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# A BOUND ON THE NUMBER OF NASH EQUILIBRIA IN A COORDINATION GAME

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Abstract

We prove that a "nondegenerate" $m \times n$ coordination game can have at most $2^M - 1$ Nash equilibria, where $M = min(m,n)$.

## 1. Introduction.

In Quint-Shubik (1994), we conjectured that a "nondegenerate" $n \times n$ bimatrix game could have at most $2^n - 1$ Nash equilibria. In this paper we prove a generalization of this result for a special class of bimatrix games. In particular, we show that a "nondegenerate" $m \times n$ coordination game (i.e., a bimatrix game in which the payoff matrices for the two players are identical) can have at most $2^M - 1$ Nash equilibria, where $M = min(m,n)$.[2]

## 2. Background.

Let there be two players in a game, denoted by I and II. Player I has $m$ pure strategies at his disposal, denoted by $I = \{1, ..., m\}$, while II has pure strategy set $J = \{1, ..., n\}$. A mixed strategy for player I is a row-vector $p$ of the $m-1$-dimensional simplex $P$, in which $p_i$ is interpreted to be the probability that he plays pure strategy $i$. Similarly, the set of mixed strategies for II are the column-vectors $q$ of the $n-1$-dimensional simplex $Q$. Given $p \in P$, define the support of $p$, or $supp(p)$, to be the set $\{i \in I : p_i > 0\}$, and define $supp(q)$ for $q \in Q$ similarly. Finally, denote by $e^i$ the mixed strategy in which I plays $i$ with probability 1, and by $e^j$ that in which II plays $j$ with probability 1.

We are also given two $m \times n$ matrices $A$ and $B$, where $a_{ij}$ and $b_{ij}$ represent the payoffs

[1] The authors wish to thank Dicky Yan for supplying a missing step in one of the proofs.

[2] When we consider the class of all bimatrix games, there are two extreme cases. They are games of coordination (where payoffs for the two players are identical) and zero-sum games (where they are diametrically opposed). It is well known that any "nondegenerate" zero-sum game has exactly 1 Nash equilibrium.

for players I and II respectively, if I plays mixed strategy $e^i$ and II plays $e^j$. Hence, if I chooses mixed strategy $p \in P$ and II chooses $q \in Q$, the expected payoff for I is $pAq$, while that for II is $pBq$. Since the two payoff matrices are sufficient to define a bimatrix game, we shall use the terminology "bimatrix game $(A, B)$".

Given $q \in Q$, $p^*$ is a best response for I against $q$ if $p^*Aq \geq pAq \; \forall p \in P$. Similarly, $q^*$ is a best response for II against $p$ if $pBq^* \geq pBq \; \forall q \in Q$. Denote by $R_I(q)$ the set of all best responses for I against $q$, and by $R_{II}(p)$ the set of all best responses for II against $p$. A Nash Equilibrium (NE) (Nash, 1950, 1953) is a pair $(p^*, q^*) \in P \times Q$ where $p^* \in R_I(q)$ and $q^* \in R_{II}(p^*)$.

In order to aid us in finding NEs, let us define the sets $\mathcal{R}_I(q)$ and $\mathcal{R}_{II}(p)$ as follows: $\mathcal{R}_I(q) = \{i \in I : e^iAq \geq e^kAq \; \forall k \in I\}$ and $\mathcal{R}_{II}(p) = \{j \in J : pBe^j \geq pBe^k \; \forall k \in J\}$. In words, $\mathcal{R}_I(q)$ is the set of best pure strategy responses for I against $q$, while a similar interpretation holds for $\mathcal{R}_{II}(p)$. The following Lemma is then readily apparent (see, e.g., Shapley (1974) or Jansen (1981)):

Lemma 1: A mixed strategy pair $(p, q)$ is a NE of bimatrix game $(A, B)$ iff $supp(p) \subseteq \mathcal{R}_I(q)$ and $supp(q) \subseteq \mathcal{R}_{II}(p)$.

In our upcoming analysis we will want to avoid having to consider the "degenerate" class of games in which there is an infinitude of NEs. To this end, we mention a version of the nondegeneracy assumption from Quint-Shubik (1994):

Nondegeneracy Assumption (NA): If $p \in P$ satisfies $|supp(p)| = z$ (the $| \;\; |$ notation denotes the cardinality of a set), then there are no more than $z$ pure strategy best reponses for II against $p$. Similarly, if $|supp(q)| = z$, we have $|\mathcal{R}_I(q)| \leq z$.

Not only does the NA assure the existence of only a finite number of NEs, but we also have the following:

Lemma 2: Suppose the NA holds, that $(p, q)$ is a NE, and that $|supp(p)| = z$. Then

a) $|supp(q)| = z$

b) $supp(p) = \mathcal{R}_I(q)$

c) $supp(q) = \mathcal{R}_{II}(p)$

d) For any other NE $(p^2, q^2)$, either $supp(p^2) \neq supp(p)$ OR $supp(q^2) \neq supp(q)$.

## 3. Coordination Games and the Theorem.

A coordination game is a bimatrix game in which $A = B$. Since in this case only one matrix is needed to define the game, we use the terminology "coordination game $A$".

Theorem: Suppose a coordination game satisfies the NA. If $(p^1, q^1)$ and $(p^2, q^2)$ are distinct NEs of the game, then a) $supp(p^1) \neq supp(p^2)$ AND b) $supp(q^1) \neq supp(q^2)$.

Remark 1: We remark that the Theorem is not necessarily true for bimatrix games which are not coordination games. For instance, in the game

$$\begin{pmatrix} (4,4) & (0,3) & (2,2) & (0,1) \\ (0,0) & (2,1) & (0,\frac{3}{2}) & (4,\frac{11}{6}) \end{pmatrix},$$

there are three NEs in which Player I uses both pure strategies with positive probability:

$p^1 = (\frac{1}{2}, \frac{1}{2}), q^1 = (\frac{1}{3}, \frac{2}{3}, 0, 0)$.

$p^2 = (\frac{1}{3}, \frac{2}{3}), q^2 = (0, \frac{1}{2}, \frac{1}{2}, 0)$.

$p^3 = (\frac{1}{4}, \frac{3}{4}), q^3 = (0, 0, \frac{2}{3}, \frac{1}{3})$.

Next, since there are only $2^m - 1$ possible "supports" for a mixed strategy $p$, and $2^n - 1$ for $q$, we have the following:

Corollary: Suppose an $m \times n$ coordination game satisfies the NA. Let $M = min(m, n)$. Then the game has no more than $2^M - 1$ NEs.

Remark 2: It is easy to construct examples of coordination games $A$ which achieve the bound expressed in the Corollary. Indeed, if $m \leq n$ (so $M = m$), define $A$ by letting its first $m$ columns define an identity matrix, and then judiciously add dominated strategies[3] to fill in the last $n - m$ columns. Likewise, if $m \geq n$, again place an $M \times M$ identity matrix in the upper left, but now add $m - n$ dominated rows.

[3] This must be done so as not to violate the NA.

Remark 3: In Eaves (1971), it was shown there is a one-to-one correspondence between NEs of coordination game $A$ and solutions to the linear complementarity problem (LCP)

$$Ix - \begin{pmatrix} 0 & A + k_1 E \\ A^T - k_2 E & 0 \end{pmatrix} y = \begin{pmatrix} -1_m \\ 1_n \end{pmatrix}, \; xy = 0, \; x, y \geq 0.$$

[$I$ is the identity matrix, $E$ is the matrix of all 1's, $k_1$ and $k_2$ are constants so that $A + k_1 E > 0$ and $A^T - k_2 E < 0$, and $1_m$ ($1_n$) is the $m$-vector ($n$-vector) of all 1's.] Hence our Theorem places an upper limit of $2^M - 1$ on the number of solutions to LCPs of a certain class in which the "M-matrix" is of dimension $(m+n) \times (m+n)$.

Proof of Theorem: We prove conclusion a) of the Theorem (the proof of part b) is similar). So suppose a) were false for some coordination game $A$. Since raising all coefficients of a bimatrix by the same amount does not change the set of NEs, we may assume $A > 0$. Let $A_j$ denote the $j$th column of $A$, and $A^i$ the $i$th row. Since we are assuming the Theorem false, there exist two NEs, $(p^1, q^1)$ and $(p^2, q^2)$, for which $supp(p^1) = supp(p^2)$. WLOG assume $|supp(p^1)| = |supp(p^2)| = z$. By relabeling if necessary, assume $supp(p^1) = supp(p^2) = \{1, ..., z\}$, i.e., both NEs "use" the first $z$ rows.

From the NA, the fact that $(p^1, q^1)$ and $(p^2, q^2)$ are NEs, and the fact that $A > 0$, we know that there exist positive constants $s, t, u$, and $v$ satisfying

$$p^1 A_j = s \text{ if } j \in supp(q^1) \tag{3.1}$$

$$p^1 A_j < s \text{ if } j \notin supp(q^1) \tag{3.2}$$

$$p^2 A_j = t \text{ if } j \in supp(q^2) \tag{3.3}$$

$$p^2 A_j < t \text{ if } j \notin supp(q^2) \tag{3.4}$$

$$A^i q^1 = u \text{ for } i \in 1, ..., z \tag{3.5}$$

$$A^i q^2 = v \text{ for } i \in 1, ..., z \tag{3.6}$$

Next, we note that the NA implies that $|supp(q^1)| = |supp(q^2)| = z$. By relabeling columns if necessary, assume $supp(q^1) = \{1, ..., z\}$ and $supp(q^2) = \{w+1, ..., w+z\}$. [The index $w$ can take on any value from 1 to $n-z$, but cannot take on the value 0 because of the NA.] Define the $z \times z$ matrix $C$ as the submatrix of $A$ defined by rows $\{1, ..., z\}$ and columns $\{1, ..., z\}$, i.e, the submatrix defined by the rows in $supp(p^1)$ and the columns of $supp(q^1)$. Similarly, define the $z \times z$ matrix $D$ as the submatrix of $A$ defined by rows $\{1, ..., z\}$ and the columns of $supp(q^2)$. Note that $C$ and $D$ will share exactly $z-w$ columns if $w < z$, and none otherwise.

We denote by $C_j$ the $j$th column of $C$, i.e., the first $z$ elements of $A_j$. Similarly, $D_j$ denotes the $j$th column of $D$, i.e., the first $z$ elements of $A_{w+j}$. Hence, if $w < z$, we have $C_{w+j} = D_j$ for $j = 1, ..., z-w$.

Claim: Matrices $C$ and $D$ are nonsingular (hence, $C^{-1}$ and $D^{-1}$ exist).

Proof of Claim: We prove the Claim for $C$; the proof for $D$ is similar. Suppose $C$ were singular. Then there exist constants $\alpha_1, ..., \alpha_z$, not all zero, such that $\alpha_1 C_1 + ... + \alpha_z C_z = 0$. Furthermore, since $C > 0$, at least one of the $\alpha_j$'s is positive and at least one is negative.

Given NE $(p^1, q^1)$, define a new mixed strategy $q^{1*}$ by

$$q_j^{1*} = \begin{cases} \frac{q_j^1 + \frac{\alpha_j}{N}}{Z} & \text{if } j \in supp(q^1) = \{1, ..., z\}; \\ 0 & \text{otherwise,} \end{cases}$$

where $N$ is a large finite number, and $Z = \sum_{j \in supp(q^1)} (q_j^1 + \frac{\alpha_j}{N}) = 1 + \frac{\sum_{j \in supp(q^1)} \alpha_j}{N}$ is a normalizing constant. Since at least one $\alpha_j$ is positive and at least one $\alpha_j$ is negative, we note that $(\alpha_1, ..., \alpha_z)$ is not a multiple of $(q_1^1, ..., q_z^1)$, and so $q^{1*}$ is distinct from $q^1$.

Now consider the pair $(p^1, q^{1*})$. The support of $q^{1*}$ is the same as that for $q^1$, so, since $supp(q^1) \subseteq \mathcal{R}_{II}(p^1)$, we have $supp(q^{1*}) \subseteq \mathcal{R}_{II}(p^1)$. Furthermore, by the construction, all pure strategies in $supp(p^1)$ pay off the same for Player I against $q^{1*}$, so, if $N$ is sufficiently large, they all will be elements of $\mathcal{R}_I(q^{1*})$. [This holds because they all were elements of $\mathcal{R}_I(q^1)$, and, if $N$ is large, $q^{1*}$ is very close to $q^1$.] Hence, $(p^1, q^{1*})$ is also a NE.

However, the fact that $(p^1, q^1)$ and $(p^1, q^{1*})$ are both NEs is a contradiction of the NA, because of Lemma 2, part d).

Define $\hat{p}^1$ as the $z$-vector consisting of the $z$ (nonzero) components of $p^1$, i.e., $\hat{p}^1 =$

$(p^1_1, ..., p^1_z)$. Define $\hat{p}^2$ similarly. Finally, define $\hat{q}^1$ and $\hat{q}^2$ as the $z$- vectors consisting of the $z$ nonzero components of $q^1$ and $q^2$ respectively.

Using the notation described above, we may rewrite conditions (3.1)-(3.6) as follows

$$\hat{p}^1 C = (s, ..., s) \Longrightarrow \hat{p}^1 = (s, ..., s)C^{-1} \tag{3.7}$$

$$\hat{p}^1 D_j \begin{cases} = s & \text{if } j \in 1, ..., z-w \text{ (and } w < z); \\ < s & \text{otherwise.} \end{cases} \tag{3.8}$$

$$\hat{p}^2 D = (t, ..., t) \Longrightarrow \hat{p}^2 = (t, ..., t)D^{-1} \tag{3.9}$$

$$\hat{p}^2 C_j \begin{cases} = t & \text{if } j \in w+1, ..., z \text{ (and } w < z); \\ < t & \text{otherwise.} \end{cases} \tag{3.10}$$

$$C\hat{q}^1 = (u, ..., u)^T \Longrightarrow \hat{q}^1 = C^{-1}(u, ..., u)^T \tag{3.11}$$

$$D\hat{q}^2 = (v, ..., v)^T \Longrightarrow \hat{q}^2 = D^{-1}(v, ..., v)^T \tag{3.12}$$

Now, since $\hat{q}^1$ is a positive probability vector, we have $\hat{q}^1_j \in (0, 1]$ for $j \in 1, ..., z$. Substituting using (3.11), we have that $C^{-1}_i(u, ..., u)^T \in (0, 1]$ for $i = 1, ..., z$. Since $u > 0$, this implies that the row sums of $C^{-1}$ are all positive. A similar argument using $\hat{q}^2$ tells us the same thing about $D^{-1}$; hence we have shown

Proposition: The row sums of $C^{-1}$ and $D^{-1}$ are all positive.

Next, substituting in (3.8) using the expression for $\hat{p}^1$ found in (3.7) gives

$$(s, ..., s)C^{-1}D_j \begin{cases} = s & \text{if } j \in 1, ..., z-w \text{ (and } w < z); \\ < s & \text{otherwise.} \end{cases}$$

This in turn implies ($I$ represents the identity matrix)

$$[(s, ..., s)(C^{-1}D - I)]_j \begin{cases} = 0 & \text{if } j \in 1, ..., z-w \text{ (and } w < z); \\ < 0 & \text{otherwise.} \end{cases}$$

Finally, since $s > 0$, this gives

$$[(1, ..., 1)(C^{-1}D - I)]_j \begin{cases} = 0 & \text{if } j \in 1, ..., z-w \text{ (and } w < z); \\ < 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

Similarly, substituting (3.9) into (3.10) gives

$$(t,...,t)D^{-1}C_j \begin{cases} = t & \text{if } j \in w+1,...,z \text{ (and } w < z); \\ < t & \text{otherwise.} \end{cases}$$

which implies

$$[(1,...,1)(D^{-1}C - I)]_j \begin{cases} = 0 & \text{if } j \in w+1,...,z \text{ (and } w < z); \\ < 0 & \text{otherwise.} \end{cases} \tag{3.14}$$

Note that in both (3.13) and (3.14), the strict inequality holds for at least one $j$, because $w \neq 0$.

The Theorem will now be proven if we can show that (3.13) and (3.14) are inconsistent. To this end, we note that (3.13) implies that

$$1 - [(1,...,1)C^{-1}D]_j \begin{cases} = 0 & \text{if } j \in 1,...,z-w \text{ (and } w < z); \\ > 0 & \text{otherwise.} \end{cases}$$

Next, by the Proposition we know that the row sums of $D^{-1}$ are positive; hence, the vector $D^{-1}(1,...,1)^T$ has all positive components. Hence

$$[(1,...,1) - (1,...,1)C^{-1}D] \times D^{-1}(1,...,1)^T > 0,$$

which gives

$$(1,...,1)D^{-1}(1,...,1)^T - (1,...,1)C^{-1}(1,...,1)^T > 0. \tag{3.15}$$

Similarly, starting with (3.14), we have

$$1 - [(1,...,1)D^{-1}C]_j \begin{cases} = 0 & \text{if } j \in w+1,...,z \text{ (and } w < z); \\ > 0 & \text{otherwise.} \end{cases}$$

Again, by the Proposition we know that the row sums of $C^{-1}$ are positive; hence, the vector $C^{-1}(1,...,1)^T$ has all positive components. Hence

$$[(1,...,1) - (1,...,1)D^{-1}C] \times C^{-1}(1,...,1)^T > 0,$$

which gives

$$(1,...,1)C^{-1}(1,...,1)^T - (1,...,1)D^{-1}(1,...,1)^T > 0. \tag{3.16}$$

Indeed, inequalities (3.15) and (3.16) are inconsistent.

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