \cdot \bar{z} = \bar{p} \cdot \bar{z} = \min_{z \in Z_+(\bar{p})} \bar{p} \cdot z$ . Since by Lemma 4,  $M(\bar{p}, \bar{p}) \leq 0$ , we must have  $\bar{z} \leq 0$  (if  $\bar{z}_i > 0$ , take  $p_i = 1$ ). Hence by Lemma 5,  $\bar{p}$  is a Walrasian equilibrium price vector. ■

### 3.4 Existence of Walras Equilibrium

We now construct an existence proof of Walras equilibrium for general quasi-concave preferences and convex production sets, that uses only the domain of prices  $S$ , and only Brouwer's fixed point theorem.

**Theorem** *Let  $E = (H, (u^h)_{h \in H}, F, (Y_f)_{f \in F}, (\theta_f^h)_{f \in F}^{h \in H})$  be a Walras economy satisfying assumptions (1)–(6). Then  $E$  has a Walras Equilibrium  $(\bar{p}, (\bar{x}^h)_{h \in H}, (\bar{y}_f)_{f \in F})$ .*

**Proof** Recalling that  $Z_+(\bar{p}) = \sum_{h \in H} D_+^h(\bar{p}) - \sum_{h \in H} e^h - \sum_{f \in F} Y_f(\bar{p})$  is the at least as good as excess demand, and that  $M(p, \bar{p}) = \min_{z \in Z_+(\bar{p})} p \cdot z$ , define  $\varphi : S \rightarrow S$  by

$$\begin{aligned} \varphi(\bar{p}) &= \arg \max_{p \in S} [M(p, \bar{p}) - \|p - \bar{p}\|^2] \\ &= \arg \max_{p \in S} \left[ \min_{z \in Z_+(\bar{p})} p \cdot z - \|p - \bar{p}\|^2 \right]. \end{aligned}$$

Since (by Lemma 3)  $M$  is concave in  $p$  for any fixed  $\bar{p}$ , and  $\|p - \bar{p}\|^2$  is quadratic, the maximand is strictly concave, so it has a unique maximum and  $\varphi(\bar{p})$  is a function. Since by Lemma 3  $M$  is continuous (equivalently, since  $Z_+(\bar{p})$  is USC and LSC),  $\varphi$  is a continuous function. Therefore by Brouwer's fixed point theorem,  $\varphi$  has a fixed point  $\bar{p}$ .

At the fixed point  $\bar{p}$ ,

$$M(\bar{p}, \bar{p}) = \max_{p \in S} [M(p, \bar{p}) - \|p - \bar{p}\|^2] = \max_{p \in S} M(p, \bar{p}).$$

where the last equality follows from the concavity of  $M$  in  $p$  and the concave perturbation lemma. By Lemma 6,  $\bar{p}$  is a Walrasian equilibrium price vector. ■

Again it is worth noting that the quasi-convexity of the  $X^h$  and the convexity of the  $Y_f$  played no role until the very last step where they guaranteed the convexity of  $Z_+(p)$  at the single point  $p = \bar{p}$ . In traditional proofs of Walrasian existence, it is important to make sure that the excess demand correspondence is convex at every point  $p$  (otherwise there might not be a fixed point).

This observation might be of some value in proving the existence of Walras equilibrium in models with a continuum of agents without the quasi-concavity hypothesis. However, I do not pursue this here.

## 4 Comparisons to Earlier Proofs: Walras Equilibrium with Strictly Convex Preferences

The main difference between the standard proofs of Walrasian existence and the proof just given in Section 3 is that the latter only requires Brouwer's fixed point theorem, applied to a domain of dimension  $L - 1$ . Another difference is that the latter proof has a natural "Lyapunov function"  $L : S \rightarrow \mathbb{R}$  given by

$$L(\bar{p}) \equiv \max_{p \in S} [M(p, \bar{p}) - \|p - \bar{p}\|^2].$$

I do not pursue the question of identifying conditions under which  $L$  declines under the dynamic  $\bar{p} \mapsto \varphi(\bar{p})$ .

Instead I turn to explaining the connection between my method of proof and the standard methods when excess demand is already a function. Taking advantage of the unicity of the excess demand, my proof can be modified to show its connection to earlier proofs.

In this section we specialize the general Walrasian economy given in Section 3 to cases where we can work with excess demand *functions*. For these cases it is already known that Brouwer's Theorem suffices to prove the existence of Walras equilibrium. But we show here that the perturbation  $-\|p - \bar{p}\|^2$  can still simplify matters.

### 4.1 Pure Exchange and Strictly Convex Technologies

Let  $S = \{p \in \mathbb{R}_+^L : \sum_{i=1}^L p_i = 1\}$  be the usual price simplex.

Let  $z$  be called an excess demand function whenever  $z : S \rightarrow \mathbb{R}^L$  is a continuous function satisfying Walras Law:  $p \cdot z(p) = 0 \forall p \in S$ .<sup>7</sup>

We define a Walras equilibrium for the excess demand function  $z$  as a price vector  $\bar{p} \in S$  satisfying

$$z(\bar{p}) \leq 0.$$

Note that by Walras Law,  $z_i(\bar{p}) = 0$  unless  $\bar{p}_i = 0$ , in which case we may have  $z_i(\bar{p}) < 0$ .

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<sup>7</sup>Suppose that, in addition to assumptions (1)–(7) from Section 2, for all  $h \in H$ ,

$$[u^h(x) \geq u^h(y)] \Rightarrow [u^h(\lambda x + (1-\lambda)y) > u^h(y)]$$

if  $0 < \lambda < 1$ ,  $x \neq y$  and  $x, y \in X^h$ , and for all  $f \in F$

$$[x \neq y \in Y_f, 0 < \lambda < 1] \Rightarrow [\exists z \in Y_f \text{ with } z \gg \lambda x + (1-\lambda)y].$$

Then  $z(p) = \sum_{h \in H} D^h(p) - \sum_{h \in H} e^h - \sum_{f \in F} \arg \max_{y_f \in Y_f} p \cdot y_f$  is a continuous function satisfying Walras Law. In the special case  $Y_f = \{0\} \forall f \in F$ , we have a pure exchange economy.

**Theorem** *Every excess demand function has a Walras equilibrium.*

**Proof** Define the map  $\psi : S \rightarrow S$  by

$$\psi(\bar{p}) \equiv \arg \max_{p \in S} [p \cdot z(\bar{p}) - \|p - \bar{p}\|^2].$$

Observe that the maximand is the sum of a linear function in  $p$  and a quadratic function in  $p$ , hence it is strictly concave and continuous in  $p$ . Since  $S$  is compact and convex,  $\psi(\bar{p})$  is a single point, and so  $\psi$  is a function. By the maximum principle,  $\psi$  is a continuous function (since the parameters  $z(\bar{p})$  and  $\bar{p}$  move continuously as  $\bar{p}$  varies).

Hence by Brouwer's Fixed Point Theorem,  $\psi$  has a fixed point  $\bar{p}$ . By the concave perturbation lemma,  $p \in S \Rightarrow p \cdot z(\bar{p}) \leq \bar{p} \cdot z(\bar{p})$ . By Walras Law,  $\bar{p} \cdot z(\bar{p}) = 0$ , which implies  $z(\bar{p}) \leq 0$ . ■

Debreu's [7] proof of Walras equilibrium uses the correspondence  $\delta(z) = \arg \max_{p \in S} p \cdot z$ . As Debreu said,  $\delta$  is motivated by the principle that when there is excess demand in some commodity,  $z_i > 0$ , prices should go up, at least where excess demand is greatest. The only drawback to Debreu's construction is that  $\psi(z)$  may be multivalued, thus forcing the use of Kakutani's Fixed Point Theorem. The function  $\psi(p)$  is obtained by a slight perturbation of Debreu's construction.

The best known continuous function for proving Walras equilibrium is obtained by imitating the Nash [19] fixed point map for matrix games:  $g_i(p) \equiv \{p_i + [z_i(p)]^+\} / \{1 + \sum_{j=1}^L [z_j(p)]^+\}$ , where  $[x]^+ = \max\{x, 0\}$ , for  $i = 1, \dots, L$ . A simple, but slightly awkward argument, using Walras law, shows that a fixed point of  $g$  is a Walras equilibrium.

The function  $\psi(p)$  is (surprisingly) identical to the map  $h(p) = \Pi_S(p + \frac{1}{2}z(p))$ , where  $\Pi_S(x)$  is the closest point in  $S$  to  $x$ .<sup>8</sup> By deriving  $\psi$  from the above maximization, one can see transparently that a fixed point is a Walrasian equilibrium. On the other hand, to show that a fixed point of  $h$  on the boundary of  $S$  is an equilibrium, the Kuhn–Tucker theorem must be invoked.

## 4.2 Production with Constant Returns-to-Scale Technologies

We now consider CRS production. A constant returns-to-scale (CRS) technology is a set  $Y \subset \mathbb{R}^L$  such that  $Y$  is a closed, convex, cone ( $y \in Y$  implies  $ty \in Y$  for all  $t \geq 0$ ; in particular,  $0 \in Y$ ). Furthermore we suppose that  $Y$  allows for free disposal;  $z \leq y$  and  $y \in Y$  implies  $z \in Y$ . Finally, we suppose there is some  $p^* \in S$  with  $p^* \cdot Y \leq 0$ , i.e.,  $p^* \cdot y \leq 0$  for all  $y \in Y$ .

A Walras equilibrium with production for an excess demand function, CRS-technology pair  $(z, Y)$  is a price  $\bar{p} \in S$  such that  $z(\bar{p}) \in Y$  and  $\bar{p}Y \leq 0$ . Note that by

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<sup>8</sup>By the Kuhn–Tucker theorem,  $\psi(\bar{p}) = \arg \max_{p \in S} [p \cdot z(\bar{p}) - \|p - \bar{p}\|^2]$  satisfies  $(\psi(\bar{p}) - \bar{p}) = \frac{1}{2}z(\bar{p}) - \lambda e + \Lambda$  where  $\Lambda \geq 0$  is a diagonal matrix with  $\Lambda_{jj} > 0$  only if  $\psi_j(\bar{p}) = 0$ . Similarly by the Kuhn–Tucker theorem  $h(\bar{p}) = \arg \min_{p \in S} \|p - [\bar{p} + \frac{1}{2}z(\bar{p})]\|^2$  satisfies the same equation.

Walras Law the production plan  $z(\bar{p})$  chosen makes zero profits, while alternatives either lose money or do no better.

The central example of a CRS-technology is an activity analysis production technology given by the matrix  $B = [-I \ A]$  where  $I$  is the  $L \times L$  identity matrix and  $A$  is an  $L \times n$  vector of activities. Each column of the  $B$  matrix represents an “activity.” Positive elements correspond to outputs, negative entries in  $B$  correspond to inputs. The first  $L$  columns of  $B$  represent pure disposal. The activity matrix  $B$  determines the CRS-technology

$$Y = \{Bx | x \in \mathbb{R}_+^{L+n}\}.$$

Clearly  $Y$  is a convex, closed cone allowing for free disposal. If for some vector  $W \gg 0$ ,  $\{x \in \mathbb{R}_+^{L+n} : Bx + W \geq 0\}$  is bounded, then there must be a  $p^* \in S$  with  $p^* \cdot Y \leq 0$ .

**Technology Lemma** *If  $Y$  is a CRS-technology and for some vector  $z \in \mathbb{R}^L$ ,  $[p \in S$  and  $pY \leq 0] \Rightarrow pz \leq 0$ , then  $z \in Y$ .*

**Proof** Suppose  $z \notin Y$ . Since  $Y$  is closed and convex, by the separating hyperplane theorem we can strictly separate  $Y$  and  $z$ , that is find some  $\bar{p} \in \mathbb{R}^L$  such that  $\bar{p} \cdot Y < \bar{p} \cdot z$ . But  $Y$  is a cone, so  $\bar{p} \cdot Y$  bounded above implies  $\bar{p} \cdot Y \leq 0$ ; also  $0 \in Y$ , so we have  $\bar{p} \cdot Y \leq 0 < \bar{p} \cdot z$ . By free disposal,  $\bar{p} \cdot Y \leq 0$  implies  $\bar{p} \geq 0$ . Scaling  $\bar{p}$ , we get  $p \in S$  and  $pY \leq 0 < p \cdot z$ , contradicting the hypothesis. ■

**Theorem** *Every excess demand function, CRS-technology pair  $(z, Y)$  has a Walras equilibrium.*

**Proof** We seek  $\bar{p} \in S_Y \equiv \{p \in S : p \cdot Y \leq 0\}$  with  $z(\bar{p}) \in Y$ . By the technology lemma, it suffices to find  $\bar{p} \in S_Y$  such that  $p \in S_Y \Rightarrow p \cdot z(\bar{p}) \leq 0 = \bar{p} \cdot z(\bar{p})$ .

By hypothesis,  $S_Y$  is nonempty. Furthermore,  $S_Y \equiv \bigcap_{y \in Y} \{p \in S : p \cdot y \leq 0\}$  is the intersection of closed and convex sets, and so is closed and convex.

Define  $\psi : S_Y \rightarrow S_Y$  by

$$\psi(\bar{p}) \equiv \arg \max_{p \in S_Y} [p \cdot z(\bar{p}) - \|p - \bar{p}\|^2].$$

As we argued earlier,  $\psi$  is a continuous function. Since  $S_Y$  is compact and convex, Brouwer’s Fixed Point Theorem guarantees  $\psi$  has a fixed point  $\bar{p}$ .

From the concave perturbation lemma, at the fixed point  $\bar{p}$ ,  $p \in S_Y \Rightarrow p \cdot z(\bar{p}) \leq \bar{p} \cdot z(\bar{p}) = 0$ . ■

The idea that Brouwer’s theorem alone can be used to prove the existence of Walras equilibrium with production is due to McKenzie [17] who also used the set  $S_Y$ . His mapping is much more elaborate than  $\psi$ , but it allows for excess demand correspondences. McKenzie [17] showed that one could always reduce convex technologies to CRS-technologies by adding  $F$  auxiliary commodities, representing the contributions of the owners to each firm. The fixed point map must then be carried



out in a simplex of dimension  $L + F - 1$ . In the above proof the domain is the original  $L - 1$  dimensional simplex.

Todd [24] suggested the map  $h(p) = \Pi_{S_Y}[p + z(p)]$ . (A similar map is in Kehoe [12].) He showed by the Kuhn–Tucker theorem that a fixed point of  $h$  must be a Walras equilibrium, when  $Y$  is given by an activity analysis technology. The map  $\psi$  is identical, its only advantage being a perhaps more transparent proof that a fixed point is a Walras equilibrium (and the incorporation of general CRS  $Y$ ).

### 4.3 Unbounded Consumption Sets, Monotonic Preferences and Boundary Behavior

In Sections 4.1 and 4.2 we assumed that the excess demand function  $z$  is continuous on all of  $S$ , including at  $p \in S$  where some prices  $p_i$  may be zero. This will be true whenever utilities  $u^h$  are strictly concave, and consumption sets  $X^h$  are compact, as we indicated in Section 3.1. Some authors prefer to skip the step where we bound the consumption sets, preferring for aesthetic reasons not to invoke Assumption (8) (see Section 3.1). In its place they make the substantive assumption of strict monotonicity. I show now that the method of proof indicated in Section 4.2 still applies. To that end, let  $S^0$  be the interior of  $S$ , and  $\partial S$  be its boundary. For every  $\varepsilon > 0$ , let  $S^\varepsilon \equiv \{p \in S : p \geq \varepsilon 1\}$  be the trimmed simplex, and  $\partial S^\varepsilon$  its boundary, where  $1 = (1, \dots, 1)$ .

We say that  $(z, Y)$  is an excess demand function, CRS-technology pair with proper boundary behavior whenever  $z : S^0 \rightarrow \mathbb{R}^L$  is a continuous function satisfying Walras Law for all  $p \in S^0$ , and such that  $\exists \varepsilon > 0$  and  $\exists p^* \in S^\varepsilon$ , satisfying

$$p^* \cdot Y \leq 0 . \tag{1}$$

$$p \in \partial S^\varepsilon \Rightarrow p^* \cdot z(p) > 0 , \tag{2}$$

When preferences are strictly monotonic,  $p \rightarrow \partial S \Rightarrow$  some  $z_i(p) \rightarrow \infty$ . Since excess demand is bounded from below by the aggregate endowment of goods, strict monotonicity implies that for any  $p^* \gg 0$ ,  $p^* \cdot z(p) > 0$  if  $p$  is close enough to the boundary. Thus proper boundary behavior is automatically satisfied by excess demand functions derived from strictly monotonic preferences, provided we can find some strictly positive prices  $p^*$  at which  $p^* \cdot Y \leq 0$ . This latter condition is trivially verified if for example there is some indispensable input like labor that is never produced.<sup>9</sup>

**Theorem** *Every monotonic excess demand function, CRS-technology pair with proper boundary behavior has a Walras equilibrium.*

**Proof**  $S^\varepsilon$  is compact and convex. Hence  $S_Y^\varepsilon \equiv S^\varepsilon \cap S_Y$  is also compact and convex. Define  $\psi : S_Y^\varepsilon \rightarrow S_Y^\varepsilon$  by

$$\psi(\bar{p}) \equiv \arg \max_{p \in S_Y^\varepsilon} [p \cdot z(\bar{p}) - \|p - \bar{p}\|^2] .$$

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<sup>9</sup>For a refinement of this boundary condition, see Neufeind [19].

As before,  $\psi$  is a continuous function, hence it has a fixed point  $\bar{p}$ . Again by the familiar argument,  $p \in S_Y^\varepsilon \Rightarrow p \cdot z(\bar{p}) \leq \bar{p} \cdot z(\bar{p}) = 0$ .

If some  $\bar{p}_i = \varepsilon$ , then by proper boundary behavior,  $p^* \cdot z(\bar{p}) > 0$ , a contradiction, since  $p^* \in S_Y^\varepsilon$ . Hence  $\bar{p} \gg \varepsilon 1$ . But then by concavity of the maximand,  $p \in S_Y \Rightarrow p \cdot z(\bar{p}) \leq 0$ . By the technology lemma,  $z(\bar{p}) \in Y$ , so  $\bar{p}$  is a Walras equilibrium. ■

## 5 The Satisficing Principle and Quasi-Concave Games

### 5.1 The Satisficing Principle

Recall that the famous Maximum Principle asserts that the best response correspondence is upper semi-continuous (USC). The USC property is the crucial hypothesis in Kakutani's fixed point theorem for correspondences. Kakutani's theorem is used instead of Brouwer precisely because the best response correspondence may not be lower semi-continuous (LSC). What I show below is that if we replace maximization with almost maximization (satisficing), then the satisficing correspondence is LSC and USC.

Let  $\mathcal{A} \subset \mathbb{R}^m$  and  $X \subset \mathbb{R}^n$ , and let  $\psi : \mathcal{A} \rightrightarrows X$  be a correspondence associating with each  $\alpha \in \mathcal{A}$  a subset  $\psi(\alpha) \subset X$ . We say that  $\psi$  is upper semi-continuous (USC) if

$$\left. \begin{array}{l} \alpha_n \rightarrow \alpha \\ x_n \rightarrow x \\ x_n \in \psi(\alpha_n) \end{array} \right\} \Rightarrow x \in \psi(\alpha)$$

for any  $\{x_n, x\} \subset X$ ,  $\{\alpha_n, \alpha\} \subset \mathcal{A}$ . We say that  $\psi$  is lower semi-continuous (LSC) iff

$$\left. \begin{array}{l} \alpha_n \rightarrow \alpha \\ x \in \psi(\alpha) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \exists x_n \rightarrow x \\ x_n \in \psi(\alpha_n) \end{array} \right.$$

for any  $\{\alpha_n, \alpha\} \subset \mathcal{A}$  and  $x \in X$ .

We say that  $\psi$  is USC or LSC at a point  $\bar{\alpha} \in \mathcal{A}$  if the above conditions hold when  $\alpha = \bar{\alpha}$ . Clearly  $\psi$  is USC or LSC if it is USC or LSC at each point  $\bar{\alpha} \in \mathcal{A}$ .

Let  $u : \beta \rightarrow \mathbb{R}$ , where  $\beta \subset \mathbb{R}^n$ . We say that  $u$  is locally nonsatiated in  $\beta$  if for any pair  $x, y \in \beta$  with  $u(x) < u(y)$ , there is a sequence  $\{x(n)\}_{n=1}^\infty \subset \beta$  with  $x(n) \rightarrow x$  and  $u(x(n)) > u(x)$  for all  $n$ .

If  $\beta$  is convex and  $u$  is quasi-concave, then it follows immediately that  $u$  is locally nonsatiated in  $\beta$ .

**Satisficing Principle** Let  $u : X \times \mathcal{A} \rightarrow \mathbb{R}$  be a continuous function, where  $X \times \mathcal{A} \subset \mathbb{R}^n \times \mathbb{R}^m$ . Let  $\beta : \mathcal{A} \rightrightarrows X$  be a nonempty, USC and LSC correspondence. For each fixed  $\alpha \in \mathcal{A}$ , let  $u(\cdot, \alpha)$  be locally nonsatiated in  $\beta(\alpha)$ . Let  $v : \mathcal{A} \rightarrow \mathbb{R} \cup \{\infty\}$  be the maximum value function defined by  $v(\alpha) \equiv \sup_{x \in \beta(\alpha)} u(x, \alpha)$ . Finally, let  $w : \mathcal{A} \rightarrow \mathbb{R}$  be continuous and satisfy  $w(\alpha) < v(\alpha)$  for all  $\alpha \in \mathcal{A}$ . Then the correspondence  $W : \mathcal{A} \rightrightarrows X$  defined by

$$W(\alpha) \equiv \{x \in \beta(\alpha) : u(x, \alpha) \geq w(\alpha)\}$$

is USC and LSC, and nonempty valued.

If in addition  $\beta(\alpha) = \beta$  for all  $\alpha \in \mathcal{A}$ , and  $u(x, \alpha) = u(x)$  for all  $(x, \alpha) \in X \times \mathcal{A}$ , then the same conclusion holds even with a weak inequality  $w(\alpha) \leq v(\alpha) \equiv v$  for all  $\alpha \in \mathcal{A}$ .

**Proof** The nonemptiness of  $W$  is evident. USC follows as in the maximum principle, and does not depend on the strict inequality  $w(\alpha) < v(\alpha)$ . Simply note that if  $\{x_n \in W(\alpha_n) \text{ for all } n, \text{ and } \alpha_n \rightarrow \alpha \text{ and } x_n \rightarrow x\}$ , then by USC of  $\beta$ ,  $x \in \beta(\alpha)$ . By hypothesis,  $u(x_n, \alpha_n) \geq w(\alpha_n)$ . Passing to the limit, and recalling the continuity of  $u$  and  $w$ ,  $u(x, \alpha) \geq w(\alpha)$ , so  $x \in W(\alpha)$ .

To prove LSC of  $W$ , let  $\alpha_n \rightarrow \alpha$ . Suppose  $\bar{x} \in W(\alpha)$  and  $u(\bar{x}, \alpha) > w(\alpha)$ . From the LSC of  $\beta$ , we can find  $\bar{x}_n \in \beta(\alpha_n)$ ,  $\bar{x}_n \rightarrow \bar{x}$ . From the continuity of  $u$  and  $w$ , for large  $n$ , say,  $n \geq N$ ,  $u(\bar{x}_n, \alpha_n) > w(\alpha_n)$ . Thus  $\bar{x}$  can be approached by  $\bar{x}_n$  in  $W(\alpha_n)$  if  $u(\bar{x}, \alpha) > w(\alpha)$ . It remains to verify that any  $\hat{x} \in W(\alpha)$  with  $u(\hat{x}, \alpha) = w(\alpha)$  can be approached. If  $v(\alpha) > w(\alpha)$ , there is some  $\bar{x} \in \beta(\alpha)$  with  $u(\bar{x}, \alpha) > w(\alpha)$ . By local nonsatiation, we can take a sequence of  $\bar{x}(k) \in \beta(\alpha)$  converging to  $\hat{x}$ , with  $u(\bar{x}(k), \alpha) > w(\alpha)$ . Since each  $\bar{x}(k)$  can be approached in  $W(\alpha_n)$ , so can  $\hat{x}$ .

If  $u(\hat{x}, \alpha) = w(\alpha) = v(\alpha)$ , then we must be in the additional case where  $\beta$  and  $u$  are independent of  $\alpha$ . In that case,  $\hat{x} \in W(\alpha_n)$  for all  $\alpha_n$ , so  $\hat{x}$  is trivially approachable. ■

The application of the satisficing principle to the Walrasian better than correspondence  $D_+(p) = \{x \in X^h : u^h(x) \geq w(p)\}$ , where  $w(p) \equiv \max\{u^h(x) : x \in X^h, p \cdot x \leq I^h(p)\}$ , is particularly simple, since then neither  $u^h$  nor  $\beta(p) = X^h$  depends on  $p$ .

**Corollary (Continuous Correspondence Lemma)** *Let  $X \subset \mathbb{R}^n$  be convex, and let  $\mathcal{A} \subset \mathbb{R}^m$ . Let  $g_i : X \times \mathcal{A} \rightarrow \mathbb{R}$  be continuous, and convex on  $X$  for each fixed  $\alpha \in \mathcal{A}$ , for all  $i = 1, \dots, k$ . Suppose that for each  $\alpha \in \mathcal{A}$ , there is  $x(\alpha) \in X$  with  $g_i(x(\alpha)) < 0$  for all  $i = 1, \dots, k$ . Then the correspondence  $B : \mathcal{A} \rightrightarrows X$  defined by*

$$B(\alpha) = \{x \in X : g_i(x, \alpha) \leq 0, \text{ for every } i = 1, \dots, k\}$$

is USC and LSC.

**Proof** Define  $u : X \times \mathcal{A} \rightarrow \mathbb{R}$  by  $u(x, \alpha) = \min_{1 \leq i \leq k} [-g_i(x, \alpha)]$ . As the minimum of concave functions,  $u$  is concave on  $X$  for each fixed  $\alpha$ , as well as continuous on  $X \times \mathcal{A}$ . Since  $B(\alpha)$  is convex,  $u_\alpha$  is nonsatiated on  $B(\alpha)$ . Furthermore,  $v(\alpha) = \sup\{u(x, \alpha) : x \in X\} \geq u(x(\alpha)) > 0$ , for all  $\alpha \in \mathcal{A}$ . Hence by the satisficing principle,  $B(\alpha) = \{x \in X : u(x, \alpha) \geq 0\}$  is USC and LSC.

## 5.2 Quasi-Concave Games

In our definition of games given in Section 1, we can weaken the hypothesis that  $u_n$  is concave in  $\sigma_n$  to the hypothesis of quasi-concavity:  $u_n(\sigma_n, \bar{\sigma}_{-n}) > u_n(\bar{\sigma}_n, \bar{\sigma}_{-n})$

implies  $u_n(\lambda\sigma_n + (1 - \lambda)\bar{\sigma}_n, \bar{\sigma}_{-n}) > u_n(\bar{\sigma}_n, \bar{\sigma}_{-n})$  for all  $0 < \lambda < 1$ . The result is called a quasi-concave game. We now use Brouwer's fixed point theorem to prove the existence of Nash equilibrium for all quasi-concave games.

**Theorem** *Every quasi-concave game has a Nash equilibrium.*

**Proof** Let  $v_n(\bar{\sigma}_{-n}) \equiv \max_{\sigma_n \in \Sigma_n} u_n(\sigma_n, \bar{\sigma}_{-n})$  define a continuous function from  $\Sigma_{-n}$  to  $\mathbb{R}$ , called the "indirect utility function." Let  $\delta_n(\bar{\sigma}) = v_n(\bar{\sigma}_{-n}) - u_n(\bar{\sigma}_n)$ , and let  $\delta(\bar{\sigma}) = \max_{n \in N} \delta_n(\bar{\sigma})$ . Clearly  $\bar{\sigma}$  is a Nash equilibrium if and only if  $\delta(\bar{\sigma}) = 0$ . Let  $\underline{\delta} = \min_{\sigma \in \Sigma} \delta(\sigma)$ . Let  $w_n(\bar{\sigma}_{-n}) = v_n(\bar{\sigma}_{-n}) - \frac{1}{2}\underline{\delta}$ , and let

$$W_n(\bar{\sigma}_{-n}) = \{\sigma_n \in \Sigma_n : u_n(\sigma_n, \bar{\sigma}_{-n}) \geq w_n(\bar{\sigma}_{-n})\}.$$

Suppose  $G$  has no Nash equilibrium. Then  $\underline{\delta} > 0$  and for all  $\bar{\sigma} \in \Sigma$  and each  $n$ ,  $w_n(\bar{\sigma}_{-n}) < v_n(\bar{\sigma}_{-n})$ . By the Satisficing Principle (which applies since  $u_n$  is quasi-concave, and thus locally non-satiated in any convex budget set),  $W_n$  is nonempty, USC, LSC, and convex-valued. Moreover, for all  $\bar{\sigma}$  there is some player  $n$  with  $u_n(\bar{\sigma}_n) < w_n(\bar{\sigma}_{-n})$ , so  $\bar{\sigma}_n \notin W_n(\bar{\sigma}_{-n})$ . Define  $\varphi_n : \Sigma_n \times \Sigma_{-n} \rightarrow \Sigma_n$  by

$$\varphi_n(\bar{\sigma}_n, \bar{\sigma}_{-n}) = \min_{\sigma_n \in W_n(\bar{\sigma}_{-n})} \|\sigma_n - \bar{\sigma}_n\|^2.$$

Clearly  $\varphi_n$  is a function, since  $W_n$  is convex-valued. Furthermore, if  $W_n$  is USC and LSC, then by the Maximum Principle,  $\varphi_n$  is a continuous function. Let  $\varphi = (\varphi_1, \dots, \varphi_N)$ . If  $G$  has no Nash equilibrium, then  $\varphi$  is a continuous function with no fixed point, a contradiction. ■

## References

- [1] Aliprantis, P., Brown, D. J.: Equilibria in markets with a Riesz space of commodities. *Journal of Mathematical Economics* **11**, 189–207 (1983).
- [2] Arrow, K., Debreu, G.: Existence of an equilibrium for a competitive economy. *Econometrica* **22**, 265–290 (1954).
- [3] Border, K.: Fixed point theorems with applications to economics and game theory. Cambridge: Cambridge University Press, 1985.
- [4] Brown, D. J.: Acyclic choice. Cowles Foundation Discussion Paper No. 360, Yale University (1973).
- [5] Debreu, G.: A social equilibrium existence theorem. *Proceedings of the National Academy of Sciences* **38**, 886–893 (1952).
- [6] Debreu, G.: Market equilibrium. *Proceedings of the National Academy of Sciences* **42**, 876–878 (1956).
- [7] Debreu, G.: The theory of value. New Haven: Yale University Press, 1959.

- [8] Fan, K.: A generalization of Tychonoff's fixed point theorem. *Mathematische Annalen* **142**, 305–310 (1961).
- [9] Gale, D.: The law of supply and demand. *Mathematica Scandinavica* **3**, 155–169 (1955).
- [10] Gale, D., A. Mas-Colell: An equilibrium existence theorem for a general model without ordered preferences. *Journal of Mathematical Economics* **2**, 9–15 (1975).
- [11] Gul, F., D. Pearce, E. Stacchetti: A bound on the proportion of pure strategy equilibria in generic games. *Mathematics of Operations Research* **18**(3), 548–552 (1993).
- [12] Kehoe, T.: An index theorem for general equilibrium models with production. *Econometrica* **48**, 1211–1232 (1980).
- [13] Kehoe, T.: The existence, stability, and uniqueness of economic equilibria. Presented at the School Celebration at Yale, 1995.
- [14] Krasa, S., N. C. Yannelis: An elementary proof of the Knaster–Kuratowski–Mazurkiewicz–Shapley theorem. *Economic Theory* **4**, 467–471.
- [15] Lemke, C. E., J. T. Howson: Equilibrium points of bimatrix games. *SIAM Journal of Applied Mathematics* **12**, 413–423 (1964).
- [16] McKenzie, L.: On equilibrium in Graham's model of world trade and other competitive systems. *Econometrica* **22**, 147–161 (1954).
- [17] McKenzie, L.: On the existence of general equilibrium for a competitive market. *Econometrica* **27**, 54–71 (1959).
- [18] Nash, J.: Equilibrium points in  $N$ -person games. *Proceedings of the National Academy of Sciences* **36**, 48–49 (1950).
- [19] Nash, J.: Non-cooperative games. *Annals of Mathematics* **54**, 286–295 (1951).
- [20] Neufeind, W.: Notes on existence of equilibrium proofs and the boundary behavior of supply. *Econometrica* **48**, 1831–1837 (1980).
- [21] Nikaido, H.: On the classical multilateral exchange problem. *Metroeconomica* **8**, 135–145 (1956).
- [22] Shafer, W. J., Sonnenschein, H.: Equilibrium in abstract economics without ordered preferences. *Journal of Mathematical Economics* **2**, 345–348 (1975).
- [23] Sonnenschein, H.: Demand theory without transitive preferences, with applications to the theory of competitive equilibrium. In J. S. Chipman, L. Hurwicz, M. K. Richter, and H. F. Sonnenschein (eds.), *Preferences, utility and demand*. New York: Harcourt Brace Jovanovich, 1971.
- [24] Todd, M.: A note on computing equilibria in economies with activity analysis models of production. *Journal of Mathematical Economics* **6**, 135–144 (1979).