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DUMB BUGS AND BRIGHT NONCOOPERATIVE PLAYERS:
GAMES, CONTEXT AND BEHAVIOR

Thomas Quint, Martin Shubik and Dicky Yan

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by T. Quint, M. Shubik, and D. Yan

Yale University, New Haven, Ct. 06520

Abstract

Consider a repeated bimatrix game. We define “bugs” as players whose “strategy” is to react myopically to whatever the opponent did on the previous iteration. We believe that in some contexts this is a more realistic model of behavior than the standard “supremely rational” noncooperative game player.

We consider possible outcome paths that can occur as the result of bugs playing a game. We also compare how bugs fare over a suitable “universe of games,” as compared with standard “Nash” players and “maximin” players.

1. WHAT IS A DECISIONMAKER?

Although we resolutely avoid the deep philosophical discussion into which we could easily become enmeshed in considering what is “really meant by a player or decisionmaker,” a few points are in order.

In much of game theory we take as a primitive concept a set of players, each with a set of strategies and each with some form of preferences. The individual agent is meant to have the perception and ability to make a selection among the strategies available. In many of the applications to date, the individual is associated with economic or political man. Frequently it is assumed that the individual is highly rational, intelligent and capable of making virtually any calculation. In some game theoretic applications to biology the individual agents are modeled more or less mechanistically, and much of the concern for chains of inference involving interaction with others is ignored.

In general, the game theory player is a creature without passions or personality, whose abilities do not appear to match closely the human species as we know it, except possibly in some specific limited contexts such as the anonymous mass market in economic life.

When we attempt to modify our version of *homo ludens* to take into account our limited abilities to perceive, to remember, to calculate, to react, or to account for our

passions, our sloth or our instinctive behavior, in general we find that we are quickly overwhelmed with the complexity of even simple modifications to our idealization of rational man. Thus we excuse ourselves for sticking with the model of *homo ludens* because it is relatively simple to work with and we can obtain some interesting results. At the other extreme the biologists studying evolutionary stable strategies may view the virtue of their model in terms of simplicity of behavior and relevance of the results.

Is a planet or an atom a decisionmaker? Could we rewrite Newtonian mechanics or quantum theory as though the basic particles were optimizing players in a game of strategy? Most of us would answer that the concept of freedom of choice is associated with life (whatever that is). We tend to rule out planets, atoms or quarks as decisionmakers. Does our view of decisionmaking begin with the cell? If so, it is plausible to postulate that many living organisms make decisions according to exceedingly simple rules.

In this paper we define “bugs” as simple myopic short-term maximizers, who act without learning in their current environment, oblivious to the actions of other bugs. They ignore the past and are unconcerned with long term prediction.

Under the set-up of a bimatrix game, we examine what might happen if the players are such bugs. Then, in an analysis of “randomly selected” 2×2 games, we determine how these creatures fare on average, compared with other types of game players such as the “standard non-cooperative” player.

2. WHAT IS A SOLUTION?

In the kingdom of the blind the one-eyed man is king!

– An old proverb.–

In the kingdom of the blind the one-eyed man is mad!

–Another old proverb.–

The set of all feasible plays of a game enables us to describe a set of all outcomes. A

solution is nothing more than a subset of these outcomes selected to satisfy the assumptions made by a solution theory. The broadest and most common approaches to an n -person game are the various cooperative and noncooperative solution theories. A basic assumption underlying most cooperative theories is that the players will always select an outcome that is efficient, but may bargain or use a fair division or some other criterion to select among the efficient outcomes. The cooperative solutions do not provide a dynamic of play and to a great extent avoid the details of process. However, implicit in them is a high level of communication and a recognition of joint interest.

In contrast the noncooperative approach makes use of the strategic or extensive form of the game. But much of the work on noncooperative game theory has been concerned with the Nash noncooperative equilibrium (often modified or qualified to limited subsets of equilibria, such as “perfect equilibrium” or “type symmetric equilibrium”) rather than the processes that may lead to such equilibria.

Virtually any rule (such as randomize over all alternatives or pick the first item you see) will provide enough instructions to “solve” any game, in the sense that it will provide a sequence of moves to develop the play. But as game theorists and economists and some biologists tend to be goal-oriented, they reject many such rules as being bad or inadequate because they appear to lead to outcomes undesirable to the player using them. In addition, one of the clear lessons from game theory is that behavior appearing advantageous to individuals may lead to joint disaster to the group. Hence, the “success” of a strategy depends on both the specific structure of the game and the strategies of others. Indeed, a plan may appear to be excellent *ex ante* but may turn out to be a disaster *ex post*.

3. CONTEXT AND GAME REPRESENTATION

The three most frequently used representations of a game are the extensive form (or game tree representation), the strategic form (or matrix representation) and the coalitional form (or characteristic function representation). We concentrate here on strategic form

games (primarily with 2 strategies for each player). These have been used considerably both for illustration of strategic problems and for both formal and informal experimentation.

In general, game theorists, when considering solution concepts to games, slip in at least two implicit assumptions about the players. The first is external symmetry, i.e., unless specifically noted in the game model it is assumed players are identical in all nonspecified properties. The second is that context does not matter and the players come to the game with a *tabula rasa*. In other words, their experience, history and knowledge is irrelevant. Both of these propositions are clearly poor approximations of reality. They may be useful in the development of some aspects of the mathematical theory of games, but they pose considerable difficulties when one tries to use games for experimental purposes. It is our belief that for a chance of success, much of experimental gaming requires the joint talents of social psychologists, game theorists and other social scientists. Fortunately, for the economist the study of mass markets in reasonably normal times requires a minimum of social psychology. Thus some economic regularity may be observed in some experiments that are studied only from the viewpoint of market-oriented economist.

If one takes the identical mathematical description of a game and tells a different story about it or even gives it a different visual representation, it is likely that different experimental results will be obtained (See for example Simon, 1967). A specific reason why mathematical game theory is replete with many solution concepts is the recognition by some game theorists that context matters and that there is no simple unique way to go from the stripped down abstraction utilized in mathematical game theory to an application to human or other organism behavior without supplying the appropriate context.

In particular von Neumann and Morgenstern (1944) were well aware of the complexity of human behavior and in the opening chapter of their book suggested that a dynamic theory might take on a completely different form from the static theory they developed. It is our belief that, in particular, any successful dynamic theory must take into account learning and inference from past actions and states of the system.

4. ON BIMATRIX GAMES

Possibly the most familiar representation of a (two-person) game for the nonspecialist is the strategic form, an example of which is shown below:

$$\begin{pmatrix} (4, 1) & (0, 0) & (2, 2) \\ (0, 0) & (9, 9) & (0, 0) \\ (1, 3) & (0, 0) & (3, 1) \end{pmatrix}$$

Figure 1 – A Bimatrix Game

The first player has three strategies, i.e., he may select row 1, 2, or 3, and the second player also has three strategies, namely selecting column 1, 2, or 3. The payoffs for each possible strategy-pair are given by the entries in each cell. Hence, for example, if Player I chooses his third strategy, and Player II her first, the outcome is that Player I earns payoff 1 and Player II earns 3.

5. THE NONCOOPERATIVE EQUILIBRIUM AND BEST RESPONSE

Both in general theory and in applications, extensive use has been made of the Nash noncooperative equilibrium point. Let $P_k(i, j)$ be the payoff to player k if Player I uses pure strategy i and Player II uses j ($k = 1, 2$). A pair (i^*, j^*) form a pure strategy Nash equilibrium (PSNE) if $\max_i P_1(i, j^*)$ is achieved when $i = i^*$ and also $\max_j P_2(i^*, j)$ is attained when $j = j^*$.

Suppose Player I has m pure strategies, and Player II has n . A mixed strategy for Player I is a probability vector (p_1, \dots, p_m) in which p_i represents the probability that he plays pure strategy i ($i = 1, \dots, m$). Similarly, a generic mixed strategy for Player II is given by an n -vector q . A mixed strategy Nash equilibrium (MSNE) is a pair (p^*, q^*) for which $p^* \in \operatorname{argmax}_p \sum_{i,j} p_i q_j^* P_1(i, j)$, $q^* \in \operatorname{argmax}_q \sum_{i,j} p_i^* q_j P_2(i, j)$, and neither p^* nor q^* is integral. A Nash Equilibrium (NE) refers to a PSNE or MSNE.

Another way to play a game is by using pessimistic maximin strategies. Mixed strategy vector p^* is a maximin strategy for Player I if it maximizes $\min_q \sum_{i,j} p_i q_j P_1(i, j)$ over all p . Similarly, a maximin strategy for II is any maximizer of $\min_p \sum_{i,j} p_i q_j P_2(i, j)$ over

all q . A player's maximin payoff is his expected payoff when both players play a maximin strategy.¹

Referring to the game in Figure 1, we may also illustrate the idea of optimal response suggested by Robert Wilson (1971). Suppose that the players were to select moves (strategies) in sequence. Given some move for one player, the other can calculate his optimal response.² For example if Player II assumes that Player I has chosen his first move, Player II's optimal response is to choose her third move. If Player I believes that Player II has chosen her second move then Player I's optimal response is to choose his second move. The optimal response calls for the simple minded "greedy" rule of selecting the largest item in a row or column. Any PSNE in a bimatrix game has the "optimal response property," i.e., at the equilibrium the action of each is the optimal response to the action of the other.

Rather than view optimal response as a computational device, we may regard it as a description of the behavior of a primitive decisionmaker (or "bug") who learns nothing, but is able to perceive the individual worth of all outcomes in a row or column. Suppose initial strategies for the two players are given. The bug's first move will be to play an optimal response against the other player's initial strategy. In general, its move in the n th repetition of the game will be an optimal response against the opponent's move in the $n - 1$ st repetition. Hence the bug player is reacting myopically to the *last* move of his opponent, *without taking into account what that opponent might be doing currently*.

We may wish to distinguish between sequential and parallel (simultaneous) motion for our bugs. The spirit of the Nash equilibrium calls for alternate one-player adjustments

¹ Note that under this definition, a player's maximin payoff may not be uniquely defined (if one or both players have multiple maximin strategies). However, using the randomly generated cell entries described in Section 9, we find that almost always it will be, and this will serve our purposes in that section.

² There are several alternatives to the assumption that the organism looks down a entire row or column of outcomes and picks out the best. For instance, we might restrict the individual to only choosing among immediate neighbors. [This implies that there is some form of metric which defines neighbors. In particular, this would mean that permuting rows or columns would change the game.] Alternatively we might regard any choice that leads to an improvement as equiprobable, etc.

(see Brown 1951); this gives rise to the notion of sequential bugs. When sequential bugs play a game, (after the initial strategies are determined) the choice to move is determined randomly, and then passed back and forth until possibly both players choose not to move. If this happens, they have arrived at a PSNE, and no further motion occurs. If this never happens, then the bugs will end up cycling (among four or more outcomes).

Another possibility is for the players to be simultaneous bugs, i.e., they move blindly and simultaneously. This has the feature that the cell to which they move may be different from what either expect (but neither is assumed to learn).

To clarify matters, consider the 3 by 3 example of Figure 1, and suppose the “initial strategies” are Player I’s first strategy and Player II’s second. If sequential bugs are playing this game, and Player I is determined to move first, then he changes to his second strategy, the PSNE payoff (9,9) is reached, and neither player has incentive to move thereafter. If Player II is determined to move first, she changes to her third strategy, obtaining payoff (2,2). Now Player I has incentive to change to his third strategy, obtaining (3,1), and then Player II changes to her first strategy, etc. Continuing this process, we see that ultimately the bugs will cycle through the four “corner” payoffs. We call this cycle a sequential bug cycle.

On the other hand, if simultaneous bugs are playing the game, then Player I’s first move (his second strategy) and Player II’s first move (her third strategy) are played *simultaneously*, resulting in the (0,0) payoff on the right of the second row. From this point, Player I switches to his third strategy, and Player II to her second, yielding the (0,0) payoff in the bottom row. Again the result is a cycle, called a simultaneous bug cycle, this time among the four (0,0) payoffs in the bimatrix.

For either type of cycle, we define its length to be the number of cells contained in it. An exception is a PSNE in the case of sequential bugs, which is considered a cycle of length two, for reasons discussed in the next section.

In this paper we consider several types of player. The first type is the standard *homo ludens*, or “Nash player”, and is wedded to the need for consistent expectations. He is capable of computing equilibrium strategies, knows his and his competitor’s payoff functions, and always plays “his part” of a Nash equilibrium. The second type is the somewhat primitive and unintelligent bug described above. The third type is intelligent but highly pessimistic. It believes that the optimal policy is to play maximin on the basis that it does not trust the other player to act rationally or nonmaliciously. The fourth type is a random player, or “Nature”, which selects among its moves with equal probability. We also consider malicious “anti-players”, who try to minimize the payoff to the other. Finally, we consider players who are not only intelligent, but are informed about the type of player (or mechanism) against whom they are playing.

6. ON CYCLES

6.1. On Cycles and MSNEs

In Figure 1 (reproduced below), we saw an example for $n = 3$ where neither sequential response nor simultaneous response may converge. The unique PSNE has payoffs of (9,9) where each player uses his second strategy. But if either type of bugs are playing the game, and they begin in one of the corner cells, they embark on a 4- cycle around the corners which misses the PSNE.

$$\begin{pmatrix} (4, 1) & (0, 0) & (2, 2) \\ (0, 0) & (9, 9) & (0, 0) \\ (1, 3) & (0, 0) & (3, 1) \end{pmatrix}$$

Figure 1 again

Note that this case, the four-cycle indicates the presence of a MSNE using both players’ first and third strategies. However, the frequencies from the MSNE ($(\frac{2}{3}, \frac{1}{3})$ for Player I and $(\frac{1}{4}, \frac{3}{4})$ for Player II) are not those produced by the cycle. [The cycle entails both players using both of their strategies with frequency $\frac{1}{2}$.]³

³ It is well known (Brown, 1951, or for a simple exposition Shubik, 1982) that for a matrix game (i.e. a constant sum game), the method of fictitious play will always yield a pure or mixed strategy NE. The players following fictitious play can be regarded as

Despite this example, we note that in general, there need not be any relation between optimal response cycles and MSNEs. The game depicted in Figure 2 has a MSNE using both strategies for both players, namely $p = (\frac{1}{2}, \frac{1}{2}), q = (\frac{1}{2}, \frac{1}{2})$. However, there is no cycle using both strategies for both players.

$$\begin{pmatrix} (1, 1) & (0, 0) \\ (0, 0) & (1, 1) \end{pmatrix}$$

Figure 2

On the other hand, Figure 3 provides an example of a game with an optimal response cycle, but no corresponding MSNE:

$$\begin{pmatrix} (10, 10) & (0, 0) & (0, 0) & (0, 0) \\ (0, 0) & (0, 0) & (1, -10) & (-10, 1) \\ (0, 0) & (-10, 1) & (0, 0) & (1, -10) \\ (0, 0) & (1, -10) & (-10, 1) & (0, 0) \end{pmatrix}$$

Figure 3

The optimal response cycle runs through the six entries which have payoffs $(-10, 1)$ or $(1, -10)$; yet there is no MSNE using the second, third, and fourth strategies for each player.

6.2 The Labelling Function

Our next objective is to characterize fully the types of movement that can occur by the two types of bugs in a bimatrix game. Specifically, we ask, “What is the longest possible cycle on which the bugs can embark?”, “How many cycles of what length can there be?”, and “How long can the bugs move before entering a cycle or PSNE?”.

The answers to these questions are dependent on a game’s labelling function. This function essentially conveys the same information as O’Neill’s (1988) “best response diagrams”, except in a form which will ultimately be more useful to us.

more intelligent bugs than the ones considered here, in the sense that they each have a memory of the frequency of plays by their opponent, choosing the best response to the mixed strategy corresponding to that frequency. Thus the players learn.

Let $R = \{R_1, \dots, R_m\}$ be the set of rows in a game, and $C = \{C_1, \dots, C_n\}$ the set of columns. Let $M = \min(m, n)$. Define the labelling function $l : R \cup C \rightarrow R \cup C$ by:

$$l(p) = \begin{cases} \text{the column Player II would move to, if } p \text{ is a row;} \\ \text{the row Player I would move to, if } p \text{ is a column.} \end{cases}$$

Hence, l is a function which maps rows to columns and vice versa. In Figure 1 above, we have $l(R_1) = C_3$, $l(R_2) = C_2$, $l(R_3) = C_1$, $l(C_1) = R_1$, $l(C_2) = R_2$, and $l(C_3) = R_3$. Note that if sequential bugs are located at (R_i, C_j) , they will next move to $(l(C_j), C_j)$ if it is Player I's turn to move, and to $(R_i, l(R_i))$ if it is Player II's turn to move. On the other hand, if simultaneous bugs are playing the game, the next move will be to $(l(C_j), l(R_i))$. Finally, we note that (R_i, C_j) constitutes a PSNE iff $l(R_i) = C_j$ and $l(C_j) = R_i$.

Next, suppose we are given a label⁴ p . Applying the labelling function recursively until we repeat⁵ some label p_1 , we will have defined a labelling cycle (LC), i.e., a sequence $\{p_1, \dots, p_K\}$ of distinct elements of $R \cup C$, in which $l(p_k) = p_{k+1}$ for $k = 1, \dots, K - 1$, and $l(p_K) = p_1$. For example, in Figure 1, $\{R_1, C_3, R_3, C_1\}$ is a LC. We note that, if $\{p_1, \dots, p_K\}$ is a LC, then

- a) For $k = 1, \dots, K$, $\{p_k, \dots, p_K, p_1, \dots, p_{k-1}\}$ is also a LC.
- b) $\frac{K}{2}$ of its labels are rows and $\frac{K}{2}$ of its labels are columns.
- c) K is even and $K \leq 2M$.
- d) None of the labels p_1, \dots, p_K can belong to any other LC.

6.3. On Cycles – Sequential Bugs

Suppose $\{p_1, \dots, p_K\}$ is a LC. Then, we see that, if a pair of sequential bugs “starts” at (p_1, p_2) , they will traverse the cells defined by $(p_1, p_2), (p_2, p_3), \dots$, and (p_K, p_1) , before returning to (p_1, p_2) . Conversely, if $(p_1, p_2), (p_2, p_3), \dots, (p_K, p_1)$ define the cells of a sequential bug cycle, it must be that $\{p_1, \dots, p_K\}$ is a LC. Hence, there is a one-to-one cor-

⁴ By a “label” we mean either a row or column.

⁵ The label repeats because there are a finite number of labels. This argument also shows that in every bimatrix game at least one labelling cycle exists.

response between sequential bug cycles and LCs. This in turn implies that *the length of a sequential bug cycle is just the value of K in the corresponding LC.*⁶

Lemma 6.1: The length of a sequential bug cycle must be an even number, between 2 and $2M$.

Proof: Follows from the sequential cycle-LC correspondence described above, together with observation c) at the end of the last subsection.

Another implication of the correspondence between sequential bug cycles and LCs (and using Observation d) above) is the fact that *no two cycles can move through the same row or column.* Hence, by permuting the rows and columns of a bimatrix, we may assume that the cycles (including PSNEs) of the game lie along the upper part of the main diagonal. In addition, we may assume that this relabeling gives each cycle the “staircase structure” shown below:

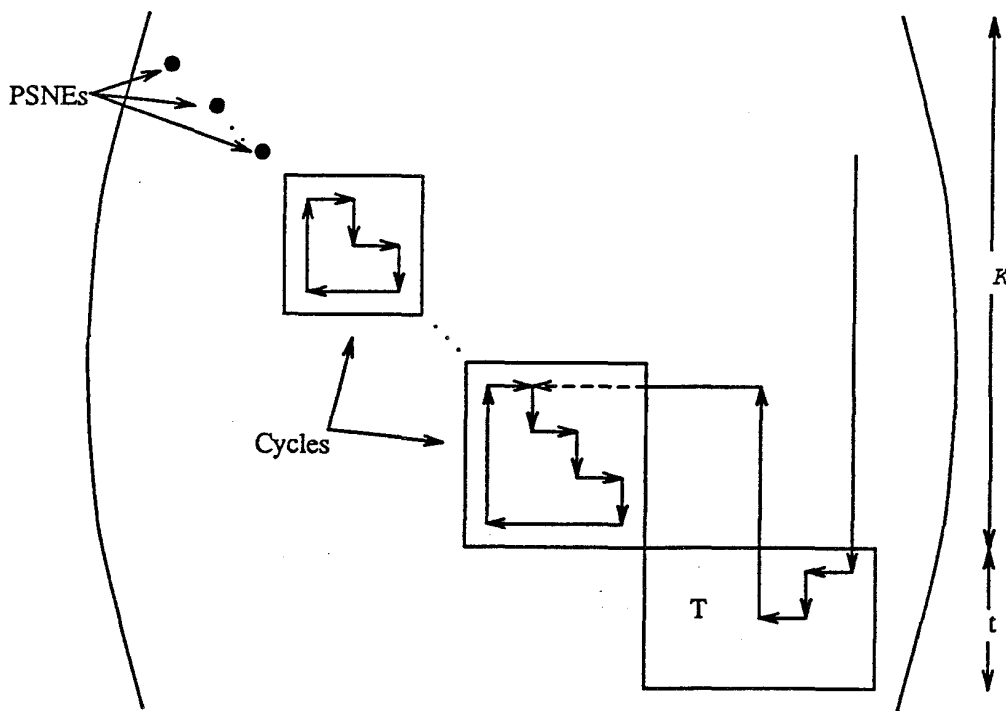


Figure 4.

⁶ If (R_i, C_j) is a PSNE, we see that $\{R_i, C_j\}$ is a LC. Hence, in order for the relationship to hold, we need to define a PSNE as a sequential bug cycle of length two.

Next, define K as half the sum of the cycle lengths, i.e, K is the number of rows contained in LCs. In general, all we can say is that this quantity is no more than M , but it may be strictly less. In that case, there is a $(m - K) \times (n - K)$ transition matrix T on the bottom right. Let $t = M - K$. Since every bimatrix has at least one LC (footnote 5), we have $K \geq 1$ and so $t \leq M - 1$.

Let (R_i, C_j) be the initial position of a pair of sequential bugs, with, say Player I to move first. If $j \leq K$, we see that Player I's first move will be into a cycle. Otherwise, if $(R_i, C_j) \in T$, at most $2t$ moves later T is exited, and then the next move is into a cycle. Finally, if $j > K$ but (R_i, C_j) is not in T , there are two possibilities. If the first move is not into T , the second move will be into a cycle. If the first move is into T , then, as before, at most $2t + 1$ moves later the bugs enter a cycle. So, we have shown:

Lemma 6.2: Using any starting point, the bugs will enter a cycle (which could be a PSNE) after at most $2t + 2$ moves.

Corollary 6.3: If the bugs are still moving after $2M$ moves, then they will keep moving forever.

Proof: Follows from Lemma 6.2 and the fact that $t \leq M - 1$.

The next result is the generalization of the fact that sequential optimal response always locates a PSNE (if there is one) in a 2 by 2 game:

Corollary 6.4: Suppose an $m \times n$ game has $M - 1$ or M PSNEs. Then the sequential bugs will locate one of them (after at most 4 moves).

Proof: Follows from Lemma 6.2, the fact that in this case there are no cycles which aren't PSNEs, and the fact that in this case $t \leq 1$.

Lemma 6.5: Suppose a game is fully labelled, i.e., there is an $M \times M$ submatrix indexed by row set \mathcal{R} and column set \mathcal{C} with $l(\mathcal{R}) = \mathcal{C}$ and $l(\mathcal{C}) = \mathcal{R}$. Then $t = 0$.

Proof: Consider the directed graph in which the node set is $\mathcal{R} \cup \mathcal{C}$, and the arc set is defined by: arc $\vec{p}q$ exists iff $l(p) = q$. Then the in-degree and out-degree of each node is one, and hence the graph is a sum of directed cycles. These cycles correspond to LCs in

the bimatrix game – hence, $\mathcal{K} = M$ and $t = 0$.

6.4 On Cycles – Simultaneous Bugs

Suppose we are given a bimatrix game, with labelling function l . If a label p is contained in some LC, we define $L(p)$ to be the set of all labels contained in that LC. Hence, in Figure 1, we have $L(R_1) = L(C_3) = L(R_3) = L(C_1) = \{R_1, R_3, C_1, C_3\}$, and $L(R_2) = L(C_2) = \{R_2, C_2\}$. If p is not a member of any LC, we write $L(p) = \emptyset$.

Proposition 6.6: Given a cell $(R_i, C_j) \in R \times C$, if $L(R_i) \cap L(C_j) \neq \emptyset$, then $L(R_i) = L(C_j)$.

Proof: Follows from the fact that it is impossible for a label to belong to more than one LC.

Now consider any simultaneous bug cycle, and suppose (R_i, C_j) is a cell contained in it. Then, the cells of the cycle are given by $\{(l^k(R_i), l^k(C_j))\}_{k=0}^{S-1}$, where S is the length of the cycle.⁷ Specifically, S is defined as the lowest positive index k satisfying either a) $l^k(R_i) = C_j$ and $l^k(C_j) = R_i$, or b) $l^k(R_i) = R_i$ and $l^k(C_j) = C_j$. If S is defined by a) above, it must necessarily be odd, and we have an odd cycle. Similarly, if S is defined by b), it is even, and we have an even cycle.

Note that in the case of an even cycle, condition b) implies that both R_i and C_j are members of LCs. For an odd cycle, we note that if a) holds, we have $l^{2k}(R_i) = R_i$ and $l^{2k}(C_j) = C_j$, so again R_i and C_j are members of LCs. Conversely, if one starts with any labels R_i and C_j which are elements of LCs, one may generate a cycle by applying the labelling function recursively. In fact, we see that if the cycle is an even one, its length will be $S = lcm(|L(R_i)|, |L(C_j)|)$, where “lcm” stands for “least common multiple”. On the other hand, if the cycle is odd, its length will certainly be no more than this amount.

Lemma 6.7: Consider an $m \times n$ bimatrix game, with $M = \min(m, n)$. Let z be the

⁷ The notation “ l^k ” is taken to mean the function l applied recursively k times, with $l^0(p)$ defined as p .

highest odd number less than or equal to M . Then A) any odd cycle has length at most z , and B) (given the right labelling function) it is possible for an odd cycle to have length exactly z .

Proof: For A), suppose (R_i, C_j) is an element contained in an odd cycle. Let S be the length of the cycle, i.e., S is the smallest positive integer k satisfying $l^k(R_i) = C_j$ and $l^k(C_j) = R_i$. Necessarily, we have $C_j \in L(R_i)$ and $R_i \in L(C_j)$; hence, by Proposition 6.6, $L(R_i) = L(C_j)$. Finally, let $K = |L(R_i)| = |L(C_j)| = lcm(|L(R_i)|, |L(C_j)|)$. We know $S \leq K$. Now, if S were strictly less than $\frac{K}{2}$, we would have $l^{2S}(R_i) = R_i$ and $l^{2S}(C_j) = C_j$, and so the length of the LC would be less than K . On the other hand, suppose S were strictly greater than $\frac{K}{2}$. Since $S \leq K$, we know that $l^k(R_i) \neq R_i$ for $k = 1, \dots, S$. In addition, since $l^{2S}(R_i) = R_i$, we must have $l^{2S-K}(R_i) = R_i$. But $2S - K$ is between 1 and S , so we have a contradiction (of the definition of K).

Hence S is exactly equal to $\frac{K}{2}$. But K is at most $2M$, so the bound follows.

Now we show B), i.e., that the bound z can be achieved. Suppose $l(R_i) = C_{i+1}$ for $i = 1, \dots, z-1$, $l(R_z) = C_1$, $l(C_j) = R_{j+1}$ for $j = 1, \dots, z-1$, $l(C_z) = R_1$, with other rows/columns labelled arbitrarily. The reader may verify that using the above labelling, and starting from location (R_1, C_1) , the bugs move down the diagonal and back to (R_1, C_1) in z moves.

Lemma 6.8: Consider an $m \times n$ bimatrix game, with $M = \min(m, n)$. Let $z = \max_{x, y \in \mathbb{Z}^+ : x+y \leq M} lcm(x, y)$. Then the maximum length even cycle has a length (S) of no more than $\max\{2M, 2z\}$.

Proof: Let (R_i, C_j) be a cell in an even cycle, and consider $L(R_i)$ and $L(C_j)$. By Proposition 6.6, there are two possibilities:

- 1) $L(R_i) = L(C_j)$. In this case, $S = lcm(|L(R_i)|, |L(C_j)|) = |L(R_i)| \leq 2M$.
- 2) $L(R_i) \cap L(C_j) = \emptyset$. Suppose WLOG that $M = m \leq n$, let $x = |R \cap L(R_i)|$, and let $y = |R \cap L(C_j)|$. Then it must be that $|L(R_i)| = 2x$, $|L(C_j)| = 2y$, and $x + y \leq M$.

Hence $lcm(|L(R_i)|, |L(C_j)|) = lcm(2x, 2y) = 2lcm(x, y)$, and this is clearly no more than $2z$ as defined in the Lemma. QED.

Lemma 6.9: The bound expressed in Lemma 6.8 can be achieved.

Proof: The $2M$ bound may be achieved using the labelling $l(C_j) = R_j$ for $j = 1, \dots, M$, $l(R_i) = C_{i+1}$ for $i = 1, \dots, M-1$, $l(R_M) = C_1$, and l defined arbitrarily for any other label in $R \cup C$. In this case, if the bugs start at (R_1, C_1) , they partake in a “staircase” cycle of $2M$ steps.

For the “ $2z$ ” bound, let x^*, y^* solve the maximization in the statement of Lemma 6.9. Now consider the following labelling of the first $x^* + y^*$ rows and columns of the matrix: $l(C_j) = R_j$ for $j = 1, \dots, x^* + y^*$, $l(R_i) = C_{i+1}$ for $i = 1, \dots, x^* - 1$ or $i = x^* + 1, \dots, x^* + y^* - 1$, $l(R_{x^*}) = C_1$, and $l(R_{x^*+y^*}) = C_{x^*+1}$. The labelling of any remaining rows or columns is arbitrary.

Under this labelling scheme, we have $L(C_1) = \{C_1, R_1, C_2, R_2, \dots, C_{x^*}, R_{x^*}\}$, and so $|L(C_1)| = 2x^*$. Meanwhile $L(R_{x^*+1}) = \{R_{x^*+1}, C_{x^*+2}, R_{x^*+2}, \dots, R_{x^*+y^*}, C_{x^*+1}\}$, so $|L(R_{x^*+1})| = 2y^*$. Hence, if we start the bugs at (R_{x^*+1}, C_1) , the resultant cycle has length $lcm(|L(R_{x^*+1})|, |L(C_1)|) = lcm(2x^*, 2y^*) = 2z$, as desired.

Putting the last three lemmas together, we have:

Theorem 6.10: Consider an $m \times n$ bimatrix game, with $M = \min(m, n)$. Then the maximum length possible for a simultaneous bug cycle is given by S , where

$$S = \begin{cases} \max\{2M, \frac{M^2-1}{2}\} & \text{if } M \text{ is odd;} \\ \max\{2M, \frac{M^2}{2} - 2\} & \text{if } \frac{M}{2} \text{ is an even integer;} \\ \max\{2M, M(\frac{M}{2} - 1), \frac{M^2}{2} - 8\} & \text{if } \frac{M}{2} \text{ is an odd integer.} \end{cases}$$

Proof: Follows from Lemmas 6.8 and 6.9, and the analysis of the optimization problem $\max_{x, y \in \mathbb{Z}^+ : x+y \leq M} lcm(x, y)$. If M is odd, the optimal x and y must be $\frac{M+1}{2}$ and $\frac{M-1}{2}$; if M is divisible by 4, they must be $\frac{M}{2} + 1$ and $\frac{M}{2} - 1$; finally, if M is even but not divisible by 4, either the pair $(\frac{M}{2}, \frac{M}{2} - 1)$ or the pair $(\frac{M}{2} - 2, \frac{M}{2} + 2)$ must be optimal.

From Theorem 6.10, we can construct the following table:

$M = \min(m, n)$	<u>Longest Possible Cycle (S)</u>
$M = 2, 3, 4, \text{ or } 6$	$2M$
$M \text{ odd, } M \geq 5$	$\frac{M^2 - 1}{2}$
$\frac{M}{2} \text{ even, } M \geq 8$	$\frac{M^2}{2} - 2$
$\frac{M}{2} \text{ odd, } M \geq 10$	$\frac{M^2}{2} - 8$

Figure 5

6.5 On Simultaneous Cycle-Packing in a Bimatrix

The last two sections have brought out an important difference between sequential and simultaneous bugs. For sequential bugs, there are at most M cycles, which cover at most $2M$ cells. All other cells are “transient”, in that it is impossible for them to be visited after $2t + 2$ moves (see Lemma 6.3). On the other hand, for simultaneous bugs, we have seen that *any cell* (R_i, C_j) *will be an element of a cycle, as long as* R_i *and* C_j *are members of LCs.* Hence, there is a $K \times K$ (see section 6.3) submatrix of “recurrent” cells. It is the decomposition of this submatrix into cycles which concerns us in this section.

We note that in the case of simultaneous bugs, a PSNE is considered as a cycle of length one.

Lemma 6.11: Let L be an LC of size $|L| = K$. Let $v = \frac{K}{2}$. Then, the $v \times v$ submatrix defined by the rows and columns of L contains exactly $\lfloor \frac{v}{2} \rfloor$ simultaneous bug cycles of length $2v$, and, if v is odd, an additional cycle of length v .

Proof: WLOG (relabelling indices if necessary), let $L = \{R_1, C_1, R_2, C_2, \dots, R_v, C_v\}$. Suppose v is even. By writing out the sequences $\{l^k(R_1), l^k(C_j)\}$, we see that a different cycle is traced by starting at location (R_1, C_j) , $j = 1, \dots, \frac{v}{2}$. On the other hand, the cycle starting from (R_1, C_{v-j+1}) is identical to that starting from (R_1, C_j) , since $l^{2j-1}(R_1) = C_j$ and $l^{2j-1}(C_{v-j+1}) = R_1$. Hence there are exactly $\frac{v}{2}$ cycles. If v is odd, there is an additional odd cycle, having length v , starting at $(R_1, C_{\frac{v+1}{2}})$.

Lemma 6.12: Let L_1 and L_2 be disjoint LCs, of sizes $2v_1$ and $2v_2$ respectively. Then,

the two $v_1 \times v_2$ submatrices defined by the cells (R_i, C_j) [$R_i \in L_1, C_j \in L_2$ OR $R_i \in L_2, C_j \in L_1$] contain exactly $\frac{v_1 v_2}{lcm(v_1, v_2)}$ cycles, each of length $2lcm(v_1, v_2)$.

Proof: Consider any cycle containing cell (R_i, C_j) as described in the statement of the Lemma. As argued in the last section, a) the cycle has length $2lcm(v_1, v_2)$ and b) the cycle is disjoint from all others. Since there are $2v_1 v_2$ such cells, the lemma follows.

7. LANDSCAPES AND ALMOST ALL GAMES

The 2×2 bimatrix game has been studied in some detail. There is a small, but useful literature on counting and characterizing matrix/bimatrix games. Rapoport, Guyer and Gordon (1972), O'Neill (1988) and Barany, Lee and Shubik (1992) provide references. The attraction of the 2×2 bimatrix game is basically that its major variants can be enumerated and studied exhaustively. Unfortunately the simplicity can also be misleading for the study of many questions we might wish to ask about general bimatrix games. In particular, because each row or column contains only two entries, the shape of the payoff surface is limited in complexity.

For the rest of this paper, we limit our analysis to the class of 2×2 games, but with some observations and qualifications for larger bimatrices.

If we regard the payoffs as being given in von Neumann Morgenstern utilities, then there are infinitely many games of any size. But these games may be classified into a finite set of classes by the ordinal aspects of the payoffs.⁸ We wish to obtain a reasonable measure for all bimatrix games of a given size and then study different forms of behavior

⁸ We know that of the 576 ordinal 2×2 games, there are 78 classes of games, each strategically different. Of these 78 classes, 66 contain 8 members, and the other 12 contain only 4 members (due to an extra symmetry). Hence, if one were to "average" over all 2×2 games, one should give 66 of the classes a weight of $\frac{2}{144}$ and give the other 12 classes a weight of $\frac{1}{144}$.

Of the 78 classes of strategically different games, 58 have a single NE that is pure, 9 have a single NE that is mixed and 11 have three NEs, two pure and one mixed.

Finally, we note that there are several other ways one might classify all games, for example by their optimal response diagrams (i.e., their labelling functions), or by the structure of their payoff sets.

over the various classes. For an $n \times n$ bimatrix we generate its payoffs by drawing $2n^2$ times from a rectangular distribution on the interval $[0,1]$. Topologically speaking, this gives us “almost all” $n \times n$ games.⁹

Limiting our concern at this point to the 2×2 bimatrix game, we draw from 8 i.i.d. uniform random variables to obtain the games and place them in the 78 ordinal categories, ignoring ties.¹⁰

If we consider large $n \times n$ bimatrices, we may observe that if we regard the n^2 payoffs to each player as describing a payoff surface, this surface may be arbitrarily rough. In biological applications and in military operations research involving duels and search problems, one may wish to consider a game between competitors played on an actual two dimensional surface. It is reasonably straightforward to show the relationship between matrix games and games played over a map.¹¹

In many applications of game theory there is a special structure to the game and its payoffs. In some instances such as in the functioning of markets, moves or strategies may have specific physical meaning and payoffs vary smoothly with small changes in strategy. For example, the amount of profit a firm may make is usually a smooth function of its sales.

Thus, although it may be desirable to consider all games in some general enquiry,

⁹ “All” games would be generated from the open set $(-\infty, \infty)$, while we have generated the games from the closed set $[0, 1]$.

¹⁰ If we permitted ties, there would be over 700 strategically different games to consider. But the games with ties will form a set of measure zero. In applications, it is dangerous to ignore ties, especially if they appear to be a fact of life and there is some feature (such as crudeness of perception) which is generating them. But if we are considering behavior over all possible games, the ignoring of ties can be justified as a close approximation.

¹¹ In particular, in some applications one may want the two dimensional surface to be the surface of a sphere. In wargaming, if a flat surface is used, the best regular paving is hexagonal. Limiting ourselves to bimatrix games, in order to avoid boundary problems it may be useful to regard the matrix as a torus where the top row is identified as a neighbor of the bottom row, and the left column is identified as a neighbor of the right column. A game with two players and k units of space where the payoff to each is a function of the land occupied by the player and whether it is occupied alone or by both players gives rise to a $k \times k$ bimatrix game.

applications tend to be constrained to special subclasses.

8. TWO USEFUL BENCHMARKS

Prior to considering the dynamics of the various players, we establish lower and upper bounds on the expected payoffs of players in the special case of the 2×2 bimatrix game where each of the eight payoffs is an iid random variable, uniformly distributed on the interval $[0, 1]$. We first consider two agents who play randomly, and then two who play completely cooperatively.

RANDOM PLAYERS: Recall that a “random” player is one who plays each of his pure strategies with equal probability. Hence, if both players play randomly every outcome is equiprobable. Since the expected value to each in a randomly selected cell is .5, the payoff from random play by all is (.5, .5).

JOINT MAXIMIZERS: To evaluate the yield from complete cooperation, we assume the players move so as to maximize the sum of their payoffs.¹² Consider a 2 by 2 matrix where the entry in cell i is a random variable X_i , ($i = 1, 2, 3, 4$), defined as the sum of two i.i.d. uniform random variables on $[0, 1]$. The payoff for either player (assuming they cooperate so as to achieve the maximum X_i and then split the payoff) is one half the random variable Y , where $Y = \max(X_1, X_2, X_3, X_4)$. The expected value of Y is calculated as follows:

1) It is easy to see that $Pr(X_i \leq x)$ is equal to 0 if $x < 0$, $\frac{x^2}{2}$ if $x \in [0, 1]$, $1 - \frac{(2-x)^2}{2}$ if $x \in [1, 2]$, and 1 if $x > 2$.

2) Hence we have

$$Pr(Y \leq x) = \begin{cases} 0, & \text{if } x < 0, \\ (\frac{x^2}{2})^4 = \frac{1}{16}x^8, & \text{if } x \in [0, 1], \\ (1 - \frac{(2-x)^2}{2})^4 = (-\frac{1}{2}x^2 + 2x - 1)^4 & \text{if } x \in [1, 2], \\ 1, & \text{if } x > 2. \end{cases}$$

¹² Hence, we are assuming *transferable utility*, i.e. that the players can transfer payoff from one to the other, and do so in a manner so as to equalize their payoffs. If we wish to consider the “nontransferable utility” case, we have a more complicated problem because there is not necessarily a unique optimal payoff in a game; instead there is in general a Pareto-optimal surface from which any point may be chosen.

3) Since $Y \geq 0$, its expectation is given by $\int_0^\infty (1 - Pr(Y \leq x))dx$ (Hoel-Port-Stone 1971, p. 193). This is equal to $\int_0^1 (1 - \frac{1}{16}x^8)dx + \int_1^2 (1 - (-\frac{1}{2}x^2 + 2x - 1)^4)dx$, which is approximately 1.424.

4) Each player expects to gain half of this, or .712.

Our results from the “random case” and the “joint maximizer” case imply that “rational” players should expect to receive a payoff of somewhere between .5 and .712. However, we should note that it is possible for the players to lower their expected payoffs below .5. In fact, it is feasible (but rather unlikely) that they could collaborate to achieve their *minimal* expected joint payoff. A calculation symmetric to the one presented above gives minimum expected payoffs of (.288, .288).

9. A COMPARISON OF PLAYERS

In this section we are concerned with comparing the expected performance of different players, playing a universe of games against a player of the same type. In all instances except for the “smart player”, we assume players are either not concerned with the type of “opponent” they face, or else they assume that opponent is like themselves. This assumption, in essence, indicates that the players do not learn from history.

In each instance it is important to be specific about the nature of the optimal response: does Player I or Player II move first, with information concerning the behavior of the other, or do they move simultaneously? When players of the same type are playing each other one can argue that if reaction is sequential, order does not matter as it will average out over all games.

9.1. Problems in Comparison

The noncooperative equilibrium solution offers no dynamics, whereas the bugs’ optimal response procedure does. It is necessary to take this into account when trying to compare the solutions. One way to do this is to average over any cycle into which bugs may settle. If they eventually “fall into” a PSNE, they are credited with its value.

In nondegenerate cases, an $n \times n$ bimatrix game can have as many as n PSNEs, and we conjecture at most $2^n - 1$ NEs in total (see Quint-Shubik, 1994a). Hence, as the theory of Nash equilibrium provides for existence, but no selection,¹³ we have several choices as to how to assign payoffs to “Nash players”. If PSNEs exist, we can rule out all MSNEs and treat all PSNEs as equiprobable, or we can treat all equilibria as equiprobable. If (as some do) we have some reservations about MSNEs, we could give them a lesser weight than PSNEs. For our comparisons we treat all PSNEs as equiprobable and all MSNEs as equiprobable, but the probability of a PSNE is not necessarily the same as the probability of a MSNE. We then consider the weighted average of the associated payoffs.

In the next section, we propose a dynamic process which produces the weighted average of NE payoffs outlined above.

9.2. Two Players of One Type vs. Two Players of Another Type

Consider a bimatrix game, in which player I is the row player and player II the column player, and where each player has two strategies. Hence the game has the following form:

$$\begin{pmatrix} (a, b) & (c, d) \\ (e, f) & (g, h) \end{pmatrix}$$

The game is to be played repeatedly. We consider five types of player:

1) A NASH PLAYER – this player always plays a (possibly mixed) strategy which is “his part” of a NE. If there are three NEs (=2 pure and 1 mixed) in the game, then this player plays each pure with probability $\frac{\alpha}{2}$ and the mix with probability $1 - \alpha$ ($\alpha \in [0, 1]$).¹⁴ If in this case he plays a particular NE and his opponent matches his expectations and plays the “other half” of it, the NE player responds to this by playing according to that NE forever. Otherwise, he re-randomizes his NE strategies on the next iteration.

¹³ Unless one wishes to follow the tracing procedure given by Harsanyi and Selten (1988). We do not do so here, but it might be of interest to evaluate the expected payoff from the equilibrium selected by tracing.

¹⁴ Hence, $\alpha = \frac{2}{3}$ represents the case where all three NE strategies are played with equal probability; $\alpha = 1$ represents the case where the PSNE strategies are each played with probability $\frac{1}{2}$ and the MSNE strategies are disregarded.

2) A SEQUENTIAL BUG PLAYER – as described in Section 5.

3) A SIMULTANEOUS BUG PLAYER – as described in Section 5.

We assume that the “initial strategies” (see Section 5) place the bugs into one of the four cells, each with probability $\frac{1}{4}$. In addition, for the sequential bugs, we assume either player “goes first” with probability $\frac{1}{2}$.

4) A MAXIMIN PLAYER – this player always plays his maximin strategy.

5) A RANDOM PLAYER – this player always plays the mixed strategy $(\frac{1}{2}, \frac{1}{2})$.

Example 9.2.1: Consider the bimatrix game

$$\begin{pmatrix} (6, 2) & (1, 1) \\ (5, 3) & (2, 6) \end{pmatrix},$$

and let us assume the parameter α is equal to $\frac{1}{2}$.

Now, if two random players were to play this game, all of the outcomes would occur with probability $\frac{1}{4}$, and so the expected payoffs would be $(\frac{7}{2}, 3)$. In addition, we note that the maximin strategy for I is to play his second pure strategy, while that for II is to play his first, so that the payoff for maximin players is $(5, 3)$.

Next, let us do the analysis for the case of two sequential bugs playing the game. In this case, all is dependent on the initial conditions. If the game “starts” in the upper left cell or the bottom right cell, neither bug will ever move, so the payoffs are in fact those in the cells. If the game begins in the upper right cell with bug player I to move, he will play his second pure strategy, giving payoff $(2, 6)$, and then neither bug has incentive to move. Similarly, if the game were to start in the lower left cell, I would play his first pure strategy, and no further moves would be made. A similar analysis would hold if player II were to make the first move. All in all, there is a 50% chance that payoffs $(6, 2)$ would be obtained forever, and a 50% chance for $(2, 6)$. Expected payoff: $(4, 4)$.

The simultaneous bug analysis is the same in the cases where the game begins in the upper left or bottom right cell. However, if the game were to begin in either of the other two cells, *both* players would then move, and the result would be endless cycling between

the outcomes (1, 1) and (5, 3). Hence, the expected payoffs for the simultaneous bugs are $\frac{1}{4}(6, 2) + \frac{1}{4}(2, 6) + \frac{1}{2} \frac{(1,1)+(5,3)}{2}$, which is equal to $(\frac{7}{2}, 3)$.

Finally, for the NE player-NE player analysis, we note that there are two PSNEs in the game, paying off (6, 2) and (2, 6). In addition, there is a MSNE which pays off $(\frac{7}{2}, \frac{9}{4})$. Since $\alpha = \frac{1}{2}$, this means that when two NE players play this game, the chances are $\frac{1}{16}$ that they both will play their first pure strategy, $\frac{1}{16}$ that they will both play their second pure strategy, and $\frac{1}{4}$ that they will both play their MSNE strategy. [The other $\frac{5}{8}$ of the time, they will be playing NE strategies that don't "match".] This 1:1:4 ratio means that in the long run, they will attain either PSNE payoff with probability $\frac{1}{6}$, and the MSNE payoff with probability $\frac{2}{3}$, giving expected longrun payoff $(\frac{22}{6}, \frac{17}{6})$.

Comparing the payoffs for Player I, we see that the maximin game is best for him, followed by the sequential bugs game, the NE-NE game, and finally the simultaneous bugs and random players game are tied. For II, we see that the sequential bugs game is best, the NE-NE game is worst, and the other three games are all tied.

To compare how the various types of players fare *over a suitable universe of games*, let us assume that the eight payoffs a, \dots, h are each iid $U[0,1]$ random variables. Define Ω as the set of all joint outcomes (a, \dots, h) . Let μ be the probability measure defined on the set of subsets of Ω , defined in the natural way with $\mu(\Omega) = 1$.

Our aim is to calculate the expected payoff for each type of player over the above universe (Ω). To this end, we will need the following simple lemmas:

Lemma 9.1: Let Ω and μ be defined as above. Let N_0, N_1, N_2 be the subsets of Ω consisting of those games with 0, 1, and 2 PSNEs respectively. Then $\mu(N_0) = \frac{1}{8}$, $\mu(N_1) = \frac{3}{4}$, and $\mu(N_2) = \frac{1}{8}$.

Lemma 9.2: Let Ω and μ be defined as above. Let MM_g^1 be the subset of Ω in which Player I has a saddlepoint at a, c, e , or g (i.e., $e < a < g, g < c < a, a < e < g$, or $c < g < e$ respectively). Similarly, let MM_g^2 be the subset of Ω in which Player II has a saddlepoint

at b, d, f , or h (i.e., $b < d < h$, $d < b < f$, $h < f < b$, or $f < h < d$ respectively).¹⁵ Then $\mu(MM_S^1) = \mu(MM_S^2) = \frac{2}{3}$.

9.2.1 Joint Maximizers

If two “joint maximizers” are playing the game, the calculation presented in Section 8 says that the expected payoff for each player is approximately .712.

9.2.2 Random Players

If two “random players” are playing the game, the calculation presented in Section 8 says that the expected payoff for each player is exactly .5.

9.2.3 Sequential Bugs

To find the expected payoff to a pair of sequential bugs playing the game, let us consider the (expected) payoffs h^1 for Player I. [By symmetry, the expected payoff for II will be the same.] We consider in turn games in N_0 , N_1 , and N_2 respectively.

Given a game is in N_0 , WLOG we may assume $a > e$, $d > b$, $g > c$, and $d > h$. In this case the bugs will cycle around the four outcomes, giving an average payoff of $\frac{a+c+e+g}{4}$ to Player I. Hence the expected payoff, given the game is an element of N_0 , is $E[\frac{a+c+e+g}{4} | a > e, g > c]$. By symmetry, this is .5.

Given a game is in N_1 , Corollary 6.4 implies that the sequential bugs attain the PSNE payoffs. WLOG assume (a, b) is the PSNE payoff; hence $a > e$ and $b > d$. In addition, either $c > g$ or $f > h$. Either way, the expected payoff to I is $E[a | a > e]$, which turns out to be $\frac{2}{3}$.

Finally, consider the case where the game is in N_2 . WLOG, $a > e$, $b > d$, $g > c$, and $h > f$. Since we assume a) the bugs “start” in any of the four cells with equal probability, and b) either player moves first with probability .5, symmetry dictates that the bugs are equally likely to end up in either PSNE. Hence the expected payoff for Player

¹⁵ The significance here is that if a player has a saddlepoint, his maximin strategy will be to play the pure strategy corresponding to that saddlepoint.

I is $E[\frac{a+g}{2} | a > e, g > c] = \frac{2}{3}$.

Putting all of this together gives that $E(h^1) = \sum_{k=0}^2 E(h^1 | \text{game in } N_k) \mu(N_k) = \frac{1}{2} \frac{1}{8} + \frac{2}{3} \frac{3}{4} + \frac{2}{3} \frac{1}{8} = \frac{31}{48} = .646$.

9.2.4. Simultaneous Bugs

For simultaneous bugs, we use a similar procedure.

For games in N_0 , it is easy to see that a pair of simultaneous bugs will act exactly as do a pair of sequential bugs, and hence the expected payoff for Player I is again $\frac{1}{2}$. For games in N_1 , one can see that the only labelling cycle (see Section 6) will contain precisely one row and one column, and the row and column will be the ones defining the PSNE. Hence, we see that the bugs must end up at the PSNE. The analysis of the previous section then gives expected value $\frac{2}{3}$ for this case.

Finally, for games in N_2 , we see that with probability .25 the bugs end up in one NE, with probability .25 they end up in the other, and with probability .5 they end up cycling between the two other cells. Hence, if we assume (WLOG) that $a > e, g > c, b > d$, and $h > f$, the expectation in this case is $E[.25a + .25g + .5(\frac{e+e}{2}) | a > e, g > c] = E[\frac{1}{4}(a + c + e + g) | a > e, g > c] = \frac{1}{2}$.

Putting this all together, we have $E(h^1) = \frac{1}{8} \frac{1}{2} + \frac{3}{4} \frac{2}{3} + \frac{1}{8} \frac{1}{2} = \frac{5}{8} = .625$.

9.2.5 Nash Players

Let us again use the same technique.

For the analysis of games in N_0 , let us assume WLOG that $a > e, d > b, g > c$, and $f > h$. The unique MSNE in this case is for Player I to play $p = (\frac{f-h}{f-h+d-b}, \frac{d-b}{f-h+d-b})$ and for Player II to play $q = (\frac{g-c}{a-e+g-c}, \frac{a-e}{a-e+g-c})$. Given a, b, c, d, e, f, g , and h , the expected payoff for Player I is then $pqa + p(1-q)c + (1-p)qe + (1-p)(1-q)g$, which turns out to be $\frac{ag-ce}{a-e+g-c}$. To find $E(h^1)$ over all games in N_0 , we evaluate $E[\frac{ag-ce}{a-e+g-c} | a > e, g > c]$, which turns out to be $\frac{1}{2}$.¹⁶

¹⁶ This is a rather complicated quadrupal integral, which we evaluated both by long-hand

For games in N_1 , it is clear that the payoffs for the players are merely those of the PSNE. Since either type of bug attains the same result, we can use the results from the previous sections to state that the expected payoff in this case is $\frac{2}{3}$.

For games in N_2 , assume WLOG that $a > e$, $b > d$, $g > c$, and $h > f$. We note that the probability is $\frac{\beta}{2}$ that either of the two PSNEs will occur, and $1 - \beta$ for the MSNE, where $\beta = \frac{\alpha^2}{2 - 4\alpha + 3\alpha^2}$.¹⁷ Hence, Player I receives payoff a with probability $\frac{\beta}{2}$, g with probability $\frac{\beta}{2}$, and $\frac{ag - ce}{a - e + g - c}$ (same analysis as in the N_0 case) with probability $1 - \beta$. In expectation this gives him $\frac{\beta}{2} \frac{2}{3} + \frac{\beta}{2} \frac{2}{3} + (1 - \beta) \frac{1}{2}$, which is equal to $\frac{1}{2} + \frac{\beta}{6}$.

Putting this all together, we get $E(h^1) = \frac{1}{8} \frac{1}{2} + \frac{3}{4} \frac{2}{3} + \frac{1}{8} (\frac{1}{2} + \frac{\beta}{6})$, which is equal to $\frac{5}{8} + \frac{\beta}{48}$. Hence the Nash players' expected payoffs range between those for the simultaneous bugs ($\frac{5}{8}$ when $\beta = 0$) and those for the sequential bugs ($\frac{31}{48}$ when $\beta = 1$).

9.2.6 Maximin Players

To analyze the case of maxjmin players, we will use the same general procedure as above, except, instead of "partitioning"¹⁸ Ω into the three sets N_0 , N_1 , and N_2 , we will break it up into the following four sets:

- 1) Games where both players have a saddlepoint, and Player II's saddlepoint is in the same column as Player I's. Using Lemma 9.2, the measure of this set is $\frac{2}{9}$.
- 2) Games where both players have a saddlepoint, but Player II's saddlepoint is not in the same column as Player I's ($\mu = \frac{2}{9}$).
- 3) Games where Player I has a saddlepoint but Player II does not ($\mu = \frac{2}{9}$).
- 4) Games where Player I has no saddlepoint ($\mu = \frac{1}{3}$).

The Analysis.

- 1) Suppose in this case WLOG that Player I's saddlepoint is a (so $e < a < c$) and so

and by using Mathematica.

¹⁷ In Example 9.2.1, where $\alpha = \frac{1}{2}$, β was equal to $\frac{1}{3}$.

¹⁸ We did not formally partition Ω , because there is a set of measure zero consisting of games with an infinite number of NEs. The same comment applies to the maximin analysis, where we disregard cases where players have more than one maximin strategy.

Player II's is in the first column. Then I's maximin payoff is a , and his expected payoff in this case is $E[a|e < a < c] = \frac{1}{2}$.

2) Suppose in this case WLOG that Player I's saddlepoint is a (so $e < a < c$) and so Player II's is in the second column. Then I's maximin payoff is c , and his expected payoff in this case is $E[c|e < a < c] = 6 \int_0^1 \int_e^1 \int_a^1 cdcdade = \frac{3}{4}$.

3) In this case we again assume I's saddlepoint is a , and that $h > b > d > f$. Then I plays $p = (1, 0)$, and II plays $q = (\frac{h-d}{h-d+b-f}, \frac{b-f}{h-d+b-f})$. Hence I's expected payoff is $\frac{h-d}{h-d+b-f}a + \frac{b-f}{h-d+b-f}c$. Since a, c, e , and g are by assumption independent of b, d, f , and h , we get that $E(h^1)$ in this case is equal to $E[a|e < a < c]E[\frac{h-d}{h-d+b-f}|h > b > d > f] + E[c|e < a < c]E[\frac{b-f}{h-d+b-f}]$. This is equal to $\frac{1}{2} \frac{1}{2} + \frac{3}{4} \frac{1}{2} = \frac{5}{8}$.

4) For this case, WLOG assume $a > g > c > e$. Then I's maximin strategy is $p = (\frac{g-e}{g-e+a-c}, \frac{a-c}{g-e+a-c})$. Suppose II's maximin strategy is $(x, 1-x)$. Then I's expected payoff is $\frac{g-e}{g-e+a-c}xa + \frac{g-e}{g-e+a-c}(1-x)c + \frac{a-c}{g-e+a-c}xe + \frac{a-c}{g-e+a-c}(1-x)g$, which is equal to $\frac{ag-ce}{g-e+a-c}$. Hence, I's expected payoff in this case is $E[\frac{ag-ce}{g-e+a-c}|a > g > c > e]$, which is equal to $\frac{1}{2}$.¹⁹

Finally putting this all together, we find that Player I's expected payoff for the entire game is $\frac{2}{9} \frac{1}{2} + \frac{2}{9} \frac{3}{4} + \frac{2}{9} \frac{5}{8} + \frac{1}{3} \frac{1}{2} = \frac{7}{12}$, or approximately .583.

9.3. "Anti-Players"

We could consider malicious, "looking-glass" versions of our players. For instance, we might consider a (sequential or simultaneous) "anti-bug", which always moves to a row (or column) which minimizes his opponent's payoff (instead of maximizing his own). Or, we might consider an "anti-maximin player", whose *modus operandi* is to minimize his opponent's maximum payoff. Finally, we can define a "anti-Nash player" similarly. Given the generation of payoffs described in Section 9.2, the following Theorem is obvious:

Theorem 9.3: The payoffs to two Anti-Bug players (resp., two anti-maximin players,

¹⁹ Again we calculated this integral using both longhand and Mathematica.

two anti-Nash players) evaluated over all environments are one minus the payoffs to two Bug players (resp., two maximin players, two Nash players).

9.4. Modelling Intelligent Players who Learn

Another type of player is assumed to have accurate information as to what type of competitor or mechanism he is playing. Our Nash player, who learns nothing from experience assumes that he confronts another Nash player also looking for mutually consistent expectations. The informed player, if he knows that his opponent is an automaton or creature of habit merely following some set strategy, will maximize against it.

If both players are such “smart players”, and are able to communicate, then, in general, they may have an incentive to cooperate. Hence the “joint maximization” case would apply. If we rule out direct communication, the way the game is played by intelligent player is not fully specified unless further qualifications are made concerning learning and social inference which may limit the strategies used.

9.5 A Summary of Expected Payoffs for Different Types of Players

Putting together our results from subsections 9.2 and 9.3, we may state a “rank ordering” on all types. The ordering^{20,21} is as follows:

1. Cooperative joint maximization (.712)
2. Sequential bugs (.646)
3. Noncooperative players (between .625 and .646)
4. Simultaneous bugs (.625)
5. Maximin players (.583)
6. Random players (.5)
7. Anti-maximin players (.417)
8. Simultaneous Anti-bugs (.375)
9. Anti-Nash players (between .354 and .375)
10. Sequential Anti-bugs (.354)
11. Cooperative joint minimization (.288).

²⁰ In Quint-Shubik (1994b), the authors ranked the player types according to the criterion: “type x” ranks higher than “type y” if the measure of the subset of Ω in which the “type x” player I does better than the “type y” Player I is larger than vice versa. Not only did this produce a complete ordering of the player types, but the ordering is exactly the same as we obtain here.

²¹ In section 9.2 our concern was with the performance of players facing an opponent of the same type. We have not covered the “mixed cases”, where a player of one type confronts a player of a different type.

10. CONCLUDING REMARKS

Neither equilibrium, nor consistent expectations are necessarily desiderata for a solution to a game. To some extent the very successes of game theory indicates its weakness. We know how to describe in detail the combinatorics required for a game such as chess or Go, but in doing so we know that humans do not play by looking at anywhere near the full strategy set available.

Cooperative game theory assumes away the question of societal rationality by assuming that intelligent individuals communicate directly and jointly optimize but bargain over splitting the proceeds. Game theory based on the strategic or extensive form does not make these assumptions.

The fashion in the last twenty to thirty years for the noncooperative equilibrium solution applied in its several variants to games in extensive or strategic form is in our opinion an unfortunate result of confusing pure and applied mathematics. It is possible that there are some broad relevant general properties which hold for all human and animal conflict and cooperation at so high a degree of abstraction that context does not matter and the model of a personality-free, intelligence normalized "rational man" can be treated as an ideal rather than an approximation which is more or less relevant depending on the application. But the evidence is against it.

Attempts such as that of Harsanyi and Selten (1988) to justify a unique equilibrium point as the solution any game, or Aumann's study of common knowledge (1976) are fascinating exercises in rationalistic philosophy, but should not be confused with the general application of game theory to problems of conflict and cooperation among humans and other organisms.

"How dumb is dumb" is at least as good a question as "How bright is bright". The biological applications of game theory do not utilize models of players with overwhelming intelligence or memory.

Cooperative theory is explicitly normative and implicitly assumes the existence of

enforceable contract, hence a context of law. Noncooperative theory is, in general, not proposed as a normative theory. The theory has apparently been reasonably fruitful in applications to mass economic markets where face-to-face communication is expensive and the economic structure provides context; the biological applications have been suggestive, but use a different model of the individual decisionmaker. The sort of “conversational game theory” used by Schelling (1960) and others in international relations lumps whole nations into a single decisionmaker but with enough context supplied provides interesting insights. Some problems in law, contracts, incentive systems and insurance can be modeled in an *ad hoc* manner where the context of the operations research application is made explicit and to some extent some variant of noncooperative equilibrium theory may offer some insights into applied problems. But further developments in game theory to develop dynamics may need to deal with limited recall, limited perception and differences in various abilities which, to date have been difficult to model and analyze.

No generally accepted, satisfactory theory of game inference and learning exists.²² At this time perhaps the two areas of application where there may be some hope to start to obtain some results of interest are mass economic markets where the *homo oeconomicus* approximation is not too bad, and simple biological mechanisms where the model of the decision unit is clearly dumb, but in some sense purposeful.

²² Even for simple organisms, there appears to be some *primitive* form of learning going on. In our model, except for the intelligent players, we assumed that all others select their mode of operation and stick to it. Hence, for example, when a Nash player plays against a bug, instead of learning that it is a bug, the player maintains the belief that it might be another Nash player.

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